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Superradiance and related phenomena

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Contents

Abstract. The current theory of superradiance is described. The effect is due to interatomic correlations (phase-locking) which arise under the action of a general electromagnetic field and are determined by the competition between the electron motion anharmonicity and the interatomic dipole – dipole interaction. The latter affects significantly the nature of the superradiance effect. A common nature for the radiations from a Dicke atomic ensemble and from collective waves in a substance (for example, cyclotron waves in a magnetized plasma) is established. Superradiance manifests itself in hot magnetically confined plasmas and accounts, among other things, for anomalous heat conductivity in tokamaks.

1. Introduction

Superradiance (SR) is cooperative radiation resulting from spontaneous origination and reinforcement of correlations between initially independent atoms (phasing) [1 – 8]. The study of mechanisms of phasing started in fact 30 years after Dicke's seminal work [8] that had put forth the concept of SR. Phasing was found to depend on two effects: the electromagnetic interaction between the atoms, and the nonlinearity of motion of electrons in an atom. Both effects are universal and fundamental; this is the reason why the seemingly greatly different phenomena as cooperative emission by a microscopic specimen, radiation of cyclotron resonance masers, gyrotrons, collective waves in a gas of dipole oscillators, etc., have a common background. Such a sufficiently general approach allows one to presume that it is the electron cyclotron waves in magnetized plasma that are primarily responsible for the anomalous energy transfer in tokamak plasma.

In the SR regime, a system of a large number N of excited atoms emits radiation for a time τ_s which is much shorter than the time τ_0 of emission of an isolated atom [1 – 8]. Another feature of SR, which is a consequence of the first, consists in that the intensity of radiation I grows in proportion with the square of the number of atoms N^2. These features sharply distinguish SR from spontaneous emission of radiation by a system of excited atoms (luminescence), for which $\tau \sim \tau_0$, $I \sim N$. The reason is that there are phase correlations between the atoms in the case of SR, such as do not exist in the case of luminescence. By itself, the dependence $I \sim N^2$ is not exotic, and by no means constitutes the main peculiarity and the main distinguishing feature of SR. Rather, it is phasing that is the main property of SR that attracts attention and raises this effect to the rank of a fundamental phenomenon (related to phase transitions). It is phasing that sets SR off from other cooperative radiation processes which are the result of correlations created in the first place by means of external

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pumping — decay of free polarization, photon echo, optical nutation, and the like. A correlated ensemble of atoms can be created, for instance, by applying a short resonant laser pulse to the system. Owing to the initial correlation, this gives rise to cooperative radiation with the above properties ($I \sim N^2$, $\tau \ll \tau_0$), which does not involve phasing. We may point to an obvious situation when the phasing of atoms is precluded, but the radiation is cooperative. Assume that at $t = 0$ we have N dipoles located within a volume with characteristic size L that start oscillating at $t = 0$, and stop at $t = \tau$. The electromagnetic field at any remote observation point is equal to the sum of signals coming from $\Delta N - Nc\tau/L$ atoms confined in the layer $c\tau$ thick. If $c\tau \ll l$, and $\Delta N \gg 1$, where $l = n^{-1/3}$ is the characteristic distance between the atoms, then the phasing is precluded but the radiation is cooperative because $I \sim \Delta N^2 \sim N^2$.

This review is devoted to the analysis of SR theory. Special attention is paid to the study of mechanisms of phasing and the role of dipole – dipole interaction between the atoms. A unified treatment is applied to the effects of SR in diverse physical systems, including small bodies (ensembles of Dicke atoms) and extended objects (coherent waves in magnetized plasma).

2. The nature of superradiance

Superradiance, one of the most interesting phenomena in the physics of cooperative electromagnetic radiation [1 – 7], was predicted by Dicke [8]. Assume that an inverse population is created in a system of $N \gg 1$ atoms (numbered $a = 1, 2, \ldots, N$) for two states $|2\rangle$ and $|1\rangle$ with the energy levels E_2 and E_1 — based, for example, on the three-level pumping scheme (Fig. 1), or by passing a short electromagnetic pulse through the system (for example, a π-pulse, Fig. 2). As a result of such pumping, over the characteristic time τ_p near $t = 0$, state $|2\rangle$ is attained by a certain number of atoms $N^* = NW$, where W is the share of inverted atoms (observe that for a three-level scheme N is the number of excited atoms). Under certain conditions that will be discussed below, SR begins. Its characteristic features are illustrated in Figs 3 and 4. At $N^* < N_c$, where N_c is the critical number of inverted

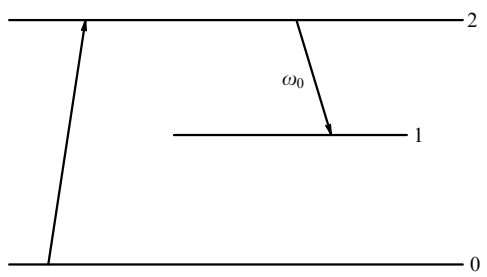

Figure 1. Three-level scheme of superradiance for the $|2\rangle \to |1\rangle$ transition. Level 0 is the ground state of atoms. In the case of pumping over the characteristic time τ_p the atoms undergo the transition $|0\rangle \to |2\rangle$.

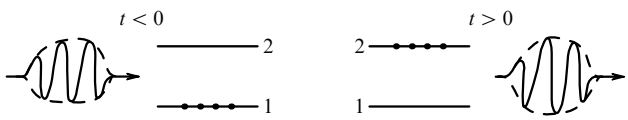

Figure 2. Pumping with the π-pulse in the two-level scheme; level 1 is the ground state.

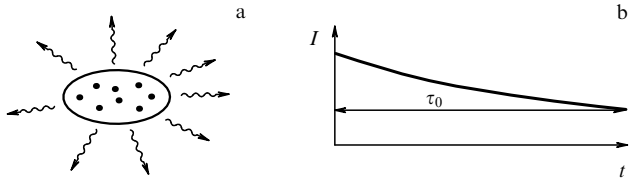

Figure 3. Fluorescence in a system of N^* inversely populated atoms (a). The atoms undergo independent spontaneous transitions over the characteristic time τ_0 with intensity $I \sim N^*$ (b). The radiation is isotropic. (Reproduced from Ref. [1]).

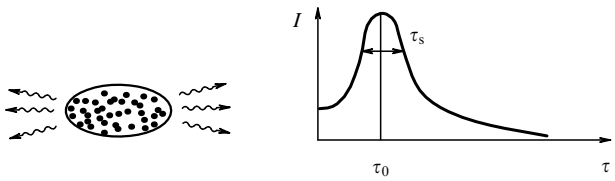

Figure 4. Superradiance occurring at $N^* > N_c$ in the Markovian regime $L < c\tau_s$. (Reproduced from Ref. [2]).

atoms (the so-called SR threshold, see below), conventional spontaneous transitions to state $|1\rangle$ take place (see Fig. 3), whose duration coincides with the time τ_0 of spontaneous transition $|2\rangle \to |1\rangle$ of an isolated atom. The characteristic intensity of emission by a system of atoms is (for the sake of simplicity we set $W \sim 1$)

$$I \sim NI_0 \,, \tag{2.1}$$

where $I_0 = \hbar\omega_0/\tau_0$ is the intensity of radiation of an isolated atom, and $\omega_0 = (E_2 - E_1)/\hbar$ is the frequency of radiation. At $N^* > N_c$, the system undergoes a phase transition which in every aspect is similar to a phase transition of the second order [1, 3]. The radiation becomes markedly anisotropic (see Fig. 4) and appears as a short powerful burst ('SR pulse') with

$$\tau_s \sim \frac{\tau_0}{N} \,, \qquad I \sim N^2 I_0 \,, \tag{2.2}$$

emitted after the time lag

$$t_0 \sim \tau_s \Lambda \,, \qquad \Lambda = \ln N \tag{2.3}$$

in the direction of the major axis of the body (i.e. in the direction of its largest thickness).

Superradiance is based on two effects which determine its essential properties:

(a) phasing of atoms, and

(b) collective intensive emission by a system of correlated atoms.

Let us illustrate this point with a simple example. Let the size L of the emitting system (which will occasionally also be referred to as a 'body') be small compared with the emitted wavelengths $\lambda = 2\pi/k$, $k = \omega_0/c$ ('the pointlike Dicke model' [8]):

$$L \ll \lambda \,. \tag{2.4}$$

We start with the classical (Lorentz) model of atoms. There are N charges e of mass m attached to springs with stiffness k. The charges occur at points with the coordinates $\mathbf{r}_a + \boldsymbol{\xi}_a$. The

other ends of the springs are fixed at points $\mathbf{r}_a$, which also host compensating charges $-e$. Hence, the dipole moments of the atoms are

$$\mathbf{d}_a = e\boldsymbol{\xi}_a = e\boldsymbol{\eta}_a \cos(\omega_0 t + \varphi_a)\,, \tag{2.5}$$

where $\omega_0 = \sqrt{k/m}$ are the eigenfrequencies of atoms – oscillators (the meaning of $\mathbf{d}_a$, $\boldsymbol{\eta}_a$, φ_a for real quantum atoms will be explained later on). The intensity of radiation averaged over the fast oscillations of dipoles is [9]

$$I = \frac{2}{3c^3}(\ddot{\mathbf{D}})^2 = I_1 + I_2\,, \tag{2.6}$$

where $\mathbf{D} = \sum_{a=1}^{N} \mathbf{d}_a$ is the dipole moment of the system,

$$I_1 = \frac{2}{3c^3}\sum_a \langle \ddot{\mathbf{d}}_a^2 \rangle\,, \qquad I_2 = \frac{2}{3c^3}\sum_{a\neq b} \sigma_{ab}\,, \tag{2.7}$$

$$\sigma_{ab} = \langle \ddot{\mathbf{d}}_a \ddot{\mathbf{d}}_b \rangle = \frac{1}{2}\, e^2 \omega_0^4 (\boldsymbol{\eta}_a \boldsymbol{\eta}_b) \cos(\varphi_a - \varphi_b)\,. \tag{2.8}$$

If the amplitudes $\boldsymbol{\eta}_a$ and phases φ_a are not correlated, and vary at random from one atom to another, then $I_2 \ll I_1$, $I \approx I_1 \sim N$ which implies the regime of spontaneous emission (2.1). If, however, there is correlation (phasing) between phases and amplitudes of different dipoles (atoms), then $I_2 \gg I_1$, $I \approx I_2 \sim N^2$, and the body emits in the regime of SR (2.2)

These arguments infer the necessary (but not sufficient) conditions for the onset of SR:

$$\tau_{\rm p} \ll \tau_{\rm s}\,, \tag{2.9}$$

i.e. brief pumping, and

$$\tau_{\rm s} \ll T_{\rm r} = \min(\tau_0, T_2)\,, \tag{2.10}$$

i.e. preserve the coherence of the atomic subsystem during the entire process. Here

$$T_2 = \min\left(\frac{1}{\Delta\omega_{\rm D}}\,, T_{\rm f}\right) \tag{2.11}$$

is the phase memory time over which the interatomic phase correlations are preserved; $\Delta\omega_{\rm D} \sim \omega_0 v_{\rm T}/c$ is the Doppler broadening of the frequency of atomic transition; $v_{\rm T}$ is the thermal velocity of atoms; $T_{\rm f}$ is the time of phase desynchronization of atoms caused by collisions [5, 10, 11] (transverse relaxation time). Since, according to Eqn (2.2), $\tau_{\rm s}$ diminishes as N increases, the condition (2.10) defines the threshold of SR — that is, the number $N_{\rm c}$.

Figure 4 relates to a small-size body:

$$L \ll c\tau_{\rm s}\,. \tag{2.12}$$

When

$$L \gg c\tau_{\rm s} \tag{2.13}$$

(the Arecchi – Courtens condition [12]), the effects of propagation of electromagnetic waves in the body become important (see below). The coherence of atoms that is necessary for SR is preserved in sufficiently small bodies:

$$L \ll cT_{\rm r}\,. \tag{2.14}$$

Note that when the density of atoms n is high, then in place of c in Eqns (2.12) – (2.14) one should use the group wave velocity $v_{\rm g}$, which may happen to be much less than c. As will be shown below, the intensity of radiation under condition (2.14) is proportional to the square of the number of atoms, and this domain therefore ought to be referred to as the range of SR. Conditions (2.9), (2.10), (2.14) will be concretized below.

The extensive literature available on SR fails (at least in the author's opinion) to offer a comprehensive account of all the most important features of SR, like the mechanism of phasing of atoms, the role of the dipole – dipole interaction between the atoms, etc. In the case of SR we are dealing with a nonequilibrium phase transition of the second order — that is, with the emergence of nonzero mean values (quasi-averages, parameters of order, etc.) of operator variables. These aspects — such as the kinetics of transition to the superfluid state [14] — have only recently received attention (see, for example, Ref. [13] and references therein). We anticipate new interesting results concerned with these aspects of SR.

The mechanism of phasing of atoms has not been explored in Ref. [8]: the prediction of SR was based on the symmetry of the Hamiltonian and wave function in the Dicke model with respect to permutation of coordinates of any pair of atoms. This symmetry allowed the wave function of the atomic system to be surmised, and the intensity of radiation to be calculated without answering the difficult (and probably the principal) question concerning the mechanism of phasing. The small size of the body postulated in Ref. [8] would seemingly preclude staging of experiments in the optical range, thus making SR of little practical interest. The feasibility of SR in extended bodies

$$L \gg \lambda\,, \tag{2.15}$$

was theoretically predicted in Refs [15 – 23] and confirmed by the first experiment [24] with SR on rotational transitions in HF molecules (note that the feasibility of SR in extended bodies had been conjectured in Ref [8] from incorrect assumptions). The same experiment revealed time oscillations of SR intensity (Fig. 5), earlier predicted theoretically in Ref. [18] (the quantitative estimate for the frequency of oscillations was obtained in Ref. [25]).

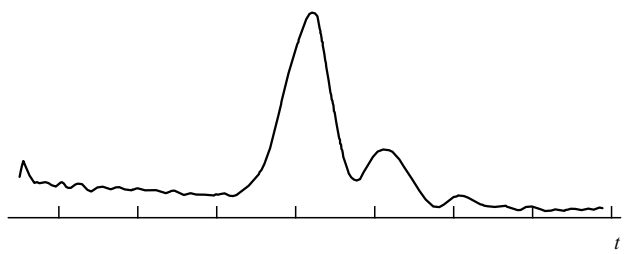

Figure 5. Oscillatory regime of superradiance [24].

In this way, by 1973 it had become clear that SR is a universal phenomenon that can be practically realized using the available technology of short powerful laser pulses required for fulfilling conditions (2.9), (2.10).

The duration $\tau_{\rm s}$ of SR for an extended specimen differs from the estimate (2.2) for small bodies compliant with (2.4). For the case of (2.12) it can be derived from simple considerations [12]. The estimate is based on the assumption

that within the time τ_s the entire energy

$$E_0 = \hbar\omega_0 nLS \tag{2.16}$$

stored in a rod with the cross section $S = \pi R^2$, radius R and length L ($R \ll L$) is emitted in the direction of the axis of the rod as electromagnetic waves with frequency ω_0 which form a train of length $c\tau_s$. From the energy balance condition ($E_0 \sim Sc\tau_s E^2/8\pi$) we find the amplitude of the electric field characteristic of the pulse of SR:

$$E \sim \left(\frac{\hbar\omega_0 nL}{c\tau_s}\right)^{1/2} . \tag{2.17}$$

Under the action of this resonant field, each atom performs a half Rabi oscillation (see, for example, Refs [5, 26]) over the time

$$\tau_s \sim \frac{\hbar}{V_{21}} , \qquad V_{21} = Ed , \tag{2.18}$$

and is found in the lower state $|1\rangle$. Here $d = |\mathbf{d}|$, $\mathbf{d} \equiv \mathbf{d}_{21} = \langle 2|\hat{\mathbf{d}}|1\rangle$ is the matrix element of the atomic transition $|2\rangle \to |1\rangle$ responsible for spontaneous emission

$$\frac{1}{\tau_0} \equiv \lambda_0 = \frac{4k^3 d^2}{3\hbar} . \tag{2.19}$$

Equations (2.18), (2.19) infer the estimates

$$\tau_s \sim \frac{\hbar}{kd^2 nL} \sim \frac{\tau_0}{\varphi} , \qquad \varphi = \frac{3\pi nL}{k^2} \tag{2.20}$$

and the main condition of realization of SR in extended bodies (2.15)

$$\varphi \gg 1 . \tag{2.21}$$

Condition (2.21) has a profound meaning that is readily understood when

$$\lambda \ll l , \tag{2.22}$$

where $l = n^{-1/3}$ is the mean distance between the atoms. Then every atom occurs within the wave zone of another atom, and the electromagnetic waves scattered by the atoms have time to form new waves before reaching the adjacent atoms. The electromagnetic field within the medium in the case of (2.22) is an assemblage of independent photons scattered by the atoms. The frequency spread of the electromagnetic field is $\Delta\omega \sim 1/\tau_s$. According to Rabi's theory, the field may be regarded as resonant if $\Delta\omega \lesssim V_{12}/\hbar$. Hence, and from Eqn (2.18), we conclude that the field is resonant. The scattering cross section of resonant photon by the atom is [9, 27, 28]

$$\sigma \sim \frac{1}{k^2} , \tag{2.23}$$

therefore, in accordance with Eqn (2.20), the main condition (2.21) is written in the form

$$\varphi \sim n\sigma L \gg 1 . \tag{2.24}$$

Hence it follows that the number of scatterings undergone by the photons over the length of the body must be large. This points to the importance of the induced radiation for the mechanism of SR.

The arguments developed above [12] based on the considerations of energy balance allow the field amplitude (2.17) to be found. In the case of SR, however, the principal role belongs to phases.

Another argument is instructive. Assume that we have an elongated body of length L (a rod or a cylinder of radius R). By the mechanism of phasing, each atom a obtains phase $\varphi_a = \mathbf{k}\mathbf{r}_a$, where the vector $\mathbf{k}$ is directed along the body, $k = \omega_0/c$. First let us consider the case of (2.22). The oscillating dipole in the quasistatic zone creates a field $E \sim d/r^3$, and $E \sim dk^2/r \exp(\mathrm{i}kr)$ in the wave zone (see Refs [9, 27]). The field E_b at the atom b is equal to the sum of fields created by all other atoms, so, with due account for the phases of dipoles, we get

$$E_b \sim \sum_{a \neq b} \frac{dk^2}{r} \exp(\mathrm{i}kr + \mathrm{i}\varphi_a) , \tag{2.25}$$

where $r \equiv r_{ab}$, $\mathbf{r}_{ab} = \mathbf{r}_a - \mathbf{r}_b$, and the dipole moment $\mathbf{d}$ includes the factor $\exp(-\mathrm{i}\omega_0 t)$. The main contribution to the sum (2.25) comes from those atoms a for which

$$|\Delta\varphi| \lesssim 1 , \tag{2.26}$$

where $\Delta\varphi = kr + \varphi_{ab} = kr_{ab} + \mathbf{k}\mathbf{r}_{ab}$, $\varphi_{ab} = \varphi_b - \varphi_a = \mathbf{k}\mathbf{r}_{ab}$. These atoms are located near the axis z that passes through atom b in the direction $\mathbf{k}$ (Fig. 6). We refer to the region (2.26) as the region of constructive interference (RCI): the signals that come to atom b from atoms a located within RCI add up — undergo constructive interference.

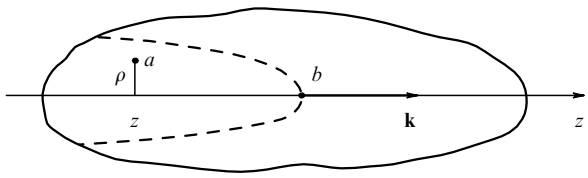

Figure 6. Region of constructive interference that determines the field at atom b (dashed line). The origin of coordinates $z = 0$ is at atom b.

Further on we shall be almost invariably concerned with the practically most interesting case

$$F \gg 1 , \tag{2.27}$$

where

$$F = \frac{kR^2}{L} \sim \frac{R^2}{\lambda L} \tag{2.28}$$

is the Fresnel number defined as the ratio of the geometric angle R/L to the diffraction angle $\theta_D = 1/kR \sim \lambda/R$, or, which is the same, as the number of Fresnel zones fitting into the end of the cylinder. From equations (2.26) and (2.27) it follows that the radius of RCI is small compared with R and L; therefore,

$$z = \frac{\mathbf{r}\mathbf{k}}{k} , \qquad \rho \ll |z| , \qquad |z| \sim L , \qquad r = \sqrt{z^2 + \rho^2} \approx |z| + \frac{\rho^2}{2|z|} ,$$

$$\Delta\varphi \approx k\big(|z| + z\big) + \frac{k\rho^2}{2|z|} . \tag{2.29}$$

The equation of the boundary of RCI $|\Delta\varphi| = 1$ is

$$z < 0\,, \qquad \rho = \rho_0|z| = \sqrt{\frac{2|z|}{k}}\,. \tag{2.30}$$

Hence, and from Eqn (2.27), it follows that the characteristic thickness of RCI,

$$\rho_0 \sim \rho_0(L) \sim \sqrt{\frac{L}{k}}\,, \tag{2.31}$$

is small compared with the transverse dimension of the body,

$$\frac{\rho_0}{R} \sim \frac{1}{\sqrt{F}} \ll 1\,. \tag{2.32}$$

The volume of RCI is

$$V_0 \sim L\rho_0^2 \sim \frac{L^2}{k}\,. \tag{2.33}$$

By virtue of Eqns (2.15) and (2.21) the number of atoms in RCI is large,

$$N_0 \sim nV_0 \sim \varphi kL \gg 1\,, \tag{2.34}$$

which allows us to replace the summation in Eqn (2.25) by integration:

$$\sum_a \to n\int \mathrm{d}^3 r_a = 2\pi\int \rho\,\mathrm{d}\rho\,\mathrm{d}z\,. \tag{2.35}$$

On account of Eqn (2.27), integration with respect to ρ can be extended to the entire interval $(0,\infty)$. Then from Eqns (2.25), (2.29), (2.35) we get

$$E_b \sim dk^2 n\int \frac{\mathrm{d}z\,\rho\,\mathrm{d}\rho}{|z|}\exp\left[\mathrm{i}k\big(|z|+z\big) + \frac{\mathrm{i}k\rho^2}{2|z|}\right]$$
$$= dk^2 n\int \frac{\mathrm{d}z}{|z|}\,\frac{\mathrm{i}|z|}{k}\exp\big[\mathrm{i}k(|z|+z)\big]\,.$$

Hence follows the estimate

$$E_b \sim ndkL\,. \tag{2.36}$$

This estimate holds [and hence does Eqn (2.20)] both in the case of (2.22) and in the opposite extreme

$$l \ll \lambda\,, \tag{2.37}$$

when the RCI features both the wave zone ($r > \lambda$) and the static zone ($r < \lambda$). The contribution to E_b from the quasistatic zone we approximate as the field inside a uniformly polarized sphere — of the order of nd. Hence, and from Eqn (2.36), it is clear that in our current case of (2.15) the contribution from the quasistatic zone is negligibly small, and the field at any atom is determined by the entire RCI associated with this atom.

Using the estimate (2.36), from Rabi's formula (2.18) we get Eqn (2.20). We can also do without formula (2.18). The intensity of radiation is equal to the flux of energy across the ends of the rod:

$$I \sim \frac{ScE_b^2}{4\pi}\,. \tag{2.38}$$

Hence, using Eqn (2.16) and $\tau_\mathrm{s} \sim E_0/I$, we again derive Eqn (2.20).

Now the overall picture is clear. After the pumping pulse, the atoms exchange electromagnetic signals. Each atom develops a fluctuating field $\delta E \sim dk^2/l$ [in the case of (2.22)]. Then, on account of the phasing discussed in sections to follow, the collective field (2.36) is established. Since $\delta E/E_b \sim kl^2/L \ll 1$ *a fortiori*, fluctuations can be neglected. Later we shall demonstrate that the fluctuative component of the field can also be neglected in the case of (2.37). As already said, phasing gives rise to amplified collective radiation.

As L increases, the duration of SR τ_s decreases [see Eqn (2.20)], so at certain values of L (or at certain values of n in experiments) the condition (2.12) is violated, and the pattern of SR becomes more complicated. We estimate the length L_0 over which this occurs, and the characteristic time of SR τ_s^0, from definition $L_0 = c\tau_\mathrm{s}^0$ and Eqn (2.20):

$$L_0 = k\left(\frac{c\tau_0}{n}\right)^{1/2}, \qquad \tau_\mathrm{s}^0 = k\left(\frac{\tau_0}{nc}\right)^{1/2}. \tag{2.39}$$

The pumping of a long rod in the case of (2.13) can be realized in either of two ways: by sending a short pulse along the body, or by simultaneously pumping all portions of the body (transverse electric discharge, transverse laser pulse, etc.). Different portions of the rod of length of about L_0 then start emitting independently of one another, since there is no correlation between adjacent portions owing to the fact that c is finite. Radiation from each portion travels out to adjacent portions and brings some of the relaxed atoms back into the excited state, the atoms emit again, and so on. An oscillatory regime (see Fig. 5) sets in an elongated body [18, 25, 29, 30], whereas other shapes host the diffusion regime, when different portions exchange energy with each other (this latter case has not been studied well enough).

Now what happens when

$$L \gg cT_\mathrm{r}? \tag{2.40}$$

This is the typical laser regime [5, 31–34] when the atomic system is stochastic, and the photon system is coherent. A photon emitted lengthwise generates an avalanche of induced photons with the characteristic length of coherence L. If the body (active medium) is placed in a high-Q resonator, we get a generator on eigenmodes — that is, a nonequilibrium phase transition in the photon subsystem [31]. The coherence length increases Q-fold (as does the monochromatism of radiation). In the case of (2.40) the nondiagonal elements of the atoms' density matrix, responsible for the coherence, are negligibly small [5]. Because of this, the atoms are described by the diagonal elements — the populations W_1, $W_2 = 1 - W_1$ (the proportions of atoms occupying states $|1\rangle$ and $|2\rangle$). The populations and the emission intensity J (the photon flux density) are described by the standard laser equations [32–34], which involve in particular the cross section of photon scattering by atoms σ. The equation for $\mathrm{d}J/\mathrm{d}x$ features the term $n\sigma JW_2$, which describes the effect of induced emission discovered by Einstein (another term, $n\sigma JW_1$, is referred to as induced absorption). Generation begins when the effective length of the active medium LQ is large compared with the photon's free path length $1/n\sigma$ [cf. Eqn (2.24)].

Let us discuss the role of induced radiation in the effect of SR. The approach depends on the definition of this phenomenon. With reference to the above-mentioned Einstein term in

the equation for intensity, it is often assumed that the atomic system must be stochastic (this is the case in Einstein's treatment which is concerned with thermal equilibrium). This is probably the reason why it is sometimes assumed that induced emission of a system of excited atoms is only possible under condition (2.40), when the atoms are subject to frequent random impacts. For this reason, this effect is of little relevance to SR. We believe that a different definition of induced radiation is more general (see, for example, Refs [35–37]), which infers the importance of the latter for the phenomenon of SR. For example, the scattering of electromagnetic waves by a charged oscillator in classical electrodynamics also ought to be classified as induced radiation processes: induced emission and absorption that in classical electrodynamics cannot be separated [38, 39]. Under the action of the wave's field the charge oscillates with an amplitude that is large compared with the amplitude of zero oscillations (since we are dealing with classical mechanics) — that is, the oscillator is excited to higher quantum states. Receiving additional acceleration from the wave, the charge emits bremsstrahlung in phase with the incident wave, with the intensity proportional to that of the incident wave J as it ought to be in the case of induced radiation. If this takes place in a resonator, then, given that the split of frequencies between the modes is large enough, the charge will emit only in a single mode, and therefore the intensity of bremsstrahlung will be proportional to the number of photons in this mode n_λ (since $J \propto n_\lambda$), in accordance with Einstein's conclusions. With this definition, the presence of induced radiation is in no way related to the frequency of atomic collisions, and therefore the induced radiation is present not only in the laser regime (2.40), but also in the regime of SR (2.14). Spontaneous radiation and induced radiation are equally important for SR.

Consider one of the atoms (labeled b). It occurs in the total field $\mathbf{E}$, and in unit time through radiation gives an energy $I = -e\dot{\boldsymbol{\xi}}\mathbf{E}$ to the field. Assume that the interaction of the atom with the field turns on at $t = 0$ — that is, the charge varies as $e(t) = e\theta(t)$. Following Ref. [38], we regard the charge as small; then $\boldsymbol{\xi} = \boldsymbol{\xi}^{(0)} + \boldsymbol{\xi}^{(1)}$, $\mathbf{E} = \mathbf{E}^{(0)} + \mathbf{E}^{(1)}$, where $\boldsymbol{\xi}^{(0)}$, $\mathbf{E}^{(0)}$ are the coordinate of the electron of atom b and the electric field at $e = 0$. Obviously, $\mathbf{E}^{(1)} = 2e/(3c^3)\dddot{\boldsymbol{\xi}}$ is the field of reaction of the radiation [9, 27]. With respect to the selected atom, $\mathbf{E}^{(0)}$ is the external field; therefore, $\mathbf{E}^{(0)} = \mathbf{E}_b$ [see Eqn (2.36)].

Thus,

$$I \approx I_1 + I_2 + I_3\,, \tag{2.41}$$

where

$$I_1 = -e\mathbf{E}^{(0)}\dot{\boldsymbol{\xi}}^{(0)}\,, \qquad I_2 = -e\mathbf{E}^{(0)}\dot{\boldsymbol{\xi}}^{(1)}\,,$$

$$I_3 = -e\mathbf{E}^{(1)}\dot{\boldsymbol{\xi}}^{(0)} = -\frac{2e^2}{3c^3}\,\dot{\boldsymbol{\xi}}^{(0)}\dddot{\boldsymbol{\xi}}^{(0)}\,.$$

If $\mathbf{E}^{(0)} = 0$, then $I_1 = I_2 = 0$. Accordingly, the first two terms ought to be regarded as induced processes (emission and absorption), and the third term I_3 as spontaneous. In the absence of interatomic correlations $\langle I_1 \rangle = 0$. Phasing results in the collective field (2.36), the scattering of which by each atom amplifies the field, thus giving rise to induced radiation: $\langle I \rangle > 0$, $\langle I \rangle \approx \langle I_1 \rangle \gg |I_{2,3}|$. This important effect was first pointed out in Ref. [36] which started a new direction in physics and engineering, leading to the development of high-power generators of electromagnetic radiation: gyrotrons, cyclotron resonance masers, and the like [37, 40, 41].

Initially the field is absent, and therefore we must treat SR as spontaneous emission by the entire atomic system (referred to in Ref. [8] as collective spontaneous radiation). Now we see that, owing to the phasing of atoms, the same radiation must be treated as the net effect of the induced emission by individual atoms. The dipole moment of atom b oscillates synchronously (but in counterphase) with the collective field (2.36). In unit time the atom gives an energy $e\mathbf{E}_b\dot{\boldsymbol{\xi}} \sim E_b\omega_0 d$ to the field, and all atoms together give N times that much, which agrees with Eqn (2.38).

The importance of induced radiation for the phenomenon of SR is further supported by the following argument. Under condition (2.27), SR is confined to the geometric solid angle $\Omega \sim (R/L)^2$ (see Section 8.1), and has a frequency spread of $\Delta\omega \sim 1/\tau_s$. Accordingly, the number of photons that cross the unit cross-sectional area of the rod per unit time is of the order of $J \sim n_p k^2 \Delta k \Omega$, where $\Delta k = \Delta\omega/c \sim 1/c\tau_s$, n_p is the occupancy of the photon states (the number of photons in one given photon state). On the other hand, $J \sim cE_b^2/\hbar\omega_0$. Hence, and from Eqns (2.20) and (2.36), it follows that $n_p \sim N/F^2$; therefore, $n_p \gg 1$ with certainty, which is a straightforward proof of the importance of induced processes. This point can also be proved by a different course of reasoning, introducing the real physical volume $(c\tau_s)^3\Omega$ occupied by the emitted photons. Boundary conditions of any kind — for example, zero boundary conditions — can be defined on the boundary of the volume. The exact type of these conditions does not matter, since photons only reach the boundaries at the end of SR. The values of n_p defined above are the occupation numbers of electromagnetic eigenmodes of this volume.

The arguments developed above allow the correction of a weak point in the reasoning of Arecchi and Courtens which brought us to Eqn (2.17): the shape of the train of emitted waves is other than cylindrical. These waves take a time τ_s to cross the end of the rod; therefore, $I\tau_s \sim E_0$, which once again brings us to Eqn (2.17).

The theory of SR has been pursued in several directions, the most important being the Schrödinger and the Heisenberg approaches, the semiclassical approximation. Each is applicable for a particular range of parameters of the system, and therefore these approaches are supplementary to one another. These directions will be discussed later; here we just point to the common methodological flaw of all these approaches: like in Ref. [8], the mechanism of phasing remains 'behind the scenes', although practically all papers on SR contain perfectly correct but methodologically useless statements like "... phase correlations between atoms arise by virtue of the common radiation field...". It would be appropriate to quote a passage from Ref. [5] (p. 188) related to the mechanism of transition from random to phased state of the atoms: "The nature of this transition, which displays certain space, time and statistical properties, is not completely known". Such a conclusion is apparently rooted in the fact that the above approaches are based on quantum concepts: operators, density matrix, averages over the quantum ensemble, and the like. Obviously, this is the most comprehensive description. Any lecturer, however, will agree that the feeling of complete understanding can only be achieved in the consideration of the quasiclassical limit. A good example is the problem of quark confinement [42], where the search for the classical solution with the string is actively pursued, notwithstanding the fact that the issue has been resolved to satisfaction by numerical quantum lattice calculations.

So one may hope that the nature of phasing of the atoms will be better understood with the aid of the classical model of superradiance (CMS) [43–46], which treats the atoms as classical Lorentz oscillators (generally speaking, anharmonic), where the field is described by the classical Maxwell equations.

According to the Schrödinger approach [19, 20, 47], detailed in Ref. [2], first constructed is the Liouville–Neumann quantum equation for the density matrix Φ of the entire system (*atoms + field*), which is then used in the Born–Markov approximation which holds under condition (2.12) for deriving the equation for the density matrix of atoms $\rho = Sp\Phi$, where the trace is taken with respect to field variables ('master equation'). In the Born approximation, which holds in the limit of Eqn (2.12), $\Phi = \rho\rho_{\rm f}$, where $\rho_{\rm f} = |0\rangle\langle 0|$ is the field density matrix in the initial (vacuum) state. This approach does not take into account the distortion of ρ and $\rho_{\rm f}$ caused by the interaction between atoms and field, which is similar to the conventional Born approximation in the scattering theory [26, 48]. Under condition (2.12), the energy of the field contained within the medium is smaller compared with the energy stored in the atoms: the ratio of these two energies is of the order of $L/c\tau_{\rm s} \ll 1$. Because of this, the back influence of the field on the state of the atomic subsystem is small, and the calculations of the behavior of the atomic subsystem may assume that the field occurs in the vacuum state. The key role of the field occurring inside the body consists in phasing the atoms, which does not require any energy expenditure. One may presume that Born's approximation becomes valid after the phasing is complete.

The Markovian approximation disregards the 'memory' effects which result from the elimination of field variables in the equation describing the behavior of the matrix of atom density ρ. The main memory effect is the retardation of electromagnetic waves, which under condition (2.12) is of little consequence. In this case the derivative $\mathrm{d}\rho/\mathrm{d}t$ is determined by the value of $\rho(t)$ at the same instant, as each subsequent jump of Brownian particle does not depend on its history.

As pointed out in Ref. [2], a drawback of the Schrödinger approach is the complexity of the master equation, and hence the lack of physical clarity. In this respect it falls short of the Heisenberg and the semiclassical approaches that will be discussed later and that have yielded more nontrivial results. The greatest advances in the Schrödinger approach have been made in the framework of the one-mode model of SR [19–21], which has a limited and not well fathomed range of applicability. Apparently (although this has yet to be proved), this model is good for thin rods with $F \ll 1$.

The idea of this review was conceived in connection with the preparation of lectures, and one of its purposes is therefore methodological. In a sense, this review is comprised of two parts: SR in atoms, and SR in the system of classical oscillators (CMS). The first part is of methodological interest, since the known results are presented in a clear and simple manner. The presentation also includes a number of original results, some of which have been published in Ref. [46]. The latter include the nonlinear effect of diminishing angular divergence of the SR beam. The second part of the review is mainly concerned with the original results (detailed calculations can be found in Ref. [46]). We start with the nonlinear mechanism of phasing of atoms in the pointlike Dicke model [43], and show that phasing disappears when the spread of atomic frequencies is sufficiently large. Then we analyze papers [49–51], which point out the inconsistency of the Dicke model which fails to include the dipole–dipole interaction. We show that the conclusions made in Refs [49–51] with regard to destruction of SR in small systems [as defined by Eqn (2.4)] by dipole–dipole interaction are not correct. This interaction has been consistently taken into account for a sufficiently general class of ellipsoidal bodies, and the feasibility of SR in such bodies has been proved, which may stimulate experiments with small bodies as defined by Eqn (2.4). Metastable nonradiative states are discovered which differ from their Dicke counterparts [8] assumed by the body as a result of dipole–dipole interaction between the atoms. There are two mechanisms of phasing of atoms: nonlinear and linear dipole–dipole mechanisms. These mechanisms compete with each other and act in opposite directions: the first mechanism assumes that each atom creates a cloud of surrounding atoms oscillating in sympathy, whereas the second holds that the surrounding atoms oscillate in antipathy. In this way, the dipole–dipole interaction results in screening of SR, a kind of coherent blocking. The dipole–dipole interaction is demonstrated to be a long-range effect, which implies that the pattern of SR depends considerably on the shape of the body. In ellipsoidal bodies the dipole fields cancel out to some extent, and SR proceeds in full agreement with Dicke's predictions [8].

This problem has three characteristic lengths: λ, L, and $l = n^{-1/3}$. Below we consider the most interesting case of $N \gg 1$; therefore, we always have $l \ll L$, and there are only three possibilities that will be discussed in this review:

$$l \ll L \ll \lambda\,, \tag{2.42a}$$

$$\lambda \ll l \ll L\,, \tag{2.42b}$$

$$l \ll \lambda \ll L\,. \tag{2.42c}$$

Along with the two-level systems where nonlinear phasing plays a crucial role, we consider weakly linear systems — for example, electrons in magnetized plasma — and perform detailed calculations in the approximation of linear CMS. We demonstrate that the transfer of collective cyclotron radiation across the magnetic force lines, which is one of the manifestations of SR, may in principle be responsible for the effect of anomalous energy transfer that defies theoretical treatment and is observed in tokamaks, necessitating an increase of dimensions of the installation.

At the end of the review we summarize the results and formulate the issues that call for further investigation.

Further on we always disregard the noncoherent component of radiation of the medium, assuming it to be small compared with the coherent part. The noncoherent emission of radiation by excited medium is well studied, and is described by the Biberman–Holstein equation (see, for example, Ref. [11]). Like coherent radiation, it is resonant.

Let us now embark on the quantitative description of the principal effects of SR.

3. Ensemble of atoms in an electromagnetic field. Quantum mechanical description

This section pursues methodological goals. Here we express the Hamiltonian of the system of atoms and field, and discuss the main approximations.

Assume that there are N atoms whose nuclei bear charge z and rest at points r_a $(a = 1, 2, \ldots, N)$. The behavior of atoms in the electromagnetic field $\mathbf{A}_\mu(\mathbf{r}, t) = (\varphi, \mathbf{A})$ is described by

the Hamiltonian (here and further $\hbar = m_e = e = 1$)

$$\hat{H} = \sum_{a,s} \left\{ \frac{1}{2} \left[-\mathrm{i}\nabla_{as} + \frac{1}{c}\,\mathbf{A}(\mathbf{r}_{as}, t) \right]^2 - \varphi(\mathbf{r}_{as}, t) \right\} + u\,,$$

where $\mathbf{r}_{as}$ are the coordinates of electrons pertaining to atom a (we disregard the exchange of electrons between the atoms); $s = 1, \ldots, z$; u is the potential energy of electrons and nuclei. The Hamiltonian is simplified in the dipole approximation ($\boldsymbol{\rho}_{as} = \mathbf{r}_{as} - \mathbf{r}_a$):

$$\mathbf{A}(\mathbf{r}_{as}, t) \approx \mathbf{A}(\mathbf{r}_a, t)\,;$$
$$\varphi(\mathbf{r}_{as}, t) \approx \varphi(\mathbf{r}_a, t) + \boldsymbol{\rho}_{as}\nabla_{\mathbf{r}_a}\varphi(\mathbf{r}_a, t)\,, \tag{3.1}$$

which holds under the condition

$$\lambda \gg a_0 = 1\ \text{a.u.}, \tag{3.2}$$

where λ is the characteristic wavelength.

In the Schrödinger equation $\mathrm{i}\,\partial\Psi/\partial t = \hat{H}\Psi$ for the wave function of electrons $\Psi(\mathbf{r}_{as}, t) \equiv \Psi(\mathbf{r}_{1s}, \ldots, \mathbf{r}_{N,s}; t)$ we carry out the gauge transformation

$$\Psi(\mathbf{r}_{as}, t) = \exp\left[-\mathrm{i}\sum_{a,s} \boldsymbol{\rho}_{as}\mathbf{A}(\mathbf{r}_a, t) - \mathrm{i}\int_{-\infty}^{t} \varphi(\mathbf{r}_a, t')\,\mathrm{d}t' \right] \Psi'.$$

By direct substitution we prove that the Schrödinger equation for the new wave function Ψ' assumes the form $\mathrm{i}\,\partial\Psi'/\partial t = \hat{H}'\Psi'$, where

$$\hat{H}' = \sum_{a,s} \left(-\frac{1}{2}\,\nabla_{as}^2 \right) + u + \hat{v}\,, \tag{3.3}$$

$$\hat{v} = -\sum_a \mathbf{d}_a \mathbf{E}(\mathbf{r}_a, t)\,, \tag{3.4}$$

where $\mathbf{d}_a = -\sum_{s=1}^{z} \boldsymbol{\rho}_{as}$ is the operator of dipole moment of atom a, and

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\varphi - \frac{1}{c}\,\frac{\partial\mathbf{A}}{\partial t}$$

is the electric field. Thus, the formulas (3.3), (3.4) hold for an arbitrarily extended system of atoms under condition (3.2) (and not under the condition $L \ll \lambda$, as it is sometimes argued).

Following Ref. [8], let us consider the approximation of two-level atoms, when only the two states with the wave functions Ψ_1 and Ψ_2, and energies E_1 and E_2 ($E_2 > E_1$) are taken into account. Levels E_1 and E_2 are assumed to be nondegenerate.

The wave function of a two-level atom may be written in the form $\psi = c_1\psi_1 + c_2\psi_2$, or as a two-component vector $e = (c_1, c_2)$. Accordingly, the operator of any physical quantity can be represented as a complex Hermitian (2×2) matrix acting upon the column e. Thus, the description of states of a two-level atom is completely similar to the description of states of the spin $s = 1/2$. In particular, the states α and β with $s_z = +1/2$ and $s_z = -1/2$ correspond to the upper and lower states of the atom. The states of the system of N two-level atoms may be interpreted as the eigenstates of operators τ^2 and τ_z, where $\boldsymbol{\tau} = \sum_{a=1}^{N} \mathbf{s}_a$ is the total (energy) spin: $\psi = |\tau, M, \alpha\rangle$. Here M is the eigenvalue of τ_z. The meaning of quantum number α is explained in Ref. [1].

At first we shall disregard collisions and Doppler broadening caused by the motion of atoms. The latter becomes important in an extended body (2.42c). It will be rigorously included in the analysis of SR in plasma.

After standard [28, 48] secondary quantization of the electromagnetic field, the Hamiltonian of the system of atoms and field becomes

$$\hat{H} = \hat{H}_0 + \hat{v}\,, \tag{3.5}$$

$$\hat{H}_0 = \sum_k \omega_k \hat{c}_k^+ \hat{c}_k + \frac{1}{2}\,\omega_0 R_3\,, \tag{3.6}$$

$$\hat{v} = \sum_k (f_k c_k + f_k^+ c_k^+)\,, \tag{3.7}$$

$$f_k = \sqrt{\frac{2\pi\omega_k}{v}}\,(\mathbf{e}_k, \mathbf{Q}_k) = g_k(R_k^+ + R_{-k}^-)\,,$$

$$\mathbf{Q}_k = \sum_a \hat{\mathbf{d}}_a \exp(\mathrm{i}\mathbf{k}\mathbf{r}_a) = \mathbf{d}(R_{\mathbf{k}}^+ + R_{-\mathbf{k}}^-)\,.$$

Here we use the notation

$$R_3 = R_0^z = \sum_a \sigma_a^z\,, \qquad R_{\mathbf{k}}^z = \sum_a \sigma_a^z \exp(\mathrm{i}\mathbf{k}\mathbf{r}_a)\,,$$
$$R_{\mathbf{k}}^{\pm} = \sum_a \sigma_a^{\pm} \exp(\pm\mathrm{i}\mathbf{k}\mathbf{r}_a)\,, \tag{3.8}$$

and also denote the operators of annihilation and creation of photons by $\hat{c}_k$, $\hat{c}_k^+$, the quantum numbers of photons (momentum, polarization, $\mu = \pm 1$) by $k = (\mathbf{k}, \mu)$; $\omega_k = ck$,

$$\sigma_a^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_a = 2s_a^z\,;$$

$$\sigma_a^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_a, \qquad \sigma_a^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_a,$$

$g_k = \sqrt{2\pi\omega_k/v}\,(\mathbf{e}_{k\mu}\mathbf{d})$; v is the normalization volume, $\mathbf{d} = \langle 2|\mathbf{d}|1\rangle$; $\mathbf{e}_{k\mu}$ is the photon polarization vector. In the transition from Eqn (3.4) to Eqn (3.7) we use the relation

$$\hat{\mathbf{d}}_a = \mathbf{d}(\sigma_a^+ + \sigma_a^-) \tag{3.9}$$

and reverse the sign of $\hat{c}_k$, $\hat{c}_k^+$, which is a canonical transformation not affecting the observables. This sign convention is used in practically all works on SR.

Operators (3.8) satisfy the commutation relations

$$[R_{\mathbf{k}}^z, R_{\mathbf{k}'}^{\pm}] = \pm 2R_{\mathbf{k}'\pm\mathbf{k}}^{\pm}\,, \qquad [R_{\mathbf{k}}^+, R_{\mathbf{k}'}^-] = R_{\mathbf{k}-\mathbf{k}'}^z\,. \tag{3.10}$$

As usual, the apparent simplicity of the Hamiltonian (3.5) for a many-body system conceals the diverse and complex phenomena that we are going to study.

4. A Dicke model

In 1954 Dicke [8] looked into spontaneous radiation by a system of atoms of small size brought at $t = 0$ into an excited state, and found that it differs dramatically from the conventional spontaneous radiation, when atoms radiate independently of one another [see Eqns (2.1)–(2.3)]. Currently this phenomenon is called superradiance, although the original term coined by Dicke — collective spontaneous radiation — is no less germane. Later it became clear that the most striking feature of SR is not that it differs from conventional spontaneous emission, but the fact that phase correlations (phasing) arise between the initially independent

atoms, which is rather similar to a phase transition of the second order. This aspect of SR will be discussed later; here we shall analyze Dicke's original work and its conventions, and consider the quantum properties of SR.

The effects of propagation of electromagnetic waves, which are important in extended bodies (2.15), bear no direct relation to the mechanisms of SR. Accordingly, and following Ref. [8], we consider the case of Eqn (2.4) — that is, Eqn (2.42a).

Dicke assumed that the main role is played by the wave vectors near the resonance,

$$k \sim k_0 = \frac{\omega_0}{c} \sim \frac{1}{\lambda}\,, \tag{4.1}$$

therefore, by virtue of Eqn (2.4), in Eqn (3.8) we may set

$$\exp(\mathrm{i}\mathbf{k}\mathbf{r}_a) \approx 1\,. \tag{4.2}$$

This, however, is not correct, since an important role in the case of (2.4) belongs to the dipole–dipole interaction of atoms. We shall see shortly that this interaction embraces the entire body — that is, it is characterized by the length L and the wave vectors

$$k \sim \frac{1}{L} \gg \frac{1}{\lambda}\,. \tag{4.3}$$

We represent the Hamiltonians (3.5)–(3.7) in the form $\hat{H} = \hat{H}_1 + \hat{H}_2$, where $\hat{H}_1$ and $\hat{H}_2$ include terms with, respectively, $k < k_1$ and $k > k_1$, where k_1 satisfies the condition $k_0 \gg k_1 \gg 1/L$. In the first sum ($\hat{H}_1$) the approximation (4.2) holds that yields Dicke's Hamiltonian $\hat{H}_\mathrm{D}$ [8]:

$$\hat{H}_1 = \hat{H}_\mathrm{D} = \frac{1}{2}\,\omega_0 R_3 + \sum_k \left[\omega_k c_k^+ c_k + g_k (R^+ c_k + R^- c_k^+)\right], \tag{4.4}$$

where $R^\pm = \sum_a \sigma_a^\pm$. Here we have dropped the nonresonant terms $R^+ c_k^+$ and $R^- c_k$. They define the Lamb shift [5] which we assume to be already included in ω_0. This is the approximation of rotating wave [1, 2, 5, 31]. In addition, in $\hat{H}_1$ we set $k_1 = \infty$. This leads to ultraviolet divergences commonly encountered in electrodynamics, which are resolved by renormalization [28] of the constants occurring in Eqn (4.4). In the term $\hat{H}_2$ the field is the fast subsystem, and the atoms are the slow one; this allows the separation of the fast variables using the Born–Oppenheimer method known in the theory of molecules [26, 48]. So, we have

$$\hat{H}_2 = \sum_{k>k_1} (\omega_k c_k^+ c_k + f_k c_k + f_k^+ c_k^+)\,.$$

According to the Born–Oppenheimer method, at the first step the atom operators must be regarded as c-numbers. In $\hat{H}_2$ we select the 'perfect square' — that is, perform the canonical transform

$$c_k = b_k - \frac{f_k}{\omega_k}\,, \tag{4.5}$$

which reduces the Hamiltonian $\hat{H}_2$ to the form

$$\hat{H}_2 = \sum_{k>k_1} \omega_k b_k^+ b_k + V_\mathrm{d}\,, \tag{4.6}$$

$$V_\mathrm{d} = -\sum_k \frac{1}{\omega_k} f_k^+ f_k = -\frac{1}{2}\sum_{a\neq b} (\mathbf{d}_a)_\alpha (\mathbf{d}_b)_\beta G_{\alpha\beta}(\mathbf{r}_{ab})$$

$$= \frac{1}{2}\sum_{a\neq b} \frac{1}{(r_{ab})^3}\left[\mathbf{d}_a\mathbf{d}_b - 3(\mathbf{d}_a\mathbf{s})(\mathbf{d}_b\mathbf{s})\right], \tag{4.7}$$

$$G_{\alpha\beta}(\mathbf{r}_{ab}) = \frac{3 s_\alpha s_\beta - \delta_{\alpha\beta}}{(r_{ab})^3}\,. \tag{4.8}$$

Here and further $\alpha, \beta, \gamma \ldots = x, y, z$; the summation is carried out with respect to repeated indices; $\mathbf{s} \equiv \mathbf{n}_{ab} = \mathbf{r}_{ab}/r_{ab}$, $\mathbf{r}_{ab} = \mathbf{r}_a - \mathbf{r}_b$. Calculation of Eqn (4.7) is based on the formulas

$$\sum_\mu (\mathbf{e}_{\mathbf{k}\mu})_\alpha (\mathbf{e}_{\mathbf{k}\mu})_\beta = \delta_{\alpha\beta} - \hat{\mathbf{k}}_\alpha \hat{\mathbf{k}}_\beta\,, \qquad \hat{\mathbf{k}} = \frac{\mathbf{k}}{k}\,,$$

$$\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, \hat{k}_\alpha \hat{k}_\beta \exp(\mathrm{i}\mathbf{k}\mathbf{r}) = \frac{-3\hat{R}_\alpha \hat{R}_\beta + \delta_{\alpha\beta}}{4\pi R^3}\,,$$

where $\hat{\mathbf{R}} = \mathbf{R}/R$, $\mathbf{R} \neq 0$. We also note that

$$\frac{1}{v}\sum_\mathbf{k} F_\mathbf{k} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, F_\mathbf{k}\,,$$

and take advantage of the fact that the main contribution to the integral over $\mathbf{k}$ comes from $k \sim 1/R \sim 1/L$, and therefore the integration can be extended to include $k < k_1$. Observe that the nonresonant terms are not important in $\hat{H}_1$, but they are important in $\hat{H}_2$. If we drop them at the very beginning in the initial expressions (3.5)–(3.7), we get a result for V_d that differs from Eqn (4.7) by a factor of 1/2.

Let us now write the final expression for the Hamiltonian of the system of small size (2.4)

$$\hat{H} = \hat{H}_\mathrm{D} + \hat{V}_\mathrm{d}\,. \tag{4.9}$$

In Eqn (4.9) we have dropped the terms of the order of $1/c^2$, which were retained in Refs [49–51]. This approximation will be proved rigorously later.

Dicke [8] disregarded the dipole interaction V_d, which is not correct [49–51, 1]. Indeed, the terms with the coupling constant g_k in Eqn (4.4) in the classical limit describe radiation friction (we shall discuss this point again later in connection with the CMS). The electric field that causes radiation damping of oscillations of dipoles $\mathbf{E}_1 = 2\dddot{\mathbf{D}}/3c^3$ is determined by the total dipole moment of the system $\mathbf{D} = \sum_a \mathbf{d}_a(t)$, and by order of magnitude is $E_1 \sim \omega_0^3 N d_a/c^3 \sim N d_a/\lambda^3$. The local electric field created by dipoles is of the order of $E_2 \sim d_a/l^3 \sim n d_a$. Accordingly, $E_1/E_2 \sim (L/\lambda)^3 \ll 1$, which proves the mistake of Ref. [8]. As will be proved below, the inclusion of the dipole–dipole interaction radically changes the situation. The author is not aware of publications that treat this interaction in a consistent way, so we shall include here detailed calculations. Ellipsoidal bodies exhibit compensation of dipole interaction; accordingly, in this section we discuss Dicke's work [8] and its implications under the assumption that $\hat{H} = \hat{H}_\mathrm{D}$:

$$\hat{H} = \hat{H}_0 + \hat{v}\,; \qquad \hat{H}_0 = \sum_k \omega_k \hat{c}_k^+ \hat{c}_k + \frac{1}{2}\,\omega_0 R_3\,;$$

$$\hat{v} = \sum_k g_k (R_+ \hat{c}_k + R_- \hat{c}_k^+)\,. \tag{4.10}$$

The total moment τ is a conserved quantum number, since $[\tau^2, \hat{H}] = 0$. It takes on the values $0 \leqslant \tau \leqslant N/2$. Operator $\hat{v}$ adds one to or subtracts one from z — projection of total spin $\tau_z = (1/2)R_3$ — $(M_\mathrm{f} - M_\mathrm{i} = \pm 1)$, and creates or annihilates one photon. If we 'switch off' $\hat{v}$, we get a continuum of field states, and an equidistant spectrum of states of matter, independent of each other:

$$E_{\tau M} = \omega_0 M\,, \qquad -\tau \leqslant M \leqslant \tau\,.$$

According to Fermi's 'golden rule', the rate of transition $(\tau, M) \to (\tau, M-1)$ is

$$\lambda_M = \sum_k \left|\langle 1k, \tau, M-1|\hat{v}|0, \tau, M\rangle\right|^2 2\pi\delta(\omega_k - \omega_0)$$
$$= (\tau - M + 1)(\tau + M)\lambda_0\,, \tag{4.11}$$

where $\lambda_0 = 1/\tau_0 = 4\omega_0^3 d^2/3c^3$ is the rate of spontaneous transition $2 \to 1$ for an isolated atom. Assume that at $t = 0$ the system of atoms occurs in the state $|\tau, M_0\rangle$. As a result of emission, the mean energy $\bar{E} = \omega_0 \bar{M}$ of the atoms decreases according to the law $\mathrm{d}\bar{E}/\mathrm{d}t = -I$, where $I = \omega_0 \lambda_M \approx I_0(\tau^2 - M^2)$ is the total intensity of radiation, $I_0 = \omega_0 \lambda_0$ is the intensity of radiation of isolated atom. Assuming the dispersion of M to be small ($\Delta M \ll \tau$, $M \approx \bar{M}$) and using Eqn (4.11), we find

$$M = -\tau \tanh\left[\lambda_\mathrm{s}\tau(t - t_0)\right]\,; \tag{4.12}$$

$$I = I_0 \frac{\tau^2}{\cosh^2\left[\lambda_0\tau(t - t_0)\right]}\,, \tag{4.13}$$

where t_0 is the pulse delay found from the initial condition

$$M_0 = \tau \tanh(\lambda_0 \tau t_0)\,. \tag{4.14}$$

Especially interesting is the case of the system of atoms completely inverted at $t = 0$:

$$\tau = \frac{N}{2} = M_0\,. \tag{4.15}$$

In order to find t_0, in Eqn (4.14) we set $M_0 = \tau - k$, $k \sim 1$ (this exposes the mistake made by replacing $M \to \bar{M}$). Then, to the logarithmic accuracy, we have

$$t_0 = \frac{1}{2\lambda_0\tau} \ln \tau\,. \tag{4.16}$$

From Eqn (4.13) we see that the delay time t_0 is followed by a strong ($I \sim I_0 N^2$) short ($\tau_\mathrm{s} \sim 1/\lambda_0 N$) pulse. Note that $t_0 \sim \tau_\mathrm{s} \ln N \gg \tau_\mathrm{s}$.

Strictly speaking, result (4.13) only holds when $k \gg 1$ [2] [the expression under the logarithm in Eqn (4.16) then must be replaced by τ/k], when the intensity fluctuations are small. In the case of $k \sim 1$ there are substantial quantum fluctuations:

$$\frac{\Delta I}{I} \sim 1\,. \tag{4.17}$$

In other words, at $k \sim 1$ we are dealing with a macroscopic ($N \gg 1$) quantum system. Indeed, according to Eqn (4.12), (4.13) the maximum of intensity occurs at the state with zero inversion ($M = 0$), attained by the system via the states $0 \leqslant M \leqslant r$, $r = N/2$. This takes the time $t_0 = \sum_{M=0}^{r} T_M$, where T_M is the time of transition $|r, M\rangle \to |r, M-1\rangle$. The dispersion of t_0 is

$$\Delta t_0 = \left[\bar{t}_0^2 - (\bar{t}_0)^2\right]^{1/2} = \left[\sum_M \overline{(\Delta t_M)^2}\right]^{1/2} \sim \left(\sum_M \lambda_M^{-2}\right)^{1/2} \sim \tau_\mathrm{s}\,, \tag{4.18}$$

whence follows Eqn (4.17). The calculation of the mean pulse delay

$$\bar{t}_0 = \sum_M \overline{T_M} = \sum_M \frac{1}{\lambda_M}$$

again brings us to Eqn (4.16).

A direct quantum mechanical calculation of the mean intensity and its dispersion is instructive. Assume that the body occurs in state $|\mathrm{i}\rangle$ with energy E_i. Then the classical formula (2.6) (see Ref. [28]) must be rewritten in the form

$$I = \sum_\mathrm{f} \frac{4}{3c^3}(E_\mathrm{f} - E_\mathrm{i})^4 \left|\langle f|\mathbf{D}^-|\mathrm{i}\rangle\right|^2\,, \tag{4.19}$$

where $\hbar = 1$, and from the total dipole moment of the system $\mathbf{D} = \mathbf{D}^- + \mathbf{D}^+$ ($\mathbf{D}^\pm = \mathbf{d}R_\pm$) we only retain the negative-frequency part $\mathbf{D}^-$. Then the sum (4.19) only includes terms corresponding to the transitions of the system of atoms from upper to lower levels, as it ought to be in the case of radiation into vacuum (see Section 1). Making use of the completeness of basis $|f\rangle$, we transform Eqn (4.19) to the form

$$\bar{I} = \langle \hat{I} \rangle\,, \tag{4.20}$$

$$\hat{I} = \frac{4}{3c^3}\,\mathbf{Q}^+\mathbf{Q} = I_0 R_+ R_-\,, \tag{4.21}$$

where $I_0 = \hbar\omega_0\lambda_0$, $Q = \left[\hat{H}_\mathrm{a}[\hat{H}_\mathrm{a}, \mathbf{D}^-]\right] = \omega_0^2 \mathbf{d}R_-$, $\hat{H}_\mathrm{a} = \omega_0 R_3/2$ is the Hamiltonian of atoms. Now we must recall that the body is not a closed system, and is therefore necessarily described by the density matrix ρ which we are presently going to calculate. Hamiltonian (4.10) commutes with operator $(1/2)R_3 + \sum_k c_k^+ c_k$; because of this, the sum of the number of excited atoms and the number of photons is conserved. Assume for definiteness that at $t = 0$ all atoms are excited, and the number of photons is zero. Then the vector of state of the system will be written in the form

$$|\psi\rangle = \sum_{M=-r}^{M=r} c_M(t)|r, M\rangle \otimes |r - M \text{ photons}\rangle\,. \tag{4.22}$$

The orthogonality of the photon states implies that matrix ρ is diagonal: $\rho = \mathrm{Sp}_\mathrm{f}\left(|\psi\rangle\langle\psi|\right) = \mathrm{diag}(\rho_M)$, where the trace (spur) is taken with respect to field variables. Numbers ρ_M on the diagonals of matrix ρ are the populations of states $|r, M\rangle$ and obey the master equation which assumes the form of a kinetic equation:

$$\dot{\rho}_M = -\lambda_M \rho_M + \lambda_{M+1}\,\rho_{M+1}\,. \tag{4.23}$$

Observe that the accuracy of this equation is very high (of the order of $1/\omega_0\tau_\mathrm{s}$). An analytical solution of Eqn (4.23) is given in Ref. [2] [see Eqn (5.33)]. It holds in the limit $N \gg 1$ and permits calculation of the mean intensity:

$$\bar{I} = \mathrm{Sp}(\rho\hat{I}) = \sum_{M=-r}^{r} \rho_M I_{MM}\,,$$

where $I_{MM} = \hbar\omega_0\lambda_M$ in accordance with Eqn (4.21), and the dispersion of the intensity $\Delta I^2 = \mathrm{Sp}\big[\rho(\hat{I}^2 - (\bar{I})^2\big]$, which once again proves the validity of Eqn (4.17).

There may be some uncertainty concerning the physical meaning of operator $\hat{I}$. Because of this, and so as to reconfirm the quantum nature of the pointlike Dicke system of atoms, we shall consider a direct experiment designed to measure the characteristics of the system. Some distance ρ away from the body, for the sake of simplicity in the wave zone ($L \ll \rho \ll \lambda$), we place the Unruh detector [52–54] which is a two-level atom with the gap ΔE between levels $|\mathrm{i}\rangle$ and $|\mathrm{f}\rangle$ (the operator of its dipole moment we denote by $\hat{\mathbf{d}}$). In the neighborhood of this atom, the Dicke system creates a field

$$\hat{\mathbf{E}} = \frac{3\hat{\boldsymbol{\rho}}(\hat{\boldsymbol{\rho}}\hat{\mathbf{D}}) - \hat{\mathbf{D}}}{\rho^3}\,, \tag{4.24}$$

which excites the atom as a result of interaction $-\hat{\mathbf{d}}\hat{\mathbf{E}}$ (which is considered sufficiently small). Observe that this interaction is included in operator $\hat{V}_{\mathrm{d}}$. Assume that the excited atom, by contrast to the non-excited one, is capable of starting some kind of chain chemical reaction. This means that it will be easy to measure the probability W of excitation of the atom. From perturbation theory we find

$$W \sim J = \iint_{-\infty}^{\infty} \mathrm{d}t\,\mathrm{d}t'\,\exp\big[\mathrm{i}\Delta E(t-t')\big]\big\langle \mathrm{i}\big|\hat{\mathbf{D}}(t)\hat{\mathbf{D}}(t')\big|\mathrm{i}\big\rangle, \tag{4.25}$$

where $\hat{\mathbf{D}}(t) = \exp(\mathrm{i}\hat{H}_{\mathrm{D}}t)\hat{\mathbf{D}}\exp(-\mathrm{i}\hat{H}_{\mathrm{D}}t)$ is the operator of dipole moment of the Dicke system in the Heisenberg representation. If the system were classical, then the operator $\hat{\mathbf{D}}$ in Eqn (4.24) could be replaced by the classical dipole moment $\mathbf{D}$; in place of Eqn (4.25) we would then get

$$J = \left|\int_{-\infty}^{\infty} \mathrm{d}t\,\exp(\mathrm{i}\Delta E t)\mathbf{D}(t)\right|^2. \tag{4.26}$$

From equations (4.25) and (4.26) we conclude that the system is classical (but made up of quantum microscopic objects) under the condition

$$\big\langle \mathbf{D}(t)\mathbf{D}(t')\big\rangle \approx \big\langle \mathbf{D}(t)\big\rangle\big\langle \mathbf{D}(t')\big\rangle\,. \tag{4.27}$$

The latter is only possible when the spread with respect to quantum numbers M is small:

$$\Delta M \ll N\,, \tag{4.28}$$

that is, in the state with small quantum fluctuations M. In such states any two physical quantities obey the relation

$$\langle AB\rangle \approx \langle A\rangle\langle B\rangle\,. \tag{4.29}$$

These are packets of the form

$$\psi = \sum_M c_M|M\rangle\,, \tag{4.30}$$

where c_M are slow functions of M localized near the mean value $\bar{M} = \sum_M |c_M|^2 M$; $1 \ll \Delta M \ll N$, $\Delta M^2 = \overline{M^2} - (\bar{M})^2$. As a matter of fact, in place of Eqn (4.30) we ought to write a more rigorous expression which follows from Eqn (4.22) upon replacement of $|r, M\rangle$ by a packet of the form (4.30).

In the state of SR (with $\bar{M} = 0$) the spread of M is large,

$$\Delta M \sim N\,, \tag{4.31}$$

and therefore relation (4.27) does not hold. This follows from Eqns (4.12), (4.18). Moreover, according to Eqn (4.22) $\langle\mathbf{D}\rangle = 0$, and therefore the right-hand side of Eqn (4.27) is zero, while the left-hand side is not.

It is interesting to observe the spreading of the system with respect to quantum number M. Time-differentiating the relations

$$\bar{M} = \sum_M M\rho_M\,, \qquad \overline{M^2} = \sum_M M^2\rho_M\,,$$

from Eqn (4.23) we get

$$\frac{\mathrm{d}\bar{M}}{\mathrm{d}t} = -\sum_M \lambda_M\rho_M\,,$$

$$\frac{\mathrm{d}M^2}{\mathrm{d}t} = -\sum_M (2M-1)\lambda_M\rho_M \approx -\sum_M 2M\lambda_M\rho_M\,,$$

$$\frac{\mathrm{d}\Delta M^2}{\mathrm{d}t} = \sum_M (2\bar{M} - 2M + 1)\lambda_M\rho_M \approx 2\sum_M (\bar{M} - M)\lambda_M\rho_M\,.$$

Hence, and from the expansion $\lambda_M \approx \lambda + \lambda'(M - \bar{M})$, where $\lambda \equiv \lambda_{\bar{M}}$, $\lambda' = \mathrm{d}\lambda_M/\mathrm{d}M$ (at $M = \bar{M}$), we get $\mathrm{d}\bar{M}/\mathrm{d}t = -\lambda$, $\mathrm{d}\Delta M^2/\mathrm{d}t = -2\lambda'\Delta M^2$. Integrating these equations, we get

$$\frac{\Delta M^2}{\Delta M_0^2} = \frac{\lambda^2}{\lambda_{M_0}^2}\,, \tag{4.32}$$

where ΔM_0^2 is the dispersion of M at $t = 0$. Assuming that in the initial state $\Delta M_0 \sim 1$, from Eqns (4.11) and (4.32) we find that $\Delta M^2 \sim (r^2 - \bar{M}^2)^2/r^2$, whence follows Eqn (4.31).

We see that the Dicke system, flung into the upper state, behaves as a typical microscopic quantum object. It differs from a classical object in the same way as a hydrogen atom differs from the hypothetical classical atom with Keplerian orbits of size of order a_0. However, it is difficult or even impossible to stage an experiment aimed at observing such behavior. The fact is that weak random external forces at the very beginning of the evolution bring the system into a classical state of type (4.30). Further evolution is determined by the classical equations of motion and is characterized by weak fluctuations in accordance with Eqn (4.32): $\Delta M/N \sim \Delta M_0/k \ll 1$, $k = r - M_0$. Here M_0 is the quantum number of the packet into which the system had collapsed as a result of the measurement occurring in the course of external interference. These ideas have been tacitly implied in Ref. [2]. The feasibility of measurements without the involvement of an observer is discussed in Refs [55–57]. According to Refs [55–57], it is the external world possessing a great number of degrees of freedom that is responsible for the irreversibility associated with measurements. Hopefully, the mechanism of irreversibility will be explained in the future.

Assume that we have staged a perfect experiment at $T \to 0$, and the external perturbations can be neglected. In such a case, the initial Dicke state is disturbed by the dipole–dipole interaction which does not conserve the cooperative quantum number τ:

$$[\tau^2, V_{\mathrm{d}}] \neq 0\,. \tag{4.33}$$

At $t = 0$ all atoms occur in state $|2\rangle$. If V_{d} is taken into account, there is a small admixture of state $|1\rangle$: $\psi = |2\rangle + \alpha|1\rangle$,

where

$$\alpha \sim \frac{1}{\hbar\omega_0}\sum_b (V_{\mathrm{d}})_{ab} \sim \frac{nd^2}{\hbar\omega_0}\,,$$

and therefore at $t = 0$

$$\langle R_-\rangle \sim \langle R_+\rangle \sim N\alpha\,. \tag{4.34}$$

Observe that the dipole – dipole interaction has fluctuating and permanent (long-range) parts (see below). The macroscopic characteristics of the body are determined by the permanent part, which is reflected in Eqn (4.34); the contribution from the fluctuating part is zero.

Let us now discuss the above-mentioned classical equations of motion and their implications that we shall need. They are most readily derived from Heisenberg's operator equations (such a derivation of classical equations is not ridiculous, since there is no such thing as a classical two-level atom). Recall that in the Heisenberg representation any operator A satisfies the equation $\mathrm{i}\hbar\,\mathrm{d}A/\mathrm{d}t = [A, H]$, where currently we have $H = H_{\mathrm{D}}$. The sought-for equations are

$$\mathrm{i}\dot{c}_k - \omega_k c_k = g_k R_-\,, \qquad \mathrm{i}\dot{c}_k^+ + \omega_k c_k^+ = g_k R_+\,,$$

$$\mathrm{i}\dot{R}_- = \omega_0 R_- - \sum_k g_k R_3 c_k\,, \qquad \mathrm{i}\dot{R}_+ = -\omega_0 R_+ + \sum_k g_k R_3 c_k^+\,,$$

$$\mathrm{i}\dot{R}_3 = 2\sum_k g_k (R_+ c_k - R_- c_k^+)\,. \tag{4.35}$$

The next step consists in replacing the operators in these equations with c-numbers — the mean of the packet (4.30) [displaying the property (4.29)]. The first two formulas in (4.35) are Maxwell equations. Noting that at $t = 0$ the field is absent, we find

$$c_k(t) = -\mathrm{i}g_k \exp(-\mathrm{i}\omega_k t)\int_0^t \exp(\mathrm{i}\omega_k t')R_-(t')\,\mathrm{d}t'\,. \tag{4.36}$$

The electric field is connected with the operators c_k, c_k^+ (now they are c-numbers):

$$\mathbf{E}(\mathbf{r}, t) = \sum_k \mathbf{e}_k \sqrt{\frac{2\pi\omega_k}{v}}\left[c_k \exp(\mathrm{i}\varphi) + c_k^+ \exp(-\mathrm{i}\varphi)\right], \tag{4.37}$$

where $\varphi = \mathbf{kr} - \omega_k t$. By contrast to c_k, c_k^+, the field is an observable variable.

We feel that a comment is due here. It is worth recalling that the 'field is force divided by charge' and to heed an original standpoint (see Ref. [54] and references therein) which argues that the observable is not the field, but rather the response of detector to this field — that is, the probabilities (4.25). From this standpoint (which essentially is operationalistic) it is easy to resolve the paradoxes contained in such questions as 'does a charge hovering above the earth radiate?', 'does a uniformly accelerated charge radiate?', etc. (these issues are discussed in Refs [39, 58, 59]). The nontrivial conclusion that follows consists in admitting non-invariance and the provisional nature of the concept of a particle [52, 60].

Obviously, the last three equations in (4.35) that describe the behavior of atoms involve the electric field within the limits of the system ($r \sim L$). Hence, after substituting Eqn (4.36) into (4.37) and carrying out summation with respect to k, we see that the main contribution to $\mathbf{E}(\mathbf{r}, t)$ comes from frequencies ω_k and times t' such that

$$|\omega_k - \omega_0| \lesssim \frac{c}{L}\,, \qquad t - t' \lesssim \frac{L}{c}\,. \tag{4.38}$$

This is the retardation of electromagnetic signals, and the above substitution yields nothing else but the formula of 'retarded potentials' for $\mathbf{E}(\mathbf{r}, t)$. In the asymptotic range $r \to \infty$, the first expression in (4.38) is changed: the principal role here is played by the resonant photons $\omega_k \approx \omega_0$, which corresponds to the fully developed diverging electromagnetic waves.

According to the third and fourth equations in (4.35),

$$R_\pm(t) = \exp(\pm\mathrm{i}\omega_0 t)F_\pm(t)\,, \tag{4.39}$$

where $F_\pm(t)$ are functions that vary slowly compared with the exponential coefficient:

$$\left|\frac{1}{F_\pm}\frac{\mathrm{d}F_\pm}{\mathrm{d}t}\right| \sim \frac{1}{\tau_{\mathrm{s}}} \ll \omega_0\,. \tag{4.40}$$

Combining Eqn (4.39) with (4.36), we get

$$c_k(t) = -\mathrm{i}g_k \exp(-\mathrm{i}\omega_k t)Q\,, \qquad Q = \int_0^t \mathrm{e}_1 F_-(t')\,\mathrm{d}t'\,, \tag{4.41}$$

where $\mathrm{e}_1 = \exp\left[\mathrm{i}(\omega_k - \omega_0)t'\right]$. Hence, from Eqn (4.38) and (4.40) we see that under condition (2.12), which is satisfied with certainty in the Dicke case (2.4), the quantity F_- in Eqn (4.41) varies slowly compared with e_1. From the second relation in (4.38) we infer that we must factor $F_-(t')$ out of the integral Q and set $t' = t$: $Q \to F_-(t)\int_0^t \mathrm{e}_1\,\mathrm{d}t'$. If this only concerns Eqn (4.41), this procedure is invalid. And yet, as explained above, it gives a correct value of the electric field.

After these manipulations and integration with respect to $\mathrm{d}t'$, we get the following replacement rule:

$$c_k(t) \to -\mathrm{i}g_k R_-(t) f(\omega_k - \omega_0)\,, \tag{4.42}$$

$$f(\Omega) = \frac{1 - \exp(-\mathrm{i}\Omega t)}{\mathrm{i}\Omega} \approx \pi\delta(\Omega) - \mathrm{i}P\left(\frac{1}{\Omega}\right), \tag{4.43}$$

where the latter approximation holds at $\omega_0 t \gg 1$ (see Appendix A in Ref. [48]). After substitution of Eqn (4.42) into (4.35), the last term in Eqn (4.43) gives a small frequency shift ω_0 (known as the collective Lamb shift). The sign of this frequency shift depends on R_3, and is therefore different in the beginning and at the end of SR (this is the effect of frequency modulation [1]). Since the shift is small compared to ω_0, in Eqn (4.43) we only retain the first term:

$$c_k(t) \to -\pi\mathrm{i}g_k R_-(t)\delta(\omega_k - \omega_0)\,,$$

$$c_k^+(t) \to \pi\mathrm{i}g_k R_+(t)\delta(\omega_k - \omega_0)\,. \tag{4.44}$$

Similar manipulations can be performed not only with c-numbers, but also with the Heisenberg operators. The operator relations (4.44) are the basis of the method of elimination of boson variables which holds, as we have seen, in the Markovian limit (2.12).

To avoid misunderstanding, the following explanation is due. We split the space of wave vectors of photons into two parts: $k < k_1$ and $k > k_1$. The latter gives the dipole – dipole interaction V_{d}. The photons from this range subsequently do

not take part in any interactions, and therefore the first term in Eqn (4.6) may be dropped, which leaves us only the photons from the first range. It is for these photons that the formulas of Eqn (4.44) hold; the second nonresonant term in Eqn (4.43) can be neglected. Alternatively [2, 51], it is possible not to define the two ranges of photons. Then the nonresonant term in Eqn (4.43) gives $V_{\rm d}$ [this is clear if only from Eqn (4.5) which corresponds to Eqn (4.43) at $\omega_k \gg \omega_0$]; the main contribution to $V_{\rm d}$ comes from the range $k > k_1$.

After substitution of Eqns (4.44) into the last three expressions in Eqns (4.35), we get the equations for the atomic subsystem with the eliminated field:

$$\dot{R}_- = -{\rm i}\omega_0 R_- + \frac{1}{2}\lambda_0 R_3 R_- ,$$

$$\dot{R}_+ = {\rm i}\omega_0 R_+ + \frac{1}{2}\lambda_0 R_3 R_+ , \qquad \dot{R}_3 = -2\lambda_0 R_+ R_- , \qquad (4.45)$$

where we have noted that

$$\lambda_0 = 2\pi \sum_k g_k^2 \delta(\omega_k - \omega_0) = \frac{4k^3 d^2}{3\hbar} . \qquad (4.46)$$

Equations (4.45) also hold for operators. The noncommutativity of R_3, $R_\pm$ introduces a negligibly small inaccuracy of the order of $1/N$.

The existence of the integral of motion τ^2 (the energy spin),

$$\tau^2 = \frac{1}{4} R_3^2 + \frac{1}{2}(R_+ R_- + R_- R_+) , \qquad (4.47)$$

allows Eqns (4.45) to be reduced to the equation $\dot{R}_3 = -2\lambda_0(\tau^2 - R_3^2/4)$ [in the case of c-numbers, the term in parentheses in Eqn (4.47) is $2R_+R_-$]. Its solution with the initial conditions (4.34) is given by Eqn (4.13), (4.13), where

$$t_0 = \frac{1}{\lambda_0 \tau} \ln \frac{1}{\alpha} . \qquad (4.48)$$

From the arguments developed above we conclude that experimental observation of the quantum behavior of the Dicke system is hardly possible, since it invariably passes into the classical state.

The phenomenon of SR has an elegant electromechanical analogy [61]. Consider a classical magnet with magnetic moment $\mathbf{M}$, which occurs at zero gravity in an external magnetic field $\mathbf{H}_0$. The magnet experiences the action of torque $\mathbf{M} \times \mathbf{H}_0$; it starts to rotate and therefore emits radiation. Losing energy by radiation, it finally arrives at the energetically most advantageous state $\mathbf{M} \parallel \mathbf{H}_0$. This process is described by equation [61] $\dot{\mathbf{M}} = \gamma \mathbf{M} \times (\mathbf{H}_0 + \mathbf{h})$, where γ is the gyromagnetic ratio, $\mathbf{h} = 2\dddot{\mathbf{M}}/3c^3 \approx -2\omega_0^2 \dot{\mathbf{M}}/3c^3$, $\omega_0 = \gamma|\mathbf{H}_0|$. This equation is easily integrable. For the projection of M_z onto $\mathbf{H}_0$ we get a formula similar to Eqn (4.12), and the intensity of radiation is given by a formula similar to Eqn (4.13).

Let us now give a brief summary of this section. Assume that at time zero there are no correlations between the atoms. Then some of the atoms undergo spontaneous transitions which lead to randomly occurring initial correlations of the phases of atoms. These correlations escalate and lead to collective emission. Observe that the mechanism of phasing is implicitly incorporated in Dicke's solution, and will be discussed specially in the sections to follow. We shall also discuss the effects of the dipole–dipole interaction of atoms $V_{\rm d}$, which was unduly neglected in Ref. [8]. We have already mentioned some of these effects in this section. Under the action of $V_{\rm d}$, and influenced by weak external forces, the Dicke ensemble of atoms behaves as a classical object. As explained above, the reason is that the dipole–dipole interaction creates small initial correlations, and together with spontaneous transitions triggers the mechanism of phasing of atoms.

5. Nonlinear mechanism of phasing of atoms

As indicated above, SR depends on the effect of phasing of atoms — that is, the inception of correlations between the initially independent atoms. In the beginning of this section we shall refine upon the meaning of these correlations and their linkage with the phase correlations in a system of classical dipoles. In Section 2 we noted that there are two competing mechanisms of phasing. In this section we use the pointlike Dicke model for analyzing the first of these — the nonlinear mechanism of phasing.

We rewrite Eqns (4.20), (4.21) in the form

$$\bar{I} = I_0(\sigma + S) , \qquad (5.1)$$

$$\sigma = \sum_a \langle \sigma_a^+ \sigma_a^- \rangle , \qquad S = \sum_{a \neq b} S_{ab} , \qquad S_{ab} = \langle \sigma_a^+ \sigma_b^- \rangle . \qquad (5.2)$$

The first term in Eqn (5.1) describes spontaneous transitions ($\sigma \sim N$). If $S_{ab} \neq 0$, then $S \sim N^2$, so the second term describes spontaneous emission.

In the Dicke model all atoms are equivalent, because $V_{\rm d}$ is dropped. Therefore, S_{ab} are the same for all (a, b), which allows the calculation of these correlators:

$$\langle R_+ R_- \rangle = (\tau + M)(\tau - M + 1) = N\langle \sigma_1^+ \sigma_1^- \rangle + N(N-1) S_{ab} .$$

Disregarding infinitesimals of the order of $1/N$, we find that

$$S_{ab} = \frac{1}{N^2}(\tau^2 - M^2) , \qquad (5.3)$$

where we have used the relations

$$\sigma^+ \sigma^- = \frac{1 + \sigma_z}{2} , \qquad \langle \sigma_1^z \rangle = \frac{1}{N}\langle R_3 \rangle = \frac{2M}{N} . \qquad (5.4)$$

If $S_{ab} \neq 0$, this does not necessarily mean that there are correlations between the atoms. For example, in the state

$$\psi = \prod_{a=1}^{N} \varphi_a , \qquad \varphi_a = c_1 |1\rangle_a + c_2 |2\rangle_a , \qquad (5.5)$$

calculation gives $\langle \sigma_a^+ \rangle = c_1^* c_2$, $\langle \sigma_a^- \rangle = c_1 c_2^*$, so $S_{ab} = \langle \sigma_a^+ \rangle \langle \sigma_b^- \rangle \neq 0$, but there are no correlations because $Q_{ab} = \big\langle (\sigma_a^+ - \langle \sigma_a^+ \rangle)(\sigma_b^- - \langle \sigma_b^- \rangle) \big\rangle = 0$.

Observe that at

$$c_1 = \cos\alpha \exp\left(\frac{{\rm i}\beta}{2}\right) , \qquad c_2 = \sin\alpha \exp\left(-\frac{{\rm i}\beta}{2}\right)$$

and $\beta = \omega_0 t + \varphi$, we have oscillating dipole moments $\langle \mathbf{d}_a \rangle$ from Eqn (2.5), whence we see correspondence between quantum and classical description of atoms. Note, however, that Eqn (5.5) is merely a special case of quantum states that agrees with the classical description.

The Dicke states $|\tau, M\rangle$ do not belong to the class of Eqn (5.5); for these states we have

$$\langle\sigma_a^+\rangle = \frac{\langle\sum_a \sigma_a^+\rangle}{N} = \frac{\langle R_+\rangle}{N} = 0\,, \qquad \langle\sigma_b^-\rangle = 0\,,$$

and therefore $S_{ab} = Q_{ab}$ — in other words, in the case of $S_{ab} \neq 0$ there are correlations between the atoms. In a sense, the Dicke state and the state of Eqn (5.5) are extreme cases. In the general case, a system of two-level atoms is described by the density matrix ρ of the format $2^N \times 2^N$. In special cases this matrix may split into fragments of lower dimension, or be diagonal, etc. In comparison with the classical case, the diagonal matrix ρ corresponds to a completely stochastic atomic system — that is, to the random initial phases φ_a.

If at $t = 0$ the system is brought into the Dicke state $|\tau, \tau\rangle$, then, according to Eqn (5.3), correlations are absent. As we have seen, with time the system passes into states $|\tau, M\rangle$ with $M < \tau$, for which $S_{ab} \neq 0$ — that is, the atoms become phased (correlated). The mechanism of phasing becomes clearer when we consider the classical limit — the CMS as described in Section 2.

This universal mechanism, proposed by A V Gaponov-Grekhov in 1960 [36], is described in full detail in Ref. [37]. In Ref. [43] it is demonstrated that this mechanism is responsible for phasing in the system of classical nonlinear oscillators, and eventually for the start of SR. It ought to be noted, however, that the dipole – dipole interaction in Ref. [43] was traditionally neglected, which is not justified. In this review we demonstrate that the dipole – dipole interaction leads to fast SR of some of the energy and to formation of a metastable state (the effect of screening). In many cases of practical interest, the nonlinearity at this stage is not manifested. The simultaneous description of dipole – dipole interaction and the effect of nonlinear phasing is a challenge for the future. One may anticipate that nonlinear phasing destroys the metastable state and results in the release of the remaining energy in the regime of SR. It is most likely that this is what happens in the system of two-level atoms.

Following Ref. [43], we disregard the dipole – dipole interaction and consider one-dimensional charged oscillators ($a = 1, 2, \ldots, N$), concentrated in the quasistatic region $L \ll \lambda$:

$$\ddot{x}_a + \omega_0^2(1 + \gamma x_a^2)x_a = -\frac{2e^2\omega_0^2}{3mc^3}\sum_b \dot{x}_b\,,$$

where the right-hand side represents the radiation friction. After substitution

$$x_a = b\big[F_a(t)\exp(-\mathrm{i}\omega t) + F_a^*(t)\exp(\mathrm{i}\omega t)\big]$$

and up to the second derivatives of F_a, we get

$$\dot{F}_a + \mathrm{i}\delta\big(|F_a|^2 - 1\big)F_a = -\frac{1}{2}\,\beta_0\sum_b F_b\,, \tag{5.6}$$

where $\omega = \omega_0 + \delta$, $\delta = 3\gamma\omega_0 b^2/2$, $\beta_0 = 2r_0\omega_0^2/3c$, $r_0 = e^2/mc^2$; and b is the characteristic amplitude of oscillations.

From components of the complex dimensionless amplitude $F_a = R_a + \mathrm{i}\tau_a$ it is convenient to construct a three-dimensional two-component vector $\mathbf{r}_a = (R_a, \tau_a, 0)$, with the z-component equal to zero. Then the motion of the system of oscillators is rather clearly represented by N points lying in the plane (x, y), and is described by the equation

$$\mathbf{v}_a = \boldsymbol{\omega}(\mathbf{r}_a) \times \mathbf{r}_a + \mathbf{f}\,, \tag{5.7}$$

where $\mathbf{v}_a \equiv \dot{\mathbf{r}}_a$, $\mathbf{f} = -\beta_0/2\sum_a \mathbf{r}_a$, $\boldsymbol{\omega}(\mathbf{r}) = \big(0, 0, -\delta(r^2 - 1)\big)$. Thus, the points perform a circular motion around the origin of coordinates ($x = y = 0$) with angular velocity $\omega = \omega(r)$. In addition, if the center of mass of the system of points $\boldsymbol{\rho}_0 = \sum_a \mathbf{r}_a/N$ does not coincide with the origin, then the system as a whole moves with the velocity $\mathbf{f}$ in the direction opposite to ρ_0.

In the case $N \gg 1$, the continuum approximation is applicable. From conservation of the number of oscillators $\partial n(\mathbf{r}, t)/\partial t + \nabla(n\mathbf{v}) = 0$, and the property $\nabla\mathbf{v} = 0$ that follows from Eqn (5.7), we conclude that the movement of points is similar to the flow of an incompressible fluid:

$$\frac{\partial n}{\partial t} + \mathbf{v}\nabla n = 0\,, \tag{5.8}$$

where $n(\mathbf{r}, t)$ is the density of points on the plane (x, y).

The velocity $\mathbf{v}(\mathbf{r}, t)$ is expressed, in accordance with Eqn (5.7), in terms of this density:

$$\mathbf{v} = \boldsymbol{\omega}(\mathbf{r}) \times \mathbf{r} + \mathbf{f}(t)\,; \tag{5.9}$$

$$\mathbf{f}(t) = -\frac{1}{2}\,\beta\int \mathrm{d}^2 r\,\mathbf{r}n(\mathbf{r}, t)\,, \tag{5.10}$$

and so Eqn (5.8) for $n(\mathbf{r}, t)$ is nonlinear:

$$\frac{\partial n(r, \varphi, t)}{\partial t} - \delta(r^2 - 1)\frac{\partial n}{\partial\varphi} + (f_x\cos\varphi + f_y\sin\varphi)\frac{\partial n}{\partial r} + (-f_x\sin\varphi + f_y\cos\varphi)\frac{1}{r}\frac{\partial n}{\partial\varphi} = 0\,, \tag{5.11}$$

where φ is the angle between $\mathbf{r}$ and the x axis.

Assume that at $t = 0$ the distribution of oscillators with respect to phase is homogeneous, $n(r, \varphi, 0) = n_0(r)$. Then from Eqn (5.11) it follows that this distribution is stationary: $n(r, \varphi, t) = n_0(r)$. Let us analyze the stability of this distribution, for which we linearize Eqn (5.11) with respect to a small perturbation $n^{(1)}$:

$$n(r, \varphi, t) = n_0(r) + n^{(1)}(r, \varphi, t)\,;$$

$$\frac{\partial n^{(1)}}{\partial t} - \delta(r^2 - 1)\frac{\partial n^{(1)}}{\partial\varphi} + (f_x\cos\varphi + f_y\sin\varphi)n_0'(r) = 0\,; \tag{5.12}$$

$$\mathbf{f} = -\frac{1}{2}\,\beta_0\int \mathrm{d}^2 r\,\mathbf{r}n^{(1)}(\mathbf{r}, t)\,. \tag{5.13}$$

Now we expand $n^{(1)}$ in eigenmodes:

$$n^{(1)} = \sum_{n=-\infty}^{\infty} n_m(r, t)\exp(\mathrm{i}m\varphi)\,, \qquad n_m = n_m^*\,.$$

The mode with $m = 0$ must be assigned to $n_0(r)$, and so $n_0 = 0$. Velocity $\mathbf{f}$ is determined by the dipole ($m = \pm 1$) modes: $f_x = (\tau + \tau^*)/2$, $f_y = \mathrm{i}(\tau - \tau^*)/2$, where

$$\tau(t) = -\pi\beta_0\int_0^\infty \mathrm{d}r\,r^2 n_1(r, t)\,.$$

Hence, and from Eqn (5.12), we get

$$\frac{\partial n_1(r,t)}{\partial t} - \mathrm{i}\delta(r^2-1)n_1(r,t) - \frac{\pi}{2}\beta_0\tau(t)n_0'(r) = 0\,; \quad (5.14)$$

$$\frac{\partial n_m(r,t)}{\partial t} - \mathrm{i}m\delta(r^2-1)n_m(r,t) = 0\,, \qquad m \neq \pm 1\,. \quad (5.15)$$

According to Eqn (5.15), we have

$$n_m(r,t) = n_m(r,0)\exp\left[\mathrm{i}m\delta(r^2-1)t\right];$$

therefore as $t \to \infty$ the modes $m \neq \pm 1$ give a contribution to $n(\mathbf{r},t)$ that oscillates faster and faster as r varies. This means that the modes $m \neq \pm 1$ give a contribution to any observable (the intensity of quadrupole radiation, etc.) that declines exponentially as $t \to \infty$.

The solution of Eqn (5.14) for the dipole mode is sought in the form $n_1(r,t) = \exp(pt)n_1(r)$, where $p = p' + \mathrm{i}p''$ is a complex number, assuming a priory the existence of instability ($p' > 0$). Then we find

$$n_1(r) = \frac{\pi\beta_0 n_0'(r)}{2\left[p - \mathrm{i}\delta(r^2-1)\right]}\,\tau_0\,; \qquad \tau_0 = \int_0^\infty \mathrm{d}r\, r^2 n_1(r)\,.$$

Hence follows the equation $\tau_0 \varDelta = 0$; therefore

$$\varDelta = 1 + \frac{\beta_0 N}{2}(\tau_1 + \mathrm{i}\delta\tau_2) = 0\,, \quad (5.16)$$

$$\tau_1 = \int_0^\infty \frac{\mathrm{d}r\, g_0(r) r}{p - \mathrm{i}\delta(r^2-1)}\,, \qquad \tau_2 = \int_0^\infty \frac{\mathrm{d}r\, g_0(r) r^2}{\left[p - \mathrm{i}\delta(r^2-1)\right]^2}\,. \quad (5.17)$$

Here we have introduced the function of a distribution of oscillators with respect to the energy $g_0(r)$: $g_0(r) = (2\pi/N)n_0(r)$, normalized from condition

$$\int_0^\infty \mathrm{d}r\, r g_0(r) = 1\,. \quad (5.18)$$

Let us consider the interesting case of the small initial (at $t = 0$) spread of oscillators with respect to the energy $\Delta E/E \ll 1$, when all points in the plane (x, y) are concentrated near the circle $r = 1$, displaced with respect to the origin of coordinates by Δy in the positive direction of the y axis. Since $|z| \sim \Delta E/E$, where $z = r - 1$, we have

$$|z| \ll 1\,. \quad (5.19)$$

Formulas (5.17), (5.18) become simpler:

$$\tau_1 = \int_{-\infty}^\infty \frac{\mathrm{d}z\, g_0(z)}{p - 2\mathrm{i}\delta z}\,, \qquad \tau_2 = \int_{-\infty}^\infty \frac{\mathrm{d}z\, g_0(z)}{(p - 2\mathrm{i}\delta z)^2}\,,$$

$$\int_{-\infty}^\infty \mathrm{d}z\, g_0(z) = 1\,.$$

Let

$$g_0(z) = \frac{\Delta z}{\pi}\,\frac{1}{z^2 + (\Delta z)^2}\,,$$

then

$$\tau_1 = \frac{1}{p + 2\delta\Delta z}\,, \qquad \tau_2 = \frac{1}{(p + 2\delta\Delta z)^2}\,.$$

Hence, and from Eqn (5.16), we get

$$p = -2\delta\Delta z - \frac{\beta_0 N}{4} + \frac{\beta_0 N}{4}\sqrt{1 - \mathrm{i}\theta}\,, \quad (5.20)$$

where

$$\theta = \frac{8\delta}{\beta_0 N}\,, \quad (5.21)$$

and the root must be taken with the positive real part. The case of $p' > 0$ is only possible when the plus sign is selected in Eqn (5.20), whence it follows that

$$p' = \frac{\beta_0 N}{4}(-\theta\Delta z - 1 + D)\,, \quad (5.22)$$

where

$$D = \left[\frac{1}{2}\left(1 + \sqrt{1+\theta^2}\right)\right]^{1/2}.$$

At $\Delta z \neq 0$ the rotation of points about the origin of coordinates is nonuniform — there is a spread $\Delta\omega$ with respect to the frequency ω. Points with $r > 1$ move clockwise, and those with $r < 1$ counterclockwise (for the sake of definiteness we consider the case $\delta > 0$). Owing to the spread of frequencies the perturbation n_1 extends in the azimuthal direction (with respect to φ) and dies out, which is described by the second term in Eqn (5.12). The third term describes the effect of phasing of oscillators. Assume that at $t = 0$ all points are concentrated in the upper part of the circle $r = 1$ (see Fig. 7). Then, as a result of emission at the initial moment, the circle as a whole will move down with the speed $\mathbf{f}$ — in the negative direction of the y axis. The initial displacement of particles along radius r depends on the initial angle φ (phase): $\Delta r \approx \Delta y \sin\varphi$. Each of these points starts rotating with the frequency $\omega = -2\delta\Delta y \sin\varphi$ counterclockwise; the points $\varphi = 0$ and $\varphi = \pi$ remain at rest and stay, as before, on the circle $r = 1$ (with accuracy to Δy^2). The points start to concentrate near $\varphi = \pi$. This is the initial stage of phase grouping (Fig. 8). Subsequently, the motion of points becomes more complicated, and no longer reduces to simple grouping with respect to angle φ.

In this way, the second and third terms in Eqn (5.22) describe the formation of clusters of particles which leads to release of the entire energy in the regime of SR. The first term

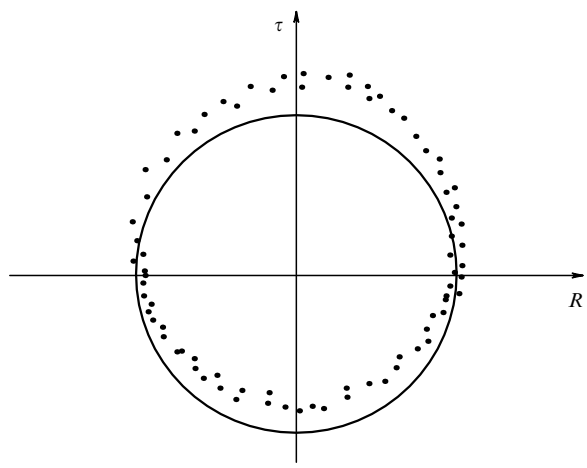

Figure 7. Phase distribution of oscillators in the initial state.

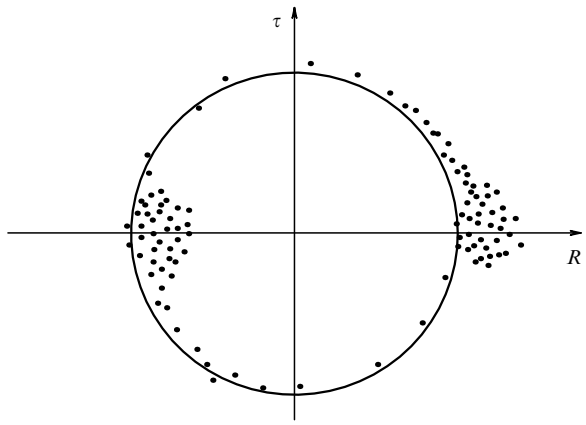

Figure 8. Nonlinear phasing of oscillators.

describes the opposite effect — dissipation of clusters, which may prevent the occurrence of SR. From equation (5.22) and the condition $p' > 0$ we conclude that nonlinear 'pumping' of SR only occurs when the energy spread of oscillators is sufficiently narrow:

$$\frac{\Delta E}{E} \sim \Delta z < \frac{D-1}{\theta} \,. \tag{5.23}$$

Recalling that the dipole – dipole interaction leads to coherent phase-conserving blocking of radiation (screening) we have to admit the complexity of phenomena that take place when all of the above effects act at the same time. This is one of the tasks for the future; our next step will consist in using the linear CMS for analyzing the role of dipole – dipole interaction of atoms.

6. Equations of the classical model of superradiance. Dipole mechanism of phasing

The quantum mechanical problem of SR is extremely complicated. For example, the Heisenberg approach requires solving the set of nonlinear operator equations. The simplifying approximations, that will be discussed in the sections to follow, have a limited and often ambiguous range of applicability. Many difficult issues can be resolved with the aid of the CMS; in this way, classical and quantum approaches supplement one another. These issues include the nonlinear mechanism of phasing, which has become clear in the previous section after we dropped the dipole – dipole interaction. To understand the role of this interaction, we are going to consider now the opposite extreme — disregarding the effects of nonlinearity, we shall analyze the Lorentzian model of atoms represented by charged harmonic oscillators.

Assume that springs are fixed at points with coordinates $\mathbf{r}_a$ $(a = 1, \ldots, N)$, which also contain the compensating charges $-e$. Attached to the other ends of springs are point masses m with charge e. The coordinate of the ath charge counted from the point where the spring is attached is ξ_a. The potential energy of the spring is assumed to be harmonic: $U_a = m\omega_0^2\xi_a^2/2$. In this way, we do not take the nonlinear effects into account, nor the motion of atoms and the associated Doppler broadening of spectral lines. This broadening is small compared with the dipole – dipole and natural broadening. As far as the latter is concerned, observe that, owing to the brevity of the process of SR, it is much greater than the ordinary natural broadening for an isolated atom [8]. For the case of (2.42c) we shall consistently include the Doppler broadening in the consideration, and prove its importance — this is due to the large dimensions of plasma to which the theory is applied. The motion of charges is considered nonrelativistic (the relativistic effects will be included at a later point), which allows us to disregard the effects of the magnetic field and to write the equations of motion in the form

$$m\ddot{\xi}_a + m\omega_0^2\xi_a = e\mathbf{E}(\mathbf{r}_a, t)\,; \tag{6.1}$$

$$\mathbf{E}(\mathbf{r}, t) = e\sum_{b=1}^{N} \boldsymbol{\nabla}_{\mathrm{r}} \times \left[\boldsymbol{\nabla}_{\mathrm{r}} \times \frac{\xi_b(t - R_b/c)}{R_b}\right], \tag{6.2}$$

where $\mathbf{R}_b = \mathbf{r} - \mathbf{r}_b$. The expression for the electric field of the dipoles is taken from Ref. [9].

Upon substitution of Eqn (6.2) into (6.1) we get an infinite term in $\mathbf{E}(\mathbf{r}_a, t)$ with $b = a$. For the correct calculation of this term one must 'smear' the charges $+e$ and $-e$ over regions of finite size. After that, $\xi_a(t - R_a/c)$ must be expanded in powers of $1/c$ down to (and including) terms of order $1/c^3$. The term of zero order gives the proper field of charges $-e$ and $+e$. Obviously, the field of charge $+e$ cannot accelerate the charge, so the corresponding term is zero. The field of charge $-e$ creates the force that acts upon charge $+e$. This force must be included in the stiffness of the spring and thus formally eliminated. Thus, the harmonic potential is the total net potential acting upon the charge $-e$. In other words, the zero-order term leads to renormalization of the stiffness of the spring. The term of first order in $1/c$ as follows from Eqn (6.2), is zero. The second-order term in $\mathbf{E}(\mathbf{r}_a, t)$ reduces to $-(4/3e)m_{\mathrm{f}}\ddot{\xi}_a$, and must be included in the first term on the left-hand side of Eqn (6.1). The quantity $m_{\mathrm{f}} = \varepsilon_{\mathrm{f}}/c^2$ (where ε_{f} is the energy of the electromagnetic cloud surrounding the dipole) is the mass of this cloud, and therefore must be included in the mass m (renormalization of the mass). The factor 4/3 at one time caused animated discussions on what was known as 'the 4/3 paradox'. The solution of this paradox, given long ago by H Poincaré and other authors (see, for example, Ref. [62]), consists in that one must take into account those forces that are not described by classical electrodynamics (chemical bonds, Van der Waals interaction, etc.) and prevent the repulsion of components of the charge. Finally, the term of the order $1/c^3$ is

$$\mathbf{E}_{\mathrm{f}} = \frac{2}{3c^3}\dddot{\mathbf{d}}_a = \frac{2e}{3c^3}\dddot{\xi}_a = -\frac{2e\omega_0^2}{3c^3}\dot{\xi}_a \,.$$

This term corresponds to radiation friction, which causes damping of dipole oscillations in an isolated atom. Observe that in a collective of atoms similar damping is caused by terms with $b \neq a$. The approximation $\dddot{\xi}_a \approx -\omega_0^2\dot{\xi}_a$ allows wrong solutions corresponding to the self-accelerated charge to be avoided [9]. Such solutions arise because of the approximate nature of the adopted procedure of elimination of electromagnetic field variables from the equations of motion (neglect of the terms of order $1/c^4$ and higher). Thus, the system (6.1), (6.2) becomes

$$\ddot{\xi}_a + \omega_0^2\xi_a + \gamma\dot{\xi}_a = \frac{e^2}{m}\sum_{b\neq a} \boldsymbol{\nabla}_a \times \left[\boldsymbol{\nabla}_a \times \frac{\xi_b(t_{ab})}{r_{ab}}\right], \tag{6.3}$$

where $\boldsymbol{\nabla}_a = \partial/\partial\mathbf{r}_a$, $\mathbf{r}_{ab} = \mathbf{r}_a - \mathbf{r}_b$, $t_{ab} = t - r_{ab}/c$, $\gamma = 2r_0\omega_0^2/3c$, $r_0 = e^2/mc^2$. Hereinafter we assume that

$\gamma \ll \omega_0$, and, moreover, $\gamma N \ll \omega_0$, which corresponds to the practically most interesting cases. Here γN is nothing else but $1/\tau_s$ — the reciprocal of the characteristic time of SR τ_s (see below).

The system (6.3) must be solved with the initial condition

$$t = 0\,, \qquad \xi_a = \boldsymbol{\eta}_a \cos\varphi_a\,, \qquad \dot{\xi}_a = -\omega_0 \boldsymbol{\eta}_a \sin\varphi_a\,, \tag{6.4}$$

where $\boldsymbol{\eta}_a$, φ_a are the initial amplitudes and phases of oscillators. Since we are interested in the solution at $t > 0$, we may assume that at $t < 0$ we have $\xi_a = 0$. The initial energy of the ath oscillator is

$$t = 0\,, \qquad E_0^{(a)} = \frac{1}{2}\, m\dot{\xi}_a^2 + \frac{1}{2}\, m\omega_0^2 \xi_a^2 = \frac{1}{2}\, m\omega_0^2 \boldsymbol{\eta}_a^2\,. \tag{6.5}$$

In the case of an isolated atom, the right-hand side in Eqn (6.3) equals zero:

$$\xi_a(t) = \boldsymbol{\eta}_a \exp\left(-\frac{\gamma t}{2}\right) \cos(\omega_0 t + \varphi_a)\,,$$
$$E_a(t) = E_0^{(a)} \exp(-\gamma t)\,,$$

whence we see that in the comparison between the classical and quantum results for a solitary atom we must take $\gamma = \lambda_0 = 1/\tau_0$.

From Eqn (6.2) it follows that as $r \to \infty$

$$\mathbf{E}(\mathbf{r}, t) \to \frac{e}{rc^2}\, \mathbf{n} \times \left[\mathbf{n} \times \sum_b \ddot{\xi}_b\left(t' + \frac{1}{c}\, \mathbf{n}\mathbf{r}_b\right)\right], \tag{6.6}$$

where $t' = t - r/c$, $\mathbf{n} = \mathbf{r}/r$, $\mathbf{r}$ is a vector drawn from the origin of coordinates (whose position is arbitrary) to the point of observation.

Calculating the Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}\left[\mathbf{E}(\mathbf{r}, t)\right]^2 \mathbf{n}\,,$$

we find the intensity of radiation into the solid angle $\mathrm{d}\Omega$:

$$\mathrm{d}I = \mathrm{d}\Omega\, \frac{e^2\omega_0^4}{4\pi c^3}\, \mathbf{Q}^2\,, \tag{6.7}$$

$$\mathbf{Q} = \mathbf{n} \times \sum_a \xi_a\left(t' + \frac{1}{c}\, \mathbf{n}\mathbf{r}_a\right), \tag{6.8}$$

where we have noted that $\ddot{\xi}_a \approx -\omega_0^2 \xi_a$.

The system (6.3) is much simplified by the substitution

$$\xi_a(t) = \mathbf{F}_a(t) \exp(-\mathrm{i}\omega_0 t) + \mathbf{F}_a^*(t) \exp(\mathrm{i}\omega_0 t)\,, \tag{6.9}$$

where $\mathbf{F}_a(t)$, in accordance with Eqn (4.40), is a slowly varying function compared with the exponential term. Dropping the terms with $\ddot{\mathbf{F}}_a(t)$, setting $\mathbf{F}_b(t - r_{ab}/c) \approx \mathbf{F}_b(t)$, which is justified in the case of Eqn (2.12) that is considered in this section (the opposite situation will be discussed at the end of Section 8.2), we get the equation for $\mathbf{F}_a$:

$$\dot{\mathbf{F}}_a + \frac{\gamma}{2}\, \mathbf{F}_a = \mathrm{i}\beta \sum_{b \neq a} \boldsymbol{\nabla}_a \times \left[\boldsymbol{\nabla}_a \frac{\exp(\mathrm{i}kr_{ab})}{r_{ab}} \times \mathbf{F}_b(t)\right], \tag{6.10}$$

where $\beta = e^2/(2m\omega_0)$, $k = \omega_0/c$. According to Eqns (6.4) and (6.9),

$$\mathbf{F}_a(0) = \mathbf{B}_a\,, \qquad \mathbf{B}_a = \exp(-\mathrm{i}\varphi^a)\, \frac{\boldsymbol{\eta}_a}{2}\,. \tag{6.11}$$

There are a number of processes in which the coherence is lost [10, 11]: the collision quenching of atoms, which results in a nonradiative inelastic transition, the break of phase of atoms due to elastic collisions, etc. In the latter case, the electron energy levels of the atom are perturbed when the atom comes close to another one $[E_n \to E_n + \Delta E_n(t)]$, and the wave function of the nth state is multiplied by the phase coefficient $\exp(-\mathrm{i}\varphi_n)$,

$$\varphi_n = \frac{1}{\hbar} \int_{-\infty}^{\infty} \Delta_n E(t)\, \mathrm{d}t\,.$$

The dipole moment of the transition $n \to n'$ is multiplied by $\exp(-\mathrm{i}\varphi_n + \mathrm{i}\varphi_{n'})$. Since these coefficients are different for different atoms, and there is no correlation between them and the coherence is destroyed. These processes are taken into account by introducing the additional damping $\gamma \to \Gamma = \gamma + \gamma_c$, where $\gamma_c = 1/T_2$ is the effective width that accounts for these processes. We assume that $\tau_s \gamma_c \ll 1$, and therefore γ_c will be dropped.

Equations of the CMS (6.3), (6.11) are linear, which means that the sum of their solutions taken with arbitrary coefficients is also a solution. This allows us to use the formalism of Green's functions. For this purpose, we must first find the solution of Eqn (6.3) $\xi_a^{(b)}(t)$ (Green's function) with the initial condition

$$t = 0\,, \qquad \xi_a^{(b)} = \boldsymbol{\eta}_b \cos(\varphi_b)\delta_{ab}\,, \qquad \dot{\xi}_a^{(b)} = -\omega_0 \boldsymbol{\eta}_b \sin(\varphi_b)\delta_{ab}\,, \tag{6.12}$$

where δ_{ab} is the Kronecker symbol ($\delta_{ab} = 0$ if $a \neq b$, and $\delta_{ab} = 1$ if $a = b$). A solution with the arbitrary initial condition (6.4) is

$$\xi_a(t) = \sum_b \xi_a^{(b)}(t)\,. \tag{6.13}$$

Let us consider the situation when at $t = 0$ the atoms are completely non-correlated:

$$\left\langle \exp\left[\mathrm{i}(\varphi_a - \varphi_b)\right]\right\rangle = 0\,. \tag{6.14}$$

Because all phases are independent, we have

$$\mathrm{d}I = \sum_b \mathrm{d}I_b\,, \tag{6.15}$$

where $\mathrm{d}I_b$ is given by Eqns (6.7), (6.8), where ξ_a ought to be replaced with $\xi_a^{(b)}$.

In subsequent sections we demonstrate that, even though there are no correlations at $t = 0$, correlations arise in the CMS at $t > 0$, and collective coherent effects develop. This is another mechanism of phasing — the dipole mechanism. Let us discuss it in greater detail.

Consider for definiteness the case of (2.42a), and evaluate the intensity I_b for the pointwise initial condition (6.12). For estimating I_b and τ_s it is sufficient to assume that atom b oscillates with constant amplitude: $\xi_b = \boldsymbol{\eta}_b \cos(\omega_0 t + \varphi_b)$. Atom b at the location of the initially nonexcited atom a creates a resonant field $\mathbf{E}_a$ that starts to jolt the oscillator a:

$$E_a \sim \frac{e\eta_b}{R_a^3} \cos(\omega_0 t + \varphi_b)\,; \qquad \xi_a \sim \frac{e^2\eta_b}{m\omega_0 R_a^3}\, t \sin(\omega_0 t + \varphi_b)\,; \tag{6.16}$$

here and further $\mathbf{R}_a \equiv \mathbf{r}_{ab} = \mathbf{r}_a - \mathbf{r}_b$.

For the purpose of making our estimates we assume that the typical distance between the atoms is

$$R_a \sim L\,. \tag{6.17}$$

After some time, the initial energy of atom b is distributed between all the atoms of active medium. Their amplitude becomes of the order of

$$\xi_a \sim \frac{1}{\sqrt{N}}\,\eta_b\,. \tag{6.18}$$

Now there is a coherent ensemble of N atoms. Coherence gives to the intensity I_b the factor N^2 (see Section 2); by virtue of Eqn (6.18), however, there is also an additional factor of $(1/\sqrt{N})^2 = 1/N$, so eventually $I_b \sim NI_0$, where I_0 is the intensity of emission by a solitary atom. Hence follows the estimate $\tau_s \sim \varepsilon_b/I_b \sim \tau_0/N$. According to Eqn (6.15), the total intensity of radiation by a medium of initially non-correlated atoms is of the order of $I \sim NI_b \sim N^2 I_0$.

These arguments are correct in the sense that they emphasize the importance of collective effects for the emission of an atom surrounded by resonant partners, but they are not exact. This is because estimate (6.17) is not correct. This becomes clear if only from Eqn (6.16): ξ_a falls off rather rapidly as R_a increases, and, as will be shown in the next section, this circumstance dramatically changes the simple picture described above. We shall show that atom b creates a cloud of dipole moments that oscillate in counter-phase. This results in shielding that hampers the escape of radiation. For this reason it would be more correct to speak of dipole antiphasing rather than phasing.

7. Effects of the dipole interaction of atoms in the Dicke model

In this section we shall apply the linear CMS with a dipole–dipole interaction to a small-sized system (2.42a), in order to refine the Dicke model. Unfortunately, the calculation can only be carried through for ellipsoidal bodies, which, however, is of some practical utility. A good deal of this section is concerned with the effect of dipole shielding. At the end of the section we give a critical analysis of the conclusions of Refs [49–51] regarding the destruction of SR by dipole–dipole interaction of atoms.

In the case $l \ll L \ll \lambda$, the expansion of the exponential in Eqn (6.10) must be carried out down to and including terms of order k^3. The term of order k is zero, the purely imaginary term of order k^2 is small compared with the imaginary term of the zero order and must be dropped. As proved at the end of this section, attempts to include the term of order k^2 lead to contradictions. The term of the order of k^3,

$$-\frac{\gamma}{2}\sum_{b\neq a}\mathbf{F}_b\,,$$

is real, and therefore must be retained. Equation (6.10) reduces to the form

$$\dot{\mathbf{F}}_a + \frac{\gamma}{2}\,\mathbf{F} = \mathrm{i}\beta\sum_{b\neq a}\frac{3\mathbf{n}_{ab}(\mathbf{n}_{ab}\mathbf{F}_b) - \mathbf{F}_b}{(r_{ab})^3}\,, \tag{7.1}$$

where $\mathbf{n}_{ab} = \mathbf{r}_{ab}/r_{ab}$,

$$\mathbf{F} = \sum_a \mathbf{F}_a(t)\,. \tag{7.2}$$

In the index notation, Eqn (7.1) becomes

$$\dot{F}_{a\alpha} + \frac{\gamma}{2}\,F_\alpha = \mathrm{i}\beta\sum_{b\neq a} G_{\alpha\beta}(\mathbf{r}_{ab})F_{b\beta}\,, \tag{7.3}$$

where

$$G_{\alpha\beta} = \frac{3S_\alpha S_\beta - \delta_{\alpha\beta}}{(r_{ab})^3}\,, \qquad \mathbf{S} \equiv \mathbf{n}_{ab}\,, \qquad \alpha, \beta = x, y, z\,.$$

In equation (7.3) and later summation is carried out over the recurring indices.

The Green function $\mathbf{F}_a^{(b)}(t)$ of Eqn (6.10) satisfies the initial condition

$$t = 0\,, \qquad \mathbf{F}_a^{(b)} = \mathbf{B}_b\delta_{ab}\,. \tag{7.4}$$

The solution with arbitrary initial condition (6.11), similarly to Eqn (6.13), is

$$\mathbf{F}_a(t) = \sum_b \mathbf{F}_a^{(b)}(t)\,. \tag{7.5}$$

Further on we assume that the medium is ellipsoidal in shape. This includes the important particular cases of a sphere, an ellipsoid of revolution, and a long thin rod. Let us first consider the 'point' solution of Eqn (7.3) $\mathbf{F}_a^{(b)}$. To simplify the notation, the superscript b will be dropped.

The intensity of radiation in the case $l \ll L \ll \lambda$ is

$$I = \frac{2}{3c^3}(\ddot{\mathbf{D}})^2 \approx \frac{2\omega_0^4}{3c^3}(\mathbf{D})^2\,, \tag{7.6}$$

where

$$\mathbf{D} = e\sum_a \boldsymbol{\xi}_a = e\big[\mathbf{F}\exp(-\mathrm{i}\omega_0 t) + \mathbf{F}^*\exp(\mathrm{i}\omega_0 t)\big]\,. \tag{7.7}$$

Carrying out summation in Eqn (7.3) with respect to a, we get

$$\dot{\mathbf{F}}_\alpha + \frac{\gamma N}{2}\,\mathbf{F}_\alpha = \mathrm{i}\beta\sum_b \tau_{\alpha\beta}(\mathbf{r}_b)F_{b\beta}\,, \tag{7.8}$$

where

$$\tau_{\alpha\beta}(\mathbf{r}_b) = \sum_a G_{\alpha\beta}(\mathbf{r}_{ab})\,. \tag{7.9}$$

The phenomenal feature possessed only by an ellipsoid is the fact that $\tau_{\alpha\beta}$ does not depend on $\mathbf{r}_b$ [63–65] (see also Appendix I). The quantity $\tau_{\alpha\beta}$ can be calculated by solving the known problem of dielectric ellipsoid in a homogeneous external electric field $\mathbf{E}_0$. The field inside the ellipsoid is

$$\mathbf{E}_\mathrm{i} = \mathbf{E}_0 + \mathbf{E}'\,,$$

where $\mathbf{E}'$ is the proper field of the ellipsoid created by all the dipole moments of the ellipsoid:

$$\mathbf{E}'_\alpha = \sum_b G_{\alpha\beta}(\mathbf{r}_b)d_{b\beta} = -4\pi Q_{\alpha\beta}P_\beta\,, \tag{7.10}$$

where $\mathbf{P}$ is the vector of polarization of the medium (dipole moment of unit volume),

$$Q_{\alpha\beta} = r^{(x)}i_\alpha i_\beta + r^{(y)}j_\alpha j_\beta + r^{(z)}K_\alpha K_\beta\,, \tag{7.11}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{K}$ are unit vectors directed along the main axes of the ellipsoid, the corresponding semiaxes are a, b, c; and $r^{(x)}$, $r^{(y)}$, $r^{(z)}$ are the depolarizing factors

$$r^{(x)} = \frac{abc}{2}\int_0^\infty \frac{\mathrm{d}S}{(S+a^2)R_\mathrm{s}}\,, \tag{7.12}$$

where $R_\mathrm{s} = \sqrt{(S+a^2)(S+b^2)(S+c^2)}$.

The formulas for $r^{(y)}$ and $r^{(z)}$ follow from Eqn (7.12) after replacing $S+a^2$ in the denominator by $S+b^2$ and $S+c^2$ respectively. Observe that

$$r^{(x)} + r^{(y)} + r^{(z)} = 1\,. \tag{7.13}$$

In particular, for a sphere

$$r^{(x)} = r^{(y)} = r^{(z)} = \frac{1}{3}\,. \tag{7.14}$$

We select a small sphere centered around atom a; then

$$\mathbf{E}' = \mathbf{E}_1 + \mathbf{E}_2\,, \tag{7.15}$$

where $\mathbf{E}_1$, $\mathbf{E}_2$ are fields created by dipoles occurring, respectively, inside and outside the sphere. For physical reasons, $\sum_{b\neq a} G_{\alpha\beta}(\mathbf{R}_b)d_{b\beta} = E_{2\alpha}$. Noting that $\mathbf{d}_b = \mathbf{d} = \mathrm{const}$ for an ellipsoid, we find that

$$E_{2\alpha} = \tau_{\alpha\beta}d_\beta = \frac{\tau_{\alpha\beta}P_\beta}{n}\,. \tag{7.16}$$

The field $\mathbf{E}_1$ is nothing other than the field inside a sphere with homogeneous polarization ($\mathbf{P} = \mathrm{const}$) [63–65]: $\mathbf{E}_1 = -4\pi\mathbf{P}/3$. Hence, from Eqns (7.10), (7.15) and (7.16) we get

$$\tau_{\alpha\beta} = 4\pi n\left(\frac{1}{3}\,\delta_{\alpha\beta} - Q_{\alpha\beta}\right). \tag{7.17}$$

As follows from Eqns (7.11), (7.14), (7.18), for a sphere we have

$$\tau_{\alpha\beta} = 0\,. \tag{7.18}$$

The above formula for $\mathbf{E}_1$ calls for special consideration. It holds for "simple, face-centered and body-centered cubic lattices, as well as for isotropic media" (quoting from Ref. [64], p. 81). The latter is of special interest for experiments with SR, since such experiments are usually designed or staged in gaseous, liquid or amorphous bodies. As argued in Ref. [63] (Section 35), a medium may be considered isotropic when the "gas molecules are chaotically distributed in space, and the position of any one practically does not depend on the positions of others". According to Ref. [63], by $\mathbf{E}_1$ one should mean the time-averaged field of dipoles located inside the selected sphere of small radius.

Formula (7.1) allows the substantiation of these statements. As a result of motion of atoms, the terms $\mathbf{E}_1$ and $\mathbf{E}_2$ in Eqn (7.15) fluctuate. Fluctuations of $\mathbf{E}_2$ may be neglected, since the relevant atoms are far away. Fluctuations of $\mathbf{E}_1$ are substantial, and are caused by the motion of atoms closest to b, which is characterized by the time l/v_T and the impact parameter $\rho \sim l$. Less frequent collisions with smaller target parameters result in dephasing. According to Eqn (2.10), they should be neglected. Since $l/v_\mathrm{T} \ll \tau_\mathrm{s}$, the second term on the left-hand side of Eqn (7.1) must now be dropped. Assuming that fluctuations of $\mathbf{F}_a$ are small, we write these vectors in the form $\mathbf{F}_a = \mathbf{F}'_a + \mathbf{F}''_a$, where $\mathbf{F}'_a$ is the constant part, and $\mathbf{F}''_a$ is the fluctuating part ($|F''| \ll |F'|$). According to Eqn (7.1), the change of F''_a over the time Δt (where $l/v_\mathrm{T} \ll \Delta t \ll \tau_\mathrm{s}$) is

$$\Delta F''_{a\alpha} \approx \mathrm{i}\beta\nu_0\Delta t\sum_{b\neq a}\langle f_{\alpha\beta}\rangle F'_{b\beta}\,,$$

where $\nu_0 \sim v_\mathrm{T}/l$ is the characteristic frequency of collisions, angle brackets denoting averaging over all possible collisions of the quantity $f_{\alpha\beta} = \int_{-\infty}^{\infty} G_{\alpha\beta}\,\mathrm{d}t$. Integration here is carried out over the straight path of atom a relative to atom b. It is easy to see that averaging over the directions of relative velocity of atoms yields zero ($\langle f_{\alpha\beta}\rangle = 0$), therefore $\Delta\mathbf{F}''_a = 0$, which was to be demonstrated. At the same time we proved that the continuum approximation (2.35) can be used in the calculation of $\mathbf{E}'$.

From arguments developed above we see that the approximation of isotropic continuous medium is valid under the condition $\beta|f_{\alpha\beta}| \ll 1$, that is,

$$\frac{e^2}{m\omega_0 v_\mathrm{T} l^2} \ll 1\,.$$

This relation is easily extended to the quantum case of two-level atoms if we recall that the matrix element of the coordinate of the oscillator is, in order of magnitude, equal to $\xi_{01} \sim \sqrt{\hbar/m\omega_0}$:

$$\frac{d^2}{\hbar v_\mathrm{T} l^2} \ll 1\,. \tag{7.19}$$

In other words, the phases φ_n (see the end of Section 6) for collisions with $\rho \sim l$ must be small. In the optical band, the criterion of isotropicity and continuity (7.19) is written in the form $n \ll 2\times 10^{21}\ \mathrm{cm}^{-3}$ and is satisfied with certainty.

From Eqn (7.8) and $\tau_{\alpha\beta}(\mathbf{r}_b) = \mathrm{const}$ it follows that

$$\dot{F}_\alpha + \frac{\gamma N}{2}\,F_\alpha = \mathrm{i}\beta\tau_{\alpha\beta}F_\beta\,. \tag{7.20}$$

According to Eqn (7.4), the initial condition for $\mathbf{F}$ is $t=0$, $\mathbf{F} = \mathbf{B}_b = B_{bx}\mathbf{i} + B_{by}\mathbf{j} + B_{bz}\mathbf{K}$. Hence, and from Eqns (7.4), (7.7), (7.11) we get

$$\mathbf{F}(t) = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{K}\,; \tag{7.21}$$

$$F_x = B_{bx}\exp\left(-\mathrm{i}\omega_x t - \frac{\gamma N}{2}\,t\right);$$

$$\mathbf{D}_b(t) = D_x^{(b)}\mathbf{i} + D_y^{(b)}\mathbf{j} + D_z^{(b)}\mathbf{K}\,; \tag{7.22}$$

$$D_x^{(b)} = e\eta_{bx}\cos\left[(\omega_0+\omega_x)t + \varphi_b\right]\exp\left(-\frac{\gamma N}{2}\,t\right),$$

where $\omega_x = \omega_\mathrm{d}\left[3n^{(x)} - 1\right]$, $\omega_\mathrm{d} = 4\pi\beta n/3$.

The expressions for F_y, F_z and D_y, D_z follow from Eqns (7.21), (7.22) after replacing the subscript x with y and z, respectively. Expressions (7.22) define the dipole moment for the pointwise initial conditions (6.2), (7.4). Because the situation is linear [see Eqn (7.5)], the dipole moment of the medium with arbitrary initial condition (6.4), (6.11) follows from summation of Eqn (7.22) with respect to all atoms b:

$$\mathbf{D} = D_x\mathbf{i} + D_y\mathbf{j} + D_z\mathbf{K}\,;$$

$$D_x = e\sum_b \eta_{bx}\cos\left[(\omega_0+\omega_x)t + \varphi_b\right]\exp\left(-\frac{\gamma N}{2}\,t\right). \tag{7.23}$$

Given that $(\mathbf{D})^2 = (D_x)^2 + (D_y)^2 + (D_z)^2$, let us average $(D_x)^2$ with respect to fast oscillations with the period $2\pi/\omega_0$:

$$\langle D_x^2 \rangle = \frac{1}{2}\, e^2 \exp(-\gamma N t) \sum_{a,b} \eta_{ax}\eta_{bx} \cos(\varphi_a - \varphi_b)\,.$$

Accordingly,

$$\langle \mathbf{D}^2 \rangle = \frac{1}{2}\, e^2 \exp(-\gamma N t) g\,,$$

where

$$g = \sum_{a,b} (\boldsymbol{\eta}_a \boldsymbol{\eta}_b) \cos(\varphi_a - \varphi_b)\,. \tag{7.24}$$

Hence, and from Eqn (7.6) we get

$$I = \frac{4e^2\omega_0^4}{3c^3}\, g \exp(-\gamma N t)\,. \tag{7.25}$$

Consequently, the total released energy is

$$E_{\mathrm{r}} = \int_0^\infty I\,\mathrm{d}t = \frac{m\omega_0^2 g}{2N}\,.$$

The initial energy of oscillators is given by Eqn (6.5):

$$E_0 = \sum_a E_0^{(a)} = \frac{1}{2}\, m\omega_0^2 \sum_a \eta_a^2\,.$$

For the ratio of these two energies we have

$$\xi = \frac{E_{\mathrm{r}}}{E_0} = \frac{g}{N\sum_a \eta_a^2}\,. \tag{7.26}$$

Let us prove that $\xi \leqslant 1$. From Eqn (7.24) we have $g = \sum_{a,b} \eta_a \eta_b q_{ab}$, where η_a, η_b are the magnitudes of the relevant vectors, $q_{ab} = \cos(\alpha_{ab}) \cos(\varphi_a - \varphi_b)$, and α_{ab} is the angle between vectors η_a and η_b. Obviously, $q_{ab} \leqslant 1$, and therefore $g \leqslant \sum_{a,b} \eta_a \eta_b$. Since $\sqrt{\eta_a \eta_b} < (\eta_a + \eta_b)/2$, we have

$$\sum_{a,b} \eta_a \eta_b \leqslant \sum_{a,b} \left(\frac{\eta_a + \eta_b}{2}\right)^2 = \frac{1}{4} \sum_{a,b} (\eta_a^2 + \eta_b^2 + 2\eta_a \eta_b)$$
$$= \frac{1}{2}\, N \sum_a \eta_a^2 + \frac{1}{2} \sum_{a,b} \eta_a \eta_b\,.$$

From this inequality it follows that $\sum_{a,b} \eta_a \eta_b \leqslant N \sum_a \eta_a^2$, and $\xi \leqslant 1$.

Now let us show that $\xi \geqslant 0$. Consider the expression $b = \left\langle \left[\sum_a \boldsymbol{\eta}_a \cos(\varphi_a + \Psi)\right]^2 \right\rangle$, where the angle brackets denote averaging with respect to a random auxiliary phase Ψ. Now we have

$$b = \sum_{a,b} \boldsymbol{\eta}_a \boldsymbol{\eta}_b \left\langle \cos(\varphi_a + \Psi) \cos(\varphi_b + \Psi) \right\rangle = \frac{g}{2}\,.$$

Given that $b \geqslant 0$, we conclude that $\xi \geqslant 0$. We see that $0 \leqslant \xi \leqslant 1$, and only part of the energy is released. The question is where the rest of the energy goes.

The reason is that, as a result of dipole–dipole interaction, the active medium goes into the metastable coherent state with $\mathbf{D} = 0$. The remaining energy is released over a much longer time as quadrupole radiation, or in the course of decay of the metastable state through interatomic collisions.

Let us consider the extreme case $N \to \infty$ (an infinite medium). In order to identify the metastable states, the solution of Eqn (7.3) must be sought in the form

$$\mathbf{F} = 0\,, \qquad \mathbf{F}_a(t) = \mathbf{S}_a \exp(-\mathrm{i}\omega t)\,. \tag{7.27}$$

For the amplitudes $\mathbf{S}_a$ we get the equation

$$-\omega S_{a\alpha} = \beta \sum_b G_{\alpha\beta}(\mathbf{r}_{ab}) S_{b\beta}\,, \tag{7.28}$$

whose solution is sought in the form

$$S_{a\alpha}^{(b)} = B_{b\alpha}\delta_{ab} + G_{\alpha\beta}(\mathbf{r}_{ab}) D_{b\beta}\,. \tag{7.29}$$

From Eqns (7.28) and (7.29), making use of the formula (see Appendix II)

$$\sum_b G_{\alpha\beta}(\mathbf{r}_{ab}) G_{\beta\gamma}(\mathbf{r}_{bc}) = -\frac{4\pi n}{3}\, G_{\alpha\gamma}(\mathbf{r}_{ac}) + \frac{32\pi^2 n^2}{9}\, \delta_{\alpha\gamma}\delta_{ac} \tag{7.30}$$

we get the set of equations

$$-\omega \mathbf{B} = \frac{8\pi n \omega_{\mathrm{d}}}{3}\, \mathbf{D}\,; \qquad -\omega \mathbf{D} = \beta \mathbf{B} - \omega_{\mathrm{d}} \mathbf{D}\,.$$

Setting its determinant equal to zero, we find two roots:

$$\omega_1 = -\omega_{\mathrm{d}}\,, \qquad \omega_2 = 2\omega_{\mathrm{d}}\,. \tag{7.31}$$

These two frequencies correspond to two solutions

$$\mathbf{F}_a^{(b)}(t) = \left[\mathbf{B}_b \delta_{ab} + \frac{3}{8\pi n}\, \frac{3\mathbf{n}_{ab}(\mathbf{n}_{ab}\mathbf{B}_b) - \mathbf{B}_b}{(r_{ab})^3}\right] \exp(\mathrm{i}\omega_{\mathrm{d}} t)\,;$$
$$\mathbf{F}_a^{(b)}(t) = \left[\mathbf{B}_b \delta_{ab} - \frac{3}{4\pi n}\, \frac{3\mathbf{n}_{ab}(\mathbf{n}_{ab}\mathbf{B}_b) - \mathbf{B}_b}{(r_{ab})^3}\right] \exp(-2\mathrm{i}\omega_{\mathrm{d}} t)\,, \tag{7.32}$$

which describe excitations of the active medium localized on atom b.

Formulas (7.30)–(7.32) are accurate (see Appendix II) down to the terms of order $1/N$, which in an infinite medium are zero. Because of this, it is not possible to use Eqn (7.32) for calculating the dipole moment of the medium, so as to prove, for example, that it equals zero. For a sphere, for example, from Eqns (7.18) and (7.32) it follows that

$$\mathbf{F} = \mathbf{B}_b \exp(-\mathrm{i}\omega t) \neq 0\,.$$

However, after we supplement $\mathbf{F}_a^{(b)}$ in Eqn (7.32) with

$$-\frac{1}{N}\, \mathbf{B}_b \exp(\mathrm{i}\omega t)$$

the dipole moment of the medium becomes to zero. Because of this, in the future it would be interesting to find the solution of Eqn (7.3) accurate down to terms of the order of $1/N$ inclusively, for a medium of arbitrary shape. In any case, the above solution for an ellipsoid, the calculation of the dipole moments, and all the inferences are true. Indeed, let us assume that in the calculation of $\tau_{\alpha\beta}$ we have made an error of the order of $1/N$. Then for an ellipsoid we would get

$$\tau_{\alpha\beta}(\mathbf{r}_b) = \tau_{\alpha\beta} + \frac{n}{N}\, Q_{\alpha\beta}(\mathbf{r}_b)\,, \tag{7.33}$$

where $\tau_{\alpha\beta}$ is our former expression (7.17), $Q_{\alpha\beta}(\mathbf{r}_b)$ are dimensionless quantities of the order of unity. From Eqns (7.8) and (7.33) then follows the equation [cf. Eqn (7.20)]

$$\dot{F}_\alpha + \frac{\gamma N}{2} F_\alpha = \mathrm{i}\beta\tau_{\alpha\beta}F_\beta + \frac{\mathrm{i}\beta n}{N}\sum_b Q_{\alpha\beta}(\mathbf{r}_b)F_{b\beta}\,.$$

For the sake of simplicity let us consider the case of a sphere: $\tau_{\alpha\beta} = 0$. It is obvious that the quantities $Q_{\alpha\beta}(\mathbf{r}_b)$ change considerably only on the characteristic scale of the medium ($r_b \sim L$). For evaluating the necessary corrections by order of magnitude it is sufficient to consider the case $Q_{\alpha\beta} = Q\delta_{\alpha\beta}$, where $Q \sim 1$. Then in place of Eqn (7.21) we get

$$\mathbf{F}(t) = \mathbf{B}_b \exp\left(-\frac{\gamma N}{2}\,t + \frac{\mathrm{i}\beta n}{N}\,Qt\right).$$

Since $N \gg 1$, these corrections may be neglected.

Solutions (7.32) are just particular cases of metastable states. Such an arbitrary state has the form $\mathbf{F}_a(t) = \sum_b \mathbf{F}_a^{(b)}$, where $\mathbf{F}_a^{(b)}$ is a linear combination of the two solutions from Eqn (7.32). The eigenmodes in Eqn (7.32) are of the same physical nature as polaritons (see Section 10); since $L \ll \lambda$, however, they do not propagate in the medium. Such oscillations could be termed quasistatic polaritons.

The case $\xi = 1$ corresponds to completely coherent pumping (with π-pulse),

$$\varphi_1 = \varphi_2 = \ldots = \varphi_N\,, \qquad \boldsymbol{\eta}_1 = \boldsymbol{\eta}_2 = \ldots = \boldsymbol{\eta}_N\,,$$

and is the classical counterpart of the problem considered by Dicke [8]. From equation (7.25) it follows that in the system of classical oscillators pumped with π-pulse there is no shielding, and SR occurs ($I \sim N^2$). A special feature is the zero time lag of the burst of SR. These characteristics indicate that Dicke's theory holds when the pumping of an ellipsoid medium is completely coherent, and also point to some kind of compensation of dipole–dipole interactions of atoms in this case. It is quite obvious that considerable deviations from Dicke's theory will be observed for media of a different shape.

This analysis allows us to make several important conclusions.

According to Ref. [8], Dicke's states $|\tau, \tau\rangle$ are metastable [this is clear from Eqn (4.11)]. This conclusion, however, is not true, because in accordance with Eqn (4.33) such states are destroyed by dipole–dipole interaction.

In this section we considered the linear CMS. At the end of Section 8.1 we prove that the conclusion regarding the compensation of dipole–dipole interaction between atoms in a body of ellipsoidal shape with homogeneous pumping also holds for the more realistic model of two-level atoms.

The effects of screening discussed in this section are characterized, as we have seen, by the time $1/\omega_\mathrm{d}$. All conclusions derived here are true in the approximation of an isotropic continuous medium, which holds when $1/\omega_\mathrm{d} \gg l/v_\mathrm{T}$. It is easy to see that this condition is equivalent to Eqn (7.19), and therefore is satisfied.

Special attention must be paid to the terms of order k^2, dropped very early in this section. According to Refs [49–51], these terms lead to decorrelation of atoms, which means that SR is not feasible in the systems (2.4). This conclusion, however, is wrong.

These terms belong to relativistic corrections ($\sim 1/c^2$) to the equations of motion of atoms. The Lagrangian of atoms to this accuracy is $L = L_0 + L_1 + L_2$ [9], where L_0 is the nonrelativistic Lagrangian (which includes V_d),

$$L_1 = \sum_a \frac{m(\dot{\xi}_a)^4}{8c^2}\,, \quad L_2 = \frac{e^2}{4c^2}\sum_{a\neq b}\frac{1}{r_{ab}}\left[\dot{\xi}_a\dot{\xi}_b + (\dot{\xi}_a\mathbf{n}_{ab})(\dot{\xi}_b\mathbf{n}_{ab})\right].$$

Only L_2 was considered in Refs [49–51], while the relativistic correction to the kinetic energy L_1 was disregarded. Since $m\dot{\xi}_a^2 \sim e^2/a_0$ in an atom, we have

$$\frac{L_2}{L_1} \sim \frac{a_0}{l} \ll 1\,. \tag{7.34}$$

This means that a much greater nonlinear term L_1 was discarded in Refs [49–51], which, in accordance with Section 5, is responsible for the correlation (phasing) of atoms. We see that the term L_2 must be discarded. Inclusion of L_1 for the systems of two-level atoms is not justified, since there is strong nonlinearity anyway. The term L_1 is of key importance in cyclotron resonance masers [37, 40, 41]. For transfer of cyclotron waves in a magnetized plasma, this term in the Lagrangian may be neglected (see Section 12).

The main result of this section consists in justification of the feasibility of SR in the case of homogeneous pumping of small bodies ($L \ll \lambda$) of ellipsoidal shape. Relevant experiments can be based on the method of electron paramagnetic resonance (EPR), using a generator of microwave pulses with wavelength 3 cm and duration 1 μs, and an electromagnetic field energy flux density of about 3 W cm^{-2}. Another result of this section is that we have duly accounted for the dipole–dipole interaction of atoms that causes the shielding effect. We also noted the strong dependence of SR on the shape of the body. The problem of SR by a body of arbitrary shape is a challenge for the future.

8. Superradiance in extended bodies in the Markovian limit

As indicated in Section 2, of special practical interest is SR in extended bodies ($L \gg \lambda$). Surprisingly, despite a considerable phase gain over the size of the body, the physics of SR in this case differs little from what happens in a small body ($L \ll \lambda$): again the phasing of atoms with a subsequent burst of SR. So, let us analyze the effect of SR in the Markovian limit $\lambda \ll L \ll c\tau_\mathrm{s}$.

First we are going to consider SR in a system of two-level atoms. The problems encountered in this quantum-mechanical approach will be resolved with the aid of a CMS.

8.1 Two-level atoms

To simplify our treatment, let us first consider the case of (2.22) when, in accordance with Sections 2 and 4, new electromagnetic waves form after each scattering by the atoms (the case $l \ll \lambda$ will be considered later). In the beginning of this section we shall be partly guided by Ref. [1].

Our aim consists in calculating the intensity of radiation

$$I = -\left\langle\frac{\mathrm{d}\hat{H}_a}{\mathrm{d}t}\right\rangle = -\frac{1}{2}\,\hbar\omega_0\langle\dot{R}_3\rangle\,. \tag{8.1}$$

This formula holds under condition (2.12), when the radiation is not delayed inside the body.

The Heisenberg equation for $\dot{R}_3$ follows from Eqns (3.5)–(3.10):

$$\mathrm{i}\dot{R}_3 = 2\sum_{\mathbf{k}} g_{\mathbf{k}}(R_{\mathbf{k}}^+ c_{\mathbf{k}} - R_{\mathbf{k}}^- c_{\mathbf{k}}^+)\,. \tag{8.2}$$

Here and further, in accordance with Eqn (2.22) and the results of Section 4, we drop the nonresonant terms like R^+c^+, R^-c, etc. Now the set of equations must be closed, for which purpose we write out the equations for photon operators:

$$\mathrm{i}\dot{c}_{\mathbf{k}} = \omega_{\mathbf{k}} c_{\mathbf{k}} + g_{\mathbf{k}} R^-_{\mathbf{k}}\,, \qquad \mathrm{i}\dot{c}^+_{\mathbf{k}} = -\omega_{\mathbf{k}} c^+_{\mathbf{k}} - g_{\mathbf{k}} R^-_{\mathbf{k}}\,,$$

where the nonresonant terms are dropped again. From this, using the method of elimination of boson variables (see Section 4), which holds in the Markovian limit (2.12), we find [cf. Eqn (4.44)]:

$$c_{\mathbf{k}} = -\pi \mathrm{i} g_{\mathbf{k}} \delta(\omega_{\mathbf{k}} - \omega_0) R^-_{\mathbf{k}}\,, \qquad c^+_{\mathbf{k}} = \pi \mathrm{i} g_{\mathbf{k}} \delta(\omega_{\mathbf{k}} - \omega_0) R^+_{\mathbf{k}}\,. \quad (8.3)$$

Hence, and from Eqns (8.1), (8.2), carrying out summation with respect to the polarization of the photons, we get [cf. Eqns (5.1), (5.2)]

$$I = \sum_{\mathbf{k}} I_{\mathbf{k}} = \frac{I_0}{4\pi} \int \mathrm{d}\Omega_{\mathbf{k}} f_{\mathbf{k}} (S_0 + S_{\mathbf{k}})\,, \quad (8.4)$$

$$I_{\mathbf{k}} = 4\hbar\omega_0 p_{\mathbf{k}} (S_0 + S_{\mathbf{k}})\,,$$

$$S_0 = \sum_a \langle \sigma^+_a \sigma^-_a \rangle = \frac{1}{2} N(1+\sigma)\,, \quad (8.5)$$

$$S_{\mathbf{k}} = \sum_{a \neq b} S_{ab} \exp\left[\mathrm{i}\mathbf{k}(\mathbf{r}_a - \mathbf{r}_b)\right]\,, \quad (8.6)$$

$$\dot{\sigma} = -\lambda_0 (1+\sigma) - \frac{\lambda_0}{2\pi N} \int \mathrm{d}\Omega_{\mathbf{k}} f_{\mathbf{k}} S_{\mathbf{k}}\,, \quad (8.7)$$

$$p_{\mathbf{k}} = \frac{2\pi^2 \omega_{\mathbf{k}}}{v} (\hat{\mathbf{k}} \times \mathbf{d})^2 \delta(\omega_{\mathbf{k}} - \omega_0)\,, \quad (8.8)$$

$$f_{\mathbf{k}} = \frac{3}{2} (\hat{\mathbf{k}} \times \hat{\mathbf{d}})^2\,; \qquad \hat{\mathbf{d}} = \frac{\mathbf{d}}{d}\,; \qquad \sigma = \frac{\langle R_3 \rangle}{N}\,. \quad (8.9)$$

In equations (8.4) and (8.7) integration is carried out with respect to directions $\mathbf{k}$. In equations (8.4)–(8.9) it is assumed that the excited atoms are distributed uniformly within the body, which implies that the quantities

$$L_{ab} \equiv \langle \sigma^z_a \sigma^z_b \rangle\,, \qquad \sigma \equiv \langle \sigma^z_a \rangle \quad (8.10)$$

do not depend on the subscripts (a, b) (since $(\sigma^z_a)^2 = 1$, we assume that $a \neq b$). The first necessary condition of homogeneity (8.10) is the uniform pumping of the atomic system, which is easily attained in the case of (2.12). The second condition of homogeneity is expressed by Eqn (2.27). Later in this section we demonstrate that SR is a narrow beam (or rather two oppositely directed beams) directed along the axis corresponding to the largest dimension of the body (we call it the main axis of the body). The RCI, associated with a given atom, is stretched along the axis of the body and is of the same shape as the similar region of another atom; therefore the atoms occur under similar conditions. The condition of homogeneity is not satisfied only for a small proportion of atoms located in the surface layer of thickness $R/\sqrt{F}$. Analysis indicates that the above condition is satisfied for the practically interesting case of considerably elongated bodies ($R \ll L$).

Along with the condition of homogeneity (8.10), we use the assumption that the atomic subsystem is classical, which was discussed in Section 4 [cf. Eqn (4.28)]:

$$\Delta R_3 \ll N \quad \text{or} \quad \langle R_3^2 \rangle \approx \langle R_3 \rangle^2\,. \quad (8.11)$$

Summing up L_{ab} over indices, from Eqns (8.10), (8.11) we get

$$\sum_{a \neq b} L_{ab} = N(N-1) L_{ab} = \sum_{a,b} \langle \sigma^z_a \sigma^z_a \rangle - \sum_a \left\langle (\sigma^z_a)^2 \right\rangle$$
$$= \langle R_3^2 \rangle - N \approx \langle R_3 \rangle^2\,; \qquad \langle \sigma^z_a \sigma^z_b \rangle \approx \langle \sigma^z_a \rangle \langle \sigma^z_b \rangle = \sigma^2\,;$$
$$\left\langle (R^z_{\mathbf{k}})^2 \right\rangle = \langle R^z_{\mathbf{k}} \rangle^2 + \sum_a (1 - \sigma^2) \exp(2\mathrm{i}\mathbf{k}\mathbf{r}_a) \approx \langle R^z_{\mathbf{k}} \rangle^2\,.$$

From these relations we conclude that the operators σ^z_a and $R^z_{\mathbf{k}}$ may be regarded as c-numbers:

$$\hat{\sigma}^z_a \approx \sigma\,, \qquad \hat{R}^z_{\mathbf{k}} \approx \sigma N p(\mathbf{k})\,, \quad (8.12)$$

where

$$p(\mathbf{k}) = \frac{1}{N} \sum_a \exp(\mathrm{i}\mathbf{k}\mathbf{r}_a)\,.$$

This validates the rule for uncoupling the operators used in Ref. [1] without proof (Section 4.1.4):

$$\langle \sigma^z_a \sigma^+_b \sigma^-_c \rangle \approx \langle \sigma^+_b \sigma^z_a \sigma^-_c \rangle \approx \langle \sigma^+_b \sigma^-_c \sigma^z_a \rangle \approx \langle \sigma^z_a \rangle \langle \sigma^+_b \sigma^-_c \rangle\,. \quad (8.13)$$

With due account for Eqn (8.12), the Heisenberg equation for σ^+_a is (the equation for σ^-_a is obtained through Hermitian conjugation):

$$\frac{\mathrm{d}\sigma^+_a}{\mathrm{d}t} = \mathrm{i}\omega_0 \sigma^+_a + \sigma \sum_{\mathbf{k}} p_{\mathbf{k}} \exp\left[\mathrm{i}\mathbf{k}(\mathbf{r}_c - \mathbf{r}_a)\right] \sigma^+_c\,. \quad (8.14)$$

Hence, after some straightforward algebra, we find [1]

$$\frac{\mathrm{d}S_{\mathbf{k}}}{\mathrm{d}t} = \frac{1}{2} \lambda_0 f_{\mathbf{k}} \gamma_0(\mathbf{k}) N^2 \sigma (1+\sigma) + \lambda_0 \varkappa(\mathbf{k}) N \sigma S_{\mathbf{k}}\,. \quad (8.15)$$

The formfactors of the medium are given by

$$\varkappa(\mathbf{k}) = f_{\mathbf{k}} \gamma(\mathbf{k})\,;$$
$$\gamma(\mathbf{k}) = \frac{1}{4\pi} \int \mathrm{d}\Omega_{\mathbf{k}'}\, p(\mathbf{k}' - \mathbf{k}) = \frac{1}{V} \int_V \mathrm{d}^3 r\, \frac{\sin kr}{kr} \exp(-\mathrm{i}\mathbf{k}\mathbf{r})\,;$$
$$\gamma_0(\mathbf{k}) = \frac{1}{4\pi} \int \mathrm{d}\Omega_{\mathbf{k}'} \left| p(\mathbf{k}' - \mathbf{k}) \right|^2\,, \quad (8.16)$$

where in the second expression we carry out integration over the volume of the body V. Owing to the resonant nature of relations (8.13), in Eqns (8.6), (8.7), (8.15) and (8.16) we have $k = k' = \omega_0/c$. The derivation of Eqn (8.15) was based on the property of formfactor $p(\mathbf{k}' - \mathbf{k})$ which states that $\mathbf{k}'$ deviates from $\mathbf{k}$ by no more than a small diffraction angle: $|\mathbf{k}' - \mathbf{k}|/k \lesssim 1/kR$. This property allows us to set $\mathbf{k}' = \mathbf{k}$ in all expressions which involve the formfactor, which can be done under condition (2.27).

Equations (8.7), (8.15), obtained in Ref. [1], comprise a closed system that describes SR in a sample with the properties $\lambda \ll L \ll c\tau_{\mathrm{s}}$, $F \gg 1$. The function $S_{\mathbf{k}}$ describes the correlations of atoms, and σ describes their inversion: the number of atoms in the upper and lower states is $N(1 \pm \sigma)/2$. Let us consider the most interesting case of noncoherent pumping

$$\sigma = 1\,, \qquad S_{\mathbf{k}} = 0 \quad \text{at} \quad t = 0\,, \quad (8.17)$$

which corresponds to the complete inversion of atoms and absence of initial correlations. From Eqn (8.15) we see that

the correlations of atoms arise because of the first term on the right-hand side (the second term at $t = 0$ is zero), which corresponds to spontaneous radiation [this is also clear from Eqn (8.7)] averaged over the initial state of the atoms. Strictly speaking, the first term is only defined with an accuracy down to a factor of $\beta \sim 1$, since earlier we neglected the noncommutativity of atomic operators when substituting Eqn (8.3) into (8.2) and other expressions containing $c_{\mathbf{k}}$ and $c_{\mathbf{k}}^{+}$. In this way, the first term describes the creation of order out of chaos as a result of the initial fluctuation. The magnitude of this fluctuation is not essential, since, as follows from the analysis of Eqn (4.16), the dependence on β is logarithmic.

The second term in Eqn (8.15) describes nonlinear phasing of two-level atoms. It only occurs in the presence of inversion ($\sigma > 0$), which points to the key importance of coherent induced processes in the phasing of atoms (see Section 2). If we take relaxation into account, the right-hand side of Eqn (8.16) will feature the term $-S_{\mathbf{k}}/T_2$, which indicates that, similarly to the point of phase transition in the case of equilibrium phase transition of the second order, there is a threshold condition of the beginning of SR:

$$\lambda_0 \varkappa_0 N \sigma > \frac{1}{T_2}\,, \tag{8.18}$$

where $\varkappa_0 \equiv \varkappa(\mathbf{k}_0)$, and $\mathbf{k} = \mathbf{k}_0$ is the direction of maximum of $\varkappa(\mathbf{k})$ along which the SR is directed (see below). Formula (8.18) specifies the threshold condition of SR as shown in Figs 3 and 4. Hereinafter we assume that $T_2 \to \infty$ in accordance with Eqn (2.10).

The author was unable to find the solution of the equations of SR (8.7), (8.15) in literature, so this solution is given below.

In the 'region of formation of SR' ($t \ll t_0$), in accordance with Eqn (8.17), from Eqn (8.15) we get

$$S_{\mathbf{k}}(t) = \frac{\gamma_0}{\gamma}\, N(\exp N\lambda_0 \varkappa t - 1)\,.$$

Therefore, at $\tau_{\mathrm{s}} \ll t \ll t_0$ we have

$$S_{\mathbf{k}}(t) = \frac{\gamma_0}{\gamma}\, N \exp N\lambda_0 \varkappa t\,. \tag{8.19}$$

In the 'region of SR' $t \gg \tau_{\mathrm{s}}$, which overlaps with the region of formation of SR, the terms with spontaneous transitions are small, and therefore Eqn (8.17) and (8.15) become

$$\dot{\sigma} = -\frac{\lambda_0}{2\pi N} \int \mathrm{d}\Omega_{\mathbf{k}} f_{\mathbf{k}} S_{\mathbf{k}}\,, \tag{8.20}$$

$$\dot{S}_{\mathbf{k}} = \lambda_0 \varkappa(\mathbf{k}) N \sigma S_{\mathbf{k}}\,. \tag{8.21}$$

The solution of Eqn (8.21) is sought in the form

$$S_{\mathbf{k}} = \exp \varkappa(\mathbf{k}) \Phi(t)\,, \tag{8.22}$$

then from Eqns (8.20), (8.21) we get

$$\sigma = \frac{1}{N\lambda_0} \frac{\mathrm{d}\Phi}{\mathrm{d}t}\,, \tag{8.23}$$

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} + \frac{\lambda_0^2}{2\pi} \int \mathrm{d}\Omega_{\mathbf{k}} f_{\mathbf{k}} \exp \varkappa(\mathbf{k}) \Phi(t) = 0\,. \tag{8.24}$$

Multiplying Eqn (8.24) by $\mathrm{d}\Phi/\mathrm{d}t$, we find its first integral

$$\left(\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)^2 + \frac{\lambda_0^2}{\pi} S = U_0\,, \tag{8.25}$$

$$S = \int \frac{\mathrm{d}\Omega_{\mathbf{k}}}{\gamma(\mathbf{k})} \exp \varkappa(\mathbf{k}) \Phi(t)\,, \tag{8.26}$$

where U_0 is the integration constant. Observe in anticipation that the small neighborhood $\Phi \approx \Phi_0 \gg 1$ is important for SR.

Assume that the body is oblong. Then in a certain direction $\mathbf{k} = \mathbf{k}_0$ the function $\varkappa(\mathbf{k})$ has an absolute maximum $\varkappa_0$. Since $\varkappa(\mathbf{k}) = \varkappa(-\mathbf{k})$, a precisely similar maximum occurs in the opposite direction. As will be proved later, in these two directions ($\pm\mathbf{k}_0$) the exponential in Eqn (8.26) exhibits sharp maxima, and practically all the radiation goes this way.

We align the z axis with $\mathbf{k}$. In the case of interest (2.27) the contribution to $\gamma(\mathbf{k})$ from Eqn (8.16) comes from the region close to the z axis [see Eqn (2.29)]. Under condition (2.27), the integration over ρ may be extended to $+\infty$, whence, by analogy with Eqn (2.36), we get

$$\gamma(\mathbf{k}) \approx \frac{\pi L(\mathbf{k})}{Vk^2}\,, \tag{8.27}$$

where $L(\mathbf{k}) \equiv L(\mathbf{k}, \mathbf{r} = 0)$ is the thickness of the body at the location where it is 'pierced' by the straight line passing through the point $\mathbf{r} = 0$ in the direction of $\mathbf{k}$. The obvious drawback of our treatment is that the physical results depend on the selection of the point $\mathbf{r} = 0$. At first sight, therefore, it may seem that our theory is only good for symmetrical bodies that have an inversion point, which from the symmetry considerations ought to be selected as the point $\mathbf{r} = 0$. A simple argument proves, however, that the theory can be applied to an arbitrary body. Take two points on the surface and connect them with a straight line. There are two points such that the length of connecting line is the greatest and corresponds to the maximum thickness of the body $L = L_0$. The line is perpendicular to the surface at these points, since otherwise we can displace one of the points on the surface and get another pair of points separated by a greater distance.

The two points selected as described above define unambiguously the direction k_0. Somewhere in the middle of the connecting line we select the point $\mathbf{r} = 0$. All our calculations will be good to logarithmic accuracy ($\sim 1/\Lambda$, $\Lambda = \ln N$). To the same accuracy, the location of the point $\mathbf{r} = 0$ on the connecting line is arbitrary. If the transitions occur between levels degenerate with respect to the direction of momentum, then the main contribution to SR will come from those states for which $f_{\mathbf{k}} = \max$ — that is, $\mathbf{d} \perp \mathbf{k}_0$; therefore, hereinafter we set $f_{\mathbf{k}} = 3/2$.

Introducing a small dimensionless deviation $\boldsymbol{\xi} = (\mathbf{k} - \mathbf{k}_0)/k_0$ ($\xi \ll 1$, $\boldsymbol{\xi}\mathbf{k} \approx \boldsymbol{\xi}\mathbf{k}_0 = 0$), we expand $L(\mathbf{k})$ and $\varkappa(\mathbf{k})$ near $\mathbf{k}_0$:

$$L(k) = L_0 - \frac{1}{2} L_1 \xi_1^2 - \frac{1}{2} L_2 \xi_2^2\,; \qquad \varkappa(\mathbf{k}) = \varkappa_0 + \Delta\varkappa\,;$$

$$\Delta\varkappa(\mathbf{k}) = -\frac{1}{2} \varkappa_1 \xi_1^2 - \frac{1}{2} \varkappa_2 \xi_2^2\,, \tag{8.28}$$

where ξ_1, ξ_2 are the projections of two-dimensional vector $\boldsymbol{\xi}$ on to the principal directions of the tensor composed from the derivatives $\partial^2 L/\partial\xi_\alpha \partial\xi_\beta$; $L_{1,2}$ are the principal values of this tensor; $\varkappa_{1,2} = 3\pi L_{1,2}/(2Vk^2)$, $\varkappa_0 = 3\gamma(\mathbf{k}_0)/2 = 3\pi L_0/(2Vk^2)$.

For an oblong ellipsoid of revolution with semiaxes a, b $(a > b)$ we have

$$L_0 = 2a\,; \qquad L_1 = L_2 = 2a\left(\frac{a^2}{b^2} - 1\right); \qquad \gamma(\mathbf{k}_0) = \frac{3}{2k^2b^2}\,;$$

$$\varkappa_0 = \frac{9}{4k^2b^2}\,; \qquad \varkappa_1 = \varkappa_2 = \varkappa_0\left(\frac{a^2}{b^2} - 1\right). \tag{8.29}$$

From equations (8.26) – (8.29), making use of the relation $\mathrm{d}\Omega_{\mathbf{k}} = \mathrm{d}\xi_1\mathrm{d}\xi_2$, we get

$$S = \exp(\varkappa_0\Phi)\,\frac{\Omega}{\gamma(\mathbf{k}_0)}\,, \tag{8.30}$$

where

$$\Omega = \frac{4\pi}{\Phi_0\sqrt{\varkappa_1\varkappa_2}} \tag{8.31}$$

is the effective solid angle into which radiation is emitted (in Ω we set $\Phi = \Phi_0$). The additional factor 2 in Eqn (8.31) accounts for the contribution from $-\mathbf{k}_0$.

In the case of a thin long rod $(L \gg R)$, the major contribution to S comes from vectors $\mathbf{k}$ that 'pierce' its ends — that is, such vectors that make an angle Θ with the axis of the rod not exceeding the geometric angle $\Theta_0 = 2R/L$. For a rod $\Omega = \pi\Theta_0^2$,

$$\varkappa \approx \varkappa_0 = \frac{3}{2k^2R^2} \quad \text{at} \quad \Theta < \Theta_0\,;$$

$$\varkappa = \frac{\varkappa_0\Theta_0}{\Theta} \quad \text{at} \quad \Theta > \Theta_0\,. \tag{8.32}$$

Strictly speaking, the integrand in Eqn (8.26) features diffraction oscillations that become important at small Fresnel numbers $(F \sim \Lambda)$. However, in the practically interesting cases we have $F \gg \Lambda$, and therefore the oscillations may be neglected.

After substitution of Eqn (8.30), equation (8.25) takes on the form of the energy conservation law for one-dimensional motion in the potential $U(\Phi) = B^2\exp\varkappa_0\Phi$, where $B^2 = \lambda_0^2\Omega/[\pi\gamma(\mathbf{K}_0)]$. Now we introduce the 'turning point' $\Phi = \Phi_0$, in which $U(\Phi_0) = U_0 = B^2\exp\varkappa_0\Phi_0$, $\mathrm{d}\Phi/\mathrm{d}t = 0$. From the analogy with mechanical motion it is clear that first $\Phi(t)$ increases, at $t = t_0$ attains its maximum $\Phi = \Phi_0$, and then decreases. The effective potential energy $U(\Phi)$ has the shape of a sharp peak located at $t = t_0$:

$$\exp\varkappa_0\Phi = g(t)\exp\varkappa_0\Phi_0\,, \qquad g(t) = \frac{1}{\cosh^2\left[(t - t_0)/\tau_{\mathrm{s}}\right]}\,, \tag{8.33}$$

where t_0, τ_{s} are the integration constants satisfying the condition

$$\frac{2}{\tau_{\mathrm{s}}} = B\varkappa_0\exp\frac{\varkappa_0\Phi_0}{2}\,, \tag{8.34}$$

which follows from Eqns (8.33) and (8.25).

From equations (8.22), (8.28) and (8.33) we find

$$S_{\mathbf{k}} = \exp\varkappa\Phi = \exp\left[(\varkappa_0 + \Delta\varkappa)\Phi\right]$$
$$\approx \exp(\varkappa_0\Phi + \Delta\varkappa\Phi_0) = g(t)\exp\varkappa\Phi_0\,. \tag{8.35}$$

Joining this formula with the asymptotic expression (8.19), and using Eqn (8.34) and the expression for $\gamma_0(\mathbf{k})$ (see below), we get

$$\tau_{\mathrm{s}} = \frac{2}{\lambda_0\varkappa_0 N}\,, \tag{8.36}$$

$$\exp\varkappa_0\Phi_0 = \frac{\lambda_0^2N^2}{B^2}\,, \tag{8.37}$$

$$\Phi_0 \approx \frac{2}{\varkappa_0}\,\Lambda\,, \qquad t_0 \approx \frac{\tau_{\mathrm{s}}}{2}\,\Lambda\,. \tag{8.38}$$

According to Eqns (8.4) and (8.35), the angular distribution of the intensity of SR is given by

$$\mathrm{d}I = \frac{3}{2}\,I_0g(t)\exp\left[\varkappa(\mathbf{k})\Phi_0\right]\mathrm{d}\Omega_{\mathbf{k}}\,, \tag{8.39}$$

whence, with due account for Eqns (8.28) and (8.32) we see that the radiation is confined within the solid angle Ω. The following formula is useful that follows from Eqns (8.27) and (8.38)

$$\exp\left[\varkappa(\mathbf{k})\Phi_0\right] = \exp(\varkappa_0\Phi_0)\exp\left[-2\Lambda\,\frac{L_0 - L(\mathbf{k})}{L_0}\right]. \tag{8.40}$$

From equation (8.40) we see that the region of SR for an arbitrary body is found from relation $L_0 - L(\mathbf{k}) < L_0/\Lambda$. In particular, the intensity of SR from a rod is practically constant within the angles $\Theta < \Theta_0$ and $\pi - \Theta < \Theta_0$ (if we disregard the above-mentioned diffraction oscillations), and falls off sharply beyond these regions. In the case of a long-drawn-out $(a \gg b)$ ellipsoid of revolution we have $\Omega \sim (b/a)^2/\Lambda$. We see that for an ellipsoid the angular divergence of the beam is much less than geometrical $(\Theta \sim b/a)$. This is because of the competition of modes, like in a laser [32, 33]. According to Eqns (8.21), (8.35), the correlations are established soonest in the directions $\mathbf{k}$ that are closest to $\pm\mathbf{k}_0$. As a consequence, practically all the energy is released in these two directions. This effect is especially important with small Fresnel numbers. For example, when $F \sim 1$ (although our theory in this case is only good by order of magnitude), the angular divergence of the beam is much less than that caused by diffraction $(\Theta \sim 1/kR\sqrt{\Lambda})$.

The intensity of SR is found by integrating Eqn (8.39) with due account for Eqn (8.37) and the additional factor of 2 [cf. Eqns (8.30) and (8.31)]:

$$I = \frac{1}{4}\,I_0\varkappa_0N^2g(t) = \frac{\hbar\omega_0}{2\tau_{\mathrm{s}}}\,Ng(t)\,. \tag{8.41}$$

Time-integrating the intensity (8.41), we make sure that the entire stored energy $\hbar\omega_0N$ is released.

Let us now calculate the formfactor γ_0 for the case of (2.27). In the last formula in Eqn (8.16) we first carry out integration over $\mathrm{d}\Omega_{\mathbf{k}'}$:

$$\gamma_0 = \frac{1}{V^2k}\int \mathrm{d}^3r\,\mathrm{d}^3r'\,\frac{\sin(kR)}{R}\exp(-\mathrm{i}\mathbf{k}\mathbf{R}_z)\,,$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ (z axis aligned with $\mathbf{k}$). From the variables $(\mathbf{r}, \mathbf{r}')$, we change to $(\mathbf{r}, \mathbf{R})$. After integration over $\mathbf{r}$, which yields the factor of V, we get an integral of the same form as in the first expression in Eqn (8.16), but over a different range (since the range of variation of $\mathbf{R}$ does not coincide with the body). Integration with respect to directions

normal to $\mathbf{k}$ gives, by virtue of Eqn (2.27), the same factor as in Eqn (2.27), but the integral with respect to R_z gives $2L(\mathbf{k})$ in place of $L(\mathbf{k})$. This means that under condition (2.27) we have $\gamma_0 = 2\gamma$.

The formalism developed in this section allows the problem of SR by two-level atoms in a small-size ellipsoidal specimen (2.4) to be resolved with due account for the dipole–dipole interaction between atoms. We consider homogeneous pumping $[\langle\sigma_a^z(t=0)\rangle \equiv \sigma_a = \sigma = \text{const}]$ in the absence of initial correlations $[S_{ab}(t=0)=0]$. The equations for $\mathrm{d}\sigma_a/\mathrm{d}t$ are obtained by averaging the Heisenberg equations with the Hamiltonian (4.9). The calculation is similar to that performed for expressing Eqns (8.7), (8.15), and therefore is not reproduced here. The fastest way to solve our equations is to assume in advance that $\sigma_a(t)$, $S_{ab}(t)$ do not depend on the numbering of atoms a, b. This assumption is confirmed by the absence of contradictions in subsequent reasoning. A rigorous approach is also possible but is more complicated. The key role in these calculations belongs to the fact that the sum (7.9) does not depend on the r_b of the ellipsoid. The final equations are

$$\dot{\sigma} = -\lambda_0(1+\sigma) - 2N\lambda_0 S\,,$$
$$\dot{S} = \frac{1}{2}\,\lambda_0(1+\sigma)\sigma + N\lambda_0\sigma S\,, \tag{8.42}$$

where $\sigma = \sigma_a(t)$, $S = S_{ab}(t)$. We see that the dipole–dipole interaction drops out of Eqn (8.42), which validates Dicke's results for ellipsoidal bodies with homogeneous pumping, in spite of the above-mentioned inconsistent rejection of the dipole–dipole interaction in the initial equations.

8.2 Classical Lorentzian atoms (classical model of superradiance)

The theory presented in Section 8.1 corresponds to the case $\lambda \ll l$ and is based on relations (8.3) which hold as long as it is possible to drop the nonresonant (second) term in Eqns (4.43), (4.42). This simplifies the equations, and the calculations can be carried through to the end. A justification of this procedure, however, cannot be found in literature. This calls for further investigation, which is the purpose of this section. Moreover, we shall demonstrate that the effects associated with the nonresonant term are important for the nature of SR, and therefore the theory presented in Section 8.1 is not complete. All that can be found in literature on the validation of this approximation reduces to the following argument [2]. The dipole–dipole interaction V_d is determined by the scale of the order of l — that is, the range of the wave vectors of photons $k \sim 1/l$. At $l \ll \lambda$ the volume of this domain of $\mathbf{k}$-space is large ($k^3 \sim 1/l^3$), and therefore V_d is large compared with the radiation friction (see Section 4). Then V_d is identified with the nonresonant term in Eqn (4.43) [in the sense of our comment after Eqn (4.44)]. Finally, the conclusion is made that, since at $\lambda \ll l$ the volume of the above-mentioned region of $\mathbf{k}$-space is small, the nonresonant term may be dropped.

The flaw in this argument is that at $\lambda \ll l$ the main role is played by the asymptotics of the electromagnetic waves scattered by atoms [see Eqn (2.25)] at $kr \gg 1$. The asymptotics are defined by the singularities of Eqn (4.43); because of this, the first and the second terms therein are equally important, they are both resonant. These terms give ($k_0 = \omega_0/c$)

$$\int \pi\delta(\omega_\mathbf{k} - \omega_0)\exp \mathrm{i}\mathbf{kr}\,\mathrm{d}^3k = \frac{4\pi^2 k_0}{cr}\sin k_0 r\,;$$
$$\mathrm{i}\fint \frac{\exp \mathrm{i}\mathbf{kr}}{\omega_\mathbf{k} - \omega_0}\,\mathrm{d}^3k = \frac{4\pi^2 k_0}{\mathrm{i}cr}\cos k_0 r\,.$$

It is only their sum that describes the correct scattered wave $r^{-1}\exp \mathrm{i}k_0 r$ (see a similar comment in Ref. [66]).

To understand the new effects brought about by the nonresonant term in Eqn (4.43), we shall analyze the linear CMS in the limit of (2.42b) and (2.12), in which this term is taken into account in an exact way. Naturally, the nonlinear effect of phasing is then lost, so it would be desirable to consider the nonlinear version of the CMS.

In the limit of (2.42b), (2.12), it is only the exponential that must be differentiated in Eqn (6.10):

$$\dot{\mathbf{F}}_a + \frac{\gamma}{2}\,\mathbf{F}_a = -\frac{\mathrm{i}r_0\omega_0}{2}\sum_{b\neq a}\frac{\exp \mathrm{i}kr_{ab}}{r_{ab}}\,\mathbf{n}_{ab}\times(\mathbf{n}_{ab}\times\mathbf{F}_b)\,, \tag{8.43}$$

where $r_0 = e^2/(mc^2)$. We make the substitution

$$\mathbf{F}_a(t) = \exp\left(-\frac{\gamma}{2}\,t\right)\mathbf{G}_a(t)\,, \tag{8.44}$$

and get an equation in $\mathbf{G}_a$ similar to Eqn (8.43), in which the second term on the left-hand side is absent.

First let us analyze the Green function of Eqn (8.43). Assuming that at $t<0$ we have $\mathbf{F}_a = 0$, and at $t=0$ the 'point' initial conditions (7.4) are satisfied, we apply the Fourier transform to the equation. For the Fourier components

$$\mathbf{N}_a = \int_0^\infty \mathbf{G}_a(t)\exp(\mathrm{i}\omega t)\,\mathrm{d}t \tag{8.45}$$

we get the equation

$$\mathbf{B}_b\delta_{ab} + \mathrm{i}\omega\mathbf{N}_a = \frac{\mathrm{i}r_0\omega_0}{2}\sum_{c\neq a}\frac{\exp \mathrm{i}kr_{ac}}{r_{ac}}\,\mathbf{n}_{ac}\times(\mathbf{n}_{ac}\times\mathbf{N}_c)\,. \tag{8.46}$$

The inverse transform is based on the formula

$$\mathbf{G}_a(t) = \int_{-\infty+\mathrm{i}0}^{+\infty+\mathrm{i}0}\frac{\mathrm{d}\omega}{2\pi}\exp(-\mathrm{i}\omega t)\mathbf{N}_a\,. \tag{8.47}$$

In this way, in Eqn (8.45) it is assumed that the frequency has a positive imaginary addition: $\omega \to \omega + \mathrm{i}0$. After substituting

$$\mathbf{N}_a = \frac{\mathrm{i}}{\omega}\,\mathbf{B}_b\delta_{ab} + \mathbf{S}_a \tag{8.48}$$

into Eqn (8.46), we get

$$-\mathrm{i}\omega\mathbf{S}_a = \frac{r_0\omega_0}{2\omega}\,\frac{\exp \mathrm{i}kr_{ab}}{r_{ab}}\,\mathbf{n}_{ab}\times(\mathbf{n}_{ab}\times\mathbf{B}_b) - \frac{\mathrm{i}r_0\omega_0}{2}\,\mathbf{\Lambda}\,;$$
$$\mathbf{\Lambda} = \sum_{c\neq a}\frac{\exp \mathrm{i}kr_{ac}}{r_{ac}}\,\mathbf{n}_{ac}\times(\mathbf{n}_{ac}\times\mathbf{S}_c)\,. \tag{8.49}$$

The solution of Eqn (8.49) is sought with the substitution

$$\mathbf{S}_a = \mathbf{M}_a\,\frac{\exp \mathrm{i}kr_{ab}}{r_{ab}}\,. \tag{8.50}$$

The term

$$\exp \mathrm{i}k(r_{ac} + r_{cb})\,, \tag{8.51}$$

that arises in the calculation of $\mathbf{\Lambda}$, remains constant for atoms c located on the line ab (Fig. 9) which connects atoms a and b. This means that the main contribution to $\mathbf{\Lambda}$ comes from the region adjacent to this line. Consider an atom c located at a distance ρ from the line ab, and draw a plane through an atom c perpendicular to line ab. This plane crosses line ab at some point which is removed to a distance z from point b, and $0 < z < r_{ab}$. At $\rho \ll r_{ab}$ we have

$$r_{cb} \approx \sqrt{z^2 + \rho^2} \approx z + \frac{\rho^2}{2z}\,; \qquad r_{ac} \approx r_{ab} - z + \frac{\rho^2}{2(r_{ab} - z)}\,;$$

$$r_{ac} + r_{cb} \approx r_{ab} + \frac{r_{ab}}{2z(r_{ab} - z)}\,\rho^2\,. \tag{8.52}$$

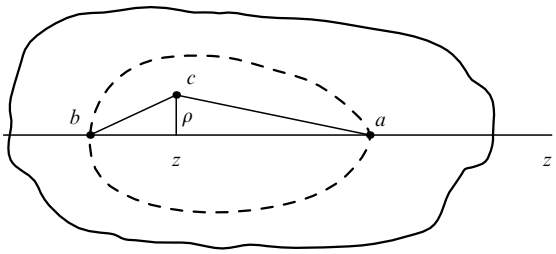

Figure 9. Region of constructive interference for two atoms a and b (dashed line). Atom b corresponds to the origin of coordinates $z = 0$.

Hence, and from Eqn (8.51), it follows that the main contribution to $\mathbf{\Lambda}$ comes from the region $0 < z < r_{ab} \sim L$, $z \sim r_{ab} \sim L$, $\rho \sim \sqrt{r_{ab}/k} \sim \sqrt{L/k}$, where we have noted that the typical value of r_{ab} is of the order of the size of the medium L (this will become clear soon). Following Section 2, we call this region the RCI. Its physical connotation is as follows. The phase of a signal moving from b to a along ab is $\varphi_1 = kr_{ab}$. Let us consider a different way, bca. Electromagnetic waves coming from b via bc pump oscillator c that starts emitting its own waves which reach a. The signal moving via bca corresponds to the phase $\varphi_2 = k(r_{bc} + r_{ca})$. The interference of the two signals is constructive if $\varphi_2 - \varphi_1 \lesssim 1$. The volume of the RCI is found from Eqn (2.33). The number of atoms within RCI is large [see Eqn (2.34)], and therefore the summation may be replaced with integration:

$$\sum_c \to n \int \mathrm{d}^3 r_c = n \int_0^{r_{ab}} \mathrm{d}z \int \mathrm{d}^2\rho\,; \tag{8.53}$$

$$\mathbf{\Lambda} \approx n \int \mathrm{d}^3 r_c\, \frac{\exp\left[\mathrm{i}k(r_{ac} + r_{bc})\right]}{(r_{ab} - z)z}\, \mathbf{Q}(z)\,, \tag{8.54}$$

where $\mathbf{Q}(z) = \mathbf{n}_{ab} \times \left[\mathbf{n}_{ab} \times \mathbf{M}(z)\right]$. Here we have taken into account that by virtue of $\rho \ll z$ in the RCI we have $\mathbf{n}_{ac} \approx \mathbf{n}_{ab}$. In addition, the coefficient $\mathbf{M}_a$ in Eqn (8.50) varies slowly compared with the exponential, and therefore in the calculation of $\mathbf{\Lambda}$ we may set $\mathbf{M}_c \equiv \mathbf{M}(z, \rho) \approx \mathbf{M}(z, 0) \equiv \mathbf{M}(z)$.

Substituting Eqn (8.52) into Eqn (8.54), we perform an integration with respect to $\mathrm{d}^2\rho = 2\pi\rho\,\mathrm{d}\rho$ [with due account for Eqn (2.27)]:

$$\mathbf{\Lambda} = \frac{2\pi \mathrm{i} n}{K}\, \frac{\exp \mathrm{i}kr_{ab}}{r_{ab}}\, \mathbf{n}_{ab} \times (\mathbf{n}_{ab} \times \mathbf{T})\,; \qquad \mathbf{T}(r_{ab}) = \int_0^{r_{ab}} \mathrm{d}z\, \mathbf{M}(z)\,.$$

Equation (8.49) becomes

$$-\mathrm{i}\omega \mathbf{M}(r) = \frac{r_0\omega_0}{2\omega}\, \mathbf{n} \times (\mathbf{n} \times \mathbf{B}_b) + \pi r_0 c n \mathbf{n} \times \left[\mathbf{n} \times \mathbf{T}(r)\right]\,, \tag{8.55}$$

where $r \equiv r_{ab}$, $\mathbf{n} \equiv \mathbf{n}_{ab} = \mathbf{r}_{ab}/r_{ab}$. The substitution

$$\mathbf{M}(r) = \mathbf{n} \times (\mathbf{n} \times \mathbf{B}_b) f(r) \tag{8.56}$$

allows us to go over from vector $\mathbf{M}$ to scalar f:

$$-\mathrm{i}\omega f(r) = \frac{r_0\omega_0}{2\omega} - \pi r_0 c n R(r)\,; \tag{8.57}$$

$$R(r) = \int_0^r \mathrm{d}r'\, f(r')\,. \tag{8.58}$$

Rewriting Eqn (8.57) in the form

$$\frac{\mathrm{d}R}{\mathrm{d}r} + \frac{\pi \mathrm{i} r_0 c n}{\omega}\, R = \frac{\mathrm{i} r_0 \omega_0}{2\omega^2} \tag{8.59}$$

and using the boundary condition $R(0) = 0$ that follows from Eqn (8.58), we find the solution of Eqn (8.59):

$$R(r) = \frac{\omega_0}{2\pi c n \omega}\left[1 - \exp\left(-\frac{\pi \mathrm{i} r_0 c n}{\omega}\, r\right)\right].$$

Hence, and from Eqns (8.47), (8.48), (8.50), (8.56), and (8.58), we find

$$f(r) = \frac{\mathrm{d}R}{\mathrm{d}r} = \frac{\mathrm{i}\omega_0 r_0}{2\omega^2}\, \exp\left(-\frac{\pi \mathrm{i} r_0 c n}{\omega}\, r\right);$$

$$\mathbf{N}_a = \frac{\mathrm{i}}{\omega}\, \mathbf{B}_b \delta_{ab} + \frac{\mathrm{i}\omega_0 r_0}{2\omega^2}\, \exp\left(-\frac{\pi \mathrm{i} r_0 c n}{\omega}\, r_{ab}\right) \times \frac{\exp \mathrm{i}kr_{ab}}{r_{ab}}\, \mathbf{n}_{ab} \times (\mathbf{n}_{ab} \times \mathbf{B}_b)\,; \tag{8.60}$$

$$\mathbf{G}_a(t) = \mathbf{B}_b \delta_{ab} + \frac{\mathrm{i}\omega_0 r_0}{4\pi}\, \frac{\exp \mathrm{i}kr_{ab}}{r_{ab}}\, \mathbf{n}_{ab} \times (\mathbf{n}_{ab} \times \mathbf{B}_b) K\,;$$

$$K = \int_{-\infty+\mathrm{i}0}^{+\infty+\mathrm{i}0} \frac{\mathrm{d}\omega}{\omega^2}\, \exp\left(-\mathrm{i}\omega t - \frac{\pi \mathrm{i} r_0 c n}{\omega}\, r_{ab}\right). \tag{8.61}$$

To calculate the integral K, we expand each exponential in the integrand in Taylor series, and apply the residue theorem:

$$K = -2\pi\, \frac{\sqrt{t}}{\beta}\, J_1\left(2\sqrt{\beta t}\right), \tag{8.62}$$

where $\beta = \pi n c r_0 r_{ab}$, and J_1 is the Bessel function. From equations (6.9), (6.11), (8.44), (8.61) and (8.62) we find

$$\xi_a(t) = \exp\left(-\frac{\gamma}{2}\, t\right)\left[\boldsymbol{\eta}_b \cos(\omega_0 t + \varphi_b)\delta_{ab} - \frac{\omega_0 r_0}{2r_{ab}}\sqrt{\frac{t}{\beta}}\, J_1\left(2\sqrt{\beta t}\right)\mathbf{n}_{ab} \times (\mathbf{n}_{ab} \times \boldsymbol{\eta}_b)\sin(\omega_0 t + \varphi_b - kr_{ab})\right]. \tag{8.63}$$

Now we embark on the calculation of the intensity of radiation using Eqns (6.7), (6.8):

$$\mathbf{Q} = \mathbf{n} \times \mathbf{b}\,; \tag{8.64}$$

$$\mathbf{b} = \sum_a \xi_a\left(t' + \frac{1}{c}\, \mathbf{n}\mathbf{r}_a\right), \tag{8.65}$$

where $t' = t - r/c$, $\mathbf{n} = \mathbf{r}/r$. We align the origin of coordinates with atom b, then r is the distance from the remote observation point to atom b. From equations (8.63), (8.65) we find

$$\mathbf{b} = \exp\left(-\frac{\gamma}{2}\,t\right)\left[\boldsymbol{\eta}_b\cos(\omega_0 t' + \varphi_b) - \frac{\omega_0 r_0}{2}\,\mathbf{A}\right]; \qquad (8.66)$$

$$\mathbf{A} = \sum_a \frac{1}{r_{ab}}\sqrt{\frac{t'}{\beta}}\,J_1\big(2\sqrt{\beta t'}\big)\mathbf{n}_{ab}\times(\mathbf{n}_{ab}\times\boldsymbol{\eta}_b) \times \sin(\omega_0 t' + \varphi_b + \Psi)\,, \qquad (8.67)$$

where $\Psi = k\mathbf{n}\mathbf{r}_{ab} - kr_{ab}$. We direct the z axis along $\mathbf{n}$; let ρ be the distance from atom a to the z axis. The main contribution to the sum $\mathbf{A}$ comes from the atoms a located within the RCI:

$$\rho \sim \sqrt{\frac{r_{ab}}{k}} \sim \sqrt{\frac{L}{k}} \ll L\,. \qquad (8.68)$$

The number of atoms in this region is given by the familiar formula (2.34). In this case the meaning of the RCI is somewhat different from what it used to be (Fig. 10). This time the interference occurs between two signals that come to the observation point. One signal comes from atom b along vector $\mathbf{r}$ — that is, along the straight line that connects atom b and observation point. The other signal is emitted by atom b and reaches atom a along a straight path. Then the signal is re-emitted by atom a, and travels to the observation point along a straight line. With this in mind, we find

$$\mathbf{n}\mathbf{r}_{ab} = z\,, \qquad r_{ab} = \sqrt{z^2+\rho^2} \approx z + \frac{\rho^2}{2z}\,;$$

$$\Psi \approx -\frac{k\rho^2}{2z}\,, \qquad \mathbf{n}_{ab} \approx \mathbf{n}\,;$$

$$\mathbf{A} \approx \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\eta}_b)D\,; \qquad (8.69)$$

$$D = n\int_0^L \frac{\mathrm{d}z}{z}\sqrt{\frac{t'}{\beta_0}}\,J_1\big(2\sqrt{\beta_0 t'}\big)P\,; \qquad \beta_0 = \pi n c r_0 z\,, \qquad (8.70)$$

$$P = \int \mathrm{d}^2\rho\,\sin\left(\omega_0 t' + \varphi_b - \frac{k\rho^2}{2z}\right) = -\frac{2\pi z}{k}\cos(\omega_0 t' + \varphi_b)\,. \qquad (8.71)$$

Here $L \equiv L(\mathbf{n})$ is the distance from atom b to the periphery of the medium along the direction of observation $\mathbf{n}$.

Further integration with respect to $\mathrm{d}z$ in Eqn (8.70) takes advantage of the equality $J_1(x) = -J_0'(x)$. For D we get

$$D = \frac{2}{\omega_0 r_0}\cos(\omega_0 t' + \varphi_b)\big[J_0(x_0) - 1\big]\,; \qquad (8.72)$$

$$x_0 = 2\sqrt{\pi n c r_0 L t'}\,. \qquad (8.73)$$

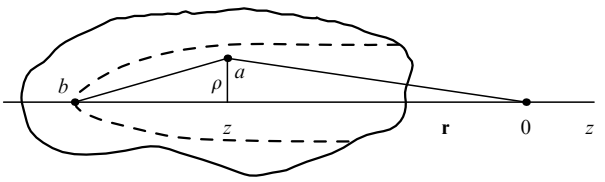

Figure 10. Region of constructive interference for radiation arriving at the observation point 0.

From equations (8.64), (8.66), (8.69) and (8.72) we get

$$\mathbf{b} = \exp\left(-\frac{\gamma}{2}\,t\right)\cos(\omega_0 t' + \varphi_b)\big[\boldsymbol{\eta}_b - \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\eta}_b)(J_0 - 1)\big]\,;$$

$$\mathbf{Q} = \exp\left(-\frac{\gamma}{2}\,t'\right)\cos(\omega_0 t' + \varphi_b)(\mathbf{n}\times\boldsymbol{\eta}_b)(1 + J_0 - 1)$$

$$= \exp\left(-\frac{\gamma}{2}\,t'\right)\cos(\omega_0 t' + \varphi_b)(\mathbf{n}\times\boldsymbol{\eta}_b)J_0(x_0)\,, \qquad (8.74)$$

where we have specially marked the phenomenal cancellation of two terms, which will be discussed in a while.

From equations (6.5), (6.7), (8.74) we get the following expression for the intensity of radiation:

$$\mathrm{d}I \equiv \mathrm{d}I_b = \mathrm{d}\Omega\,\frac{3\gamma E_0^{(b)}}{8\pi}\sin^2(\alpha)J_0^2(x_0)\exp(-\gamma t)\,, \qquad (8.75)$$

where α is the angle between vectors $\mathbf{n}$ and $\boldsymbol{\eta}_b$. Time-integrating the intensity (8.75) between the limits $(0,\infty)$, we find the energy emitted into the solid angle $\mathrm{d}\Omega$:

$$\mathrm{d}E_{\mathrm{r}} = E_0^{(b)}\,\mathrm{d}\Omega\,\frac{3}{8\pi}\,F(\varphi)\sin^2\alpha\,, \qquad (8.76)$$

$$F(\varphi) = \exp(-\varphi)I_0(\varphi)\,, \qquad (8.77)$$

where φ is defined in Eqn (2.20), and I_0 is a modified Bessel function. Integration with respect to time is based on the formula [67]

$$\int_0^\infty \mathrm{d}t\,\exp(-\gamma t)J_0\big(\alpha\sqrt{t}\big)J_0\big(\beta\sqrt{t}\big) = \frac{1}{\gamma}\,I_0\left(\frac{\alpha\beta}{2\gamma}\right)\exp\left(-\frac{\alpha^2+\beta^2}{4\gamma}\right).$$

Let us discuss the physics behind these results. First we adopt the 'primitive' standpoint described in Section 6. After some time, the energy of atom b is distributed between N atoms of the medium. This prepares a coherent ensemble of oscillators that vibrate with the amplitudes (6.18) and phases $\varphi_a = kr_{ab}$ as determined by the signals coming from atom b. The amplitude of the field at arbitrary observation point r far beyond the limits of the medium is [cf. Eqn (2.25)]

$$E \sim \sum_a \frac{e\xi_0}{\sqrt{N}}\,\frac{k^2}{R_a}\exp(\mathrm{i}kR_a + \mathrm{i}\varphi_a)\,,$$

where $R_a = |\mathbf{r} - \mathbf{r}_a| \approx r - \widehat{\mathbf{r}}\mathbf{r}_a$. The origin is aligned with atom b. Replacing summation with integration, we get

$$E \sim \sum_a \frac{e\xi_0 k}{\sqrt{N}}\,\frac{\exp\mathrm{i}kr}{r}\,J\,,$$

where $J = n\int \mathrm{d}^3 r_a\,\exp(-\mathrm{i}k\widehat{\mathbf{r}}\mathbf{r}_a + \mathrm{i}kr_a) \sim N_0$. We see that out of N atoms it is only N_0 that determines the field at an arbitrary remote point. The fields of other atoms cancel out through destructive interference.

According to the formula obtained above, the amplitude of the field is N_0 times the amplitude that would be created by a solitary atom oscillating with the amplitude $\xi_0/\sqrt{N}$. Since such an atom emits radiation with intensity I_0/N, the intensity of radiation by the medium is

$$I \sim \frac{I_0}{N}(N_0)^2 \sim I_0\,\frac{n^2L^4}{Nk^2} \sim I_0\varphi\,, \qquad (8.78)$$

where we have noted that $nL^3 \sim N$.

In reality, like in Section 7, everything is much more complicated. Instead of SR, the coherent cloud of atoms of the medium prepared by atom b give rise to the opposite effect of screening. This is clear from the cancellation of terms in Eqn (8.74) above, and from the property $J_0^2 < 1$ and expression (8.75) which we now rewrite in the form

$$\mathrm{d}I = \mathrm{d}I_\mathrm{s}\, J_0^2(x_0)\,, \tag{8.79}$$

where $\mathrm{d}I_\mathrm{s}$ is the intensity of spontaneous radiation by a solitary atom b. Thus, $\mathrm{d}I < \mathrm{d}I_\mathrm{s}$, which means that the medium shields even the spontaneous emission by atom b. The extent of shielding is clear from Eqns (8.76), (8.77), (8.79). By virtue of Eqn (2.21), we have $F(\varphi) \approx 1/\sqrt{2\pi\varphi} \ll 1$, that is, $E_\mathrm{r}/E_0^{(b)} \sim 1/\sqrt{\varphi} \ll 1$. Expression (8.79) features two characteristic times: $\tau_0 = 1/\gamma$, $\tau_\mathrm{s} = 1/(ncr_0L) \sim \tau_0/\varphi$. At $t < \tau_\mathrm{s}$ we have $\mathrm{d}I \approx \mathrm{d}I_\mathrm{s}$, and at $t \gg \tau_\mathrm{s}$ $\mathrm{d}I \sim \mathrm{d}I_\mathrm{s}\,\tau_\mathrm{s}/t$.

Now let us consider the arbitrary boundary condition (6.4), (6.11). The intensity of radiation is now given by Eqn (6.7), in which, in accordance with Eqns (6.13), (8.74), we must define $\mathbf{Q}$ as

$$\mathbf{Q} = \sum_b \exp\left(-\frac{\gamma t_b}{2}\right)\cos(\omega_0 t_b + \varphi_b)(\mathbf{n}\times\boldsymbol{\eta}_b)J_0(x_b)\,. \tag{8.80}$$

Here

$$t_b = t - \frac{1}{c}\,|\mathbf{r}-\mathbf{r}_b| \approx t' + \frac{1}{c}\,\mathbf{n}\mathbf{r}_b\,, \qquad t' = t - \frac{r}{c}\,,$$

$\mathbf{r}$, $\mathbf{r}_b$ are radius vectors drawn to the observation point and atom b respectively, from the origin of coordinates located at the arbitrary point: $x_b = 2\sqrt{\pi ncr_0L_bt_b}$, where $L_b \equiv L_b(\mathbf{r}_b)$. The exponential term in Eqn (8.80) and the function $J_0(x_b)$ depend but weakly on the coordinates of atom b, so we may set $\exp(-\gamma t_b/2) \approx \exp(-\gamma t'/2)$, $x_b \approx 2\sqrt{\pi ncr_0L_bt'}$. If the initial phases φ_b of different atoms are not correlated, then, in accordance with Eqns (6.15), (8.75), shielding occurs in this case as well. Assume now that $\varphi_1 = \varphi_2 = \ldots = \varphi_N$. Then from Eqn (8.80) we see that, owing to the fast variation of the cosine and to the lack of correlation between the positions of atoms $\mathbf{r}_b$, Eqn (6.15) will hold again, and shielding will occur.

Now let us consider the case of pumping of the medium by a running wave with the wave vector $\mathbf{k}$ — for example, by a short laser pulse. The pulse passes atom b at the time $T_b = z_b/c = \mathbf{k}\mathbf{r}_b/\omega_0$, and therefore the dipole of this atom vibrates by the law $\Theta(t-T_b)\times\exp\left[-\mathrm{i}\omega_0(t-T_b)\right]$. Comparing the real part of this expression with Eqn (2.5), we get

$$\varphi_b = -\mathbf{k}\mathbf{r}_b\,; \tag{8.81}$$

$$\mathbf{Q} = \exp\left(-\frac{\gamma t'}{2}\right)\sum_b \cos(\omega_0 t' + k\mathbf{n}\mathbf{r}_b - \mathbf{k}\mathbf{r}_b)\mathbf{m}(\mathbf{r}_b)\,;$$

$$\mathbf{m}(\mathbf{r}_b) = (\mathbf{n}\times\boldsymbol{\eta}_b)J_0(x_b)\,. \tag{8.82}$$

In this case the emission by the medium will occur at small angles $(1/kL)$, close to the direction of the vector $\mathbf{n}_0 = \mathbf{k}/k$. We set therefore $\mathbf{n} = \mathbf{n}_0 + \boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is a small two-dimensional vector, $\boldsymbol{\delta}\mathbf{n}_0 = 0$, $\delta \ll 1$. Summation in Eqn (8.82) may be replaced with integration [see Eqn (2.35)]. Averaging the intensity (6.7) over the fast oscillations (with the period $T = 2\pi/\omega$), and integrating with respect to angles using the formula $\mathrm{d}\Omega \approx \mathrm{d}^2\delta$, we get the following expression for the total intensity (further on we set $\mathbf{r}_b \equiv \mathbf{r}$, $\mathbf{r}_{b'} \equiv \mathbf{r}'$):

$$I = \frac{e^2\omega_0^4}{4\pi c^3}\exp(-\gamma t')n^2\int \mathrm{d}^3r\,\mathrm{d}^3r'\,\mathbf{m}(\mathbf{r})\mathbf{m}(\mathbf{r}')A_0\,;$$

$$A_0 = \int \mathrm{d}^2\delta\,\left\langle\cos(\omega_0 t' + k\mathbf{r}_\perp\boldsymbol{\delta})\cos(\omega_0 t' + k\mathbf{r}'_\perp\boldsymbol{\delta})\right\rangle$$

$$= \frac{1}{2}\int \mathrm{d}^2\delta\cos\left[k(\mathbf{r}_\perp - \mathbf{r}'_\perp)\boldsymbol{\delta}\right] = \frac{1}{2}(2\pi)^2\delta^{(2)}\left[k(\mathbf{r}_\perp - \mathbf{r}'_\perp)\right]$$

$$= \frac{2\pi^2}{k^2}\,\delta^{(2)}(\mathbf{r}_\perp - \mathbf{r}'_\perp)\,,$$

where $\mathbf{r}_\perp$ is the component of vector $\mathbf{r}_b$ normal to $\mathbf{n}_0$. Thus,

$$I = \frac{\pi e^2\omega_0^2 n^2}{2c}\exp(-\gamma t')\int \mathrm{d}z\,\mathrm{d}z'\,\mathrm{d}^2r_\perp\,(\boldsymbol{\eta}_b\boldsymbol{\eta}_{b'})J_0(x_b)J_0(x_{b'})\,. \tag{8.83}$$

Here z is the component of $\mathbf{r}_b$ collinear with $\mathbf{n}_0$. In Eqn (8.83) we have noted the transverse character of the electromagnetic pumping wave: $\mathbf{n}_0\boldsymbol{\eta}_b = 0$. If the damping of the pulse over the length of the medium is small, in Eqn (8.83) we should set $\boldsymbol{\eta}_b = \boldsymbol{\eta} = \text{const}$, $\boldsymbol{\eta}_b\boldsymbol{\eta}_{b'} = \eta^2$.

Equation (8.83) can be rewritten as

$$I = \frac{\pi e^2\omega_0^2 n^2}{2c}\,\eta^2\exp(-\gamma t')\int \mathrm{d}^2r_\perp\,\Phi^2\,, \tag{8.84}$$

where $\Phi = \int_0^L \mathrm{d}z\,J_0(x)$, $x = 2\sqrt{\pi ncr_0t'z}$. Here $L \equiv L(\mathbf{n}_0, \mathbf{r}_\perp)$ is the thickness of the body at the point where it is pierced by the straight line passing through point $\mathbf{r}_\perp$ in the direction $\mathbf{n}_0$. Using formula $xJ_0(x) = \left[xJ_1(x)\right]'$, we calculate the integral Φ:

$$\Phi = \frac{1}{2\pi ncr_0t'}\,x_0J_1(x_0)\,,$$

where $x_0 = 2\sqrt{\pi ncr_0t'L}$. Using Eqn (6.5), we reduce the expression (8.84) to the form

$$I = \frac{E_0}{t'}\exp(-\gamma t')Q(t')\,, \tag{8.85}$$

where $Q(t') = (1/V)\int \mathrm{d}^2r_\perp\,LJ_1^2(x_0)$, V is the volume of the body, $E_0 = m\omega_0^2\eta^2N/2$.

From Eqn (8.85) it follows that in the case of (8.81) there is SR. Indeed, at $t \lesssim \tau_\mathrm{s}$ we have $I \sim I_0\varphi N$, and there is no retardation. At $t \gtrsim \tau_\mathrm{s}$, the completion of the main pulse is followed by a number of pulses whose amplitude fades as t^{-2}. Since $\gamma\tau_\mathrm{s} \sim 1/\varphi \ll 1$, the exponential in Eqn (8.85) may be set equal to unity.

Using the integral

$$\int_0^\infty \frac{\mathrm{d}x}{x}\,J_1^2(x) = \frac{1}{2}\,,$$

it is easy to show that the entire stored energy is emitted:

$$E_\mathrm{r} = \int_0^\infty I\,\mathrm{d}t = E_0 = \frac{1}{2}\,m\omega_0^2\eta^2N\,.$$

In the calculations we must take advantage of the formula $\int \mathrm{d}^2r_\perp\,L = V$. Thus, like in Section 7, the completely coherent state in this case is not shielded: the clouds that shield different dipoles kind of cancel out.

Both the quantum (Section 8.1) and classical (this section) models of SR are not perfect, and supplement one another. The quantum model disregards the nonresonant term [the first term in Eqn (4.43)], but includes the nonlinearity exhibited by two-level atoms. The linear CMS disregards the nonlinearity and the associated mechanism of phasing (as indicated if only by the absence of delay of the pulse of SR), but the nonresonant term is consistently taken into account. Now we may conclude that the main effect caused by this term is shielding. In the case of $l \ll \lambda$ (Section 7) the shielding is caused by the static dipole–dipole interaction of atoms, and by the lagging dipole – dipole interaction in the case of $l \gg \lambda$.

If at $t = 0$ there is no correlation between the atoms, then from Eqn (6.15) we find that the total intensity is

$$I \sim N I_b \,, \tag{8.86}$$

where I_b is the intensity of emission by one excited atom surrounded by nonexcited atoms. Each atom prepares its own coherent ensemble. In accordance with the linear CMS, there is dipole shielding. As opposed to the linear classical model, the system of two-level atoms displays phasing of the ensemble by the nonlinear mechanism. Then the intensity I_b is evaluated from Eqn (8.78), and Eqn (8.86) agrees with Eqn (8.41). The latter implies that the entire energy is released in the regime of SR. Currently it is not yet clear what is going on in reality. The experiments indicate that SR arises with noncoherent pumping; we believe, however, that a comprehensive picture can only be given by theory. It is possible that because of shielding it is only part of the energy that is released in the regime of SR. The answer to these questions can be supplied by numerical solution of the nonlinear CMS — the left-hand sides of Eqn (6.3) must then include terms nonlinear with respect to the amplitudes of dipoles. No such treatment has been performed so far.

The effect of shielding in the system of linear charged oscillators is rather self-evident, and it is strange that it was not discovered earlier. As a matter of fact, it is implied in Ref. [68] which is an elaboration of Ref. [69]. Namely, if we integrate Eqn (8.29) in Ref. [68] with respect to frequencies, we find that a medium of identical nuclei that contains one excited nucleus will release only a small fraction of the energy stored by this nucleus (the idealization with fixed nuclei was considered). This omission is possibly explained by the fact that Refs [68, 69] were concerned with finding the width of the emitted spectral line.

The results of Section 8.2 are easily extended to the case of Eqn (2.13) (and in general to any arbitrary relationship between L and $c\tau_s$). For this purpose, we substitute Eqn (6.9) into (6.3), drop the second derivatives, and keep the retarded time on the right-hand side. The resulting equations are solved using the Fourier transform, similar to Eqn (6.10). The absence of delay of the pulse of SR is a considerable drawback of the linear CMS (lagging is present in the nonlinear version of this model [43]). We do not reproduce here the corresponding calculation, which is rather interesting by itself. A more advanced model of SR in the case of (2.13) is discussed in the next section.

9. Oscillatory regime of superradiance

The oscillatory regime is realized in sufficiently extended bodies, $c\tau_s \leqslant L < cT_r$. This regime was discovered in the very first experiment on SR [24]. In a long specimen, the radiation from portions of the body passes through other portions and causes their coherent re-excitation. Secondary emission by these regions gives rise to oscillations of intensity. As a matter of fact, the effect of oscillations of intensity has already been discussed at the end of Section 8.2. Electromagnetic waves, passing through the portions that have already released their energy, excite them again. Then these oscillators emit again. The oscillatory mechanism allows the identification of another mechanism that acts simultaneously. Naturally, it also works in the more realistic case of two-level atoms. This mechanism consists essentially in the nonstationary shielding described by the Bessel function in the second term in Eqn (8.63). We see that the exciting atom b is surrounded by collapsing spherical layers of matter with the opposite phase of dipole moments. This phenomenon is caused by polaritons traveling in the body (see Section 10).

In the case of (2.13), the phenomenon of SR in the system of two-level atoms is most simply and adequately described in the framework of semiclassical theory (see, for example, Refs [1, 2, 5, 31, 33, 34]. In this theory, the two-level atoms are described by the one-particle density matrix ρ of the (2×2) format, and the self-consistent classical field is found from the Maxwell equations whose right-hand sides carry the polarization of the medium $\mathbf{P} = n\,\mathrm{Sp}(\rho\hat{\mathbf{d}})$, where $\hat{\mathbf{d}}$ is defined by Eqn (3.9). As done at the end of Section 8.2, let us calculate the SR that results from pumping with a short laser π pulse running along the axis of the rod of length L and radius R (the z axis is aligned with the axis of the rod). The polarization $\mathbf{P}$ and electric field $\mathbf{E}$ are directed along the y axis. Since the electric induction is $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$, from the Maxwell equation it follows that

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 P}{\partial t^2} \approx -4\pi k^2 P \,, \tag{9.1}$$

where $P = \mathrm{d}(\rho_{12} + \rho_{21})$, $\rho_{12} = \rho_{21}^*$. The density matrix obeys the Bloch equations — that is, the Liouville–Neumann equations with the relaxation terms that may be dropped on the strength of Eqn (2.14):

$$\mathrm{i}\hbar\,\frac{\partial\sigma}{\partial t} = 2\mathrm{d}E(\rho_{21} - \rho_{12}) \,, \qquad \mathrm{i}\hbar\,\frac{\partial\rho_{21}}{\partial t} = \mathrm{d}E\,\sigma + \hbar\omega_0\rho_{21} \,, \tag{9.2}$$

where $\sigma = \rho_{22} - \rho_{11}$, $\rho_{11} + \rho_{22} = 1$. In Eqn (9.1) we have dropped the terms that describe the propagation of field transversely to the rod (diffraction), and thus the field is regarded as homogeneous across the rod. From the preceding sections we know that this homogeneity is violated near the side walls of the rod to the depth of about $R/\sqrt{F}$ (the transverse dimension of the RCI), and therefore Eqn (9.1) holds under the condition (2.27).

Now we select the quickly varying terms

$$E = \frac{\mathrm{i}}{2}\left[E_0 \exp \mathrm{i}\varphi - E_0^* \exp(-\mathrm{i}\varphi)\right] \,, \qquad \rho_{21} = S \exp \mathrm{i}\varphi \,,$$

$\varphi = kz - \omega_0 t$, $k = \omega_0/c$. Dropping the second derivatives from E_0, the fast-oscillating terms (approximation of rotating wave), and making the substitution $\sigma = \cos\Theta$, $S = (1/2)\sin\Theta$, we get from Eqns (9.1) and (9.2)

$$\frac{\partial E_0}{\partial z} + \frac{1}{c}\frac{\partial E_0}{\partial t} = 2\pi n d k \sin\Theta \,, \qquad \frac{\partial\Theta}{\partial t} = \frac{1}{\hbar}\,dE_0 \,,$$

whence follows the equation for Bloch's angle Θ [15, 21]

$$\frac{\partial^2\Theta}{\partial t\,\partial z}+\frac{1}{c}\frac{\partial^2\Theta}{\partial t^2}=\frac{2}{L\tau_{\rm s}}\sin\Theta\,,$$

where $\tau_{\rm s}=\hbar/(\pi knLd^2)$ [cf. Eqn (2.20)]. In the variables $\tau=2t'/\tau_{\rm s}$, $x=z/L$, where $t'=t-z/c$ is the retarded time, this equation becomes simpler:

$$\frac{\partial^2\Theta}{\partial\tau\,\partial x}=\sin\Theta\,. \tag{9.3}$$

The atom in the ground state corresponds to $\sigma=-1$, that is, $\Theta=\pi$. The short pumping π-pulse travels along the path $\tau=0$ $(z=ct)$ and brings the atoms into the state $\sigma=1$, that is, $\Theta=0$. Spontaneous transitions, which are disregarded in the semiclassical theory, give rise to a small amount of initial phasing (see Section 8.1). In addition, at $t=0$ we have $E_0=0$. Accordingly, the boundary conditions for this problem are

$$\frac{\partial\Theta}{\partial\tau}=0\,,\qquad \Theta=\Theta_0\quad\text{at}\quad\tau=0\,, \tag{9.4}$$

where $\Theta_0\ll 1$.

Equation (9.3) with the boundary conditions (9.4) is satisfied by the self-similar solution

$$\Theta=\Theta(\eta)\,,\qquad \eta=2\sqrt{\tau x}=2\sqrt{\frac{2t'z}{\tau_{\rm s}L}}\,, \tag{9.5}$$

where $\Theta(\eta)$ is found from the equation

$$\Theta''(\eta)+\frac{1}{\eta}\,\Theta'(\eta)-\sin\Theta=0\,. \tag{9.6}$$

Multiplying this by Θ', we represent it in a form admitting a simple mechanical interpretation:

$$\frac{{\rm d}E}{{\rm d}\eta}=-\frac{1}{\eta}(\Theta')^2\,,\qquad E=\frac{1}{2}(\Theta')^2+U(\Theta)\,. \tag{9.7}$$

This equation describes the one-dimensional motion of a particle of mass $m=1$ in the potential field (Fig. 11) $U(\Theta)=\cos\Theta-1=-2\sin^2(\Theta/2)$ in the presence of friction $-\Theta'/\eta$. According to Eqn (9.7), because of dissipation the particle sinks to the bottom of the potential pit $(\Theta=\pi)$.

The intensity of SR is

$$I=\pi R^2C\left\langle\frac{E^2}{4\pi}\bigg|_{z=L}\right\rangle \tag{9.8}$$

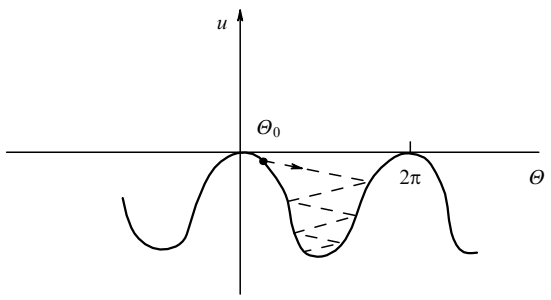

Figure 11. Mechanical analogy of superradiance in oscillatory regime.

and is thus proportional to the expression

$$I\sim\left(\frac{\partial\Theta}{\partial t}\right)_{Z=L}\sim\left[\frac{\Theta'(\beta)}{\beta}\right]^2, \tag{9.9}$$

where

$$\beta=2\sqrt{\frac{2z(t-L/c)}{L\tau_{\rm s}}}\,.$$

From equation (9.9) and Fig. 11 we see that the time dependence of intensity agrees with the experiment (see Fig. 5), and appears as a sequence of pulses of decreasing amplitude. The first and the strongest pulse is delayed from the pumping pulse by the lagging time $t_0\sim\Lambda_0^2\tau_{\rm s}$, $\Lambda_0=\ln(1/\Theta_0)$. The characteristic number of pulses which release the major part of the stored energy is Λ_0; the characteristic length of each is $\Lambda_0\tau_{\rm s}$.

From equation (9.8) it is easy to see that $I/I_0\sim N^2/k^2R^2$. The fact that the intensity is proportional to the square of the number of particles indicates that SR occurs over the entire range of (2.14).

The results of this section agree with the CMS (Section 8.2) at $\Lambda_0\sim 1$. Then there is no lagging in both cases — and hence no nonlinear phasing, and the number of oscillations is decreased. It is likely that this case was realized in the experiment of Ref. [24] with elevated density.

10. Superradiance in a dense extended medium

For completeness of our analysis, we now ought to consider SR in the case of (2.42c), which is interesting by itself since it may be realized on polaritons in a solid [1, 7, 10], or in magnetized plasma (see below). Now the dipoles are close to one another, and therefore, as explained in the beginning of Section 8.2, the most adequate description of SR is given by the CMS (currently, like in Ref. [46], in the linear version).

At $l\ll\lambda\ll L$, the physical parameters vary little over the interatomic distance, so the approximation of a continuous medium may be used: $\mathbf{r}_a\to\mathbf{r}$, $\xi_a(t)\to\xi(\mathbf{r},t)$; $\mathbf{E}(\mathbf{r}_a,t)\to\mathbf{E}(\mathbf{r},t)$, etc. Magnetization of the medium is one of the effects small in the first order in v_a/c that have been disregarded from the outset. For this reason we assume that the medium is nonmagnetic, and set $\mathbf{B}=\mathbf{H}$.

The Maxwell equations together with the equation of motion form a closed set in the CMS in the case of (2.42c):

$$\nabla\cdot\mathbf{D}=0\,; \tag{10.1}$$

$$\mathbf{D}=\mathbf{E}+4\pi en\xi\,; \tag{10.2}$$

$$\nabla\cdot\mathbf{H}=0\,; \tag{10.3}$$

$$\nabla\times\mathbf{E}=-\frac{1}{c}\,\dot{\mathbf{H}}\,; \tag{10.4}$$

$$\nabla\times\mathbf{H}=\frac{1}{c}\,\dot{\mathbf{D}}\,; \tag{10.5}$$

$$\ddot{\xi}+\omega_0^2\xi=\frac{e}{m}\,\mathbf{E}\,, \tag{10.6}$$

where the dot denotes the partial time derivative $\partial/\partial t$.

Let V be a fixed volume within the medium. Using Eqns (10.1) – (10.6) to transform the derivative

$$\frac{{\rm d}\varepsilon_a}{{\rm d}t}=\int_V {\rm d}^3r\,n\dot{\xi}\,m(\ddot{\xi}+\omega_0^2\xi)\,,$$

where

$$\varepsilon_a = \int \mathrm{d}^3 r\, n \left(\frac{m\dot{\xi}^2}{2} + \frac{m\omega_0^2 \xi^2}{2} \right)$$

is the energy of atoms in the CMS, we get the equation of energy balance:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon_a + \varepsilon_\mathrm{f}) = -\int_\Gamma \mathrm{d}\mathbf{f}\mathbf{S}\,, \tag{10.7}$$

where Γ is the boundary of the volume V, $\mathrm{d}\mathbf{f}$ is the external element of the surface of this boundary, and

$$\varepsilon_\mathrm{f} = \int_V \mathrm{d}^3 r\, W\,, \qquad W = \frac{E^2 + H^2}{8\pi}\,, \tag{10.8}$$

$$\mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{H})$$

is the density of energy flux (the Poynting vector). The effects of spontaneous emission and collisions in this section are considered negligible, and are not included in Eqns (10.1)–(10.6), since we are concerned with the applications of the CMS to a rarefied medium like a tokamak plasma. Shortly we shall prove that these effects are actually negligible.

The medium in our approach is a dielectric. Substituting into Eqns (10.1)–(10.6) ξ, $\mathbf{E}$, $\mathbf{D}$ as $\xi(\mathbf{r}, t) = \xi(\mathbf{r}) \exp(-\mathrm{i}\omega t)$; $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}) \exp(-\mathrm{i}\omega t)$; $\mathbf{D}(\mathbf{r}, t) = \mathbf{D}(\mathbf{r}) \exp(-\mathrm{i}\omega t)$; $\mathbf{D}(\mathbf{r}) = \varepsilon(\omega)\mathbf{E}(\mathbf{r})$, we find the permittivity at frequency ω:

$$\varepsilon(\omega) = \frac{\omega_0^2 + \omega_\mathrm{p}^2 - \omega^2}{\omega_0^2 - \omega^2}\,. \tag{10.9}$$

The physical essence of SR in the case of (2.42c) is as follows. Assume that at $t = 0$ there is a certain excitation in the medium, with the initial condition

$$t = 0\,, \qquad \dot{\xi} = \mathbf{f}(\mathbf{r})\,, \qquad \xi = 0\,, \qquad \mathbf{E} = 0\,, \qquad \mathbf{H} = 0\,. \tag{10.10}$$

This initial condition obviously corresponds to the case when at $t = 0$ the masses that had been initially at rest are accelerated to the velocity $\dot{\xi}_a = \mathbf{f}(\mathbf{r}_a)$. If this excitation is localized somewhere inside the medium, then collective waves start in the medium, and the excitation propagates. On the periphery of the medium these waves convert into electromagnetic waves and leave the medium.

Let us find the form of such collective waves in an infinite homogeneous ($n = \mathrm{const}$) medium. With this purpose, we seek a solution of Eqns (10.1)–(10.6) in the form $\xi(\mathbf{r}, t) = \xi \exp(\mathrm{i}\mathbf{pr} - \mathrm{i}\omega t)$; $\mathbf{E}(\mathbf{r}, t) = \mathbf{E} \exp(\mathrm{i}\mathbf{pr} - \mathrm{i}\omega t)$; etc., where ξ, $\mathbf{E}$ are constant amplitudes. We get

$$\xi = \frac{e}{m(\omega_0^2 - \omega^2)}\,\mathbf{E}\,; \tag{10.11}$$

$$\mathbf{p} \times \mathbf{E} = \frac{\omega}{c}\,\mathbf{H}\,; \tag{10.12}$$

$$\mathbf{p} \times \mathbf{H} = -\frac{\omega}{c}\,\varepsilon(\omega)\mathbf{E}\,.$$

We align the z axis with $\mathbf{p}$. Then from Eqn (10.12) we get

$$\begin{cases} -pE_y = \dfrac{\omega}{c}\,H_x\,, \\ pH_x = -\dfrac{\omega}{c}\,\varepsilon E_y\,; \end{cases} \tag{10.13}$$

$$\begin{cases} pE_x = \dfrac{\omega}{c}\,H_y\,, \\ pH_y = \dfrac{\omega}{c}\,\varepsilon E_x\,; \end{cases} \tag{10.14}$$

$$\begin{cases} \varepsilon(\omega)E_z = 0\,, \\ H_z = 0\,. \end{cases} \tag{10.15}$$

Thus, the field $\mathbf{H}$ is transverse ($H_z = 0$). The field $\mathbf{E}$ can be either transverse ($E_z = 0$) or longitudinal ($E_x = E_y = 0$, $E_z \neq 0$). Transverse waves are described by the systems (10.14), (10.15), which correspond to two different polarizations. From a condition of compatibility for each of these systems follows the dispersion relation for transverse waves:

$$\mathbf{p}^2 = \frac{\omega^2}{c^2}\,\varepsilon(\omega)\,. \tag{10.16}$$

Hence, and from Eqn (10.9), it follows that there are two branches of transverse collective waves: the fast (or ordinary) wave

$$\omega_1^2 = B + \sqrt{A}\,, \tag{10.17}$$

and the slow (or extraordinary) wave

$$\omega_2^2 = B - \sqrt{A}\,, \tag{10.18}$$

where $B = (\omega_0^2 + \omega_\mathrm{p}^2 + p^2c^2)/2$, $A = B^2 - \omega_0^2 p^2 c^2$. The magnitude of the wave vector $\mathbf{p}$ varies over the interval $(0, +\infty)$. The value of ω_1 increases monotonically from $\omega = \omega_a = (\omega_0^2 + \omega_\mathrm{p}^2)^{1/2}$ to $\omega = +\infty$, and ω_2 also increases monotonically from $\omega = 0$ to $\omega = \omega_0$. The waves with frequencies $\omega_0 < \omega < \omega_a$ fade soon, since for them $\varepsilon < 0$. At $p \to 0$ we have $\omega_1 \to \omega_a$, $\omega_2 \approx pc\omega_0/\omega_a$. At $p \to \infty$ we have $\omega_1 \approx pc$, $\omega_2 \approx \omega_0$.

In this way, at high frequencies, when the displacements of oscillators are negligibly small ($\xi_a \approx 0$), the medium carries the ordinary electromagnetic wave ($\omega_1 \approx pc$). The second branch (ω_2) at $p \to \infty$ describes the practically independent vibrations of oscillators, slightly modified by the dipole–dipole interaction. Following, for example, Ref. [70], we shall refer to such transverse waves as polaritons. They are a kind of hybrid of mechanical and electromagnetic oscillations.

Longitudinal waves are the proper oscillations of the medium. At $E_z \neq 0$ from Eqn (10.15) we see that $\varepsilon(\omega) = 0$, $\mathbf{D} = 0$ — that is,

$$\omega = \omega_a\,. \tag{10.19}$$

In this case, according to Eqns (10.13), (10.14), we have $E_x = E_y = H_x = H_y = 0$, and according to Eqn (10.11),

$$\xi = -\frac{1}{4\pi ne}\,\mathbf{E}\,, \qquad \mathbf{H} = 0\,. \tag{10.20}$$

Longitudinal waves do not travel in plasma, since, according to Eqn (10.19), their group velocity is zero: $\mathbf{v}_\mathrm{g} = \partial\omega/\partial\mathbf{p} = 0$. In the limit $\omega_0 \to 0$, corresponding to a completely ionized ideal plasma, these waves become conventional plasma oscillations.

Let us now solve the following problem. Assume that in an infinite homogeneous medium at $t = 0$ excitation (10.10) is created for which $\varepsilon_{\rm f} = 0$, $\varepsilon_a = (1/2)\mathbf{n}m\int \mathrm{d}^3 r\, \mathbf{f}^2(\mathbf{r})$. In this way, the initial energy of the excitation is

$$E_0 = \varepsilon_a + \varepsilon_{\rm f} = \frac{1}{2}\,\mathbf{n}m\int \mathrm{d}^3 r\, \mathbf{f}^2(\mathbf{r})\,. \tag{10.21}$$

Let us calculate the energy carried away by the collective waves:

$$\begin{aligned} E_{\rm r} &= \lim_{r\to\infty} r^2 \int \mathrm{d}\Omega_{\rm r} \int_{-\infty}^{\infty} \mathrm{d}t\, \mathbf{n}\mathbf{S}(t,\mathbf{r}) \\ &= \frac{c}{8\pi^2}\lim_{r\to\infty} r^2 \int \mathrm{d}\Omega_{\rm r} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathbf{n}(\mathbf{E}_{-\omega}\times\mathbf{H}_{\omega}) \\ &= \frac{c}{8\pi^2}\lim_{r\to\infty} r^2 \int \mathrm{d}\Omega_{\rm r} \int_{0}^{\infty} \mathrm{d}\omega\, \mathbf{n}\big[(\mathbf{E}^*_{\omega}\times\mathbf{H}_{\omega}) + (\mathbf{E}_{\omega}\times\mathbf{H}^*_{\omega})\big]\,, \end{aligned} \tag{10.22}$$

where $\mathbf{n} = \mathbf{r}/r$, and where we have used the properties $\mathbf{E}_{-\omega} = \mathbf{E}^*_{\omega}$, $\mathbf{H}_{-\omega} = \mathbf{H}^*_{\omega}$ that follow from the fact that $\mathbf{E}$, $\mathbf{H}$ are real. Since at $t < 0$ all physical variables are zero ($\boldsymbol{\xi} = 0$, $\dot{\boldsymbol{\xi}} = 0$, $\mathbf{E} = \mathbf{H} = 0$), Eqns (10.1)–(10.6) admit a Fourier transform (8.45), (8.47). At the same time, we carry out the Fourier transform with respect to the space variables of the form

$$\mathbf{f}(\mathbf{p}) = \int \mathrm{d}^3 r\, \exp(-\mathrm{i}\mathbf{p}\mathbf{r})\mathbf{f}(\mathbf{r})\,, \qquad \mathbf{f}(\mathbf{r}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\exp(\mathrm{i}\mathbf{p}\mathbf{r})\mathbf{f}(\mathbf{p})\,.$$

The equations for the Fourier components with the initial condition (10.10) are

$$\begin{cases} \mathbf{p}\times\mathbf{E}_{\omega p} = \dfrac{\omega}{c}\,\mathbf{H}_{\omega p}\,, \\ \mathbf{p}\times\mathbf{H}_{\omega p} = -\dfrac{\omega}{c}\,\varepsilon(\omega)\mathbf{E}_{\omega p} - \dfrac{4\pi n e\omega}{c^2(\omega_0^2-\omega^2)}\,\mathbf{f}(\mathbf{p})\,, \\ (\omega_0^2-\omega^2)\boldsymbol{\xi}_{\omega p} = \dfrac{e}{m}\,\mathbf{E}_{\omega p} + \mathbf{f}(\mathbf{p})\,. \end{cases}$$

Hence we find

$$\mathbf{H}_{\omega p} = \frac{4\pi n e\omega}{c(\omega^2-\omega_0^2)d}\big[\mathbf{p}\times\mathbf{f}(\mathbf{p})\big]\,,$$

where

$$d = \frac{\omega^2}{c^2}\,\varepsilon(\omega) - p^2\,.$$

At $r\to\infty$ the main contribution to the integral

$$\mathbf{H}_{\omega}(\mathbf{r}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\exp(\mathrm{i}\mathbf{p}\mathbf{r})\mathbf{H}_{\omega p}$$

comes from the singularity $1/d$ (when integration is carried out with respect to $\mathrm{d}p$), and from the directions $\mathbf{p}$ close to $\mathbf{n} = \mathbf{r}/r$ (when integration is carried out with respect to the directions $\mathbf{p}$). Because of this, in the nonsingular terms in the integrand we may set

$$\mathbf{p} = \mathbf{q} = q\mathbf{n}\,, \qquad q = \frac{\omega}{c}\,\sqrt{\varepsilon(\omega)}\,, \tag{10.23}$$

and factor them out of the integral. The resulting integral is calculated by the formula

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\,\frac{4\pi}{p^2-q^2-\mathrm{i}0}\,\exp\mathrm{i}\mathbf{p}\mathbf{r} = \frac{\exp\mathrm{i}qr}{r}\,.$$

The rule of circuiting the poles used here follows from the comment related to Eqns (8.45), (8.46) to the effect that ω is actually the complex variable $\omega + \mathrm{i}0$. The result is

$$\mathbf{H}_{\omega}(\mathbf{r}) = -\frac{ne\omega}{c(\omega^2-\omega_0^2)}\big[\mathbf{q}\times\mathbf{f}(\mathbf{q})\big]\theta\,\frac{\exp\mathrm{i}qr}{r}\,, \tag{10.24}$$

where the coefficient $\theta = \theta\big[(\omega-\omega_0)(\omega-\omega_a)\big]$ arises because at $\omega_0 < \omega < \omega_a$ we have $\varepsilon < 0$, so q is purely imaginary, and $\mathbf{H}(\mathbf{r})$ fades exponentially. Now we apply a Fourier transform of the form (8.45), (8.47) to Eqns (10.2), (10.5) and (10.6):

$$\boldsymbol{\nabla}\times\mathbf{H}_{\omega} = -\frac{\mathrm{i}\omega}{c}\,\varepsilon\mathbf{E}_{\omega}\,. \tag{10.25}$$

Hence, and from Eqn (10.24), we get

$$\mathbf{E}_{\omega} = \frac{ne}{(\omega^2-\omega_0^2)\varepsilon}\,\mathbf{q}\times\big[\mathbf{q}\times\mathbf{f}(\mathbf{q})\big]\,\frac{\exp\mathrm{i}qr}{r}\,\theta\,. \tag{10.26}$$

Then from Eqns (10.22), (10.24) we get

$$E_{\rm r} = \frac{n^2e^2}{4\pi^2c^3}\int \mathrm{d}\Omega_{\rm n}\int_0^{\infty}\mathrm{d}\omega\,\rho(\omega)(\mathbf{n}\times\mathbf{f})^2\theta = \frac{n^2e^2}{\pi c^3}(\tau_1+\tau_2)\,,$$

where

$$\rho(\omega) = \frac{\omega^4\sqrt{\varepsilon}}{(\omega^2-\omega_0^2)^2}\,; \qquad \tau_1 = \int_{\omega_a}^{\infty}\mathrm{d}\omega\,\rho(\omega)S(q)\,;$$

$$\tau_2 = \int_0^{\omega_0}\mathrm{d}\omega\,\rho(\omega)S(q)\,; \qquad S(q) = \big\langle\big[\mathbf{n}\times\mathbf{f}(\mathbf{q})\big]^2\big\rangle\,;$$

the angle brackets denote averaging with respect to directions $\mathbf{r}$. In each of the intervals $\tau_{1,2}$ we replace the integration variable ω with q, and in the transformations of τ_1 and τ_2 we use, respectively, branches ω_1 and ω_2 with the replacement $p\to q$ in Eqns (10.17), (10.18). In this way, τ_2 defines the energy carried away by the extraordinary waves, and τ_1 defines the energy carried away by the ordinary waves. Differentiating the relation (10.23), we get $\rho(\omega)\,\mathrm{d}\omega = c^3F(\omega)q^2\,\mathrm{d}q$, where

$$F(\omega) = \frac{\omega^2}{(\omega_a^2-\omega^2)(\omega_0^2-\omega^2)+\omega_{\rm p}^2\omega^2}\,.$$

Therefore,

$$E_{\rm r} = \frac{n^2e^2}{\pi}\int_0^{\infty}\mathrm{d}q\,q^2S(q)K(q)\,, \tag{10.27}$$

where $K(q) = F(\omega_1) + F(\omega_2)$. The factor $K(q)$ is a symmetrical algebraic expression with respect to the roots of biquadratic equation

$$\omega^4 - (\omega_a^2+q^2c^2)\omega^2 + q^2c^2\omega_0^2 = 0\,,$$

that follows from Eqn (10.23). We transform this expression using the Vièta theorem:

$$\omega_1^2+\omega_2^2 = \omega_a^2+q^2c^2\,, \qquad \omega_1^2\omega_2^2 = q^2c^2\omega_0^2\,.$$

The result is $K(q) = 1/\omega_{\rm p}^2$. Hence, and from Eqn (10.27), we get

$$E_{\rm r} = \frac{nm}{4\pi^2}\int_0^\infty {\rm d}q\, q^2 S(q)\,. \tag{10.28}$$

Formula (10.21) for the initial energy after transition to the Fourier components becomes

$$E_0 = \frac{nm}{4\pi^2}\int_0^\infty {\rm d}q\, q^2 M(q)\,, \tag{10.29}$$

where $M(q) = \langle f^2(q\mathbf{n})\rangle$.

Since $(\mathbf{n}\times\mathbf{f})^2 \leqslant f^2$, then from Eqns (10.28), (10.29) it follows that $E_{\rm r} \leqslant E_0$. If, for example, at $t = 0$ the electrons are pushed in a certain direction, and the arrangement of excited atoms is spherically symmetrical [$\mathbf{f}(\mathbf{r}) = \mathbf{k}f(r)$, $|\mathbf{k}| = 1$, $\mathbf{k} = \text{const}$], then $E_{\rm r} = 2E_0/3$. In this way, in the case of (2.42c) we also get non-emitting dipole states. The nature of these states becomes clear from the analysis of formula (10.28), according to which $E_{\rm r} = 0$ at $\mathbf{q}\times\mathbf{f}(\mathbf{q}) = 0$ — that is, at

$$\boldsymbol{\nabla}\times\mathbf{f}(\mathbf{r}) = 0\,. \tag{10.30}$$

From equations (10.4), (10.6) and (10.10) we get

$$\boldsymbol{\nabla}\times\mathbf{E}_\omega = \frac{{\rm i}\omega}{c}\,\mathbf{H}_\omega\,; \tag{10.31}$$

$$(\omega_0^2 - \omega^2)\boldsymbol{\xi}_\omega = \frac{e}{m}\,\mathbf{E}_\omega + \mathbf{f}(\mathbf{r})\,, \tag{10.32}$$

and from Eqns (10.30) and (10.32) follows

$$(\omega_0^2 - \omega^2)(\boldsymbol{\nabla}\times\boldsymbol{\xi}_\omega) = \frac{e}{m}(\boldsymbol{\nabla}\times\mathbf{E}_\omega)\,. \tag{10.33}$$

Now we take the rotor of Eqn (10.25) and use Eqn (10.31):

$$\left(\boldsymbol{\nabla}^2 + \frac{\omega^2}{c^2}\,\varepsilon\right)\mathbf{H}_\omega = 0\,. \tag{10.34}$$

Equations (10.31), (10.33), (10.34) form a closed system in three variables: $\boldsymbol{\nabla}\times\mathbf{E}_\omega$, $\mathbf{H}_\omega$, $\boldsymbol{\nabla}\times\boldsymbol{\xi}_\omega$. Since they are homogeneous [cf. Eqn (10.32)], and the medium is infinite, the solution is $\boldsymbol{\nabla}\times\mathbf{E}_\omega = 0$, $\mathbf{H}_\omega = 0$, $\boldsymbol{\nabla}\times\boldsymbol{\xi}_\omega = 0$. Therefore,

$$\boldsymbol{\nabla}\times\mathbf{D}_\omega = \boldsymbol{\nabla}\times(\mathbf{E}_\omega + 4\pi re\boldsymbol{\xi}_\omega) = 0\,.$$

Hence, and from Eqn (10.1) rewritten as $\boldsymbol{\nabla}\cdot\mathbf{D}_\omega = 0$, it follows that $\mathbf{D}_\omega = 0$, and therefore $\mathbf{D} = 0$, whence follows Eqn (10.20). From $\mathbf{H}_\omega = 0$ and Eqn (10.25) we conclude that $\varepsilon = 0$ — that is, the frequency ω is found from Eqn (10.19). Thus, the counterpart of the dipole nonemitting states in the case of (2.42c) are the longitudinal waves discussed above. Some energy of the excitation is tied up by these states for a long time. Eventually such coherent states are destroyed by atomic collisions, and their energy is either released or transferred into other degrees of freedom.

Let us now consider the counterpart of the fundamental solution of Eqn (5.12). In the approximation of a continuous medium we must make the replacement

$$\delta_{ab} \to n\delta^{(3)}(\mathbf{r}_a - \mathbf{r}_b)\,, \tag{10.35}$$

which agrees with the rule (2.35). As follows from Eqn (10.21), however, such a replacement implies that $E_0 = \infty$, which is physically absurd. The reason is clear: the initial condition of the form (10.35) disagrees with the criterion of applicability of the approximation of a continuous medium:

$$l \ll L_0\,, \qquad q_0 l \ll 1\,, \tag{10.36}$$

where L_0 is the characteristic length of the initial excitation, and $q_0 = 1/L_0$. This complication is actually not important, since in the case of (2.42c) there is no need to find the fundamental solution of Eqn (5.12). Since $\lambda \ll L$, the problem of evolution of any initial excitation of the medium can be solved by other techniques: numerical, Fourier transform, approximation of geometrical optics, etc.

In Section 12 we shall prove that all practically interesting cases can be easily analyzed by considering the analog of the fundamental solution for which

$$\frac{c}{\omega_{\rm p}} \ll L_0 \ll L\,, \tag{10.37}$$

and in all such cases

$$\omega_0 \gg \omega_{\rm p}\,, \tag{10.38}$$

that is, the medium is sufficiently rare. Under conditions (10.37), (10.38) we have

$$\omega_1 \approx qc\,, \qquad \omega_2 \approx \omega_0 - \frac{\omega_0\omega_{\rm p}^2}{2c^2q^2}\,; \tag{10.39}$$

$$\frac{\tau_1}{\tau_2} \sim \left(\frac{c}{L_0\omega_{\rm p}}\right)^2 \ll 1\,.$$

This means that such highly enough localized initial excitation is represented mainly by the slow extraordinary waves with small λ, for which the dispersion law (10.18) reduces to a simpler expression (10.39).

The characteristic rate of dissipation of such an initial excitation depends on the group velocity of such waves, which, according to Eqn (10.39), is

$$v_{\rm g} = \frac{{\rm d}\omega_2}{{\rm d}q} = \frac{\omega_0\omega_{\rm p}^2}{c^2q^3} \sim \frac{\omega_0\omega_{\rm p}^2 L_0^3}{c^2}\,. \tag{10.40}$$

Comparing the conclusions of this section with those of Section 8, we do not see any big difference between the physics of SR in cases (2.42b) and (2.42c): in both cases SR is promoted by collective waves. Only the speed of propagation of these waves is different: close to c in the case of (2.42b), and much lower in the case of (2.42c), because of the dipole–dipole interaction between atoms. Owing to this interaction, long-range order sets in the system of atoms, which is necessary for the propagation of coherent waves. The fluctuating short-range part of the dipole–dipole interaction is not important under condition (7.19), which holds for all cases of practical interest.

11. Damping of collective waves

In the preceding section we considered SR in the dense medium of Lorentz atoms represented as charged harmonic oscillators with the same eigenfrequency. Let us now study the effects related to the frequency spread of oscillators. This section is included mainly for methodological purposes, and ought to be viewed as an introduction to the study of SR in a magnetized plasma.

Consider a rarefied medium, in which the main mechanism of damping is Doppler broadening. Owing to this broadening, the eigenfrequency of oscillators in the laboratory frame, which we denote by Ω, will be not the same. The distribution function with respect to Ω we denote by $f(\Omega)$. The following relations hold:

$$\int_0^\infty \mathrm{d}\Omega\, f(\Omega) = 1\,, \qquad \int_0^\infty \mathrm{d}\Omega\, \Omega f(\Omega) = \omega_0\,.$$

Let $\Delta\Omega$ be the characteristic width of the distribution $f(\Omega)$. In the course of time, the phase difference between individual oscillators builds up, and coherence is destroyed over the characteristic time $\tau \sim 1/\Delta\Omega$ — this is the effect of fading of coherent waves. Naturally, the energy of excitation of the medium remains unchanged. In the absence of coherence, however, this energy will be released over a much longer period of spontaneous emission, when individual atoms emit independently of one another. Further on we shall disregard the small intensity of spontaneous radiation. With due account for broadening, the oscillator coordinate will now be a function of three variables: Ω, $\mathbf{r}$, and t: $\boldsymbol{\xi} = \boldsymbol{\xi}_\Omega(\mathbf{r}, t)$.

In place of Eqn (10.6) we now have

$$\ddot{\boldsymbol{\xi}}_\Omega + \Omega^2 \boldsymbol{\xi}_\Omega = \frac{e}{m}\,\mathbf{E}(\mathbf{r}, t)\,. \tag{11.1}$$

The vector of polarization of the medium is

$$\mathbf{P}(\mathbf{r}, t) = ne \int_0^\infty \mathrm{d}\Omega\, f(\Omega) \boldsymbol{\xi}_\Omega(\mathbf{r}, t)\,,$$

therefore, for $\boldsymbol{\xi}$ in Eqn (10.2) we must use

$$\boldsymbol{\xi}(\mathbf{r}, t) = \int_0^\infty \mathrm{d}\Omega\, f(\Omega) \boldsymbol{\xi}_\Omega(\mathbf{r}, t)\,. \tag{11.2}$$

Let us find the damping length for collective waves. With this purpose we seek the solution of Eqns (10.1)–(10.5), (11.1), (11.2) in the form $\boldsymbol{\xi}_\Omega(\mathbf{r}, t) = \boldsymbol{\xi}_\Omega \exp(-\mathrm{i}\omega t + \mathrm{i}pz)$, $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}\exp(-\mathrm{i}\omega t + \mathrm{i}pz)$, $\mathbf{H}(\mathbf{r}, t) = \mathbf{H}\exp(\mathrm{i}\omega t + \mathrm{i}pz)$. We get Eqns (10.13)–(10.15), in which

$$\varepsilon(\omega) = 1 + \omega_\mathrm{p}^2 \int_0^\infty \mathrm{d}\Omega\, f(\Omega) \frac{1}{\Omega^2 - (\omega + \mathrm{i}0)^2} = \varepsilon'(\omega) + \mathrm{i}\varepsilon''(\omega)\,;$$

$$\varepsilon'(\omega) = 1 + \omega_\mathrm{p}^2 \int_0^\infty \mathrm{d}\Omega\, f(\Omega) \frac{1}{\Omega^2 - \omega^2}\,; \tag{11.3}$$

$$\varepsilon''(\omega) = \pi\omega_\mathrm{p}^2 \operatorname{sign}\omega \int_0^\infty \mathrm{d}\Omega\, f(\Omega)\delta(\Omega^2 - \omega^2) = \frac{\pi\omega_\mathrm{p}^2}{2\omega} f\big(|\omega|\big)\,,$$

where we have used the Sokhotsky formula

$$\frac{1}{a \pm \mathrm{i}\delta} = p\left(\frac{1}{a}\right) \mp \pi\mathrm{i}\delta(a)\,.$$

In particular, when the broadening is Lorentzian,

$$f(\Omega) = \frac{\Delta\Omega}{2\pi} \frac{1}{(\Omega - \omega_0)^2 + (\Delta\Omega/2)^2}\,,$$

the permittivity is

$$\varepsilon(\omega) = 1 - \frac{\omega_\mathrm{p}^2}{(\omega + \mathrm{i}\Delta\Omega/2)^2 - \omega_0^2} = \frac{\omega_a^2 - (\omega + \mathrm{i}\Delta\Omega/2)^2}{\omega_0^2 - (\omega + \mathrm{i}\Delta\Omega/2)^2}\,. \tag{11.4}$$

We see that the longitudinal oscillations for which $\varepsilon(\omega) = 0$ fade out as $\xi \sim \exp(-\Delta\Omega t/2)$.

To find the damping length for transverse waves, we represent p as $p = q + \mathrm{i}p''$. The amplitude of the waves decreases as $\xi \sim \exp(-p''z)$; therefore, the damping length is $L_\mathrm{d} = 1/(2p'')$. At $\omega \approx \omega_0$, from Eqn (11.4) follows

$$\varepsilon(\omega) = \frac{\omega_\mathrm{p}^2}{2\omega_0(\omega_0 - \omega - \mathrm{i}\Delta\Omega/2)}\,.$$

Hence, and from Eqn (10.16), assuming that the damping is small, we get

$$q = \frac{\sqrt{\omega_0}\,\omega_\mathrm{p}}{c\sqrt{2(\omega_0 - \omega)}}\,, \qquad L_\mathrm{d} = \frac{\omega_0\omega_\mathrm{p}^2}{c^2 q^3 \Delta\Omega} = \frac{v_\mathrm{g}}{\Delta\Omega}\,, \tag{11.5}$$

where v_g is defined in Eqn (10.40).

12. Application of the classical model to plasma physics

In this and subsequent sections we consider cyclotron waves in a plasma perhaps from a somewhat unconventional standpoint — namely, as a particular case (2.42c) of SR. This approach allows interesting conclusions to be made.

Consider a plasma cylinder placed in a longitudinal magnetic field B_0. The plasma parameters selected are typical for tokamaks:

$$n_\mathrm{e} = n_\mathrm{i} = n \sim 10^{14}\ \mathrm{cm}^{-3}\,, \qquad T \sim 10^4\ \mathrm{eV},$$
$$B_0 \sim 5 \times 10^4\ \mathrm{G}, \qquad a \sim 3\ \mathrm{m}, \tag{12.1}$$

where a is the radius of the plasma cylinder (like in the ITER Project [71]). Under conditions (12.1), the plasma is strongly magnetized [72, 73]:

$$\omega_H \tau_\mathrm{c} \gg 1\,, \tag{12.2}$$

where $\omega_H = \omega_B/\gamma$ is the cyclotron frequency of electrons, $\omega_B = eB_0/(mc) \sim 10^{12}\ \mathrm{s}^{-1}$, $\gamma = (1 - v^2/c^2)^{-1/2} \approx 1$ is the electron relativistic factor, and v is the electron velocity,

$$\tau_\mathrm{c} \sim 10^{-4}\ \mathrm{s} \tag{12.3}$$

is the characteristic time of electron – electron and electron – ion collisions. In this way, the electrons perform a mostly circular motion around the magnetic lines of force, and after about 10^8 revolutions experience collisions which sharply change their paths. Naturally, in such circumstances the role of collisions is minor, and the coherent effects must come to the fore.

The Larmor orbit of electrons is

$$r_H \sim \frac{v_\mathrm{T}}{\omega_H} \sim 3 \times 10^{-3}\ \mathrm{cm}, \tag{12.4}$$

where $v_\mathrm{T} \sim 3 \times 10^9\ \mathrm{cm\ c}^{-1}$ is the thermal velocity of electrons. Since $r_H \ll a$, the magnetic field on the small scale is practically uniform; therefore the electrons engaged in cyclotron motion are in resonance with one another. As a result of the dipole – dipole interaction, correlations between the orbits of different electrons are established: each electron slightly disturbs the cyclotron motion of other electrons, and this disturbance is in phase with the rotation of the selected electron. From this we conclude that cyclotron radiation

must be dominated by the collective coherent effects which may strongly reinforce this radiation. Let us evaluate the intensity of this radiation and the characteristic time of energy loss by the plasma τ_E. The problem must be formulated as follows. At $t = 0$ we 'turn off' all sources of energy in the plasma — that is, fusion and external heating. For the sake of simplicity we also turn off the exchange of energy between electrons and ions. Then the decrease of energy of electrons $\varepsilon \sim T$ by radiation will obey the law

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{1}{\tau_E}\, T\,. \tag{12.5}$$

Since the initial (at $t = 0$) phases of cyclotron motion of different electrons are independent, the total intensity I of radiation by the plasma must be calculated by formula (8.86), where I_b is the intensity of radiation by one electron with the initial energy $\varepsilon \sim T$ surrounded by other electrons with zero initial energy. Naturally, we refer to the energy stored in the oscillatory (cyclotron) degrees of freedom of electrons. It is these degrees of freedom that are in resonance with one another and display collective effects. Differently from previous sections, the oscillators in a magnetized plasma are two-dimensional: only the components d_x and d_y of the dipole moment vary with time according to a harmonic law (the z axis is aligned with $\mathbf{B}$). This means that at $t = 0$ for all electrons except one we must set $\varepsilon_\perp = (m/2)(\dot{x}^2 + \dot{y}^2) = 0$. At the same time, there are no restrictions on the motion along the z axis, which is not a resonant degree of freedom. We come to the conclusion that I_b must be calculated under the condition

$$T_\perp = 0\,, \qquad T_\parallel = T\,. \tag{12.6}$$

We shall soon see that condition (12.6) that follows from a relatively trivial relation (8.86) leads to rather nontrivial conclusions. Observe that for the model of three-dimensional isotropic oscillators considered in the previous sections the condition used for the calculation of I_b has a form simpler than (12.6): $T = 0$. In other respects, the nature of physical effects in magnetized plasma is the same as in the case of isotropic oscillators — owing to the dipole–dipole interaction between the cyclotron orbits of electrons, collective waves travel in the plasma, and again there are two types of wave: the fast and the slow cyclotron waves.

So, at $t = 0$ we have one electron with $\varepsilon_\perp \neq 0$, $\varepsilon_\perp \sim \varepsilon_\parallel \sim T$; for all other electrons we have $\varepsilon_\perp = 0$, $\varepsilon_\parallel \sim T$ — at $t = 0$ they only move along the z axis. Our selected electron revolves in circular orbit of radius r_H. Hence, the characteristic value of the wave vector q is:

$$q \sim \frac{1}{r_H}\,. \tag{12.7}$$

From formulas (10.40) and (12.4), where we now set $L_0 \sim r_H$, $\omega_0 = \omega_H$, we find

$$v_{\mathrm{g}} \sim 10^7 \text{ cm s}^{-1}\,. \tag{12.8}$$

First (within a time r_H/v_{g}), the selected electron transfers the energy to other electrons that occur at a distance of r_H from it by the Coulomb interaction — that is, within the Larmor orbit. This fast stage in the plasma is followed by excitation with energy T in a region of size r_H, which throws more light on Eqn (12.7).

Since

$$\frac{qc}{\omega_0} \equiv \frac{qc}{\omega_H} \sim \frac{c}{r_H\omega_H} \sim \frac{c}{v_{\mathrm{T}}} \gg 1\,, \tag{12.9}$$

this excitation is represented mainly by the slow cyclotron waves. Then the waves start to travel out at the velocity v_{g}, and the excitation 'dissolves'. The motion of electrons along the z axis gives rise to Doppler broadening, which leads to fast damping of the majority of such waves — that is, to the conversion of coherent oscillations into noncoherent ones, whose energy, as noted in Section 11, is released very slowly. There is, however, a small share ρ of undamped waves, which leave the plasma and carry away the energy.

The physical meaning of expression (11.3) for ε'' consists in that the δ-function in the integrand selects a group of electrons which are in resonance with the cyclotron wave of frequency ω, which absorb the energy of this wave. Consider the electrons that move along the line of force of magnetic field at the velocity v_z (as already said, $v_x = v_y = 0$). The wave in resonance with this electron in the electron's frame has frequency $\omega_0 = \omega_B$ and wave vector $\mathbf{q}'$, and in the laboratory frame Ω and $\mathbf{q}$. Since $(\Omega, \mathbf{q})$ is a four-vector, we have the relation $\omega_B = \gamma(\Omega - v_z q_z)$, where $\gamma = (1 - v_z^2/c^2)^{-1/2}$. Hence it follows that at nonrelativistic velocities the frequency of the wave in resonance with the electron in the laboratory frame is

$$\Omega = \frac{\omega_B}{\gamma} + q_z v_z \approx \omega_B - \frac{\omega_B}{2c^2}\, v_z^2 + q_z v_z\,. \tag{12.10}$$

Equating this frequency to the frequency $\omega = \omega(\mathbf{q})$ of the wave traveling in plasma, we get the condition of resonance — that is, the condition of absorption of energy:

$$\Omega = \omega(\mathbf{q})\,, \tag{12.11}$$

which we rewrite in the form

$$v_z^2 - 2\,\frac{c^2 q_z}{\omega_B}\, v_z + 2c^2\,\frac{\omega - \omega_B}{\omega_B} = 0\,.$$

Hence we find the velocities of resonant electrons

$$(v_z)_{1,2} = \frac{c}{\omega_B}\left(cq_z \pm \sqrt{D}\,\right), \qquad D = c^2 q_z^2 - 2\omega_B\left[\omega(\mathbf{q}) - \omega_B\right]. \tag{12.12}$$

Obviously, the condition of absence of absorption is

$$D < 0\,. \tag{12.13}$$

Let us show that there exist waves that satisfy this condition, and therefore are not absorbed. For such waves the vector $\mathbf{q}$ is real, as assumed in the derivation of Eqn (12.13). The wave vectors $\mathbf{q}$ of such waves are almost perpendicular to $\mathbf{B}$ — that is, $|q_z| \ll q$. Since in Eqn (12.12) both terms on the right-hand side are of the same order of magnitude, and q_z is small, we may set $q_z = 0$ in the second term — that is, in place of $\omega(\mathbf{q})$ we use the dispersion law $\omega(q)$ for the waves with $\mathbf{q} \perp \mathbf{B}$, which substantially simplifies the calculation. For the same reason in the dispersion law $\omega(\mathbf{q})$ in Eqn (12.12) we may set $T_\perp = T_\parallel = 0$. A more rigorous calculation can be found in Appendix III.

The set of equations describing a nonrelativistic plasma in a magnetic field $\mathbf{B}_0$ directed along the z axis is [cf. Eqns

(10.1) – (10.6)]:

$$m\ddot{\boldsymbol{\xi}}_\perp = \frac{e}{c}\,\dot{\boldsymbol{\xi}}_\perp \times \mathbf{B}_0 + e\mathbf{E}_\perp\,; \qquad m\ddot{\boldsymbol{\xi}}_z = eE_z\,; \tag{12.14}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\,\frac{\partial \mathbf{B}}{\partial t}\,; \qquad \nabla \times \mathbf{B} = \frac{4\pi}{c}\,ne\dot{\boldsymbol{\xi}} + \frac{1}{c}\,\frac{\partial \mathbf{E}}{\partial t} \equiv \frac{1}{c}\,\frac{\partial \mathbf{D}}{\partial t}\,,$$

where the subscript $\perp$ marks the components perpendicular to $\mathbf{B}_0$. Assuming that the dependence on $\mathbf{r}$ and t has the form $\exp(\mathrm{i}\mathbf{q}\mathbf{r} - \mathrm{i}\omega t)$, for the amplitude of the wave from Eqn (12.14) we get the system

$$-\omega^2\boldsymbol{\xi}_\perp = -\mathrm{i}\omega_B\omega\boldsymbol{\xi}_\perp \times \mathbf{K} + \frac{e}{m}\,\mathbf{E}_\perp\,; \qquad -\omega^2\xi_z = \frac{e}{m}\,E_z\,; \tag{12.15}$$

$$\mathbf{q} \times \mathbf{E} = \frac{\omega}{c}\,\mathbf{B}\,; \qquad \mathbf{q} \times \mathbf{B} = -\frac{\omega}{c}\,\mathbf{D}\,,$$

where $\mathbf{B}$, $\mathbf{E}$ are the fields in the cyclotron wave, and $\mathbf{K} = \mathbf{B}_0/B_0$. Hence, and from Eqn (10.2), follow the equations

$$D_\alpha = \varepsilon_{\alpha\beta}E_\beta\,; \tag{12.16}$$

$$\mathbf{q}(\mathbf{q}\cdot\mathbf{E}) - q^2\mathbf{E} + \frac{\omega^2}{c^2}\,\mathbf{D} = 0\,, \tag{12.17}$$

where

$$\varepsilon_{xx} = \varepsilon_{yy} \equiv \varepsilon = \frac{\omega_B^2 + \omega_\mathrm{p}^2 - \omega^2}{\omega_B^2 - \omega^2}\,;$$

$$\varepsilon_{xy} = -\varepsilon_{yx} \equiv -\mathrm{i}f\,, \qquad f = \frac{\omega_B\,\omega_\mathrm{p}^2}{\omega(\omega^2 - \omega_B^2)}\,;$$

$$\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zx} = \varepsilon_{zy} = 0\,, \qquad \varepsilon_{zz} \equiv \eta = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\,.$$

Setting the determinant of Eqn (12.17) equal to zero, and making use of Eqn (12.16), we get the dispersion equation for finding the functions $\omega = \omega(\mathbf{q})$ for collective waves [74]:

$$\det\left[\frac{\omega^2}{c^2}\,\varepsilon_{\alpha\beta}(\omega) + q_\alpha q_\beta - q^2\delta_{\alpha\beta}\right] = 0\,. \tag{12.18}$$

At $\mathbf{q} \perp \mathbf{B}_0$, from Eqn (12.18) follows the equation $F_1(\omega)F_2(\omega) = 0$, where

$$F_1 = \frac{\omega^2}{c^2}\,\eta - q^2\,, \qquad F_2 = \varepsilon(\varepsilon - N^2) - f^2\,,$$

$N = qc/\omega$ is the index of refraction. We align the x axis with $\mathbf{q}$. By virtue of Eqn (12.15), $B_x = 0$. From equation $F_1 = 0$ follows the relation $\omega^2 = q^2c^2 + \omega_\mathrm{p}^2$. The latter defines the wave for which $E_x = E_y = B_z = 0$, $\xi_x = \xi_y = 0$. Equation $F_2 = 0$ is easily reduced to

$$\omega^4 - (\omega_B^2 + 2\omega_\mathrm{p}^2 + q^2c^2)\omega^2 + \omega_\mathrm{p}^4 + q^2c^2(\omega_B^2 + \omega_\mathrm{p}^2) = 0\,. \tag{12.19}$$

This describes the fast and the slow cyclotron waves, for which $E_z = 0$, $\xi_z = 0$, $B_x = B_y = 0$. For the fast wave $\omega^2 = A + \sqrt{D} \equiv \omega_\mathrm{f}^2$, where

$$A = \omega_\mathrm{p}^2 + \frac{\omega_B^2 + q^2c^2}{2}\,, \qquad D = \frac{1}{4}(q^2c^2 - \omega_B^2)^2 + \omega_\mathrm{p}^2\omega_B^2\,.$$

For the slow wave $\omega^2 = A - \sqrt{D} \equiv \omega_\mathrm{s}^2$.

At $q^2c^2 - \omega_B^2 \gg \omega_\mathrm{p}\omega_B$, we have

$$\omega_\mathrm{f}^2 \approx \omega_\mathrm{p}^2 + q^2c^2 + \frac{\omega_\mathrm{p}^2\omega_B^2}{q^2c^2 - \omega_B^2}\,; \qquad \omega_\mathrm{s}^2 \approx \omega_B^2 + \omega_\mathrm{p}^2 - \frac{\omega_\mathrm{p}^2\omega_B^2}{q^2c^2 - \omega_B^2}\,.$$

In particular, in the most practically interesting case (12.9), we get

$$\omega_\mathrm{s}^2 \approx \omega_B^2 + \omega_\mathrm{p}^2 - \frac{\omega_\mathrm{p}^2\omega_B^2}{q^2c^2}\,.$$

Since the condition (10.38) is satisfied for parameters (12.1), we find that

$$\omega_\mathrm{s} \approx \omega_B + \frac{\omega_\mathrm{p}^2}{2\omega_B} - \frac{\omega_\mathrm{p}^2\omega_B}{2q^2c^2}\,. \tag{12.20}$$

The comparison between Eqns (10.40) and (12.20) reconfirms our earlier estimate (12.8).

When substituting Eqn (12.20) into (12.12), (12.13), we may discard the last term in Eqn (12.20). The condition of nonabsorption (12.13) becomes

$$|q_z| < \frac{\omega_\mathrm{p}}{c}\,. \tag{12.21}$$

We see that those waves for which the angles between wave vectors $\mathbf{q}$ and the plane perpendicular to the magnetic field $\mathbf{B}_0$ are limited to

$$|\alpha| < \alpha_\mathrm{c}\,, \tag{12.22}$$

are not absorbed; here

$$\alpha_\mathrm{c} = \frac{\omega_\mathrm{p}}{qc} \sim \frac{\omega_\mathrm{p} r_H}{c} \sim \frac{\omega_\mathrm{p} v_\mathrm{T}}{\omega_B c} \sim 0{,}04\,. \tag{12.23}$$

In degrees, $\alpha_\mathrm{c} \approx 3^\circ$, and the full angle of the transparency window (12.22) is $2\alpha_\mathrm{c} \approx 6^\circ$. On a unit sphere the region (12.22) forms an annular region (a ring) of area $4\pi\alpha_\mathrm{c}$, therefore, after the initial excitation dissolves [see our comment after Eqn (12.8)], all these collective waves are absorbed with the exception of a small share

$$\rho = \frac{4\pi\alpha_\mathrm{c}}{4\pi} = \alpha_\mathrm{c}\,. \tag{12.24}$$

Let us now analyze the resulting picture. The equation of energy balance (12.5) has the form

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -I_b\,, \tag{12.25}$$

where I_b is the intensity of radiation by one electron excited at $t = 0$. The undamped waves carry an energy ρT, and after a time $\tau_1 \sim a/v_\mathrm{g} \sim 10^{-5}$ s reach the periphery of the plasma and leave the system as electromagnetic waves (see Appendix IV). Since $\tau_\mathrm{c} \gg \tau_1$ [see Eqn (12.3)], the coherence is not destroyed during this motion. After the time τ_c collisions occur, and the system returns to *status quo* — the same as at $t = 0$ but with entirely different phases of oscillators. Then each electron releases the energy ρT from the plasma, and so on. Thus,

$$I_b \sim \frac{\rho T}{\tau_\mathrm{c}}\,. \tag{12.26}$$

From comparison of Eqns (12.25), (12.26) with (12.5) we get the estimate $\tau_E \sim \tau_\mathrm{c}/\rho \sim 3 \times 10^{-3}$ s.

Such will be the lifetime of a plasma not surrounded by a metallic container. The cyclotron waves in the millimeter range emitted by electrons are effectively reflected by the first wall of the tokamak; however, the coherence is lost after the very first reflection because the large phases are practically completely random. The energy of these waves is transferred to the noncoherent excitation of the plasma, which, as indicated above, is practically not emitted at all. The share of energy absorbed after the first reflection is $R \sim \sqrt{\omega/\sigma} \approx \sqrt{\omega_B/\sigma} \sim 3 \times 10^{-3}$, where $\sigma \sim 10^{17}\ \mathrm{s}^{-1}$ is the conductivity of the wall of the tokamak. This means that the intensity (12.26) must be multiplied by R. Finally, we get $\tau_E \sim \tau_c/(\rho R) \sim 1$ s, which agrees with the experimental time measured with TFTR and JET [71]. This agreement, however, should not be taken as proof of the theory developed in this section, since we have left several effects out of the consideration. They will be discussed in sections to follow.

To end this section, let us briefly discuss formulas (12.10), (12.13). Equation (12.10) includes the relativistic correction v_z^2/c^2, which turns out to be important in the range of small angles α. This means that the absence of absorption of waves at small α is a relativistic effect. There may be some doubt concerning the fact that the inclusion of terms quadratic with respect to velocity may violate the linearity of the CMS that was an important assumption in our reasoning. The linearity, however, is not violated, because the oscillatory resonant variable is $\xi_\perp$, with respect to which the equations of CMS remain linear, whereas the terms included are quadratic with respect to v_z (see Appendix III).

13. The diffusion regime of transfer of coherent cyclotron radiation in an inhomogeneous plasma

In the previous section we considered a highly idealized case of a homogeneous plasma. Assume that λ is the characteristic length of inhomogeneity in a magnetized plasma (for example, a magnetic islet [75]). Since $r_H \ll \lambda$, the main effect identified in Section 12 remains: the coherent cyclotron radiation is transported in a plasma across the lines of force; differently from Section 12, however, this time it cannot travel further than a distance λ. In this way, regions of size λ (hereinafter referred to as cells) exchange energy, and the entire process appears as a diffusion of energy towards the periphery of the plasma. This does not mean that the plasma will radiate mainly at the cyclotron frequency: in the course of diffusion, the ordered collective energy may be converted into the energy of chaotic thermal motion of plasma particles, into conventional bremsstrahlung, etc. These issues call for thorough investigation, and the ideas presented in this section must be regarded as just a hypothesis that the diffusion of coherent cyclotron oscillations is the main mechanism of energy transfer in a tokamak.

The density of energy flux is evaluated from the relation $q\lambda^2 \sim I_b n \lambda^3$, where $I_b \sim T\rho/\tau_c$ is the intensity of radiation per electron (see Section 12). Accordingly,

$$q \sim \frac{nT\lambda}{\tau_c}\,\rho\,.$$

The total large-scale energy flow is

$$q_D \sim \frac{nT\lambda}{\tau_c}\,\Delta\rho\,,$$

where $\Delta\rho$ is the difference in ρ for adjacent cells; $\Delta\rho \sim (\lambda/a)\rho_0$, ρ_0 is the value of ρ in the midst of plasma [see Eqns (12.23), (12.24)]. Thus,

$$q_D \sim \frac{nT\lambda^2\rho_0}{\tau_c a}\,.$$

The lifetime of energy τ_E is estimated from the equation of energy balance for unit length of plasma cylinder:

$$\frac{nTa^2}{\tau_E} \sim q_D a\,.$$

Hence

$$\tau_E \sim \frac{\tau_c}{\rho_0}\left(\frac{a}{\lambda}\right)^2.$$

We see that, compared to τ_E for a homogeneous plasma (see Section 12), the lifetime of energy in inhomogeneous plasma is $(a/\lambda)^2$ times longer. Substituting the typical parameters of a tokamak, we find (in seconds) $\tau_E \sim 3 \times 10^{-3}(a/\lambda)^2$. This result agrees with the experiment when $a/\lambda \sim 10$.

14. Conclusions

Superradiance is defined as a phenomenon that involves two mechanisms: the development of correlations between the initially independent excited atoms (phasing) and the subsequent reinforced collective emission by a system of atoms as a result of phasing. From this definition we see that SR is quite a universal phenomenon. If we consider a system of inverted-population atoms, it is an essentially quantum effect related to laser generation. It had been regarded as such starting with Dicke's seminal work [8], in which the correlation between atoms arose as an elegant and simple consequence of the symmetry of Dicke's Hamiltonian and the complete wave function. If we speak of classical systems — for example, electrons revolving in magnetic field, clusters of electrons in wigglers, cathode ray tubes for microwaves, etc. — then the corresponding radiation is also SR, this time represented by slowly propagating waves since the energy is shared between the particles and the field.

It would be impossible to discuss all the aspects of SR in one review, which, if we adopt the above definition, were applied in practice long before the publication of Ref. [8]. We have concentrated on a narrower task, which consists in sorting out the main theoretical approaches, identifying their strengths and weaknesses, presenting the theory at different levels of complexity, improving where possible, and resolving or trying to resolve the obscure points.

We started with the general terms discussion of the main publications on SR, and the analysis of Dicke's theory from different angles. Then we considered in detail the nonlinear mechanism of phasing and its role in SR. For elucidating the role of the dipole – dipole interaction we introduced the CMS, which turned out quite useful. Such interaction in a small-size specimen ($L \ll \lambda$) was found to result in shielding which prevents radiation from escaping from the sample. Shielding creates localized excitation which has a zero dipole moment and therefore does not emit. Subsequently, this coherent excitation is destroyed by atomic collisions, and the excess energy slowly leaves the specimen as noncoherent radiation. We go on to demonstrate that in an ellipsoidal body the dipole – dipole interaction is kind of canceled out, which does not happen in specimens of other shapes. This interaction

forms the basis of another (dipole) mechanism of phasing, concurrent and competing with the nonlinear mechanism. It is found that the conclusions of some authors concerning the dipole dephasing of atoms and suppression of SR are incorrect, which may stimulate experiments on SR in small objects.

In Section 8 we reviewed the results on SR in extended bodies ($\lambda \ll L \ll c\tau_s$), obtained a new solution for the known equations of SR, and discovered the effect of nonlinear narrowing of the angular divergence of the beam of SR. We used the linear CMS to account for the retarded dipole–dipole interaction, which again leads to shielding. A nonlinear CMS is proposed which may prove the most adequate model of SR. For the case of a rod pumped with a π-pulse, we discuss the oscillatory regime of SR observed in experiments. In Sections 10–13 we consider SR in a dense medium ($l \ll \lambda$), where it takes the form of collective waves that travel out to the periphery and escape as electromagnetic waves. A model of diffusion transfer of cyclotron waves is proposed which may explain the effect of anomalous heat conduction in a magnetized plasma [76–79].

In conclusion, let us touch upon some interesting problems for the future. Further development of the theory of SR requires numerical simulation of SR based on the nonlinear CMS which simultaneously includes the nonlinear effects and the dipole–dipole interaction. This must be done for both $L \lesssim \lambda$ and $L \gtrsim \lambda$. Of particular interest is the calculation of SR for a spherical body including terms of the order of $1/c^2$ in the Lagrangian (see Section 7). One could think of staging an experiment on SR in small objects based on electron paramagnetic resonance (see Section 7). The hypothesis of cyclotron energy transfer in magnetized plasma is well worth further consideration. Another interesting problem is related to astrophysics, where the high radiative luminance of pulsars could be attributed to the collective effects of SR [79, 80]. The development of the theory of SR will also advance our knowledge of nonequilibrium phase transitions. To be certain, there are many fundamental problems awaiting their solution in the theory of SR and its applications.

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15. Appendices

I. Direct calculation of $\tau_{\alpha\beta}$

The main contribution to the tensor $\tau_{\alpha\beta}(\mathbf{r}_b)$, as defined by Eqn (7.9), comes from the regions far from atom b. Indeed, let us select a point on the surface of the medium closest to b, and draw a tangential sphere centered at point b. Then the sum (7.9) will split into two parts corresponding to atoms a inside this sphere and outside of it. As explained in Section 7, the medium may be considered isotropic under condition (7.19), which is satisfied in all cases of practical interest. By virtue of spherical symmetry, the first term is zero, since the quantities $G_{\alpha\beta}(\mathbf{r}_{ab})$ are composed of spherical functions corresponding to the moment $l = 2$. Thus, only the second term is left, whence follows the above statement.

Since the main contribution to $\tau_{\alpha\beta}$ comes from the regions of the medium whose size is of the same order as that of the medium itself, the sum over a in Eqn (7.9) may be replaced with integration [see Eqn (2.35)], which means going over to the continuum approximation. Consequently,

$$\tau_{\alpha\beta}(\mathbf{r}_a) \approx n \int \mathrm{d}^3 r_b \, G_{\alpha\beta}(\mathbf{r}_{ab}) \,. \tag{I.1}$$

According to Eqn (7.3), $b \neq a$. This means that at $\mathbf{r} = 0$ there is no singularity, and we must define $G_{\alpha\beta}(\mathbf{r}_{ab})$ at $a = b$: $G_{\alpha\beta}(0) = 0$. Hence we conclude that in the continuum limit

$$G_{\alpha\beta}(\mathbf{r}_{ab}) = (\nabla_a)_\alpha (\nabla_a)_\beta \left(\frac{1}{r_{ab}}\right) + \frac{4\pi}{3}\, \delta_{\alpha\beta}\delta(\mathbf{r}_{ab}) \,. \tag{I.2}$$

The singular term in Eqn (I.2) cancels out the corresponding term that arises in the calculation of the first term, and therefore there is no singularity in Eqn (I.2) at $\mathbf{r}_{ab} = 0$, as ought to be expected. From Eqns (I.1) and (I.2) we find

$$\tau_{\alpha\beta}(\mathbf{r}) = n\nabla_\alpha \nabla_\beta \varphi(\mathbf{r}) + \frac{4\pi}{3}\, n\delta_{\alpha\beta} \,; \qquad \varphi(\mathbf{r}) = \int_v \mathrm{d}^3 r' \, \frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

from where it is easy to express Eqn (7.17)

II. Proof of formula (7.30)

Consider an infinite medium ($N \to \infty$). Similarly to the quantity $\tau_{\alpha\beta}$ (see Appendix I), the main contribution to the sum over b in Eqn (7.30) comes from the range of variation of $\mathbf{r}_b$ with the characteristic size of the order of L, and therefore the summation may be replaced with integration. Of course, the terms of order of $1/N$ will be lost, which in an infinite medium are zero. Using Eqn (I.2), we get

$$\begin{aligned}
Q_{\alpha\gamma} &\equiv \sum_b G_{\alpha\beta}(\mathbf{r}_{ab}) G_{\beta\gamma}(\mathbf{r}_{bc}) \\
&= n \int \mathrm{d}^3 r_b \left[(\nabla_a)_\alpha (\nabla_a)_\beta \left(\frac{1}{r_{ab}}\right) + \frac{4\pi}{3}\, \delta_{\alpha\beta}\, \delta(\mathbf{r}_{ab}) \right] \\
&\times \left[(\nabla_b)_\beta (\nabla_b)_\gamma \left(\frac{1}{r_{bc}}\right) + \frac{4\pi}{3}\, \delta_{\beta\gamma}\, \delta(\mathbf{r}_{bc}) \right] \\
&= n \int \mathrm{d}^3 r_b \, (\nabla_a)_\alpha (\nabla_a)_\beta \left(\frac{1}{r_{ab}}\right) (\nabla_b)_\beta (\nabla_b)_\gamma \left(\frac{1}{r_{bc}}\right) \\
&+ \frac{4\pi n}{3} (\nabla_a)_\alpha (\nabla_a)_\gamma \left(\frac{1}{r_{ac}}\right) + \frac{4\pi n}{3}\, G_{\alpha\gamma}(\mathbf{r}_{ac}) \,.
\end{aligned}$$

The first term in this expression reduces to

$$n(\nabla_a)_\alpha (\nabla_c)_\gamma \int \mathrm{d}^3 r_b \, (\nabla_b)_\beta \left[\frac{1}{r_{ab}} (\nabla_b)_\beta \frac{1}{r_{bc}} \right] - n(\nabla_a)_\alpha (\nabla_c)_\gamma \int \mathrm{d}^3 r_b \, \frac{1}{r_{ab}} \, \nabla_b^2 \left(\frac{1}{r_{bc}}\right)$$

and the first term here reduces to an integral over the surface surrounding the medium. Since the medium is infinite, this integral is zero (the terms of order $1/N$ are lost again). Using the relation $\nabla^2(1/r) = -4\pi\delta(\mathbf{r})$, after some straightforward algebra we get Eqn (7.30). In doing this we use the replacement rule that holds for the transition to the continuum limit $\delta_{ab} \to n\delta(\mathbf{r}_{ab})$.

III. Waves in a magnetized plasma with $T_\perp = 0$, $T_\parallel = T$

Let us demonstrate that waves moving nearly normal to the magnetic field in a plasma with $T_\perp = 0$ do not fade.

Following Sections 10–12, we use the approximation of a continuous medium. At $T_\parallel \neq 0$ we take care of the motion of

electrons along the lines of force by replacing vector $\xi_\perp(\mathbf{r}, t)$ in Eqn (12.14) by $\xi_v(\mathbf{r}, t)$, where $v \equiv v_z$, $\xi_v \perp \mathbf{B}$. In addition, $\dot{\xi}_\perp$ must be replaced with

$$\left(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z}\right)\xi_v .$$

We also take into account the weak relativism with respect to motion along the z axis, by including the relativistic factor γ in the left-hand side of the first equation in (12.14):

$$\gamma = \left(1 - \frac{v_z^2}{c^2}\right)^{-1/2} \approx 1 + \frac{v_z^2}{2c^2} .$$

Then in place of Eqn (12.14) we get

$$m\gamma\left(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z}\right)^2 \xi_v = \frac{e}{c}\left(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z}\right)\xi_v \times \mathbf{B}_0 + e\mathbf{E} ; \quad \text{(III.1)}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} , \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} ; \qquad \mathbf{D} = \mathbf{E} + 4\pi r e \xi ;$$

$$\xi = \int_{-\infty}^{\infty} \mathrm{d}v_z \, \xi_v f(v_z) , \quad \text{(III.2)}$$

where $f(v_z)$ is the distribution function of electrons with respect to v_z, $\int_{-\infty}^{\infty} \mathrm{d}v_z \, f(v_z) = 1$.

Now our task is to calculate the permittivity of the plasma and the damping length of the waves in the plasma. Similarly to Eqn (12.15), we get

$$-(\omega - q_z v_z)\xi_v \approx -\frac{\mathrm{i}\omega_B}{\gamma}(\xi_v \times \mathbf{K}) + \frac{e}{m\omega_B}\mathbf{E} ,$$

which we express component-wise

$$\xi_{vx} \approx -\frac{e}{m\Delta_0}(E_x + \mathrm{i}E_y) ; \qquad \xi_{vy} \approx \frac{\mathrm{i}e}{m\Delta_0}(E_x + \mathrm{i}E_y) ;$$

$$\xi_{vz} \approx -\frac{e}{m\omega^2} E_z ,$$

where

$$\Delta_0 = (\omega - q_z v_z)^2 - \frac{\omega_B^2}{\gamma^2} \approx 2\omega_B(\omega - \omega_B) - 2q_z v_z + \frac{\omega_B^2 v_z^2}{c^2} .$$

Hence, and from Eqns (III.1), (III.2) for the nonzero components we get [compare with the components in Eqn (12.16) corresponding to $T_\perp = T_\parallel = 0$]:

$$\varepsilon_{xx} = \varepsilon_{yy} = 1 - \omega_\mathrm{p}^2 A \equiv \varepsilon ;$$

$$\varepsilon_{xy} = -\varepsilon_{yx} = -\mathrm{i}f , \qquad f = \omega_\mathrm{p}^2 A ; \qquad \varepsilon_{zz} = \eta ,$$

where $A = \int_{-\infty}^{\infty} \mathrm{d}v_z \, f(v_z)/\Delta_0$. When calculating A, we must make the conventional replacement $\omega \to \omega + \mathrm{i}\delta$, $\delta \to +0$ [73, 74], which gives us the damping

$$A = A' + \mathrm{i}A'' ; \qquad A' \approx \frac{1}{\omega^2 - \omega_B^2} ;$$

$$A'' = -\pi \int_{-\infty}^{\infty} \mathrm{d}v_z \, f(v_z)\delta\left[(\omega - q_z v_z)^2 - \frac{\omega_B^2}{\gamma^2}\right]$$

$$\approx -\frac{\pi}{2\omega_B}\int_{-\infty}^{\infty} \mathrm{d}v_z \, f(v_z)\delta\left(\omega - q_z v_z - \frac{\omega_B}{\gamma}\right)$$

$$\approx -\frac{\pi}{2\omega_B}\int_{-\infty}^{\infty} \mathrm{d}v_z \, f(v_z)\delta\left(\omega - \omega_B - q_z v_z + \frac{\omega_B v_z^2}{2c^2}\right) . \quad \text{(III.3)}$$

Since we are interested in the waves that travel almost transversely to the magnetic field $\mathbf{B}_0$, in the dispersion relation (12.18) everywhere except in A we may set $q_z = 0$, which considerably simplifies the problem. From Eqn (12.18) we get

$$(1 - \omega_\mathrm{p}^2 A)\left(1 - \omega_\mathrm{p}^2 A - \frac{q^2 c^2}{\omega^2}\right) - A^2 = 0 . \quad \text{(III.4)}$$

The absence of damping for the waves from the intervals (12.21), (12.22) follows from Eqns (III.3), (III.4). The damping length $L_\mathrm{d} = 1/(2q'')$ is found by substituting $q = q' + \mathrm{i}q''$ into Eqn (III.4). We see the condition of the total absence of damping; therefore we assume that $|A''| \ll |A'|$, $q'' \ll q'$. Further on we denote q' by q again.

Setting the real part of Eqn (III.4) equal to zero, we get the dispersion relation (12.20), and the damping is found by equating the imaginary part to zero:

$$q'' = \frac{q^3 \omega_\mathrm{p}^2 c^2}{4\omega_B^4}\,\delta ;$$

$$\delta = \pi \int_{-\infty}^{\infty} \mathrm{d}v_z \, f(v_z)\delta\left(\frac{v_z^2}{2c^2} - \frac{q_z}{\omega_B} v_z + \frac{\omega_\mathrm{p}^2}{2\omega_B^2}\right).$$

In this calculation, in Eqn (III.3) in accordance with Eqn (12.20) we have made the replacement

$$\omega - \omega_B = \frac{\omega_\mathrm{p}^2}{2\omega_B} - \frac{\omega_\mathrm{p}^2 \omega_B}{2q^2 c^2} \approx \frac{\omega_\mathrm{p}^2}{2\omega_B} .$$

IV. Passage of a slow cyclotron wave across the plasma boundary

Let us demonstrate that the wave passes across the plasma boundary practically without reflection. For this purpose we assume that the boundary is flat, and the wave falls at right angles to this plane: $\mathbf{q} = (q, 0, 0)$. The boundary is the plane $x = 0$.

According to Section 12, the solution of Eqn (12.14) is sought in the form

$$\mathbf{E} = \big(E_x(x), E_y(x), 0\big)\exp(-\mathrm{i}\omega t) ;$$

$$\mathbf{B} = \big(0, 0, B_z(x)\big)\exp(-\mathrm{i}\omega t) ;$$

$$\xi = \big(\xi_x(x), \xi_y(x), 0\big)\exp(-\mathrm{i}\omega t) .$$

Hence, and from Eqn (12.14), we get Eqn (12.16), and also equations

$$E_x = \frac{\mathrm{i}f}{\varepsilon} E_y ; \qquad B_z = -\frac{\mathrm{i}c}{\omega} E_y'(x) ;$$

$$E_y''(x) + q^2(x)E_y(x) = 0 ;$$

$$q^2(x) = \frac{\omega^4 - (\omega_B^2 + 2\omega_\mathrm{p}^2)\omega^2 + \omega_\mathrm{p}^4}{c^2(\omega^2 - \omega_B^2 - \omega_\mathrm{p}^2)} .$$

At $\omega_\mathrm{p} \ll \omega_B$ we have

$$q^2(x) \approx \frac{\omega^2(\omega_B^2 + 2\omega_\mathrm{p}^2 - \omega^2)}{c^2(\omega_B^2 + \omega_\mathrm{p}^2 - \omega^2)} .$$

Inside and outside of the plasma the quantity $q(x)$ in accordance with Eqn (12.7) is q_i and $q = q_\mathrm{e} = \omega/c \approx \omega_B/c$, respectively. According to Ref. [72], the width of the plasma

boundary is

$$d \sim \frac{c}{\omega_{\rm p}} \,. \qquad \text{(IV.1)}$$

If the phase gain on the boundary is large, the movement of the wave is classical, and therefore the reflection is small. Since

$$q_i d \sim \frac{c}{v_{\rm T}} \frac{\omega_B}{\omega_{\rm p}} \sim 10 \,, \qquad q_{\rm e} d \sim \frac{\omega_B}{\omega_{\rm p}} \sim 2 \,,$$

the motion of the wave across the boundary is indeed quasiclassical. Observe that the actual value of d is much greater than the estimate of Eqn (IV.1), because the density of plasma decreases gradually to zero from the middle to the periphery [78]. Accordingly, in reality we have $d \sim a$, $qa \gg 1$, and the reflection of waves may be disregarded.

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