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The relation of Thomas precession to Ishlinskiĭ's theorem as applied to the rotating image of a relativistically moving body

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Abstract. It is shown that for a solid body following a curvilinear trajectory its rotation angle due to the effect of the special theory of relativity (Thomson precession) is numerically equal to the rest-frame-observed solid angle through which the body-fixed axis turns as a consequence of the rotation change the body image undergoes due to Lorentz length contraction and the retardation of the light emitted by various portions of the body. In classical mechanics, the same relation connects the solid-body rotation angle to the actual solid angle that the body-fixed axis describes as the body performs a conical motion — which is a consequence of Ishlinskiĭ's theorem.

1. Introduction

Thomas precession [1, 2] is a relativistic kinematic effect in which the axis of a gyroscope (point-like compass) turns (precesses) when its point of support moves along a curvilinear trajectory [3, 4]. The aim of the present paper is to show that a deep physical analogy exists between Thomas precession and the classical mechanics phenomenon of a solid-body turn during conical movement. The turning angle is numerically equal to the solid angle described by the body axis, which is a consequence of the Ishlinskiĭ theorem [5 – 9]. In other terms, Thomas precession can be interpreted as a consequence of the rotation (change in orientation) of the solid-body image in a rest system of reference when the body moves along a curvilinear path. This turn is caused by the relativistic contraction of length and the retardation of the light emitted by different parts of the body [10 – 14].

It should be noted that the apparent turn of an image of relativistically moving body, observed in the rest frame, does not imply that the body changes its orientation in space [13, 14]. The effect considered is a manifestation of relativistic aberration [14, 15].

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2. Thomas precession

Let us consider Thomas precession in more detail and its physical consequences. As noted above, this effect leads to the turn of a gyroscope moving along a curvilinear trajectory. In the general case, the *gyroscope* means some solid body or material particle which determine a certain direction in space. An example is provided by a spherically symmetric (about the center of gravity) solid body moving in gravitational field in a curvilinear trajectory, circular or elliptical, or a parabolic orbit, which conserves its spatial orientation due to the law of inertia. Another example is a material particle with a spin, like an electron, neutron, etc.

The angular velocity of the precession in the laboratory frame has the form [4]

$$\Omega_T = \left(1 - \frac{1}{\gamma}\right) \frac{[\mathbf{v}\,\mathbf{a}]}{v^2}\,, \tag{1}$$

where $\mathbf{v}$ and $\mathbf{a}$ are the velocity and acceleration in the laboratory frame, respectively; $\gamma = (1 - v^2/c^2)^{-1/2}$, and c is the speed of light. In the particular case of motion in a circular orbit with radius r and angular velocity $\omega = v/r$ (see Fig. 1) one gets

$$\Omega_T = \omega\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right). \tag{2}$$

In the comoving frame of reference the angular velocity of Thomas precession is γ times as high as in the laboratory frame (1), (2).

The body turns in one revolution through the angle

$$\alpha = 2\pi\frac{\Omega_T}{\omega} = 2\pi\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right). \tag{3}$$

In quantum physics, the Thomas precession appears as an oscillation in the complex probability amplitudes of the particle spin states in a force field [16].

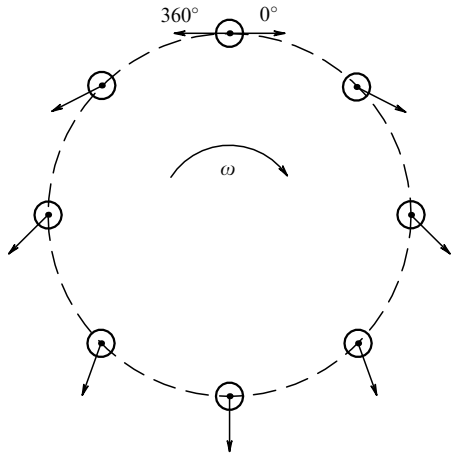

Figure 1. Turn of axis of a solid body caused by Thomas precession during a circular motion with $\gamma = 2$.

Thomas precession takes into account precession-induced corrections for calculation of the spin-orbit interaction effect on fine structure of atomic spectra [1, 2, 17], explains the anomalous Zeeman effect [1, 2], and provides a qualitative explanation for nucleon interactions and the reason for the doublet 'inversion' inside a nucleus [4].

Thomas precession causes an additional shift in the zero of the interference pattern formed by counter de Broglie waves of material particles (electrons, neutrons, atoms, etc.) in interferometric angular velocity gauges [18, 19] based on the Sagnac effect [20]. The reason for this zero shift, which does not relate to the interferometer rotation, is as follows. Let the particles have the same polarization at the interferometer input, i.e. the same spin orientation. Then at the interferometer output the Thomas precession for particles propagating in opposite directions causes an angular turn of the spin orientation with the same absolute values but with opposite signs, so the polarization states of interfering particles become different. The interference pattern zero shift of counter de Broglie waves is similar to the polarization nonreciprocity in fiber ring interferometers [21] and can, in principle, be excluded by using spinless material particles, for example, π-mesons [22].

Thomas precession for muons was directly registered twenty years ago at the ring accelerator in CERN [23, 24].

About forty years ago L Schiff (Stanford University, California) suggested an experiment [25, 26] for the discovery of the effect of Thomas precession and two effects of the general theory of relativity — geodetic precession [27, 28] and the Lense – Thirring effect [28 – 30], using a one-axis mechanical gyroscope mounted on a satellite moving along a drift-free polar orbit around the Earth. To increase the measurement accuracy, two identical oppositely orbiting satellites were presumed to be launched. In this case Thomas precession and effects of the general theory of relativity lead to opposite turns of the gyroscope axes on the satellites. The experiment is assumed to last from one to several years.

Equation (3) implies that for a satellite velocity lying between the minimum orbital and escape velocities, the rotation angle due to Thomas precession of gyroscope axes in one orbital revolution is about 3×10^{-9} rad. In this case, the angular velocity of the gyroscope axis turn may amount to 0.2 angle second per year. The contributions of the effects of the general theory of relativity to the gyroscopic axis turn are about 7 and 0.05 angle second per year from geodetic precession and the Lense – Thirring effect, respectively [28].

The three above effects on the gyroscope axis turn can be separated by mounting three gyroscopes on the satellite, with one of them being directed along the orbital normal, the second along the binormal, and the third tangential to the orbit.

So far such an experiment has not been realized, although the construction of a one-axis mechanical gyroscope with the required precision has been continuously pursued at Stanford since 1964 [28, 31]. According to estimates [31], the present accuracy of such a gyroscope designed there, which consists of a quartz sphere 5 cm in diameter coated with a thin niobium superconducting film and suspended in an electrostatic field, should be better than 3×10^{-6} angle second per year in microgravity, which is quite enough for the experimental purposes.

It should be noted, however, that the main goal of the Stanford experiment is not the detection of Thomas or geodetic precession (which in particular manifests itself as the perihelion precession of the planet orbits and was discovered in 1859 by French astronomer Leverrier [32] for the example of Mercury orbit perihelion precession and later adequately explained by general relativity [27]), but measurement of the Lense – Thirring effect. The latter is analogous to the electromagnetic effect of mutual induction of two turns with currents and appears as an interaction of two rotating masses; so far this effect has not been detected because of its smallness. The measurement of the Lense – Thirring effect will allow some predictions of general relativity to be tested [28]. Therefore, if experiments considered in Refs [25, 26, 28, 31] are realized, it is unclear how the axis of a satellite-mounted gyroscope will be oriented in space and whether Thomas precession will be measured using a mechanical gyroscope in the near future.

Notice that recently some papers have appeared (see, for example, Ref. [33]) arguing that Thomas precession for material particles with both mechanical and magnetic quantum momenta (in particular, for electrons) breaks the principle of relativity. As shown in Ref. [34], however, such statements are erroneous.

3. The Ishlinskiĭ theorem and its applications

Consider now a kinematic effect in classical mechanics which has, as will be shown below, much in common with Thomas precession. In the beginning of 1950s, A Yu Ishlinskiĭ proved a theorem, also called the *solid-angle theorem* [5, 6] (see also Refs [7 – 9]), which can be formulated as follows [9]: if some axis in a solid body with three degrees of freedom has described a closed conical surface when executing a motion and the projection of the body's angular velocity onto this axis has been zero, then after the axis has returned to its initial position, the body turns around this axis by an angle numerically equal to the solid angle of the circumscribed cone (see Fig. 2). Note here that this equality is valid to factor $2\pi N$, where N is an integer [35, 36]. The translational movement of the axis is of no importance in the process.

We shall consider a particular example of this effect in classical mechanics. Let a wheel initially at rest be settled on an axis without friction. If the axis describes in space some solid angle, the wheel turns through this angle after the axis have returned to the initial position. In our paper [37] this additional angle by which the body turns during its space

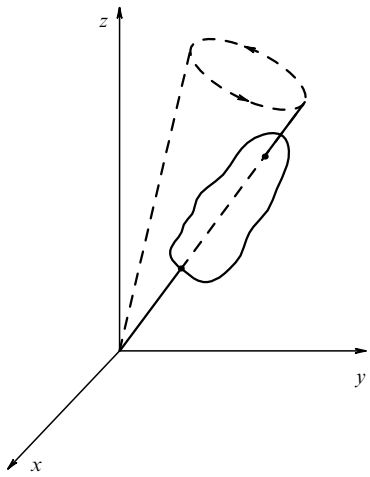

Figure 2. Solid angle described by the axis of a solid body in a conical motion.

evolution was called the *Ishlinskiĭ angle*. As shown in Refs [37, 38], the Ishlinskiĭ angle is the manifestation of the geometrical (topological) phase in classical mechanics, often called the Berry phase [39] (see Refs [40–44] about manifestations of the geometrical phase in various physical phenomena). If the axis describes some conical surface in opposite directions, the absolute value of the Ishlinskiĭ angle is the same for opposite displacements but its sign is different.

The Ishlinskiĭ angle is independent of the initial and final positions of the axis in space but depends on the path the axis describes when being in motion, so the accumulation of the Ishlinskiĭ angle is a nonholonomic phenomenon [45]. The latter also follows from the fact that, as was shown in Refs [5, 6], the accumulation of the Ishlinskiĭ angle of a mechanical system during its spatial evolution occurs when nonholonomic constrains exist in the system.

The Ishlinskiĭ theorem finds use in gyroscope theory. In particular, it explains the appearance of the angular error in a space gyrocompass — a gyroframe, inside which a pair of connected mechanical gyroscopes with parallel axes is mounted, and in gyroscopes with strong correction. This error is due to the change of the space orientation of the vertical axis, around which the gyroframe or correspondingly the external shell of the strongly corrected gyroscope are free to rotate during the gyrocompass movement across the Earth surface [5, 6].

Notice that the effect considered is closely related to the so-called parallel vector translation in Riemann geometry [5, 6].

4. The apparent turn of an object rapidly moving in a circular orbit and Thomas precession

Over the more than 50 years since special theory of relativity appeared, the size of a rapidly moving body as seen by an observer at rest has been considered to be squeezed γ times in the direction of motion. However, in 1959, Penrose [10] and Terrell [11] noted that the light quanta simultaneously reaching the observer are emitted by different points of a body at different times — the points located further away from the observer emit light earlier than closer ones. This causes a compensation of the Lorentzian contraction and in the case where the size of the object is much smaller than the distance to it, the object or, more precisely speaking, its image on the retina of the observer or on the camera film looks undistorted but rotated through some angle. Presently, there are a lot of papers on this subject; the most detailed study can be found in Refs [12–14].

In our case we are interested in the change of the orientation angle (registered by the observer at rest) of a body moving along a circular path. This angle can be determined by the shape of the body, for example, by one of its faces if the body takes on a polyhedral form or the body axis. We recall once again that the apparent turn of the image of a relativistically moving body as observed in the rest frame does not imply that the body changes its orientation in space [13, 14].

In the simple case of rectilinear motion of an object with velocity $\mathbf{v}$, the angle Θ' determining some direction in the comoving frame of reference relates to the angle Θ as observed in the laboratory (rest) frame by the well-known relativistic aberration formula [14, 15]:

$$\sin\Theta = \frac{\sqrt{1 - v^2/c^2}\,\sin\Theta'}{1 + \cos\Theta' \cdot v/c}\,. \tag{4}$$

The observer at rest sees the object turned by the aberration angle $\Delta\Theta = \Theta - \Theta'$. Consider an object in rectilinear motion in the plane perpendicular to the line of sight. Assume that the axis setting the direction also lies in a given plane, i.e. $\Theta' = 90°$, then from (4) we find

$$\Delta\Theta = \Theta - 90°\,, \quad \cos(\Delta\Theta) = \sqrt{1 - \frac{v^2}{c^2}}\,.$$

Consider now an object in circular motion in this plane with $\Theta' = 90°$ as above. In this case the apparent orientation of the axis (registered by the observer at rest) will change: the image of the object will turn such that in one revolution its axis describes a cone with the vertex angle $2\Delta\Theta$ (see Fig. 3). The solid angle contained by the cone is numerically equal to the area on a unit-radius sphere enclosed by the element of the cone with the vertex at the center of the sphere [46]. From here it is easy to derive the expression relating the solid angle with the vertex angle of a cone:

$$\chi = 4\pi\sin^2\left(\frac{\Delta\Theta}{2}\right) = 2\pi\big(1 - \cos(\Delta\Theta)\big) = 2\pi\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right). \tag{5}$$

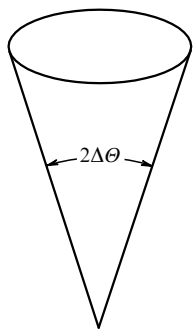

Figure 3. Apparent cone described in the rest frame by the image of a solid-body axis in a relativistic circular motion. $\Delta\Theta$ is the relativistic aberration angle.

This effect can be illustrated by the example of images (observed at rest) of a die (cube) rotating along a circular path, which conserves its space orientation (Figs 4, 5). The die rotates in a plane normal to the line connecting the observer's pupil with the center of the circle, with the distance to the plane being much larger than the diameter of the circle. At the top point of the circle (position *1* in Figs 4, 5) the die is oriented in such a manner that the observer sees the face 'six', the 'four' is on the leading side, the 'three' is on the receding face, the 'five' is on the top face, the 'two' is on the bottom face, and the 'one' is on the opposite face. The direction of motion at each point is marked with an arrow. If the velocity of the die is $v \ll c$, the observer at rest sees the face 'six' at every moment and the image orientation does not change during the motion (Fig. 4). When $v \sim c$, the observer sees the die differently turned to him at different points of the circle (Fig. 5). Thus the observer watches that some axis related to the die (for example, one of its edges) describes some solid angle in one revolution. The image in Fig. 5 corresponds to $\gamma = 2$.

Comparing Eqns (3) and (5) we obtain $\alpha \equiv \chi$, i.e. the turning angle of the body due to Thomas precession is equal to the Ishlinskiĭ angle by which the body turns when moving in a circular orbit, if the actual change in its orientation angle is equal to the turning angle of a body relativistically moving along a curvilinear path as viewed in the laboratory frame. Thus, Thomas precession can be interpreted as a consequence of the formal application of the Ishlinskiĭ theorem to the solid angle corresponding to a change in the apparent turn of the image of a solid body in its motion along a curvilinear path relative to an observer at rest.

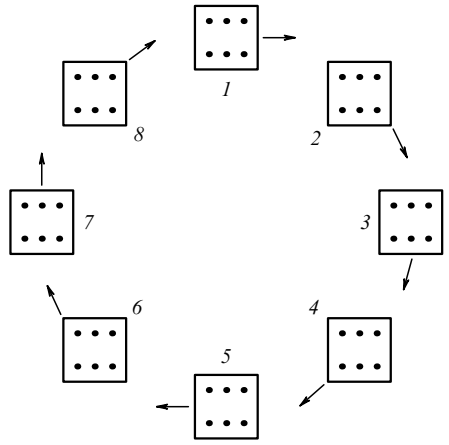

Figure 4. Image of a die in circular motion as seen at rest, for the case $v \ll c$.

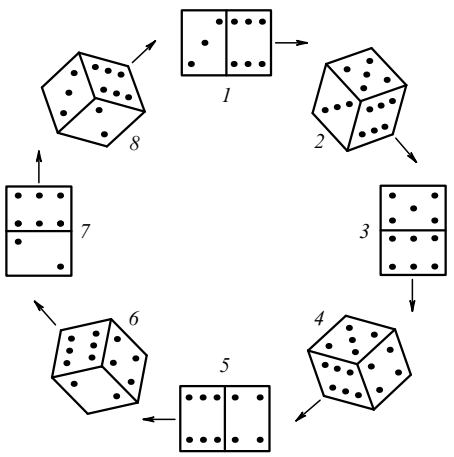

Figure 5. Image of a die in circular motion as seen at rest, for the case of relativistic motion with velocity $v = 0.865\,c$ ($\gamma = 2$).

It should be emphasized once again that in this instance we do not lead our conversation towards a real solid angle described by the body-related axis but we mean an apparent (observable) solid angle corresponding to the change of the solid-body image turn during its movement along the curvilinear trajectory.

Note here that Thomas precession arises not because the body (or some axis in the body) is observed in the laboratory frame as turned through some angle, but because this angle changes in the course of the body's motion along a curvilinear path, which leads to the apparent axis describing the solid angle.

5. Physical sense of Thomas precession and the Ishlinskiĭ angle

Let us next consider the physical reasons giving rise to the discussed effects. Thomas precession is explained by the relativity of the notion of curvilinear translational movement of a system of material points. If in one inertial system **K** all points of the body at instant t have the same velocity, they will differ in another inertial frame $\mathbf{K}'$ at instant t' for accelerated motion of the body [4].

The effect described by the Ishlinskiĭ theorem is due to the kinematics of a solid body as a system of material points in classical mechanics not reducing to the kinematics of a single point. Kinematic equations of a solid body written in any form have much more complicated structure than those of a material point. If the projection of the velocity of the material point on some axis is zero, the corresponding coordinate does not change. This is not the case for a solid body. If the projection of the angular velocity of the body on some axis is zero, the body does not remain at rest with respect to this axis [9]. Thus, both effects considered in special theory of relativity and in classical mechanics are caused by the specifics of the curvilinear motion of the solid body as a system of material points.

An analogy between the two effects considered is drawn by the relativistic (also called quadratic) and classical Doppler effects. These effects have different causes — the relativistic time dilation in a moving object emitting a wave (e.g., electromagnetic wave) relative to an observer at rest, and by the receding of the object with some velocity away from the observer, respectively. However, the results are the same in both cases — the observer registers a reduction of the emitting frequency. Notice that when $v \ll c$, Thomas precession, like the relativistic Doppler effect, depends quadratically on the velocity [see Eqn (2)].

Note here that both Thomas precession and the Ishlinskiĭ angle are manifestations of the geometrical (topological) phase (angle) in relativistic and classical mechanics, respectively. For example, papers [47, 48] showed that the Thomas precession effect on spin-orbit interaction appears as the Berry phase in quantum mechanics. Here we should emphasize the paper by Sommerfeld [49] (see also Ref. [50]), in which the expression for Thomas precession is derived using the constructions on a sphere with an imaginary radius. Such

constructions are typical for calculating the geometrical phase [37]. As noted above, the Ishlinskiĭ angle is also the manifestation of the geometrical phase but, in this instance, in classical mechanics [37, 38].

6. Conclusions

The main results of the present paper can be summarized as follows:

1. A physical analogy is shown to exist between two different kinematic effects, i.e. Thomas precession in special theory of relativity and the effect described by the Ishlinskiĭ theorem in classical mechanics.

2. The reasons for the change of the solid-body orientation in space are different for both effects: for the first, this is measured in the laboratory frame as an apparent change in the rotation (change in turn) of a solid body (axis of the body) in its curvilinear motion; for the second, this is the result of a real turn of the body (axis of the body) in its conical motion. However, both effects lead to the same consequences for the solid body: after its axis has returned to the original position, the body is found to be turned by some angle numerically equal to the solid angle described by the axis.

3. Both effects considered in special theory of relativity and classical mechanics are caused by the specifics of the curvilinear motion of a solid body as a system of material points.

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References

1. Thomas L H *Nature* (London) **117** 514 (1926)
2. Thomas L H *Philos. Mag.* **3** 1 (1927)
3. Møller C *The Theory of Relativity* (Oxford: Clarendon Press, 1972) [Translated into Russian (Moscow: Atomizdat, 1975)]
4. Malykin G B, Permitin G V "Thomas precession", in *Fizicheskaya Éntsiklopediya* (Physical Encyclopedia) Vol. 5 (Moscow: Rossiĭskaya Entsiklopediya 1998) p. 108
5. Ishlinskiĭ A Yu *Mekhanika Spetsial'nykh Giroskopicheskikh Sistem* (Mechanics of Special Gyroscopic Systems) (Kiev: Izd. AN USSR, 1952)
6. Ishlinskiĭ A Yu *Mekhanika Giroskopicheskikh Sistem* (Mechanics of Gyroscopic Systems) (Moscow: Izd. AN SSSR, 1963)
7. Zhbanov Yu K, Zhuravlev V F *Izv. Akad. Nauk SSSR Mekh. Tverd. Tela* (1) 9 (1978)
8. Zhuravlev V F *Prikl. Mat. Mekh.* **60** 323 (1996)
9. Zhuravlev V F *Osnovy Teoreticheskoĭ Mekhaniki* (Principles of Theoretical Mechanics) (Moscow: Nauka, Fizmatlit, 1997)
10. Penrose L *Proc. Cambridge Philos. Soc.* **55** (1) 137 (1959)
11. Terrell J *Phys. Rev.* **116** 1041 (1959)
12. Weisskopf V F *Physics in Twentieth Century* (Cambridge: MIT Press, 1972) [Translated into Russian (Moscow: Atomizdat, 1977)]
13. Bolotovskiĭ B M, in *Èĭnshteĭnovskiĭ Sbornik 1980–1982* (Einstein's Collection 1980–1982) (Ed. I Yu Kobzarev) (Moscow: Nauka, 1985) p. 142
14. Bolotovskiĭ B M, in *Èĭnshteĭnovskiĭ Sbornik 1986–1990* (Einstein's Collection 1986–1990) (Ed. I Yu Kobzarev) (Moscow: Nauka, 1990) p. 279
15. Landau L D, Lifshitz E M *The Classical Theory of Fields* (Oxford: Pergamon Press, 1975) [Translated from Russian edition (Moscow: Nauka, 1973)]
16. Schiff L I *Quantum Mechanics* (New York: McGraw-Hill, 1955) [Translated into Russian (Moscow: IL, 1959)]
17. Jackson J D *Classical Electrodynamics* (New York: Wiley, 1962) [Translated into Russian (Moscow: Mir, 1965)]
18. Anandan J *Phys. Rev. D* **24** 338 (1981)
19. Mashhoon B *Phys. Rev. Lett.* **61** 2639 (1988)
20. Malykin G B *Usp. Fiz. Nauk* **167** 337 (1997) [*Phys. Usp.* **40** 317 (1997)]
21. Malykin G B *Opt. Spektrosk.* **76** 540 (1994) [*Opt. Spectrosc.* **76** 484 (1994)]
22. Malykin G B *Izv. Vyssh. Uchebn. Zaved., Radiofizika* **41** 767 (1998)
23. Field J H, Picasso E, Combley F *Tests of Fundamental Physical Theories from Measurements of Free Charged Leptons* (Geneva: CERN, 1978)
24. Field J H, Picasso E, Combley F *Usp. Fiz. Nauk* **127** 553 (1979) [*Sov. Phys. Usp.* **22** 199 (1979)]
25. Schiff L I *Phys. Rev. Lett.* **4** 215 (1960)
26. Schiff L I *Nuovo Cimento* **17** 124 (1960)
27. Einstein A *Sitzungsber. preuss. Acad. Wiss.* **47** 831 (1915) [Translated into Russian in *Sobranie Nauchnykh Trudov* T. 1 (Moscow: Nauka, 1965) p. 439]
28. Shapiro I, in *Astrofizica e Cosmologia Gravitazione Quanti e Relativita* (Fireneze: Giunti Barbera, 1979) [Translated into Russian (Moscow: Mir, 1982) p. 215]
29. Lense J, Thirring H *Phys. Z.* **19** 156 (1918)
30. Thirring H *Phys. Z.* **22** 29 (1921)
31. Lange B *Phys. Rev. Lett.* **74** 1904 (1995)
32. Leverrier U J *Ann. Observ. Imper.* (Paris) **5** 19 (1859)
33. Neganov B S *Hadr. J.* **14** 377 (1991)
34. Perelstein E A, Perelstein M E *Found. Phys. Lett.* **8** 449 (1995)
35. Goodman L E, Robinson A R *J. Appl. Mech.* **25** 210 (1958)
36. Goodman L E, Robinson A R, in sb. *Mekhanika* (Mechanics Collection) N 5 (57) (Moscow: IL 1959) p. 133
37. Malykin G B *Izv. Vyssh. Uchebn. Zaved., Radiofizika* **40** 265 (1997)
38. Malykin G B *Opt. Spektrosk.* **81** 474 (1996) [*Opt. Spectrosc.* **81** 431 (1996)]
39. Berry M *Proc. R. Soc. London Ser. A* **392** 45 (1984)
40. Vinitskiĭ S I, Derbov V L, Dubovik V N, Markovskiĭ B L, Stepanovskiĭ Yu P *Usp. Fiz. Nauk* **160** (6) 1 (1990) [*Sov. Phys. Usp.* **33** 403 (1990)]
41. Berry M *Phys. Today* **43** 34 (1990)
42. Anandan J *Nature* (London) **360** 307 (1992)
43. Klyshko D N *Usp. Fiz. Nauk* **163** (11) 1 (1993) [*Phys. Usp.* **36** 1005 (1993)]
44. Bodnarchuk V I, Davtyan L S, Korneev D A *Usp. Fiz. Nauk* **166** 185 (1996) [*Phys. Usp.* **39** 169 (1996)]
45. Neĭmark Yu I, Fufaev N A *Dinamika Negolonomnykh Sistem* (Dynamics of Nonholonomical Systems) (Moscow: Nauka, 1967)
46. Vygodskiĭ M Ya *Spravochnik po Èlementarnoĭ Matematike* (Elementary Mathematics Handbook) (Moscow: Nauka, 1982)
47. Mathur H *Phys. Rev. Lett.* **67** 3325 (1991)
48. Shankar R, Mathur H *Phys. Rev. Lett.* **73** 1565 (1994)
49. Sommerfeld A *Atombau und Spektrallinien* Bd. I (Braunschweig: Fiedr. Vieweg & Sohn, 1951) [Translated into Russian (Moscow: GITL, 1956) p. 579]
50. Belloni L, Reina R, in *Èĭnshteĭnovskiĭ Sbornik 1984–1985* (Einstein's Collection 1984–1985) (Ed. I Yu Kobzarev) (Moscow: Nauka, 1988) p. 201