# Coherent phenomena in stochastic dynamical systems 

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#### Abstract

Basic ideas of the statistical topography of random processes and fields are presented, which are used in the analysis of coherent phenomena in simple dynamical systems. Such phenomena take place with probability one, and provide links between individual realizations and statistical characteristics of systems at large. We confine ourselves to several examples: transfer phenomena in singular dynamic systems under the action of random forces; dynamic localization of plane waves in randomly stratified media; clustering of randomly advected passive tracers; and formation of caustic structures for wave fields in randomly inhomogeneous media. All these phenomena are studied based on the analysis of one-point (space-time) probability distribution functions.


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## 1. Introduction

Many physical processes take place in complex media, whose parameters may be viewed as space-time realizations of chaotic (or stochastic) fields. Such dynamic problems are too complex to allow an explicit mathematical solution for specific realizations of the media. However, one is often interested in generic features of random solutions, rather than particular details. So one is naturally inclined to adopt the well developed machinery of random fields and processes, that is to replace individual realizations with statistical (ensemble) averages. Nowadays, such an approach is commonly used in many problems of atmospheric and oceanic physics.

The randomness of a medium gives rise to stochastic physical (solution) fields. Thus a typical realization of, say 2D scalar (density) fields $\rho(\mathbf{R}, t)$ with $\mathbf{R}=\{x, y\}$ would resemble a complex mountain terrain with randomly distributed peaks, valleys, passes, etc. But the standard statistical tools, like means $\langle\rho(\mathbf{R}, t)\rangle$, and moments $\left\langle\rho\left(\mathbf{R}_{1}, t\right) \rho\left(\mathbf{R}_{2}, t\right)\right\rangle$, where $\langle\ldots\rangle$ indicates ensemble averaging over random parameters, often smooth out some important qualitative features of individual realizations.

So the resulting 'mean fields' would bear little resemblance to a typical (individual ) realization, and sometimes give conflicting predictions. For instance, ensemble average of a randomly advected passive tracer often yields a diffusive process, that smooths out the mean-(solution) field, whereas
individual realizations tend to evolve a very rugged and fragmented shape.

Thus, standard statistical means could reasonably predict some 'global' spatial-temporal scales and parameters of solutions, but tell little about the fine (small scale) structure and details of evolution. Such details (for an advected tracer) may strongly depend on some special properties of random velocities, for instance, compressibility vs. incompressibility.

In the former case the turbulent transport would typically bring about the formation of clusters - regions of high tracer concentration surrounded by low density 'voids' $[1,2]$. However, all statistical moments of the particle's position (or separation-function) show an exponential increase in time, which implies 'statistical dispersion' of particles [2-4]. Similarly, optical rays in random media diverge exponentially in the 'mean' [3-5], yet the caustics are formed at finite distances within the layer with probability one [6-9].

Another example of the kind is the dynamic localization of plane waves by randomly layered media. When a plane wave is incident upon a randomly layered half-space, its intensity decreases exponentially with the distance from the boundary, for almost all realizations. Yet all statistical moments show an exponential increase [10-13].

We shall call physical phenomena that occur with probability one and characterize 'typical realizations' coherent. Such statistical coherence, could be viewed as a way to 'organize dynamic complexity', and identify its 'statistically stable' properties, by analogy with the usual notion of coherence, as self-organization of complex, multi-component systems, arising from their chaotic interaction (cf. Ref. [14]).

Although our notion of coherence differs from the standard usage, we find it natural. In general there is no simple way to assert that a given phenomenon occurs with probability 1. However, it becomes possible to do it theoretically for certain problems with simple models of fluctuating parameters. In other cases one could do it by numeric modeling, or analyzing physical experiments. Of course, our notion of coherence makes it more a mathematical problem, rather than a physical one.

Let us also remark that in many cases we don't have a full understanding of the physical causes that lead to coherence. For instance, the above clustering of Lagrangian particles advected by random potential velocities represents by itself a purely kinematic phenomenon in the absence of any real particle interaction.

The complete statistics (e.g. all $n$-point moments) would allow in principle a complete description of the dynamical system. But in practice one could handle only a few simple statistics, typically expressed through the one-point probability distributions (PDF) in space-time variables. The natural problem then is to deduce some important qualitative and quantitative characteristics of individual realizations from such limited data. It takes on a particular significance for atmospheric and oceanic problems, where 'statistical ensembles' do no exist in the strict sense, or require a long temporal exposure (time series) and experimentalists deal most often with individual realizations.

A possible answer to the problem is suggested by the methods of statistical topography. The name statistical topography was first coined in the book [15], though the main ideas go back to papers [16-19] (see also survey [20], for a complete bibliography). The early works applied statistical topography of random fields to the statistical analysis of the
rough sea surface, and the radar and TV images. Applications of statistical topography to the turbulent transport problem came later [2, 21-23], while [24] adopted these ideas for the wave propagation in random media.

The methods of statistical topography call into question the basic 'philosophy' of statistical analysis of stochastic dynamical systems. We believe such an approach could be useful for experimentalists who apply statistical tools to large experimental data.

In the present paper we shall exploit some basic ideas of statistical topography of random fields and processes to analyze coherent phenomena in a few model examples, chosen from a great variety of such systems, namely:

- transfer phenomena in (nonlinear) dynamical systems with 'singular solutions' and random forcing. Similar transfer phenomena for a general class of randomly forced dynamical systems with finite/discrete set of stable/unstable equilibria, are well known and discussed in many textbooks;
- dynamic localization of plane waves in randomly stratified media;
- clustering of passive tracers by random (turbulent) velocity fields;
- formation of caustics for wave propagation in random media.

All these phenomena may be approached through a unified method based on the analysis of the one-point probability distribution functions (PDF), that result from their dynamic evolution.

We shall start with the dynamic description of model systems, and discuss specifics of their solutions in the presence of random parameters. The statistical analysis will follow.

Though our examples are drawn mostly from statistical hydrodynamics, radio-physics and acoustics, - the areas of interest to the authors, we believe the methods developed in the paper could find applications in other areas of physics.

## 2. Examples of dynamical systems, problem formulation, and special features of solutions

### 2.1 Particles in random velocity and force fields

A particle moving in a (random) velocity field is described by an ordinary differential system

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \mathbf{r}(t)=\mathbf{U}(\mathbf{r}, t), \quad \mathbf{r}\left(t_{0}\right)=\mathbf{r}_{0} \tag{2.1}
\end{equation*}
$$

Here $\mathbf{U}(\mathbf{r}, t)=\mathbf{u}_{0}(\mathbf{r}, t)+\mathbf{u}(\mathbf{r}, t)$ is made of the deterministic (mean) component $\mathbf{u}_{0}(\mathbf{r}, t)$ and the random perturbation $\mathbf{u}(\mathbf{r}, t)$. In the absence of randomness $(\mathbf{u}=0)$, and constant $\mathbf{u}_{0}$ we get a simple rectilinear motion

$$
\mathbf{r}(t)=\mathbf{r}_{0}+\mathbf{u}_{0}\left(t-t_{0}\right)
$$

The same system (2.1) could also describe the particle motion under the action of random forces. Indeed, a simple linear friction law yields

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} t} \mathbf{r}(t)=\mathbf{v}(t), \quad \frac{\mathrm{d}}{\mathrm{~d} t} \mathbf{v}(t)=-\lambda \mathbf{v}(t)+\mathbf{f}(\mathbf{r}, t), \\
& \mathbf{r}(0)=\mathbf{r}_{0}, \quad \mathbf{v}(0)=\mathbf{v}_{0} . \tag{2.2}
\end{align*}
$$

Once again in the absence of friction and forcing we get a simple rectilinear motion

$$
\mathbf{v}(t)=\mathbf{v}_{0}, \quad \mathbf{r}(t)=\mathbf{r}_{0}+\mathbf{v}_{0} t
$$

Let us discuss some qualitative features of stochastic system (2.1) in the absence of the mean flow. Formally equation (2.1) describes the motion of independent particles, as no interaction takes place. If however, field $\mathbf{u}(\mathbf{r}, t)$ has finite correlation radius $l_{\text {cor }}$, then particles within $l_{\text {cor }}$-proximity of each other lie in the common domain of influence of velocity $\mathbf{u}(\mathbf{r}, t)$. Hence, they could exhibit a collective behavior.

In general, the velocity field $\mathbf{u}(\mathbf{r}, t)$ is made of solenoidal [such that $\operatorname{div} \mathbf{u}(\mathbf{r}, t)=\nabla \mathbf{u}(\mathbf{r}, t)=0$ ] plus potential [such that $\mathbf{u}(\mathbf{r}, t)=\nabla \psi(\mathbf{r}, t)]$ components. Numeric simulations [22, 25] of multiparticle systems show a marked difference between the two cases. Figure 1a shows a divergent-free random field $\mathbf{u}(\mathbf{r}, t)$ advecting a uniformly distributed set of particles over the disk. Here the total area enclosed by the deformed contours is conserved, and the particles fill the area in the (approximately) 'uniform' manner. Observe, however, that the contours become increasingly more rugged and 'fractallike'.


Figure 1. Particle dynamics in solenoidal (a) and potential (b) velocity fields.

In the presence of potential component $(\operatorname{div} \mathbf{u}(\mathbf{r}, t) \neq 0)$, the initial uniform distribution of particles (over the square) evolves into clusters - compact regions of high concentration amidst low-density voids. The results of numeric simulations are shown in Fig. 1b. Let us stress here the kinematic nature of this effect. Indeed, the ensemble averaging over velocity realizations could completely obliterate it. Let us also note, that numeric simulations of Fig. 1 were conducted for stationary (time-independent) fields $\mathbf{u}(\mathbf{r})$.

Such clustering of particle systems was first observed in papers [26,27], via computer simulation of a simple model of atmospheric dynamics, based on the so-called EOLE experiment. This global experiment was conducted in Argentina in 1970-1971, and involved launching 500 air balloons of constant density, that spread over the entire Southern hemisphere at an altitude of roughly 12 km . Figure 2 shows a numeric simulation of the distribution of balloons 105 days after the beginning [27], and clearly exhibits their convergence into clusterized groups.

Now let us turn attention to another stochastic feature of randomly stirred dynamic systems (2.1), the so-called transfer


Figure 2. Distribution of air balloons in the atmosphere 105 days after the launch.
phenomena. The well known examples of transfer phenomena involve systems with finitely many stable equilibria (see, for instance, Refs [3, 28]). Here we shall confine ourselves to a simple case, that arises in the statistical theory of wave propagation (below), and exhibits singular solutions in time:

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} x(t)=-\lambda x^{2}(t)+f(t), \quad x(0)=x_{0}, \quad \lambda>0 \tag{2.3}
\end{equation*}
$$

Here $f(t)$ stands for a random function of time. In the absence of forcing $(f(t)=0)$ it has an exact solution of the form

$$
x(t)=\frac{1}{\lambda\left(t-t_{0}\right)}, \quad t_{0}=-\frac{1}{\lambda x_{0}}
$$

If the initial point $x_{0}>0$, then $t_{0}<0$, and $x(t)$ converges monotonically to 0 as $t \rightarrow \infty$. If $x_{0}<0$, then solution $x(t)$ approaches $-\infty$, i.e. 'blows up' in a finite time $t_{0}=-1 /\left(\lambda x_{0}\right)$. In this case random forcing would play no significant role. It becomes significant only for positive $x_{0}$.

Indeed, in this case $x(t)$ first slightly fluctuating decrease with time, and when it becomes sufficiently small a random force $f(t)$ will 'transfer' it ('kick over') into the negative halfline where it would be dragged to $-\infty$ in a finite time.

Thus, stochastic forcing may turn any initial state $x_{0}$ into an unstable ('explosive') solution, that reaches $-\infty$ in a finite time $t_{0}$. Figure 3 gives a schematic view of a particular solution realization $x(t)$, for various times: $t<t_{0}, t>t_{0}$, that exhibits a 'quasi-periodic' pattern. The principal coherent phenomenon here is the explosive character of realizations of the process.

The above example involves additive random noise $f$. The simplest case of a multiplicative noise is given by the stochastic parametric resonance model - a second order differential equation

$$
\begin{align*}
& \frac{\mathrm{d}^{2}}{\mathrm{~d} t^{2}} x(t)+\omega_{0}^{2}[1+z(t)] x(t)=0 \\
& x(0)=x_{0}, \quad \frac{\mathrm{~d}}{\mathrm{~d} t} x(0)=v_{0} \tag{2.4}
\end{align*}
$$



Figure 3. Typical realization of solutions of equation (1.3).
with random potential $z(t)$. This appears in many areas of physics. From the physical standpoint, dynamic system (2.4) is subjected to a parametric excitation, since process $z(t)$ may contain harmonics of all frequencies, including $2 \omega_{0} / n$, $n=1,2,4, \ldots$ Those modes may produce a parametric resonance, as for the well known periodic (Mathieu) case $z(t)$ (see, for instance, Refs [3, 5, 10]).

### 2.2 Plane waves in randomly layered media

In the previous section we considered two examples of the initial value problems described by ordinary differential equations. Next we shall review a simple boundary value problem, namely the 1D stationary wave problem.

Let us consider a inhomogeneous layered medium occupying a strip $L_{0}<x<L$. A plane wave of unit amplitude $u_{0}(x)=\exp [-\mathrm{i} k(x-L)]$ is incident upon it from the right half-space $x>L$ (Fig. 4a). The wave field in the strip obeys the Helmholtz equation

$$
\begin{equation*}
\frac{\mathrm{d}^{2}}{\mathrm{~d} x^{2}} u(x)+k^{2}[1+\varepsilon(x)] u(x)=0 \tag{2.5}
\end{equation*}
$$

with function $\varepsilon(x)$ representing inhomogeneities of the medium. We assume $\varepsilon(x)=0$ outside the strip, and $\varepsilon(x)=\varepsilon_{1}(x)+\mathrm{i} \gamma$ within, the real part $\varepsilon_{1}(x)$ responsible for the wave scattering, while the imaginary part $\gamma \ll 1$ describes wave attenuation by the medium.

In the right half space $(x>L)$ the wave field is made up of the incident and reflected components, $u(x)=$ $\exp [-\mathrm{i} k(x-L)]+R_{L} \exp [\mathrm{i} k(x-L)]$, where $R_{L}$ is the (complex) reflection coefficient. In the left half $x<L_{0}$ we have $u(x)=T_{L} \exp \left[i k\left(L_{0}-x_{0}\right)\right]$, with the (complex) transmission coefficient $T_{L}$. The boundary conditions for Eqn (2.5) are
continuity relations for $u(x)$ and its derivative $\mathrm{d} u(x) / \mathrm{d} x$ at $x=L$ and $x=L_{0}$ :

$$
\begin{equation*}
\frac{\mathrm{i}}{k} \frac{\mathrm{~d}}{\mathrm{~d} x} u(x)+\left.u(x)\right|_{x=L}=2, \quad \frac{\mathrm{i}}{k} \frac{\mathrm{~d}}{\mathrm{~d} x} u(x)-\left.u(x)\right|_{x=L_{0}}=0 \tag{2.6}
\end{equation*}
$$

So the wave field in the inhomogeneous medium is determined by the boundary value problem (2.5), (2.6), as opposed to the initial value problem (2.4) with the same differential equation.

If parameter $\varepsilon_{1}$ is random, one is interested in the statistics of the reflection and transmission coefficients: $R_{L}=u(L)-1$, and $T_{L}=u\left(L_{0}\right)$, that depend on the boundary values of the wave-field (2.5), (2.6), as well as the field intensity $I(x)=|u(x)|^{2}$ within the layer (statistical radiative transport).

Equation (2.5) implies the energy conservation (dissipation) law at $x<L$

$$
k \gamma I(x)=\frac{\mathrm{d}}{\mathrm{~d} x} S(x)
$$

where $S(x)$ denotes the energy-density flux

$$
S(x)=\frac{1}{2 \mathrm{i} k}\left[u ( x ) \frac { \mathrm { d } } { \mathrm { d } x } u ^ { * } \left(x\left(-u^{*}(x) \frac{\mathrm{d}}{\mathrm{~d} x} u(x)\right] .\right.\right.
$$

Furthermore, one has $S(L)=1-\left|R_{L}\right|^{2}, S\left(L_{0}\right)=\left|T_{L}\right|^{2}$.
If the medium does not dissipate waves $(\gamma=0)$, then the energy-conservation yields

$$
\begin{equation*}
\left|R_{L}\right|^{2}+\left|T_{L}\right|^{2}=1 \tag{2.7}
\end{equation*}
$$

Let us turn to some special features of the stochastic boundary value problem (2.5), (2.6). In the absence of medium fluctuations $\varepsilon_{1}(x)=0$, and for sufficiently small attenuation $\gamma$ the field intensity decays exponentially inside the layer as

$$
\begin{equation*}
I(x)=|u(x)|^{2}=\exp [-k \gamma(L-x)] . \tag{2.8}
\end{equation*}
$$

Figure 5 shows numeric simulations of two wave intensities in a sufficiently thick layer, that come from two different realizations of the medium [29]. Skipping further details and parameters of the problem, let us only note a clearly perceived exponential fall-off trend accompanied by large intensity fluctuations, directed both ways (to zero and to infinity). They result from the multiple scattering processes in randomly inhomogeneous media, and demonstrate the socalled dynamic localization.

Similarly a point-source located inside the strip is described by the boundary value Green's function of the


Figure 4. Incident plane wave on a layer of random medium (a), and the wave-source inside the medium (b).


Figure 5. Numeric modeling of the dynamic localization for two realizations of random media.

Helmholtz equation

$$
\begin{align*}
& \frac{\mathrm{d}^{2}}{\mathrm{~d} x^{2}} G\left(x ; x_{0}\right)+k^{2}[1+\varepsilon(x)] G\left(x ; x_{0}\right)=2 \mathrm{i} k \delta\left(x-x_{0}\right), \\
& \frac{\mathrm{i}}{k} \frac{\mathrm{~d}}{\mathrm{~d} x} G\left(x ; x_{0}\right)+\left.G\left(x ; x_{0}\right)\right|_{x=L}=0, \\
& \frac{\mathrm{i}}{k} \frac{\mathrm{~d}}{\mathrm{~d} x} G\left(x ; x_{0}\right)-\left.G\left(x ; x_{0}\right)\right|_{x=L_{0}}=0 . \tag{2.9}
\end{align*}
$$

Here the exterior wave field (outside the layer) consists of outgoing waves (Fig. 4b) with the transmission coefficients $T_{1}=G\left(L ; x_{0}\right), T_{2}=G\left(L_{0} ; x_{0}\right)$. Moreover, the wave field in the left half-space $x<x_{0}$ is proportional to the plane wave incident from the half-space $x>x_{0}$ upon the layer ( $L_{0}, x_{0}$ ) [10], i.e.

$$
G\left(x ; x_{0}\right) \sim u\left(x ; x_{0}\right), \quad x \leqslant x_{0} .
$$

Let us also remark that the scattering problem (2.5), (2.6) corresponds to putting the source (2.9) on the boundary $x_{0}=L$, i.e. taking $u(x)=G(x ; L)$.

Boundary value problems (2.5), (2.6) and (2.9) could be solved by the embedding method of Refs [30-32], that reformulates them as initial value problems in parameter $L$ - the right boundary end of the strip [11]. Thus the reflection coefficient $R_{L}$ of Eqns (2.5), (2.6) obeys the Riccati equation in $L$,

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} L} R_{L}=2 \mathrm{i} k R_{L}+\frac{\mathrm{i} k}{2} \varepsilon(L)\left(1+R_{L}\right)^{2}, \quad R_{L_{0}}=0 \tag{2.10}
\end{equation*}
$$

whereas field $u(x) \equiv u(x ; L)$ inside the layer obeys the linear equation

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} L} u(x ; L)=\mathrm{i} k u(x ; L)+\frac{\mathrm{i} k}{2} \varepsilon(L)\left(1+R_{L}\right) u(x ; L), \\
& u(x ; x)=1+R_{x} . \tag{2.11}
\end{align*}
$$

Hence follows the equation for the squared modulus of the reflection coefficient $W_{L}=\left|R_{L}\right|^{2}$ :
$\frac{\mathrm{d}}{\mathrm{d} L} W_{L}=-2 k \gamma W_{L}-\frac{\mathrm{i} k}{2} \varepsilon_{1}(L)\left(R_{L}-R_{L}^{*}\right)\left(1-W_{L}\right), W_{L_{0}}=0$.

If boundary $L_{0}$ is completely reflective, the initial condition becomes $W_{L_{0}}=1$. So in the absence of damping $(\gamma=0)$ the incident wave is fully reflected, $W_{L}=1$. Hence the reflection coefficient $R_{L}=\exp \left\{\mathrm{i} \phi_{L}\right\}$, and Eqn (2.10) would imply the following evolution of the 'reflection phase'

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} L} \phi_{L}=2 k+k \varepsilon_{1}(L)\left(1+\cos \phi_{L}\right), \tag{2.13}
\end{equation*}
$$

valid over the entire range $(-\infty, \infty)$ of variable $\phi_{L}$. On the other hand Eqn (2.11) for the wave-field $u$ involves only trigonometric functions of $\phi_{L}$. In order to make a transition from $(-\infty, \infty)$ to the natural range $(-\pi, \pi)$ of $\phi_{L}$, we introduce another function [13, 33]

$$
\begin{equation*}
z_{L}=\tan \left(\frac{\phi_{L}}{2}\right), \tag{2.14}
\end{equation*}
$$

that obeys a nonlinear dynamic evolution of type (2.3),

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} L} z_{L}=k\left(1+z_{L}^{2}\right)+k \varepsilon_{1}(L) \tag{2.15}
\end{equation*}
$$

with singular-type (exploding) solutions.
When the left boundary is let arbitrary far $\left(L_{0} \rightarrow-\infty\right)$, and the media is non-dissipative $(\gamma=0)$, there exists a 'stationary' solution $W_{L}=1$, independent of $L$, which corresponds to the complete reflection of incident waves. Such solution will be shown to appear in the stochastic problem with probability one [10].

Researchers often deal with multidimensional situations, when certain wave-types generate other types due to spatial inhomogeneity of the medium. In some cases such media could be approximately divided into a discrete set of distinct strata, with continuously changing parameters within each stratum. As an example we mention large scale/low frequency oscillations in the atmosphere and ocean, known as Rossby waves. These modes are derived in the context of the quasigeostrophic model, where the atmosphere (or ocean) is viewed as an aggregation of thin multi-layer films, stratified by density variations and thicknesses [34]. The role of 'localizing media' for the Rossby waves is played by the (inhomogeneous) bottom topography. The simplest onelayer model (for the so-called barotropic modes) could be reduced to the Helmholtz equation, while the two-layer system could account for the baroclinic effects [35-37].

We shall consider a simple two-layer wave-model given by system [38]

$$
\begin{align*}
& \frac{\mathrm{d}^{2}}{\mathrm{~d} x^{2}} \psi_{1}+k^{2} \psi_{1}-\alpha_{1} F\left(\psi_{1}-\psi_{2}\right)=0 \\
& \frac{\mathrm{~d}^{2}}{\mathrm{~d} x^{2}} \psi_{2}+k^{2}[1+\varepsilon(x)] \psi_{2}+\alpha_{2} F\left(\psi_{1}-\psi_{2}\right)=0 \tag{2.16}
\end{align*}
$$

Here parameters $\alpha_{1}=1 / H_{1}, \alpha_{2}=1 / H_{2}$ designate characterizes waves interaction, while coupling constant $F$ characterizes waves interaction.

As before we assume the random function $\varepsilon(x)$ to vanish outside the finite interval $\left(L_{0}, L\right)$. The schematic geometry of problem (2.16) is shown in Fig. 6. The boundary


Figure 6. Two-layer medium.
conditions include the radiation condition at $\infty$, and the continuity of wave fields along with their derivatives at boundary points $L_{0}$ and $L$.

Parameter $F$ encodes the vertical stratification, and gives the horizontal length scale for the 'cross-mode' generation. The specific form of Eqn (2.16), like coefficients $\alpha_{i}$ measuring layers' thicknesses etc., arise in the geophysical fluid setup [35-37]. The basic equations and parameters could change, depending on the physical system in question. But one essential feature should remain, namely the linear form of the wave interactions.

System (2.16) could be formally reduced to a single-layer case by setting $F=0, \psi_{1}=0$, and the corresponding wave equation takes on the 'Helmholtz form' (2.5). Another way to reduce the system to a 'single-layer' is via limit $H_{1} \rightarrow 0$, hence $\psi_{1}=\psi_{2}$. Let us remark however, that the order of two limiting procedures, $L_{0} \rightarrow-\infty$ (half-space) and $H_{i} \rightarrow 0$ (single layer), cannot be interchanged in the statistical problem [35]. The layers' relative thickness could be made arbitrarily small, but should remain nonzero.

Let us consider the equations for the Green's function,

$$
\begin{align*}
& \frac{\mathrm{d}^{2}}{\mathrm{~d} x^{2}} \psi_{1}+k^{2} \psi_{1}-\alpha_{1} F\left(\psi_{1}-\psi_{2}\right)=-v_{1} \delta\left(x-x_{0}\right) \\
& \frac{\mathrm{d}^{2}}{\mathrm{~d} x^{2}} \psi_{2}+k^{2}[1+\varepsilon(x)] \psi_{2}+\alpha_{2} F\left(\psi_{1}-\psi_{2}\right)=-v_{2} \delta\left(x-x_{0}\right) \tag{2.17}
\end{align*}
$$

with sources in either the upper or the lower layer, respectively. Introducing vector field $\boldsymbol{\psi}\left(x ; x_{0}\right)=\left\{\psi_{1}\left(x ; x_{0}\right), \psi_{2}\left(x ; x_{0}\right)\right\}$, and vector $\mathbf{v}=\left\{v_{1}, v_{2}\right\}$, we can recast Eqn (2.17) in the vector form

$$
\begin{equation*}
\left[\frac{\mathrm{d}^{2}}{\mathrm{~d} x^{2}}+A^{2}+k^{2} \varepsilon(x) \Gamma\right] \boldsymbol{\Psi}\left(x ; x_{0}\right)=-\mathbf{v} \delta\left(x-x_{0}\right), \tag{2.18}
\end{equation*}
$$

where

$$
\begin{align*}
& A^{2}=\left(\begin{array}{cc}
k^{2}-\alpha_{1} F & \alpha_{1} F \\
\alpha_{2} F & k^{2}-\alpha_{2} F
\end{array}\right), \\
& A=k\left(\begin{array}{cc}
\tilde{\alpha}_{2}+\lambda \tilde{\alpha}_{1} & (1-\lambda) \tilde{\alpha}_{1} \\
(1-\lambda) \tilde{\alpha}_{2} & \tilde{\alpha}_{1}+\lambda \tilde{\alpha}_{2}
\end{array}\right), \quad \Gamma=\left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right) ; \tag{2.19}
\end{align*}
$$

here we introduce the parameter

$$
\lambda^{2}=\left[1-\left(\alpha_{1}+\alpha_{2}\right) \frac{F}{k^{2}}\right]
$$

describing the mode we call the ' $\lambda$-wave' (taking $\lambda^{2}>0$ ), and the relative layer thicknesses

$$
\tilde{\alpha}_{1}=\frac{\alpha_{1}}{\alpha_{1}+\alpha_{2}}=\frac{H_{2}}{H_{0}}, \quad \tilde{\alpha}_{2}=\frac{\alpha_{2}}{\alpha_{1}+\alpha_{2}}=\frac{H_{1}}{H_{0}}, \quad \tilde{\alpha}_{1}+\tilde{\alpha}_{2}=1 .
$$

This resembles the scalar Helmholtz equation (2.18). Here matrix $A$ plays the role of a 'uniform background' (constant refraction index), while $\varepsilon \Gamma$ represents medium inhomogeneities.

Next we consider the fundamental matrix-solution $\Psi$

$$
\begin{equation*}
\left[\frac{\mathrm{d}^{2}}{\mathrm{~d} x^{2}}+A^{2}+k^{2} \varepsilon(x) \Gamma\right] \Psi\left(x ; x_{0}\right)=-E \delta\left(x-x_{0}\right) \tag{2.20}
\end{equation*}
$$

that gives all other 'vector-solutions' $\boldsymbol{\psi}\left(x ; x_{0}\right)$ as

$$
\begin{equation*}
\boldsymbol{\psi}\left(x ; x_{0}\right)=\Psi\left(x ; x_{0}\right) \mathbf{v} . \tag{2.21}
\end{equation*}
$$

The columns $\left\{\psi_{11}, \psi_{21}\right\}$ and $\left\{\psi_{12}, \psi_{22}\right\}$ of matrix $\Psi$ describe waves generated by the point sources $\left\{v_{1}, 0\right\}$ and $\left\{0, v_{2}\right\}$ in the upper and lower layers, respectively. The fundamental matrix $\Psi$ satisfies boundary conditions

$$
\begin{align*}
& \left.\left(\frac{\mathrm{d}}{\mathrm{~d} x}-\mathrm{i} A\right) \Psi\left(x ; x_{0}\right)\right|_{x=L}=0 \\
& \left.\left(\frac{\mathrm{~d}}{\mathrm{~d} x}+\mathrm{i} A\right) \Psi\left(x ; x_{0}\right)\right|_{x=L_{0}}=0 . \tag{2.22}
\end{align*}
$$

Following Ref. [38] we shall place the wave source on the boundary $x_{0}=L$. The corresponding boundary value problem, with the 'jump-condition' at the source gives

$$
\begin{align*}
& {\left[\frac{\mathrm{d}^{2}}{\mathrm{~d} x^{2}}+A^{2}+k^{2} \varepsilon(x) \Gamma\right] \Psi(x ; L)=0,} \\
& \left.\left(\frac{\mathrm{~d}}{\mathrm{~d} x}-\mathrm{i} A\right) \Psi(x ; L)\right|_{x=L}=E, \\
& \left.\left(\frac{\mathrm{~d}}{\mathrm{~d} x}+\mathrm{i} A\right) \Psi(x ; L)\right|_{x=L_{0}}=0 . \tag{2.23}
\end{align*}
$$

The latter could be further simplified by diagonalizing the constant matrix-coefficient $A$ (2.19) with the help of matrices

$$
K=\left[\begin{array}{cc}
1 & -1 \\
\tilde{\alpha}_{2} & \tilde{\alpha}_{1}
\end{array}\right], \quad K^{-1}=\left[\begin{array}{cc}
\tilde{\alpha}_{1} & 1 \\
-\tilde{\alpha}_{2} & 1
\end{array}\right] .
$$

The transformed coefficients $A$ and $\Gamma$ become

$$
B=k\left[\begin{array}{ll}
\lambda & 0 \\
0 & 1
\end{array}\right], \quad \widetilde{\Gamma}=K \Gamma K^{-1}=\left[\begin{array}{cc}
\tilde{\alpha}_{2} & -1 \\
-\tilde{\alpha}_{1} \tilde{\alpha}_{2} & \tilde{\alpha}_{1}
\end{array}\right],
$$

and the transformed $\Psi$,

$$
\begin{equation*}
U(x ; L)=-2 \mathrm{i} K \Psi(x ; L) K^{-1} B \tag{2.24}
\end{equation*}
$$

obeys the equation

$$
\begin{align*}
& {\left[\frac{\mathrm{d}^{2}}{\mathrm{~d} x^{2}}+B^{2}+k^{2} \varepsilon(x) \widetilde{\Gamma}\right] U(x ; L)=0} \\
& \left.\left(\frac{\mathrm{~d}}{\mathrm{~d} x}-\mathrm{i} B\right) U(x ; L)\right|_{x=L}=-2 \mathrm{i} B \\
& \left.\left(\frac{\mathrm{~d}}{\mathrm{~d} x}+\mathrm{i} B\right) U(x ; L)\right|_{x=L_{0}}=0 \tag{2.25}
\end{align*}
$$

Boundary-value problem Eqn (2.25) describes the interaction and generation of $k$ - and $\lambda$ - waves of unit amplitude (labeled according to the diagonal entries of $B$ ). Here the incident $\lambda$-wave $U_{11}$ generates $k$-wave $U_{21}$, whereas incident $k$-wave $U_{22}$ generates $\lambda$-wave $U_{12}$.

It follows from Eqn (2.25) that the amplitude of the generated $k$-wave $U_{21}$ is proportional to
$\delta=\lambda \tilde{\alpha}_{1} \tilde{\alpha}_{2}=\lambda H_{1} H_{2} / H_{0}^{2}$. Parameter $\delta$ is always less than $\lambda / 4$. In real (geophysical) media $\delta$ becomes a small parameter, as $\tilde{\alpha}_{1} \tilde{\alpha}_{2} \ll 1$. Indeed, in the atmosphere $H_{2} \ll H_{1}$, hence $\tilde{\alpha}_{1} \ll 1$, $\tilde{\alpha}_{2} \cong 1$, whereas in the ocean $H_{1} \ll H_{2}$, so $\tilde{\alpha}_{2} \ll 1, \tilde{\alpha}_{1} \cong 1$. Should it happen that the relative depths of two layers are comparable $\left(H_{2} / H_{1} \cong 1\right)$, parameter $\delta$ becomes small, provided $\lambda$ is small.

To continue the discussion of wave systems we introduce the reflection and transmission matrices $R(L)=U(L ; L)-E$ and $T(L)=U\left(L_{0} ; L\right)$, whose (complex) entries $R_{i j}, T_{i j}$ give the reflection and transmission coefficients of $\lambda$ and $k$ incident and scattered modes.

System (2.25) has two conserved integrals for the energy current of the $k$ - and $\lambda$-modes

$$
\begin{aligned}
& \tilde{\alpha}_{1} \tilde{\alpha}_{2} {\left[U_{11}^{*}(x) \frac{\mathrm{d}}{\mathrm{~d} x} U_{11}(x)-U_{11}(x) \frac{\mathrm{d}}{\mathrm{~d} x} U_{11}^{*}(x)\right]+} \\
& \quad+U_{21}^{*}(x) \frac{\mathrm{d}}{\mathrm{~d} x} U_{21}(x)-U_{21}(x) \frac{\mathrm{d}}{\mathrm{~d} x} U_{21}^{*}(x)=\mathrm{const} \\
& \tilde{\alpha}_{1} \tilde{\alpha}_{2}[ \left.U_{12}^{*}(x) \frac{\mathrm{d}}{\mathrm{~d} x} U_{12}(x)-U_{12}(x) \frac{\mathrm{d}}{\mathrm{~d} x} U_{12}^{*}(x)\right]+ \\
& \quad+U_{22}^{*}(x) \frac{\mathrm{d}}{\mathrm{~d} x} U_{22}(x)-U_{22}(x) \frac{\mathrm{d}}{\mathrm{~d} x} U_{22}^{*}(x)=\text { const } .
\end{aligned}
$$

The latter could be recast in terms of the reflection and transmission coefficients as

$$
\begin{align*}
& \delta\left[1-\left|R_{11}\right|^{2}-\left|T_{11}\right|^{2}\right]=\left|R_{21}\right|^{2}+\left|T_{21}\right|^{2}, \\
& 1-\left|R_{22}\right|^{2}-\left|T_{22}\right|^{2}=\delta\left[\left|R_{12}\right|^{2}+\left|T_{12}\right|^{2}\right] . \tag{2.26}
\end{align*}
$$

Complete localization in the band $\left(L_{0}, L\right)$ implies that the transmission coefficients $T_{i j}$ converge to zero, as the bandwidth increases.

Equations (2.26) establish certain algebraic relations between the reflection and transmission coefficients. Next we apply the embedding method $[37,38]$ to produce a closed system of differential equations for the coefficients. The embedding method allows a boundary value problem for the matrix function $U(x ; L)$ to be converted to an initial value problem for $U(x ; L)$ and $U(L ; L)$, as functions of variable $L(x$ is now viewed as a parameter):

$$
\begin{gather*}
\frac{\partial}{\partial L} U(x ; L)=\mathrm{i} U(x ; L) B+\frac{\mathrm{i}}{2} k^{2} \varepsilon(L) U(x ; L) B^{-1} \widetilde{\Gamma} U(L ; L), \\
\left.U(x ; L)\right|_{L=x}=U(x ; x), \\
\frac{\mathrm{d}}{\mathrm{~d} L} U(L ; L)=-2 \mathrm{i} B+\mathrm{i}[U(L ; L) B+B U(L ; L)] \\
\quad+\frac{\mathrm{i}}{2} k^{2} \varepsilon(L) U(L ; L) B^{-1} \widetilde{\Gamma} U(L ; L),\left.\quad U(L ; L)\right|_{L=L_{0}}=E . \tag{2.27}
\end{gather*}
$$

The latter gives the matrix Riccati equation for the reflection matrix $R(L)=U(L ; L)-E$,

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} L} R(L)=\mathrm{i}[R(L) B+B R(L)] \\
& +\frac{\mathrm{i}}{2} k^{2} \varepsilon(L)[E+R(L)] B^{-1} \widetilde{\Gamma}[E+R(L)],\left.\quad R(L)\right|_{L=L_{0}}=0 . \tag{2.28}
\end{align*}
$$

Expanding Eqn (2.28) in terms of matrix entries $R_{i j}$, one could derive another relation for the reflection coefficients $R_{21}=\delta R_{12}$, which reduces the system to 3 unknown quantities $R_{11}, R_{12}, R_{22}$.

So far we have discussed the dynamic evolution of finite dimensional systems given by ODE's. Next we turn to dynamical fields described by partial differential equations.

### 2.3 Passive tracer advection by random velocities

The simplest example of the kind is a linear continuity equation for a tracer density advected by the velocity field $\mathbf{U}(\mathbf{r}, t)$ :

$$
\begin{equation*}
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \mathbf{r}} \mathbf{U}(\mathbf{r}, t)\right) \rho(\mathbf{r}, t)=0, \quad \rho(\mathbf{r}, 0)=\rho_{0}(\mathbf{r}) . \tag{2.29}
\end{equation*}
$$

It conserves the total mass

$$
M=M(t)=\int \mathrm{d} \mathbf{r} \rho(\mathbf{r}, t)=\int \mathrm{d} \mathbf{r} \rho_{0}(\mathbf{r}) .
$$

The first order linear PDE (2.29) is solved by the method of characteristics, a family of ODE solutions

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \mathbf{r}(t)=\mathbf{U}(\mathbf{r}, t), \quad \mathbf{r}(0)=\xi \tag{2.30}
\end{equation*}
$$

that describe the evolution of $\rho$ :

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \rho(t)=-\frac{\partial \mathbf{U}(\mathbf{r}, t)}{\partial \mathbf{r}} \rho(t), \quad \rho(0)=\rho_{0}(\xi) . \tag{2.31}
\end{equation*}
$$

The latter gives the so-called Lagrangian formulation of the original Eulerian PDE (2.29). It depends on the characteristic parameter (initial point) $\xi$. Notice that equation (2.30) has the same form as the particle evolution (2.1) driven by random velocities.

The dependence of solution (2.30), (2.31) on the initial point $\xi$ will be designated here and henceforth by a vertical bar,

$$
\begin{equation*}
\mathbf{r}(t)=\mathbf{r}(t \mid \xi), \quad \rho(t)=\rho(t \mid \xi) . \tag{2.32}
\end{equation*}
$$

The first equation (2.32) gives an algebraic relation between $\xi$ and r , which could be solved (for a fixed $t$ ) to find

$$
\xi=\xi(\mathbf{r}, t)
$$

provided its Jacobian $j(t \mid \xi)=\operatorname{Det}\left|j_{j k}(t \mid \xi)\right|$, $j_{i k}(t \mid \xi)=\partial r_{i}(t \mid \xi) / \partial \xi_{k}$, is nonzero. This would yield the Eulerian density (2.29) in terms of $\xi(\mathrm{r}, t)$ :

$$
\begin{equation*}
\rho(\mathbf{r}, t)=\rho(t \mid \xi(\mathbf{r}, t)) . \tag{2.33}
\end{equation*}
$$

The Jacobian $j(t \mid \xi)$ itself solves a first order ODE along the characteristics:

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} j(t \mid \xi)=\frac{\partial \mathbf{U}(\mathbf{r}, t)}{\partial \mathbf{r}} j(t \mid \xi), \quad j(0)=1 . \tag{2.34}
\end{equation*}
$$

So the transported density in the Lagrangian variables becomes

$$
\rho(t \mid \xi)=\frac{\rho_{0}(\xi)}{j(t \mid \xi)}
$$

Combining it with Eqn (2.33), we get the Eulerian density $\rho$ in the form of integral

$$
\begin{equation*}
\rho(\mathbf{r}, t)=\int \mathrm{d} \xi \rho_{0}(\xi) \delta(\mathbf{r}(t \mid \xi)-\mathbf{r}) . \tag{2.35}
\end{equation*}
$$

The latter provides an explicit connection between the Lagrangian and Eulerian characteristics. The delta-function on the RHS of Eqn (2.35) becomes an indicator of the
position of a Lagrangian particle (see next section). So performing ensemble averaging of Eqn (2.35) we get the well known relation of the mean Eulerian density to the one-point Lagrangian $\operatorname{PDF} P(t \mid \xi)=\langle\delta(\mathbf{r}(t \mid \xi)-\mathbf{r})\rangle$ (see for instance, Ref. [39]):

$$
\langle\rho(\mathbf{r}, t)\rangle=\int \mathrm{d} \xi \rho_{0}(\xi) P(t, \mathbf{r} \mid \xi) .
$$

In the case of the divergent-free velocity $(\operatorname{div} \mathbf{U}(\mathbf{r}, t)=0)$, the 'particle Jacobian' and the Eulerian density are conserved along the characteristics:

$$
j(t \mid \xi)=1, \quad \rho(t \mid \xi)=\rho_{0}(\xi) .
$$

Hence the solution

$$
\rho(\mathbf{r}, t)=\rho_{0}(\xi(\mathbf{r}, t))
$$

maintains its initial value along the path.
Let us dwell now on the stochastic features of the transport problem (2.29). For the divergent-free velocities the tracer's iso-contours $\rho(\mathbf{r}, t)=$ const evolve along the particle trajectories, as described in Section 2.1, and illustrated in Fig. 1a. Here the total area enclosed by the contour is conserved, but as evidenced from the plot, the contour grows increasingly rugged, with sharpening gradients and evolving small scale structures. At the other extreme (potential velocities) the enclosed area goes to zero, as the tracer density concentrates in small clusters. We refer to Ref. [25] for further results and numeric simulations, and remark that the ensemble averaging would typically obliterate all these dynamic features.

The above discussion clearly indicates that dynamic evolution (2.29) gives an adequate physical picture only within a limited time range. A more complete description of the system should involve also the tracer gradient, that obeys a system of PDEs

$$
\begin{align*}
& \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \mathbf{r}} \mathbf{U}(\mathbf{r}, t)\right) p_{i}(\mathbf{r}, t)=-p_{k}(\mathbf{r}, t) \frac{\partial U_{k}(\mathbf{r}, t)}{\partial r_{i}} \\
& -\rho(\mathbf{r}, t) \frac{\partial^{2} U_{k}(\mathbf{r}, t)}{\partial r_{i} \partial r_{k}}, \quad \mathbf{p}(\mathbf{r}, 0)=\mathbf{p}_{0}(\mathbf{r})=\nabla \rho_{0}(\mathbf{r}) . \tag{2.36}
\end{align*}
$$

Furthermore at some stage, one needs to bring into play the molecular diffusivity (with coefficient $x$ ), that obeys the second order linear PDE

$$
\begin{equation*}
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \mathbf{r}} \mathbf{U}(\mathbf{r}, t)\right) \rho(\mathbf{r}, t)=\varkappa \nabla \rho(\mathbf{r}, t), \quad \rho(\mathbf{r}, 0)=\rho_{0}(\mathbf{r}) \tag{2.37}
\end{equation*}
$$

and would flatten sharp gradients.

### 2.4 Waves in random media

We shall discuss wave propagation in random 2D and 3D media within the so-called scalar parabolic approximation [5, 40, 41]. It holds for large scale inhomogeneities and relatively short waves, hence small scattering angles:

$$
\begin{gather*}
\frac{\partial}{\partial x} u(x, \mathbf{R})=\frac{\mathrm{i}}{2 k} \Delta_{\mathbf{R}} u(x, \mathbf{R})+\frac{\mathrm{i} k}{2} \varepsilon(x, \mathbf{R}) u(x, \mathbf{R}), \\
u(0, \mathbf{R})=u_{0}(\mathbf{R}) . \tag{2.38}
\end{gather*}
$$

Here $x$ denotes the preferred direction of wave propagation, $\mathbf{R}$ - transverse variable (s), and $\varepsilon(x, \mathbf{R})$ - the deviation of the
dielectric permeability from its uniform value 1. This equation is clearly an approximation.

The complete 3D-Hemlholtz equation with the preferred direction $z$

$$
\left\{\frac{\partial^{2}}{\partial z^{2}}+\Delta_{\mathbf{p}}+k^{2}[1+\varepsilon(z)]\right\} G\left(z, \boldsymbol{p} ; z_{0}\right)=\delta\left(z-z_{0}\right) \delta(\mathbf{p})
$$

is not parabolic. However in special cases, like stratified medium $\varepsilon=\varepsilon(z)$, the Green's function of the 3D Helmholtz could be represented through the 2D parabolic propagator

$$
\begin{gathered}
\frac{\partial}{\partial \tau} \psi\left(\tau, z ; z_{0}\right)=\frac{\mathrm{i}}{2 k}\left[\frac{\partial^{2}}{\partial z^{2}}+k^{2} \varepsilon(z)\right] \psi\left(\tau, z ; z_{0}\right), \\
\psi\left(0, z ; z_{0}\right)=\delta\left(z-z_{0}\right)
\end{gathered}
$$

in the auxiliary variable $\tau$. Namely, $G$ is a superposition of parabolic solutions $\psi[13,42]$

$$
G\left(z, \boldsymbol{p} ; z_{0}\right)=-\frac{1}{4 \pi} \int_{0}^{\infty} \frac{\mathrm{d} \tau}{\tau} \exp \left[\mathrm{i} k\left(\frac{\rho^{2}+\tau^{2}}{2 \tau}\right)\right] \psi\left(\tau, z ; z_{0}\right)
$$

Next we introduce the complex phase for wave-field $u$ of (2.38),
$u(x, \mathbf{R})=A(x, \mathbf{R}) \exp \{\mathrm{i} S(x, \mathbf{R})\}=\exp \{\chi(x, \mathbf{R})+\mathrm{i} S(x, \mathbf{R})\}$, where $\chi(x, \mathbf{R})=\ln A(x, \mathbf{R})$ gives the so-called amplitude level of wave-field $u$, while $S(x, \mathbf{R})$ is the standard real phase, and shows that the field-intensity $I(x, \mathbf{R})=u(x, \mathbf{R}) u^{*}(x, \mathbf{R})$ obeys the transport equation

$$
\begin{equation*}
\frac{\partial}{\partial x} I(x, \mathbf{R})+\frac{1}{k} \nabla_{\mathbf{R}}\left\{\nabla_{\mathbf{R}} S(x, \mathbf{R}) I(x, \mathbf{R})\right\}=0, I(0, \mathbf{R})=I_{0}(\mathbf{R}) . \tag{2.39}
\end{equation*}
$$

Hence follows the conservation of the wave-field power in the transverse planes $x=$ const:

$$
E_{0}=\int \mathrm{d} \mathbf{R} I(x, \mathbf{R})=\int \mathrm{d} \mathbf{R} I_{0}(\mathbf{R})
$$

Equation (2.39) closely resembles Eqn (2.29), and could be viewed as the 'tracer transport' by the potential velocity $(\mathbf{U}=\nabla S)$. However, tracer $I$ could be considered passive only in the geometrical optics approximation, when the phase evolution is uncoupled from that of the amplitude/ intensity. Then phase $S$, its gradient $\mathbf{p}(x, \mathbf{R})=$ $(1 / k) \nabla_{\mathbf{R}} S(x, \mathbf{R})$, and the second-derivative (curvature) matrix

$$
u_{i j}(x, \mathbf{R})=\frac{1}{k} \frac{\partial^{2}}{\partial R_{i} \partial R_{j}} S(x, \mathbf{R})
$$

of the wave-front $S(x, \mathbf{R})=$ const, all evolve according to a closed system of differential equations:

$$
\begin{align*}
& \frac{\partial}{\partial x} S(x, \mathbf{R})+\frac{k}{2} \mathbf{p}^{2}(x, \mathbf{R})=\frac{k}{2} \varepsilon(x, \mathbf{R}) \\
& \begin{aligned}
&\left(\frac{\partial}{\partial x}+\mathbf{p}(x, \mathbf{R}) \nabla_{\mathbf{R}}\right) \mathbf{p}(x, \mathbf{R})=\frac{1}{2} \nabla_{\mathbf{R}} \varepsilon(x, \mathbf{R}) \\
&\left(\frac{\partial}{\partial x}+\mathbf{p}(x, \mathbf{R}) \nabla_{\mathbf{R}}\right) u_{i j}(x, \mathbf{R})+u_{i k}(x, \mathbf{R}) u_{k j}(x, \mathbf{R}) \\
&=\frac{1}{2} \frac{\partial^{2}}{\partial R_{i} \partial R_{j}} \varepsilon(x, \mathbf{R})
\end{aligned}
\end{align*}
$$

In general, one has to account for the diffraction, and that would make $I$ an 'active tracer'.

As we mentioned earlier the realizations of the intensity field should cluster into the caustic structures. Indeed, the cover page of book [41] shows a cross sectional photograph of a laser beam propagating through a turbulent atmosphere. The fragment, shown in Fig. 7, clearly demonstrates the appearance of such caustic structures (see Refs [43-45] for further experimental results and numeric simulations). Figure 8 shows a swimming pool with clear caustic structures at the bottom. The latter arises due to the refraction and reflection of light by the perturbed water surface, so the light is scattered by the so-called phase screen.


Figure 7. Cross-section of a laser beam in a turbulent medium.


Figure 8. Caustic structure in a swimming pool.

Following Refs [5, 13], we shall consider the example of a Gaussian beam

$$
\begin{equation*}
u_{0}(R)=u_{0} \exp \left[-\frac{R^{2}}{2 a^{2}}\right] \tag{2.41}
\end{equation*}
$$

propagating through a random parabolic wave-guide

$$
\begin{equation*}
\varepsilon(x, R)=-\left[\alpha^{2}-z(x)\right] R^{2} \tag{2.42}
\end{equation*}
$$

Here $\rho$ designates the radial variable in the transverse plane, $a$ - beam's width, $\alpha$ - a deterministic (mean) component of the refractive index curvature at the center of the wave-guide, and $z(x)$ - its random variation along the axis of the wave-guide. If the beam's wave-number $k$ agrees
with the wave-guide parameters,

$$
\begin{equation*}
k \alpha a^{2}=1, \tag{2.43}
\end{equation*}
$$

and dielectric fluctuations $z$ are absent, then function (2.41) is the transverse eigenmode, and equation (2.38) has an exact solution

$$
u_{0}(x, R)=u_{0} \exp \left[-\frac{R^{2}}{2 a^{2}}-\mathrm{i} \alpha x\right]
$$

as the beam remains strictly parallel with constant amplitude along $x$-rays.

In the presence of dielectric fluctuations of the wave-guide curvature we look for solutions of the form

$$
u(x, R)=u_{0} \exp \left[-\frac{R^{2}}{2 a^{2}} A(x)+B(x)\right]
$$

with complex variable coefficients $A(x)$ and $B(x)$, which give the relative deviations of the complex phase from its constant (mean) values $A_{0}=1, B_{0}=i \alpha x$. The coefficients $A, B$ obey a nonlinear differential system

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} x} A(x) & =-\frac{\mathrm{i}}{k a^{2}}\left[A^{2}(x)-\alpha^{2} k^{2} a^{4}\right]-\mathrm{i} k a^{2} z(x), \quad A(0)=1 \\
\frac{\mathrm{~d}}{\mathrm{~d} x} B(x) & =-\frac{\mathrm{i}}{k a^{2}} A(x), \quad B(0)=0 \tag{2.44}
\end{align*}
$$

Assuming an initial field-intensity $I=\left|u_{0}\right|^{2}=1$ one could derive from Eqn (2.44) a closed form solution

$$
\begin{equation*}
I(x, R)=I(x, 0) \exp \left[-\frac{R^{2}}{a^{2}} I(x, 0)\right] \tag{2.45}
\end{equation*}
$$

in terms of the level-function on the wave-guide axis (see Refs $[5,13])$ :

$$
\begin{equation*}
I(x, 0)=\frac{1}{2}\left[A(x)+A^{*}(x)\right] . \tag{2.46}
\end{equation*}
$$

Later on we shall see that the randomly inhomogeneous wave-guide could effectively localize Gaussian beams at finite distances in the direction of propagation.

Let us now take a close view of the geometrical optics approximation for a general parabolic equation (2.38). Here the transverse gradient obeys a quasilinear partial differential equation (2.40), solved by the method of characteristics (see for instance Ref. [46]). The characteristic curves (rays) $\mathrm{R}(t)$ obey the differential system

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} x} \mathbf{R}(x)=\mathbf{p}(x), \quad \frac{\mathrm{d}}{\mathrm{~d} x} \mathbf{p}(x)=\nabla_{\mathbf{R}} \varepsilon(x, \mathbf{R}), \tag{2.47}
\end{equation*}
$$

while the wave-intensity $I$ and the second derivative matrix $u_{i j}(x)=\partial^{2} S / \partial R_{i} \partial R_{j}$ of the phase-function evolve along the characteristics according to

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} x} I(x)=-I(x) u_{i i}(x), \\
& \frac{\mathrm{d}}{\mathrm{~d} x} u_{i j}(x)+u_{i k}(x) u_{k j}(x)=\frac{1}{2} \frac{\partial^{2}}{\partial R_{i} \partial R_{j}} \varepsilon(x, \mathbf{R}) . \tag{2.48}
\end{align*}
$$

The Hamiltonian system (2.47) could be formally viewed as frictionless motion of a particle (2.4) in a potential force $\nabla \mathrm{R} \varepsilon(x, \mathrm{R})$.

Equations (2.47) and (2.48) are further simplified in the 2 D case ( $R=y$ ):

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} x} y(x) & =p(x), \quad \frac{\mathrm{d}}{\mathrm{~d} x} p(x)=\frac{\partial}{\partial y} \varepsilon(x, y) \\
\frac{\mathrm{d}}{\mathrm{~d} x} I(x) & =-I(x) u(x), \quad \frac{\mathrm{d}}{\mathrm{~d} x} u(x)+u^{2}(x)=\frac{1}{2} \frac{\partial^{2}}{\partial y^{2}} \varepsilon(x, y) . \tag{2.49}
\end{align*}
$$

The last equation in $u(x)$ is similar to equation (2.3) with singular ('blow up' type) solutions, the only difference being the more complicated random (forcing) term. Yet solutions of stochastic problem (2.49) would still blow up at finite distances, where the curvature $u(x)$ goes to $-\infty$, while the intensity $I$ grows to $+\infty$. This singularity manifests itself by (random) focusing of the wave field and formation of caustics [6-9].

### 2.5 Equations of geophysical fluid dynamics

We shall consider hydrodynamic flows on the rotating Earth in the so-called quasigeostrophic approximation [34]. In the simplest case of a single layer fluid its state is described by the 2D potential vorticity field in variables $\mathbf{r}=(x, y)$, that obeys equation

$$
\begin{gather*}
\frac{\partial}{\partial t} \Delta \psi(\mathbf{r}, t)+\beta_{0} \frac{\partial}{\partial x} \psi(\mathbf{r}, t)=J(\Delta \psi(\mathbf{r}, t)+h(\mathbf{r}) ; \psi(\mathbf{r}, t)), \\
\psi(\mathbf{r}, 0)=\psi_{0}(\mathbf{r}), \tag{2.50}
\end{gather*}
$$

where $J(\psi, \varphi)$ is the Jacobian of two functions, $\beta_{0}$ the latitudinal derivative of the local Coriolis parameter $f_{0}$, and $h(r)=f_{0} \tilde{h}(\mathbf{r}) / H_{0}$ takes into accounts the deviation of the bottom topography $\tilde{h}(\mathbf{r})$ from the uniform (mean) depth $H_{0}$. The velocity field is computed from $\psi$ via

$$
\mathbf{v}(\mathbf{r}, t)=\left(-\frac{\partial \psi(\mathbf{r}, t)}{\partial y}, \frac{\partial \psi(\mathbf{r}, t)}{\partial x}\right)
$$

Let us remark that neglecting the Coriolis and topographic factors would reduce Eqn (2.50) to the standard Eulerian 2D hydrodynamics [47].

The scalar equation (2.50) describes so-called barotropic fluid motion, where density stratification plays no role. The simplest way to incorporate the baroclinic effects (due to density stratification) involves a two-layer model, made of two coupled PDEs [34]:

$$
\begin{align*}
& \frac{\partial}{\partial t}\left[\Delta \psi_{1}-\alpha_{1} F\left(\psi_{1}-\psi_{2}\right)\right]+\beta_{0} \frac{\partial}{\partial x} \psi_{1} \\
& =J\left(\Delta \psi_{1}-\alpha_{1} F\left(\psi_{1}-\psi_{2}\right) ; \psi_{1}\right), \\
& \frac{\partial}{\partial t}\left[\Delta \psi_{2}-\alpha_{2} F\left(\psi_{2}-\psi_{1}\right)\right]+\beta_{0} \frac{\partial}{\partial x} \psi_{2} \\
& =J\left(\Delta \psi_{2}-\alpha_{2} F\left(\psi_{2}-\psi_{1}\right)+f_{0} \alpha_{2} h ; \psi_{2}\right) . \tag{2.51}
\end{align*}
$$

Additional parameters appear in Eqn (2.51): $\alpha_{1}=1 / H_{1}$ and $\alpha_{2}=1 / H_{2}$ are reciprocals of the layers' depths, $F=f_{0}^{2} \rho / g(\Delta \rho)$ depends on the local Coriolis parameter, and $\Delta \rho / \rho=\left(\rho_{2}-\rho_{1}\right) / \rho_{0}>0$ is the relative density variation.

A special case of Eqns (2.50) and (2.51) arises when one drops the Earth rotation (2D Eulerian fluids), but takes into account the topography and density stratification. Such linearized equations would describe the effect of topography on propagation of Rossby waves. Let us remark that in all the above geophysical models the topography enters the dynamic equations in the form of derivatives.

### 2.6 Solution dependence on initial parameters and coefficients of equations

We have considered several examples of dynamical systems, both ordinary and partial differential ones. In many applications, like the statistical analysis of chapter 4, one needs to know how solutions of such dynamical systems depend on the initial/boundary parameters, as well as their (functional) dependence on coefficients of the equations. There are two common features for all such dependencies, important for their statistical analysis. We shall illustrate them for the simplest problem given by an ordinary differential system of type (2.1), that describes the particle dynamics in a (random) velocity field $\mathbf{U}(\mathbf{r}, \tau)$. It could be recast in the (equivalent) integral form

$$
\begin{equation*}
\mathbf{r}(t)=\mathbf{r}_{0}+\int_{t_{0}}^{t} \mathrm{~d} \tau \mathbf{U}(\mathbf{r}(\tau), \tau) \tag{2.52}
\end{equation*}
$$

The solution of Eqn (2.52) has a (functional) dependence on the vector-field $\mathbf{U}\left(\mathbf{r}^{\prime}, \tau\right)$, and the initial parameters $\mathbf{r}_{0}, t_{0}$.
2.6.1 Dynamic causality principle. Let us take a variational derivative of solution (2.52) over the vector field $\mathbf{U}(\mathbf{r}, t)$. Assuming an initial condition $\mathbf{r}_{0}$, we get a linear equation for the variational derivative,

$$
\begin{align*}
\frac{\delta r_{i}(t)}{\delta U_{j}\left(\mathbf{y}, t^{\prime}\right)} & =\delta_{i j} \delta\left(\mathbf{y}-\mathbf{r}\left(t^{\prime}\right)\right) \theta\left(t^{\prime}-t_{0}\right) \theta\left(t-t^{\prime}\right) \\
& +\int_{t_{0}}^{t} \mathrm{~d} \tau \frac{\partial U_{i}(\mathbf{r}(\tau), \tau)}{\partial r_{k}} \frac{\delta r_{k}(t)}{\delta U_{j}\left(\mathbf{y}, t^{\prime}\right)} \tag{2.53}
\end{align*}
$$

where $\delta\left(\mathbf{y}-\mathbf{y}_{0}\right)$ designates the Dirac delta, while $\theta(z)$ stands for the Havyside step-function. From Eqn (2.53) it follows that

$$
\begin{equation*}
\frac{\delta \mathbf{r}_{i}(t)}{\delta U_{j}\left(\mathbf{y}, t^{\prime}\right)}=0, \quad \text { if } t^{\prime}>t \quad \text { or } \quad t^{\prime}<t_{0} \tag{2.54}
\end{equation*}
$$

so the solution $\mathbf{r}(t)$ of Eqn (2.52) considered as a functional of field $\mathbf{U}\left(\mathbf{y}, t^{\prime}\right)$ depends only the values of $\mathbf{U}\left(\mathbf{y}, t^{\prime}\right)$ in the range $t_{0}<t^{\prime}<t$. So function $\mathbf{r}(t)$ remains independent of the variations of $\mathbf{U}\left(\mathbf{y}, t^{\prime}\right)$ outside the interval $\left(t_{0}, t^{\prime}\right)$. Condition (2.54) is called the dynamic causality principle.

Taking into account dynamic causality we can recast Eqn (2.53) in the form

$$
\begin{aligned}
\frac{\delta r_{i}(t)}{\delta U_{j}\left(\mathbf{y}, t^{\prime}\right)} & =\delta_{i j} \delta\left(\mathbf{y}-\mathbf{r}\left(t^{\prime}\right)\right) \theta\left(t^{\prime}-t_{0}\right) \theta\left(t-t^{\prime}\right) \\
& +\int_{t^{\prime}}^{t} \mathrm{~d} \tau \frac{\partial U_{i}(\mathbf{r}(\tau), \tau)}{\partial r_{k}} \frac{\delta r_{k}(t)}{\delta U_{j}\left(\mathbf{y}, t^{\prime}\right)}
\end{aligned}
$$

which yields in the limit $t \rightarrow t^{\prime}+0$

$$
\begin{equation*}
\left.\frac{\delta r_{i}(t)}{\delta U_{j}\left(\mathbf{y}, t^{\prime}\right)}\right|_{t=t^{\prime}+0}=\delta_{i j} \delta\left(\mathbf{y}-\mathbf{r}\left(t^{\prime}\right)\right) . \tag{2.55}
\end{equation*}
$$

Integral equation (2.53) for the variational derivative has an equivalent differential form

$$
\begin{align*}
\frac{\partial}{\partial t} \frac{\delta r_{i}(t)}{\delta U_{j}\left(\mathbf{y}, t^{\prime}\right)} & =\frac{\partial U_{i}(\mathbf{r}, t)}{\partial r_{k}} \frac{\delta r_{k}(t)}{\delta U_{j}\left(\mathbf{y}, t^{\prime}\right)}, \\
\left.\frac{\delta r_{i}(t)}{\delta U_{j}\left(\mathbf{y}, t^{\prime}\right)}\right|_{t=t^{\prime}} & =\delta_{i j} \delta\left(\mathbf{y}-\mathbf{r}\left(t^{\prime}\right)\right) . \tag{2.56}
\end{align*}
$$

Dynamic causality is a common feature of all initial value problems, but not boundary value ones. So for problem (2.5), (2.6) for a plane wave in the randomly stratified medium, the wave-field $u(x)$ at point $x$ as well as the the reflectiontransmission coefficients have a functional dependence on the refractive index $\varepsilon(x)$ over the entire range $\left(L_{0}, L\right)$. However, the embedding method allows this problem to be recast as an initial value one in the auxiliary variable $L$, and apply the dynamic causality for the 'embedded problem'.
2.6.2 Dependence on initial conditions. Next we shall consider the dependence of the solution $\mathbf{r}(t)$ of $\operatorname{Eqn}$ (2.52) on the initial parameters $\mathbf{r}_{0}, t_{0}$ and designate this dependence by the vertical bar:

$$
\begin{equation*}
\mathbf{r}(t)=\mathbf{r}\left(t \mid \mathbf{r}_{0}, t_{0}\right), \quad \mathbf{r}_{0}=\mathbf{r}\left(t_{0} \mid \mathbf{r}_{0}, t_{0}\right) \tag{2.57}
\end{equation*}
$$

Taking derivatives of Eqn (2.52) in $r_{0 k}$ and $t_{0}$, we get the linear integral equations for the Jacobian matrix $\partial r_{i}\left(t \mid \mathbf{r}_{0}, t_{0}\right) / \partial r_{0 k}$ and vector $\partial r_{i}\left(t \mid \mathbf{r}_{0}, t_{0}\right) / \partial t_{0}$ :
$\frac{\partial r_{i}\left(t \mid \mathbf{r}_{0}, t_{0}\right)}{\partial r_{0 k}}=\delta_{i k}+\int_{t_{0}}^{t} \mathrm{~d} \tau \frac{\partial U_{i}(\mathbf{r}(\tau), \tau)}{\partial r_{j}} \frac{\partial r_{j}\left(\tau \mid \mathbf{r}_{0}, t_{0}\right)}{\partial r_{0 k}}$,
$\frac{\partial r_{i}\left(t \mid \mathbf{r}_{0}, t_{0}\right)}{\partial t_{0}}=-U_{i}\left(\mathbf{r}_{0}\left(t_{0}\right), t_{0}\right)+\int_{t_{0}}^{t} \mathrm{~d} \tau \frac{\partial U_{i}(\mathbf{r}(\tau), \tau)}{\partial r_{j}} \frac{\partial r_{j}\left(\tau \mid \mathbf{r}_{0}, t_{0}\right)}{\partial t_{0}}$.

Multiplying the first by $U_{k}\left(\mathbf{r}_{0}\left(t_{0}\right), t_{0}\right)$, summing over $k$ and adding the second one, we obtain a linear integral relation for the vector-function

$$
F_{i}\left(t \mid \mathbf{r}_{0}, t_{0}\right)=\left(\frac{\partial}{\partial t_{0}}+\mathbf{U}\left(\mathbf{r}_{0}, t_{0}\right) \frac{\partial}{\partial \mathbf{r}_{0}}\right) r_{i}\left(t \mid \mathbf{r}_{0}, t_{0}\right)
$$

namely,

$$
F_{i}\left(t \mid \mathbf{r}_{0}, t_{0}\right)=\int_{t_{0}}^{t} \mathrm{~d} \tau \frac{\partial U_{i}(\mathbf{r}(\tau), \tau)}{\partial r_{j}} F_{j}\left(\tau \mid \mathbf{r}_{0}, t_{0}\right)
$$

whose only solution is $\mathbf{F}\left(t \mid \mathbf{r}_{0}, t_{0}\right) \equiv 0$. Hence follows the first order linear PDE for $r_{i}$ in the 'initial variables',

$$
\begin{equation*}
\left(\frac{\partial}{\partial t_{0}}+\mathbf{U}\left(\mathbf{r}_{0}, t_{0}\right) \frac{\partial}{\partial \mathbf{r}_{0}}\right) r_{i}\left(t \mid \mathbf{r}_{0}, t_{0}\right)=0 \tag{2.59}
\end{equation*}
$$

with the terminal condition at $t_{0}=t$

$$
\begin{equation*}
\mathbf{r}\left(t \mid \mathbf{r}_{0}, t\right)=\mathbf{r}_{0} \tag{2.60}
\end{equation*}
$$

Here the variable $t$ enters the problem (2.59), (2.60) as the parameter.

Equation (2.59) is solved backward in time relative to Eqn (2.1), and is called the backward problem.

Notice that the 'terminal value' problem (2.59), (2.60) also satisfies the dynamic causality in $t_{0}$ and

$$
\frac{\delta}{\delta U_{j}\left(\mathbf{y}, t^{\prime}\right)} r_{i}\left(t \mid \mathbf{r}_{0}, t_{0}\right)=0, \text { if } t^{\prime}>t \text { or } t^{\prime}<t_{0}
$$

and in this case, as follows from Eqn (2.59), the following holds:

$$
\begin{equation*}
\left.\frac{\delta}{\delta U_{j}\left(\mathbf{y}, t^{\prime}\right)} \mathbf{r}\left(t \mid \mathbf{r}_{0}, t_{0}\right)\right|_{t^{\prime}=t_{0}+0}=\delta\left(\mathbf{r}_{0}-\mathbf{y}\right) \frac{\partial}{\partial r_{0 j}} \mathbf{r}\left(t \mid \mathbf{r}_{0}, t_{0}\right) \tag{2.61}
\end{equation*}
$$

## 3. Indicator function and the Liouville equation

The modern theory of random processes allows one to get a closed form description of stochastic systems, if those are described by either linear differential equations, or certain integral equations [5], and obey the dynamic causality principle. In general, nonlinear dynamical system could be transformed into equivalent linear PDEs by means of indicator functions [5]. Of course, such a transition would typically increase the number of variables. We shall outline this method for the models of the previous section.

### 3.1 Ordinary differential equations

Consider the stochastic system (2.1), and introduce the scalar (distributional) function

$$
\begin{equation*}
\Phi(t ; \mathbf{r})=\delta(\mathbf{r}(t)-\mathbf{r}), \tag{3.1}
\end{equation*}
$$

supported at the cross-section of random process $\mathbf{r}(t)$ by the plane $\mathbf{r}=$ const, and called the indicator function.

Differentiating Eqn (3.1) in time $t$ and applying dynamic system (2.1), we get a linear $\operatorname{PDE}[3,5]$

$$
\begin{equation*}
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \mathbf{r}} \mathbf{U}(\mathbf{r}, t)\right) \Phi(t ; \mathbf{r})=0, \quad \Phi\left(t_{0} ; \mathbf{r}\right)=\delta\left(\mathbf{r}_{0}-\mathbf{r}\right) \tag{3.2}
\end{equation*}
$$

which is equivalent to the original system and is called the Liouville equation.

Transition from an ODE (2.1) to a PDE (3.2) enlarges the phase-space, but the number of variables remains finite. Let us remark that Eqn (3.2) coincides with the transport equation (2.29), the only difference being their initial conditions.

Solutions of both Eqns (2.1) and (3.2) depend on the initial data. Thus the position-function $\mathbf{r}(t)=\mathbf{r}\left(t \mid \mathbf{r}_{0}, t_{0}\right)$ solves a linear PDE (2.59) in variables $\mathbf{r}_{0}$, $t_{0}$. Similar derivation works for the indicator function, indeed, $t_{0}$-derivative of Eqn (3.1) along with PDE (2.59) yields a linear equation

$$
\begin{align*}
\left(\frac{\partial}{\partial t_{0}}+\mathbf{U}\left(\mathbf{r}_{0}, t_{0}\right) \frac{\partial}{\partial \mathbf{r}_{0}}\right) & \Phi\left(t ; \mathbf{r} \mid t_{0}, \mathbf{r}_{0}\right)=0 \\
& \Phi\left(t ; \mathbf{r} \mid t, \mathbf{r}_{0}\right)=\delta\left(\mathbf{r}_{0}-\mathbf{r}\right), \tag{3.3}
\end{align*}
$$

called the backward Liouville equation.

### 3.2 First order partial differential equations

If the original dynamic evolution is given by PDEs, we could pass to an equivalent description using variational derivatives in the infinite-D space (of functions), called the Hopf equation [5, 39, 48]. However, in special cases such infiniteD linear transport could be further simplified.

For instance, if the original dynamic system is described by a first order PDE, either linear, like a 'passive tracer' (2.29), or quasilinear of type (2.40) - for the cross-sectional phase gradient in random media, then certain indicator functions (e.g. iso-contour indicators in the 'passive transport') live on the reduced (finite-D) phase-space [3, 5]. This link is established by the method of characteristics for first order PDEs.

Let us elaborate it for the 'passive transport' (2.30), (2.31) in random velocities. The state of the transported tracer could be described either by the Lagrangian field $\rho(t \mid \xi)$ ( $\xi$ Lagrangian label), or in the Eulerian form by $\rho(\mathrm{r}, t)$.

The Lagrangian indicator-function

$$
\begin{equation*}
\Phi_{\mathrm{Lag}}(t ; \mathbf{r}, \rho \mid \xi)=\delta(\mathbf{r}(t \mid \xi)-\mathbf{r}) \delta(\rho(t \mid \xi)-\rho) \tag{3.4}
\end{equation*}
$$

could be shown to obey a linear Liouville equation

$$
\begin{gather*}
\frac{\partial}{\partial t} \Phi_{\mathrm{Lag}}(t ; \mathbf{r}, \rho \mid \xi)=\left[-\frac{\partial}{\partial \mathbf{r}} \mathbf{U}(\mathbf{r}, t)+\frac{\partial \mathbf{U}(\mathbf{r}, t)}{\partial \mathbf{r}} \frac{\partial}{\partial \rho} \rho\right] \Phi_{\mathrm{Lag}}(t ; \mathbf{r}, \rho \mid \xi) \\
\Phi_{\mathrm{Lag}}(0 ; \mathbf{r}, \rho \mid \xi)=\delta(\xi-\mathbf{r}) \delta\left(\rho_{0}(\xi)-\rho\right) \tag{3.5}
\end{gather*}
$$

that incorporates explicit dependence on label $\xi$ - the initial position of a parcel.

To pass from the Lagrangian to Eulerian description one could employ the Jacobian matrix $j_{i k}(t \mid \xi)=\partial r_{i}(t \mid \xi) / \partial \xi_{k}$, whose determinant $j(t \mid \xi)$ measures the rate of volume compression/expansion along the Lagrangian path, and satisfies the linear transport equation (2.34). Augmenting the indicator-function $\Phi_{\mathrm{Lag}}(t ; \mathbf{R}, \rho \mid \xi)$ with the $j$-factor,

$$
\begin{equation*}
\Phi_{\mathrm{Lag}}(t ; \mathbf{r}, \rho, j \mid \xi)=\delta(\mathbf{r}(t \mid \xi)-\mathbf{r}) \delta(\rho(t \mid \xi)-\rho) \delta(j(t \mid \xi)-j), \tag{3.6}
\end{equation*}
$$

one could write a Liouville equation similar to Eqn (3.5):

$$
\begin{align*}
\frac{\partial}{\partial t} \Phi_{\mathrm{Lag}}(t ; \mathbf{r}, \rho, j \mid \xi)=\left[-\frac{\partial}{\partial \mathbf{r}} \mathbf{U}(\mathbf{r}, t)+\right. & \left.\frac{\partial \mathbf{U}(\mathbf{r}, t)}{\partial \mathbf{r}}\left(\frac{\partial}{\partial \rho} \rho-\frac{\partial}{\partial j} j\right)\right] \\
& \times \Phi_{\mathrm{Lag}}(t ; \mathbf{r}, \rho, j \mid \xi) \\
\Phi_{\mathrm{Lag}}(0 ; \mathbf{r}, \rho, j \mid \xi)= & \delta(\xi-\mathbf{r}) \delta\left(\rho_{0}(\xi)-\rho\right) \delta(j-1) . \tag{3.7}
\end{align*}
$$

The Eulerian indicator

$$
\begin{equation*}
\Phi(t, \mathbf{r} ; \rho)=\delta(\rho(t, \mathbf{r})-\rho), \tag{3.8}
\end{equation*}
$$

is supported on the iso-surfaces (or 2D iso-contours) of level $\rho(\mathbf{r}, t)=\rho=$ const. The evolution could be derived either from Eqn (2.29) as in Ref. [5], or from the Lagrangian Liouville equation (3.7). Indeed, taking into account the equality

$$
\begin{aligned}
\delta(\mathbf{r}(t \mid \xi)-\mathbf{r}) & =\frac{1}{\left|\partial r_{\alpha} / \partial \xi_{\beta}\right|} \delta(\xi-\xi(t, \mathbf{r})) \\
& =\frac{1}{j(t \mid \xi)} \delta(\xi-\xi(t, \mathbf{r}))
\end{aligned}
$$

we recast Eqn (3.6) as

$$
\begin{equation*}
\Phi_{\mathrm{Lag}}(t ; \mathbf{r}, \rho, j \mid \xi)=\frac{1}{j} \delta(\xi-\xi(t, \mathbf{r})) \delta(j(t \mid \xi)-j) \Phi(t, \mathbf{r} ; \rho) . \tag{3.9}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\Phi(t, \mathbf{r} ; \rho)=\int \mathrm{d} \xi \int_{0}^{\infty} j \mathrm{~d} j \Phi_{\mathrm{Lag}}(t ; \mathbf{r}, \rho, j \mid \xi) \tag{3.10}
\end{equation*}
$$

Multiplying Eqn (3.7) by $j$ and integrating over $j$ and $\xi$, we get the 'Eulerian' equation

$$
\begin{align*}
& \left(\frac{\partial}{\partial t}+\mathbf{U}(\mathbf{r}, t) \frac{\partial}{\partial \mathbf{r}}\right) \Phi(t, \mathbf{r} ; \rho)=\frac{\partial \mathbf{U}(\mathbf{r}, t)}{\partial \mathbf{r}} \frac{\partial}{\partial \rho} \rho \Phi(t, \mathbf{r} ; \rho), \\
& \Phi(0, \mathbf{r} ; \rho)=\delta\left(\rho_{0}(\mathbf{r})-\rho\right) \tag{3.11}
\end{align*}
$$

For divergent-free velocities $\mathrm{U}(\mathrm{r}, t)$ all three equations (3.3), (3.5), (3.11) coincide, their differences are due to compressible (potential) velocity component.

The indicator functions have many applications, for instance they yield one-point probability distributions for stochastic processes generated by dynamical systems via ensemble averaging of corresponding indicator functions:

$$
P(t ; \mathbf{r}, \rho, j \mid \xi)=\left\langle\Phi_{\mathrm{Lag}}(t ; \mathbf{r}, \rho, j \mid \xi)\right\rangle, \quad P(t, \mathbf{r} ; \rho)=\langle\Phi(t ; \mathbf{r} ; \rho)\rangle .
$$

That explains their importance in the 'statistical' dynamical theory.

Furthermore, indicator functions carry some important qualitative and quantitative geometric information about random fields. Let us elaborate the last point.

### 3.3 Statistical topography of random fields

The main subject of statistical topography, like the usual one (i.e. topographic maps of 'mountain terrains'), is the set of iso-contours in 2D (or 3D iso-surfaces) of constant density $\rho$,

$$
\begin{equation*}
\rho(\mathbf{r}, t)=\rho=\mathrm{const} . \tag{3.12}
\end{equation*}
$$

To analyze such contours (for the sake of the presentation we shall talk about the 2D case) let us introduce a (singular) indicator function of level $\rho$, viewed as a 'functional' of the media parameters. Such a function (3.8) yields several geometric characteristics of contours. Those include the total area, enclosed by $\rho(\mathbf{r}, t)=\rho$,

$$
\begin{equation*}
S(t, \rho)=\int \theta(\rho(\mathbf{r}, t)-\rho) \mathrm{d} \mathbf{r}=\int_{\rho}^{\infty} \mathrm{d} \tilde{\rho} \int \mathrm{~d} \mathbf{r} \Phi(t, \mathbf{r} ; \tilde{\rho}), \tag{3.13}
\end{equation*}
$$

the total mass inside the region,

$$
\begin{equation*}
M(t, \rho)=\int \rho(\mathbf{r}, t) \theta(\rho(\mathbf{r}, t)-\rho) \mathrm{d} \mathbf{r}=\int_{\rho}^{\infty} \tilde{\rho} \mathrm{d} \tilde{\rho} \int \mathrm{~d} \mathbf{r} \Phi(t, \mathbf{r} ; \tilde{\rho}) \tag{3.14}
\end{equation*}
$$

etc., all expressed through the indicator $\Phi(t, \mathrm{r} ; \rho)$.
Functionals (313), (3.14) obey the time-evolutions, derived from the Liouville's equation (3.11) for $\Phi(t, \mathrm{r} ; \rho)$, that take the form

$$
\begin{aligned}
\frac{\partial}{\partial t} S(t, \rho) & =\int \mathrm{d} \mathbf{r} \int_{\rho}^{\infty} \mathrm{d} \tilde{\rho} \frac{\partial}{\partial t} \Phi(t, \mathbf{r} ; \tilde{\rho}) \\
& =\int \mathrm{d} \mathbf{r} \int_{\rho}^{\infty} \mathrm{d} \tilde{\rho} \frac{\partial \mathbf{U}(\mathbf{r}, t)}{\partial \mathbf{r}}\left(\frac{\partial}{\partial \tilde{\rho}} \tilde{\rho}+1\right) \Phi(t, \mathbf{r} ; \tilde{\rho}) \\
\frac{\partial}{\partial t} M(t, \rho) & =\int \mathrm{d} \mathbf{r} \int_{\rho}^{\infty} \tilde{\rho} \mathrm{d} \tilde{\rho} \frac{\partial}{\partial t} \Phi(t, \mathbf{r} ; \tilde{\rho}) \\
& =\int \mathrm{d} \mathbf{r} \int_{\rho}^{\infty} \mathrm{d} \tilde{\rho} \frac{\partial \mathbf{U}(\mathbf{r}, t)}{\partial \mathbf{r}} \tilde{\rho}\left(\frac{\partial}{\partial \tilde{\rho}} \tilde{\rho}+1\right) \Phi(t, \mathbf{r} ; \tilde{\rho})
\end{aligned}
$$

In the special case of divergent-free velocities the total area bounded by contour $\rho$, and the total mass inside the region are conserved. Such flows also conserve the number $N$ of $\rho$ contours. Indeed, the contours could not arise or disappear, but only evolve in time starting from their initial distribution on the slice $\rho_{0}(\mathbf{r})=\rho=$ const.

Of course, compressible flows with nonzero potential component of $u$, would have both quantities evolving in time.

As we already mentioned, the ensemble average of indicator function (3.8) gives one-point PDFs of the tracer density

$$
\begin{equation*}
P(t, \mathbf{r} ; \rho)=\langle\Phi(t, \mathbf{r} ; \rho)\rangle=\langle\delta(\rho(\mathbf{r}, t)-\rho)\rangle . \tag{3.15}
\end{equation*}
$$

Hence one could also get statistical means of geometric invariants (3.13), (3.14) determined by $\Phi$ and the one-point PDFs.

Additional geometric information about density contours could be obtained from values of $\rho(\mathbf{r}, t)$ combined with its spatial gradient $\mathbf{p}(\mathbf{r}, t)=\nabla \rho(\mathbf{r}, t)$. For instance, integral

$$
\begin{equation*}
l(t, \rho)=\int \mathrm{d} \mathbf{r}|\mathbf{p}(\mathbf{r}, t)| \delta(\rho(\mathbf{r}, t)-\rho)=\oint \mathrm{d} l \tag{3.16}
\end{equation*}
$$

gives the total contour length at level $\rho[18-22]$.
The augmented indicator-function for Eqn (3.16) has extra variables $\mathbf{p}$,

$$
\begin{equation*}
\Phi(t, \mathbf{r} ; \rho, \mathbf{p})=\delta(\rho(\mathbf{r}, t)-\rho) \delta(\mathbf{p}(\mathbf{r}, t)-\mathbf{p}), \tag{3.17}
\end{equation*}
$$

and satisfies the extended Liouville equation, that follows from Eqns (2.29), (2.36)

$$
\left.\begin{array}{l}
\left(\frac{\partial}{\partial t}+\mathbf{U}(\mathbf{r}, t) \frac{\partial}{\partial \mathbf{r}}\right) \Phi(t, \mathbf{r} ; \rho, \mathbf{p}) \\
= \\
=\left[\frac{\partial U_{k}(\mathbf{r}, t)}{\partial r_{i}} \frac{\partial}{\partial p_{i}} p_{k}+\frac{\partial \mathbf{U}(\mathbf{r}, t)}{\partial \mathbf{r}}\left(\frac{\partial}{\partial \rho} \rho+\frac{\partial}{\partial \mathbf{p}} \mathbf{p}\right)\right. \\
+  \tag{3.18}\\
\left.+\frac{\partial^{2} U_{k}(\mathbf{r}, t)}{\partial r_{i} \partial r_{k}} \frac{\partial}{\partial p_{i}} \rho\right] \Phi(t, \mathbf{r} ; \rho, \mathbf{p}), \\
\Phi(0, \mathbf{r} ; \rho, \mathbf{p})
\end{array}\right) \delta\left(\rho_{0}(\mathbf{r})-\rho\right) \delta\left(\mathbf{p}_{0}(\mathbf{r})-\mathbf{p}\right) . ~ \$ ~ \$ 3.18,
$$

Equation (3.18) yields the evolution of the total contour length,

$$
\begin{align*}
\frac{\partial}{\partial t} l(t, \rho) & =\int \mathrm{d} \mathbf{r} \int \mathrm{~d} \mathbf{p} p \frac{\partial}{\partial t} \Phi(t, \mathbf{r} ; \rho, \mathbf{p}) \\
& =\int \mathrm{d} \mathbf{r} \int \mathrm{~d} \mathbf{p}\left[-\frac{\partial U_{k}(\mathbf{r}, t)}{\partial r_{i}} \frac{p_{i} p_{k}}{p}+\frac{\partial U_{k}(\mathbf{r}, t)}{\partial r_{k}} p \frac{\partial}{\partial \rho} \rho\right. \\
& \left.-\frac{\partial^{2} U_{k}(\mathbf{r}, t)}{\partial r_{i} \partial r_{k}} \frac{p_{i}}{p} \rho\right] \Phi(t, \mathbf{r} ; \rho, \mathbf{p}), \tag{3.19}
\end{align*}
$$

whose RHS is typically nonzero (positive) in all cases, including divergent-free velocities.

The mean values of Eqns (3.17)-(3.19) are related to the joint PDF of random fields $\rho(\mathbf{r}, t)$ and $\mathbf{p}(\mathbf{r}, t)$ via statistical (ensemble) averaging of the indicator function (3.17):

$$
\begin{equation*}
P(t, \mathbf{r} ; \rho, \mathbf{p})=\langle\delta(\rho(\mathbf{r}, t)-\rho) \delta(\mathbf{p}(\mathbf{r}, t)-\mathbf{p}\rangle . \tag{3.20}
\end{equation*}
$$

Higher derivatives of $\rho$ (e.g. second order) furnish additional geometric information, like the total number of closed contours at a given level $\rho(\mathbf{r}, t)=\rho=$ const. The latter could be approximately expressed (excluding non-closed ones) by the formula [19]

$$
\begin{align*}
N(t, \rho) & =N_{\text {in }}(t, \rho)-N_{\text {out }}(t, \rho) \\
& =\frac{1}{2 \pi} \int \mathrm{~d} \mathbf{r} \chi(t, \mathbf{r} ; \rho)|\mathbf{p}(\mathbf{r}, t)| \delta(\rho(\mathbf{r}, t)-\rho) . \tag{3.21}
\end{align*}
$$

Here $\chi(t, \mathbf{r} ; \rho)$ denotes the curvature along the contour, while $N_{\text {in }}(t, \rho), N_{\text {out }}(t, \rho)$ count contours with inward or outward pointing gradient $\mathbf{p}$.

In the case of the statistically homogeneous random field $\rho(\mathbf{r}, t)$, one-point PDFs $P(t, \mathbf{r} ; \rho)$ and $P(t, \mathbf{r} ; \rho, \mathbf{p})$ are independent of $\mathbf{r}$. So dropping the $\mathbf{r}$-integration but taking statistical
(ensemble) averages we could produce the corresponding specific quantities (per unit area/volume), whenever appropriate.

Next we shall proceed to the statistical analysis of the above problems.

## 4. Statistical analysis of dynamical systems

### 4.1 The forward and backward Fokker - Planck equation

We consider a general dynamical system

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \mathbf{x}(t)=\mathbf{v}(\mathbf{x}, t)+\mathbf{f}(\mathbf{x}, t), \quad \mathbf{x}(0)=\mathbf{x}_{0} \tag{4.1}
\end{equation*}
$$

for the vector-function $\mathbf{x}(t)=\left\{x_{1}(t), x_{2}(t), \ldots, x_{N}(t)\right\}$, where $\mathbf{v}(\mathbf{x}, t)$ and $\mathbf{f}(\mathbf{x}, t)$ denote the deterministic and random Gaussian vector-fields. The latter is assumed to have zero mean, $\langle\mathbf{f}(\mathbf{x}, t)\rangle=0$, and the correlation tensor

$$
B_{i j}\left(\mathbf{x}, t ; \mathbf{x}^{\prime}, t^{\prime}\right)=\left\langle f_{i}(\mathbf{x}, t) f_{j}\left(\mathbf{x}^{\prime}, t^{\prime}\right)\right\rangle .
$$

As before angular brackets $\langle\ldots\rangle$ indicate averaging over the ensemble of all realizations of field $\mathbf{f}(\mathbf{x}, t)$. The Liouville equation for the indicator-function $\Phi(t, \mathbf{x})=\delta(\mathbf{x}(t)-\mathbf{x})$ takes the form

$$
\begin{align*}
& \frac{\partial}{\partial t} \Phi(t, \mathbf{x})=-\frac{\partial}{\partial \mathbf{x}}[\mathbf{v}(\mathbf{x}, t)+\mathbf{f}(\mathbf{x}, t)] \Phi(t, \mathbf{x}), \\
& \Phi(0, \mathbf{x})=\delta\left(\mathbf{x}-\mathbf{x}_{0}\right) . \tag{4.2}
\end{align*}
$$

After averaging Eqn (4.2) over the $\mathbf{f}$-ensemble, we get a non-closed equation for the one-point $\operatorname{PDF} P(t, \mathbf{x})=$ $\langle\Phi(t, \mathbf{x})\rangle$,

$$
\begin{align*}
& \frac{\partial}{\partial t} P(t, \mathbf{x})=-\frac{\partial}{\partial \mathbf{x}} \mathbf{v}(\mathbf{x}, t) P(t, \mathbf{x})-\frac{\partial}{\partial \mathbf{x}}\langle\mathbf{f}(\mathbf{x}, t) \Phi(t, \mathbf{x})\rangle, \\
& P(0, \mathbf{x})=\delta\left(\mathbf{x}-\mathbf{x}_{0}\right), \tag{4.3}
\end{align*}
$$

where the cross-correlations of $\mathbf{f}(\mathbf{x}, t)$ and $\Phi(t, \mathbf{x})$ appear on the right-hand side, the latter being a functional of $\mathbf{f}(\mathbf{x}, t)$.

The correlation splitting procedures depend on the type of random field $\mathbf{f}(\mathbf{x}, t)$. For Gaussian fields and their functionals one could use the so-called Furutsu - Novikov formula [49, 50] (see also Refs [5, 39]),

$$
\begin{align*}
\left\langle f_{k}(\mathbf{x}, t) R[\mathbf{f}]\right\rangle & =\int \mathrm{d} \mathbf{x}^{\prime} \int \mathrm{d} t^{\prime} B_{k l}\left(\mathbf{x}, t ; \mathbf{x}^{\prime}, t^{\prime}\right) \\
& \times\left\langle\frac{\delta}{\delta f_{l}\left(\mathbf{x}^{\prime}, t^{\prime}\right)} R[\mathbf{f}]\right\rangle, \tag{4.4}
\end{align*}
$$

that holds for an arbitrary functional $R[\mathbf{f}]$ of random Gaussian field $\mathbf{f}(\mathbf{x}, t)$ and could be viewed as an 'integration by parts formula' in functional spaces [51]. Applying it to Eqn (4.3) and using the dynamic causality principle (2.54) we get

$$
\begin{aligned}
& \frac{\partial}{\partial t} P(t, \mathbf{x})=-\frac{\partial}{\partial \mathbf{x}} \mathbf{v}(\mathbf{x}, t) P(t, \mathbf{x}) \\
& \quad-\frac{\partial}{\partial x_{i}} \int \mathrm{~d} \mathbf{x}^{\prime} \int_{0}^{t} \mathrm{~d} t^{\prime} B_{i j}\left(\mathbf{x}, t ; \mathbf{x}^{\prime}, t^{\prime}\right)\left\langle\frac{\delta}{\delta f_{j}\left(\mathbf{x}^{\prime}, t^{\prime}\right)} \Phi(t, \mathbf{x})\right\rangle
\end{aligned}
$$

$$
\begin{equation*}
P(0, \mathbf{x})=\delta\left(\mathbf{x}-\mathbf{x}_{0}\right) \tag{4.5}
\end{equation*}
$$

Equation (4.5) is still non-closed in general.
It could be closed for delta-correlated fields $\mathbf{f}(\mathbf{x}, t)$. To this end we introduce the effective correlation tensor (see, for
instance Ref. [5])

$$
\begin{equation*}
B_{i j}^{\mathrm{eff}}\left(\mathbf{x}, t ; \mathbf{x}^{\prime}, t^{\prime}\right)=2 \delta\left(t-t^{\prime}\right) F_{i j}\left(\mathbf{x}, \mathbf{x}^{\prime} ; t\right) \tag{4.6}
\end{equation*}
$$

whose spatial component $F_{i j}$ is given by the time-integral

$$
F_{i j}\left(\mathbf{x}, \mathbf{x}^{\prime} ; t\right)=\frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d} t^{\prime} B_{i j}\left(\mathbf{x}, t ; \mathbf{x}^{\prime}, t^{\prime}\right)
$$

Inserting such a delta-correlated field $\mathbf{f}(\mathbf{x}, t)$ in formula (4.6) we get

$$
\begin{align*}
\frac{\partial}{\partial t} P(t, \mathbf{x})= & -\frac{\partial}{\partial \mathbf{x}} \mathbf{v}(\mathbf{x}, t) P(t, \mathbf{x})-\frac{\partial}{\partial x_{i}} \int \mathrm{~d} \mathbf{x}^{\prime} F_{i j}\left(\mathbf{x}, \mathbf{x}^{\prime} ; t^{\prime}\right) \\
& \times\left\langle\frac{\delta}{\delta f_{j}\left(\mathbf{x}^{\prime}, t\right)} \Phi(t, \mathbf{x})\right\rangle \\
P(0, \mathbf{x})= & \delta\left(\mathbf{x}-\mathbf{x}_{0}\right) \tag{4.7}
\end{align*}
$$

The variational derivative in Eqn (4.7) should be understood as the limit $\delta / \delta \mathbf{f}\left(\mathbf{x}^{\prime}, t-0\right)$.

Taking into account Liouville transport (4.2) we obtain the variational derivative

$$
\frac{\delta}{\delta f_{j}\left(\mathbf{x}^{\prime}, t\right)} \Phi(t, \mathbf{x})=-\frac{\partial}{\partial x_{j}}\left\{\delta\left(\mathbf{x}-\mathbf{x}^{\prime}\right) \Phi(t, \mathbf{x})\right\}
$$

expressed through $\Phi$ itself. Hence Eqn (4.5) turns into the standard Fokker - Planck equation

$$
\begin{align*}
& \frac{\partial}{\partial t} P(t, \mathbf{x})+\frac{\partial}{\partial x_{k}} {\left[v_{k}(\mathbf{x}, t)+A_{k}(\mathbf{x}, t)\right] P(t, \mathbf{x}) } \\
&=\frac{\partial^{2}}{\partial x_{k} \partial x_{l}}\left[F_{k l}(\mathbf{x}, \mathbf{x} ; t) P(t, \mathbf{x})\right] \\
& P(0, \mathbf{x})=\delta\left(\mathbf{x}-\mathbf{x}_{0}\right) \tag{4.8}
\end{align*}
$$

whose coefficients depend on the mean (transport) field $\mathbf{v}$, and the effective correlation tensor $F_{i j}$ :

$$
A_{k}(\mathbf{x}, t)=\left.\frac{\partial}{\partial x_{l}^{\prime}} F_{k l}\left(\mathbf{x}, \mathbf{x}^{\prime} ; t\right)\right|_{\mathbf{x}^{\prime}=\mathbf{x}}
$$

So the $\delta$-correlated approximation in Eqn (4.1) gives rise to a Markov process, a system without 'memory' [such memory would be encoded in the integral term of (4.5)]. Its transitional probabilities

$$
p\left(\mathbf{x}, t \mid \mathbf{x}_{0}, t_{0}\right)=\left\langle\delta(\mathbf{x}(t)-\mathbf{x}) \mid \mathbf{x}\left(t_{0}\right)=\mathbf{x}_{0}\right\rangle
$$

obey the FP-equation (4.8), subject to the initial condition

$$
\left.p\left(\mathbf{x}, t \mid \mathbf{x}_{0}, t_{0}\right)\right|_{t \rightarrow t_{0}}=\delta\left(\mathbf{x}-\mathbf{x}_{0}\right) .
$$

The partial differential equation (4.8) is often called the forward Fokker-Planck equation. Its solutions depend on the type of initial/boundary conditions, determined by the physical conditions.

The delta-correlated approximation applies when the time correlation radius $\tau_{0}$ of random field $\mathbf{f}(\mathbf{x}, t)$ is short compared to the typical time scales of the process, that is $\tau_{0} \ll \tau_{1}=L / v$, or $L / \sqrt{\left\langle f^{2}\right\rangle}$, where $L$ is the typical lengthscale. The latter could involve the flow-properties, e.g. a typical eddy size $L=v /|\nabla v|$, or the scale $L=\rho /|\nabla \rho|$ of tracer distribution. In either case, the time-evolution of the flow-transport dynamics would typically bring the length
scale down by creating 'small-scale’ structures. At this step one has to take into account the finite value $\tau_{0}$ of the correlation time.

One way to accommodate the finite correlation radius is given by the so-called diffusion approximation (see for instance, Refs [1, 23, 52]). The 'integral-type' approximations used here are more natural and physically relevant than the mathematical abstraction of 'delta-correlated' random fields. The basic assumption here is that random sources have a negligible effect on the dynamics of Eqn (4.1) on time scales of order $\tau_{0}$, so the system would follow the deterministic (mean-field) evolution.

The diffusion approximation starts with the exact relation (4.5), and computes the variational derivative $\delta \Phi(t, \mathrm{x}) \delta f_{i}\left(\mathrm{x}^{\prime}, t^{\prime}\right)$ by solving the mean field transport for $\Phi$ and $\delta \Phi / \delta f$,

$$
\begin{aligned}
& \frac{\partial}{\partial t} \Phi(t, \mathbf{x})=-\frac{\partial}{\partial \mathbf{x}} \mathbf{v}(\mathbf{x}, t) \Phi(t, \mathbf{x}) \\
& \frac{\partial}{\partial t} \frac{\delta \Phi(t, \mathbf{x})}{\delta f_{i}\left(\mathbf{x}^{\prime}, t^{\prime}\right)}=-\frac{\partial}{\partial \mathbf{x}}\left[\mathbf{v}(\mathbf{x}, t) \frac{\delta \Phi(t, \mathbf{x})}{\delta f_{i}\left(\mathbf{x}^{\prime}, t^{\prime}\right)}\right],
\end{aligned}
$$

subject to the initial conditions

$$
\begin{aligned}
& \left.\Phi(t, \mathbf{x})\right|_{t \rightarrow t^{\prime}}=\Phi\left(t^{\prime}, \mathbf{x}\right), \\
& \left.\frac{\delta \Phi(t, \mathbf{x})}{\delta f_{i}\left(\mathbf{x}^{\prime}, t^{\prime}\right)}\right|_{t \rightarrow t^{\prime}}=-\frac{\partial}{\partial x_{j}}\left\{\delta\left(\mathbf{x}-\mathbf{x}^{\prime}\right) \Phi\left(t^{\prime}, \mathbf{x}\right\} .\right.
\end{aligned}
$$

The latter condition couples the variation $\frac{\delta \Phi}{\delta f}$ to the value of functional $\Phi$ at the initial moment $t^{\prime}$.

Let us remark that the large scale limit, $t \gg \tau_{0}$, of the diffusion approximation to Eqn (4.1) would turn into another (approximately) Markovian process.

Averaging the backward Liouville equation (3.3) over the ensemble of realizations, we get the backward FP-equation that describes transitional probabilities as functions of the initial parameters $t_{0}, \mathbf{x}_{0}$ (see, for instance Ref. [28]):

$$
\begin{align*}
\frac{\partial}{\partial t_{0}} p\left(\mathbf{x}, t \mid \mathbf{x}_{0}, t_{0}\right) & +\left[v_{k}\left(\mathbf{x}_{0}, t_{0}\right)+A_{k}\left(\mathbf{x}_{0}, t_{0}\right)\right] \frac{\partial}{\partial x_{0 k}} p\left(\mathbf{x}, t \mid \mathbf{x}_{0}, t_{0}\right) \\
& =-F_{k l}\left(\mathbf{x}_{0}, \mathbf{x}_{0} ; t_{0}\right) \frac{\partial^{2}}{\partial x_{0 k} \partial x_{0 l}} p\left(\mathbf{x}, t \mid \mathbf{x}_{0}, t_{0}\right) . \tag{4.9}
\end{align*}
$$

The forward and backward FP-equations are equivalent. The forward problem is convenient for studying the temporal evolution of the basic statistics (means, moments) of system (4.1), while the backward one yields such properties as the occupation time of the process in a particular phase-space region, the arrival time at the boundary of the region, etc. Indeed, the probability of finding particle $\mathbf{x}(t)$ within region $V$ is given by the integral

$$
G\left(t ; \mathbf{x}_{0}, t_{0}\right)=\int \mathrm{d} \mathbf{x} p\left(\mathbf{x}, t \mid \mathbf{x}_{0}, t_{0}\right)
$$

which is described due to Eqn (4.9) by the closed form equation

$$
\begin{align*}
& \frac{\partial}{\partial t_{0}} G\left(t ; \mathbf{x}_{0}, t_{0}\right)+ {\left[v_{k}\left(\mathbf{x}_{0}, t_{0}\right)+A_{k}\left(\mathbf{x}_{0}, t_{0}\right)\right] \frac{\partial}{\partial x_{0 k}} G\left(t ; \mathbf{x}_{0}, t_{0}\right) } \\
&=-F_{k l}\left(\mathbf{x}_{0}, \mathbf{x}_{0} ; t_{0}\right) \frac{\partial^{2}}{\partial x_{0 k} \partial x_{0 l}} G\left(t ; \mathbf{x}_{0}, t_{0}\right), \\
& G\left(t ; \mathbf{x}_{0}, t_{0}\right)=\left\{\begin{array}{lll}
1, & \text { if } & \mathbf{x}_{0} \in V \\
0, & \text { if } & \mathbf{x}_{0} \notin V .
\end{array}\right. \tag{4.10}
\end{align*}
$$

One also needs additional boundary conditions, that depend on the properties of $V$, and the 'physics' at the boundary (absorption, reflection et al.).

As we mentioned in the introduction, solutions of many stochastic dynamical problems exhibit large fluctuations about special deterministic curves, that determine the 'largescale dynamics' of the system over the entire time-interval. We shall call such curves typical realizations, and define them through one-point PDFs of the process.

### 4.2 Typical realizations of random processes

Let $y(\tau)$ be a random process with a one-point PDF

$$
P(\tau ; y)=\langle\delta(y(\tau)-y)\rangle,
$$

and a dimensionless 'time' parameter $\tau$. The integral distribution function of the process is defined by averaging the Havyside step-function $\Theta(y)$, equal to 1 for $y>0$ and 0 for $y<0$

$$
F(\tau, y)=P(y(\tau)<y)=\langle\boldsymbol{\Theta}(y-y(\tau))\rangle .
$$

We call a typical realization of the process a deterministic median curve $y^{*}(\tau)$ computed from an algebraic equation

$$
F\left(\tau, y^{*}(\tau)\right)=\frac{1}{2}
$$

The motivation for this definition comes from the properties of the median. Namely, for any time-interval $\left(\tau_{1}, \tau_{2}\right)$ process $y(\tau)$ 'winds around' the median in such a way that it spends on average half of the time above it, $y(\tau)>y^{*}(\tau)$, and half below, $y(\tau)<y^{*}(\tau)$ (see Fig. 9 and Ref. [12, 13]):

$$
\left\langle T_{y(\tau)>y^{*}(\tau)}\right\rangle=\left\langle T_{y(\tau)<y^{*}(\tau)}\right\rangle=\frac{1}{2}\left(\tau_{2}-\tau_{1}\right) .
$$

Of course, such $y^{*}(\tau)$ would bear little resemblance to any particular realization of the process, and say nothing about the scope and size of fluctuations.

Evidently, the typical realization $y^{*}(\tau)$ of random process $y(\tau)$ is well defined over the entire time range $\tau \in(0, \infty)$.

For special random processes one could get additional information about the fluctuations of $y(\tau)$ around $y^{*}(\tau)$.

One such class of examples consists of log-normal (positive valued) processes, that arise in a large number of physical applications, and have some remarkable properties [12, 13]. They manifest themselves in such important coherent phenomena as localization and clustering.


Figure 9. Typical realization of a random process.

### 4.3 Log-normal processes

We define a log-normal random process as

$$
\begin{equation*}
y(\tau)=\exp \left\{-\tau+\int_{0}^{\tau} \mathrm{d} \xi z(\xi)\right\} \tag{4.11}
\end{equation*}
$$

where $z(\xi)$ stands for Gaussian white noise:

$$
\langle z(\xi)\rangle=0, \quad\left\langle z(\xi) z\left(\xi^{\prime}\right)\right\rangle=2 \delta\left(\xi-\xi^{\prime}\right) .
$$

It could be described by the stochastic equation

$$
\frac{\mathrm{d}}{\mathrm{~d} \tau} y(\tau)=\{-1+z(\tau)\} y(\tau), \quad y(0)=1
$$

and has the following properties.

1. The log-normal process is Markovian, and its one-point PDF solves the FP-equation

$$
\begin{align*}
& \frac{\partial}{\partial \tau} P(\tau ; y)=\frac{\partial}{\partial y} y P(\tau ; y)+\frac{\partial}{\partial y} y \frac{\partial}{\partial y} y P(\tau ; y), \\
& P(0 ; y)=\delta(y-1) . \tag{4.12}
\end{align*}
$$

Solution of Eqn (4.12),

$$
\begin{equation*}
P(\tau ; y)=\frac{1}{2 \sqrt{\pi \tau} y} \exp \left\{-\frac{1}{4 \tau} \ln ^{2}\left(y \mathrm{e}^{\tau}\right)\right\}, \tag{4.13}
\end{equation*}
$$

has long tail which shows that the one-point statistics of $y(\tau)$ are dominated by large deviations (Fig. 10). The integral distribution function

$$
F(\tau ; y)=P(y(\tau)<y)=\Phi\left[\frac{1}{(2 \tau)^{1 / 2}} \ln ^{2}\left(y \mathrm{e}^{\tau}\right)\right]
$$

is expressed through the standard error-function

$$
\Phi(z)=\frac{1}{(2 \pi)^{1 / 2}} \int_{-\infty}^{z} \mathrm{~d} y \exp \left(-\frac{y^{2}}{2}\right) .
$$

2. The moments of $y$ increase exponentially in time:

$$
\begin{gather*}
\left\langle y^{n}(\tau)\right\rangle=\exp \{n(n-1) \tau\}, \quad\left\langle\frac{1}{y^{n}(\tau)}\right\rangle=\exp \{n(n+1) \tau\}, \\
n=1,2, \ldots, \tag{4.14}
\end{gather*}
$$

3. Yet the typical realization of the process falls off exponentially

$$
\begin{equation*}
y^{*}(\tau)=\mathrm{e}^{-\tau} . \tag{4.15}
\end{equation*}
$$



Figure 10. Probability distribution function of a log-normal process.

Hence, exponential growth of the moments is due to large fluctuations about $y^{*}(\tau)$ on both sides (large and small values of $y$ ). However, the probability of $y<1$, for large times $\tau \gg 1$, goes rapidly to 1

$$
P(y(\tau)<1)=F(\tau, 1)=1-\frac{1}{(\pi \tau)^{1 / 2}} \mathrm{e}^{-\tau / 4} .
$$

4. For any probability $0<p<1$ there exists a oneparameter family of exponentially decaying curves,

$$
\begin{equation*}
Y_{p}(\tau, \beta)=(1-p)^{-1 / \beta} \exp \{-(1-\beta) \tau\} \quad(0 \leqslant \beta \leqslant 1), \tag{4.16}
\end{equation*}
$$

that dominate the process in the sense that a sizable fraction of realizations lie below $Y_{p}(\tau, \beta)$, i.e.

$$
P\left\{y(\tau)<Y_{p}(\tau, \beta) \text { for all } \tau \in(0, \infty)\right\}=p
$$

In particular, 'half of realizations' (probability $p=1 / 2$ ) satisfy

$$
\begin{equation*}
y(\tau)<4 \exp \left\{-\frac{\tau}{2}\right\} \tag{4.17}
\end{equation*}
$$

over the entire time-range $\tau \in(0, \infty)$.
5. Random variables $S_{n}=\int_{0}^{\infty} \mathrm{d} \tau y^{n}(\tau)$ that characterize large deviations, have finite (stationary) probability distributions

$$
\begin{equation*}
P_{n}(S)=\frac{n^{-2 / n}}{\Gamma(1 / n)} \frac{1}{S^{1+1 / n}} \exp \left\{-\frac{1}{n^{2} S}\right\} \tag{4.18}
\end{equation*}
$$

with polynomial fall off at large $S$. In particular, for the 'area integral' $S_{1}=\int_{0}^{\infty} \mathrm{d} \tau y(\tau)$ we get a PDF and integral distribution of the form

$$
P_{1}(S)=\frac{1}{S^{2}} \exp \left\{-\frac{1}{S}\right\}, \quad F(S)=\exp \left\{-\frac{1}{S}\right\} .
$$

This means that 'large deviations' of process $y$ enclose 'small areas'.

All the above properties are characteristic of log-normal processes, and follow from their one-point PDFs (4.13).

Now we can proceed to the statistical analysis of specific problems formulated in the previous part .

## 5. Transfer phenomena in dynamical systems with singularities

### 5.1 Simple case

We start with the stochastic equation (2.3) written for $\lambda=1$,

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} x(t)=-x^{2}(t)+f(t), \quad x(0)=x_{0} \tag{5.1}
\end{equation*}
$$

and assume the random process $f(t)$ to be Gaussian and deltacorrelated:

$$
\langle f(t)\rangle=0, \quad\left\langle f(t) f\left(t^{\prime}\right)\right\rangle=2 D \delta\left(t-t^{\prime}\right) .
$$

In the absence of sources, Eqn (5.1) has the exact solution

$$
x(t)=\frac{1}{t-t_{0}}, \quad t_{0}=-\frac{1}{x_{0}} .
$$

So if the initial point $x_{0}>0$, solution goes monotonically to zero, but in the case of $x_{0}<0$ it explodes to $-\infty$ in finite time $t_{0}$.

Solution of the stochastic problem (5.1) is determined by the forward and backward Fokker - Planck equation in terms of the difference $t-t_{0}$, that we shall relabel as $t$ :

$$
\begin{aligned}
& \frac{\partial}{\partial t} p\left(x, t \mid x_{0}\right)=\frac{\partial}{\partial x} x^{2} p\left(x, t \mid x_{0}\right)+D \frac{\partial^{2}}{\partial x^{2}} p\left(x, t \mid x_{0}\right), \\
& \frac{\partial}{\partial t} p\left(x, t \mid x_{0}\right)=-x_{0}^{2} \frac{\partial}{\partial x_{0}} p\left(x, t \mid x_{0}\right)+D \frac{\partial^{2}}{\partial x_{0}^{2}} p\left(x, t \mid x_{0}\right),
\end{aligned}
$$

$$
\begin{equation*}
p\left(x, 0 \mid x_{0}\right)=\delta\left(x-x_{0}\right) \tag{5.2}
\end{equation*}
$$

Here the variables $x, p\left(x, t \mid x_{0}\right)$ and $D$ have dimensions $[x]=t^{-1},[D]=t^{-3},[p]=t$, that allows Eqn (5.2) to be brought to the dimensionless form

$$
\begin{align*}
& \frac{\partial}{\partial t} p\left(x, t \mid x_{0}\right)=\frac{\partial}{\partial x} x^{2} p\left(x, t \mid x_{0}\right)+\frac{\partial^{2}}{\partial x^{2}} p\left(x, t \mid x_{0}\right) \\
& \frac{\partial}{\partial t} p\left(x, t \mid x_{0}\right)=-x_{0}^{2} \frac{\partial}{\partial x_{0}} p\left(x, t \mid x_{0}\right)+\frac{\partial^{2}}{\partial x_{0}^{2}} p\left(x, t \mid x_{0}\right) \\
& p\left(x, 0 \mid x_{0}\right)=\delta\left(x-x_{0}\right) \tag{5.3}
\end{align*}
$$

Next we set up boundary conditions for Eqn (5.3) at singular points $\pm \infty$. Two kinds of problems are of special interest to us.

The first type appears, if curve $x(t)$ is assumed to terminate at $t_{0}$ by turning to $-\infty$. It corresponds to vanishing of the PDF's flux density

$$
\begin{equation*}
J(t, x)=x^{2} p\left(x, t \mid x_{0}\right)+\frac{\partial}{\partial x} p\left(x, t \mid x_{0}\right) \tag{5.4}
\end{equation*}
$$

at $x \rightarrow \infty$. Thus we get the boundary conditions

$$
J(t, x) \rightarrow 0 \text { as } x \rightarrow \infty ; \quad p\left(x, t \mid x_{0}\right) \rightarrow 0 \text { as } x \rightarrow-\infty .
$$

In this case the total space-integral

$$
G\left(t \mid x_{0}\right)=\int_{-\infty}^{\infty} \mathrm{d} x p\left(x, t \mid x_{0}\right) \neq 1
$$

defines the probability that function $x(t)$ would stay finite over the entire range $(-\infty, \infty)$, i.e. it gives the probability $P\left(t<t_{0}\right)$ that there is no singularity at time $t$. So the probability of reaching the singularity in finite time $t$ is given by

$$
P\left(t>t_{0}\right)=1-\int_{-\infty}^{\infty} \mathrm{d} x p\left(x, t \mid x_{0}\right)
$$

(see, for instance Ref. [28]). Its density function

$$
\begin{equation*}
p\left(t \mid x_{0}\right)=\frac{\partial}{\partial t} P\left(t>t_{0}\right)=-\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \mathrm{d} x p\left(x, t \mid x_{0}\right) \tag{5.5}
\end{equation*}
$$

obeys the backward FP-equation that follows from Eqn (5.3):

$$
\begin{equation*}
\frac{\partial}{\partial t} p\left(t \mid x_{0}\right)=-x_{0}^{2} \frac{\partial}{\partial x_{0}} p\left(t \mid x_{0}\right)+\frac{\partial^{2}}{\partial x_{0}^{2}} p\left(t \mid x_{0}\right) . \tag{5.6}
\end{equation*}
$$

Let us estimate the mean 'blow-up' time for the system to transfer from state $x_{0}$ to $-\infty,\left\langle T\left(x_{0}\right)\right\rangle=\int_{0}^{\infty} t \mathrm{~d} t p\left(t \mid x_{0}\right)$. This function solves the time-independent DE

$$
\begin{equation*}
-1=-x_{0}^{2} \frac{\mathrm{~d}}{\mathrm{~d} x_{0}}\left\langle T\left(x_{0}\right)\right\rangle+\frac{\mathrm{d}^{2}}{\mathrm{~d} x_{0}^{2}}\left\langle T\left(x_{0}\right)\right\rangle, \tag{5.7}
\end{equation*}
$$

a consequence of Eqn (5.6), with the boundary condition $\left\langle T\left(x_{0}\right)\right\rangle \rightarrow 0$ at $x_{0} \rightarrow-\infty$, and the requirement that $\left\langle T\left(x_{0}\right)\right\rangle$ is bounded at $x_{0} \rightarrow \infty$. This problem can be is easily integrated:

$$
\begin{equation*}
\left\langle T\left(x_{0}\right)\right\rangle=\int_{-\infty}^{x_{0}} \mathrm{~d} \xi \int_{\xi}^{\infty} \mathrm{d} \eta \exp \left\{\frac{1}{3}\left(\xi^{3}-\eta^{3}\right)\right\} . \tag{5.8}
\end{equation*}
$$

Thus we get from Eqn (5.8) the mean time between two subsequent singular points (the 'explosive' events of Fig. 3):

$$
\langle T(\infty)\rangle=\sqrt{\pi} \frac{12^{1 / 6}}{3} \Gamma\left(\frac{1}{6}\right) \approx 4.976 .
$$

Let us also remark that $\langle T(0)\rangle=(2 / 3)\langle T(\infty)\rangle$ corresponds to a transition from $x_{0}=0$ to $x_{0}=-\infty$.

A different set of boundary conditions arises if one lets $x(t)$ be discontinuous, and considers it over the entire range of $t$, so as $x(t)$ reaches $-\infty$ at time $t \rightarrow t_{0}-0$, it immediately reappears at $\infty$ at $t \rightarrow t_{0}+0$. The corresponding boundary conditions for Eqn (5.3) impose continuity of the PDF flux at $\pm \infty$ :

$$
\left.J(t, x)\right|_{x \rightarrow-\infty}=\left.J(t, x)\right|_{x \rightarrow \infty}
$$

In this case the limiting $(t \rightarrow \infty)$ stationary probability distribution

$$
\begin{equation*}
P(x)=J \int_{-\infty}^{x} \mathrm{~d} \xi \exp \left\{\frac{1}{3}\left(\xi^{3}-x^{3}\right)\right\} \tag{5.9}
\end{equation*}
$$

is independent of the initial state $x_{0}$ and has the normalising coefficient $J=1 /\langle T(\infty)\rangle$ - the stationary probability flux .

From Eqn (5.4) we get an asymptotic formula for $P$ at large $x$ :

$$
\begin{equation*}
P(x)=\frac{1}{\langle T(\infty)\rangle x^{2}} . \tag{5.10}
\end{equation*}
$$

This asymptote is produced by the distribution of pole-like jumps of $x(t)$,

$$
x(t)=\frac{1}{t-t_{k}},
$$

and randomness plays little role here. Indeed, given sufficiently large $t \gg\langle T(\infty)\rangle$ and $x$, one finds the PDF

$$
\begin{aligned}
p\left(x, t \mid x_{0}\right) & =\sum_{k=0}^{\infty}\left\langle\delta\left(x-\frac{1}{t-t_{k}}\right)\right\rangle=\frac{1}{x^{2}} \sum_{k=0}^{\infty}\left\langle\delta\left(t-t_{k}\right)\right\rangle \\
& =\frac{1}{2 \pi x^{2}} \int_{-\infty}^{\infty} \mathrm{d} \chi \exp (-\mathrm{i} \varkappa t) \sum_{k=0}^{\infty}\left\langle\exp \left(\mathrm{i} \varkappa t_{k}\right)\right\rangle \\
& =\frac{1}{2 \pi x^{2}} \int_{-\infty}^{\infty} \mathrm{d} \chi \exp (-\mathrm{i} \varkappa t) \frac{\Phi_{0}(\varkappa)}{1-\Phi(\varkappa)}
\end{aligned}
$$

expressed in terms of two characteristic functions: $\Phi_{0}(\varkappa)=\left\langle\exp \left(\mathrm{i} \chi t_{0}\right)\right\rangle-$ the first 'explosive' point, and $\Phi(\varkappa)=\langle\exp (\mathrm{i} \varkappa T)\rangle$ - the characteristic function of time gaps between two subsequent explosives. As $t \rightarrow \infty$ we get an asymptotic formula

$$
P(x)=-\frac{1}{2 \pi \mathrm{i} x^{2}\langle T(\infty)\rangle} \int_{-\infty}^{\infty} \mathrm{d} \chi \exp (-\mathrm{i} \nsim t) \frac{1}{\varkappa+i 0}=\frac{J}{x^{2}}
$$

that coincides with Eqn (5.10).

### 5.2 Caustics in random media

The problem of caustics in random media is similar to the previous one. Indeed, in the 2 D case the curvature of the phase surface in $(x, y)$ plane is described by the equation

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} x} u(x)=-u^{2}(x)+f(x), \quad u(0)=u_{0}, \tag{5.11}
\end{equation*}
$$

where $f(x)=(1 / 2) \partial^{2} \varepsilon(x, y(x)) / \partial y^{2}$, while the vertical displacement $y(x)$ obeys the differential system (2.49), both determined by variations of the refraction index $\varepsilon$ along the ray. If the field $\varepsilon(x, y)$ is Gaussian, homogeneous, isotropic and delta-correlated,

$$
\left\langle\varepsilon(x, y) \varepsilon\left(x^{\prime}, y^{\prime}\right)\right\rangle=\delta\left(x-x^{\prime}\right) A\left(y-y^{\prime}\right),
$$

the one-point PDF of the curvature-function is independent of the vertical displacement and obeys the FP-equation:

$$
\begin{equation*}
\frac{\partial}{\partial x} P(x ; u)=\frac{\partial}{\partial u} J(x ; u), \quad P(0 ; u)=\delta\left(u-u_{0}\right) . \tag{5.12}
\end{equation*}
$$

Here the probability flux density and the diffusion coefficient are given by

$$
\begin{aligned}
& J(x ; u)=u^{2} P(x ; u)+\frac{D}{2} \frac{\partial}{\partial u} P(x ; u), \\
& D=\frac{1}{4} \frac{\partial^{4}}{\partial y^{4}} A(0)=\pi \int_{0}^{\infty} \chi^{4} \mathrm{~d} \chi \Phi_{\varepsilon}(0, \chi),
\end{aligned}
$$

where $\Phi_{\varepsilon}(0, \chi)$ denotes the 2D spectral density of random field $\varepsilon(x, y)$.

Equation (5.12) was studied in the previous section. We have shown the random process $u(x)$ to be discontinuous and drop to $-\infty$ at finite distances $x_{0}=x\left(u_{0}\right)$ determined by the initial condition $u_{0}$ (here $x$ plays the role of time $t$ of the previous section). This 'blow-up' means the focusing of optical rays by randomly inhomogeneous medium. The mean focusing distance $\left\langle x\left(u_{0}\right)\right\rangle$ is given by the integral

$$
\begin{equation*}
\left\langle x\left(u_{0}\right)\right\rangle=\frac{2}{D} \int_{-\infty}^{u_{0}} \mathrm{~d} \xi \int_{\xi}^{\infty} \mathrm{d} \eta \exp \left\{\frac{2}{3 D}\left(\xi^{3}-\eta^{3}\right)\right\}, \tag{5.13}
\end{equation*}
$$

hence [6-8]

$$
D^{1 / 3}\langle x(\infty)\rangle \approx 6.27, \quad D^{1 / 3}\langle x(0)\rangle=\frac{2}{3} D^{1 / 3}\langle x(\infty)\rangle \approx 4.18
$$

Here quantity $\langle x(0)\rangle$ gives the mean 'focusing distance' of the incident plane wave (zero curvature), while $\langle x(\infty)\rangle$ measures the mean distance between two follow-up focusing events.

In the method of smooth perturbations [5, 40, 41] one utilizes the level function of intensity $\chi(x, y)=\ln I(x, y)$ to study its fluctuations. All the statistical characteristics of $\chi$ depend on its variance $\sigma^{2}(x)$, which is approximately (to first order) cubic: $\sigma^{2}(x) \cong D x^{3}$. One distinguishes two regions of intensity fluctuations: $\sigma^{2}(x) \ll 1$ (weak fluctuations), and $\sigma^{2}(x) \gg 1$ (strong ones). As formula (5.13) shows, random focusing occurs in the region of strong fluctuations.

Equation (5.12) has a limiting stationary PDF, as $x \rightarrow \infty$, that corresponds to the constant probability flux density, and looks similar to Eqn (5.9),

$$
\begin{equation*}
P(u)=J \int_{-\infty}^{u} \mathrm{~d} \xi \exp \left\{\frac{2}{3 D}\left(\xi^{3}-u^{3}\right)\right\}, \tag{5.14}
\end{equation*}
$$

where $J=1 /\langle x(\infty)\rangle$. From Eqn (5.12) we get large- $u$ asymptotes of $P$,

$$
P(u)=\frac{1}{\langle x(\infty)\rangle u^{2}},
$$

which indicates that the stationary statistics are determined by the behavior of $u(x)$ in the vicinity of jumps

$$
u(x)=\frac{1}{x-x_{k}}
$$

The wave-field intensity has a similar structure due to Eqn (2.49),

$$
I(x)=\frac{x_{k}}{\left|x-x_{k}\right|},
$$

the asymptotes of its PDF at large $I$ and $x$ takes the form

$$
P(x, I)=\frac{2 x}{\langle x(\infty)\rangle I^{2}}
$$

depending on the distance $x$ traveled by the wave.
The probability of focusing at a distance $x$, found in the previous section, was

$$
P\left(x>x_{0}\right)=1-\int_{-\infty}^{\infty} \mathrm{d} u P(x, u) .
$$

Hence its PDF is related to the flux density by the following equation [6-8]:

$$
p(x)=\frac{\partial}{\partial x} P\left(x>x_{0}\right)=-\frac{\partial}{\partial x} \int_{-\infty}^{\infty} \mathrm{d} u P(x, u)=\lim _{u \rightarrow-\infty} J(x, u) .
$$

To find the asymptotic dependence of $p(x)$ on the small parameter $(D \rightarrow 0)$, we adopt the standard analytic techniques for parabolic problems with a small leading coefficient (see, for instance, Ref. [28]). The solution of Eqn (5.12) is represented as exponential

$$
\begin{equation*}
P(x, u)=C(D) \exp \left\{-\frac{1}{D} A(x, u)-B(x, u)\right\} . \tag{5.15}
\end{equation*}
$$

After substitution of Eqn (5.15) into Eqn (5.12) we select terms of order $D^{0}$ and $D^{-1}$ and get PDEs for the unknown functions $A(x, u)$ and $B(x, u)$. The constant factor $C(D)$ is determined by the known probability distribution (of the plane wave) as $x \rightarrow 0$

$$
P(x, u)=\frac{1}{(2 \pi D x)^{1 / 2}} \exp \left\{-\frac{u^{2}}{2 D x}\right\} .
$$

That gives an estimate $C(D) \cong D^{-1 / 2}$. Substituting Eqn (5.15) in the expression for probability of 'focusing', we get

$$
\begin{equation*}
p(x, u)=\lim _{u \rightarrow-\infty} P(x, u)\left[u^{2}-\frac{1}{2} \frac{\partial}{\partial u} A(x, u)\right] . \tag{5.16}
\end{equation*}
$$

The exponential form (5.15) along with the probability distribution of 'focusing' immediately yield the structural dependence of $p(x)$ on $x$ from the dimensional analysis [9]. Indeed, $u, D$ and $P(x, u)$ have dimensions

$$
[u]=x^{-1}, \quad[D]=x^{-3}, \quad[P(x, u)]=x .
$$

Hence from Eqns (5.15) and (5.16) we get the scaling law

$$
p(x)=C_{1} D^{-1 / 2} x^{-5 / 2} \exp \left\{-\frac{C_{2}}{D x^{3}}\right\}
$$

with unknown coefficients $C_{1}, C_{2}$. The latter were computed in Ref. [6] and the final result takes the form

$$
\begin{equation*}
p(x)=3 \alpha^{2}(2 \pi D)^{-1 / 2} x^{-5 / 2} \exp \left\{-\frac{\alpha^{4}}{6 D x^{3}}\right\} \tag{5.17}
\end{equation*}
$$

with $\alpha=1.85$.
The applicability condition for Eqn (5.17) is $D x^{3} \ll 1$. However, numeric simulations [6] show that Eqn (5.17) also gives an accurate prediction of PDF $p$ for $D x^{3} \sim 1$.

In the 3D case one could apply the dimensional analysis to compute the PDF of caustic formation [9],

$$
p(x)=\alpha D^{-1} x^{-4} \exp \left\{-\frac{\beta}{D x^{3}}\right\}
$$

with certain constants $\alpha, \beta$. This scaling law with $\alpha=1.74$, $\beta=0.66$ was obtained in Ref. [7].

### 5.3 The reflection phase for plane waves in layered media

In some cases the above discontinuous random processes prove to be useful in stochastic systems with no apparent discontinuity of solutions. One such example are fluctuations of phase-functions in wave-propagation. For instance, we have shown in Section 2.2 that the incident plane wave in randomly layered medium has a reflection coefficient with phase function $\phi_{L}$ that obeys equation (2.13). We want to find the distribution law of random variable $\phi_{L}$.

The problem with solution (2.13) is that it gives $\phi_{L}$ as a real-valued function over the entire line $\phi_{L}(-\infty, \infty)$. However, in applications we need the distribution of $\phi_{L}$ over its natural domain $(-\pi, \pi)$. Furthermore, the half-space limit of $\phi_{L}$ should be independent of $L$ (uniform distribution). To get the finite-interval distribution we replace the phase $\phi_{L}$ with a trigonometric function $z_{L}=\tan \left(\phi_{L} / 2\right)$, that possesses singular poles. The dynamic equation for $z_{L}$ has the form (2.15). We assume the random coefficient $\varepsilon_{1}(L)$ (the real part of the complex refraction index) to be Gaussian, deltacorrelated with variance $\sigma_{\varepsilon}^{2}$ and correlation radius $l_{0}$ :

$$
\left\langle\varepsilon_{1}(L)\right\rangle=0, \quad\left\langle\varepsilon_{1}(L) \varepsilon_{1}\left(L^{\prime}\right)\right\rangle=2 \sigma_{\varepsilon}^{2} l_{0} \delta\left(L-L^{\prime}\right) .
$$

Then the stationary ( $L$-independent) half-space limiting PDF,

$$
P(z)=\lim _{L_{0} \rightarrow-\infty} P(L, z)
$$

solves the equation

$$
\begin{equation*}
-\chi \frac{\mathrm{d}}{\mathrm{~d} z}\left(1+z^{2}\right) P(z)+\frac{\mathrm{d}^{2}}{\mathrm{~d} z^{2}} P(z)=0 \tag{5.18}
\end{equation*}
$$

whose coefficients, $x=k / 2 D$, and $D=k^{2} \sigma_{\varepsilon}^{2} l_{0} / 2$ depend on the incident wave number $k$ and the media parameters. The solution of Eqn (5.18) with constant probability flux at the boundary $(\infty)$ has the following form [11, 33]

$$
\begin{align*}
& P(z)=J(\varkappa) \int_{z}^{\infty} \mathrm{d} \xi \exp \left\{-\chi \xi\left[1+\frac{\xi^{2}}{3}+z(z+\xi)\right]\right\} \\
& J^{-1}(\varkappa)=\left(\frac{\pi}{\chi}\right)^{1 / 2} \int_{0}^{\infty} \xi^{-1 / 2} \mathrm{~d} \xi \exp \left\{-\chi\left(\xi+\frac{\xi^{3}}{12}\right)\right\} \cdot(\xi \tag{5.19}
\end{align*}
$$

The PDF of such a phase-function over the finite interval ( $-\pi$, $\pi$ ),

$$
P(\phi)=\frac{1+z^{2}}{2} P(z)_{z=\tan (\phi / 2)},
$$

is shown in Fig. 11 for different values of parameter $\varkappa$ : $\varkappa=0.1$ (A), $x=1$ (B), and $x=10$ (C).

As $x$ grows large, $x \gg 1$, we get the asymptotic expansion $P(z)=1 / \pi\left(1+z^{2}\right)$ over $-\infty<z<\infty$, that corresponds to a uniformly distributed phase

$$
P(\phi)=\frac{1}{2 \pi}, \quad-\pi<\phi<\pi
$$



Figure 11. PDF of the plane wave phase in stratified media.

## 6. Localization of plane waves in randomly stratified media

Our main goal here is the wave-propagation in single and multi-layer media.

The issues of wave localization by randomly stratified media with or without dissipation have been actively pursued and debated over the last few years (see review articles [1113]). However, different physical systems and conditions produced inconclusive and often inconsistent results.

The reason is largely due to the complex spatial structure of field realizations inside the media. Namely, the general decay of intensity away from the source is accompanied by increasingly strong and rare bursts due to the coherent superposition of multiple scattering events (Fig. 5). As a result one could observe dynamic localization for almost all individual realizations, whereas statistical averages (like the mean intensity and higher moments) show no statistical (energy) localization on the level of ensembles of realizations.

### 6.1 Single layer medium.

The one-layer, 1D model of wave propagation and scattering was set up in Section 2.2. Here the statistical problem has two
parts: statistical analysis of the reflection and transmission coefficients on the boundary of random layer, and statistical analysis of the field intensity inside the layer.
6.1.1 The reflection and transmission coefficients. The embedding method gives closed form equations (2.10), (2.12) for the squared modulus of the reflection coefficient as function of the (right) moving boundary parameter $L$. Once again we assume the random component of the medium refraction coefficient $\varepsilon_{1}(x)$ to be Gaussian, and use the delta-correlation approximation with variance $\sigma_{\varepsilon}^{2}$ and correlation radius $l_{0}$ :
$\left\langle\varepsilon_{1}(L)\right\rangle=0, \quad\left\langle\varepsilon_{1}(L) \varepsilon_{1}\left(L^{\prime}\right)\right\rangle=B_{\varepsilon}\left(L-L^{\prime}\right)=2 \sigma_{\varepsilon}^{2} l_{0} \delta\left(L-L^{\prime}\right)$.
The resulting Fokker-Planck equation for the PDF, $P(L, W)=\left\langle\delta\left(W_{L}-W\right)\right\rangle$, has the form [10]

$$
\begin{align*}
& \frac{\partial}{\partial L} P(L, W)=2 k \gamma \frac{\partial}{\partial W}(W P)-2 D \frac{\partial}{\partial W}[W(1-W) P] \\
& \quad+D \frac{\partial}{\partial W}(1-W)^{2} W \frac{\partial}{\partial W} P, \quad P\left(L_{0}, W\right)=\delta(W) \tag{6.1}
\end{align*}
$$

with the diffusion coefficient $D=k^{2} \sigma_{\varepsilon}^{2} l_{0} / 2$. Derivation of Eqn (6.1) exploits averaging over fast oscillations of the 'reflection phase', and is valid under an additional constraint $k / D \gg 1$ (see Section 5.3).

Taking into account the finite correlation radius $l_{0}$ of $\varepsilon_{1}(L)$ and using the diffusion approximation, the FP-equation (6.1) doesn't change its form, but the diffusion coefficient becomes

$$
\begin{equation*}
D=\frac{k^{2}}{4} \int_{-\infty}^{\infty} \mathrm{d} \xi B_{\varepsilon}(\xi) \cos (2 k \xi)=\frac{k^{2}}{4} \Phi_{\varepsilon}(2 k), \tag{6.2}
\end{equation*}
$$

where $\Phi_{\varepsilon}(2 k)$ denotes the spectral function of random process $\varepsilon(L)$. The diffusion approximation assumes that $\varepsilon_{1}(x)$ has negligible effect on the wave-field on the scale of the correlation radius $l_{0}$, that is $D l_{0} \ll 1$.

In the absence of dissipation $(\gamma=0)$ the FP-equation (6.1) could be integrated [10]. Assuming the random layer $\left[L_{0}, L\right]$ to be sufficiently wide, $\tau=D\left(L-L_{0}\right) \gg 1$, we get the following asymptotes for the moments of the transmission coefficient $\left|T_{L}\right|^{2}=1-W_{L}$ :

$$
\left.\left.\langle | T_{L}\right|^{2 n}\right\rangle \approx \frac{[(2 n-3)!!]^{2} \pi^{2} \sqrt{\pi}}{2^{2 n-1}(n-1)!} \tau^{-3 / 2} \exp \left(-\frac{\tau}{4}\right) .
$$

Thus all the statistical moments of the transmission/ reflection coefficients have a universal asymptotic dependence on the (dimensionless) layer width $\tau$, but their coefficients vary with $n$. As all the statistical moments of $|T|$ converge to zero with increasing $\tau$, we get the reflection modulus $|R| \rightarrow 1$ with probability one. Hence the randomly stratified half-space is fully reflective.

The nonzero dissipation $\gamma$ does not allow a closed form solution of Eqn (6.1) for a finite width interval. However, the half-space limit ( $L_{0} \rightarrow-\infty, \tau \rightarrow \infty$ ) has stationary $L$ and $\tau$ independent distribution for $W_{L}=\left|R_{L}\right|^{2}$,

$$
\begin{equation*}
P(W)=\frac{2 \beta}{(1-W)^{2}} \exp \left(-\frac{2 \beta W}{1-W}\right), \tag{6.3}
\end{equation*}
$$

where $\beta=k \gamma / D$ represents the dimensionless absorption coefficient. PDF (6.3) has an obvious physical meaning. It gives the statistical properties of the reflection coefficient for a
stochastically inhomogeneous layer wide enough that an incident wave is unable to cross it because of dynamic absorption by the medium.

PDF (6.3) yields all moments of $W_{L}=\left|R_{L}\right|^{2}$, in particular it gives asymptotic formula for the mean value

$$
\langle W(L)\rangle= \begin{cases}1-2 \beta \ln \left(\frac{1}{\beta}\right), & \beta \ll 1,  \tag{6.4}\\ \frac{1}{2 \beta}, & \beta \gg 1,\end{cases}
$$

as well as the recurrence for higher moments:

$$
\begin{equation*}
n\left\langle W^{n+1}\right\rangle-2(\beta+n)\left\langle W^{n}\right\rangle+n\left\langle W^{n-1}\right\rangle=0 \quad(n=1,2, \ldots) . \tag{6.5}
\end{equation*}
$$

If we place the source inside the stratified media, the wavefield solves the boundary value problem (2.9). In the infinite space limit ( $L_{0} \rightarrow-\infty, L \rightarrow \infty$ ), the mean intensity at the source becomes $[5,10]$

$$
\left\langle I\left(x_{0} ; x_{0}\right)\right\rangle=1+\beta^{-1} .
$$

Its unlimited growth, as $\beta \rightarrow 0$, indicates the accumulation of energy by randomly stratified media.
6.1.2 Wave intensity within a random layer. Let us study the wave-field structure inside a chaotically disordered layer. If layer $\left[L_{0}, L\right]$ is sufficiently wide $\left(\tau=D\left(L-L_{0}\right) \gg 1\right)$, and dissipation-free $(\gamma=0)$, one has the so-called stochastic parametric resonance, manifested by the initial exponential growth of all moments $\left\langle I^{n}(x ; L)\right\rangle$ of the intensity divided by $\left\langle I^{n}(L ; L)\right\rangle\left(I(x ; L)=|u(x ; L)|^{2}\right)$, in variable $L[10]$. They then reach maximum somewhere in the middle of the layer (Fig. 12). In the half-space limit $\left(L_{0}=-\infty\right)$ the range of exponential (explosive) growth extends through the entire half-line, but the mean intensity remains fixed, $\langle I(x ; L)\rangle=2$.

In this case the realizations of wave-intensity have the form

$$
\begin{equation*}
I(x ; L)=2 W(x ; L)\left(1+\cos \phi_{L}\right), \tag{6.6}
\end{equation*}
$$



Figure 12. Stochastic wave parametric resonance.
where the phase-function $\phi_{L}$ solves a stochastic equation (2.13) (see Section 5.3). The amplitude-function

$$
W(x ; L)=\exp \{-[q(L)-q(x)]\}
$$

is expressed through $q(L)$ that solves the related stochastic equation

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} L} q(L)=k \varepsilon_{1}(L) \sin \phi_{L} \tag{6.7}
\end{equation*}
$$

The initial condition for Eqn (6.7) is imposed at a fardistant point $B$ and reads $q(B)=0$, and the random variable $\phi_{B}$ is uniformly distributed [10].

The random functions $q(L)$ and $\phi_{L}$ are statistically independent here. Averaging the PDF $P(L, q)=$ $\langle\delta(q(L)-1)\rangle$ over the fast oscillations, we get the FokkerPlanck equation

$$
\begin{align*}
& \frac{\partial}{\partial L} P(L, q)=-D \frac{\partial}{\partial q} P(L, q)+D \frac{\partial^{2}}{\partial q^{2}} P(L, q), \\
& P(B, q)=\delta(q) \tag{6.8}
\end{align*}
$$

Hence the random function $q(L)$ is Gaussian, while the distribution $W(x ; L)$ is log-normal, having all moments starting with the second one growing exponentially inside the random layer:

$$
\begin{equation*}
\langle W(x ; L)\rangle=1, \quad\left\langle W^{n}(x ; L)\right\rangle=\exp \{n(n-1) D(L-x)\} . \tag{6.9}
\end{equation*}
$$

Its typical realization, due to log-normality, becomes

$$
\begin{equation*}
W^{*}(x ; L)=\exp \{-D(L-x)\} . \tag{6.10}
\end{equation*}
$$

So its realizations meet the inequality

$$
W(x ; L)<4 \exp \left\{-\frac{D(L-x)}{2}\right\}
$$

with probability $1 / 2$, and the latter is valid in the half-space limit.

In the physics of disordered systems such an exponential fall-off in variable $\xi=D(L-x)$ for a typical realization (6.10) is associated with dynamical localization [11,53,54], the localization length being $l_{\mathrm{loc}}=1 / D$. However, the energy is not localized in the statistical (mean) sense here.

To conclude: one-point PDFs yield the detailed evolution of the field intensity on the level of individual realizations, and allow one to estimate the parameters of evolution through the statistics of the media.

Dissipation arrests the exponential growth of the moments (6.9), and at large $\xi \gg 4(n-1 / 2) \ln (n / \beta)$ they revert to the decaying asymptotic law of [13]:

$$
\langle W(\xi)\rangle \cong A_{n} \beta^{-(n-1 / 2)} \ln \left(\frac{1}{\beta}\right) \xi^{-3 / 2} \exp \left\{-\frac{\xi}{4}\right\} .
$$

As for the mean field intensity of a point source in the layer, one finds [13]

$$
\left\langle I\left(x ; x_{0}\right)\right\rangle=\frac{\pi^{5 / 2}}{8 \beta} \xi^{-3 / 2} \exp \left\{-\frac{\xi}{4}\right\}
$$

at large distance from the source $\xi=D\left(x-x_{0}\right) \geqslant 1$. So one has both statistical and dynamical localization in this case.

### 6.2 Stratified two-layer model

The simplest model of wave propagation in stratified (twolayer) media was outlined in Section 2.2. It satisfies the differential system (2.17).

The diffusion approximation technique was used in papers [37, 38] to derive the Fokker - Planck equations for PDFs of the intensities of reflection coefficients $W_{i j}=\left|R_{i j}\right|^{2}$. Unlike the one-layer case, they now involve four diffusion coefficients expressed through the spectral function of the random process $\varepsilon(x)$ as follows:

$$
\begin{align*}
& D_{1}=\left(\frac{k}{2 \lambda} \frac{H_{1}}{H_{0}}\right)^{2} \Phi_{\varepsilon}(2 \lambda k), \quad D_{2}=\left(\frac{k}{2} \frac{H_{2}}{H_{0}}\right)^{2} \Phi_{\varepsilon}(2 k), \\
& D_{3}=\left(\frac{k}{2 \lambda}\right)^{2} \Phi_{\varepsilon}(k(1+\lambda)), \quad D_{4}=\left(\frac{k}{2 \lambda}\right)^{2} \Phi_{\varepsilon}(k(1-\lambda)) . \tag{6.11}
\end{align*}
$$

For small-scale (relative to the wave length) inhomogeneities such, that $k l_{0} \ll 1$, all diffusion coefficients are proportional to a single coefficient $D$ determined by Eqn (6.2). Namely,

$$
\begin{equation*}
D_{1}=\left(\frac{1}{\lambda} \frac{H_{1}}{H_{0}}\right)^{2} D, \quad D_{2}=\left(\frac{H_{2}}{H_{0}}\right)^{2} D, \quad D_{3}=D_{4}=\frac{1}{\lambda^{2}} D . \tag{6.12}
\end{equation*}
$$

Let us also notice that the diffusion approximation requires $D l_{0} \ll 1$.

As we mentioned in Section 2 the problem has a small parameter $\delta$ that could be used to expand and simplify it. We write the perturbation expansion in $\delta$ and maintain only the first order terms, i.e. neglect secondary radiation effects. In such an approximation the squared moduli of the diagonal reflection coefficients $W_{11}$ and $W_{22}$ are statistically independent and their PDFs $P\left(L, W_{11}\right)$ and $P\left(L, W_{22}\right)$ obey

$$
\begin{align*}
\frac{\partial}{\partial L} P\left(L, W_{11}\right) & =\left\{\frac{\partial}{\partial W_{11}}\left[-D_{1}\left(1-W_{11}\right)^{2}+2 \delta\left(D_{3}+D_{4}\right) W_{11}\right]\right. \\
& \left.+D_{1} \frac{\partial^{2}}{\partial W_{11}^{2}}\left(1-W_{11}\right)^{2} W_{11}\right\} P\left(L, W_{11}\right), \\
\frac{\partial}{\partial L} P\left(L, W_{22}\right) & =\left\{\frac{\partial}{\partial W_{22}}\left[-D_{2}\left(1-W_{22}\right)^{2}+2 \delta\left(D_{3}+D_{4}\right) W_{22}\right]\right. \\
& \left.+D_{2} \frac{\partial^{2}}{\partial W_{22}^{2}}\left(1-W_{22}\right)^{2} W_{22}\right\} P\left(L, W_{22}\right) . \tag{6.13}
\end{align*}
$$

Those resemble equation (6.1) of the one-layer model, but have an extra term $2 \delta\left(D_{3}+D_{4}\right) \partial[W P(L, W)] / \partial W$. This means that the generation of cross-modes ( $k$-incident to $\lambda$ scattered, or $\lambda$-incident to $k$-scattered) is statistically equivalent to the effective damping (dissipation) in the original stochastic problem for incident wave-fields $U_{11}, U_{22}$. In other words the dissipation-free refraction index $\varepsilon(x)$ should be replaced by the complex one, $\varepsilon(x) \rightarrow \varepsilon(x)+\mathrm{i} \delta\left(D_{3}+D_{4}\right)$.

Furthermore, the half-space limit $\left(L_{0} \rightarrow-\infty\right)$ has stationary ( $L$-independent) solutions of (6.13),

$$
\begin{align*}
& P\left(W_{11}\right)=\frac{2 \gamma_{1}}{\left(1-W_{11}\right)^{2}} \exp \left\{-\frac{2 \gamma_{1} W_{11}}{1-W_{11}}\right\} \\
& P\left(W_{22}\right)=\frac{2 \gamma_{2}}{\left(1-W_{22}\right)^{2}} \exp \left\{-\frac{2 \gamma_{2} W_{22}}{1-W_{22}}\right\}, \tag{6.14}
\end{align*}
$$

with parameters

$$
\begin{equation*}
\gamma_{1}=\delta \frac{D_{3}+D_{4}}{D_{1}}, \quad \gamma_{2}=\delta \frac{D_{3}+D_{4}}{D_{2}} \tag{6.15}
\end{equation*}
$$

These parameters measure the 'effective dissipation' due to cross-generation of the wave-modes, as opposed to the 'effective wave diffusion' resulting from multiple scattering by random obstacles.

For small scale inhomogeneities the damping parameters

$$
\begin{equation*}
\gamma_{1}=2 \lambda \frac{H_{2}}{H_{1}}, \quad \gamma_{2}=\frac{2}{\lambda} \frac{H_{1}}{H_{2}} \tag{6.16}
\end{equation*}
$$

are determined by the relative thicknesses of two strata (for a fixed wave length $\lambda$ ), and bear no other dependence on the statistics of the media. Furthermore, one has $\gamma_{1} \gamma_{2}=4$, so two $\gamma$ 's are reciprocals of each other.

The probability distributions (6.14) yield the statistics of the reflection coefficients of incident waves. It follows from our discussion that a sufficiently wide random band ( $L_{0}, L$ ) (or the limiting half-space) has the transmission rates $\left|T_{11}\right|^{2}$ and $\left|T_{22}\right|^{2}$ almost surely 0 . So the incident $\lambda$ - and $k$-waves are localized. Their localization length are determined either by the 'diffusion' or 'dissipation' coefficients, whichever is stronger.

So case $\gamma_{1} \ll 1\left(\gamma_{2} \gg 1\right)$ yields the localization length for two modes proportional to the localization length $l_{\text {loc }}$ of the one-layer model:

$$
l_{\mathrm{loc}}^{(1)}=\frac{1}{D_{1}}=\left(\frac{\lambda H_{0}}{H_{1}}\right)^{2} l_{\mathrm{loc}}, l_{\mathrm{loc}}^{(2)}=\frac{1}{2 \delta\left(D_{3}+D_{4}\right)}=\frac{\lambda H_{0}}{4 H_{1} H_{2}} l_{\mathrm{loc}} .
$$

In the opposite case $\gamma_{1} \gg 1\left(\gamma_{2} \ll 1\right)$, we get

$$
l_{\mathrm{loc}}^{(1)}=\frac{1}{2 \delta\left(D_{3}+D_{4}\right)}=\frac{\lambda H_{0}}{4 H_{1} H_{2}} l_{\mathrm{loc}}, \quad l_{\mathrm{loc}}^{(2)}=\frac{1}{D_{2}}=\left(\frac{H_{0}}{H_{2}}\right)^{2} l_{\mathrm{loc}} .
$$

The statistics of the 'off-diagonal' modes $W_{12}$ poses a more challenging problem, since it involves coupling $W_{12}$ with $W_{11}$ and $W_{22}$. The analysis of the corresponding Fokker - Planck equations shows that for the special combinations $T_{1}=1-W_{11}-\delta W_{12}$ and $T_{2}=1-W_{22}-\delta W_{12}$ that arise in Eqn (2.26) and determine the transmission rates of the generated cross-modes, there are no stationary (in halfspace) solutions of the form $P\left(T_{i}\right)=\delta\left(T_{i}\right)$. This means the absence of localization for the generated cross-modes [38].

Coming back to the source-problem in the upper or lower strata (at the boundary $x_{0}=L$ of the random band), one could show that the transmission coefficients differ from zero in both strata (the upper and the lower), so there is no localization whatsoever [38].

The basic model could be further compounded by introducing inhomogeneities in both strata, or by taking different types of wave-coupling, or by changing fluctuating parameters, e.g. $\varepsilon(x)$ to $\mathrm{d} \varepsilon(x) / \mathrm{d} x$, etc., all of them relevant to certain geophysical problems [35-37]. Such changes could further complicate and modify the FokkerPlanck equations, as well as their dependence on the basic geometry and statistics of the problem. But the main conclusion - the absence of the dynamic localization should still hold.

### 6.3 Localization in parabolic wave-guides

A wave beam propagating in the parabolic wave-guide with randomly varying curvature is described by system (2.44).

Function $A(x)$ could be represented in the form

$$
A(x)=k \alpha a^{2} \frac{1+\psi(x) \exp \{-2 \mathrm{i} \alpha x\}}{1-\psi(x) \exp \{-2 \mathrm{i} \alpha x\}},
$$

where $\psi(x)$ obeys an equation derived from Eqn (2.44):

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} x} \psi(x)=-\frac{\mathrm{i}}{2 \alpha k}[\exp (\mathrm{i} \alpha x)-\psi \exp (-i \alpha x)]^{2} z(x), \\
& \psi(0)=\frac{1-k \alpha a^{2}}{1+k \alpha a^{2}} .
\end{aligned}
$$

Let us introduce the phase and amplitude of $\psi(x)$ via

$$
\psi(x)=\sqrt{\frac{w(x)-1}{w(x)+1}} \exp \{\mathrm{i}(\phi(x)-2 \alpha x)\}, \quad w \geqslant 1 .
$$

Then we get a system of equations for functions $w(x)$ and $\phi(x)$ :

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} x} w(x)=-\frac{1}{\alpha k} z(x) \sqrt{w^{2}(x)-1} \sin \{\phi(x)-2 \alpha x\}, \\
& w(0)=\frac{1}{2 k \alpha a^{2}}\left[1+k^{2} \alpha^{2} a^{4}\right], \\
& \frac{\mathrm{d}}{\mathrm{~d} x} \phi(x)=\frac{1}{\alpha k} z(x)\left[1-\frac{w}{\sqrt{w^{2}(x)-1}} \cos \{\phi(x)-2 \alpha x\}\right], \\
& \phi(0)=0 . \tag{6.17}
\end{align*}
$$

Hence the wave-field intensity on the wave-guide axis (2.46) assumes the form

$$
\begin{equation*}
I(x, 0)=\frac{\alpha k a^{2}}{w(x)+\sqrt{w^{2}(x)-1} \cos \{\phi(x)-2 \alpha x\}} . \tag{6.18}
\end{equation*}
$$

As before we assume $z(x)$ to be Gaussian and deltacorrelated with parameters

$$
\langle z(x)\rangle=0, \quad\left\langle z(x) z\left(x^{\prime}\right)\right\rangle=2 \sigma^{2} l \delta\left(x-x^{\prime}\right) .
$$

Furthermore, we assume the variance of $z$-fluctuations to be sufficiently small, $\sigma^{2} \ll 1$. Then the statistical properties of $w(x)$ and $\phi(x)$ change slowly on scales of order $1 / \alpha$. Hence to determine the statistics of the wave intensity (6.18) we could average it over rapid oscillations, regarding the functions $w(x)$ and $\phi(x)$ as statistically independent, and over the uniformly distributed phase $\phi(x)$. The resulting PDF $P(x, w)=\langle\delta(w(x)-w)\rangle$ of $w(x)$ obeys the Fokker-Planck equation

$$
\begin{align*}
& \frac{\partial}{\partial x} P(x, w)=D \frac{\partial}{\partial w}\left(w^{2}-1\right) \frac{\partial}{\partial w} P(x, w), \\
& P(0, w)=\delta(w(0)-w) \tag{6.19}
\end{align*}
$$

with the diffusion coefficient $D=\sigma^{2} l /\left(2 \alpha^{2} k^{2}\right)$.
Under the same assumptions we could find the moments of intensity $\left\langle I^{n}(x, 0)\right\rangle$ on the wave-guide axis. Here the averaging works in two stages. In the first step we average over fast phase-oscillations and get

$$
\begin{equation*}
\left\langle\left(\frac{I}{\alpha k a^{2}}\right)^{n}\right\rangle_{\phi}=P_{n-1}(w), \tag{6.20}
\end{equation*}
$$

with $P_{n}(w)$ being the $n$-th Legendre polynomial. In the second step we average Eqn (6.20) over the distribution (6.19) of $w$. The final result $[5,10]$ becomes

$$
\begin{equation*}
\left\langle\left(\frac{I}{\alpha k a^{2}}\right)^{n}\right\rangle=P_{n-1}\left(w_{0}\right) \exp \{\operatorname{Dn}(n-1) x\} . \tag{6.21}
\end{equation*}
$$

If the wave-beam parameters are adjusted to the waveguide [see Eqn (2.43)], then $w_{0}=1$ and formula (6.20) becomes

$$
\begin{equation*}
\left\langle I^{n}(x, 0)\right\rangle=\exp \{D n(n-1) x\} . \tag{6.22}
\end{equation*}
$$

The latter means that $I(x, 0)$ is distributed according to a lognormal law. However, the typical realization of process $I(x, 0)$ decays exponentially inside the medium:

$$
I^{*}(x, 0)=\exp \{-D x\} .
$$

So the radiation should spread in the cross-sectional directions (away from the axis) for specific realizations, which means dynamic localization in $x$. The typical realization gives the standard Gaussian cross-sectional intensity (2.45) modulated along the axis:

$$
I^{*}(x, \mathbf{R})=I^{*}(x, 0) \exp \left\{-\frac{\mathbf{R}^{2}}{a^{2}} I^{*}(x, 0)\right\} .
$$

## 7. Passive tracers in random velocity fields

Now we shall consider the statistical problem of passive tracer diffusion in the random velocity field introduced in Section 2.3.

We shall study general compressible velocity fields $(\operatorname{div} \mathbf{u}(\mathbf{r}, t) \neq 0)$, assumed to be Gaussian homogeneous, isotropic in space, stationary in time with correlationfunction and spectral tensor (assuming $\langle\mathbf{u}(\mathbf{r}, t)\rangle \equiv 0$ ) of the form

$$
\begin{align*}
& \left\langle\mathbf{u}_{i}(\mathbf{r}, t) u_{j}\left(\mathbf{r}^{\prime}, t^{\prime}\right)\right\rangle=B_{i j}\left(\mathbf{r}-\mathbf{r}^{\prime}, t-t^{\prime}\right) \\
& E_{i j}(\mathbf{k}, t)=\frac{1}{(2 \pi)^{N}} \int \mathrm{~d} \mathbf{r} B_{i j}(\mathbf{r}, t) \exp (-\mathrm{i} \mathbf{k}) \tag{7.1}
\end{align*}
$$

$N$ being the space dimension. The spectral tensor may be decomposed into the usual solenoidal $E^{s}(k, t)$ plus potential $E^{p}(k, t)$ components:

$$
\begin{equation*}
E_{i j}(\mathbf{k}, t)=E^{s}(k, t)\left(\delta_{i j}-\frac{k_{i} k_{j}}{k^{2}}\right)+E^{p}(k, t) \frac{k_{i} k_{j}}{k^{2}} . \tag{7.2}
\end{equation*}
$$

The practically important cases include
— incompressible fluid flow: $\operatorname{div} \mathbf{u}(\mathbf{r}, t)=0\left(E^{p}(k, t)=0\right)$;
— potential velocities: $\left(E^{s}(k, t)=0\right)$;

- the mixed case.

An example of the latter is a floating tracer [2, 22]. Indeed, if such a tracer moves over the surface $z=0$ of an incompressible 3D fluid, with horizontal and vertical velocity components ( $\operatorname{div} \mathbf{u}(\mathbf{r}, t)=0)$, then the surface elevation $\mathbf{u}=(\mathbf{U}, w)$ creates an effective compressible horizontal flow with the divergence $\nabla_{\mathbf{R}} \mathbf{U}(\mathbf{R}, t)=-\partial w /\left.\partial z\right|_{z=0}$.

We assume the entire 3D spectral tensor of $\mathbf{u}(\mathbf{r}, t)$ to be

$$
\begin{equation*}
E_{i j}(\mathbf{k}, t)=E(k, t)\left(\delta_{i j}-\frac{k_{i} k_{j}}{k^{2}}\right) \tag{7.3}
\end{equation*}
$$

and write the tracer density as

$$
\rho(\mathbf{r}, t)=\rho(\mathbf{R}, t) \delta(z), \quad \mathbf{r}=(\mathbf{R}, z), \quad \mathbf{R}=(x, y) .
$$

After substitution into Eqn (2.9) and integration over the variable $z$ the reduced 2D system would evolve according to

$$
\begin{equation*}
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \mathbf{R}} \mathbf{U}(\mathbf{R}, t)\right) \rho(\mathbf{R}, t)=0, \quad \rho(\mathbf{R}, 0)=\rho_{0}(\mathbf{R}) \tag{7.4}
\end{equation*}
$$

The resulting horizontal (compressible) field $\mathbf{U}(\mathbf{R}, t)$ is clearly Gaussian, homogeneous, and isotropic with spectral tensor

$$
\begin{gather*}
E_{\alpha \beta}\left(\mathbf{k}_{\perp}, t\right)=\int_{-\infty}^{\infty} \mathrm{d} k_{z} E\left(\mathbf{k}_{\perp}^{2}+k_{z}^{2}, t\right)\left(\delta_{\alpha \beta}-\frac{k_{\perp \perp} k_{\perp \beta}}{\mathbf{k}_{\perp}^{2}+k_{z}^{2}}\right), \\
\alpha, \beta=1,2 \tag{7.5}
\end{gather*}
$$

expressed in terms of the horizontal and vertical wavenumbers $\mathrm{k}=\left(\mathrm{k}_{\perp}, k_{z}\right)$.

Comparing Eqn (7.5) with Eqn (7.2), we get the solenoidal and potential velocity components on the $z=0$ plane [2]:

$$
\begin{align*}
& E^{s}\left(\mathbf{k}_{\perp}, t\right)=\int_{-\infty}^{\infty} \mathrm{d} k_{z} E\left(\mathbf{k}_{\perp}^{2}+k_{z}^{2}, t\right), \\
& E^{p}\left(\mathbf{k}_{\perp}, t\right)=\int_{-\infty}^{\infty} \mathrm{d} k_{z} E\left(\mathbf{k}_{\perp}^{2}+k_{z}^{2}, t\right) \frac{k_{z}^{2}}{\mathbf{k}_{\perp}^{2}+k_{z}^{2}} . \tag{7.6}
\end{align*}
$$

Let us now go back to the general case. In what follows we shall assume the velocity $\mathbf{u}(\mathbf{r}, t)$ to be delta-correlated in time, and approximate its space-time correlations

$$
\begin{equation*}
B_{i j}(\mathbf{r}, t)=2 B_{i j}^{\mathrm{eff}}(\mathbf{r}) \delta(t) \tag{7.7}
\end{equation*}
$$

with the effective correlation-tensor

$$
B_{i j}^{\text {eff }}(\mathbf{r})=\frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d} t B_{i j}(\mathbf{r}, t)=\int_{0}^{\infty} \mathrm{d} t B_{i j}(\mathbf{r}, t)
$$

The homogeneity and isotropy of $\mathbf{u}$ yield the following standard relations for the matrix-function $B^{\text {eff }}$, along with its first, second and fourth derivatives:

$$
\begin{align*}
B_{k l}^{\text {eff }}(0)=\frac{D_{0}}{N} \delta_{k l}, & \frac{\partial}{\partial r_{i}} B_{k l}^{\text {eff }}(0)=0, \frac{\partial^{4} B_{k l}^{\text {eff }}(0)}{\partial r_{i} \partial r_{k} \partial r_{j} \partial r_{l}}=\frac{D_{4}^{p}}{N} \delta_{i j}, \\
-\frac{\partial^{2} B_{k l}^{\text {eff }}(0)}{\partial r_{i} \partial r_{j}} & =\frac{D_{2}^{s}}{N(N+2)}\left[(N+1) \delta_{k l} \delta_{i j}-\delta_{k i} \delta_{l j}-\delta_{k j} \delta_{l i}\right] \\
& +\frac{D_{2}^{p}}{N(N+2)}\left[\delta_{k l} \delta_{i j}+\delta_{k i} \delta_{l j}+\delta_{k j} \delta_{l i}\right] . \tag{7.8}
\end{align*}
$$

Here we have used the standard convention of summing over the repeated indices.

We shall also define the effective spectral densities $\left[\widetilde{E}^{s}(k)=\int_{0}^{\infty} \mathrm{d} t E^{s}(k, t), \quad \widetilde{E}^{p}(k)=\int_{0}^{\infty} \mathrm{d} t E^{p}(k, t)\right]$ of the solenoidal and potential components, and introduce the following 'diffusion coefficients':

$$
\begin{align*}
& D_{0}=\int \mathrm{d} \mathbf{k}\left[(N-1) \widetilde{E}^{s}(k)+\widetilde{E}^{p}(k)\right], \quad D_{2}^{s}=\int \mathrm{d} \mathbf{k} k^{2} \widetilde{E}^{s}(k), \\
& D_{2}^{p}=\int \mathrm{d} \mathbf{k} k^{2} \widetilde{E}^{p}(k), \quad D_{4}^{p}=\int \mathrm{d} \mathbf{k} k^{4} \widetilde{E}^{p}(k) . \tag{7.9}
\end{align*}
$$

For the sake of presentation we shall confine ourselves to 2D motion, i.e. consider equation (7.4).

As we mentioned in Section 2.3, first order PDEs of type (2.30) are solved by the method of characteristics. Introducing characteristic rays $\mathbf{R}(t)$ (2.30) ('particle trajectories'), we
can rewrite this in the form (2.31) which in our case turns into a differential system

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} t} \mathbf{R}(t \mid \boldsymbol{\xi}) & =\mathbf{U}(\mathbf{R}, t), \quad \mathbf{R}(0 \mid \xi)=\xi \\
\frac{\mathrm{d}}{\mathrm{~d} t} \rho(t \mid \xi) & =-\frac{\partial \mathbf{U}(\mathbf{R}, t)}{\partial \mathbf{R}} \rho(t \mid \xi), \quad \rho(0 \mid \xi)=\rho_{0}(\xi) \tag{7.10}
\end{align*}
$$

The above equations describe the Lagrangian transport of tracer particles. When supplemented by the evolution of the Jacobian divergence,

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} j(t \mid \boldsymbol{\xi})=\frac{\partial \mathbf{U}(\mathbf{R}, t)}{\partial \mathbf{R}} j(t \mid \xi), \quad j(0 \mid \boldsymbol{\xi})=1, \tag{7.11}
\end{equation*}
$$

they yield the solution

$$
\begin{equation*}
\rho(t \mid \xi) \equiv \frac{\rho_{0}(\xi)}{j(t \mid \xi)} . \tag{7.12}
\end{equation*}
$$

### 7.1 Lagrangian description

The Lagrangian indicator function

$$
\Phi_{\text {Lag }}(t ; \mathbf{R}, \rho, j \mid \xi)=\delta(\mathbf{R}(t \mid \xi)-\mathbf{R}) \delta(\rho(t \mid \xi)-\rho) \delta(j(t \mid \xi)-j)
$$

satisfies the Liouville equation

$$
\begin{gather*}
\frac{\partial}{\partial t} \Phi_{\mathrm{Lag}}(t ; \mathbf{R}, \rho, j \mid \xi)=\left[-\frac{\partial}{\partial \mathbf{R}} \mathbf{U}(\mathbf{R}, t)+\frac{\partial \mathbf{U}(\mathbf{R}, t)}{\partial \mathbf{R}}\right. \\
\left.\times\left(\frac{\partial}{\partial \rho} \rho-\frac{\partial}{\partial j} j\right)\right] \Phi_{\mathrm{Lag}}(t ; \mathbf{R}, \rho, j \mid \xi)  \tag{7.13}\\
\Phi_{\mathrm{Lag}}(0 ; \mathbf{R}, \rho, j \mid \xi)=\delta(\xi-\mathbf{R}) \delta\left(\rho_{0}(\xi)-\rho\right) \delta(j-1)
\end{gather*}
$$

We shall start our analysis with such important statistics as particle positions and density. Averaging Eqn (7.13) over the ensemble $\{\mathbf{U}\}$ of velocity realizations gives the FokkerPlanck equation for the one-point Lagrangian PDF $P(t ; \mathbf{R}, \rho, j \mid \xi)=\left\langle\Phi_{\text {Lag }}(t ; \mathbf{R}, \rho, j \mid \xi)\right\rangle($ see Refs $[2,23]):$

$$
\begin{aligned}
& \frac{\partial}{\partial t} P(t ; \mathbf{R}, \rho, j \mid \xi)=\left\{\frac{1}{2} D_{0} \frac{\partial^{2}}{\partial \mathbf{R}^{2}}\right. \\
& \left.\quad+D_{2}^{p}\left(\frac{\partial}{\partial \rho} \rho^{2} \frac{\partial}{\partial \rho}-2 \frac{\partial^{2}}{\partial \rho \partial j} \rho j+\frac{\partial^{2}}{\partial j^{2}} j^{2}\right)\right\} P(t ; \mathbf{R}, \rho, j \mid \xi)
\end{aligned}
$$

$$
\begin{equation*}
P(0 ; \mathbf{R}, \rho, j \mid \boldsymbol{\xi})=\delta(\boldsymbol{\xi}-\mathbf{R}) \delta\left(\rho_{0}(\boldsymbol{\xi})-\rho\right) \delta(j-1) . \tag{7.14}
\end{equation*}
$$

Here the initial state could be visualized as a highly localized tracer (near the source $\xi$ ) of the mean specific concentration $\rho$. Accordingly, the solution of Eqn (7.14) could be factored into the product of three terms:

$$
\begin{equation*}
P(t ; \mathbf{R}, \rho, j \mid \xi)=P(t ; \mathbf{R} \mid \xi) P(t ; j \mid \xi) \delta\left(\rho-\frac{\rho_{0}(\xi)}{j}\right) . \tag{7.15}
\end{equation*}
$$

The first is the standard Gaussian propagator in the variable $\mathbf{R}$, for a single randomly diffusing particle initiated at point $\mathbf{R}^{\prime}$ with the effective diffusivity $D_{0}$ :

$$
\begin{align*}
P\left(t ; \mathbf{R} \mid \mathbf{R}^{\prime}\right) & =\exp \left\{\frac{1}{2} D_{0} t \frac{\partial^{2}}{\partial \mathbf{R}^{2}}\right\} \delta\left(\mathbf{R}-\mathbf{R}^{\prime}\right) \\
& =\frac{1}{2 \pi D_{0} t} \exp \left\{-\frac{\left(\mathbf{R}-\mathbf{R}^{\prime}\right)^{2}}{2 D_{0} t}\right\} . \tag{7.16}
\end{align*}
$$

The second factor,

$$
\begin{align*}
P(t ; j \mid \boldsymbol{\xi}) & =\exp \left\{D_{2}^{p} t \frac{\partial^{2}}{\partial j^{2}} j^{2}\right\} \delta(j-1) \\
& =\frac{1}{2 j \sqrt{\pi \tau}} \exp \left\{-\frac{\ln ^{2}\left(j \mathrm{e}^{\tau}\right)}{4 \tau}\right\}, \tag{7.17}
\end{align*}
$$

describes a similar evolution of the Jacobian density (its PDF) generated by the differential operator $\partial^{2} /\left(\partial j^{2}\right) j^{2}$ on the halfline $0 \leqslant j<\infty$.

Here and henceforth we shall often use the dimensionless time $\tau=D_{2}^{p} t$.

The product form of solution (7.15) implies statistical independence of the particle's position $\mathbf{R}(t \mid \xi)$ and the Jacobian (flow divergence) $j(t \mid \xi)$. Furthermore, the Gaussian distribution (7.16) corresponds to standard Brownian motion with parameters

$$
\begin{align*}
& \langle\mathbf{R}(t \mid \xi)\rangle=\xi \\
& \sigma_{\alpha \beta}^{2}(t)=\left\langle\left[R_{\alpha}(t)-\xi_{\alpha}\right]\left[R_{\beta}(t)-\xi_{\beta}\right]\right\rangle=D_{0} \delta_{\alpha \beta} t, \tag{7.18}
\end{align*}
$$

whereas the Jacobian (divergence factor) $j$ has a log-normal distribution, with constant mean $\langle j(t \mid \xi)\rangle=1$ and exponentially increasing higher moments

$$
\begin{equation*}
\left\langle j^{n}(t \mid \xi)\right\rangle=\exp \{n(n-1) \tau\}, \quad n= \pm 1, \pm 2, \ldots \tag{7.19}
\end{equation*}
$$

The moments of the Lagrangian density $\rho$ have similar exponential growth due to Eqn (7.12):

$$
\begin{equation*}
\left\langle\rho^{n}(t \boldsymbol{\xi})\right\rangle=\rho_{0}^{n}(\xi) \exp \{n(n+1) \tau\} . \tag{7.20}
\end{equation*}
$$

So both the (Lagrangian) mean density $\langle\rho\rangle$ and its higher moments increase in time. Furthermore, the joint PDF of $\rho$ and $j$ has the form

$$
\begin{equation*}
P(t ; \rho, j \mid \xi)=P(t ; j \mid \xi) \delta\left(\rho-\frac{\rho_{0}(\xi)}{j}\right), \tag{7.21}
\end{equation*}
$$

with the log-normal $P(t ; j \mid \xi)$ of Eqn (7.17).
Integrating out variable $j$ from Eqn (7.21), we get the PDF of the Lagrangian density alone:

$$
\begin{equation*}
P(t ; \rho \mid \xi)=\frac{1}{2 \rho \sqrt{\pi \tau}} \exp \left\{-\frac{\ln ^{2}\left(\rho \mathrm{e}^{-\tau} / \rho_{0}(\xi)\right)}{4 \tau}\right\} . \tag{7.22}
\end{equation*}
$$

The same solution could also be obtained by direct reduction of the general Fokker - Planck equation (7.14):

$$
\begin{align*}
& \frac{\partial}{\partial t} P(t ; \rho \mid \xi)=D_{2}^{p} \frac{\partial}{\partial \rho} \rho^{2} \frac{\partial}{\partial \rho} P(t ; \mathbf{R}, \rho \mid \xi), \\
& P(0 ; \rho \mid \xi)=\delta\left(\rho_{0}(\xi)-\rho\right) \tag{7.23}
\end{align*}
$$

Some unexpected statistical features of the tracer and Jacobian densities, like the exponential increase of their moments, etc., find a simple explanation in terms of the lognormality. Thus the 'typical realization' of random Jacobian density is given by an exponential curve

$$
\begin{equation*}
j^{*}(t)=\exp (-\tau) . \tag{7.24}
\end{equation*}
$$

Furthermore, the random process $j(t \mid \xi)$ admits majorant (envelope) estimates, like

$$
j(t \mid \xi)<4 \exp \left(-\frac{\tau}{2}\right)
$$

holding for all $t \in(0, \infty)$ with probability $1 / 2$. As for the tracer density it has an increasing 'typical median', and admits minorant estimates from below:

$$
\rho^{*}(t)=\rho_{0} \mathrm{e}^{\tau}, \quad \rho(t \mid \xi)>\frac{\rho_{0}}{4} \exp \left(\frac{\tau}{2}\right) .
$$

Let us notice the marked differences of Lagrangian statistics in the compressible and incompressible cases. The latter have $j(t \mid \xi) \equiv 1$, so the density remains constant along (characteristic) particle trajectories $\rho(t \mid \boldsymbol{\xi})=\rho_{0}(\xi)=$ const.

The above results confirm our main thesis on the dominant role of large fluctuations from the 'typical behavior', that shape the essential statistics of random processes $j(t \mid \xi)$ and $\rho(t \mid \xi)$. As for the random 'particle position' process neither case (compressible or incompressible) shows much difference.

Next we shall study the relative displacement of two particles. The separation-function $\quad \mathbf{I}(t)=\mathbf{R}_{1}(t)-\mathbf{R}_{2}(t)$ obeys the dynamic evolution

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \mathbf{I}(t)=\mathbf{U}\left(\mathbf{R}_{1}, t\right)-\mathbf{U}\left(\mathbf{R}_{2}, t\right), \quad \mathbf{I}(0)=\mathbf{I}_{0}
$$

and the corresponding Fokker-Planck equation
$\frac{\partial}{\partial t} P(t ; \mathbf{I})=\frac{\partial^{2}}{\partial l_{\alpha} \partial l_{\beta}} D_{\alpha \beta}(\mathbf{I}) P(t ; \mathbf{I}), \quad P(0 ; \mathbf{I})=\delta\left(\mathbf{I}-\mathbf{I}_{0}\right)$.
The diffusion tensor $D_{\alpha \beta}(\mathbf{I})=2\left[B_{\alpha \beta}^{\mathrm{eff}}(0)-B_{\alpha \beta}^{\mathrm{eff}}(\mathbf{I})\right]$ involves the structure matrix of the vector field $\mathbf{U}(\mathbf{R}, t)$ at two different points. In general equation (7.25) has no closed form solution.

However, for a short initial displacement (relative to the correlation radius of $l_{0} \ll l_{\text {cor }}$ ) the coefficients $D_{\alpha \beta}(\mathbf{I})$ could be expanded in a Taylor series in $\mathbf{I}$, to get the first order approximation

$$
D_{\alpha \beta}(\mathbf{I})=-\left.\frac{\partial^{2} B_{\alpha \beta}^{\mathrm{eff}}(\mathbf{I})}{\partial l_{\gamma} \partial l_{\delta}}\right|_{\mathbf{I}=0} l_{\gamma} l_{\delta} .
$$

The corresponding diffusion tensor $D_{\alpha \beta}(\mathbf{I})$ could be further simplified using the symmetries of the problem (homogeneity and isotropy), that are expressed through the effective diffusion rates (7.8):

$$
\begin{equation*}
D_{\alpha \beta}(\mathbf{I})=\frac{1}{8}\left[3 D_{2}^{s}+D_{2}^{p}\right] \mathbf{I}^{2} \delta_{\alpha \beta}-\frac{1}{4}\left[D_{2}^{s}-D_{2}^{p}\right] l_{\alpha} l_{\beta} \tag{7.26}
\end{equation*}
$$

To get a closed form equation for the moments of $l$, we substitute Eqn (7.26) into Eqn (7.25), multiply by $l^{n}$ and integrate over $l$. This yields a simple linear evolution [2],

$$
\frac{\mathrm{d}}{\mathrm{~d} t}\left\langle l^{n}(t)\right\rangle=\frac{1}{8} n\left[n\left(D_{2}^{s}+3 D_{2}^{p}\right)+2\left(D_{2}^{s}-D_{2}^{p}\right)\right]\left\langle l^{n}(t)\right\rangle,
$$

whose solutions grow exponentially in time for all $(n=1,2, \ldots)$. Furthermore, the random process $l(t) / l_{0}$ has log-normal PDF with parameters

$$
l^{*}(t)=l_{0} \exp \left\{\frac{1}{4}\left(D_{2}^{s}-D_{2}^{p}\right) t\right\} .
$$

Hence, a typical realization of the 'particle-separation' grows or decreases exponentially in time, depending on the sign of $\left(D_{2}^{s}-D_{2}^{p}\right)$. In particular, incompressible flows $\left(D_{2}^{p}=0\right)$ have
exponentially increasing typical realizations, which means an exponential divergence of particles at short distances and times. This result holds as long as

$$
\frac{1}{4} D_{2}^{s} t \ll \ln \left(\frac{l_{\mathrm{cor}}}{l_{0}}\right),
$$

i.e. the Taylor expansion (7.26) remains valid.

At the opposite end stand pure potential velocities $\left(D_{2}^{s}=0\right)$. Here a typical realization of particle-separation would exponentially decrease, so the particles tend to coalesce. Such a tendency of the flow to 'bring particles together' could lead to the formation of clusters - regions of high tracer concentration interspersed within large voids. Indeed, our conclusion is consistent with some numeric studies (illustrated in Fig. 1b), although our model of random velocities is different from those used in computations.

This suggests that clustering is largely insensitive to the specifics of the model, which qualifies it as a coherent phenomenon.

Let us remark that the 'Eulerian clustering' (to be discussed in the next section) is possible even in the case $\left(D_{2}^{s}-D_{2}^{p}\right)>0$, as long as velocities have a potential component.

### 7.2 Eulerian description

In the Eulerian form of turbulent transport we take the indicator function $\Phi_{\text {Eul }}(t, \mathbf{R} ; \rho)=\delta(\rho(t, \mathbf{R})-\rho)$, that obeys the Liouville equation (3.11):

$$
\begin{align*}
& \begin{array}{l}
\left.\frac{\partial}{\partial t}+\mathbf{U}(\mathbf{R}, t) \frac{\partial}{\partial \mathbf{R}}\right) \Phi_{\mathrm{Eul}}(t, \mathbf{R} ; \rho) \\
\quad=\frac{\partial \mathbf{U}(\mathbf{R}, t)}{\partial R} \frac{\partial}{\partial \rho} \rho \Phi_{\mathrm{Eul}}(t, \mathbf{R} ; \rho), \\
\Phi_{\mathrm{Eul}}(0, \mathbf{R} ; \rho)=\delta\left(\rho_{0}(\mathbf{R})-\rho\right) .
\end{array} .
\end{align*}
$$

Then we get the PDF for the Eulerian tracer-field,

$$
P(t, \mathbf{R} ; \rho)=\left\langle\Phi_{\mathrm{Eul}}(t, \mathbf{R} ; \rho)\right\rangle,
$$

by averaging Eqn (7.27) over the ensemble, or alternatively using Eqn (3.10), as in Refs [2, 22]. The resulting FP equation takes the form

$$
\begin{aligned}
& \left(\frac{\partial}{\partial t}-\frac{1}{2} D_{0} \frac{\partial^{2}}{\partial \mathbf{R}^{2}}\right) P(t, \mathbf{R} ; \rho)=D_{2}^{p} \frac{\partial^{2}}{\partial \rho^{2}} \rho^{2} P(t, \mathbf{R} ; \rho), \\
& P(0, \mathbf{R} ; \rho)=\delta\left(\rho_{0}(\mathbf{R})-\rho\right) .
\end{aligned}
$$

The equation for moments follows directly from Eqn (7.28):

$$
\begin{align*}
\left(\frac{\partial}{\partial t}-\frac{1}{2} D_{0} \frac{\partial^{2}}{\partial \mathbf{R}^{2}}\right)\left\langle\rho^{n}(\mathbf{R}, t)\right\rangle & =D_{2}^{p} n(n-1)\left\langle\rho^{n}(\mathbf{R}, t)\right\rangle, \\
\left\langle\rho^{n}(\mathbf{R}, 0)\right\rangle & =\rho_{0}^{n}(\mathbf{R}) . \tag{7.29}
\end{align*}
$$

In particular, the mean tracer density obeys the evolution

$$
\begin{equation*}
\frac{\partial}{\partial t}\langle\rho(\mathbf{R}, t)\rangle=\frac{1}{2} D_{0} \frac{\partial^{2}}{\partial \mathbf{R}^{2}}\langle\rho(\mathbf{R}, t)\rangle \tag{7.30}
\end{equation*}
$$

consistent with that of a single-particle PDF.

Notice however, that the 'mean' diffusion coefficient $D_{0}$ (7.14) gives only global (large-scale) characteristics of the tracer distribution, and carries little information about the fine (local) structure and details of realizations.

Solutions of the moment equations (7.29) are given by the convolution with the standard Gaussian propagator,

$$
\begin{equation*}
\left\langle\rho^{n}(\mathbf{R}, t)\right\rangle=\exp \{n(n-1) \tau\} \int \mathrm{d} \mathbf{R}^{\prime} P\left(t ; \mathbf{R} \mid \mathbf{R}^{\prime}\right) \rho_{0}^{n}\left(\mathbf{R}^{\prime}\right) \tag{7.31}
\end{equation*}
$$

whereas the FP equation (7.28) combines the Gaussian diffusion in variable $\mathbf{R}$ with 'log-normal diffusion' in $\rho \in(0, \infty)$. In particular, the initial uniform density, $\rho_{0}(\mathbf{R})=\rho_{0}=$ const, yields a log-normal PDF $P(t ; \rho)$, independent of $\mathbf{R}$, along with its integrated probability distribution

$$
\begin{align*}
& P(t ; \rho)=\frac{1}{2 \rho \sqrt{\pi \tau}} \exp \left\{-\frac{\ln ^{2}\left(\rho \mathrm{e}^{\tau} / \rho_{0}\right)}{4 \tau}\right\}, \\
& F(t ; \rho)=\Phi\left(\frac{\ln \left(\rho \mathrm{e}^{\tau} / \rho_{0}\right)}{2 \sqrt{\tau}}\right) . \tag{7.32}
\end{align*}
$$

The corresponding mean-field and moments grow in $\tau$ as

$$
\begin{equation*}
\langle\rho(\mathbf{R}, t)\rangle=\rho_{0}, \quad\left\langle\rho^{n}(\mathbf{R}, t)\right\rangle=\rho_{0}^{n} \exp \{n(n-1) \tau\}, \tag{7.33}
\end{equation*}
$$

whereas a typical realization falls off exponentially

$$
\begin{equation*}
\rho^{*}(t)=\rho_{0} \exp (-\tau) \tag{7.34}
\end{equation*}
$$

This shows that the Eulerian statistics are also caused by large fluctuations about a typical realization (7.34), and suggests the clustering of tracer density for compressible flows.

So far we have studied the one-point PDF of the tracer concentration $\rho$, and deduced some of its spatial-temporal properties. There are a few other fine-scale structures of realizations that could be gleaned from the FP equation (7.34) and its PDF-solutions.

In Section 3.3 we mentioned some important geometric characteristics of $\rho$ related to iso-contours

$$
\rho(\mathbf{R}, t)=\rho=\mathrm{const}
$$

and the corresponding statistical means expressed through its PDF (7.28). Examples include the mean area enclosed by contours $\rho(\mathbf{R}, t)>\rho$, and the mean 'enclosed mass':

$$
\begin{align*}
& \langle S(t, \rho)\rangle=\int_{\rho}^{\infty} \mathrm{d} \tilde{\rho} \int \mathrm{~d} \mathbf{R} P(t, \mathbf{R} ; \tilde{\rho}), \\
& \langle M(t, \rho)\rangle=\int_{\rho}^{\infty} \tilde{\rho} \mathrm{d} \tilde{\rho} \int \mathrm{~d} \mathbf{R} P(t, \mathbf{R} ; \tilde{\rho}) . \tag{7.35}
\end{align*}
$$

One could easily derive the closed form equations for both quantities, and find exact solutions [2]:

$$
\begin{align*}
& \langle S(t, \rho)\rangle=\int \mathrm{d} \mathbf{R} \Phi\left(\frac{\ln \left(\rho_{0}(\mathbf{R}) \mathrm{e}^{-\tau} / \rho\right)}{2 \sqrt{\tau}}\right), \\
& \langle M(t, \rho)\rangle=\int \rho_{0}(\mathbf{R}) \mathrm{d} \mathbf{R} \Phi\left(\frac{\ln \left(\rho_{0}(\mathbf{R}) \mathrm{e}^{\tau} / \rho\right)}{2 \sqrt{\tau}}\right) . \tag{7.36}
\end{align*}
$$

At large time $\tau \gg 1$ the total area of high density concentration (above $\rho$ ) decreases in time according to

$$
\begin{equation*}
\langle S(t, \rho)\rangle=\frac{1}{\sqrt{\pi \tau \rho}} \exp \left(-\frac{\tau}{4}\right) \int \sqrt{\rho_{0}(\mathbf{R})} \mathrm{d} \mathbf{R} \tag{7.37}
\end{equation*}
$$

whereas the enclosed mass within the $\rho$-area,

$$
\begin{equation*}
\langle M(t, \rho)\rangle=M-\sqrt{\frac{\rho}{\pi \tau}} \exp \left(-\frac{\tau}{4}\right) \int \sqrt{\rho_{0}(\mathbf{R})} \mathrm{d} \mathbf{R} \tag{7.38}
\end{equation*}
$$

converges monotonically to the total mass of the system

$$
M=\int \rho_{0}(\mathbf{R}) \mathrm{d} \mathbf{R}
$$

The last result confirms our earlier conclusion regarding the clustering of tracers in the tightly bounded regions of high density.

Let us illustrate a few dynamic features of the clustering process with an example of a uniformly distributed density $\rho_{0}(\mathbf{R})=\rho_{0}=$ const. In this case specific area (per unit ' 2 D volume') with concentration $\rho(\mathbf{R}, t)>\rho$ is equal to

$$
\begin{equation*}
s(t, \rho)=\int_{\rho}^{\infty} P(t ; \tilde{\rho}) \mathrm{d} \tilde{\rho}=\Phi\left(\frac{\ln \left(\rho_{0} \mathrm{e}^{-\tau} / \rho\right)}{2 \sqrt{\tau}}\right), \tag{7.39}
\end{equation*}
$$

where $P(t ; \rho)$ is $\mathbf{R}$-independent solution of $\operatorname{Eqn}$ (7.28) given by Eqn (7.32). At the same time, the specific mass (per unit ' 2 D volume') enclosed in such regions is

$$
\begin{equation*}
m(t, \rho)=\frac{1}{\rho_{0}} \int_{\rho}^{\infty} \tilde{\rho} P(t ; \tilde{\rho}) \mathrm{d} \tilde{\rho}=\Phi\left(\frac{\ln \left(\rho_{0} \mathrm{e}^{\tau} / \rho\right)}{2 \sqrt{\tau}}\right) . \tag{7.40}
\end{equation*}
$$

In follows from Eqns (7.39) and (7.40) that the specific area drops off exponentially,

$$
\begin{equation*}
s\left(t, \rho_{0}\right)=\Phi\left(-\frac{\sqrt{\tau}}{2}\right) \approx \frac{1}{\sqrt{\pi \tau}} \exp \left(-\frac{\tau}{4}\right) \tag{7.41}
\end{equation*}
$$

whereas the tracers' mass aggregates (almost entirely) when $\tau \rightarrow \infty$ :

$$
\begin{equation*}
m\left(t, \rho_{0}\right)=\Phi\left(\frac{\sqrt{\tau}}{2}\right) \approx 1-\frac{1}{\sqrt{\pi \tau}} \exp \left(-\frac{\tau}{4}\right) \tag{7.42}
\end{equation*}
$$

Furthermore, their large-time asymptotics are independent of the ratio $\rho / \rho_{0}$.

However, the evolution leading to the asymptotic clustered state crucially depends on this ratio $\rho / \rho_{0}$. If we take a low 'test-level' $\rho / \rho_{0}<1$, which corresponds to the initial values of specific quantities, $s(0, \rho)=1$ and $m(0, \rho)=1$, and let the system evolve, we first see small regions forming, where concentration drops below $\rho(\mathbf{R}, t)<\rho$. Initially they cover a small area, but as time goes on those regions rapidly grow in size, while the enclosed mass 'leaks away' to the outer (clustered) regions, eventually reaching the asymptotic state (7.41), (7.42). At some particular moment $\tau^{*}=\ln \left(\rho_{0} / \rho\right)$ the specific area drops to $s\left(\tau^{*}, \rho\right)=1 / 2$.

In the opposite case $\rho / \rho_{0}>1$ (high test-level $\rho$ ), the initial values are $s(0, \rho)=0$ and $m(0, \rho)=0$. Here the evolution first creates a few cluster regions of high concentration, $\rho(\mathbf{R}, t)>\rho$. These continue 'sucking in' a sizable portion of the tracer mass, as their area contracts, but the enclosed mass grows, and gradually passes to the same asymptotic state (7.41), (7.42) (see Fig. 13).

As was pointed out in Section 2.3 to get further (fine scale) geometric structure of the tracer concentration we need to consider its gradient $\mathbf{p}(\mathbf{R}, t)=\nabla \rho(\mathbf{R}, t)$ and higher derivatives. The tracer gradient evolves according to Eqn (2.36),


Figure 13. Dynamics of cluster formation for $\rho / \rho_{0}=1.5$ (a), and $\rho / \rho_{0}=10$ (b).
hence its indicator function

$$
\Phi(t, \mathbf{R} ; \rho, \mathbf{p})=\delta(\rho(\mathbf{R}, t)-\rho) \delta(\mathbf{p}(\mathbf{R}, t)-\mathbf{p})
$$

satisfies equation (3.18), where $\mathbf{r}$ has been replaced by $\mathbf{R}$. Averaging Eqn (3.18) over the velocity ensemble we get the FP-equation for the joint PDF of the tracer concentration and its gradient $P(t, \mathbf{R} ; \rho, \mathbf{p})=\langle\Phi(t, \mathbf{R} ; \rho, \mathbf{p})\rangle$. Namely,

$$
\begin{align*}
\frac{\partial}{\partial t} P(t, \mathbf{R} ; \rho, \mathbf{p}) & =\left[\frac{1}{2} D_{0} \frac{\partial^{2}}{\partial \mathbf{R}^{2}}+D_{2}^{p} \frac{\partial^{2}}{\partial \mathbf{R} \partial \mathbf{p}} \rho+D_{2}^{p} \frac{\partial^{2}}{\partial \rho^{2}} \rho^{2}\right. \\
+ & \frac{1}{8} D_{2}^{s} \widehat{L}^{s}(\mathbf{p})+\frac{1}{8} D_{2}^{p} \widehat{L}^{p}(\mathbf{p})+3 D_{2}^{p} \frac{\partial}{\partial \mathbf{p}} \mathbf{p} \frac{\partial}{\partial \rho} \rho \\
+ & \left.\frac{1}{2} D_{4}^{p} \frac{\partial^{2}}{\partial \mathbf{p}^{2}} \rho^{2}\right] P(t, \mathbf{R} ; \rho, \mathbf{p}) \\
P(0, \mathbf{R} ; \rho, \mathbf{p}) & =\delta\left(\rho_{0}(\mathbf{R})-\rho\right) \delta(\mathbf{p}(\mathbf{R})-\mathbf{p}) \tag{7.43}
\end{align*}
$$

Here we have introduced differential operators in the 'gradient variable' p,
$\widehat{L}^{s}(\mathbf{p})=3 \frac{\partial^{2}}{\partial \mathbf{p}^{2}} \mathbf{p}^{2}-2 \frac{\partial}{\partial \mathbf{p}} \mathbf{p}-2\left(\frac{\partial}{\partial \mathbf{p}} \mathbf{p}\right)^{2}=3 \frac{\partial^{2}}{\partial \mathbf{p}^{2}} \mathbf{p}^{2}-2 \frac{\partial^{2}}{\partial p_{k} p_{l}} p_{k} p_{l}$,
$\widehat{L}^{p}(\mathbf{p})=\frac{\partial^{2}}{\partial \mathbf{p}^{2}} \mathbf{p}^{2}+18\left(\frac{\partial}{\partial \mathbf{p}} \mathbf{p}\right)^{2}+10 \frac{\partial}{\partial \mathbf{p}} \mathbf{p}$,
which represent the contributions of the solenoidal and potential components (superscripts $s$ and $p$ ).

In general, equation (7.43) has no closed-form solution. In the case of divergence-free velocities it could be reduced to [21-23]

$$
\begin{equation*}
\frac{\partial}{\partial t} P(t, \mathbf{R} ; \rho, \mathbf{p})=\left[\frac{1}{2} D_{0} \frac{\partial^{2}}{\partial \mathbf{R}^{2}}+\frac{1}{8} D_{2}^{s} \widehat{L}^{s}(\mathbf{p})\right] P(t, \mathbf{R} ; \rho, \mathbf{p}) \tag{7.45}
\end{equation*}
$$

One could show that evolution conserves the mean value of the gradient, while the gradient's norm has a log-normal PDF. So a typical realization as well as higher moments grow exponentially in time. Furthermore,

$$
\left\langle\mathbf{p}^{2}(\mathbf{R}, t)\right\rangle \sim \mathbf{p}_{0}^{2} \exp \left\{D_{2}^{s} t\right\}
$$

Besides, the average contour $\rho(\mathbf{R}, t)=\rho=$ const length also grows exponentially as

$$
\langle L(t, \rho)\rangle=l_{0} \exp \left\{D_{2}^{s} t\right\},
$$

starting from the initial length $l_{0}$, as a consequence of Eqns (7.45) and (3.16).

Let us remark that the divergent-free velocities conserve the number of contours, as those cannot appear or disappear, but only evolve in time from their initial value and distribution in space.

In the process of evolution, the initially smooth tracer distribution acquires an increasingly complicated spatial structure, its gradients grow and sharpen, and the contours undergo distortion and fractalization. We have shown the schematic view of the process in Fig. 1a, based on numeric simulations for a somewhat different velocity field. Once again we see, that the qualitative features of the process are insensitive to the specifics of the model.

We have studied the statistics of equation (7.4) in the Lagrangian and Eulerian formulation, and showed the clustering of the tracer in the presence of the potential velocity component, both on the level of particles and the (Eulerian) fields. Alongside the dynamic equation (7.4), one is sometimes interested in non-conservative tracer transport (see, for instance, Ref. [47])

$$
\left(\frac{\partial}{\partial t}+\mathbf{U}(\mathbf{R}, t) \frac{\partial}{\partial \mathbf{R}}\right) \rho(\mathbf{R}, t)=0, \quad \rho(\mathbf{R}, 0)=\rho_{0}(\mathbf{R})
$$

Equation (2.59) is of the same type. Since the Lagrangian equations for particles coincide with Eqn (7.10), the clustering occurs on the level of particles. However, continuous Eulerian fields do not cluster here. This case is similar to a divergentfree one in that it conserves the mean contour number, the mean area, and the tracer mass $\int \mathrm{d} S \rho(\mathbf{R}, t)$ enclosed inside the contours $\rho(\mathbf{R}, t)>\rho$.

### 7.3 Additional factors

So far we have studied the transport problem by random velocities in the absence of a mean flow and molecular diffusion. Furthermore, we have used delta-correlated approximations in time for the random velocities. While such simplifications are justifiable under certain conditions, those missing factors should be taken into account at some stage, as should spatial-temporal scales. Their incorporation brings about some new features and physical effects. In this section we shall briefly outline a few examples that include an additional factor for divergent-free velocities.
7.3.1 Linear shear flow. We shall consider velocity profiles that combine a linear (horizontal) shear

$$
v_{x}=\alpha y, \quad v_{y}=0
$$

with the random component [23]. In this case equation (7.45) is transformed into

$$
\begin{align*}
& \frac{\partial}{\partial t} P(t, \mathbf{R} ; \rho, \mathbf{p})=\left[-\alpha y \frac{\partial}{\partial x}+\frac{1}{2} D_{0} \frac{\partial^{2}}{\partial \mathbf{R}^{2}}\right] P(t, \mathbf{R} ; \rho, \mathbf{p}) \\
& \quad+\left\{\alpha p_{x} \frac{\partial}{\partial p_{y}}+\frac{1}{8} D_{2}^{s}\left[3 \frac{\partial^{2}}{\partial \mathbf{p}^{2}} \mathbf{p}^{2}-2\left(\frac{\partial}{\partial \mathbf{p}} \mathbf{p}\right)^{2}-2 \frac{\partial}{\partial \mathbf{p}} \mathbf{p}\right]\right\} \\
& \quad \times P(t, \mathbf{R} ; \rho, \mathbf{p}) \tag{7.46}
\end{align*}
$$

To derive the first and second moments of $\mathbf{p}$ we observe that the ensemble means $\left\langle p_{i}\right\rangle,\left\langle p_{i} p_{j}\right\rangle$, etc. are computed from the one-point PDF (7.46) by integrating out variables $\mathbf{p}, \rho$. Here the mean gradient is no longer conserved, but solves the equation without the fluctuating velocity component, hence

$$
\left\langle p_{x}(t)\right\rangle=p_{x}(0), \quad\left\langle p_{y}(t)\right\rangle=p_{y}(0)-\alpha p_{x}(0) t .
$$

The second moments of gradient satisfy the following differential system:

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} t}\left\langle\mathbf{p}^{2}\right\rangle=D_{2}^{s}\left\langle\mathbf{p}^{2}\right\rangle-2 \alpha\left\langle p_{x} p_{y}\right\rangle, \\
& \frac{\mathrm{d}}{\mathrm{~d} t}\left\langle p_{x} p_{y}\right\rangle=-\frac{1}{2} D_{2}^{s}\left\langle p_{x} p_{y}\right\rangle-\alpha\left\langle p_{x}^{2}\right\rangle, \\
& \frac{\mathrm{d}}{\mathrm{~d} t}\left\langle p_{x}^{2}\right\rangle=\frac{3}{4} D_{2}^{s}\left\langle\mathbf{p}^{2}\right\rangle-\frac{1}{2}\left\langle p_{x}^{2}\right\rangle . \tag{7.47}
\end{align*}
$$

A linear ODE system (7.47) has exponential eigenmode solutions $\exp \{\lambda t\}$ with eigenvalues $\lambda$ determined by the characteristic equation

$$
\begin{equation*}
\left(\lambda+\frac{1}{2} D_{2}^{s}\right)^{2}\left(\lambda-D_{2}^{s}\right)=\frac{3}{2} \alpha^{2} D_{2}^{s}, \tag{7.48}
\end{equation*}
$$

whose roots strongly depend on the parameter $\alpha / D_{2}^{s}$.
For small $\alpha / D_{2}^{s} \ll 1$, the roots of Eqn (7.48) are approximately equal to

$$
\lambda_{1}=D_{2}^{s}+\frac{2 \alpha^{2}}{3 D_{2}^{s}}, \quad \lambda_{2}=-\frac{1}{2} D_{2}^{s}+i|\alpha|, \quad \lambda_{3}=-\frac{1}{2} D_{2}^{s}-i|\alpha| .
$$

Hence, on the time scale $D_{2}^{s} t \gg 2$ the solutions are completely determined by random factors. This means that the effects of fluctuating velocity components dominate the effect of a weak linear shear.

At the other extreme, $\alpha / D_{2}^{s} \ll 1$, equation (7.48) has the approximate roots

$$
\begin{aligned}
& \lambda_{1}=\left(\frac{3}{2} \alpha^{2} D_{2}^{s}\right)^{1 / 3}, \quad \lambda_{2}=\left(\frac{3}{2} \alpha^{2} D_{2}^{s}\right)^{1 / 3} \exp \left\{\mathrm{i} \frac{2}{3} \pi\right\}, \\
& \lambda_{3}=\left(\frac{3}{2} \alpha^{2} D_{2}^{s}\right)^{1 / 3} \exp \left\{-\mathrm{i} \frac{2}{3} \pi\right\} .
\end{aligned}
$$

Since $\lambda_{2}$ and $\lambda_{3}$ have negative real parts, we get an asymptotic solution to Eqn (7.47) of the form

$$
\left\langle\mathbf{p}^{2}(t)\right\rangle \sim \exp \left\{\left(\frac{3}{2} \alpha^{2} D_{2}^{s}\right)^{1 / 3} t\right\}
$$

valid on time scales $\left(3 / 2 \alpha^{2} D_{2}^{s}\right)^{1 / 3} t \gtrdot 1$. It shows that even small velocity fluctuations become significant for a large shear gradient.
7.3.2 Molecular diffusivity. As we mentioned earlier, random velocity fluctuations cause the initially smooth tracer density to develop small scale structures with steepening gradients. In a real physical situation the molecular diffusion flux would tend to smooth out small scales, so the above pure 'transportdynamics' could last only for a limited time.

From the mathematical standpoint molecular diffusion would turn the problem into a second order stochastic partial differential equation (2.37), whose one-point PDFs have no closed-form FP equation.

Here we shall confine ourselves to estimating the time scale of the initial ('explosive') phase, and follow the exposition of [23].

To this end we consider powers $\rho^{n}(\mathbf{R}, t), n=1,2, \ldots$ that obey a nonclosed system of equations that follow from Eqn (2.37):

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \mathbf{R}} \mathbf{U}(\mathbf{R}, t)\right) & \rho^{n}(\mathbf{R}, t)=\varkappa \Delta \rho^{n}(\mathbf{R}, t)- \\
& -\varkappa n(n-1) \rho^{n-2}(\mathbf{R}, t) \mathbf{p}^{2}(\mathbf{R}, t) .
\end{aligned}
$$

The ensemble average over random velocities yields (nonclosed) equations for statistical moments:

$$
\begin{align*}
\frac{\partial}{\partial t}\left\langle\rho^{n}(\mathbf{R}, t)\right\rangle & =\left(\frac{1}{2} D_{0}+x\right) \Delta\left\langle\rho^{n}(\mathbf{R}, t)\right\rangle- \\
& -\varkappa n(n-1)\left\langle\rho^{n-2}(\mathbf{R}, t) \mathbf{p}^{2}(\mathbf{R}, t)\right\rangle . \tag{7.49}
\end{align*}
$$

Assuming $x \ll D_{0}$ we could write Eqn (7.49) in the integral form

$$
\begin{align*}
\left\langle\rho^{n}(\mathbf{R}, t)\right\rangle & =\exp \left\{\frac{1}{2} D_{0} t \frac{\partial^{2}}{\partial \mathbf{R}^{2}}\right\} \rho_{0}^{n}(\mathbf{R}) \\
& -\varkappa n(n-1) \int_{0}^{t} \mathrm{~d} \tau \exp \left\{\frac{1}{2} D_{0}(t-\tau) \frac{\partial^{2}}{\partial \mathbf{R}^{2}}\right\} \\
& \times\left\langle\rho^{n-2}(\mathbf{R}, \tau) \mathbf{p}^{2}(\mathbf{R}, \tau)\right\rangle \tag{7.50}
\end{align*}
$$

To estimate the last term of Eqn (7.50) we exploit the FP equation (7.45) derived for zero molecular diffusivity. As a result we get an (approximate) closed-form equation for the moments $\left\langle\rho^{n-2} \mathbf{p}^{2}\right\rangle$, whose solution is

$$
\begin{equation*}
\left\langle\rho^{n-2}(\mathbf{R}, t) \mathbf{p}^{2}(\mathbf{R}, t)\right\rangle=\exp \left\{D_{2}^{s} t+\frac{1}{2} D_{0} t \frac{\partial^{2}}{\partial \mathbf{R}^{2}}\right\} \rho_{0}^{n-2}(\mathbf{R}) \mathbf{p}_{0}^{2}(\mathbf{R}) . \tag{7.51}
\end{equation*}
$$

Substitution of Eqn (7.51) into Eqn (7.50) yields the conditions where the last term on the RHS of Eqn (7.50) gives a negligible contribution to the evolution of $\left\langle\rho^{n}\right\rangle$, and could be dropped. They impose certain constraints on the spatial scale $\mathbf{R}_{0}$ of the initial tracer concentration $\rho_{0}$, and the time range. Namely, [23]

$$
D_{2}^{s} \mathbf{R}_{0}^{2} \gtrdot x n(n-1), \quad D_{2}^{s} t \ll \ln \frac{D_{2}^{s} \mathbf{R}_{0}^{2}}{\psi n^{2}} .
$$

More detailed analysis could be given in special cases, e.g. for constant mean gradient of $\rho$, see Ref. [55] This case corresponds to solving equations (2.29), (2.37) with the initial data

$$
\rho_{0}(\mathbf{R})=\mathbf{G R}, \quad \mathbf{p}_{0}(\mathbf{R})=\mathbf{G} .
$$

As above we shall confine ourselves to the 2D case. Breaking $\rho$ into its mean and fluctuating components,

$$
\rho(\mathbf{R}, t)=\mathbf{G R}+\tilde{\rho}(\mathbf{R}, t),
$$

we get the equation for $\tilde{\rho}(\mathbf{R}, t)$ :

$$
\begin{gather*}
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \mathbf{R}} \mathbf{U}(\mathbf{R}, t)\right) \tilde{\rho}(\mathbf{R}, t)=-\mathbf{G} \mathbf{U}(\mathbf{R}, t)+\varkappa \Delta \tilde{\rho}(\mathbf{R}, t) \\
\tilde{\rho}(\mathbf{R}, 0)=0 \tag{7.52}
\end{gather*}
$$

Unlike the previous problems, this one has a stationary (limiting) probability distribution as $t \rightarrow \infty$, both for the tracer and its gradient. Lately this problem has drawn considerable attention both from the theoretical and experimental standpoint, see papers [56-62]. Their authors have used computer simulations and phenomenology to derive among other things the 'long exponential tails' of the 'gradient's PDF', while paper [62] showed a 'slowly decaying tail' for $\rho$ itself in its limiting (stationary) state.

Coming back to Eqn (7.52) let us notice that the variance of the gradient fluctuations $\tilde{\mathbf{p}}(\mathbf{R}, t)=\nabla \tilde{\rho}(\mathbf{R}, t)$ satisfies

$$
\begin{equation*}
\lim _{t \rightarrow \infty}\left\langle\tilde{\mathbf{p}}^{2}(\mathbf{R}, t)\right\rangle=\frac{D_{0} \mathbf{G}^{2}}{2 \chi} \tag{7.53}
\end{equation*}
$$

whereas the variance of the tracer concentration is given by

$$
\begin{equation*}
\left\langle\tilde{\rho}^{2}(\mathbf{R}, t)\right\rangle=D_{0} \mathbf{G}^{2} \int_{0}^{t} \mathrm{~d} \tau\left\{1-\frac{2 \varkappa}{D_{0} \mathbf{G}^{2}}\left\langle\tilde{\mathbf{p}}^{2}(\mathbf{R}, \tau)\right\rangle\right\} \tag{7.54}
\end{equation*}
$$

Formulae (7.53), (7.54) allow one to estimate the relaxation time (to a stationary regime),

$$
D_{2}^{s} T_{0} \sim \ln \left[\frac{D_{0}+2 \chi}{2 \varkappa}\right],
$$

as well as the variance of the stationary (limiting) state:

$$
\lim _{t \rightarrow \infty}\left\langle\tilde{\rho}^{2}(\mathbf{R}, t)\right\rangle \sim \frac{D_{0}}{D_{2}^{s}} \mathbf{G}^{2} \ln \left[\frac{D_{0}+2 \chi}{2 \varkappa}\right] .
$$

Here the coefficients $D_{0} \sim \sigma_{u}^{2} t_{0}$ and $D_{0} / D_{2}^{s} \sim l_{0}^{2}$ are expressed through statistics of random velocities, namely its variance $\sigma_{u}^{2}$, and spatial and temporal correlation radii, $t_{0}$ and $l_{0}$. So we get the relaxation time $T_{0}$ logarithmically dependent on the molecular diffusivity $\chi$. As for the stationary variance of $\widetilde{\rho}$ we get an estimate

$$
\left\langle\tilde{\rho}^{2}\right\rangle \sim \mathbf{G}^{2} l_{0}^{2} \ln \left[\frac{\sigma_{u}^{2} t_{0}}{x}\right] \text { at } x \ll \sigma_{u}^{2} t_{0}
$$

provided $x \ll \sigma_{u}^{2} t_{0}$.
7.3.3 Velocity fluctuations with finite temporal correlation. One way to account for the finite temporal correlation radius was mentioned in Section 4.1, based on the diffusion approximation method. This method still requires some constraints on the correlation radius, though less restrictive than the 'delta-correlated' ones. One could also discover some new physical phenomena, due to the finiteness of temporal correlations. These would be illustrated by two problems: sedimentation [23, 63], and the floating tracer on a random surface $z(\mathbf{R}, t)$ with statistically independent 'driving velocities'.

The first problem has manifestly anisotropic diffusion tensor relative to the sedimentation direction. We shall denote the sedimentation velocity by $v$, and assume its fluctuating component to have the energy spectrum

$$
E(k, t)=E(k) \exp \left\{-\frac{|t|}{\tau_{0}}\right\}
$$

falling off exponentially in time with the correlation radius $\tau_{0}$. The limiting diffusion tensor, as $t \rightarrow \infty$, was computed in
$\operatorname{Refs}[23,63]$,

$$
\begin{aligned}
D_{i j}(v) & =\frac{1}{v} \int \mathrm{~d} \mathbf{k} E(k) \Delta_{i j}(\mathbf{k}) \frac{p}{k} \frac{1}{1+p^{2}(\mathbf{k} \mathbf{v})^{2} /\left(k^{2} v^{2}\right)}, \\
\Delta_{i j}(\mathbf{k}) & =\left(\delta_{i j}-\frac{k_{i} k_{j}}{k^{2}}\right),
\end{aligned}
$$

where $\cos \theta=\mathbf{k v} /(k v)$ and

$$
p(\mathbf{k}, \mathbf{v})=\frac{\mathbf{k v} \tau_{0}}{1+\varkappa k^{2} \tau_{0}}
$$

After we project tensor $D_{i j}(\mathbf{v})$ onto the vertical (sedimentation) direction marked by $\|$, and the transverse (perpendicular) plane $\perp$, the corresponding diffusion rates

$$
\begin{aligned}
& D_{\|}(v)=\frac{4 \pi}{v} \int_{0}^{\infty} k \mathrm{~d} k E(k) f_{\|}(k, v) \\
& D_{\perp}(v)=\frac{4 \pi}{v} \int_{0}^{\infty} k \mathrm{~d} k E(k) f_{\perp}(k, v)
\end{aligned}
$$

may be expressed through the pair of functions

$$
\begin{aligned}
& f_{\|}(k, v)=\left[\arctan (p)+\frac{1}{p}\left(\frac{1}{p} \arctan (p)-1\right)\right] \\
& f_{\perp}(k, v)=\left[\arctan (p)-\frac{1}{p}\left(\frac{1}{p} \arctan (p)-1\right)\right] .
\end{aligned}
$$

For small values $p$ (i.e. when $v<l_{0} / \tau_{0}$, where $l_{0}$ is the correlation length of the velocity field) both functions $f_{\|}(k, v)$ and $f_{\perp}(k, v)$ approach values close to $2 p / 3$, corresponding to isotropic diffusion unaffected by the sedimentation drift $v$. For large values of $p\left(v \tau_{0} \gg l_{0}\right)$ the strong anisotropy shows up $f_{\|}(k, v)=2 f_{\perp}(k, v) \cong \pi / 2$. Let us stress that the resulting anisotropy is due to the nonvanishing radius of temporal velocity correlations, and would disappear in the deltacorrelated case.

Let us turn to the second problem. Here the basic transport equation (7.4) has to be modified to include an additional random parameter $z$ :

$$
\begin{array}{r}
\frac{\partial}{\partial t} \rho(\mathbf{R}, t)+\frac{\partial}{\partial R_{\alpha}}\left\{U_{\alpha}(\mathbf{R}, z(\mathbf{R}, t) ; t) \rho(\mathbf{R}, t)\right\}=0, \\
\mathbf{R}=\{x, y\} \tag{7.55}
\end{array}
$$

We normalise the total mass:

$$
\int \mathrm{d} \mathbf{R} \rho(\mathbf{R}, t)=1
$$

First we shall average over random velocities $\mathbf{U}(\mathbf{R}, z ; t)$ and use the diffusion approximation. This yields the FP equation for the mean concentration,

$$
\begin{align*}
& \frac{\partial}{\partial t} \mathfrak{R}(\mathbf{R}, t)=\frac{\partial^{2}}{\partial R_{\alpha} \partial R_{\beta}} \int_{0}^{t} \mathrm{~d} t^{\prime} \int \mathrm{d} \mathbf{k}_{\perp} \\
& \times \int_{-\infty}^{\infty} \mathrm{d} k_{z} E_{\alpha \beta}\left(\mathbf{k}_{\perp}^{2}+k_{z}^{2}, t-t^{\prime}\right) \exp \left\{\mathrm{i} k_{z} Z\left(\mathbf{R}, t, t^{\prime}\right)\right\} \mathfrak{R}(\mathbf{R}, t) \\
& -\mathrm{i} \frac{\partial}{\partial R_{\alpha}} \int \mathrm{d} \mathbf{k}_{\perp} \int_{0}^{t} \mathrm{~d} t^{\prime} \int_{-\infty}^{\infty} k_{z} \mathrm{~d} k_{z} E_{\alpha \beta}\left(\mathbf{k}_{\perp}^{2}+k_{z}^{2}, t-t^{\prime}\right) \\
& \times \exp \left\{i k_{z} Z\left(\mathbf{R}, t, t^{\prime}\right)\right\} \frac{\partial(\mathbf{R}, t)}{\partial R_{\beta}} \mathfrak{R}(\mathbf{R}, t), \tag{7.56}
\end{align*}
$$

where the velocity spectral tensor $E_{\alpha, \beta}$ is a function of the horizontal and vertical wave-vectors,

$$
E_{\alpha \beta}\left(\mathbf{k}_{\perp}^{2}+k_{z}^{2}, t\right)=E\left(\mathbf{k}_{\perp}^{2}+k_{z}^{2}, t\right)\left(\delta_{\alpha \beta}-\frac{k_{\perp \alpha} k_{\perp \beta}}{\mathbf{k}_{\perp}^{2}+k_{z}^{2}}\right)
$$

while

$$
Z\left(\mathbf{R}, t, t^{\prime}\right)=z(\mathbf{R}, t)-z\left(\mathbf{R}, t^{\prime}\right), \quad \mathfrak{R}(\mathbf{R}, t)=\langle\rho(\mathbf{R}, t)\rangle_{u}
$$

measures surface variation at different times $t, t^{\prime}$. We view Eqn (7.3) as a stochastic equation in the random parameter $Z\left(\mathbf{R}, t, t^{\prime}\right)$.

The surface elevation $z$ is assumed to be a Gaussian field with the following mean and correlation functions:

$$
\langle z(\mathbf{R}, t)\rangle=0, \quad\left\langle z(\mathbf{R}, t) z\left(\mathbf{R}^{\prime}, t^{\prime}\right)\right\rangle=B_{z}\left(\mathbf{R}-\mathbf{R}^{\prime}, t-t^{\prime}\right) .
$$

Introducing the new function

$$
\begin{aligned}
& F\left(\mathbf{R} ; t, t^{\prime} ; k_{z}\right)=\exp \left\{k_{z}^{2} D_{z}\left(t-t^{\prime}\right)+\mathrm{i} k_{z} Z\left(\mathbf{R}, t, t^{\prime}\right)\right\} \mathfrak{R}(\mathbf{R}, t), \\
& \mathfrak{R}(\mathbf{R}, t)=\exp \left\{-k_{z}^{2} D_{z}\left(t-t^{\prime}\right)-\mathrm{i} k_{z} Z\left(\mathbf{R}, t, t^{\prime}\right)\right\} F\left(\mathbf{R} ; t, t^{\prime} ; k_{z}\right),
\end{aligned}
$$

where $D_{z}(t)=B_{z}(0,0)-B_{z}(0, t)$, we rewrite the FP equation (7.3) in the following the form:

$$
\begin{align*}
& \frac{\partial}{\partial t} \mathfrak{R}(\mathbf{R}, t)=\frac{\partial^{2}}{\partial R_{\alpha} \partial R_{\beta}} \int_{0}^{t} \mathrm{~d} t^{\prime} \int \mathrm{d} \mathbf{k}_{\perp} \int_{-\infty}^{\infty} \mathrm{d} k_{z} \\
& \times E_{\alpha \beta}\left(\mathbf{k}_{\perp}^{2}+k_{z}^{2}, t-t^{\prime}\right) \exp \left\{-k_{z}^{2} D_{z}\left(t-t^{\prime}\right)\right\} \\
& \times F\left(\mathbf{R} ; t, t^{\prime} ; k_{z}\right)-\mathrm{i} \frac{\partial}{\partial R_{\alpha}} \int_{0}^{t} \mathrm{~d} t^{\prime} \int \mathrm{d} \mathbf{k}_{\perp} \int_{-\infty}^{\infty} k_{z} \mathrm{~d} k_{z} \\
& \times E_{\alpha \beta}\left(\mathbf{k}_{\perp}^{2}+k_{z}^{2}, t-t^{\prime}\right) \exp \left\{\mathrm{i} k_{z} Z\left(\mathbf{R}, t, t^{\prime}\right)\right\} \frac{\partial z(\mathbf{R}, t)}{\partial R_{\beta}} \mathfrak{R}(\mathbf{R}, t) . \tag{7.57}
\end{align*}
$$

The field $\mathfrak{R}(\mathbf{R}, t)$ is a functional of the random surface elevation,

$$
\mathfrak{R}(\mathbf{R}, t)=\mathfrak{R}[\mathbf{R}, t ; z(\widetilde{\mathbf{R}}, \tilde{\tau})],
$$

and the mean value of the functional $F\left(\mathbf{R} ; t, t^{\prime} ; k_{z}\right)$ over the random field $z(\widetilde{\mathbf{R}}, \tilde{\tau})$ is computed via

$$
\begin{aligned}
& \left\langle F\left(\mathbf{R} ; t, t^{\prime} ; k_{z}\right)\right\rangle_{z}=\left\langle\operatorname { e x p } \left\{ k_{z}^{2} D_{z}\left(t-t^{\prime}\right)\right.\right. \\
& \left.\left.+\mathrm{i} k_{z} Z\left(\mathbf{R}, t, t^{\prime}\right)\right\} \mathfrak{R}(\mathbf{R}, t)\right\rangle_{z}=\langle\mathfrak{R}[\mathbf{R}, t ; z(\widetilde{\mathbf{R}}, \tilde{\tau}) \\
& \left.\left.+\mathrm{i} k_{z}\left\{B_{z}(\widetilde{\mathbf{R}}-\mathbf{R}, \tilde{\tau}-t)-B_{z}\left(\widetilde{\mathbf{R}}-\mathbf{R}, \tilde{\tau}-t^{\prime}\right)\right\}\right]\right\rangle_{z} .
\end{aligned}
$$

So it depends on the mean density subjected to a 'functional shift'.

We are interested in the dispersion tensor

$$
\Sigma_{\alpha \beta}(t)=\left\langle\int R_{\alpha} R_{\beta} \mathfrak{R}(\mathbf{R}, t) \mathrm{d} \mathbf{R}\right\rangle_{z}
$$

which is equal to $1 / 2 \sigma^{2}(t) \delta_{\alpha \beta}$ due to isotropy. It is connected with the mean value $\langle\mathfrak{R}(\mathbf{R}, t)\rangle$.

To this end we need to average equation (7.57). Dropping $\sigma_{z}^{4}$ and other terms of higher order, one could get the following expression for the diffusion coefficient:

$$
\begin{aligned}
D(t) & =\frac{\partial}{\partial t} \sigma^{2}(t)=2 \int_{0}^{t} \mathrm{~d} \tau \int \mathrm{~d} \mathbf{k}_{\perp} \int_{-\infty}^{\infty} \mathrm{d} k_{z} E_{\alpha \alpha}\left(\mathbf{k}_{\perp}^{2}+k_{z}^{2}, \tau\right) \\
& \times \exp \left\{-k_{z}^{2} D_{z}(\tau)\right\} \int \mathrm{d} \mathbf{R}\left\langle F\left(\mathbf{R} ; t, t-\tau ; k_{z}\right)\right\rangle_{z} .
\end{aligned}
$$

Thus the surface elevation has a double effect on the tracer diffusion. On the one hand it changes the effective (random) velocity spectrum, on the other hand it transforms the tracer density. Both effects are due to the nonvanishing temporal correlation radius. As for the delta-correlated case the tracer statistics become independent of the surface elevation and have the same form as for the tracer in an ideal fluid flow.

## 8. Caustic structure of wave-fields in random inhomogeneous media

We have studied wave propagation in random media in Section 2.4 using a parabolic approximation (2.38). From the mathematical stand-point this problem is similar to the passive tracer diffusion by potential velocity fields. We have discovered a fundamental property of potential flows - the clustering of tracers. For wave problems, clustering (of wave intensity) shows up as caustic structures brought about by random focusing and defocusing of wave fields by the media.

The basic techniques of the delta-correlated approximation, used in the previous section for the statistical analysis of 'cluster structures', is not applicable anymore. Indeed, the longitudinal correlation-radius (in the beam-direction) for the beam phase is now comparable to the beam track [5, 40, 41]. So the dynamic equation (2.39) for the wave intensity has little practical value for the analysis of random media.

However, the one-point PDF of the wave intensity, derived from Eqn (2.38), allows one to adopt some ideas of the statistical topography [24]. In this regard let us mention the pioneering papers [64, 65] (see also Ref. [66]), that applied the theory of large deviations to the analysis of wave propagation in turbulent media.

We consider an iso-contour $I(x, \mathbf{R})=I=$ const of constant intensity in the fixed cross-sectional plane $x=$ const, and its singular indicator

$$
\begin{equation*}
\Phi(x, \mathbf{R} ; I)=\delta(I(x, \mathbf{R})-I) \tag{8.1}
\end{equation*}
$$

as a 'functional of the media'. The ensemble-mean of Eqn (8.1) defines the one-point PDF

$$
\begin{equation*}
P(x, \mathbf{R} ; I)=\langle\Phi(x, \mathbf{R} ; I)\rangle=\langle\delta(I(x, \mathbf{R})-I\rangle . \tag{8.2}
\end{equation*}
$$

Other quantities could be expressed through Eqn (8.2), like the mean area of domains with $I(x, \mathbf{R})>I$,

$$
\begin{equation*}
\langle S(x, I)\rangle=\int_{I}^{\infty} \mathrm{d} \widetilde{I} \int \mathrm{~d} \mathbf{R} P(x, \mathbf{R} ; \widetilde{I}), \tag{8.3}
\end{equation*}
$$

and their mean field power

$$
\begin{equation*}
\langle E(x, I)\rangle=\int_{I}^{\infty} \widetilde{I} \mathrm{~d} \widetilde{I} \int \mathrm{~d} \mathbf{R} P(x, \mathbf{R} ; \widetilde{I}) \tag{8.4}
\end{equation*}
$$

Further information on the field intensity and its fine structure could be derived from the cross-sectional gradient $\mathbf{p}(x, \mathbf{R})=\nabla_{\mathbf{R}} I(x, \mathbf{R})$. For instance,

$$
\begin{equation*}
\langle L(x, I)\rangle=\int \mathrm{d} \mathbf{R} \int \mathrm{~d} \mathbf{p}|\mathbf{p}(x, \mathbf{R})| P(x, \mathbf{R} ; I, \mathbf{p}) \tag{8.5}
\end{equation*}
$$

where

$$
P(x, R ; I, \mathbf{p})=\langle\delta(I(x, \mathbf{R})-I) \delta(\mathbf{p}(x, \mathbf{R})-\mathbf{p})\rangle
$$

is the joint PDF of $I(x, \mathbf{R})$ and $\mathbf{p}(x, \mathbf{R})$, gives the mean contour length at level $I(x, \mathbf{R})=I=$ const.

The higher (second) derivatives of $I$ with the aid of formula

$$
\begin{align*}
\langle N(x, I)\rangle & =\left\langle N_{\text {in }}(x, I)\right\rangle-\left\langle N_{\text {out }}(x, I)\right\rangle \\
= & \frac{1}{2 \pi} \int \mathrm{~d} \mathbf{R}\langle\varkappa(x, \mathbf{R} ; I)| \mathbf{p}(x, \mathbf{R})|\delta(I(x, \mathbf{R})-I)\rangle \tag{8.6}
\end{align*}
$$

would yield an estimate (modulo non-closed contours) of the mean number of closed contours at level $I$. Here $N_{\text {in }}(x, I)$ and $N_{\text {out }}(x, I)$ count contours with inward- and outward-looking gradient $\mathbf{p}$, (i.e. 'mountain peaks' and closed 'valleys' of topography $I(x, \mathrm{R})$, truncated at the level $I)$, and $\varkappa$ designates the curvature of the contour:

$$
\begin{align*}
x(x, \mathbf{R} ; I) & =\left[-p_{y}^{2}(x, \mathbf{R}) \frac{\partial^{2} I(x, \mathbf{R})}{\partial z^{2}}-p_{z}^{2}(x, \mathbf{R}) \frac{\partial^{2} I(x, \mathbf{R})}{\partial y^{2}}\right. \\
& \left.+2 p_{y}(x, \mathbf{R}) p_{z}(x, \mathbf{R}) \frac{\partial^{2} I(x, \mathbf{R})}{\partial y \partial z}\right] p^{-3}(x, \mathbf{R}) \tag{8.7}
\end{align*}
$$

Here and henceforth we shall consider a plane incident wave, whose one-point PDFs will be independent of $\mathbf{R}$ due to the (transverse) spatial homogeneity. Thus the statistical averages (8.3)-(8.7) could be viewed as specific (per unit area) values of the relevant quantities.

The natural length scale in the cross-sectional plane $x=$ const independent of the media is the size of the first Fresnel zone, $L_{f}(x)=\sqrt{x / k}$, that measures the light-shadow diffraction region of a non-transparent screen (see for instance, Refs [40, 41]).

Then the mean specific values of the contour lengths and the number of contours are given by the following dimensionless expressions:

$$
\begin{equation*}
\langle l(x, I)\rangle=L_{f}(x)\langle | \mathbf{p}(x, \mathbf{R})|\delta(I(x, \mathbf{R})-I)\rangle \tag{8.8}
\end{equation*}
$$

$$
\begin{equation*}
\langle n(x, I)\rangle=\frac{1}{2 \pi} L_{f}^{2}(x)\langle\chi(x, \mathbf{R} ; I)| \mathbf{p}(x, \mathbf{R})|\delta(I(x, \mathbf{R})-I)\rangle . \tag{8.9}
\end{equation*}
$$

In fact, formula (8.9) gives the mean excess of contours with the opposite orientation of normal vectors within the first Fresnel zone.

As above we assume the random field $\varepsilon(x, \mathbf{R})$ to be Gaussian, homogeneous, and isotropic with the correlation and spectral functions

$$
\begin{align*}
& B_{\varepsilon}\left(x_{1}-x_{2}, \mathbf{R}_{1}-\mathbf{R}_{2}\right)= \\
& \left.=\int_{-\infty}^{\infty} \mathrm{d}\left(x_{1}, \mathbf{R}_{1}\right) \varepsilon\left(x_{2}, \mathbf{R}_{2}\right)\right\rangle \\
& \mathrm{d} \mathbf{q} \Phi_{\varepsilon}\left(q_{x}, \mathbf{q}\right) \exp \left\{\mathrm{i} q_{x}\left(x_{1}-x_{2}\right)\right. \\
&  \tag{8.10}\\
& \left.\quad+i \mathbf{q}\left(\mathbf{R}_{1}-\mathbf{R}_{2}\right)\right\}, \\
& \Phi_{\varepsilon}\left(q_{x}, \mathbf{q}\right)=\frac{1}{(2 \pi)^{3}} \int_{-\infty}^{\infty} \mathrm{d} x \int \mathrm{~d} \mathbf{R} B_{\varepsilon}(x, \mathbf{R}) \exp \left\{-\mathrm{i} q_{x} x-i \mathbf{q} \mathbf{R}\right\} .
\end{align*}
$$

The $x$-axis can be roughly divided into three regions, depending on the properties of the field intensity:

- the region of weak (intensity) fluctuations;
- the region of strong focusing;
- the region of strong fluctuations.


### 8.1 Region of weak fluctuations

In the region of weak fluctuations the statistics of intensity are well described by the method of smooth perturbations. Here the amplitude

$$
\chi(x, \mathbf{R})=\frac{1}{2} \ln I(x, \mathbf{R})
$$

is described by a perturbation series and can be viewed as a Gaussian random field. In this case the mean and the variance of the amplitude level are related:

$$
\left\langle\chi(x, \mathbf{R}\rangle=-\sigma_{\chi}^{2}(x) .\right.
$$

Let us introduce an important parameter, called the scintillation index (see, for instance, Refs [40, 41]): $\beta_{0}(x)=4 \sigma_{\chi}^{2}(x)$. For small $\beta_{0}(x) \ll 1$ the variance

$$
\sigma_{I}^{2}(x)=\left\langle I^{2}(x, \mathbf{R})\right\rangle-1=\beta_{0}(x)
$$

of the wave intensity is approximated by $\beta_{0}$, so the intensity is a log-normal random process with the one-point PDF

$$
\begin{equation*}
P(x ; I)=\frac{1}{I \sqrt{2 \pi \beta_{0}(x)}} \exp \left\{-\frac{1}{2 \beta_{0}(x)} \ln ^{2}\left(\mathrm{I}^{\beta_{0}(x) / 2}\right)\right\} \tag{8.11}
\end{equation*}
$$

The region of weak fluctuations refers to $\beta_{0}(x) \leqslant 1$. Here the typical realization of the log-normal process (8.11) falls off exponentially,

$$
I^{*}(x)=\exp \left\{-\frac{1}{2} \beta_{0}(x)\right\}
$$

and the essential statistics (like moments $\left.\left\langle I^{n}(x, \mathbf{R})\right\rangle\right)$ are formed by large deviations of $I(x, \mathbf{R})$ about $I^{*}(x)$. Furthermore, the random process $I(x)$ admits the majorant estimates, for instance, half of all realizations (probability $1 / 2$ ) obey the inequality

$$
I(x)<4 \exp \left\{-\frac{1}{4} \beta_{0}(x)\right\}
$$

at any distance $x$ along the beam. This clearly suggest the onset of 'clustering' of the wave intensity.

The knowledge of PDF (8.11) yields some quantitative characteristics of the 'cluster structure', such as the specific mean area $\langle s(x, I)\rangle$ and specific mean power $\langle\mathrm{e}(x, I)\rangle$ enclosed in domains with $I(x, \mathbf{R})>I$, see Ref. [24].

The evolving cluster-structure follows the changing scintillation index $\beta_{0}(x)$, and depends strongly on the level $I$. In the most interesting (high intensity) case $I>1$ we have $\langle s(0, I)\rangle=0,\langle\mathrm{e}(0, I)\rangle=0$ at the initial plane. As $\beta_{0}(x)$ grows, small cluster regions of high intensity, $I(x, \mathbf{R})>1$, are formed that persist over certain distances and rapidly absorb a significant fraction of the wave power. Further down the path, their area contracts, as $\beta_{0}(x)$ increases, but the entrailed power keeps growing, creating bright spots within dark regions. So one observes the effect of random focusing of radiation by inhomogeneous patches of the media.

In the region of weak fluctuations the gradient of the amplitude level $\nabla_{\mathbf{R}} \chi(x, \mathbf{R})$ is statistically independent of $\chi(x, \mathbf{R})$ itself. This allows one to compute the specific mean contour length at levels $I(x, \mathbf{R})=I$, and to estimate the specific mean number of contours. Indeed, in the region of weak fluctuations the gradient $\mathbf{q}(x, \mathbf{R})=\nabla_{\mathbf{R}} \chi(x, \mathbf{R})$ has a Gaussian PDF, and one has [24]

$$
\begin{align*}
\langle l(x, I)\rangle & =2 L_{f}(x)\langle | \mathbf{q}(x, \mathbf{R})| \rangle I P(x, I) \\
& =L_{f}(x) \sqrt{\pi \sigma_{\mathbf{q}}^{2}(x)} I P(x, I),  \tag{8.12}\\
\langle n(x, I)\rangle & =-\frac{1}{\pi} L_{f}^{2}(x)\left\langle\mathbf{q}^{2}(x, \mathbf{R})\right\rangle I \frac{\partial}{\partial I} I P(x, I) \\
& =\frac{L_{f}^{2}(x) \sigma_{\mathbf{q}}^{2}(x)}{\pi \beta_{0}(x)} \ln \left(I \mathrm{e}^{\beta_{0}(x) / 2}\right) I P(x, I) . \tag{8.13}
\end{align*}
$$

Let us observe that formula (8.13) turns to zero for the typical realization $I=I_{0}(x)=\exp \left\{-\beta_{0}(x) / 2\right\}$. This means that the 'typical intensity' has statistically equal numbers of contours $I(x, \mathbf{R})=I_{0}$ with different orientations (bright 'peaks' and dark 'valleys').

The above discussion was based on the fundamental parameter $\beta_{0}(x)$ determined by the medium.

If one assumes delta-correlated refraction $\varepsilon(x, \mathbf{R})$ in $x$, the correlation function (8.10) is approximately given by

$$
B_{\varepsilon}(x, \mathbf{R})=\delta(x) A(\mathbf{R}), \quad A(\mathbf{R})=\int_{-\infty}^{\infty} \mathrm{d} x B_{\varepsilon}(x, \mathbf{R})
$$

(see, for instance, Refs [5, 40, 41]).
In turbulent media $[5,40,41]$ one could compute $\beta_{0}(x)$ in terms of the 3D spectral function $\Phi_{\varepsilon}(0, \mathbf{q})=\Phi_{\varepsilon}(q)$ of the planar wave-vector $\mathbf{q}$,

$$
\begin{align*}
\beta_{0}(x)=4 \sigma_{\chi}^{2}(x) & =2 k^{2} \pi^{2} x \int_{0}^{\infty} \mathrm{d} q q \Phi_{\varepsilon}(q) \\
& \times\left[1-\frac{k}{q^{2} x} \sin \left(\frac{q^{2}}{k} x\right)\right] \tag{8.14}
\end{align*}
$$

provided the turbulence 'fills in' the entire space. If the random inhomogeneous $\varepsilon(x, \mathbf{R})$ is concentrated in a narrow band $\Delta x \ll x$ (random phase screen), then

$$
\begin{align*}
\beta_{0}(x)=4 \sigma_{\chi}^{2}(x) & =2 k^{2} \pi^{2} \Delta x \int_{0}^{\infty} \mathrm{d} q q \Phi_{\varepsilon}(q) \times \\
& \times\left[1-\cos \left(\frac{q^{2}}{k} x\right)\right] \tag{8.15}
\end{align*}
$$

The dielectric permeability $\varepsilon$ in Eqns (8.14), (8.15) has a spectral function

$$
\Phi_{\varepsilon}(\mathbf{q})=A C_{\varepsilon}^{2} q^{-11 / 3} \exp \left\{-\left(\frac{\mathbf{q}^{2}}{\chi_{m}^{2}}\right)\right\},
$$

with $A=0.033$, coefficient $C_{\varepsilon}^{2}$ depending on the ambient flow parameters, and the turbulence microscale wavenumber $\varkappa_{m}$.

Assuming a large value of the so-called wave parameter $D(x)=\chi_{m}^{2} x / k \gg 1$, we derive the following scaling law for $\beta_{0}(x)$ in two cases:

$$
\begin{align*}
& \beta_{0}(x)=0.307 C_{\varepsilon}^{2} k^{7 / 6} x^{11 / 6} \quad(\Delta x=x) \\
& \beta_{0}(x)=0.563 C_{\varepsilon}^{2} k^{7 / 6} x^{5 / 6} \Delta x \quad(\Delta x \ll x) . \tag{8.16}
\end{align*}
$$

Hence follows the variance of the amplitude levelgradient:

$$
\begin{gathered}
\sigma_{\mathbf{q}}^{2}(x)=\frac{k^{2} \pi^{2} x}{2} \int_{0}^{\infty} \mathrm{d} q q^{3} \Phi_{\varepsilon}(q)\left[1-\frac{k}{q^{2} x} \sin \left(\frac{q^{2}}{k} x\right)\right] \\
=\frac{1.476}{L_{f}^{2}(x)} D^{1 / 6}(x) \beta_{0}(x)
\end{gathered}
$$

Finally, one could find the dependence of the mean (specific) length and contour-number $\langle l(x, I)\rangle,\langle n(x, I)\rangle$, Eqns (8.12) and (8.13), on the parameters $\beta_{0}(x)$ and $D(x)$. Their dependence on the turbulent microscale indicates the presence of small ripples superimposed on the large-scale random landscape of intensity. The ripples do not affect the distribution of high concentration areas and the wave-power within them, but they could bring about the roughening and fragmentation of iso-contours.

As we mentioned earlier, the above description holds for small scintillation values $\beta_{0}(x) \leqslant 1$. As $\beta_{0}(x)$ grows larger (with $x$ ), the method of smooth perturbations ceases to work, and one has to deal with the fully nonlinear 'complex phase' equation. This region of strong focusing poses the difficult analytic problem, and we won't discuss it here.

The further growth of $\beta_{0}(x)$, e.g. $\beta_{0}(x) \geqslant 10$, would take one into the region of strong intensity fluctuations, where the statistics of intensity saturate.

### 8.2 Strong intensity fluctuations

In the region of strong fluctuations the intensity PDF may be approximated by

$$
\begin{align*}
P(x, I) & =\frac{1}{\sqrt{\pi(\beta(x)-1)}} \int_{0}^{\infty} \mathrm{d} z \\
& \times \exp \left\{-z I-\frac{(\ln z-(\beta(x)-1) / 4)^{2}}{\beta(x)-1}\right\} \tag{8.17}
\end{align*}
$$

(see, for instance, Refs [24, 67, 68]), with a different scintillation index $\beta(x)=\left\langle I^{2}(x)\right\rangle-1$.

In turbulent media, either continuous or localized within the phase screen, one finds

$$
\begin{array}{ll}
\beta(x)=1+0.861 \beta_{0}^{-2 / 5}(x) & (\Delta x=x) \\
\beta(x)=1+0.429 \beta_{0}^{-2 / 5}(x) & (\Delta x \ll x) \tag{8.18}
\end{array}
$$

(see, for instance, Ref. [5]), with $\beta_{0}(x)$ given by Eqn (8.16).
Let us note the gradual 'slide down' of $\beta(x)$ along with the increase of $\beta_{0}(x)$. Indeed, the long-distance limit $\beta_{0}(x) \rightarrow \infty$ yields $\beta(x)=1$, while the moderate value $\beta_{0}(x)=1$ corresponds to $\beta(x)=1.861$.

Let us also remark that PDF (8.17) is not applicable in the narrow vicinity of $I \sim 0$ : the larger $\beta_{0}(x)$, the smaller the window about 0 . This has to do with infinitely large values of the 'negative' moments $1 / I(x, \mathbf{R})$, that follow from Eqn (8.17). However, the finite values of $\beta_{0}(x)$, no mater how large, give finite moments $\left\langle I^{n}(x, \mathbf{R})\right\rangle$, hence the vanishing probability of $I=0, P(x, 0)=0$. The existence of the narrow band near $I \sim 0$ has no effect on the behavior of 'positive' moments $\left\langle I^{n}(x, \mathbf{R})\right\rangle$ at large $\beta_{0}(x)$.

The asymptotic formulae (8.17) and (8.18) describe a transition into the region of saturated intensity fluctuations, $\beta(x) \rightarrow 1$. In this region the mean specific area of high intensity $I(x, \mathbf{R})>I$, and the specific power enclosed, are given by

$$
\begin{equation*}
P(I)=\mathrm{e}^{-I}, \quad\langle s(I)\rangle=\mathrm{e}^{-I}, \quad\langle\mathrm{e}(I)\rangle=(I+1) \mathrm{e}^{-I} . \tag{8.19}
\end{equation*}
$$

Hence the fraction of the mean area and the mean power depend only on the level $I$. For large $I$, these fractions are insignificant.

The exponential distribution of PDF (8.19) implies that the complex random field $u(x, \mathbf{R})$ is Gaussian. Furthermore,
the gradient of the cross-sectional amplitude is statistically independent of the wave field intensity, and is also Gaussian [24]. Besides, it has no statistical dependence on the second derivatives of intensity in cross-variables. In this case, one could compute the mean specific contour length and the contour number explicitly,

$$
\begin{align*}
& \langle l(x, I)\rangle=L_{f}(x) \sqrt{2 \pi \sigma_{\mathbf{q}}^{2}(x)} I \mathrm{e}^{-I}, \\
& \langle n(x, I)\rangle=\frac{2 L_{f}^{2}(x) \sigma_{\mathbf{q}}^{2}(x)}{\pi}\left(I-\frac{1}{2}\right) \mathrm{e}^{-I}, \tag{8.20}
\end{align*}
$$

where

$$
\begin{equation*}
\sigma_{\mathbf{q}}^{2}(x)=\frac{1,476}{L_{f}^{2}(x)} D^{1 / 6}(x) \beta_{0}(x) \tag{8.21}
\end{equation*}
$$

Let us remark that formula (8.20) does not apply in the narrow vicinity of zero intensity $I=0$. For the latter we expect $\langle n(x, 0)\rangle=0$.

Formula (8.20) shows the mean contour length and contour number continue to grow with $\beta_{0}(x)$, though the total contour-area and the power enclosed therein, remain constant. Such a process leads to increasing roughness and fragmentation of iso-contours. The explanation lies in the dominant role played by the interference of partial waves coming from various directions.

The dynamics of iso-contours depend strongly on the balance between focusing and defocusing of radiation by different patches of turbulent media [69]. The focusing by large scale inhomogeneities results in high intensity peaks on the random landscape of $I$. In the regime of maximal focusing $\beta_{0}(x) \sim 1$ about half of the total power is concentrated in such high narrow peaks. As $\beta_{0}(x)$ increases further the defocusing prevails, which tends to smooth out high peaks and create highly fragmented (interferential) landscape with large number of small peaks near $I \sim 1$.

Besides the scintillation parameter $\beta_{0}(x)$ the mean contour length and the number of contours depend on the wave parameter $D(x)$, hence both would grow as the microscale of inhomogeneities goes down. This process reflects the superposition of the large scale intensity landscape with small ripples, due to wave scattering on small scale inhomogeneities.

In this section we have attempted to give a qualitative explanation of the cluster (caustic) structure of the wave field in the cross-sectional plane, for an incident plane wave in a turbulent medium, and to quantify and estimate some of its statistical topography. In general, such problems have many parameters. However, when confined to a fixed cross-section, the solution is described by a single parameter $\beta_{0}$ - the variance of the intensity-level in the region of weak fluctuations.

We have analyzed two extreme asymptotic cases, that correspond to the weak and strong (saturated) fluctuations of intensity. We should remark that the above asymptotic formulae could apply only certain limits depending on the intensity level $I$. As $I$ goes down the applicability range of these formulae should extend.

The most interesting case from the standpoint of applications involves the region of developed caustic structure. A detailed analysis would require knowledge of the joint PDFs of $I$ and cross-sectional gradients at arbitrary distances along the beam. Such an analysis could be done either using
approximate methods for all possible values of essential parameters [61], or using numeric simulation as in papers [44, 45].

## 9. Conclusions

We conclude with some general 'philosophical' comments, that follow from our work.

- For many dynamical systems statistical (mean) characteristics show little resemblance to individual realizations, and sometimes gives contradictory results. For such systems the traditional 'moment-based' description carries little information. Instead they require a statistical description in terms of PDFs (at least 'single-point' or 'single-level').
- However, such stochastic dynamics could often exhibit some statistically coherent physical phenomena, that occur with probability one, i.e. for almost all realizations of the process. One could suspect coherent phenomena to be abundant in nature, as they appear in the most simple systems described by ordinary differential equations like (2.1). No other physical model would beat it for simplicity! The basic statistical model for positive-valued parameters of coherent processes is furnished by the log-normal process.
- Coherent phenomena are largely independent of the specific model of fluctuating parameters. In some cases they allow a complete characterization in terms of single-point (in space-time) PDFs of the process, which could be deduced by the method of statistical topography. Of course, certain parameters of a particular phenomenon (like the characteristic time and space scales for clustering) could depend on the specific random model.
- Coherent phenomena could be particularly relevant to physical problems where the 'ensemble means' (and 'ensembles' themselves) are not available, and the experimentalist most often has to deal with specific realizations. This applies, in particular to the physics of the atmosphere and ocean.

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