Physics – Uspekhi 42 (6) 603 – 605 (1999)

METHODOLOGICAL NOTES

PACS numbers: **52.25.-b**, 52.25.Dg, **52.90.+z**

On the quantum description of the linear kinetics of a collisionless plasma

M V Kuzelev, A A Rukhadze

Abstract. **It is demonstrated that the linear kinetics of a collisionless quantum plasma can be described in a simple and effective way by means of a self-consistent-field scheme in which the quantum hydrodynamic equations are derived directly from the Schrödinger equation.**

1. We show that the known system of equations of cold hydrodynamics in the Eulerian form [1] [1]

$$\frac{\partial n}{\partial t} + \nabla(n\mathbf{V}) = 0\,,$$

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\times\nabla)\mathbf{V} = \frac{e}{m}\left\{\mathbf{E} + \frac{1}{c}[\mathbf{V}\times\mathbf{B}]\right\} \tag{1}$$

can also be used with profit, at least in the linear approximation, for describing the kinetic properties of a plasma with a thermal scatter in the particle velocities (a Vlasov plasma).

Let some group of particles with number density n in a homogeneous isotropic plasma (without an external magnetic field $\mathbf{B}_0$) possess a velocity $\mathbf{V}$. A small perturbation of this state by a weak electromagnetic field $\mathbf{E}$, $\mathbf{B}$ will give rise to perturbations of density δn and velocity $\delta\mathbf{V}$, which are found from the linearized system (1). Since n and $\mathbf{V}$ are constant, the quantities δn and $\delta\mathbf{V}$ can be sought as $\exp(-\mathrm{i}\omega t + \mathrm{i}\mathbf{kr})$. On determining δn, $\delta\mathbf{V}$ and then the current density

$$j_i = en\delta V_i + e\delta n V_i = \sigma_{ij}(\omega,\mathbf{k})E_j\,, \tag{2}$$

we shall find the conductivity σ_{ij} and the dielectric constant of the particle group under consideration:

$$\begin{aligned}\varepsilon_{ij}(\omega,\mathbf{k}) &= \delta_{ij} + \frac{4\pi\mathrm{i}}{\omega}\sigma_{ij}(\omega,\mathbf{k})\\ &= \delta_{ij} - \frac{4\pi e^2 n}{m\omega^2}\left[\delta_{ij} + \frac{k^2 V_i V_j}{(\omega-\mathbf{kV})^2} + \frac{k_i V_j + V_i k_j}{\omega - \mathbf{k}V}\right]. \end{aligned}\tag{3}$$

[1] For brevity of the following presentation, we consider only one plasma component, for instance, the electron component.

M V Kuzelev Moscow State University of Publishing,
ul. Pryanishnikova 2a, 127550 Moscow, Russian Federation
Tel. (7-095) 976 40 88. Fax (7-095) 976 06 35
E-mail: mgup@mail.cnt.ru
A A Rukhadze Institute of General Physics, Russian Academy of Sciences,
ul. Vavilova 38, 117942 Moscow, Russian Federation
Tel. (7- 095) 135 02 47. Fax (7- 095) 135 80 11
E-mail: rukh@fpl.gpi.ru

Received 5 March 1999, revised 16 March 1999
Uspekhi Fizicheskikh Nauk **169** (6) 687 – 689 (1999)
Translated by E K Ragozin; edited by A Radzig

Now we can go over from a group of particles to the entire plasma by averaging over the momentum distribution function $f_0(p)$ with the substitution $n \to f_0(p)\,\mathrm{d}p$ and subsequent integration

$$n \to \int \mathrm{d}\mathbf{p} f_0(p)[...]\,. \tag{4}$$

The last bracketed factor of the integrand stands for the factor in expression (3) enclosed in square brackets. In consequence we find the known expression for the permittivity tensor for an isotropic plasma, which is usually obtained by solving the kinetic Vlasov equation [2]:

$$\varepsilon_{ij}(\omega,\mathbf{k}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\varepsilon^{\mathrm{tr}}(\omega,\mathbf{k}) + \frac{k_i k_j}{k^2}\varepsilon^{\mathrm{l}}(\omega,\mathbf{k})\,,$$

where

$$\varepsilon^{\mathrm{l}}(\omega,\mathbf{k}) = 1 - \frac{4\pi e^2}{k^2}\int\frac{\mathbf{k}\,\partial f_0/\partial\mathbf{p}}{\omega-\mathbf{kV}}\,\mathrm{d}\mathbf{p}\,, \tag{5}$$

$$\varepsilon^{\mathrm{tr}}(\omega,\mathbf{k}) = 1 - \frac{4\pi e^2}{m\omega^2}\int\mathrm{d}\mathbf{p}\left[f_0(p) + \frac{mV_\perp^2\mathbf{k}\,\partial f_0/\partial\mathbf{p}}{2(\omega-\mathbf{kV})}\right].$$

Naturally, the outlined method is applicable not only for calculating the dielectric constant of an isotropic plasma. The substitution (4) is appropriate whenever the plasma with a thermal velocity scatter can be treated as a collection of groups of particles described by Eqns (1). In this case, the square brackets under the integral in (4) should enclose all expressions dependent on the hydrodynamic characteristics of each group of particles.

2. We shall generalize the outlined method to the case of a quantum plasma. In doing this, we proceed from the Schrödinger equation for the electrons without a spin, following Ref. [3] in the derivation:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi = \left[-\frac{\hbar^2}{2m}\Delta + \mathrm{i}\hbar\frac{e}{mc}\mathbf{A}\nabla + \frac{e^2}{2mc^2}\mathbf{A}^2 + e\varphi\right]\psi\,. \tag{6}$$

Here $\mathbf{A}$ and φ are the vector and scalar potentials of the fields $\mathbf{E}$ and $\mathbf{B}$, with

$$\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla\varphi,\quad \mathbf{B} = [\nabla\times\mathbf{A}],\quad (\nabla\mathbf{A}) = 0\,. \tag{7}$$

We represent the wave function as

$$\psi = a(\mathbf{r},t)\exp\left[\frac{\mathrm{i}}{\hbar}S(\mathbf{r},t)\right] \tag{8}$$

and draw on the definitions of charge and current densities

$$\rho = en = e|\psi|^2 = ea^2 ,$$

$$\mathbf{j} = en\mathbf{V} = \frac{\mathrm{i}e\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi) - \frac{e^2}{mc}\mathbf{A}\psi\psi^*$$

$$= \frac{ea^2}{m}\left(\nabla S - \frac{e}{c}\mathbf{A}\right), \tag{9}$$

to obtain the system of equations

$$\frac{\partial n}{\partial t} + \nabla(n\mathbf{V}) = 0 ,$$

$$\frac{\partial n}{\partial t} + (\mathbf{V}\times\nabla)\mathbf{V} = \frac{e}{m}\left\{\mathbf{E} + \frac{1}{c}[\mathbf{V}\times\mathbf{B}]\right\}$$

$$+ \frac{\hbar^2}{4m^2}\nabla\left\{\frac{1}{n}\left[\Delta n - \frac{1}{2n}(\nabla n)^2\right]\right\} \tag{10}$$

from the Schrödinger equation (6).

The first of these equations coincides with the equation of continuity, and the second with the Euler equation of system (1). Therefore, by analogy with system (1), system (10) will be referred to as the quantum equations of cold plasma hydrodynamics.

Eqns (10) differ from Eqns (1) in that the Euler equation includes the quantum force resulting from the Heisenberg uncertainty principle. This is easily verified by considering small perturbations of the uniform state with $n = \mathrm{const}$ and $\mathbf{V} = 0$. In the limit $n \to 0$ when the self-consistent fields $\mathbf{E}$ and $\mathbf{B}$ can be neglected, for solutions of the type $\exp(-\mathrm{i}\omega t + \mathrm{i}\mathbf{kr})$ the linearized system (10) yields the dispersion relation

$$\omega = \frac{\hbar k^2}{2m} \equiv \omega_{\mathrm{q}} , \tag{11}$$

which describes the oscillations of a single electron. This expression relates the temporal (proportional to $1/\omega$) and spatial (proportional to $1/k$) domains of localization of a free electron, or the energy $\hbar\omega$ and the momentum $\hbar\mathbf{k}$. The quantity (11) is the frequency of the quantum oscillations of a free electron.

Following the outlined procedure, we can now derive the dielectric constant of a quantum isotropic plasma with a thermal scatter in electron velocities. First, for any group of plasma particles we obtain the corresponding quantum dielectric constant, i.e. the quantum analog of tensor (3). Assuming the perturbed quantities to be of the form $\exp(-\mathrm{i}\omega t + \mathrm{i}\mathbf{kr})$, from Eqn (10) it follows that

$$\varepsilon^{\mathrm{q}}_{ij}(\omega,\mathbf{k}) = \varepsilon^{\mathrm{cl}}_{ij}(\omega,\mathbf{k}) - \frac{\omega_{\mathrm{q}}^2}{\omega_{\mathrm{L_e}}^2}\,\delta\varepsilon^{\mathrm{cl}}_{i\mu}(\omega,\mathbf{k})\frac{k_\mu k_\nu}{k^2}\delta\varepsilon^{\mathrm{cl}}_{i\nu}(\omega,\mathbf{k})$$

$$\times\left[1 + \frac{\omega_{\mathrm{q}}^2}{\omega_{\mathrm{L_e}}^2}\frac{k_\mu k_\nu}{k^2}\delta\varepsilon^{\mathrm{cl}}_{\mu\nu}(\omega,\mathbf{k})\right]^{-1}, \tag{12}$$

where $\omega_{\mathrm{L_e}} = \sqrt{4\pi e^2 n/m}$ is the electron Langmuir frequency, and $\varepsilon^{\mathrm{cl}}_{ij} = \delta_{ij} + \delta\varepsilon^{\mathrm{cl}}_{ij}$ is the classical dielectric constant tensor defined by expression (3). In the derivation of expression (12), we drew on the obvious substitution

$$\mathbf{E}^{\mathrm{q}} = \mathbf{E}^{\mathrm{cl}} - \mathrm{i}\mathbf{k}\frac{\hbar k^2}{4me}\frac{\delta n}{n} , \tag{13}$$

which follows in the linear approximation from the Euler equation (10).

We next substitute expression (3) into (12) and pass on to the kinetic description with the help of change (4) to obtain by straightforward calculations the known expressions for the quantum longitudinal and transverse dielectric constants of an isotropic plasma [2]

$$\varepsilon^{\mathrm{l}}(\omega,\mathbf{k}) = 1 + \frac{4\pi e^2}{\hbar k^2}\int\frac{\mathrm{d}\mathbf{p}}{\omega - \mathbf{kV}}\left[f_0\left(\mathbf{p} + \frac{\hbar\mathbf{k}}{2}\right) - f_0\left(\mathbf{p} - \frac{\hbar\mathbf{k}}{2}\right)\right],$$

$$\varepsilon^{\mathrm{tr}}(\omega,\mathbf{k}) = 1 - \frac{\omega_{\mathrm{L_e}}^2}{\omega^2} + \frac{2\pi e^2}{\hbar\omega^2}$$

$$\times\int\frac{\mathrm{d}\mathbf{p}}{\omega - \mathbf{kV}}V_\perp^2\left[f_0\left(\mathbf{p} + \frac{\hbar\mathbf{k}}{2}\right) - f_0\left(\mathbf{p} - \frac{\hbar\mathbf{k}}{2}\right)\right]. \tag{14}$$

Notice that in Ref. [2] expressions (14) were derived by solving the Wigner quantum kinetic equation, which involved tedious calculations. In the limit $\hbar \to 0$, formulas (14) obviously transform to formulas (5).

3. Now consider a homogeneous magnetoactive plasma. Let the external magnetic field $\mathbf{B}_0$ be aligned with the OZ-axis. For simplicity, we shall restrict our consideration to the case of a potential field $\mathbf{E} = -\nabla\varphi$, $\mathbf{A} = 0$. As above, we consider a group of particles with number density n, which possess longitudinal velocity V_z and rotate about the magnetic lines of force with the Larmor frequency $\Omega = eB_0/mc$ and the Larmor radius $R_{\mathrm{L}} = V_\perp/\Omega$. The longitudinal dielectric constant of this classical cold plasma (group of particles) is easy to obtain from the general formula given in Ref. [1]. It is of the form

$$\varepsilon(\omega,\mathbf{k}) = \frac{k_i k_j}{k^2}\varepsilon_{ij}(\omega,\mathbf{k}) = 1 - \frac{\omega_{\mathrm{L_e}}^2}{k^2}$$

$$\times\sum_s\left[\frac{k_z^2 J_s^2(z)}{(\omega - k_z V_z - s\Omega)^2} + \frac{2sk_\perp^2 J_s(z)J_s'(z)}{z\Omega(\omega - k_z V_z - s\Omega)}\right], \tag{15}$$

where $J_s(z)$ is the Bessel function of the real argument $z = k_\perp R_{\mathrm{L}}$.

We average expression (15) over the distribution function $f_0(\mathbf{p})$ according to the above recipe (4) to obtain the known expression for the longitudinal dielectric constant of a classical magnetoactive plasma [1]

$$\varepsilon(\omega,\mathbf{k}) = 1 + \frac{4\pi e^2}{mk^2}\int\mathrm{d}\mathbf{p}\sum_s\frac{J_s^2(z)}{\omega - k_z V_z - s\Omega}$$

$$\times\left(k_z\frac{\partial f_0}{\partial V_z} + \frac{s\Omega}{V_\perp}\frac{\partial f_0}{\partial V_\perp}\right). \tag{16}$$

It is also an easy matter to write out the longitudinal dielectric constant of a quantum magnetoactive plasma. To accomplish this, it should be recognized that the total force in the right-hand part of the Euler equation (10) does not depend on the type of plasma at all. Consequently, relation (13) is universal in character too, and with it formula (12). Hence, the longitudinal dielectric constant is given by

$$\varepsilon^{\mathrm{q}}(\omega,\mathbf{k}) = \varepsilon^{\mathrm{cl}}(\omega,\mathbf{k}) - \frac{\omega_{\mathrm{q}}^2}{\omega_{\mathrm{L_e}}^2}\frac{\delta\varepsilon^{\mathrm{cl}}(\omega,\mathbf{k})\delta\varepsilon^{\mathrm{cl}}(\omega,\mathbf{k})}{1 + (\omega_{\mathrm{q}}^2/\omega_{\mathrm{L_e}}^2)\delta\varepsilon^{\mathrm{cl}}(\omega,\mathbf{k})} , \tag{17}$$

where $\varepsilon^{\mathrm{cl}} = 1 + \delta\varepsilon^{\mathrm{cl}}$ is defined by expression (15).

Therefore, expression (17) refers to the longitudinal dielectric constant of a cold quantum plasma. As above, the passage to the kinetic description is accomplished by averaging expression (17) over the distribution function $f_0(\mathbf{p})$ with the help of substitution (4). Substitution of expression (15) into (17) with subsequent averaging results in cumbersome expressions, which we omit here.

It is more expedient to address the question of the $f_0(\mathbf{p})$ distribution itself over which the averaging is performed. The point is that, in general, account must be taken of the energy of quantization of the transverse electron motion in a magnetoactive plasma. This has no effect on the magnitude of ω_{q} but substantially affects the shape of the distribution function $f_0(\mathbf{p})$.

In the case of Maxwellian statistics (nondegenerate electrons) [4], one obtains

$$f_0(\mathbf{p}) = \frac{n}{(2\pi m)^{3/2} T^{1/2} E_\perp} \exp\left(-\frac{p_z^2}{2mT} - \frac{p_\perp^2}{2mE_\perp}\right), \qquad (18)$$

where

$$E_\perp = \frac{\hbar\Omega}{2} \coth\frac{\hbar\Omega}{2T} \approx \begin{cases} T, & \hbar\Omega \ll 2T, \\ \hbar\Omega/2, & \hbar\Omega \gg 2T \end{cases} \qquad (19)$$

is the average energy of the transverse electron motion. The condition for nondegeneracy is written as

$$E_{\mathrm{F}} \ll T^{1/3} E_\perp^{2/3}, \qquad (20)$$

where $E_{\mathrm{F}} = (3\pi^2)^{2/3} \hbar^2 n^{2/3}$ is the Fermi energy for $B_0 = 0$.

When inequality (20) is violated, the degeneracy should be taken into consideration and the function $f_0(\mathbf{p})$ becomes more complicated. Nevertheless, in the Hartree approximation it has the simple form [5]

$$f_0(\mathbf{p}) = \frac{2}{(2\pi\hbar)^3} \sum_s (-1)^s$$

$$\times \frac{L_s\left(p_\perp^2/m\hbar\Omega\right) \exp\left(-p_\perp^2/m\hbar\Omega\right)}{1 + \exp\left\{T^{-1}\left[p_z^2/2m + \hbar\Omega(s+1/2) - \zeta\right]\right\}}, \qquad (21)$$

where $L_s(x)$ is the Laguerre function, and ζ is the chemical potential, which coincides with the Fermi energy E_{F} for free electrons. The summation in expression (21) is extended over all the Landau levels s.

Notice that the extension of the results derived in the foregoing to a multicomponent plasma medium is apparent and reduces to a simple summation over the components in formulas (3), (5), (12), (14)–(17). It is significant that the plasma dielectric constant in a quantized magnetic field can equally be derived through the direct solution of the Wigner equation with the distributions (18) or (21). However, this procedure is found to be very complicated owing to the arduous mathematical treatment [6, 7]. The application of formulas (16) and (17) may prove to be preferable.

Thus, with the appropriate averaging over the distribution function, the simple cold hydrodynamic model describes the kinetic properties of a quantum plasma as fully as of a classical one. This was demonstrated above in the linear approximation. But nonlinear processes call for special consideration.

References

1. Aleksandrov A F, Bogdankevich L S, Rukhadze A A *Osnovy Elektrodinamiki Plazmy* (Foundations of Electrodynamics of Plasma) 2nd ed. (Moscow: Vysshaya Shkola, 1988) [Translated into English (Berlin: Springer, 1984)]
2. Silin V P, Rukhadze A A *Elektromagnitnye Svoĭstva Plazmy i Plazmopodobnykh Sred* (Electromagnetic Properties of Plasmas and Plasma-like Media) (Moscow: Gosatomizdat, 1961)
3. Blokhintsev D I *Osnovy Kvantovoĭ Mekhaniki* (Principles of Quantum Mechanics) (Moscow: Vysshaya Shkola, 1963) [Translated into English as *Quantum Mechanics* (Dordrecht: Reidel, 1964)]
4. Rukhadze A A, Shafer V Yu *Kratk. Soobshch. Fiz.* (10) 41 (1980)
5. Zil'berman P S *Fiz. Tverd. Tela* (Leningrad) **12** 1697 (1970)
6. Zyryanov P S, Kalashnikov V P *Zh. Eksp. Teor. Fiz.* **40** 1119 (1961) [*Sov. Phys. JETP* **13** 894 (1961)]
7. Eleonskiĭ V M, Zyryanov P S, Silin V P *Zh. Eksp. Teor. Fiz.* **42** 896 (1962) [*Sov. Phys. JETP* **15** 804 (1962)]