Physics – Uspekhi **42** (4) 385 – 390 (1999)

METHODOLOGICAL NOTES PACS number: 41.20.Cv

On the interaction of two electrically charged conducting balls

V A Saranin

Contents

Abstract. Departures from Coulomb's law displayed by two electrostatically interacting conducting balls are examined in detail. By computing forces on the balls as a function of the ball separation, it is found that at short separations a switch from repulsion to attraction occurs in the general case of arbitrary, likely charged balls. The only exception is the case in which the charges of the balls are related as the squares of their radii: such balls always repel each other. For identical balls of equal charges in magnitude, asymptotic short-separation relations for energies and forces are found. For most of the results obtained the self-similarity property is shown to apply.

1. Introduction

The problem of the interaction between two conducting charged balls is classical and has been considered in a number of monographs and manuals [1 – 6]. However, the data there are presented rather formally (for example, as infinite series), which does not enable one to get answers to simple but scientifically and methodically important questions. Thus it is known that at small distances between the balls this interaction cannot be described by Coulomb's law. This may bring up the questions of what is the character of deviations from Coulomb's law, or what error the Coulomb approximation yields (i.e. replacement of the balls by equivalent point charges located at their centers) in calculations of the force and the energy of the interaction? These questions are not discussed in the scientific and methodical literature, though the problem of the interaction between two charged balls can be found in virtually every school or student's problem book on physics. The reason is probably that the cumbersome final expressions for the energy and the force of interaction do not enable one to investigate the problem analytically in a simple manner. However, some special methods and the use of computers make such an investigation possible.

In this paper prominence is given to the derivation and discussion of final results, detailed calculations can be found, for example, in Ref. [1].

V A Saranin V G Korolenko Glazov State Pedagogical Institute,
ul. Pervomaĭskaya 25, 427600 Glazov, Russian Federation
Tel. (341-41) 4 77 82
E-mail: saranin@ggpi.glazov.udm.net

Received 28 October 1998
Uspekhi Fizicheskikh Nauk **169** (4) 453 – 458 (1999)
Translated by G N Chuev; edited by A Radzig

2. Statement of the problem and the method of electrical images

Let us consider two conducting balls of radii R_1 and R_2 with charges q_1 and q_2, whose centers are separated by a distance $l \geqslant R_1 + R_2$ (Fig. 1) The sought-for potential energy of the interaction between the balls and the force acting on each ball can be presented as

$$
\begin{aligned}
W &= \frac{1}{2}\left(q_1^2 s_{11} + 2 s_{12} q_1 q_2 + q_2^2 s_{22}\right), \\
F_l &= -\frac{\partial W}{\partial l}.
\end{aligned} \tag{1}
$$

Here s_{11}, s_{12}, s_{22} are the potential coefficients. It is simpler to find capacitive coefficients instead of the potential ones. Therefore, we turn to the capacitive coefficients in Eqn (1)

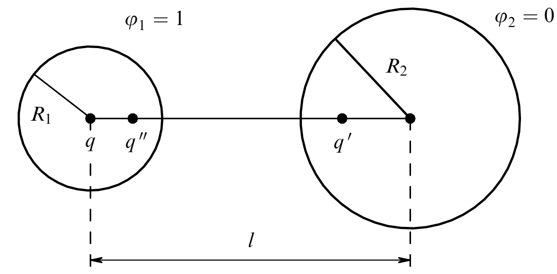

Figure 1. Defining capacitive coefficients of two conducting balls.

by the relationships

$$s_{11} = \frac{c_{22}}{c_{11}c_{22} - c_{12}^2}\,,$$
$$s_{22} = s_{11}\frac{c_{11}}{c_{22}}\,,$$
$$s_{12} = -\frac{c_{12}}{c_{11}c_{22} - c_{12}^2}\,. \tag{2}$$

Using the method of electrical images, we briefly outline the procedure of deriving expressions for the capacitive coefficients. By the definition of the coefficients the charges of the balls can be written as

$$q_1 = c_{11}\varphi_1 + c_{21}\varphi_2\,,$$
$$q_2 = c_{12}\varphi_1 + c_{22}\varphi_2\,.$$

Here c_{11} stands for the charge of the first ball, while c_{12} is the same of the second ball, providing that the potential of the first ball is equal to unity and the potential of the second ball is sustained at a zero value. Mentally excluding for the time being the charge q_1 from the consideration, we place a charge $q = 4\pi\varepsilon_0 R_1$ at the center of the first ball. Then the potential of the first ball is equal to unity, but an image arises at the second ball, i.e. the charge q'. It can be shown that the image is located at the distance R_2^2/l from the center of the second ball and if its value is equal to $q' = -4\pi\varepsilon_0 R_1 R_2/l$, the potential of the second ball remains zero. However, the image q' is an origin for an image q'' at the first ball. Fitting the value of q'', we can make the potential of the first ball to be equal to unity. Thus, it is seen that there arises an infinite series of image charges at each ball. Fitting their values we can make the potentials of the first and second balls equal to unity and zero, respectively. The total charges of each ball will be calculated as

$$c_{11} = 4\pi\varepsilon_0 R_1 + \sum_{n=1}^{\infty} q^{(2n)}\,,$$
$$c_{12} = \sum_{n=1}^{\infty} q^{(2n-1)}\,. \tag{3}$$

The coefficient c_{22} can easily be obtained from the expression for c_{11} due to the symmetry of the problem.

3. Interaction of a point charge and charged conducting ball

The simplest limiting case is the interaction between a charged conducting ball of radius R and a point charge q_2. In this case only a single image charge arises at the ball, therefore the solution to the problem can be found analytically in the explicit form [3, 7]:

$$W = \frac{kq_1q_2}{l} - \frac{kq_2^2R^3}{2l^2(l^2 - R^2)}\,,$$
$$F_l = \frac{kq_1q_2}{l^2} - \frac{kq_2^2R^3(2l^2 - R^2)}{l^3(l^2 - R^2)^2}\,, \tag{4}$$

where $k = 1/(4\pi\varepsilon_0)$. We may write these expressions in a dimensionless form, choosing R as the unit of length, $kq_1^2/(2R)$ as the energy unit, and $kq_1^2/(2R^2)$ as the force unit. Then we arrive at

$$W = \alpha\left[\frac{2}{x} - \frac{\alpha}{x^2(x^2-1)}\right],$$
$$F_x = 2\alpha\left[\frac{1}{x^2} - \frac{\alpha(2x^2-1)}{x^3(x^2-1)^2}\right], \tag{5}$$

where $x = l/R$, and $\alpha = q_2/q_1$. The most interesting effect of this interaction is that the sign of the force changes at $\alpha > 0$, i.e. when the charges are like: the repulsion between them can change to attraction. This occurs at a distance x_0 satisfying the equation

$$\alpha = \frac{x_0(x_0^2-1)^2}{2x_0^2 - 1}\,. \tag{6}$$

Figure 2 plots the line $x_0(\alpha)$ at which the force is zero. The region below the line corresponds to attraction of the ball and the point charge, while the region above the line corresponds to repulsion. The attraction is seen to take place when the point charge is rather large or when the distances between the charge and the ball are relatively small. It is easily verified that the potential energy peaks on the equilibrium line so that, according to the Earnshaw theorem, the equilibrium is not stable.

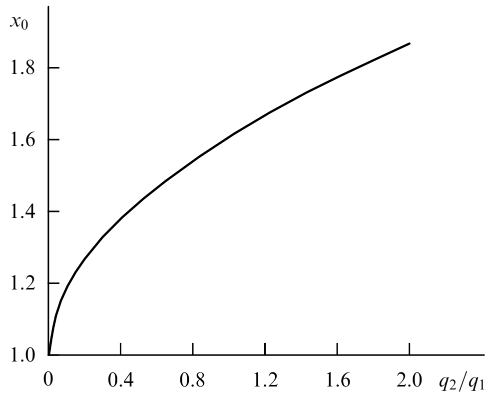

Figure 2. Line of zero force for the interaction between a point charge and likely charged ball. The region below the line corresponds to attraction, while the region above the line corresponds to repulsion.

4. Interaction of conducting balls with arbitrary charges

The calculations of capacitive coefficients using (3), which can be found, for example, in Ref. [1], yield the following expressions:

$$c_{11} = 4\pi\varepsilon_0 R_1\gamma\sinh\beta\sum_{n=1}^{\infty}\Big\{\gamma\sinh(n\beta) + \sinh\big[(n-1)\beta\big]\Big\}^{-1},$$
$$c_{12} = -4\pi\varepsilon_0 R_1\gamma\frac{\sinh\beta}{(1+\gamma)x}\sum_{n=1}^{\infty}\big[\sinh(n\beta)\big]^{-1},$$
$$c_{22} = 4\pi\varepsilon_0 R_1\gamma\sinh\beta\sum_{n=1}^{\infty}\Big\{\sinh(n\beta) + \gamma\sinh\big[(n-1)\beta\big]\Big\}^{-1}. \tag{7}$$

Here $\gamma = R_2/R_1$, the unit length is chosen to be $R_1 + R_2$ so that the dimensionless distance x between the centers is equal to $x = l/(R_1 + R_2)$. The parameter β is related to this distance by the expression

$$\cosh\beta = \frac{x^2(1+\gamma)^2 - (1+\gamma^2)}{2\gamma} \equiv y\,. \qquad (8)$$

We use $k|q_1 q_2|/(R_1 + R_2)$ as a measurement unit of energy. Then, the dimensionless energy of the interaction between the balls is

$$W = \frac{1+\gamma}{2\alpha}\,\frac{\alpha^2 c_{11} - 2\alpha c_{12} + c_{22}}{c_{11}c_{22} - c_{12}^2}\,. \qquad (9)$$

Here, the dimensionless coefficients c_{11}, c_{12}, and c_{22} are expressed using (7) without the multiplier $4\pi\varepsilon_0 R_1$.

The expression for the force is obtained by differentiation of (9) with respect to the distance x between the centers of the balls. In units of $k|q_1 q_2|/(R_1 + R_2)^2$ it is written as

$$F_x = -\frac{\partial W}{\partial x} = -\frac{x(1+\gamma)^3}{2\alpha\gamma\sinh\beta} f(c_{ik}, c'_{ik})\,, \quad i,k = 1,2\,. \qquad (10)$$

Here f denotes the derivative of the second fraction in (9) with respect to β. The derivatives c'_{ik} of capacitive coefficients involved in f with respect to β can easily be found; we present neither these nor f because of their cumbersome form.

As mentioned above, an analytical solution to the problem of the interaction between two balls can hardly be obtained in the general case, hence we should use numerical calculations. For this purpose we express all the terms of all series via the variable $z = \exp(-\beta)$. Then, to program and compute the coefficient c_{11}, we can conveniently write it in the form

$$c_{11} = 2\gamma\sqrt{y^2-1} \times \sum_{n=1}^{\infty} \frac{z^n}{(1-z^{2n})\left[(1+\gamma y) - \gamma\sqrt{y^2-1}(1+z^{2n})/(1-z^{2n})\right]}\,.$$

Other coefficients are written similarly. The definition of y is given by (8). As n increases, z^n and z^{2n} decrease, when the value of z^n becomes zero to the accuracy of computations, the calculation of the sum is complete.

Let us consider more carefully two cases of the interaction between charged conducting balls of different sizes, which are widely met in practice and theory: (a) the potentials of the balls are equal with respect to infinity, and (b) the charges of the balls are related as the squares of their radii.

(a) In this case $q_2/q_1 = R_2/R_1$, i.e. $\alpha = \gamma$. Such situation arises when before approaching each other the balls were charged by the same voltage source or when their potentials became equal due to the corona current. At $\alpha = \gamma \neq 1$, the force at small distances between the balls is attractive, and grows indefinitely as the balls approach each other. When the distance between the balls rises the force changes sign, becomes repulsive, reaches a maximum, and then decreases tending asymptotically to Coulomb's law as the balls become farther apart. The typical dependence of the force on the dimensionless distance between the centers of the balls [in the units $(R_1 + R_2)$] is depicted in Fig. 3 (curve *3*) for the case $\alpha = \gamma = 4, 3$. Curve *4* corresponds to the interaction between a point charge and the ball at $\alpha = 1$, and curve *1* to Coulomb's law. The coordinates of zero force and the maximum of force depend on the value of α (or γ) and vary over the ranges $1 \leqslant x_0 \leqslant 1.08$ and $x_0 < x_m \leqslant 1.27$, respectively.

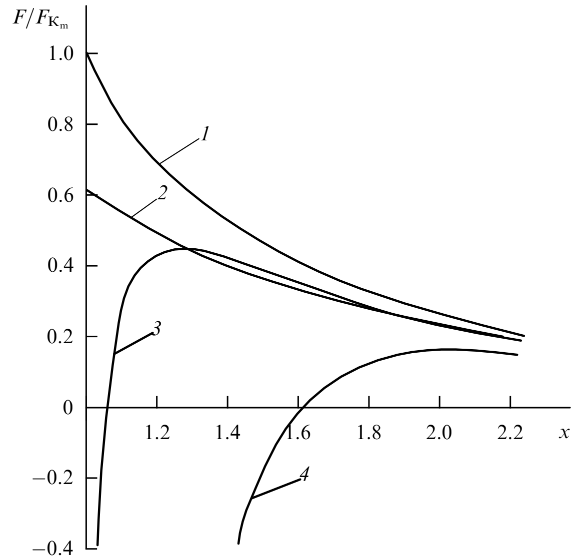

Figure 3. Typical dependences of the force of interaction between two likely charged balls on the distance between the balls at various ratios of their radii and charges: *1* — Coulomb approximation; *2* — force of interaction between two identical balls; *3* — that for two balls with equal potentials; *4* — force of interaction between a point charge and a ball with the same charge. The force is normalized with respect to the maximum F_{K_m} calculated in the Coulomb approximation.

From the physical standpoint this result can be explained as follows. The point charge is attracted to the uncharged conducting ball due to the fact that the induced charges of the same sign as the point one are located at the ball farther from the point charge, while unlike charges reside closer to it. If we place a small charge of the same sign as the point one on the ball, the attraction effect can remain, otherwise we can bring the point charge closer to the ball. The situation for two conducting balls of different radii is similar.

(b) Let the charges of the balls be related as the squares of their radii, i.e. $\alpha = \gamma^2$. This can occur in practice, when the balls are charged inductively in an external electric field. In this case the maximal charge at the ball of radius R is known to be equal to

$$q = c_{\mathrm{a}} E_0 R^2\,,$$

where c_{a} is a constant depending on the particular mechanism of charging, and E_0 is the strength of the external field. It surprisingly turns out that the force behaves invariantly, being repulsive for all distances between the balls. At any ratio γ, the curve of the distance dependence of the force lies in the region between the curve *1* and the curve *2* corresponding to the identical balls with $\gamma = \alpha = 1$ (see Fig. 3).

5. Interaction between charged conducting balls of the same size

It is just this interaction which is most often dealt with in applied and training problems, therefore we shall consider it in greater detail.

Let there be two identical conducting balls of radius R, with charge q, the distance between their centers being l. Then according to Ref. [1] we have

$$c_{11} = c_{22} = 4\pi\varepsilon_0 R \sinh\beta \sum_{n=1}^{\infty} \left\{ \sinh\left[(2n-1)\beta\right] \right\}^{-1},$$

$$c_{12} = -4\pi\varepsilon_0 R \sinh\beta \sum_{n=1}^{\infty} \left[\sinh(2n\beta)\right]^{-1},$$

where the parameter β is determined by the relation $\cosh\beta = l/(2R) \equiv x$. Let us choose the ball diameter $2R$ as a unit of length, and $W_{\max} = kq^2/(2R)$ as a unit of energy. Then, the expression for the dimensionless energy is written as

$$W = \frac{2}{\sinh\beta \sum_{n=1}^{\infty} (-1)^{\varkappa_n} / \sinh(n\beta)}\,, \qquad \varkappa_n = n+1\,. \tag{11}$$

As one would expect in the case of identical balls with charges equal in magnitude, the interaction problem is self-similar.

The potential energy is found within an additional constant, hence we may scale the obtained expression so that the energy of interaction between the balls will tend to zero as $x \to \infty$. We obtain $\sinh\beta \approx \exp\beta$ at $\beta \gg 1$ and the denominator of expression (11) can be written as

$$z_1 \approx 1 + \exp(-\beta)\,.$$

Thus, at $\beta \gg 1$ we have $z_1 \approx 1$, hence the energy of the interaction between the balls, scaled in the indicated way, is equal to

$$W = 2\left[\frac{1}{\sinh\beta \sum_{n=1}^{\infty} (-1)^{\varkappa_n} / \sinh(n\beta)} - 1\right]. \tag{12}$$

Of special interest is obviously the limiting value of the energy at $x \to 1$ ($\beta \to 0$), i.e. when the balls are in contact, since the approximation of ball charges by point ones located at the centers of the balls gives rise to most dramatic changes of the results in this case. Expanding the denominator by its Taylor series expansion with respect to β, we obtain an alternating harmonic series, whose sum is known be equal to $\ln 2$. Therefore, the maximum of the energy of interaction between the balls is

$$W = 2\left(\frac{1}{\ln 2} - 1\right) \approx 0.885\,.$$

Due to the self-similar properties of the problem, the result obtained is universal and means that the approximation of identical ball charges by point charges located at the centers of the balls yields a correction for the interaction energy less than $\sim 12\%$. Figure 4 plots the dependence of the actual interaction energy rationalized with respect to the Coulomb energy on the dimensionless distance between the centers of the balls (curve *1*), calculated numerically from expression (12) multiplied by x.

Now let us find the force acting on each ball.

We have

$$F_x = -\frac{\partial W}{\partial x} = \frac{\partial \beta}{\partial x}\,\frac{\partial c_{11}/\partial\beta + \partial c_{12}/\partial\beta}{(c_{11}+c_{12})^2}\,.$$

Omitting all derivations, we present here only the final expression written in the force units $F_{\mathrm{m}} = kq^2/(4R^2)$:

$$F_x = 2\,\frac{\sum_{n=1}^{\infty} (-1)^{\varkappa_n} \left[\coth\beta - n\coth(n\beta)\right]/\sinh(n\beta)}{\left[\sinh\beta \sum_{n=1}^{\infty} (-1)^{\varkappa_n}/\sinh(n\beta)\right]^2}\,. \tag{13}$$

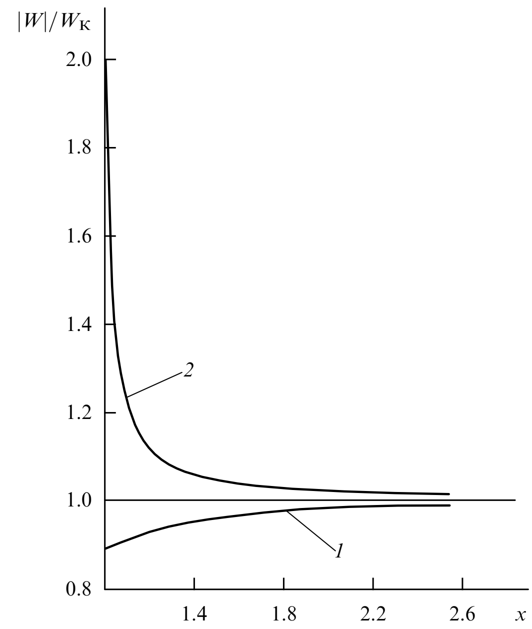

Figure 4. Ratio of the actual energy of interaction between identically charged balls to that calculated in the Coulomb approximation versus the distance between the centers of the balls: *1* — like charges; *2* — opposite charges.

To find the ratio between the actual force and the Coulomb one acting between the point charges located at the centers of the balls, we multiply expression (13) by x^2. The result was computed and is depicted in Fig. 5 (curve *1*). As the distance between the balls decreases to $x = 1.001$, the magnitude of the actual force acting on each ball tends to ~ 0.6157 of the Coulomb force. The first three significant digits are retained when the distance decreases to $x = 1.00001$, but to calculate

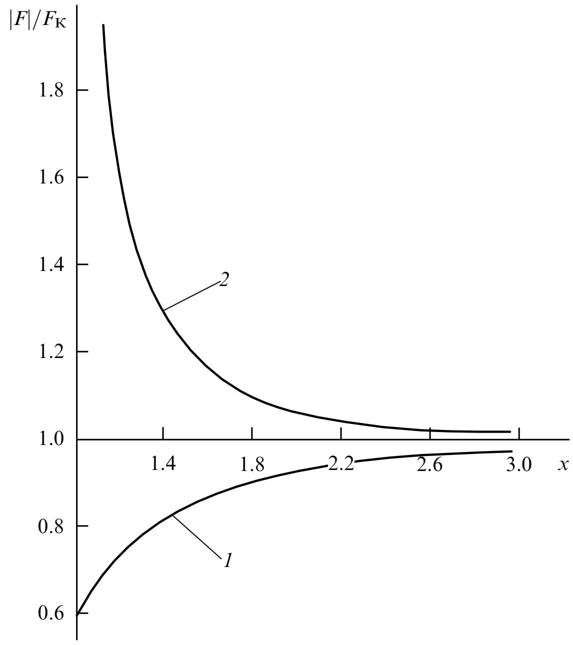

Figure 5. Ratio of the actual force of interaction between identically charged balls to that calculated in the Coulomb approximation versus the distance between the centers of the balls: *1* — like charges; *2* — opposite charges.

the sum accurately, we should take into account terms up to 10^{-60}. The accuracy of the calculations is also confirmed by the fact that the combined numerical calculations of the force and energy yield a ratio for energies equal to ~ 0.8857, coinciding up to the first three significant digits with the analytical estimate.

Thus, the approximation of contacted charged balls by point charges located at their centers results in a nearly 39% error for the calculated force acting on each ball whatever their radii and the magnitude of their charges may be. But the error of the Coulomb approximation decreases rapidly as the distance between the balls rises and becomes less than 10% at a distance of double the ball diameter.

Let us consider the interaction of two unlikely charged identical conducting balls. In this case the dimensionless energy of the interaction between the balls is given by Eqn (11), where $\varkappa_n = 0$, so that all the terms of the series have the same sign. At large values of β, the denominator of expression for W anew tends to unity, hence the dimensionless potential energy takes on the form (12), where $\varkappa_n = 0$. Since at small β the denominator constitutes the sum of the harmonic series, which is divergent, then at $\beta \ll 1$ $(x \to 1)$ we have $W = -2$, i.e. the energy of the short-range interaction between two unlikely charged balls is twice that of two equivalent point charges located at the centers of the balls.

Differentiating expression (12) with respect to distance x, we obtain an expression for the force acting on each ball from the side of the other in the form (13), where $\varkappa_n = 0$. Figures 4 and 5 plot the data of numerical calculations of the energy and force of interaction between the balls, curves *2* corresponding to the ratios of actual forces and energies to that resulted from the Coulomb approximation. It is seen that as the distance between the balls decreases, the attractive force acting on each ball tends to infinity.

6. Interaction between a ball and a conducting plane

In this case we have for the capacitive coefficients [1]:

$$c_{11} = -c_{12} = 4\pi\varepsilon_0 R \sinh\beta \sum_{n=1}^{\infty} \frac{1}{\sinh(n\beta)}\,, \quad c_{22} \to \infty\,.$$

Here the parameter β is related to the distance d between the center of the ball and the plane by the expression $\cosh\beta = d/R \equiv x$. Then, we find from (1) the relationship for the interaction energy $W = q^2/(2c_{11})$, where q is the charge of the ball. The interaction force is given by

$$F_x = -\frac{\partial W}{\partial d} = \frac{q^2}{2Rc_{11}^2 \sinh\beta}\,\frac{\partial c_{11}}{\partial \beta}\,.$$

Differentiating this expression, one can find that the resulting expression for the force coincides with that for two unlikely charged identical balls [formula (13) at $\varkappa_n = 0$].

Thus, the conducting plane induces an electrical image of the charged ball so that the interaction force between the plane and the ball is equal to that between two unlikely charged identical balls, whose centers are separated by a distance $2d$.

7. Asymptotic relations for forces

In order to arrive at asymptotic expressions for forces at small distances between the balls, we make use of the Euler–Maclaurin formula [8]. There we consider only the first three terms giving the main contribution. We have

$$\sum_{k=0}^{m} f(k) \approx \int_0^m f(t)\,\mathrm{d}t + \frac{1}{2}\left[f(0) + f(m)\right]\,. \tag{14}$$

Here $f(t)$ is the same function for the series and the integral. Calculating the numerator in (13) at $\varkappa_n = 0$, $k = n - 1$, $m = \infty$ by (14), we express it as I_1:

$$I_1 = -\frac{1}{\beta^2}\left[(\beta\coth\beta - 1)\ln\left(\tanh\frac{\beta}{2}\right) + \frac{\beta}{\sinh\beta}\right]\,.$$

At $\beta \ll 1$ $(x = \cosh\beta)$ we get

$$I_1 \approx -\frac{1}{2(x-1)}\,.$$

Similar calculations of the denominator in (13) at $\varkappa_n = 0$ yield

$$I_2 \approx \ln\frac{2}{\beta} + \frac{1}{2} = \frac{1}{2}\left(\ln\frac{2}{x-1} + 1\right)\,.$$

The final asymptotic expression for the force acting on closely spaced unlikely charged balls takes on the form

$$F_x = -\frac{4}{(x-1)\left\{\ln[2/(x-1)] + 1\right\}^2}\,. \tag{15}$$

Numerical calculations using this formula demonstrate that the interaction force tends to infinity as $x \to 1$. At $x = 1.001$, we have $F_x \approx -54.1$ in the units of force calculated in the Coulomb approximation. Numerical calculation of the force ratio, taking into account 1985 terms in the series, yields $F_x \approx -52.0$. The difference between the data is close to 4%; as x tends to unity this difference still further decreases.

In a similar manner we find the asymptote of the force acting on closely spaced, likely charged identical balls. To do it, we put $\varkappa_n = n + 1$ in (13) and apply the Euler–Maclaurin formula (14) to the numerator in (13) twice: for odd and even series. Omitting derivations, we present the final expression for the numerator in (13):

$$I_3 = \frac{1}{2\beta^2}\left\{(\beta\coth\beta - 1)\ln\left[\frac{\tanh\beta}{\tanh(\beta/2)}\right] - \left(\frac{\beta}{\sinh\beta} - \frac{2\beta}{\sinh 2\beta}\right)\right\} - \frac{1}{2}\,\frac{\coth\beta - 2\coth 2\beta}{\sinh 2\beta}\,.$$

At $\beta \ll 1$ we have

$$I_3 \approx \frac{\ln 2}{6} = 0.116$$

to the accuracy of terms quadratic in β. Numerical calculations taking into account 1985 terms of the corresponding series at $x = 1.001$ yield $I_3 \approx 0.148$.

Since at $\varkappa_n = n + 1$ and $\beta \ll 1$ the series entering the denominator of expression (13) for force is again alternating and harmonic, the denominator tends asymptotically to $\ln^2 2$, while the ratio between the actual and Coulomb forces tends to

$$F_x \approx \frac{1}{3\ln 2} = 0.481\,.$$

As noted above, the numerical calculation yields ~ 0.615 for the corresponding asymptotic value. Hence, the difference is, in this instance, about 20%.

8. Conclusions

Our study suggests that though the attraction between likely charged conducting bodies is virtually not discussed in scientific and methodical literature, probably believed to be rather exotic, this effect is as common as repulsion. In other words, we can always indicate scenarios at which two arbitrary likely charged balls attract. The discussed effect has a deep analogy in microphysics, namely, the van der Waals attraction of neutral spherically symmetric particles [10].

Within the framework of the considered electrostatic interaction between macroscopic balls the only exception is the case when the charges of the balls are related as the squares of their radii: such balls always repel each other. But the repulsion may switch to attraction in this case too if the van der Waals attraction appears to be competitive with the Coulomb repulsion.

Of special interest is the interaction of likely charged identical balls which also always repel. At small distances between the balls the Coulomb approximation, which is often used to calculate the force acting on the balls, yields a 39% error, whatever the sizes of the balls and magnitude of their charges may be. The Coulomb approximation employed to calculate the interaction energy between the balls yields an error of up to 12%.

In the case of unlikely charged conducting balls the interaction force rises to infinity as the distance between the balls decreases, while the interaction energy remains finite. For contacting balls the force is twice that calculated by the Coulomb approximation. The result obtained is also independent of the sizes of the balls and their charges, i.e. it is self-similar.

A conducting plane induces an electrical mirror image of the charged ball, resulting in an interaction between the image and the actual ball.

Using the Euler – Maclaurin formula for sums, we have found analytically the asymptotic relationships for the force acting between the identical balls at small distances, which are in good agreement with precise numerical computations.

References

1. Smythe W R *Static and Dynamic Electricity* (New York: McGraw-Hill, 1950) [Translated into Russian (Moscow: IL, 1954)]
2. Buchholz H *Elektrische und Magnetische Potentialfelder* (W. Berlin: Springer-Verlag, 1957) [Translated into Russian (Moscow: IL, 1961)]
3. Batygin V V, Toptygin I N *Sbornik Zadach po Elektrodinamike* (Problems in Electrodynamics) (Moscow: Nauka, 1970) [Translated into English (London: Academic Press, 1978)]
4. Govorkov V A *Elektricheskie i Magnitnye Polya* (Electric and Magnetic Fields) (Moscow: Energiya, 1968)
5. Govorkov V A, Kupalyan S D *Teoriya Elektromagnitnogo Polya v Uprazhneniyakh i Zadachakh* (Theory of Electromagnetic Field in Exercises) (Moscow: Vysshaya Shkola, 1970)
6. Mirolyubov N N et al. *Metody Rascheta Elektrostaticheskikh Poleĭ* (Methods for Calculations of Electrostatic Fields) (Moscow: Vysshaya Shkola, 1963)
7. Landau L D, Lifshitz E M *Elektrodinamika Sploshnykh Sred* (Electrodynamics of Continuous Media) (Moscow: Nauka, 1982) [Translated into English (Oxford: Pergamon Press, 1985)]
8. Vereshchagin I P et al. *Osnovy Elektrodinamiki Dispersnykh Sistem* (Foundations of Electrodynamics of Dispersed Systems) (Moscow: Energiya, 1974)
9. Madelung E *Die Mathematischen Hilfsmittel des Physikers* (Mathematical Guide to Physics) (Berlin: Springer-Verlag, 1957) [Translated into Russian (Moscow: Nauka, 1968)]
10. Barash Yu S *Sily Van-der-Vaal'sa* (Van der Waals Forces) (Moscow: Nauka, 1988)