Physics – Uspekhi **42** (4) 375 – 384 (1999)

METHODOLOGICAL NOTES PACS numbers: 03.65.Bz, **03.67.–a**, **05.65.+c**, **89.70.+c**

Entropy and information of open systems

Yu L Klimontovich

Contents

Abstract. **Of the two definitions of 'information' given by Shannon and employed in the communication theory, one is identical to that of Boltzmann's entropy and gives in fact a measure of statistical uncertainty. The other involves the difference of unconditional and conditional entropies and, if properly specified, allows the introduction of a measure of information for an open system depending on the values of the system's control parameters. Two classes of systems are identified. For those in the first class, an equilibrium state is possible and the law of conversation of information and entropy holds. When at equilibrium, such systems have zero information and maximum entropy. In self-organization processes, information increases away from the equilibrium state. For the systems of the other class, the equilibrium state is impossible. For these, the so-called 'chaoticity norm' is introduced and also two kinds of self-organization processes are considered and the concept of information is appropriately defined. Common information definitions are applied to classical and quantum physical systems as well as to medical and biological systems.**

1. Introduction

In 1997, the publishers of 'Physics – Uspekhi (Advances in Physical Sciences)' released the first edition of the book written by Boris Borisovich Kadomtsev and entitled *Dynamics and Information* [1]. The value of this book consists primarily in the insight it offers into a number of fundamental concepts and issues of classical and quantum physics. The fundamental conceptions include the notion of information whose importance is emphasized by the title of the book itself. The concepts of information, information coupling, information-wise open systems, information exchange, information content of the wave function are discussed throughout the book. It is important that the book by B B Kadomtsev stimulates the reader to look for alternative solutions to the problems under study whenever the reader to some extent or other disagrees with the author. Sadly, it is no longer possible to discuss with B B Kadomtsev those questions that arise after reading his book. Never again shall we hear and appreciate the pertinence of his elegant physical arguments in defence of his reasoning.

In this paper, for the first time, we summarize the known results of information theory that comprises an important chapter of the general theory of communication, and give a review of the recent papers concerned with the possibility of defining the information of both passive and active open systems depending on the values of control parameters.

The foundations of the modern theory of communication were laid by the classical works of Claude Shannon [2, 3]. He gives two definitions of information. The first actually coincides with Boltzmann's definition of entropy. This information, like Boltzmann's entropy, is the measure of uncertainty at the selected level of statistical description of the system. Because of this, we call it S-information.

Such a definition of information, being widely used in literature, is nevertheless not sufficient for the purposes of

Yu L Klimontovich M V Lomonosov Moscow State University, Physics Department, Vorob'evy Gory, 119899 Moscow, Russian Federation
E-mail: ylklim@hklim.phys.msu.su

Received 19 March 1998, revised 25 October 1998
Uspekhi Fizicheskikh Nauk **169** (4) 443 – 452 (1999)
Translated by A S Dobroslavskiĭ; edited by A Radzig

description of open systems. More adequate in this case is another definition of information, also suggested by C Shannon and running essentially as follows.

Assume there is a distribution function $f(X, Y)$ of the duplex set of variables of the system in question. This permits a definition of information about the object X with respect to Y, and vice versa.

In both cases, the information is determined by the difference between unconditional and conditional entropies, and is thus associated with the corresponding change in the uncertainty concerning the state of the selected system.

A N Kolmogorov gave full credit to the importance of Shannon's works for the development of the theory of information in his preface to the Russian edition of collected *Papers on the Theory of Information and Cybernetics* [3] . He wrote:

"The importance of Shannon's works for pure mathematics was not fully appreciated from the outset. I recall that at the international congress of mathematicians in Amsterdam (in 1951) my American colleagues, specialists in the theory of information, regarded my interest in Shannon's work as somewhat exaggerated, since this after all was more engineering than mathematics. Today such opinions hardly need to be refuted.

Admittedly, in all more or less complicated cases Shannon left the rigorous mathematical 'proof' of his ideas to his successors. His mathematical intuition, however, is remarkably precise."

The works of C Shannon stimulated the appearance of publications that laid a solid mathematical foundation of the theory of information. We shall only refer to the first papers of this kind published on the pages of 'Soviet Mathematical Surveys' and 'Doklady Akademii Nauk SSSR' [4, 5]. The former contains proofs of the main theorems of the theory of information for the discrete case, and the latter gives a most general definition of entropy and information for continuous distributions.

In the stream of books on the theory of information that followed, a special place belongs to the book by R L Stratonovich [6]. Along with the traditional presentation of the main ideas of Shannon's information theory, this book also contains the theory of the value of information developed by Stratonovich. One is also impressed by the deep analogy of the mathematical methods of the theory of information and statistical thermodynamics.

This paper is a review of works concerned with further development of the theory of information, and in particular its applications to the theory of open systems [7 – 9].

Open systems may exchange energy, matter, and, which is also very important, information with their environment. Here we shall only consider macroscopic open systems. They may consist of many elements of diverse nature.

Various structures may arise spontaneously in open systems. Dissipation plays a constructive role in their formation. To emphasize this circumstance, I Prigogine has coined a very apt and comprehensive term 'dissipative structures' [10 – 12], which may be divided into three classes: temporal, spatial, and space – time dissipative structures. Autowaves are one example of the latter [13].

The complexity of macroscopic systems offers ample opportunities for the manifestation of cooperative phenomena. To emphasize the role of collective interactions in the formation of dissipative structures, which result from non-equilibrium phase transitions forming the self-organization processes, H Haken introduced the term 'synergetics', which translates as 'joint, cooperative effort' [14 – 17].

In many cases macroscopic systems may be regarded as continuous media. The transition to a continuous medium completely overturns the conception of the system as consisting of separate, macroscopic or microscopic but 'small' elements. To avoid the associated fundamental difficulties, it is necessary to define the physically infinitesimal scales that determine, in particular, the size of a point in a 'continuous' medium [7, 8].

For defining the information of open systems, the general Shannon formula must be transformed so as to reveal the dependence of information on the control parameters. To ensure that information is always positive, one must introduce the additional condition of constancy of the mean energy (effective energy in the general case) when finding the difference of entropies that defines information. We select a class of systems for which the constancy of the mean energy in the course of evolution is an intrinsic property (like in the case of a Boltzmann gas). For such systems the expression for information coincides with the Lyapunov functional, which is also defined as the difference between unconditional entropy (for example, the entropy of the equilibrium state) and conditional entropy (for example, that for the non-equilibrium states in the course of time evolution). Such systems obey the law of conservation of information and entropy.

For systems of another class, the mean energy changes in the processes of time evolution or in the course of evolution of steady states in the space of control parameters. In such a case there are two possibilities for defining the information. One (which is currently used extensively for physical and biological systems) is based on the criterion of the relative degree of order (the S-theorem); the other is used for Brownian-type systems, when the system under consideration occurs in a fluctuating medium with a given noise intensity. Information for Brownian systems can also be defined as a Lyapunov functional, which is this time determined by the difference in free energies rather than the difference in entropies [7 – 9].

General definitions of information of open systems are accommodated for classical and quantum system, as well as for medical and biological studies based on the statistical analysis of cardiograms.

This paper is the first review of the results in the theory of information of open systems. Some topical reviews that have appeared so far (see, for example, Ref. [27]), fail to reflect the specifics of the concept of information of open systems. The present paper is intended to fill this gap.

2. Entropy and information

2.1 S-information

Once again, there are two statistical definitions of the concept of 'information'. One is an extension of Boltzmann's definition of entropy to the case of arbitrary systems, when the model of mechanical motion of the components of an open system cannot be used.

Let $f(X)$ be a certain dimensionless distribution function of values of a dimensionless random quantity X. The latter may also be a set of variables comprising a vector. The information and entropy are then given by

$$I[X] = S[X] = -\int f(X)\ln f(X)\,\mathrm{d}X\,, \quad \int f(X)\,\mathrm{d}X = 1\,. \quad (2.1)$$

The corresponding definition for the case of a discrete variable then takes the form

$$I[n] = S[n] = -\sum_n f_n \ln f_n\,, \quad \sum_n f_n = 1\,. \tag{2.2}$$

Some authors (see, for example, Refs [1, 18]) adduce arguments in favor of existence of the law of conservation of the sum of entropy and information. In our notation this law is expressed as

$$I[X] + S[X] = \text{const}\,. \tag{2.3}$$

Equation (2.3), however, does not follow from our definitions of S-information. We shall see that under certain conditions the sum of entropy and information is indeed conserved. This equality, however, may be derived only for a more general definition of Shannon's entropy.

2.2 Shannon's entropy

For the definition of information it is more natural to use a differential characteristic — the difference between unconditional entropy (Boltzmann's entropy) and conditional entropy [2–9]:

$$I[X, Y] = S[X] - S[X|Y]\,. \tag{2.4}$$

Here $S[X]$ is the conventional (unconditional) Boltzmann–Shannon entropy

$$S[X] = -\int f(X) \ln f(X)\, \mathrm{d}X\,, \tag{2.5}$$

and $S[X|Y]$ is the conditional entropy. The latter is defined via the relevant conditional distribution function $f[X|Y]$ ($f(X, Y) = f[X|Y]f(Y)$) in the following manner:

$$S[X|Y] = -\int f(X, Y) \ln f(X|Y)\, \mathrm{d}X\, \mathrm{d}Y\,. \tag{2.6}$$

Expression (2.4) may be rewritten in an explicitly symmetric form

$$I[X, Y] = I[Y, X] = \int \ln \frac{f(X, Y)}{f(X)f(Y)} f(X, Y)\, \mathrm{d}X\, \mathrm{d}Y \geqslant 0\,. \tag{2.7}$$

The equality sign corresponds to the case when the quantities X and Y are statistically independent. Because of this, the function $I[X, Y]$ may be referred to as 'correlation information'. This function characterizes the information about the state with the duplex set of variables X, Y, as determined by the statistical correlation of the latter.

2.3 Information of open systems

Now let us render concrete Shannon's general expression for the correlation information with a view to exposing the dependence on the control parameters. The simplest approach to this problem consists in the following [9].

We break the symmetry of Shannon's formula by assuming that the distribution function $f(Y)$ of the set of variables Y is completely characterized by the corresponding set of the first moments:

$$f(Y) = \delta(Y - a)\,, \quad \langle Y \rangle = a\,. \tag{2.8}$$

We accept the set of parameters or at least one of them as the control parameter(s). Substituting the last expression into Shannon's formula, we carry out integration with respect to Y. As a result, we get the expression for information about the set of variables X at a given value of the control parameter a:

$$I[X|a] = S[X] - S[X|a] \equiv S[X] + \int f(X|a) \ln f(X|a)\, \mathrm{d}X\,. \tag{2.9}$$

Observe that this definition of information cannot be used in all cases. Indeed, information by definition is a positive quantity. Our last expression, however, may generally assume negative values. To make it always positive — that is, to ensure that $I[X|a] \geqslant 0$ — we must introduce an additional condition. The essence of this additional condition is best explained with the aid of a concrete example of an open system. For such example we shall use a rarefied gas of structureless particles — a Boltzmann gas.

3. Boltzmann's H-theorem

3.1 Lyapunov functional Λ_S

The term 'H-theorem' (where H stands for 'heat') was first proposed by the British physicist Barberry in 1894, and later approved by Boltzmann.

Many textbooks on statistical physics assert that Boltzmann's H-theorem only holds for closed systems. This statement, however, needs a certain correction which is important for the formulation of the criterion of self-organization — the S-theorem.

Boltzmann's H-theorem holds that the entropy of a closed system increases in the course of time evolution towards equilibrium, and remains constant in the state of equilibrium:

$$\frac{\mathrm{d}S}{\mathrm{d}t} \geqslant 0\,. \tag{3.1}$$

It is important that it is not the energy of the system that remains constant in the course of evolution towards the state of equilibrium, as would be the case in mechanics, but only its mean value

$$\langle E \rangle \equiv \left\langle \frac{p^2}{2m} \right\rangle = \text{const}\,. \tag{3.2}$$

Under this condition, fluctuations are possible. Therefore, the description of evolution of the system according to the Boltzmann equation (given that the system is externally closed) is not complete. This is natural, since Boltzmann's equation is based on the model of a continuous medium and is therefore approximate in the sense that information about the motion of particles within the points of the continuous medium is lost.

Let us estimate the number of degrees of freedom that are employed and disregarded upon transition from the system of particles to the approximation of a continuous medium. For Boltzmann's gas of N structureless atoms, the total number of degrees of freedom is $6N$. Let N_{ph} be the average number of particles within a physically infinitesimal volume — the 'point' of the continuous medium. The number of degrees of freedom used in the description according to the Boltzmann equation can be estimated as N/N_{ph}. Then the number of 'unclaimed' degrees of freedom (the number of degrees of freedom of the 'thermostat' or buffer) is $6N(N_{\rm ph}-1)/N_{\rm ph}$, and obviously constitutes the lion's share of the total number of degrees of freedom of a Boltzmann gas. This fact characterizes the internal nonclosure associated with the use of the Boltzmann equation.

Given the condition $\langle E\rangle = \text{const}$, Boltzmann's H-theorem may be reformulated in terms of the Lyapunov functional which is defined by the difference in the entropies of the equilibrium and nonequilibrium states of the gas [7–9]:

$$\Lambda_S = S_0 - S(t) = k_{\rm B}\int\left(\ln\frac{f(r,p,t)}{f_0(r,p)}\right)f(r,p,t)\,\frac{{\rm d}r\,{\rm d}p}{(2\pi\hbar)^3}\geqslant 0\,, \tag{3.3}$$

$$\frac{{\rm d}\Lambda_S}{{\rm d}t} = \frac{\rm d}{{\rm d}t}\big(S_0 - S(t)\big)\leqslant 0\,. \tag{3.4}$$

Inequality (3.3) was derived using the condition $\langle E\rangle = \text{const}$, and (3.4) using the H-theorem in the form of Eqn (3.1).

3.2 Information and the Lyapunov functional Λ_S. Law of conservation of the sum of information and entropy

Now we return to expression (2.9). Let the control parameter a assume only positive values, and the unconditional entropy $S[X]$ correspond to its zero value:

$$a\geqslant 0\,,\quad S[X] = S[X|a=0]\,. \tag{3.5}$$

Under these conditions, the information of the equilibrium state is

$$I[X|a=0] = 0 \tag{3.6}$$

and the unconditional entropy $S[X]$ coincides with the entropy of the equilibrium state.

We shall use the general definition (2.9) for the information of the Boltzmann gas. Then we may use the current time t for the control parameter. The values of the control parameter then belong to the interval $0\leqslant t\leqslant\infty$, the value of $t=\infty$ corresponding to the state of equilibrium. The information of the Boltzmann gas is then expressed as

$$I[r,p,t] = \Lambda_S = S_0 - S(t) = \\ = k_{\rm B}\int\left(\ln\frac{f(r,p,t)}{f_0(r,p)}\right)f(r,p,t)\,\frac{{\rm d}r\,{\rm d}p}{(2\pi\hbar)^3}\geqslant 0\,. \tag{3.7}$$

According to arguments developed above, the positivity of information is ensured by the condition of constant mean energy $\langle E\rangle = \text{const}$. It is a consequence of the structure of the Boltzmann collision integral, and is therefore a natural (not an additional) condition.

We see that the sum of information and entropy of a Boltzmann gas in the course of time evolution to the state of equilibrium remains constant:

$$I[r,p|t] + S(t) = S_0 = \text{const}\,. \tag{3.8}$$

The constant here is determined by the entropy of the equilibrium state. The information of the equilibrium state is

$$I[r,p|t=\infty] = \Lambda_S(t=\infty) = 0\,. \tag{3.9}$$

So, for a Boltzmann gas the positivity of information is a natural property of the system. The fulfillment of the inequality $I[X|a]\geqslant 0$ depends, as we shall see, on a particular additional condition.

4. S-theorem and the law of conservation of the sum of information and entropy

We know that, among all thermodynamic functions, it is only the entropy S that features the combination of properties which permit using it as the measure of uncertainty (chaoticity) in the statistical description of processes in macroscopic systems [7, 8, 19]. To formulate the criterion known as the 'S-theorem', the entropy of the more chaotic state must be renormalized in such a way as to make sure that the comparison of states of the open system in the course of evolution is carried out at the same values of the mean effective energy.

As a relatively simple example, let us consider the process of evolution of the steady states of a van der Pol oscillator as the feedback parameter a is varied. We shall compare the relative degrees of order using the criterion of the 'S-theorem', which was first formulated for concrete examples in Refs [20, 21].

We select two states corresponding to the following values of the feedback parameter: $a=0$ (equilibrium state of the oscillatory circuit), and $a=a_1$ (a steady but nonequilibrium state of generation).

By X we denote the selected characteristic of the stationary state. In the case of an oscillator, the energy of oscillations E plays the role of the quantity X. We also use the appropriate notation for the distribution functions f_0, f_1, and the corresponding entropies S_0, S_1. Renormalization to the selected value of the mean energy for this system reduces to replacing the temperature of the equilibrium state T by its effective value $\widetilde{T}$, found by solving the equation

$$k_{\rm B}\widetilde{T} = \int E\widetilde{f}_0(E,a=0)\,{\rm d}E = \int Ef_1(E,a=a_1)\,{\rm d}E\,, \tag{4.1}$$

which serves as an additional condition ensuring the positivity of information. The solution of this equation satisfies the inequality

$$\widetilde{T}(a)\geqslant T\,, \tag{4.2}$$

where the equality sign corresponds to the case $a=a_0=0$.

Hence it follows that for equalizing the values of the mean energy the state with $a=0$ must be 'heated'.

By $\widetilde{S}_0$ we denote the corresponding renormalized value of entropy. Since the two states under consideration now have the same mean energy, the difference of entropies $\widetilde{S}_0$, S_1 can serve as a measure of the relative degree of order of the selected states. Using the condition

$$\langle E\rangle = \text{const}\,, \tag{4.3}$$

the difference of entropies may be expressed as

$$\widetilde{S}_0 - S_1 = \int\left(\ln\frac{f_1(E)}{\widetilde{f}_0(E)}\right)f_1(E)\,{\rm d}E\geqslant 0\,. \tag{4.4}$$

So, the calculation of the relative degree of order of the two selected states is based on two formulae. Formula (4.2) justifies the selection of the equilibrium state $a=0$ as the more chaotic state, whereas Eqn (4.4) gives a quantitative measure of their relative order.

Using the general expression (2.9), we can define the information $\widetilde{I}(E)$ of the steady states of the oscillator for all values of the parameter of order as

$$\widetilde{I}(E) = \widetilde{S}_0 - S_1 = \int \left(\ln \frac{f_1(E)}{\tilde{f}_0(E)} \right) f_1(E)\, \mathrm{d}E \geqslant 0\,. \tag{4.5}$$

Hence it follows that when the feedback parameter is zero, the state of the system coincides with the equilibrium state, and the information is equal to zero.

In the same way, the S-theorem may be used for finding the information for a sequence of steady states in transition from the laminar to turbulent flow regime. The control parameter is the Reynolds number. Computation reveals that information grows with increasing Reynolds number. This is another proof that the steady turbulent flow is more ordered compared with the corresponding laminar flow [8, 21].

5. Van der Pol oscillator. S-information and Shannon information

We have used the example of a van der Pol oscillator to demonstrate how Shannon's information varies in the course of evolution of stationary states as the feedback parameter a_f is varied slowly enough. The relative degree of order of these states was evaluated on the basis of the S-theorem criterion [7, 8].

Let us now use the example of the van der Pol oscillator for showing the fundamental difference between the two definitions of information given at the beginning of this paper: S-information and Shannon's information.

Running ahead, we turn to the Fokker–Planck equation (see below Section 11) for the distribution function of the values of the energy of oscillations. Its stationary solution holds for all values of the feedback parameter. We once again select two characteristic steady states:

(1) $a_f = 0$. In this case, the distribution coincides with the equilibrium Boltzmann distribution.

(2) $a_f \gg \gamma$. In this case we have the Gaussian distribution

$$f_1 = \sqrt{\frac{1}{2\pi \langle \delta E \rangle^2}} \exp\left\{ \frac{(E - a/b)^2}{2 \langle \delta E \rangle^2} \right\}, \quad \langle \delta E \rangle^2 = \frac{D}{b}\,. \tag{5.1}$$

For these selected states we can get the expressions for the values of S-information — the entropy. We see that both entropy and S-information increase as we pass from the equilibrium state to the regime of well-developed generation:

$$S_0 \leqslant S_1\,, \quad I_0 \leqslant I_1\,. \tag{5.2}$$

The first inequality may be interpreted as the decrease of the degree of order as the generation develops. Intuitively, however, it is clear that the degree of order must increase upon transition to the regime of generation.

At the same time, the second inequality indicates that upon transition to the regime of generation the S-information increases. These results violate the law of conservation of entropy and information. This means that the calculations of entropy and S-information cannot be used for finding the relative degree of order and information content of the selected states. What is cause of such a contradiction?

The reason is that the mean energy of oscillations increases as generation sets in:

$$\langle E \rangle_0 < \langle E \rangle_1\,. \tag{5.3}$$

At the same time, the S-theorem criterion stipulates that the entropies should be compared at the same values of the mean energy. As we know, this requires carrying out an appropriate renormalization.

The arguments developed above lead to the conclusion that the physically more meaningful results concerning the change of information of open systems as the control parameters are varied can only be obtained using Shannon's definition of information. This is natural, since it is only then that the information is represented as a difference characteristic. Calculations based on S-information, as we have seen for a concrete example, do not lead to physically sensible results in the analysis of the relative degree of order of the open system states.

6. Evaluation of the information and relative degree of order from experimental data

Practical applications of the S-theorem require knowing the effective Hamilton function. It is not a major problem to find it as long as we have a mathematical model of the process. In many cases, however, an adequate mathematical model of an open system cannot be constructed even for physical systems. This task is even more complicated when dealing with biological, sociological or economical objects. Because of this, it would be good to be able to express the relative degree of order of the states of open systems directly from experimental data. The appropriate procedure is as follows.

(1) Select the control parameters for the system under consideration. Select two states of the system corresponding to different values of the control parameter: a_0 and $a_0 + \Delta a$.

(2) For the selected parameters of the system, obtain sufficiently long experimental time-domain realizations

$$X_0(t, a_0)\,, \quad X(t, a_0 + \Delta a)\,. \tag{6.1}$$

Enter these data into a computer and construct the relevant distribution functions

$$f_0(X, a_0)\,, \quad f(X, a_0 + \Delta a)\,. \tag{6.2}$$

Both distributions are normalized to unity.

(3) Take one of the states (for example, a_0) for the state of physical chaos, and find the effective Hamilton function

$$H_{\mathrm{eff}} = -\ln f_0(X, a_0)\,. \tag{6.3}$$

In this way, the effective Hamilton function is found directly from the experimental data. The name of 'the effective Hamilton function' is justified by the fact that the distribution function renormalized to the given mean value $\langle H_{\mathrm{eff}} \rangle$ has the form of the canonical Gibbs distribution

$$\tilde{f}_0(X) = \exp\left\{ \frac{F_{\mathrm{eff}}(T) - H_{\mathrm{eff}}(X)}{kT} \right\}. \tag{6.4}$$

Here T is the effective temperature. For the state of physical chaos, $T = 1$.

The effective free energy as a function of T is found from the condition of normalization of the function f_0. The effective temperature as a function of the control parameter Δa is determined, as before, from the condition of constancy of the mean effective energy

$$\int H_{\text{eff}}\tilde{f}_0(X, a_0)\,\mathrm{d}X = \int H_{\text{eff}} f(X, a_0 + \Delta a)\,\mathrm{d}X\,. \tag{6.5}$$

If the solution of this equation has the form of Eqn (4.2) (here $T \geqslant 1$), then the selection of the state of physical chaos is correct. Calculation of the relative degree of order is again based on Eqn (4.4).

For the zero level of information we take the state of physical chaos — the state $a = a_0$. Then the 'redundant information' gained upon transition to the more ordered state $a = a_0 + \Delta a$ (at the same value of the mean effective energy) is given by

$$\widetilde{I}(X) = \widetilde{S}_0 - S = \int \left(\ln \frac{f(X, a_0 + \Delta a)}{\tilde{f}_0(X, a_0)} \right) f(X, a_0 + \Delta a)\,\mathrm{d}X \geqslant 0\,. \tag{6.6}$$

The equality sign corresponds to $\Delta a = 0$.

7. Information of medico-biological and complex physical objects

From the above discussion we may conclude that two classes of phenomena in open systems may be distinguished in the calculations of both entropy and information. The first includes such systems and processes that admit the state of thermal equilibrium. In such cases, as we have seen, the information may be measured with respect to the most chaotic (equilibrium) state. In the van der Pol oscillator, for example, an increase of the feedback parameter leads to the transition from thermal oscillations in electrical circuit to the regime of developed generation. If the states are compared at the same value of the mean energy of oscillations, then, as the generation develops (the system recedes from the state of equilibrium), the entropy decreases, and the information increases. This allows the development of generation to be regarded as a process of self-organization. Accordingly, we may define the process of self-organization in such systems as a transition from a more chaotic to a less chaotic state, or as a transition from a state with zero information (equilibrium state) to a state with nonzero information (nonequilibrium state). In the terminology of I Prigogine, we may say that the process of generation gives rise to a time-domain dissipative structure.

A similar increase of information accompanies the transition from a laminar flow in a pipe to turbulent flow, as the pressure differential (the Reynolds number) increases. For the origin of an equilibrium state here we can also take the equilibrium state of the liquid when the pressure drop is zero (the zero value of the control parameter). In such a case hydrodynamic motion is absent, and there only is the chaotic motion of molecules of the liquid. This motion is the most chaotic, and hence the least informative.

The concept of 'self-organization' as the transition from chaotic to a more ordered state is the cornerstone of the theory of dissipative structures. The first systematic presentation of the theory of self-organization was given in the well-known works of I Prigogine and G Nicolis [10 – 12]. This theory is based on the ideas and results of I Prigogine concerned with the thermodynamics of irreversible nonequilibrium processes.

This traditional definition of self-organization, however, is not general. As a matter of fact, there is a broad class of systems (for the most part biological) that do not admit of states of either total chaos (thermal equilibrium) or complete order. In the state of total chaos such systems simply will not function.

More appropriate for such systems is the concept of the 'norm of chaoticity', which may be likened to the concept of 'health'. Then the *process of self-organization* may be regarded as the process of self-recuperation.

Let us look at some studies of response of men and women to stress [22 – 24]. The state after stress we agree to call *sickness*. For women, the *transition to the 'norm of chaoticity'* (*self-recuperation*), which we agreed to call self-organization, is the transition from the more chaotic to the less chaotic state. By contrast, for men the state after stress (sickness) corresponds to the more ordered state.

The process of self-organization may be defined as self-recuperation. For women, the process of self-organization correlates with the transition from the more chaotic to the more ordered state, which corresponds to the increase of information. For men, however, the process of recuperation (and thus self-organization) develops with the increase of chaoticity, or decrease of information.

By $\widetilde{I}_W$ and $\widetilde{I}_M$ we denote the amounts of information normalized to a certain value of the mean effective energy and obtained, respectively, for women and men from their electrocardiograms. From experiments described above it follows that the more chaotic states are the states *after* the stress for women, and *before* the stress for men. Accordingly, $\widetilde{S}^{(W)}_{\text{after}}$ is the renormalized entropy for women after the stress, and $\widetilde{S}^{(M)}_{\text{before}}$ is the renormalized entropy for men before the stress (in the state corresponding to the norm of chaoticity). In this notation, we have the following definitions of information:

$$\widetilde{I}_W = \widetilde{S}^{(W)}_{\text{after}} - S^{(W)}_{\text{before}} \geqslant 0\,, \quad \widetilde{I}_M = \widetilde{S}^{(M)}_{\text{before}} - S^{(M)}_{\text{after}} \geqslant 0\,. \tag{7.1}$$

These formulae allow using the cardiograms for evaluating the information increase for women after the stress and the information decrease for men in the process of recuperation.

Let us summarize the results. In complicated situations (such as the transition from one turbulent regime to another, or when dealing with biological systems), it is only possible to distinguish the processes of degradation or self-organization using a criterion of the relative degree of order in the states of open systems. The concept of self-organization as the formation of structures or as the transition from a less ordered to a more ordered state is no longer sufficient. More adequate is the concept of the 'norm of chaoticity' that may be established for biological systems from experimental data with the aid of the S-theorem criterion. Changed accordingly is the concept of information, which is defined as the difference of unconditional and conditional entropies used for defining the relative degree of order in the states of open systems.

Let us now turn our attention to some examples of finding the information of quantum systems.

8. Information of quantum systems

8.1 Oscillatory form of Heisenberg's uncertainty principle
It is known that Heisenberg's principle of uncertainty can be established from the following obvious inequality [25, 26]:

$$\int\left|\frac{x}{L}\psi + L\frac{\mathrm{d}\psi}{\mathrm{d}x}\right|^2\frac{\mathrm{d}x}{L} \geqslant 0\,, \quad \int|\psi|^2\frac{\mathrm{d}x}{L} = 1\,, \tag{8.1}$$

where L is a certain length parameter.

Let $f(x,p,t)$ be a quantum distribution function — the Wigner function. Then the above inequality can be rewritten in the equivalent form

$$\int\left(\frac{x^2}{L^2} + \frac{L^2p^2}{\hbar^2}\right)f(x,p,t)\frac{\mathrm{d}x\,\mathrm{d}p}{2\pi\hbar} \geqslant 1\,. \tag{8.2}$$

The left-hand side of this inequality may be interpreted as the mean energy of the harmonic oscillator with eigenfrequency

$$\omega_0 = \frac{\hbar}{mL^2}\,, \quad \frac{\hbar^2}{2mL^2} = \frac{1}{2}\hbar\omega_0\,. \tag{8.3}$$

Given this, we come to the inequality

$$\int\left(\frac{m\omega_0^2x^2}{2} + \frac{p^2}{2m}\right)f(x,p,t)\frac{\mathrm{d}x\,\mathrm{d}p}{2\pi\hbar} \geqslant \frac{1}{2}\hbar\omega_0\,. \tag{8.4}$$

This implies that the mean energy of a harmonic oscillator in any nonequilibrium state cannot be less than the corresponding zero energy:

$$\frac{m\omega_0^2\langle x^2\rangle}{2} + \frac{\langle p^2\rangle}{2m} \geqslant \frac{1}{2}\hbar\omega_0\,. \tag{8.5}$$

Finally, the above inequalities can be rewritten using the expressions for the variance of coordinate and momentum as

$$L^4 - \frac{\hbar^2}{\langle p^2\rangle}L^2 + \hbar^2\frac{\langle x^2\rangle}{\langle p^2\rangle} \geqslant 0\,, \tag{8.6}$$

$$\omega_0^2 - \frac{\hbar}{m\langle x^2\rangle}\omega_0 + \frac{\langle p^2\rangle}{m^2\langle x^2\rangle} \geqslant 0\,. \tag{8.7}$$

Hence follows Heisenberg's uncertainty relation

$$\langle x^2\rangle\langle p^2\rangle \geqslant \frac{\hbar^2}{4}\,. \tag{8.8}$$

In the general case, the parameters L and ω_0 may take on arbitrary values. It is only for the equality sign that they are linked with the variances of coordinate and momentum:

$$L^2 = \frac{\hbar}{m\omega_0} = 2\langle x^2\rangle = \frac{\hbar^2}{2\langle p^2\rangle}\,, \tag{8.9}$$

or, in an alternative form, one obtains

$$\frac{\langle p^2\rangle}{m} = m\omega_0^2\langle x^2\rangle = \frac{\hbar^2}{2mL^2} = \frac{1}{2}\hbar\omega_0\,. \tag{8.10}$$

8.2 Distribution function $f(x, p)$ with the sign =
With the equality sign, equation (8.1) has the following solution

$$|\psi(x)|^2 = \frac{1}{\sqrt{2\pi\langle x^2\rangle}}\exp\left(-\frac{x^2}{2\langle x^2\rangle}\right). \tag{8.11}$$

For equation (8.2), the corresponding solution has the form of Wigner distribution for a harmonic oscillator with the eigenfrequency ω_0:

$$f(x,p) = \frac{\hbar}{\sqrt{\langle x^2\rangle\langle p^2\rangle}}\exp\left(-\frac{x^2}{2\langle x^2\rangle} - \frac{p^2}{2\langle p^2\rangle}\right). \tag{8.12}$$

The variances $\langle x^2\rangle$, $\langle p^2\rangle$ are then given by Eqn (8.10).

9. Relative degree of order of the states with the signs =, >

Let us consider the problem of the relative degree of order of the states corresponding to the signs =, > in Heisenberg's uncertainty relation. By assumption, let the quantum state corresponding to the sign = be the most chaotic. This assumption will be justified later. For the states with sign =, the quantum distribution function $f_0 = (x,p)$ is given by Eqn (8.12). The corresponding expression for the entropy is

$$\begin{aligned} S_0[x,p] &= -\int f_0(x,p)\big(\ln f_0(x,p)\big)\frac{\mathrm{d}x\,\mathrm{d}p}{2\pi\hbar} \\ &= -\int f_0(x)\big(\ln f_0(x)\big)\frac{\mathrm{d}x}{L} - \int f_0(p)\big(\ln f_0(p)\big)\frac{L\,\mathrm{d}p}{2\pi\hbar} \\ &\equiv S_0[x] + S_0[p]\,. \end{aligned} \tag{9.1}$$

Under this condition, the mean energy coincides with the zero energy

$$\langle E\rangle = \frac{m\omega_0^2\langle x^2\rangle}{2} + \frac{\langle p^2\rangle}{2m} = \frac{1}{2}\hbar\omega_0\,. \tag{9.2}$$

In the case of inequality (8.5), however, the value of the mean energy is larger.

According to the S-theorem, to assess the relative degrees of order we have to compare the entropies at the same value of mean energy. To satisfy this requirement, we must, like in the classical case, renormalize the distribution function:

$$f_0(x,p) \to \tilde{f}_0(x,p)\,. \tag{9.3}$$

Function $\tilde{f}_0(x,p)$ is also the Gibbs distribution with the renormalized values of the variances $\langle\tilde{x}^2\rangle$, $\langle\tilde{p}^2\rangle$:

$$\tilde{f}_0(x,p) = \frac{\hbar}{\sqrt{\langle\tilde{x}^2\rangle\langle\tilde{p}^2\rangle}}\exp\left(-\frac{x^2}{2\langle\tilde{x}^2\rangle} - \frac{p^2}{2\langle\tilde{p}^2\rangle}\right) \geqslant 0\,. \tag{9.4}$$

To equalize the mean energy values, we shall 'warm up' the initial state — consider the case of nonzero temperature T:

$$\frac{\langle\tilde{p}^2\rangle}{m} = m\omega_0^2\langle\tilde{x}^2\rangle = k_\mathrm{B}T_{\omega_0} = \frac{1}{2}\hbar\omega_0\coth\frac{\hbar\omega_0}{2k_\mathrm{B}T} \geqslant \frac{1}{2}\hbar\omega_0\,. \tag{9.5}$$

Let us now consider the quantum distribution function — the Wigner function $f(x,p,t)$. It may characterize the nonequilibrium steady and unsteady states. They correspond to the greater sign $>$ in Heisenberg's relation.

Quantum distribution functions $f(x,p,t)$ may take on negative values, but the corresponding distributions separately for coordinate and momentum are always positive:

$$\int f(x,p,t)\,\frac{L\,\mathrm{d}p}{2\pi\hbar}=f(p,t)\geqslant 0,\quad \int f(x,p,t)\,\frac{\mathrm{d}x}{L}=f(x,t)\geqslant 0\,. \tag{9.6}$$

At equilibrium, $f_0(x,p)=f_0(x)f_0(p)$. The corresponding expression for the entropy has the form of Eqn (9.1).

For the stationary distributions $f(x,p)$, the value of the temperature to be used for renormalizing the distribution function is found by solving the equation

$$\begin{aligned}\int\left(\frac{m\omega_0^2x^2}{2}+\frac{p^2}{2m}\right)\tilde{f}_0(x,p)\,\frac{\mathrm{d}x\,\mathrm{d}p}{2\pi\hbar}\\ =\int\left(\frac{m\omega_0^2x^2}{2}+\frac{p^2}{2m}\right)f(x,p)\,\frac{\mathrm{d}x\,\mathrm{d}p}{2\pi\hbar}\\ \equiv\int\frac{m\omega_0^2x^2}{2}\,f(x)\,\frac{\mathrm{d}x}{L}+\int\frac{p^2}{2m}f(p)\,\frac{L\,\mathrm{d}p}{2\pi\hbar}\,.\end{aligned} \tag{9.7}$$

The solution of this equation is such that the following inequality holds:

$$T\geqslant 0\,. \tag{9.8}$$

This result justifies the selection of the state with the equality sign $=$ in Heisenberg's uncertainty relation for the most chaotic state.

Using expression (9.4) for the renormalized distribution $\tilde{f}_0(x,p)$, and condition (9.7) for the mean energy, the expressions for the difference in entropies with respect to x and p, corresponding to the signs $=$ and $>$, respectively, may be represented as inequalities

$$\begin{aligned}\widetilde{S}_0[x]-S[x]&=\int f(x,t)\left(\ln\frac{f(x,t)}{\tilde{f}_0(x)}\right)\frac{\mathrm{d}x}{L}\,,\\ \widetilde{S}_0[p]-S[p]&=\int f(p,t)\left(\ln\frac{f(p,t)}{\tilde{f}_0(p)}\right)\frac{L\,\mathrm{d}p}{2\pi\hbar}\geqslant 0\,.\end{aligned} \tag{9.9}$$

Thus, the S-theorem criterion tells us that the state with the sign $=$ in Heisenberg's uncertainty relation is the most chaotic state. The last expressions give a quantitative measure for the relative degree of order of the most chaotic (equality sign $=$) and arbitrary (greater sign $>$) quantum states with regard to x and p variables, respectively.

Recall that the oscillatory model used above generally bears no relation to real oscillators. The parameter L, for instance, is some generalized length parameter. If L is the size of the system, then, given the linkage between L and ω_0, the oscillatory model can be used for the description of free motion. A comment to this effect can be found in the book by B B Kadomtsev [1].

10. Entropy and information of open quantum systems

Let us return to the definition of information (2.9), and consider its counterpart for a quantum system. Let

$$S[X]\to S_0[x,p]=S_0[x]+S_0[p]$$

be the unconditional entropy for the state corresponding to the sign $=$ in Heisenberg's uncertainty relation. It is the ground state at the temperature $T=0$. Assume also that

$$\widetilde{S}_0[x,p]=\widetilde{S}_0[x]+\tilde{S}_0[p]$$

is the renormalized entropy for the state corresponding to the sign $=$ in Heisenberg's uncertainty relation but at a temperature $T>0$. Finally, let

$$S[x,t],\,S[p,t]$$

be the entropies (with regard to x and p, respectively) for selected steady or nonequilibrium states. Then the amounts of the information (with regard to x and p, respectively) are given by

$$\begin{aligned}\widetilde{I}[x,t]&=\widetilde{S}_0[x]-S[x,t]\geqslant 0\,,\\ \widetilde{I}[p,t]&=\widetilde{S}_0[p]-S[p,t]\geqslant 0\,.\end{aligned} \tag{10.1}$$

We see that, according to the S-theorem criterion, any excited (steady or nonequilibrium) state is more informative than the renormalized ground state — the state with the sign $=$ in Heisenberg's uncertainty relation. Shannon information for the ground state of a quantum system is zero.

Observe in this connection that Boltzmann information (S-information) is nonzero for the ground state of a quantum system as well. The value of S_0 determines the relevant constant in the Nernst law — the entropy of the system at zero temperature.

So far the information was defined in terms of the difference between unconditional and conditional entropies. This is possible as long as the mean energy of the selected states has one and the same value. For a Boltzmann gas this does not constitute an additional condition, since the mean energy remains constant in the course of time evolution of a closed system. We saw, however, that in most cases the appropriate renormalization is necessary to ensure that the mean energy of open systems under study remains the same. In such a case, the unconditional entropy is also renormalized.

It is possible to do without the renormalization procedure if we change the definition of information itself. Namely, we may define information as the difference of free energies rather than the difference of unconditional and conditional entropies. The feasibility of such a definition is discussed in the next section.

11. Information, free energy and the Lyapunov functional Λ_F for Brownian motion

Now for the Boltzmann gas it is possible to define information by formulae (3.7). This reveals the linkage between information and the Lyapunov functional Λ_S which is defined by the difference of entropies of equilibrium and nonequilibrium states. In those cases when the mean energy is not conserved in the course of evolution, it is necessary to renormalize the distribution functions and the entropy for the state of physical chaos to the selected value of mean energy.

Let us consider another possible definition of information for systems that do not conserve the mean energy in the course of evolution. We shall use the example of a van der Pol oscillator, when the energy of oscillations plays the role of a Brownian particle.

The Fokker – Planck equation for the energy distribution function $f(E, t)$ is [7, 8]

$$\frac{\partial f}{\partial t} = D \frac{\partial}{\partial E}\left(E \frac{\partial f}{\partial E}\right) + \frac{\partial}{\partial E}\left[(-a + bE)Ef\right] . \tag{11.1}$$

Here D is the intensity of noise; $a = a_f - \gamma$, a_f is the feedback parameter; γ and b are the coefficients of linear and nonlinear friction, respectively.

We may write the stationary solution of this equation $f_0(E)$ by analogy with the canonical distribution in the form

$$f_0(E) = \exp\left(\frac{F_0 - H(E)}{D}\right), \quad H(E) = -aE + \frac{1}{2}E^2 . \tag{11.2}$$

Here we have used the notation for the effective Hamilton function $H(E)$; F_0 and S_0 are the free energy and the entropy:

$$F_0 = \langle H(E) \rangle_0 - DS_0 . \tag{11.3}$$

The quantity D plays the role of the effective temperature.

Let us define the nonequilibrium free energy and entropy in terms of the distribution function at the current time $f(E, t)$ ($\int f \, \mathrm{d}E = \int f_0 \, \mathrm{d}E = 1$). Then

$$F(t) = \langle H(E) \rangle_t - DS(t) . \tag{11.4}$$

Now we can define the Lyapunov functional for Brownian motion as the difference of free energies $F(t)$, F_0. It satisfies the following inequalities [7, 8]:

$$\Lambda_F = F(t) - F_0 = D \int_0^\infty \left(\ln \frac{f(E, t)}{F_0}\right) f(E, t) \, \mathrm{d}E \geqslant 0 , \tag{11.5}$$

$$\frac{\mathrm{d}\Lambda_F}{\mathrm{d}t} = \frac{\mathrm{d}\big(F(t) - F_0\big)}{\mathrm{d}t} \leqslant 0 . \tag{11.6}$$

This result is similar to Boltzmann's H-theorem in the form of Eqns (3.3), (3.4), expressed in terms of the Lyapunov functional $\Lambda_S = S_0 - S(t)$.

In the case of time evolution according to the Fokker – Planck equation (with the given noise intensity or the effective temperature), the mean energy is not conserved. Because of this, the part of Λ_S is now played by the Lyapunov functional Λ_F defined by the difference of free energies.

Boltzmann's result is more valuable for characterizing the relative degree of order, since out of all thermodynamic potentials it is only the entropy that features all the properties necessary for the measure of uncertainty in the statistical description. This is the reason why the definition of information as the difference of unconditional and conditional entropies (with constant mean energy) is the most natural choice in the physics of open systems.

By analogy with Shannon information, we may introduce the measure of information in terms of the Lyapunov functional Λ_F:

$$I_F[E|t] = \Lambda_F = F[E|t] - F_0[E] = D \int_0^\infty \left(\ln \frac{f(E, t)}{F_0}\right) f(E, t) \, \mathrm{d}E \geqslant 0 . \tag{11.7}$$

The quantity $I_F[E|t]$ is the measure of information regarding the departure of the nonequilibrium state at the current time from the stationary state at a given value of the feedback parameter. The information defined in such way decreases on approaching the stationary state, and remains constant when the stationary state is attained.

The last expression implies a kind of *conservation law*: the difference of the free energy of nonequilibrium state $F[E|t]$ and information $I_F[E|t]$ in the course of time evolution at a fixed value of the feedback parameter remains constant, namely

$$F[E|t] - I_F[E|t] = F_0[E] . \tag{11.8}$$

The constant at any fixed value of the feedback parameter is determined by the magnitude of the free energy. It is lowest when the feedback parameter is zero — that is, in the state of equilibrium. This result is similar to the conservation law (3.8) derived earlier.

12. Conclusions

Of the problems of the theory of information, treated in the book by B B Kadomtsev [1] and enumerated in the Introduction, we have only touched upon those that involve transformations of Shannon's formula with the purpose of defining the information of classical and quantum open systems, depending on the values of control parameters. Other fundamental problems and concepts (information coupling, information-wise open systems, information exchange, information content of the wave function) discussed in the book call for special treatment and analysis.

One of the main tasks of this review consists in exposing the difference between S-information (which in fact only serves as the measure of uncertainty in a statistical description) and Shannon information (which with appropriate concretization may serve as the measure of information of open systems in the processes of time evolution and evolution of steady states in the space of control parameters). The efficiency of concretization of Shannon's formula has been proved here for a number of classical and quantum systems, and for a medico-biological system analyzed on the basis of special statistical processing of cardiograms using the S-theorem criterion [22 – 24].

Observe finally that the monograph by B B Kadomtsev [1] was sold out quite quickly, and the publishers of 'Physics – Uspekhi (Advances in Physical Sciences)' released the second edition of this remarkable book (see subsection 'New books' at www.ufn.ru) in March 1999.

References

1. Kadomtsev B B *Dinamika i Informatsiya* (Dynamics and Information) (Moscow: Red. Zhurn. Usp. Fiz. Nauk, 1997); 2nd ed. (Moscow: Red. Zhurn. Usp. Fiz. Nauk, 1999)
2. Shannon C *Bell Syst. Tech. J.* **27** 379 (1948)
3. Shannon C *Raboty po Teorii Informatsii i Kibernetike* (Papers on the Theory of Information and Cybernetics) (Eds R L Dobrushin, O B Lupanov) (Moscow: Inostrannaya Literatura, 1963)
4. Khinchin A Ya *Usp. Mat. Nauk* **11** (1) 17 (1956)
5. Gel'fand I M, Kolmogorov A N, Yaglom A M *Dokl. Akad. Nauk SSSR* **111** 745 (1956)
6. Stratonovich R L *Teoriya Informatsii* (Information Theory) (Moscow: Sovetskoe Radio, 1975)
7. Klimontovich Yu L *Statisticheskaya Fizika* (Statistical Physics) (Moscow: Nauka, 1982) [Translated into English (New York: Harwood Acad. Publ., 1986)]
8. Klimontovich Yu L *Statisticheskaya Teoriya Otkrytykh Sistem* (Statistical Theory of Open Systems) (Moscow: Yanus, 1995) [Translated into English (Dordrecht: Kluwer Acad. Publ., 1995)]
9. Klimontovich Yu L *Phys. Scripta* **58** 54 (1998)

10. Nicolis G, Prigogine I *Self-Organization in Nonequilibrium Systems* (New York: Wiley, 1977)
11. Prigogine I *From Being to Becoming* (San Francisco: W H Freeman, 1980)
12. Prigogine I, Stengers I *Order out of Chaos* (Toronto: Bantam Books, 1984)
13. Vasil'ev V A, Romanovskiĭ Yu M, Yakhno V G *Avtovolnovye Protsessy* (Autowave Processes) (Moscow: Nauka, 1987)
14. Haken H *Synergetics* 2d enl. ed. (Berlin: Springer-Verlag, 1978)
15. Haken H *Advanced Synergetics* (Springer Series in Synergetics, Vol. 20) (Berlin: Springer-Verlag, 1983)
16. Haken H *Information and Self-Organization* (Springer Series in Synergetics, Vol. 40) (Berlin: Springer-Verlag, 1988)
17. Haken H *Principles of Brain Functioning* (Springer Series in Synergetics, Vol. 67) (Berlin: Springer-Verlag, 1996)
18. Vol'kenshteĭn M V *Éntropiya i Informatsiya* (Entropy and Information) (Moscow: Nauka, 1986)
19. Stratonovich R L *Nelineĭnaya Neravnovesnaya Termodinamika* (Nonlinear Nonequilibrium Thermodynamics) (Moscow: Nauka, 1985) [Translated into English (Berlin: Springer-Verlag, 1992 – 1994)]
20. Klimontovich Yu L *Pis'ma Zh. Tekh. Fiz.* **9** 1089 (1983) [*Sov. Tech. Phys. Lett.* **9** 467 (1983)]
21. Klimontovich Yu L *Pis'ma Zh. Tekh. Fiz.* **10** 80 (1984) [*Sov. Tech. Phys. Lett.* **10** 33 (1984)]
22. Anishchenko V S, Saparin P I, Anishchenko T G *Zh.Tekh. Fiz.* **64** (11) 1 (1994) [*Tech. Phys.* **39** 1091 (1994)]
23. Anishchenko V S et al. *Izv. Vyssh. Uchebn. Zaved. Ser. Prikladnaya Nelineĭnaya Dinamika* **2** (3 – 4) 55 (1994)
24. Klimontovich Yu L *Usp. Fiz. Nauk* **166** 1231 (1996) [*Phys. Usp.* **39** 1169 (1996)]
25. Landau L D, Lifshitz E M *Kvantovaya Mekhanika. Nerelyativistskaya Teoriya* 3 izd. (Quantum Mechanics: Non-Relativistic Theory, 3rd ed.) (Moscow: Nauka, 1974) [Translated into English (Oxford: Pergamon Press, 1977)]
26. Sokolov A A, Ternov I M, Zhukovskiĭ V Ch *Kvantovaya Mekhanika* (Quantum Mechanics) (Moscow: Nauka, 1979) [Translated into English (Moscow: Mir, 1984)]
27. Ebeling W, Freund J, Schweitzer F *Komplexe Strukturen: Entropic und Information* (Stuttgart, Leipzig: B G Teubner, 1998)