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Fresnel formulas and the principle of causality

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Contents

Abstract. A method is considered whereby Fresnel formulas for transparent, absorbing, and amplifying linear media are uniquely determined by passing to the plane-monochromatic-wave limit of an electromagnetic pulse, using the principle of causality and employing the analytical properties of the amplitude reflection coefficient of the Fourier components of the light pulse.

"*It is not uncommon that the shortest path between two truths in the real domain passes through the complex domain.*"

J Hadamard

1. Introduction

The Fresnel formulas for plane electromagnetic waves were first obtained in 1823 and underlie the Fourier technique in solving the problem of the reflection of electromagnetic radiation from a flat boundary surface between two linear homogeneous dielectric media. The modern-day derivation of these formulas invokes two boundary conditions which express the continuity of the tangential components of the vectors of electric and magnetic radiation field strengths at the interface between the media. More often than not, the derivation is extremely simplified and reduced to the formal solution of a system of two algebraic equations [1, 2]. However, it is common knowledge that these boundary conditions are inadequate to uniquely determine the Fresnel formulas and therefore should be complemented with selection rules for the refracted wave. This rule is typically reduced to the requirement that the resultant refracted wave should be limited in amplitude and transfer energy away from the boundary surface between the media [3].

Experiments on the reflection of light from amplifying media have set the task of generalizing the Fresnel formulas to nonequilibrium reflecting media. Both the excitation of waves with an exponentially growing amplitude and the formation of an energy flux towards the media interface are possible in an amplifying medium. That is why the usual selection rule for the refracted wave does not apply in this case. Different attempts to obtain the Fresnel formulas for an amplifying medium starting from new selection rules have led to inconsistent results [4 – 10].

One way to overcome these difficulties is to solve the boundary problem employing a more adequate model of incident radiation in the form of a pulse with an amplitude leading edge [7 – 9]. For linear media without spatial dispersion, the boundary conditions complemented with the causality principle in terms of requirements on the direction of propagation of the amplitude leading edges of reflected and refracted radiation relative to the media interface make it possible to obtain a unique solution.

However, the uniqueness of definition of the amplitude reflection coefficient for an individual Fourier component of the pulse, i.e. for a plane monochromatic wave, in no way follows from the uniqueness of solution of the boundary problem for a pulse. The point is that the causality principle prescribes only the asymptotic behavior of the amplitude reflection coefficient in the domain of infinitely high frequencies which determine the velocity of the amplitude leading edge. That is why the reflected radiation can be described employing a continuum of equivalent Fourier representations, in which one and the same Fourier component appearing in different Fourier representations is characterized by various reflection coefficients.

As shown below, the local amplitude reflection coefficient equal to the ratio between the tangential field components of the reflected and incident pulses at a given point of the media interface at a fixed instant of time, is uniquely determined. For transparent and absorbing reflecting media, the local amplitude reflection coefficient for a quasi-monochromatic square pulse approaches a limit equal to the Fresnel reflection coefficient for plane monochromatic waves as the pulse

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length and width tend to infinity. In this case, the reflected pulse of finite length and width can always be written through the use of at least two Fourier representations, for which its central Fourier component has various reflection coefficients.

Therefore, the Fresnel formulas determine the limiting values of the local amplitude reflection coefficient rather than the 'correct' values of the amplitude reflection coefficient for plane monochromatic waves. The formulas are obtained in the solution of a different boundary problem invoking the causality principle. This approach is based on obtaining a unique solution of the boundary problem on the incident pulse with an amplitude leading edge and on effecting the passage from a pulse to the limit of a plane monochromatic wave. The approach makes it possible to uniquely determine the Fresnel formulas for linear media from a unified standpoint and eliminate the contradictions which emerged in the description of the light reflection from amplifying media.

2. Lack of uniqueness in the solution of the boundary problem for a plane monochromatic wave

Let there be a plane interface between two linear homogeneous and transparent media void of spatial dispersion, whose permeabilities are taken to be unity. The coordinate x- and y-axes are selected so as to be parallel, and the coordinate z-axis perpendicular to the interface. A plane monochromatic wave with frequency ω polarized perpendicular to the plane of incidence $x0z$ travels from a medium with permittivity $\varepsilon_1 > 0$, which occupies the half-space $z < 0$. The electric field $E(x, z, t)$ of the incident wave is aligned with the y-axis and is of the form

$$E(x, z, t) = \mathcal{E} \exp\left[\mathrm{i}(k_x x + k_{1z} z - \omega t)\right] , \tag{2.1}$$

where $\mathcal{E}$ is a constant complex amplitude, k_x and k_{1z} are the projections of the wave vector on the x- and z-axes, respectively, t is time, and $\mathrm{i} = \sqrt{-1}$.

According to the boundary conditions at the $z = 0$ interface between the media, the electric fields $E_{\mathrm{r}}(x, z, t)$ of the reflected and $E_{\mathrm{tr}}(x, z, t)$ of the transmitted waves are also aligned with the y-axis and are written as follows

$$E_{\mathrm{r}}(x, z, t) = \mathcal{E}_{\mathrm{r}} \exp\left[\mathrm{i}(k_x x - k_{1z} z - \omega t)\right] , \tag{2.2}$$

$$E_{\mathrm{tr}}(x, z, t) = \mathcal{E}_{\mathrm{tr}} \exp\left[\mathrm{i}(k_x x - \omega t)\right] , \tag{2.3}$$

where $\mathcal{E}_{\mathrm{r}}$ is a constant complex amplitude, and the $\mathcal{E}_{\mathrm{tr}}(z)$ function is a solution of the equation

$$\frac{\mathrm{d}^2 \mathcal{E}_{\mathrm{tr}}}{\mathrm{d}z^2} + \left(\varepsilon_2 \frac{\omega^2}{c^2} - k_x^2\right) \mathcal{E}_{\mathrm{tr}} = 0 . \tag{2.4}$$

Here, $\varepsilon_2 > 0$ is the permittivity of the reflecting medium, and c is the speed of light in vacuum.

The general solution of Eqn (2.4) can be written as

$$\mathcal{E}_{\mathrm{tr}}(z) = C_1 \exp(\mathrm{i}k_{2z} z) + C_2 \exp(-\mathrm{i}k_{2z} z) , \tag{2.5}$$

where $k_{2z}(k_x, \omega) = \sqrt{\varepsilon_2 \omega^2 / c^2 - k_x^2}$, and the complex constants C_1 and C_2 should satisfy the relationship

$$2\mathrm{i}k_{1z}\mathcal{E} = \mathrm{i}k_{1z}\mathcal{E}_{\mathrm{tr}} + \frac{\mathrm{d}\mathcal{E}_{\mathrm{tr}}}{\mathrm{d}z} , \qquad z = 0 , \tag{2.6}$$

which was obtained by eliminating $\mathcal{E}_{\mathrm{r}}$ from the boundary conditions.

Relationship (2.6) does not permit the constants C_1 and C_2 to be determined uniquely, and therefore the amplitude reflection coefficient of a plane monochromatic wave, namely

$$R_{\perp}(k_x, \omega) = \frac{E_{\mathrm{r}}(x, z = 0, t)}{E(x, z = 0, t)} = \frac{\mathcal{E}_{\mathrm{r}}}{\mathcal{E}} , \tag{2.7}$$

can assume a continuum of values, depending on the arbitrary choice of one of the complex constants. In particular, for $C_2 = 0$, when the refracted wave propagates away from the media interface, one obtains

$$R_{\perp}(k_x, \omega) = R_{\perp +}(k_x, \omega) = \frac{k_{1z} - k_{2z}}{k_{1z} + k_{2z}} . \tag{2.8}$$

For $C_1 = 0$, when the refracted wave propagates towards the media interface, we get

$$R_{\perp}(k_x, \omega) = R_{\perp -}(k_x, \omega) = \frac{k_{1z} + k_{2z}}{k_{1z} - k_{2z}} = \frac{1}{R_{\perp +}(k_x, \omega)} . \tag{2.9}$$

Consider the transmission range of the reflecting medium, where $k_x^2 < \varepsilon_2 \omega^2 / c^2$ and k_{2z} is a real positive quantity. As is generally accepted, in this range that solution which describes the energy transfer away from the media interface is realized. According to this selection rule for the refracted wave, we get $C_2 = 0$ and $R_{\perp}(k_x, \omega) = R_{\perp +}(k_x, \omega)$ [11].

With this formulation of the selection rule for the refracted wave, essential use is made of the general solution in the form of expression (2.5). A more universal formulation of the selection rule, not related to the specific form of representation of the general solution, invokes two integrals of Eqn (2.4) [12]:

$$J_1 = \mathrm{i}\left(\mathcal{E}_{\mathrm{tr}} \frac{\mathrm{d}\mathcal{E}_{\mathrm{tr}}^*}{\mathrm{d}z} - \mathcal{E}_{\mathrm{tr}}^* \frac{\mathrm{d}\mathcal{E}_{\mathrm{tr}}}{\mathrm{d}z}\right) ,$$

$$J_2 = \left|\frac{\mathrm{d}\mathcal{E}_{\mathrm{tr}}}{\mathrm{d}z}\right|^2 + \left(\varepsilon_2 \frac{\omega^2}{c^2} - k_x^2\right) |\mathcal{E}_{\mathrm{tr}}|^2 , \tag{2.10}$$

where '$*$' signifies complex conjugate quantity. The J_1 quantity describes the energy flux density transferred by the refracted wave along the z-axis, and the J_2 integral is related to the conservation of the flux density of the zth component of the momentum transferred by the wave along the z-axis.

As is readily shown employing expression (2.5), the J_1/J_2 ratio for the refracted wave to be realized ($C_2 = 0$) assumes a maximum value equal to $1/k_{2z}$. This result leads to a new formulation of the selection rule: the ratio of integrals J_1/J_2 for the refracted wave attains its maximum value. The formulation allows generalization to the case of a nonlinear reflecting medium [12].

In the nonpropagation domain of the reflecting medium, where $k_x^2 > \varepsilon_2 \omega^2 / c^2$ and $k_{2z}(k_x, \omega) = \mathrm{i}\left|k_{2z}(k_x, \omega)\right|$, the selection rule consists in the requirement that the amplitude of the refracted wave should be limited for all $z > 0$. Hence it directly follows that $C_2 = 0$, and the unique solution of the boundary problem is thus determined.

If the absorption of the reflecting medium is taken into account, when $\varepsilon_2 = \varepsilon_2' + \mathrm{i}\varepsilon_2''$, $\varepsilon_2'' > 0$ and $k_{2z} = k_{2z}' + \mathrm{i}k_{2z}''$, $k_{2z}' > 0$, $k_{2z}'' > 0$, the requirement that the energy is transferred away from the media interface leads to the necessity of putting $C_2 = 0$ for all real k_x. The corresponding solutions for a transparent reflecting medium can be obtained in the passage to the limit, when $\varepsilon_2'' \to 0$.

All the results outlined above pertain equally to a wave polarized parallel to the plane of incidence, when the amplitude reflection coefficient is determined by the ratio between the tangential components of the vectors of magnetic field strengths for the reflected and incident waves.

3. Causality principle for the electromagnetic radiation at the interface between two media

Plane monochromatic waves are nonexistent in nature and are invoked only as the Fourier components in the solution of linear problems by the Fourier technique. In this connection the requirement of uniqueness on the solution of the boundary problem for plane monochromatic waves is not exactly correct. The more so as the properties of individual Fourier components are not necessarily identical to the properties of the pulse as a whole concerning the amplitude behavior and the energy transfer [13]. Only when a pulse is incident whose field at the media interface is nonzero for $0 < t < \tau$ and $|x| < \sigma$, where τ is the pulse length and 2σ is the pulse width, can the boundary conditions be supplemented with the causality principle and a unique solution be obtained.

For a pulse polarized perpendicular to the plane of incidence, the electric field $E_{\mathrm{r}}(x,t)$ of the reflected radiation for $z = 0$, according to the Fourier method, assumes the form

$$E_{\mathrm{r}}(x,t) = \frac{1}{(2\pi)^2}\iint_{\Gamma(k_x,\omega)} R_\perp(k_x,\omega)\tilde{E}(k_x,\omega) \times \exp\left[\mathrm{i}(k_x x - \omega t)\right] \mathrm{d}k_x\,\mathrm{d}\omega = \iint_{-\infty}^{\infty} \mathcal{G}_{\mathrm{R}\perp}(x',t')E(x-x',t-t')\,\mathrm{d}x'\,\mathrm{d}t'\,, \quad (3.1)$$

where

$$\mathcal{G}_{\mathrm{R}\perp}(x',t') = \frac{1}{(2\pi)^2}\iint_{\Gamma(k_x,\omega)} R_\perp(k_x,\omega) \times \exp\left[\mathrm{i}(k_x x' - \omega t')\right] \mathrm{d}k_x\,\mathrm{d}\omega \quad (3.2)$$

is the surface Green function for the reflected radiation, $\tilde{E}(k_x,\omega)$ is the Fourier transform of the field $E(x,t)$ of the incident light pulse at $z = 0$, and $\Gamma(k_x,\omega)$ is the surface of integration in the space of complex Fourier variables $k_x = k_x' + \mathrm{i}k_x''$ and $\omega = \omega' + \mathrm{i}\omega''$.

In Eqns (3.1) and (3.2), the amplitude reflection coefficient $R_\perp(k_x,\omega)$ is as yet undefined. Its unambiguous definition is effected with the use of the causality principle. In the space of Fourier variables, the causality principle is realized as a succession of operations in the theory of functions of complex variables, which includes adoption of the rule for detour around singular points, making cuts, and selection of the Riemann surface sheets. On accomplishing these operations, $R_\perp(k_x,\omega)$ becomes a regular function of the spatial k_x and time ω frequencies.

According to the causality principle, $E_{\mathrm{r}}(x,t) = 0$ for all $t < 0$, and therefore the line of intersection of the $\Gamma(k_x,\omega)$ surface with the complex ω surface should lie above all the singular points $R_\perp(0,\omega)$. If this line is adopted as the straight line $\omega = \omega' + \mathrm{i}\omega_\Gamma''$ ($\omega_\Gamma'' = \mathrm{const}$) parallel to the axis of real ω values, then the $\Gamma(k_x,\omega)$ surface will lie in the 3D space of variables $\omega' + \mathrm{i}\omega_\Gamma''$, k_x', and k_x''.

For every frequency $\omega' + \mathrm{i}\omega_\Gamma''$, the $R_\perp(k_x,\omega)$ function has four branch points in the complex plane k_x, which are solutions of the equations $k_{1z}(k_x,\omega) = 0$ and $k_{2z}(k_x,\omega) = 0$:

$$k_{\mathrm{br}1,2}(\omega) = \pm\sqrt{\varepsilon_1(\omega)}\,\frac{\omega}{c} = \pm k_1(\omega)\,,$$
$$k_{\mathrm{br}3,4}(\omega) = \pm\sqrt{\varepsilon_2(\omega)}\,\frac{\omega}{c} = \pm k_2(\omega)\,. \quad (3.3)$$

The branch points (3.3) define the asymptotic behavior of the reflected monochromatic beam with frequency ω in the $|x| \gg \sigma$ domain, where four side waves are formed [14, 16].

For all media $\varepsilon(\omega) \to 1 - \Omega^2/\omega^2$ as $|\omega| \to \infty$, where Ω is the electron plasma frequency of the medium. Therefore the limiting phase velocities of the side waves

$$v_{1,3}(\infty) = \lim_{|\omega'|\to\infty} \frac{\omega}{k_{\mathrm{br}1,3}} = c\,,$$
$$v_{2,4}(\infty) = \lim_{|\omega'|\to\infty} \frac{\omega}{k_{\mathrm{br}2,4}} = -c \quad (3.4)$$

coincide with the velocities of the amplitude leading edges of these waves along the media interface.

In order that the amplitude fronts of the side waves propagate, in accordance with the causality principle, away from the region of the light pulse incidence, for $|\omega'| \to \infty$ it is necessary to detour around the $k_{\mathrm{br}1,3}$ branch points in the complex k_x plane from below, and the $k_{\mathrm{br}2,4}$ branch points from above. The $\Gamma(k_x,\omega)$ surface is nowhere intersected by the curves of the branch points, and therefore this asymptotic rule determines the detour around the branch points for all integration frequencies.

To determine $R_\perp(k_x,\omega)$ uniquely, cuts should be made to connect the branch points in pairs or to connect them to infinitely distant points. The detour rule adopted above can be satisfied if the cuts connect the points $k_{\mathrm{br}1}$ and $k_{\mathrm{br}3}$ in pairs, along with the points $k_{\mathrm{br}2}$ and $k_{\mathrm{br}4}$. Moreover, cuts may be drawn to connect the points $k_{\mathrm{br}1}$ and $k_{\mathrm{br}3}$ to infinitely distant points in the $k_x'' > 0$ domain, and the points $k_{\mathrm{br}2}$ and $k_{\mathrm{br}4}$ to infinitely distant points in the $k_x'' < 0$ domain.

Upon making cuts, one of the two sheets of the Riemann surface of the two-valued function $k_{2z}(k_x,\omega)/k_{1z}(k_x,\omega)$ should be chosen (the cuts connect the branch points in pairs). Optionally, one of the two sheets of the Riemann surfaces of the two-valued functions $k_{1z}(k_x,\omega)$ and $k_{2z}(k_x,\omega)$ should be chosen individually (the cuts connect the branch points to infinitely distant points).

In the former case, on one sheet of the Riemann surface of the $k_{2z}(k_x,\omega)/k_{1z}(k_x,\omega)$ function one obtains

$$\lim_{|\omega'|\to\infty} \frac{k_{2z}(k_x,\omega)}{k_{1z}(k_x,\omega)} = 1\,, \qquad \lim_{|\omega'|\to\infty} R_\perp(k_x,\omega) = 0\,, \quad (3.5)$$

and on the other sheet

$$\lim_{|\omega'|\to\infty} \frac{k_{2z}(k_x,\omega)}{k_{1z}(k_x,\omega)} = -1\,, \qquad \lim_{|\omega'|\to\infty} R_\perp(k_x,\omega) = \infty\,. \quad (3.6)$$

In accordance with the causality principle, the amplitude leading edges of the incident and refracted radiation should propagate in one direction along the z-axis, and therefore preference should be given to that sheet of the Riemann surface where the limits (3.5) hold.

In the latter case, when the cuts connect the branch points to infinitely distant points, the decision between the sheets of the Riemann surface for $k_{1z}(k_x, \omega)$ and $k_{2z}(k_x, \omega)$ is based on the requirement that the amplitude leading edges of the incident and refracted radiation should travel in the positive direction relative to the z-axis:

$$v_{1z}(\infty) = \lim_{|\omega'| \to \infty} \frac{\omega}{k_{1z}(\omega)} = v_{2z}(\infty) = \lim_{|\omega'| \to \infty} \frac{\omega}{k_{2z}(\omega)} = c\,. \tag{3.7}$$

As a result, the limits (3.5) regain validity in the chosen sheets of the Riemann surfaces.

At any point of the $\Gamma(k_x, \omega)$ surface, the $R_\perp(k_x, \omega)$ quantity is determined by analytic continuation from the zero value in the $|\omega'| \to \infty$ domain to a given point along the chosen surface of integration. According to Eqns (3.5) and (3.7), the causality principle imposes limitations only on the asymptotics of the amplitude reflection coefficient of the Fourier components, and therefore the value of $R_\perp(k_x, \omega)$ can always be redefined at any finite point by deforming the cuts. Hence it follows that the complete set of Fourier components for the refracted radiation, which permits the use of an arbitrary Fourier representation, incorporates both independent solutions in expression (2.5). This conclusion applies equally to the Fourier representations of the incident and reflected radiation as regards the independent solutions of the corresponding wave equation.

According to the Cauchy theorem, local redefinitions of an $R_\perp(k_x, \omega)$ quantity of this type have no effect on the spacetime distribution of the reflected radiation field $E_{\mathrm{r}}(x, t)$, because all possible Fourier representations can be transformed into one another in the continuous deformation of the cuts and the surface of integration.

The case when the incident light pulse is polarized parallel to the plane of incidence and the amplitude reflection coefficient of a plane monochromatic wave is of the form [11]

$$R_\parallel(k_x, \omega) = \frac{\varepsilon_2 k_{1z} - \varepsilon_1 k_{2z}}{\varepsilon_2 k_{1z} + \varepsilon_1 k_{2z}} \tag{3.8}$$

is treated in an analogous way. The only distinction is associated with the Brewster effect, when function (3.8) at the points $k_x = \pm k_{\mathrm{B}} = \pm\sqrt{\varepsilon_1\varepsilon_2/(\varepsilon_1+\varepsilon_2)}\,\omega/c$ is zero in one sheet of the Riemann surface, and has a pole in the other one. If drawing the cuts results in the occurrence of a pole at the $k_x = \pm k_{\mathrm{B}}$ points, the contribution of this pole to the reflected radiation will be zero. During integration, this pole can be bypassed only an even number of times, the number of detours in one direction always being equal to the number of detours in the opposite one.

A consequence of the causality principle for the reflection of electromagnetic radiation from the boundary surface between two linear media is the fulfillment of the Kramers – Kronig relations for $R_{\perp,\parallel}(k_x, \omega) = R'_{\perp,\parallel} + \mathrm{i}R''_{\perp,\parallel}$ in the plane of real k_x and ω, which does not meet the cuts [15]. Once for real k_x the $R_{\perp,\parallel}(k_x, \omega)$ functions have no singular points in the $\omega'' > 0$ domain, for real ω it follows

$$R'_{\perp,\parallel}(k_x, \omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{R''_{\perp,\parallel}(k_x, u)}{u - \omega}\,\mathrm{d}u\,,$$
$$R''_{\perp,\parallel}(k_x, \omega) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{R'_{\perp,\parallel}(k_x, u)}{u - \omega}\,\mathrm{d}u\,, \tag{3.9}$$

where the integrals are taken in the sense of the principal value.

Relationships (3.9) can be fulfilled for transparent media, because $R''_{\perp,\parallel} \neq 0$ in the nonpropagation domain of a reflecting medium (total reflection). The Kramers – Kronig relations make it possible to obtain the sum rule for $R_{\perp,\parallel}(k_x, \omega)$, which is convenient for studying the structure of reflected radiation near its leading edge [15]. The integral nature of these relations emphasizes once again that the fulfillment of the causality principle is related to the analytic properties of the $R_{\perp,\parallel}(k_x, \omega)$ functions as a whole rather than to their definition at an individual point.

The surface Green function $\mathcal{G}_{R\perp,\parallel}(x, t)$ for the reflected radiation is uniquely determined and is independent of the choice of cuts. By displacing the surface of integration $\Gamma(k_x, \omega)$ to the domain where $\omega'' \to \infty$, $\varepsilon_1(\omega) \to 1$, and $\varepsilon_2(\omega) \to 1$ and performing the integration as was done in Refs [15, 16], it can be shown that

$$\mathcal{G}_{R\perp,\parallel}(x, t) = 0\,, \qquad \text{if} \qquad t < \frac{|x|}{c}\,. \tag{3.10}$$

In the passage through the points $x = \pm ct$ the value of $\mathcal{G}_{R\perp,\parallel}$ changes in jumps. Therefore, the amplitude leading edge of the surface Green function travels at the speed of light in vacuum c along the media interface from the point of incidence $x = 0$ of a delta-like light pulse.

4. Passage from a light pulse to the limit of a plane monochromatic wave

The causality principle, combined with the boundary conditions, uniquely determines the local amplitude reflection coefficient for a light pulse:

$$\rho_\perp(x, t; \sigma, \tau) = \frac{E_{\mathrm{r}}(x, z=0, t)}{E(x, z=0, t)}\,, \qquad 0 < t < \tau\,, \qquad |x| < \sigma\,. \tag{4.1}$$

To establish the relation between $\rho_\perp(x, t; \sigma, \tau)$ and $R_\perp(k_x, \omega)$, we consider the passage from a quasi-monochromatic light pulse with a square amplitude distribution

$$E(x, t) = \begin{cases} \mathcal{E}\exp\left[\mathrm{i}(k_{1x}x - \omega_1 t)\right], & 0 < t < \tau\,, \quad |x| < \sigma\,, \\ 0\,, & t < 0\,,\ t > \tau\,, \quad |x| > \sigma \end{cases} \tag{4.2}$$

($\mathcal{E}$ is the complex amplitude, $k_{1x} = \sin\theta\sqrt{\varepsilon_1}\,\omega_1/c$, θ is the angle of pulse incidence, and ω_1 is the frequency of the central Fourier component of the pulse) to the limit of a plane monochromatic wave, putting $\tau \to \infty$ and $\sigma \to \infty$.

According to expressions (3.2), (3.10), (4.1), and (4.2), we get

$$\rho_\perp(x, t; \sigma, \tau) = \int_{x-\sigma}^{x+\sigma}\int_0^t \mathcal{G}_{R\perp}(x', t') \times \exp\left[-\mathrm{i}(k_{1x}x' - \omega_1 t')\right]\mathrm{d}x'\,\mathrm{d}t'\,, \tag{4.3}$$

and therefore it follows from the properties of direct and inverse Fourier transformations that

$$\lim_{\tau\to\infty,\,\sigma\to\infty} \rho_\perp(x, t=\tau; \sigma, \tau) = \rho_{\perp\infty} = \int_{-\infty}^{+\infty}\int_0^\infty \mathcal{G}_{R\perp}(x', t') \times \exp\left[-\mathrm{i}(k_{1x}x' - \omega_1 t')\right]\mathrm{d}x'\,\mathrm{d}t' = R_\perp(k_{1x}, \omega_1)\,, \tag{4.4}$$

if the entire surface of integration $\Gamma(k_x, \omega)$ can be brought into coincidence with the plane of real values of k_x and ω, with the exception of the branch points.

The feasibility of this coincidence depends on the positions of the singular points $R_\perp(k_x, \omega)$ and the choice of cuts. In the case of absorbing media, where all the singular points of $\varepsilon_1(\omega)$ and $\varepsilon_2(\omega)$ lie in the $\omega'' < 0$ domain, in deciding on the cuts not intersecting the plane of real values, the $\Gamma(k_x, \omega)$ surface can always be brought into coincidence with this plane. If the cuts adopted intersect the plane of real values, for individual domains of real k_{1x} and ω_1 values one gets $\rho_{\perp\infty} \neq R_\perp(k_{1x}, \omega_1)$. As regards the use of the Fourier technique, all possible cuts are totally equivalent. However, those which fulfill equality (4.4) are more convenient for the description of reflection of broad long-lasting light pulses.

As noted in Section 3, the $R_\perp(k_x, \omega)$ quantity in the right-hand part of equality (4.4) is obtained by analytic continuation from the zero value in the $|\omega'| \to \infty$ domain to a given point (k_x, ω) along the plane of real values. The result of this analytic continuation may be one of the two possible amplitude reflection coefficient of a plane monochromatic wave (2.8) or (2.9). The analytic continuation along the real frequency axis has certain physical grounds, because the current frequency of the electromagnetic field at any point of the medium changes from infinitely high values to the central, or carrier, pulse frequency on arrival of the amplitude leading edge.

By way of example let us consider the case of normal incidence of radiation from vacuum ($\varepsilon_1 = 1$), with the light pulse having a constant amplitude along the entire media interface. In this case, $k_{1x} = 0$ and $\sigma \to \infty$ in expression (4.2). If all the singular points and the zeros of $\varepsilon_2(\omega)$ reside in the $\omega'' < 0$ domain, the path of integration $\Gamma(\omega)$ can be brought into coincidence with the real axis and the entire procedure of determining $R_\perp(\omega) = R_\parallel(\omega) = R(\omega)$ reduces to finding the $k_{2z}(\omega)/k_{1z}(\omega) = \sqrt{\varepsilon_2(\omega)}$ ratio on this axis.

As ω changes from ∞ to 0, the behavior of the $\sqrt{\varepsilon_2(\omega)}$ function is conveniently considered employing a vector in the complex plane $z = u + \mathrm{i}v$. For $\omega \to \infty$, we get $\varepsilon_2(\omega) \to 1$ and therefore the imaging vector has a unit length and is aligned with the positive u-semiaxis (Fig. 1). In the case of absorption at all frequencies $0 < \omega < \infty$, $\varepsilon_2''(\omega) > 0$ and $0 < \arg \varepsilon_2 < \pi$, and therefore the imaging vector can reside only in quadrant I irrespective of the law of dispersion. Hence, $k_{2z}'(\omega) > 0$ and $|R(\omega)| < 1$ for all the frequencies $0 < \omega < \infty$.

Let us assume that amplification with $\varepsilon_2''(\omega) < 0$ is produced over some frequency band in the nonpropagation domain of the reflecting medium, where $\varepsilon_2'(\omega) < 0$. Then, in this frequency band $\pi < \arg \varepsilon_2(\omega) < 3\pi/2$ and the imaging vector will find itself in quadrant II (see Fig. 1). Therefore, in the nonpropagation domain of a medium with amplification $k_{2z}'(\omega) < 0$, $k_{2z}''(\omega) > 0$, and $|R(\omega)| > 1$. The corresponding Fourier components of the transmitted radiation propagate towards the media interface and their amplitudes decay exponentially with distance from this interface. As a consequence, the amplification of the reflected quasi-monochromatic light pulse is made possible if its frequency falls into the amplification band of the reflecting medium.

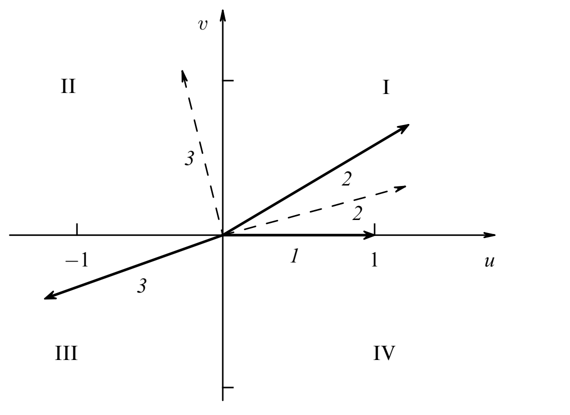

Figure 1. Vectorial representation of the quantities $\varepsilon_2(\omega)$ (solid line) and $\sqrt{\varepsilon_2(\omega)}$ (dashed line) on the complex plane $\omega = u + \mathrm{i}v$: *1* — $\varepsilon_2(\infty) = \sqrt{\varepsilon_2(\infty)} = 1$; *2* — $\varepsilon_2'(\omega) > 0$, $\varepsilon_2''(\omega) > 0$; *3* — $\varepsilon_2'(\omega) < 0$, $\varepsilon_2''(\omega) < 0$.

If the $\varepsilon_2(\omega)$ function has zeros or poles on the real axis, in detouring around them $\arg \varepsilon_2(\omega)$ changes by a magnitude which depends on the nature of the singular point, and the behavior of the imaging vector becomes more complex. A similar situation occurs when the singular points reside in the $\omega'' > 0$ domain, and the cuts being made intersect the real axis. The issues related to the application of the causality principle to the propagation of electromagnetic radiation through a transparent medium, when the singular points lie on the real axis, were considered in Ref. [17].

The known Fresnel formulas for transparent and absorbing media, obtained employing the selection rules for the refracted wave, pertain in essence to the solution of the boundary problem for a light pulse. They determine the limiting local amplitude reflection coefficient, which is found using the Fourier transformation of the surface Green function [see expression (4.4)]. The spatial and temporal dimensions of the incident pulse for which equality (4.4) is fulfilled depend on the requisite accuracy and the rate of decay of the side waves.

Using expression (4.2), for transparent media it can be shown that the half-width σ of a monochromatic light beam ($\tau \to \infty$) and the length τ of a quasi-monochromatic light pulse incident normally from vacuum ($k_{1x} = 0$, $\sigma \to \infty$, $\varepsilon_1 = 1$) should satisfy the inequalities [15, 16]

$$\sigma > \frac{1}{\beta^{\gamma_\sigma(k_{1x})}} |k_1 - k_2|^{-1}\,, \qquad \tau > \frac{1}{\beta^{\gamma_\tau(\omega_1)}} |\omega_{\mathrm{sn}} - \omega_{\mathrm{sp}}|^{-1}\,, \quad (4.5)$$

where β is the requisite relative accuracy equal to $|(\rho_\perp(x = 0; \sigma) - R_\perp(k_{1x}, \omega_1))/R_\perp(k_{1x}, \omega_1)|$ for a beam and to $|(\rho_\perp(t = \tau; \tau) - R_\perp(\omega_1))/R_\perp(\omega_1)|$ for a pulse; the functions $\gamma_\sigma(k_{1x})$ and $\gamma_\tau(\omega_1)$ assume values from 2/3 to 2, depending on the proximity of the k_{1x} value to k_1 or k_2 and of the ω_1 value to ω_{sn} or ω_{sp}. The frequencies ω_{sn} and ω_{sp} define the zero and the pole of the permittivity of the reflecting medium with allowance made only for that ensemble of harmonic oscillators which make the largest contribution to $\varepsilon_2(\omega)$ for $\omega = \omega_1$ [15].

The minimum width and length of the incident radiation are determined by the dimensions of the nonpropagation domains of the reflecting medium for spatial $\Delta k_x = |k_1 - k_2|$ and temporal $\Delta\omega = |\omega_{\mathrm{sn}} - \omega_{\mathrm{sp}}|$ frequencies. The inclusion of absorption enhances the damping of side waves and reduces the requisite values of the width and the duration of the incident radiation.

5. Reflection of light from an amplifying medium

Let us consider the reflection of a monochromatic light beam of frequency ω_1 incident from a transparent medium with $\varepsilon_1(\omega_1) > 0$ on a flat surface $z = 0$ of an amplifying medium with $\varepsilon_2 = \varepsilon_2' + i\varepsilon_2''$, $0 < \varepsilon_2'(\omega_1) < \varepsilon_1(\omega_1)$, and $\varepsilon_2''(\omega_1) < 0$. This extremely simplified model makes it possible to consider the problem of generalizing the Fresnel formulas to the case of an amplifying medium and eliminate the contradictions in Refs [4–10] from a unified standpoint outlined in Sections 3 and 4.

With expressions (3.1), (4.2), and (4.3), the field $E_r(x, t)$ of the reflected beam can be written in three ways:

$$E_r(x,t) = \frac{1}{2\pi}\int_{\Gamma(k_x)} R_\perp(k_x)\tilde{E}(k_x)\exp\left[i(k_x x - \omega_1 t)\right] dk_x$$
$$= \int_{-\infty}^{\infty} \mathcal{G}_{R\perp}(x_1) E(x - x_1, t)\, dx_1$$
$$= \rho_\perp(x;\sigma)\mathcal{E}\exp\left[i(k_{1x}x - \omega_1 t)\right] , \qquad (5.1)$$

where

$$\mathcal{G}_{R\perp}(x_1) = \frac{1}{2\pi}\int_{\Gamma(k_x)} R_\perp(k_x)\exp(ik_x x_1)\, dk_x , \qquad (5.2)$$

$$\rho_\perp(x;\sigma) = \int_{x-\sigma}^{x+\sigma} \mathcal{G}_{R\perp}(x_1)\exp(-ik_{1x}x_1)\, dx_1 , \qquad (5.3)$$

$\tilde{E}(k_x)$ is the Fourier transform of the $\mathcal{E}\exp(ik_{1x}x)$ function, $\mathcal{E} = \text{const}$, and $\Gamma(k_x)$ is the path of integration in the complex k_x plane.

The behavior of the $R_\perp(k_x)$ function has been studied most thoroughly on the axis of real k_x values. Plotted in Fig. 2 are three $|R_\perp(\theta)|$ dependences (θ is the angle of incidence) obtained employing different approaches to the solution of the problem of the reflection of a plane monochromatic wave from an amplifying medium. As shown below, all three versions do not contradict each other and are equivalent as regards the Fourier technique, because they lead to similar field distributions of the reflected beam.

For $\varepsilon_2'' < 0$, the $k_{\rm br3}$ branch point is shifted to the lower half-plane, and the $k_{\rm br4}$ branch point to the upper one. Therefore, the cuts which provide a correct detour around the branch points do not make it possible to bring the entire $\Gamma(k_x)$ path into coincidence with the real axis. The amplification of the side waves determined by the branch points $k_{\rm br3}$ and $k_{\rm br4}$ generates the need for inclusion of the Fourier components with complex k_x. We draw cuts parallel to the

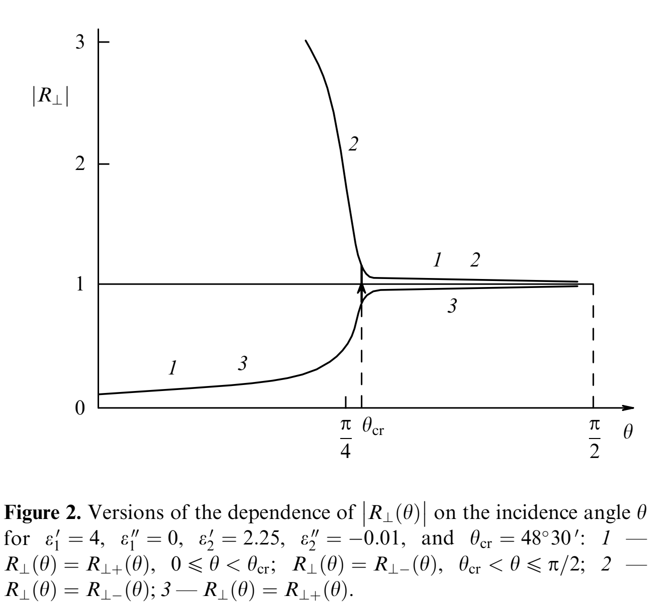

Figure 2. Versions of the dependence of $|R_\perp(\theta)|$ on the incidence angle θ for $\varepsilon_1' = 4$, $\varepsilon_1'' = 0$, $\varepsilon_2' = 2.25$, $\varepsilon_2'' = -0.01$, and $\theta_{\rm cr} = 48°30'$: *1* — $R_\perp(\theta) = R_{\perp+}(\theta)$, $0 \leqslant \theta < \theta_{\rm cr}$; $R_\perp(\theta) = R_{\perp-}(\theta)$, $\theta_{\rm cr} < \theta \leqslant \pi/2$; *2* — $R_\perp(\theta) = R_{\perp-}(\theta)$; *3* — $R_\perp(\theta) = R_{\perp+}(\theta)$.

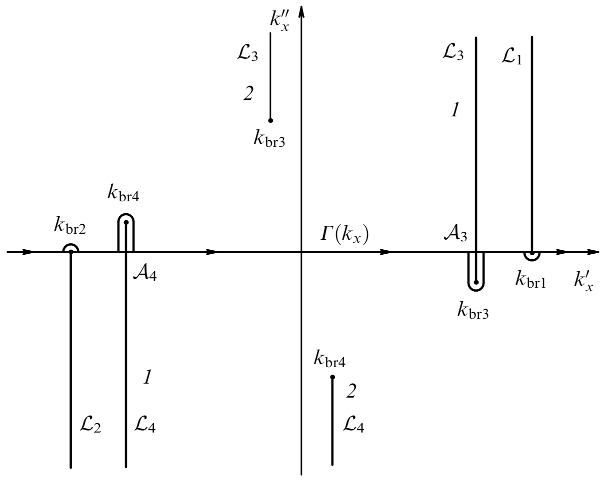

Figure 3. Location of the $k_{\rm br1,2}$ branch points for $\varepsilon_1'(\omega) > 0$, $\varepsilon_1''(\omega) = 0$ and of $k_{\rm br3,4}$ for $\varepsilon_1'(\omega) > \varepsilon_2'(\omega) > 0$, $\varepsilon_2''(\omega) < 0$ (*1*) and $\varepsilon_2'(\omega) < 0$, $\varepsilon_2''(\omega) < 0$ (*2*).

imaginary axis to connect the branch points $k_{\rm br1}$ and $k_{\rm br3}$ to infinitely distant points in the upper half-plane, and the branch points $k_{\rm br2}$ and $k_{\rm br4}$ to those in the lower half-plane (Fig. 3). The path of integration passes along the entire length of the real axis and along the edges of the cuts $\mathcal{L}_3$ and $\mathcal{L}_4$.

The decision upon the sheet of the Riemann surface for $R_\perp(k_x, \omega_1)$ in expression (5.1) is made using the asymptotic conditions

$$\lim_{|k_x'|\to\infty} \frac{k_{2z}(k_x,\omega_1)}{k_{1z}(k_x,\omega_1)} = 1 , \qquad \lim_{|k_x'|\to\infty} R_\perp(k_x,\omega_1) = 0 , \qquad (5.4)$$

which follow from the accepted rule for detouring around the branch points and from the limiting conditions (3.5). In this case, according to the causality principle and the convergence requirement on the Fourier integral for the incident beam, we obtain $k_{1z}(k_x,\omega_1) \to i|k_x'|$, when $|k_x'| \to \infty$.

We consider the behavior of the functions k_{2z} and $R_\perp$ on the axis of real k_x values. In the transmission range of the reflecting medium, where $-k_2' < k_x < k_2'$, one has $k_{2z}'(k_x) > 0$, $k_{2z}''(k_x) < 0$, and $|R_\perp(k_x)| < 1$. The Fourier components of the refracted beam propagate away from the media interface and their amplitudes grow exponentially with z. The corresponding Fourier components of the reflected beam are not amplified. In the nonpropagation domain, where $k_2' < |k_x| < k_1$, we find $k_{2z}'(k_x) < 0$, $k_{2z}''(k_x) > 0$, and $|R_\perp(k_x)| > 1$. The Fourier components of the refracted beam propagate towards the boundary surface between the media and their amplitudes decrease exponentially with z. In the nonpropagation domain, amplification of the Fourier components of the reflected beam occurs. At points $\mathcal{A}_3$ and $\mathcal{A}_4$, where the cuts $\mathcal{L}_3$ and $\mathcal{L}_4$ meet the real axis and where $k_x = \pm k_2'$ and $\theta = \theta_{\rm cr} = \arcsin(k_2'/k_1)$, the $R_\perp(k_x)$ values experience jump-like changes owing to the passage from one Riemann sheet to the other. This dependence is described by curve *1* in Fig. 2 and was obtained in Refs [4, 7, 9].

If the cuts $\mathcal{L}_3$ and $\mathcal{L}_4$ are deformed in such a way that the points $\mathcal{A}_3$ and $\mathcal{A}_4$ shift directly to $k_x = 0$, then for all real $k_x \neq 0$ we get $k_{2z}'(k_x) < 0$, $k_{2z}''(k_x) > 0$, and $|R_\perp(k_x)| > 1$. The Fourier components of the refracted beam travel to the media interface and their amplitudes decrease exponentially with z. This results in the amplification of the Fourier components of the reflected beam, which have real $k_x \neq 0$. The corresponding $|R_\perp(\theta)|$ dependence obtained in Ref. [6] is depicted by curve *2* in Fig. 2.

In the case when the $\mathcal{A}_4$ point is shifted to the $k_x \to -\infty$ domain, and the $\mathcal{A}_3$ point to the $k_x \to +\infty$ domain, for all the Fourier components of the refracted beam with real k_x one gets $k'_{2z}(k_x) > 0$, $k''_{2z}(k_x) < 0$, and $|R_\perp(k_x)| < 1$. The Fourier components of the refracted beam propagate away from the media interface and their amplitudes grow exponentially with z. The corresponding Fourier components of the reflected beam are not amplified. In this case, the $|R_\perp(\theta)|$ dependence obtained in Ref. [8] is represented by curve *3* in Fig. 2.

Clearly, the deformation of cuts and the displacement of the points of intersection of the cuts with the real axis have no effect on the Fourier integral but change only the relative contribution of the Fourier components with real and complex k_x. Hence, all the three versions of determining the amplitude reflection coefficient of plane monochromatic waves are equivalent. In just the same way it is possible to consider different equivalent determinations of the $R_\perp(k_x)$ function on the real axis for transparent and absorbing reflecting media. To this end, it would suffice to adopt cuts intersecting the real axis.

If the amplification is weak and $|k''_2|\sigma \lesssim 1$, the field $E_{\mathrm{r}}(x,t)$ at the boundary surface between the media is spatially separated into the field of the reflected beam in the $|x| < \sigma$ domain and the field of side waves, which is represented by the expression [16]

$$E_{\mathrm{r}}(x,t) = \frac{1}{\sqrt{\pi(k_1-k_2)}}\left\{\frac{\tilde{E}(k_1)}{x^{3/2}}\exp\left[\mathrm{i}\left(k_1 x - \omega_1 t + \frac{3\pi}{4}\right)\right] + \frac{\tilde{E}(k_2)}{x^{3/2}}\exp\left[\mathrm{i}\left(k_2 x - \omega_1 t + \frac{\pi}{4}\right)\right]\right\} \tag{5.5}$$

in the domain $x \gg \sigma$, $1/|k_1-k_2|$. In the domain $x \ll -\sigma$, $-1/|k_1-k_2|$, two similar side waves arise with amplitudes proportional to $\tilde{E}(-k_1)$ and $\tilde{E}(-k_2)$, which travel away from the region of beam incidence. Notice that expression (5.5) is valid for transparent ($k''_2 = 0$) and absorbing ($k''_2 > 0$) reflecting media. The asymptotics of the surface Green function (5.2) are also described by similar expressions, with $\tilde{E}(k_1) = \tilde{E}(k_2) = \tilde{E}(-k_1) = \tilde{E}(-k_2) = 1$.

For an amplifying reflecting medium, the passage to the limit of a plane monochromatic wave fails, because the amplitudes of the side waves, which have phase velocities $\pm\omega_1/k_2$, grow exponentially with increasing $|x|$ and the Fourier integral (5.3) diverges if $\sigma \to \infty$. This brings up the question of how to decide upon the most convenient Fourier representation. The version of determining $R_\perp(k_x)$ represented by curve *1* in Fig. 2 is preferable, because in this case the field of the reflected beam is described with the requisite accuracy by the Fourier integral taken along the axis of real k_x, and the asymptotics in the form of amplified side waves by the Fourier integral taken along the edges of the cuts $\mathcal{L}_3$ and $\mathcal{L}_4$.

Outside small domains of size $\Delta k_x \approx |k''_2|$ centered at the points $k_x = \pm k'_2$ (in the vicinity of the critical angle of total reflection) as well as outside the domain of grazing angles for light beams with $k_x \approx k_1$, it may be assumed with sufficient accuracy that

$$\rho_\perp\left(x = 0; \sigma \approx \frac{1}{|k''_2|}\right) = R_\perp(k_{1x})\,. \tag{5.6}$$

In the region of total reflection, amplification of the reflected beam is a possibility for $|x| < \sigma$, the maximum value of $|\rho_\perp(x=0;\sigma\approx 1/|k''_2|)|$ not exceeding the maximum value of $|R_\perp(k_{1x})|$, where $k'_2 < |k_{1x}| < k_1$.

For an amplifying reflecting medium with $\varepsilon'_2(\omega_1) < 0$, the $k_{\mathrm{br}3}(\omega_1)$ branch point lies in the $k'_x < 0$ domain, and the $k_{\mathrm{br}4}(\omega_1)$ branch point in the $k'_x > 0$ domain, as shown in Fig. 3. The side waves governed by these branch points propagate towards the region of beam incidence and their amplitudes decay exponentially with $|x|$. This result becomes evident when it is realized that the laws of dispersion for the side waves involved and for the refracted waves with $k_x = 0$ coincide (see Section 4).

For an amplifying medium with $\varepsilon'_2 > 0$, for real $k_x \neq 0$ the $k_{2z}(k_x,\omega)$ function has branch points in the $\omega'' > 0$ domain, and therefore the Kramers–Kronig relations (3.9) are not fulfilled. In this case, the reflection of radiation with the field distribution at the media interface

$$E(x,t) = \mathcal{E}(t)\exp(\mathrm{i}k_x x)\,, \tag{5.7}$$

where k_x is a real quantity, shows an absolutely unstable process. Taking into account the boundedness of radiation along the media interface turns the absolute instability into a convective one, which is represented by the amplified side waves [16].

Experiments on the light reflection from amplifying media have shown that the amplification of reflected light in the region of total reflection is indeed possible [18–22]. Under specific conditions, the energy reflection coefficient ranged up to about 10^3, which is two orders of magnitude higher than the maximum value of $|R_{\perp,\parallel}|^2$ for plane monochromatic waves in the case of practical significance when $|\varepsilon''_2| \ll \varepsilon'_2$ [10]. The discrepancy observed in this case is due to inhomogeneity of the reflecting medium, when the reflection of transmitted light occurs at some distance from the boundary surface between the media [7, 10, 23]. The physical processes responsible for the inhomogeneity of the reflecting medium subject to optical pump were considered in Ref. [23].

In the description of the reflection of plane monochromatic waves from a medium inhomogeneous along the z-axis, precisely the same problems arise as in the case of a homogeneous reflecting medium. For instance, for

$$\varepsilon_2(z) = \varepsilon_2(\infty) + \Delta\varepsilon_2\exp\left(-\frac{z}{h}\right) \tag{5.8}$$

(where $0 < \varepsilon_2(\infty) < \varepsilon_1$, $\Delta\varepsilon_2 = \Delta\varepsilon'_2 + \mathrm{i}\Delta\varepsilon''_2$, $\Delta\varepsilon'_2$ and $\Delta\varepsilon''_2 < 0$ are real constants, and h is the characteristic inhomogeneity length), the amplitude reflection coefficient of a plane monochromatic wave polarized perpendicular to the plane of incidence takes the form [10]

$$R_\perp(k_x,\omega) = \frac{k_{1z} - k_{2z}(\infty) + \mathrm{i}\sqrt{\Delta\varepsilon_2}\,\omega p/c}{k_{1z} + k_{2z}(\infty) - \mathrm{i}\sqrt{\Delta\varepsilon_2}\,\omega p/c}\,. \tag{5.9}$$

Here, we put

$$p = \frac{J_{\nu+1}(2\sqrt{\Delta\varepsilon_2}\,\omega h/c)}{J_\nu(2\sqrt{\Delta\varepsilon_2}\,\omega h/c)}\,, \tag{5.10}$$

$J_{\nu+1}$ and J_ν are Bessel functions of the first kind with complex indices, viz.

$$\nu = -2\mathrm{i}k_{2z}(\infty)h \quad\text{and}\quad k_{2z}(\infty) = \left[\varepsilon_2(\infty)\omega^2/c^2 - k_x^2\right]^{1/2}.$$

To define $R_\perp(k_x, \omega)$ uniquely, we need to adopt the rule for detour around the singular points, draw cuts connecting the k_{1z} and $k_{2z}(\infty)$ branch points to infinitely distant points, and decide upon the sheets of the Riemann surface for the two-valued functions k_{1z} and $k_{2z}(\infty)$. The complex indices of the Bessel functions, which specify the asymptotics of the refracted wave for $z \to \infty$, are not uniquely defined, either, until the cuts are drawn and the decision is made upon the sheet of the Riemann surface for $k_{2z}(\infty)$. All the operations in the complex space of Fourier variables are performed on the basis of the causality principle in the form of conditions imposed on the propagation of the amplitude leading edges of the incident, reflected, and refracted waves along the normal and along the boundary surface between the media.

By selection of the quantities $\Delta\varepsilon_2'$, $\Delta\varepsilon_2'' < 0$, and h, it is possible to obtain any value of $|R_\perp(k_x, \omega_1)|$. For a strong inhomogeneity, when $\omega_1 h/c \approx 1$, the denominator of expression (5.9) vanishes for some real k_x, signifying the onset of the generation of light in the subsurface layer of the reflecting medium [10]. The corresponding angles of incidence lie in a small neighborhood of the critical angle of total reflection $\theta_{\rm cr}$. Irrespective of the polarization of light, these angles are always smaller than $\theta_{\rm cr}$ for $\Delta\varepsilon_2' < 0$, and always larger than $\theta_{\rm cr}$ for $\Delta\varepsilon_2' > 0$. The critical angle of total reflection is distinguished in the sense that in its vicinity the light generation threshold is reached for the minimum optical gain.

6. Directivity of the atomic stimulated emission in the medium nonpropagation domain

The amplification of light reflected from a homogeneous amplifying medium is due to the stimulated emission by excited atoms of the medium. In the nonpropagation domain of the reflecting medium, where $\varepsilon_2'(\omega) < 0$ or $\omega^2\varepsilon_2'(\omega)/c^2 < k_x^2$, one finds $\varepsilon_2''(\omega) > 0$ and excited atoms interact with exponentially decaying waves, which are coupled to the media interface and are not free radiation. The spatial directivity of the stimulated emission by an atom in the wave field of this kind calls for special consideration.

In phenomenological electrodynamics, the energy, momentum, and angular momentum exchange between atoms and radiation field is described employing complex atomic polarizability $\alpha = \alpha' + {\rm i}\alpha''$, where $\alpha'' > 0$ for the ground state, and $\alpha'' < 0$ for excited atomic states. The total flux of Poynting's vector $\mathbf{S}$ through an arbitrary closed surface F, which encloses an atom with the dipole moment $\mathbf{p} = \alpha\mathbf{E}_0 \exp(-{\rm i}\omega t)$ induced by a radiation field of frequency ω, is made up of three terms. The first term depends only on the field of the incident wave and is zero in the stationary case. The second is exclusively determined by the field of the dipole and is proportional to $|\alpha|^2$. Neither term is related to the stimulated emission of the atom.

The third term $I_{\rm int}$ by its nature is interference, because it is determined both by the field of the incidence wave and by the field of the dipole. According to Poynting's theorem, we have

$$I_{\rm int} = \oint_F \mathbf{S}_{\rm int}\,{\rm d}\mathbf{F} = -\frac{1}{2}\,\alpha''\omega|\mathbf{E}_0|^2\,, \tag{6.1}$$

where the electric field $\mathbf{E}_0$ of the incident wave is taken at the point of the atom's location. According to expression (6.1), the atom – wave energy exchange is described by the interference flux $I_{\rm int}$, and therefore the spatial directivity of stimulated emission results from the anisotropy of the distribution of the interference component $\mathbf{S}_{\rm int}$ of Poynting's vector.

Assume that a plane monochromatic wave with a wave vector $\mathbf{k} = (\omega/c, 0, 0)$, which propagates in the positive direction of the x-axis, is incident on an atom residing at the point $x = y = z = 0$ in vacuum. Employing formulas for the wave-zone field of a dipole and assuming that the wave has arbitrarily large but finite lateral dimensions, it can be analytically shown that the entire interference energy flux $I_{\rm int}$ passes through the $x = {\rm const} > 0$ plane and is described by expression (6.1) [24, 25]. For an unexcited atom, one has $\alpha'' > 0$, and therefore $I_{\rm int} < 0$. The total energy flux along the x-axis transferred by the summary radiation decreases, which signifies that the atom absorbs a part of the incident radiation. For an excited atom, we have $\alpha'' < 0$, and therefore $I_{\rm int} > 0$. The total energy flux along the x-axis increases, which corresponds to directional stimulated emission by the atom.

The momentum and angular momentum exchange between atoms and radiation field is described by the interference components of the Maxwell stress tensor for the summary field of the incident wave and the dipole [26]. For the plane monochromatic wave considered above, the ratio between the momentum P_x and the energy W, which are transferred by the interference flux, is of the form [26]

$$\frac{P_x}{W} = \frac{k_x}{\omega}\,, \tag{6.2}$$

where $k_x = \omega/c$, and coincides with the ratio of these quantities for the incident wave. It is precisely this circumstance that points to the emission of a photon of a plane monochromatic wave by an atom. Therefore, the hypothesis that the radiation emitted by an atom is 'spiky' becomes unnecessary, because the spatial directivity of its stimulated emission is determined by the spatial structure of the incident wave.

Now let an atom be exposed to a plane monochromatic wave which propagates in the positive direction of the x-axis and exponentially decays along the z-axis. The real projection of the wave vector onto the x-axis satisfies the condition $k_x > \omega/c$, and in this case the projection of the wave vector onto the z-axis, $k_z = {\rm i}(k_x^2 - \omega^2/c^2)^{1/2}$, is imaginary. Calculations show that the interference flux described by formula (6.1) passes entirely through the $z = {\rm const} < 0$ plane, namely, it is directed perpendicular to the energy flux transferred by the incident wave along the x-axis [24, 25]. The interference energy flux is aligned with the positive direction of the z-axis for $\alpha'' > 0$, and is opposed to it for $\alpha'' < 0$, viz. it points toward the exponential growth of the amplitude of the incident wave.

In the region of total reflection from an absorbing medium, the interference energy flux is directed away from the media interface into the depths of a reflecting medium, while in the case of an amplifying medium toward the interface (amplifies the reflected radiation). This conclusion is consistent with the result obtained in Section 5 on the basis of the causality principle. It is notable that in the region of total reflection the interference flux transfers the P_x momentum component which satisfies relationship (6.2), where $k_x > \omega/c$, through the media boundary surface [26]. Measurements of the recoil momentum of an atom interacting with an exponentially decaying wave under the total reflection of light are consistent with formula (6.2) [27, 28]. The transfer of angular momentum by the interference flux in the region of total reflection was considered in Ref. [26] for light of different polarizations.

In a medium with negative permittivity, the field of a dipole executing harmonic oscillations decays exponentially with distance and does not transfer energy. However, the situation reverses when the atomic dipole moment is induced by the refracted wave excited at the normal incidence of a plane monochromatic wave. Taken alone, the refracted wave exponentially decays with the normal distance from the media interface and does not transfer energy. Nevertheless, the interference of the reactive components of the field of the dipole and the refracted wave is responsible for an energy flux, which is described by expression (6.1) as before [30].

If $\alpha'' > 0$, the interference energy flux is directed toward the atom and points toward the exponential decay of the refracted wave. For $\alpha'' < 0$, the interference energy flux is directed away from the atom and points toward the exponential growth of the refracted wave, i.e. toward the media boundary surface, amplifying the reflected radiation. As regards the mechanism of transfer of the energy of the electromagnetic field, this case of stimulated emission bears similarity to the process of radiationless energy transfer between atoms and may be referred to as tunneling stimulated emission [30]. Notice that ratio (6.2) vanishes in this case. An alternative possibility of the emission by excited atoms occurring in a medium with negative permittivity was considered in Ref. [29].

In the course of radiation reflection, the energy flux through the media boundary surface is determined by the excitation of the transmitted wave and by the energy exchange between the refracted wave and the reflecting medium. In the nonpropagation domain of the reflecting medium, the refracted wave occupies a layer of finite thickness, and therefore it is formed in a finite time. Once the refracted wave has formed, the energy flux through the media boundary surface is caused by the energy exchange between the refracted wave and the medium atoms, its direction depending on the states of the atoms and the spacetime structure of the refracted wave.

By changing the boundary conditions at the media interface, it is possible to control the spacetime structure of the transmitted wave and, accordingly, the directivity of the stimulated emission by the excited atoms of the reflecting medium. In this way it is possible to realize the amplification of refracted, reflected, and side waves as well as to formally attain the absolute instability of the reflection process [16]. The stimulated emission by atoms in the field of exponentially decaying refracted waves coupled to the media boundary surface proceeds invariably in the direction of the exponential growth of the amplitude of these waves and not in the direction of energy transfer. It is precisely this directivity of the stimulated emission that is responsible for the formation of the refracted wave which propagates to the media interface.

7. Conclusions

The problem of the reflection of electromagnetic radiation is posed on the plane of spatial x and temporal t real variables. Formulated in the plane are the boundary conditions and the causality principle, which determine the directions of propagation of the amplitude leading edges of the incident, reflected, and refracted waves. Only when the incident radiation is bounded both in x and in t at the media interface can the causality principle be used and a unique solution of the boundary problem be obtained for linear media without spatial dispersion. This solution defines the field of reflected radiation, the local amplitude reflection coefficient, and the surface Green function.

The Fourier technique transfers the solution of this boundary problem to the 4D space of complex Fourier variables k_x and ω, where the singular points of the amplitude reflection coefficient of the Fourier components make immediately obvious the salient features of reflection associated with the excitation of surface waves. The unambiguous relationship between the Fourier components of the incident and reflected radiation at an arbitrary individual point (k_x, ω) does not follow from the boundary conditions at the media interface. The principle of causality determines uniquely only the asymptotic behavior of the amplitude reflection coefficient of the Fourier components for $|\omega| \to \infty$ and does not eliminate the existing ambiguity. As a consequence, the unique solution of the boundary problem can be expressed employing a continuum of equivalent Fourier representations, with a particular Fourier component having different amplitude reflection coefficients in various Fourier representations.

The derivation of the continuum of equivalent Fourier representations reduces to the determination of the amplitude Fourier-component reflection coefficient as a regular function in the space of Fourier variables, where adoption of the rule for bypassing the singular points, making cuts, and deciding upon the sheet of the Riemann surface are performed in accordance with the principle of causality. The amplitude reflection coefficient of the Fourier components at any point of the surface of integration is found by analytic continuation from the zero value to the domains where $|\omega'| \to \infty$. Hence it follows that the selection rule for the Fourier components of the refracted radiation at every point of the surface of integration should be replaced by the criterion for choosing the most convenient Fourier representation.

If the local amplitude reflection coefficient tends to the limit ρ_∞ with unlimited increase in the light pulse length and width, it coincides with one of the two possible amplitude reflection coefficients of plane monochromatic waves R_+ or R_- admitted by the boundary conditions. In this case, a reasonable choice is of that Fourier representation for which the amplitude reflection coefficient of the central Fourier component of the light pulse coincides with ρ_∞. The Fourier representations in common use for transparent and absorbing reflecting media possess precisely this property. The experimentally determined reflectivities described by the Fresnel formulas should be associated with the limiting local amplitude pulse reflectivity determined unambiguously by the causality principle and the boundary conditions.

The discussion about the reflection of light from amplifying as well as from nonlinear media, for which the resolution of the issue of whether hysteretic effects exist has proved to be impossible in the context of the model of a plane monochromatic wave [31], has furnished an opportunity to view the Fresnel formulas from a new standpoint and to offer a more exact physical interpretation of them. This approach rests on the causality principle, which determines the direction and the velocity of propagation of the amplitude leading edge of electromagnetic radiation — a fundamental characteristic of radiation introduced by L Brillouin and A Sommerfeld. In essence, the uniqueness of the solution of the problem of light pulse reflection is attained by reconciling the directions of propagation of the amplitude leading edges of all the three light pulses originating at the media interface.

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