 ©1999 Uspekhi Fizicheskikh Nauk, Russian Academy of Sciences

TO THE 275th ANNIVERSARY OF THE RUSSIAN ACADEMY OF SCIENCES

PACS numbers: **01.70.+w**,

Mathematics and physics: mother and daughter or sisters?

V I Arnol'd

Contents

__Abstract.__ **The unity of mathematics and physics is emphasized in a paper previously presented, in part, in 1998 at a meeting of the French Mathematical Society, and at the Russian Academy of Sciences (seminar at the Institute of Theoretical and Experimental Physics and V L Ginzburg's seminar at the Physics Institute).**

I loved and continue to love mathematics for the sake of itself as something that does not tolerate hypocrisy and vagueness which are abominable for me.
Stendhal [1]

Let us put $\alpha = 0$ though it does not make any sense and is not quite correct from the viewpoint of quantum mechanics.
E Schrödinger "Statistical Thermodynamics"

1. Introduction

The statement that mathematics is the part of theoretical physics in which experiments are very cheap [1] immediately invited numerous attacks from both sides, including even one parody (written by A M Vershik) [2].

Permit me to start from terminology. The word 'mathematics' is foreign almost for all languages — it is an ancient Greek loan-word meaning 'exact knowledge'. Among modern countries only Netherlands seem to substitute foreign word 'mathematics' with native 'knowledge' (wiskunde). Probably, it happened due to Stevin who protested upon the whole against cluttering up native language with international terms. For everybody who speaks Russian 'triangle' is more clear than 'rhombus'. Making our children suffer from alien 'files' or 'bites', we automatically create a medium of 'bucks' and 'killers' followed by preconditions of technological backwardness which, probably, could be paid with the fate of Yugoslavia.

Stevin's striking success in creating science and culture in the Netherlands tells till now — this country sharply stands out though it is not very large. Not only Amsterdam which so caught the fancy of Peter the Great, but Utrecht, Leiden, Saardam and Groningen are among the greatest mathematical centers of Europe.

Originally mathematics was created for solving real practical problems (in the case of the Netherlands — first of all hydraulic and hydrotechnical problems; there were plans to flood the entire country under the threat of fascist invasion, but it seems that the technology let it down).

I'll try to describe as many applications as I can of the fundamental mathematics itself not bounding myself with technical details. At first sight mathematics seems to be a set of handicrafts. But I shall try to show that here the question is always of the same art — the art of mathematical description of the world.

The mathematical description of the world is based on a subtle interaction between continuous (smooth) and discrete (jump-like) phenomena. For example, a function, say, $y = x^2$, has the derivative $\mathrm{d}y/\mathrm{d}x = 2x$, describing the rate of smooth change of y under smooth change of x, and a critical point ($x = 0$), where the function takes its minimal value.

In modern mathematics a vast *theory of singularities* is created which generalizes the theory of critical points of a function to the case when several functions of several arguments are considered simultaneously. For the reader who wants to feel the flavor of this science immediately it is

[1] In: L I Volpert *Pushkin and Stendhal* Ch.12 (Moscow, 1998).

[2] **Example of a mathematical problem.** There are two volumes of Pushkin on the bookshelf. Each volume is 2 cm thick without the cover; each cover is 2 mm thick. A worm gnawed through the books from the first page of the first volume to the last page of the second volume normally to the pages. Trough what distance did it gnaw?

Example of a physical problem. A boatman went against the current along the Neva and lost his hat under the Kirovskiĭ bridge. Under the Liteĭnyĭ bridge he realized and 30 minutes later caught up with it under the Dvortsovyĭ bridge. Find the speed of the current of the Neva. (Physics enters the ancient solution: the distance from the Kirovskiĭ bridge to the Dvortsovyĭ bridge is 1 mile.)

V I Arnol'd V A Steklov Mathematical Institute, Russian Academy of Sciences, ul. Gubkina 6, 117966 Moscow, Russian Federation
Tel. (7-095) 938 39 80. Fax (7-095) 135 05 55
E-mail: arnold@genezis.mian.ras.ru

Received 14 September 1999
Uspekhi Fizicheskikh Nauk **169** (12) 1311 – 1323 (1999)
Translated by Yu A Danilov; edited by A Yaremchuk

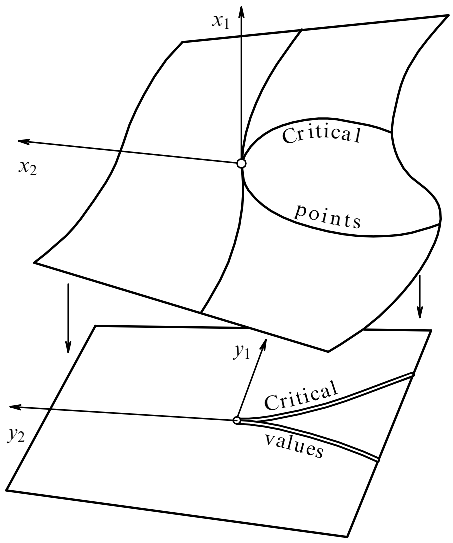


Figure 1. Whitney mapping.

instructive to compare the mapping

$$y_1 = x_1^3 + x_1 x_2\,, \qquad y_2 = x_2 \tag{1}$$

of the plane with coordinates (x_1, x_2) onto the plane with coordinates (y_1, y_2), studied by American mathematician H Whitney [2], with the mapping

$$y_1 = x_1^2 - x_2^2\,, \qquad y_2 = 2x_1 x_2 \tag{2}$$

(Fig. 1).

A surprising phenomenon here is the following one: the mapping (1) is stable (or *structurally stable*) in the sense that any enough close mapping has (in the neighborhood of the origin) singularities similar to those of the mapping (1).

For the mapping (2) it is not true as one can see from the example of mapping

$$y_1 = x_1^2 - x_2^2 + a x_1\,, \qquad y_2 = 2x_1 x_2 - a x_2\,, \tag{3}$$

where a is very small (however in the example under consideration it is not necessary).

For a function of one variable usually the point of maximum or minimum (say, $x = 0$ for the function $y = x^2$) is structurally stable. Under small deformation one obtains another function for which $x = 0$ is not the point of minimum. But there is always a point of minimum (for example, $y = x^2 + ax$, where a is small) in the vicinity of the origin.

On the contrary, the critical point $x = 0$ of the function $y = x^3$ is not structurally stable (for example in case of $y = x^3 + ax$ the critical point divides into two [3] or disappears depending on the sign of a) (Fig. 2).

The study of the case $y = x^4$ enables our readers to make progress in the matter for themselves (there is no disappearance, but bifurcations are possible).

It is also instructive to find the critical points x and critical values y for the mapping (3). In this case the equation

$$\begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\[2mm] \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{vmatrix} = 0$$

[3] This phenomenon of division is called *bifurcation* of the critical point.

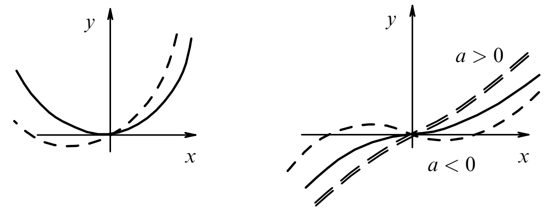


Figure 2. Structurally stable and unstable critical points.

of critical points is as follows:

$$\begin{vmatrix} 2x_1 + a & -2x_2 \\ 2x_2 & 2x_1 - a \end{vmatrix} = 0\,, \qquad 4(x_1^2 + x_2^2) = a^2\,.$$

Consequently, the critical points form a circle $2x_1 = a\cos\varphi$, $2x_2 = a\sin\varphi$. For the critical values, we obtain from Eqn (3)

$$y_1 = \frac{1}{4}\,a^2\cos 2\varphi + \frac{1}{2}\,a^2\cos\varphi\,,$$

$$y_2 = \frac{1}{4}\,a^2\sin 2\varphi - \frac{1}{2}\,a^2\sin\varphi\,.$$

Thus, vector y is a sum of two uniformly rotating vectors, the twice lesser vector rotating twice more rapidly. From this fact one can easily derive the conclusion that the set of critical values of the mapping (3) is a small (for small a) hypocycloid with three cusps (Fig. 3).

It is worth mentioning that for $a = 0$ this mapping can be considered as an analytic complex function $w = z^2$ $(z = x_1 + \mathrm{i}x_2,\ w = y_1 + \mathrm{i}y_2)$ with a simple critical point. If the perturbations do not break the complex-analytical nature of the mapping, then the character of the branching will remain unchanged: there will be a single critical point and it will be stable in respect to complex stirring.

If real stirrings are admissible [for example, such as (3)], then, as we know, the structural stability will be lost and around a given critical point a critical curve arises whose image under the mapping has three cusps.

It turns out that this phenomenon is already stable — under other stirrings of the general situation the same division of a complex critical point (2) into three cusps of the line of critical values which links them occurs.

The theory of singularities, in which all these facts are proved, unifies the most abstract parts of mathematics (number theory, Lie groups and algebras theory, the theory of groups of Coxeter reflections, algebraic and symplectic geometry and topology, calculus of variations and complex

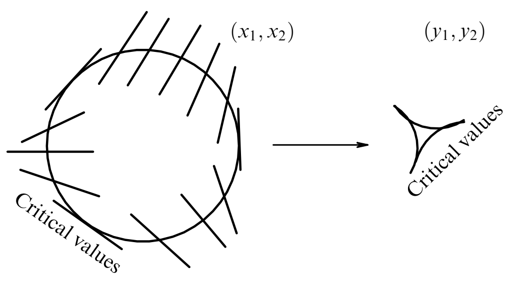


Figure 3. Critical points and critical values of the mapping (3).

analysis) with such applied domains as tomography, optimal control, stationary phase and saddle-point asymptotic methods, wave propagation, optics, classical, celestial and quantum mechanics, quantum field theory and so on.

A detailed exposition of all this can be found in books (see, for example, Refs [3–6]).

2. Unifying power of mathematics

Here I shall show by examples how the same object [it can be considered to be the line of critical values for the mapping (1)] arises in quite different problems of the applications of mathematics.

Wave fronts

Let us consider a wave starting to propagate from a curve on a plane with velocity equal to 1. After time t the wave front becomes an equidistant of the initial curve. Let us focus our attention, for example, on the case when the initial curve is an ellipse, and the perturbation propagates inside the ellipse (Fig. 4). After some experimenting, you can convince yourself that though for small t the t-equidistant inside the ellipse is smooth, for larger values of t semi-cubic type cusps can arise [in the neighborhood of such a point the curve is given by equation $p^2 = q^3$ in a corresponding system of smooth coordinates (p, q) on a plane].

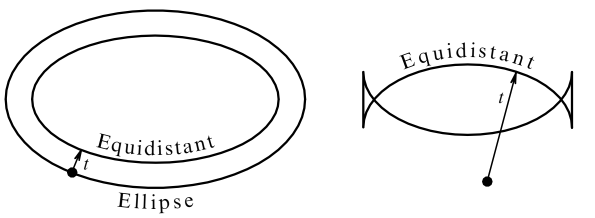


Figure 4. Propagation of the wave front inside ellipse.

Explanation. Let us consider a *cylinder C*, which is the product of the initial curve and the time axis t. The wave propagation is described by a pair of mappings

$$\mathbf{R} \xleftarrow{T} C \xrightarrow{F} \mathbf{R}^2,$$

the first of which maps the point (c, t) of the cylinder into the instant t and the second one into the point of the plane which is located at distance t from c in the direction of the normal to the initial curve at the point c.

The curves $T^{-1}(t)$ are isochrones on the cylinder. The mapping F maps isochrones onto equidistants. The singularities of the mapping F turn smooth isochrones into equidistants with cusps.

To make certain of this, one should consider the case $F(p, q) = (p^2, q)$, $T(p, q) = q - p^3$. Then the isochrone $T = 1$ turns into the planar curve $(p^2,\ 1 + p^3)$ with the semi-cubic singularity $p = 0$.

One can check up that for general F and T almost the same folding of a smooth isochrone into an equidistant with a cusp occurs.

The semi-cubic singularities on equidistants are not the distinctive property of an ellipse. Substituting another curve for the ellipse, we obtain similar singularities of the wave front, as a rule, semi-cubic ones. But the number of singularities is not necessary equal to four.

Caustics

Continuing the discussion of wave propagation from a curve, let us consider a system of *rays* on a plane. The ray coming out from the point c of initial curve is an image of the whole straight line $\{c, t\}$, where c is fixed and t is arbitrary, under the mapping $F : C \to \mathbf{R}^2$ of our cylinder into the plane. It consists of those points of the equidistants of the initial curve which are located on the normals to the latter at the point c (of course, in the simplest case of an Euclidean plane with metric independent of time, all these normals just coincide; but if one employs the little more cautious terminology introduced above, then our definition of rays remains valid in a far more general situation, for example, in the relativistic or Finslerian cases or in the control theory).

Draw rays perpendicular to an ellipse (directing them inside). You will discover the envelope of this system of straight lines (Fig. 5). This is called a *caustic* (from the word 'burning', because the energy of the propagating process concentrates on it). If you draw accurately, them you will notice that the caustic has four cusps (where it 'burns' more intensively).

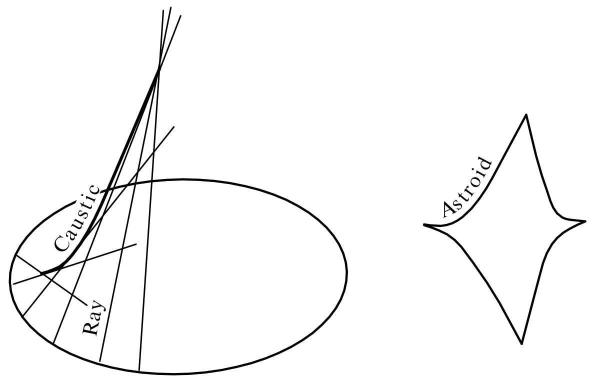


Figure 5. System of rays normal to an ellipse.

The exact mathematical definition is as follows: a *caustic is a set of critical values* of the mapping $F : C \to \mathbf{R}^2$ (evidently, it is possible to consider more general cases of propagation of perturbation from the source X to the screen Y given by the mapping $F : C \times \mathbf{R} \to Y$ for arbitrary dimensions of the manifolds X and Y; the caustic is located in Y and usually has a dimension which is less by one than that of Y).

Explanation. For an ellipse the mapping $F : S^1 \times \mathbf{R} \to \mathbf{R}^2$ has four singularities of type (1). There are four semi-cubic type cusps corresponding to them, connected with the star-like curve — the image of the 'circle' of critical point under the mapping F.

Remark. The calculations (given below) show that a *caustic is an envelope of the system of normals to ellipse — an astroid* with four cusps.

An astroid can be defined as follows:

(**1**) *an astroid is an image of a circle* $|z| = 1$ *under the mapping* $z \mapsto z^3 + 3\bar{z}$ (*i.e. an astroid is a hypocycloid with four cusps*);

(**2**) *an astroid is a curve projectively dual to an 'anticircle', i.e. the curve given by the equation*

$$\frac{1}{u^2} + \frac{1}{v^2} = 1\,;$$

(**3**) *an astroid is a curve, given in appropriate affine coordinates by equation*

$$x^{2/3} + y^{2/3} = 1 .$$

Simple calculations can serve as a check-up of the fact that all these calculations lead (up to projective or affine transformation) to one curve. For example, for $z = \exp(\mathrm{i}\varphi)$ the point $z^3 + 3\bar{z}$ runs along a hypocycloid with four cusps because its motion is the rolling of a circle of radius 1 inside a fixed circle of radius 4. This motion is rolling because the angular speed of the smaller circle is three times more than that of the point of contact and for this reason there is no slip.

When z is multiplied by i, $z^3 + 3\bar{z}$ is multiplied by $-\mathrm{i}$. As a result, the image has symmetry of order 4. It can be easily calculated that a semicubic cusp on the curve $z^3 + 3\bar{z}$, $z = \exp(\mathrm{i}\varphi)$, corresponds to the point $z = 1$ $(\varphi = 0)$:

$$\mathrm{Re}(z^3 + 3\bar{z}) = \cos 3\varphi + 3\cos\varphi = 4 - 6\varphi^2 + O(\varphi^4) ,$$
$$\mathrm{Im}(z^3 + 3\bar{z}) = \sin 3\varphi - 3\sin\varphi = -4\varphi^3 + O(\varphi^5) .$$

Therefore, the singularity is semicubic. It is worth mentioning the following formulas [stemming from the Moivre formula $\cos 3\varphi + \mathrm{i}\sin 3\varphi = (\cos\varphi + \mathrm{i}\sin\varphi)^3$]:

$$\cos 3\varphi + 3\cos\varphi = 4\cos^3\varphi ,$$
$$\sin 3\varphi - 3\sin\varphi = -4\sin^3\varphi . \qquad (4)$$

It can be seen from these relations as well that the singularity is semicubic.

Property (3) of a hypocycloid with four cusps follows from the derived formulas. Really, if $z = p + \mathrm{i}q$, then $p^2 + q^2 = 1$ on the circle $z = \exp(\mathrm{i}\varphi)$, where $p = \cos\varphi$, $q = \sin\varphi$.

Denoting $z^3 + 3\bar{z}$ as $x + \mathrm{i}y$, we obtain from Eqn (4)

$$x = 4p^3 , \qquad y = -4q^3 ,$$

and the equation $p^2 + q^2 = 1$ of circle turns into the equation of an astroid (**3**). Differentiating (**3**), we get (**2**).

Let us show that the caustic of an ellipse in fact is an astroid (up to affine transformations).

According to the property (**2**) proved above this comes from the following fact.

Lemma. *A set of normals to an ellipse, considered as a curve in dual plane, is affinely equivalent to anticircle.*

Let us write ellipse equation as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 . \qquad (5)$$

The normal $px + qy = 1$ passes through the point (x_0, y_0) of the ellipse and is parallel there to a gradient of quadratic form (5):

$$px_0 + qy_0 = 1 , \quad q = \frac{\lambda x_0}{a^2} , \quad -p = \frac{\lambda y_0}{b^2} , \quad \frac{px_0}{a^2} + \frac{qy_0}{b^2} = 0 .$$

From these equations we find that

$$x_0 = \frac{A}{p} , \qquad y_0 = \frac{B}{q} ,$$

where

$$A = \frac{a^2}{a^2 - b^2} , \qquad B = -\frac{b^2}{a^2 - b^2} .$$

Now the condition that the point (x_0, y_0) belongs to the ellipse gives that (p, q) is on the standard curve antidual to the circle (p and q should be rescaled appropriately).

Remark 1. It seems to me this striking algebraic lemma can be linked with the known geometric property of an ellipse: the bases of normals to the tangents of the ellipse, dropped from the focal point, form a circle.

Remark 2. Although it was proved above that the caustic of an ellipse is an astroid, the focal points in the problem of wave propagation inside an ellipse (cusps of astroid) are not in general foci of the ellipse.

The propagation of waves inside ellipse can easily be observed if we let a drop fall near the center of a cup. After reflection from the walls the waves gather in the opposite point where even a splash can be observed.

The matter is that waves, coming from focal point of an ellipse, gather in another focal point. A cup with a point near the center can be considered as an ellipse with a focus because the influence of the small eccentricity e of the ellipse on the ratio of semiaxes is of second order of smallness $[b = a\sqrt{1 - e^2} = a(1 - e^2/2 + \ldots)]$, so the nonellipticity of a cup can be neglected.

The semicubic singularities on a caustic are not an intrinsic property of an ellipse. Changing an ellipse for another curve, we obtain similar singularities of caustic, as a rule, again semicubic. But the number of singularities is not necessary 4.

Remark. Aristpohanes in *Clouds* (circa 450 B.C.) mentions a practical (legal) application of a caustic [4], created by lenses, which could be bought in drug stores (according to the witness of Socrates). The application consisted in burning the document in the hands of opponents during a trial session. Two centuries later Archimedian caustics were coarser.

Groups of reflections

By *reflection* in Euclidean space one means an orthogonal transformation such that the set of fixed points is a hyperplane.

A set of some hyperplanes determines a set of reflections. A group generated by them is called *an Euclidean group of reflections*, if it is finite.

For example, two straight lines on the plane determine the group of reflections [denoted as $I_2(p)$], iff the angle between the straight lines is π/p, where p is an integer. This group $I_2(p)$ is group of symmetries of regular p-gon on the plane.

All groups of reflections in Euclidean spaces are enumerated, and this is one of the main achievements of modern mathematics. It is surprising, but the answer turned out to be connected with all kinds of other important mathematical objects, for example, with simple and closely related (complex and compact) Lie algebras like $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, $Spin(n)$, $Sp(n)$ and so on.

Let us consider the simplest case of the symmetry group $I_2(3) = A_2$ [corresponding to the Lie algebra $SU(2)$] of a regular triangle (Fig. 6).

So, on a real Euclidean plane let three straight lines pairwise intersect each other at the origin at angles 120°. From the abstract viewpoint it is simply a permutation group of three elements. (A good idea is to turn to a model of reflection group A_{n-1} each time you have to deal with permutations of n elements.)

[4] The author is very obliged to Mrs. F Aicardi who found this.

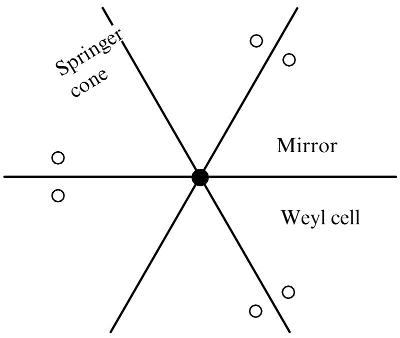


Figure 6. Mirrors of the reflection group A_2.

The orbits of reflection group $I_2(3)$ consist of either six elements (not laying on the mirrors) or of three elements (common points on the mirrors), and one orbit more is the origin.

The manifold of the orbits was studied by the founder of algebra F Viet. Let us consider the points of the Euclidean plane to be complex numbers. We can introduce in the space $\mathbf{C}^3$ with coordinates (z_1, z_2, z_3) reflections in the mirrors $z_i = z_j$ (which transpose coordinates z_i and z_j). As a result, we get the action of $I_2(3)$ on $\mathbf{C}^3$. The diagonal $z_1 = z_2 = z_3$ remains fixed. Therefore, the Hermitian-orthogonal plane $z_1 + z_2 + z_3 = 0$ remains fixed as well.

We extended the action of $I_2(3)$ from $\mathbf{R}^2$ to $\mathbf{C}^2$ and will use complex coordinates z_k.

Polynomials $\sigma_2 = z_1z_2 + z_2z_3 + z_3z_1$ and $\sigma_3 = z_1z_2z_3$ are invariant with respect to the action of the group as well as $\sigma_1 = 0$. It follows from the main theorem on symmetric functions that the manifold of the orbits of our action is the plane $\mathbf{C}^2$ with coordinates (σ_2, σ_3). It can be identified with the plane of polynomials $\lambda^3 + \sigma_2\lambda - \sigma_3$.

The manifold of regular orbits (each of which consists of six elements) can be identified with a domain in $\mathbf{C}^2$ formed by polynomials free of multiple roots. Singular orbits form the curve $\lambda^3 + \sigma_2\lambda - \sigma_3 = (\lambda - u)^2(\lambda + 2u)$ of polynomials with multiple roots.

Along this curve $\sigma_3 = -2u^3$, $\sigma_2 = -3u^2$, therefore *the manifold of singular orbits of group* $I_2(3)$ *is a semicubic parabola* (Fig. 7)

It turns out that interrelations between the group of reflections, caustics and wave fronts is not exhausted by the

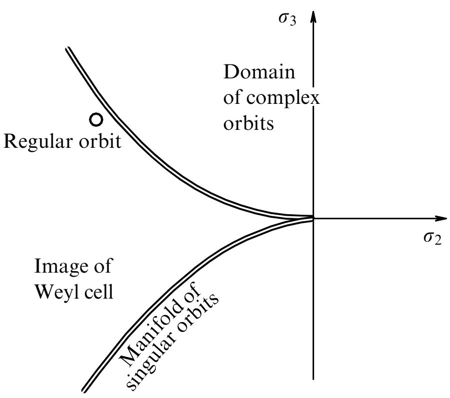


Figure 7. Manifold of the orbits of the reflection group A_2.

semicubic singularity $A_2 = I_2(3)$. This fact allows us to employ powerful algebraic methods of the theory of reflection groups for the study of caustics and wave fronts in multidimensional spaces (see, for example, Ref. [6]).

Oscillating integrals

In talking with me, N N Bogolyubov insisted that mathematicians should work in quantum physics. "A good article will be read by 1000 readers whether you publish it as mathematical or physical one, — he said. — But those readers who are mathematicians, will read it for a whole century, but physicists will forget it after 100 days, even if they constantly use it."

The theory of oscillating integrals is a quantum counterpart of caustics and wave fronts, but it arose (together with the 'method of stationary phase' and 'quasiclassical asymptotics') in the works of Carlucci and Jacobi dating back to the beginning of the XIX century. These works are devoted to the asymptotics of integrals which are necessary for the calculations of perturbations in celestial mechanics [5].

Definition. *Oscillating integral* with phase S, amplitude a and wave length h is defined as

$$I(h) = \int_{-\infty}^{\infty} \exp\left[\frac{\mathrm{i}S(x)}{h}\right] a(x)\,\mathrm{d}x\,.$$

Often one considers the case when the phase, amplitude and integral depend on parameters and the argument x is multidimensional ($x \in \mathbf{R}^n$) (Fig. 8).

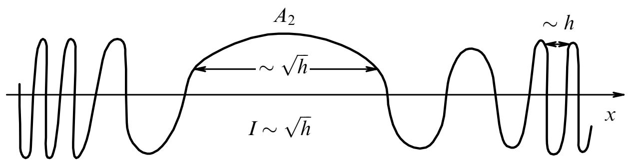


Figure 8. Integral of oscillating function is small.

Example 1. *Fresnel integral*

$$I(h) = \int_{-\infty}^{\infty} \cos\left(\frac{x^2}{h}\right) \mathrm{d}x \approx Ch^{n/2}\,.$$

If the phase does not have critical points, then the integral decreases (for $h \to 0$) faster than any power of h due to the interference of the contributions from close points x. Around the usual (non-degenerated) critical point there arises a 'domain of stationary phase', of radius of order $\sqrt{h}$, where S changes little and the interference is weakened. In the n-dimensional case this leads to an integral of order $h^{n/2}$. This can be strictly proved by reduction to the sum (or difference) of quadrates using the substitution of variables x (the so-called *Morse lemma*).

Example 2. *Airy integral.* If the phase depends on a parameter, then for some values of the parameter the critical point of the phase of S may be degenerate, and the Morse lemma then becomes not valid. The simplest case of this kind is the phase $S(x) = x^3 + \lambda x$ depending on parameter λ. When λ passes through 0, two simple critical points merge.

The critical point of the phase $S(x) = x^3$ arising at critical value of parameter $\lambda = 0$ is called *double* and denoted A_2 (usual non-degenerate critical point is denoted A_1).

[5] The author is very obliged to Mr. S Graffi who indicated these works.

The corresponding integral

$$I_{\alpha,\beta}(h) = \int_{-\infty}^{\infty} \exp\left[\frac{\mathrm{i}(x^3+\alpha x+\beta)}{h}\right] \mathrm{d}x$$

is called the *Airy integral*.

It describes the behavior of light near a caustic (A_1 corresponds to the light in usual point).

More exactly, let us consider a source of light Y and a screen X. Let $S(x,y)$ be the optical length of the path from point y of the source to the point x of the screen. The source and the screen can be considered at the beginning as submanifolds in Euclidean space. In the case under consideration $S(x,y)$ is just the Euclidean distance from x to y.

The influence of the light from the source y at the point x of the observation is proportional to $\exp(\mathrm{i}S/h)$; the proportionality coefficient depends on the intensity of the source at the point y and on the weakening of the propagating light due to the divergence of rays. This leads to the amplitude factor $a(x,y)$.

The analysis of the Airy integral leads to the conclusion that in the case of an n-dimensional source $I_{0,\beta}(h) \approx h^{n/2-1/6}$, i.e. when approaching the caustic ($\alpha = 0$) the intensity of light is substantially ($h^{-1/6}$ times) higher than usual.

The exponent 1/6 is called the *exponent of singularity* (e.s.). At the points near the caustic the integral is all the same evaluated by magnitude $Ch^{n/2-1/6}$, where the coefficient C is uniformly bound in the neighborhood of a point of a caustic of type A_2.

Similar evaluations can be obtained for all general one-parametric families of the phase functions. The matter is that they can be brought to the form $x^3 + \alpha x$ by change of coordinate x. If the number of parameters $k > 1$, then more complex singularities of caustics are possible, where the intensity of light is even higher:

$$k=2\,,\quad A_3 \quad S = x^4 + \alpha x^2 + \beta x, \qquad \text{e.s.} = 1/4\,,$$

$$k=3\,,\quad A_4 \quad S = x^5 + \alpha x^3 + \beta x^2 + \gamma x, \qquad \text{e.s.} = 3/10\,,$$

$$k=3\,,\quad D_4^+ \quad S = x_1^2 x_2 + x_2^3 + \alpha x_2^2 + \beta x_1 + \gamma x_2, \qquad \text{e.s.} = 1/3\,,$$

$$k=3\,,\quad D_4^- \quad S = x_1^2 x_2 - x_2^3 + \alpha x_2^2 + \beta x_1 + \gamma x_2, \qquad \text{e.s.} = 1/3\,.$$

It is evident that the value of the integral itself has asymptote $Ch^{n/2-(\text{e.s.})}$ depending on the dimension n of the source; for example, for $n = 3$ the true normal form of A_3 is

$$S = x_1^4 \pm x_2^2 \pm x_3^2 + \alpha x_1^2 + \beta x_1\,.$$

The exponent of singularity takes into account only *how many times* [in $h^{-(\text{e.s.})}$] the asymptotes increase approaching the caustic and its singular points.

For the 3-dimensional manifold of observation ($k = 3$) for the general typical family of rays from an n-dimensional source there are no other singularities besides those which are listed above (we shall describe now how the form of the function S affects the form of the caustic).

Therefore, the exponent of singularity in this case never exceeds 1/3. As will be shown now, the exponent takes the values 1/3 and 3/10 only at single points (D_4 and A_4) and the value 1/4 on the line A_3 (line of singularities of the caustic).

This result, achieved in the note [7], became the basis of the modern theory of caustics and wave fronts in their connection with reflection groups.

In this note some models of heating of electronic circuits were studied. To know the highest peak of heating localized at a point is very important in this problem, and that is why this question was proposed to me by V P Maslov from the Moscow Institute of Electronic Machine-Building in the framework of a commercial agreement.

Success was achieved when I noticed a sudden link of the mysterious exponents of singularity of typical singular points with the 'Coxeter numbers' of the reflection groups (whose definition will be given later).

Singularities of caustics

Let us consider a light system given by the optical length of the path $S(x,y)$ from a point (x) of the source to a point (y) of the manifold of observation. The greatest contribution in the illumination in the point y is given by stationary points of the phase considered as a function of x, where $\partial S/\partial x = 0$.

A *caustic* consists of those points y, where at least one of the stationary points is degenerate, i.e. the Hessian $\det(\partial^2 S/\partial x^2)$ at this point is equal to zero:

$$C = \left\{ y : \exists x : \ \frac{\partial S}{\partial x} = 0\,, \qquad \det\left(\frac{\partial^2 S}{\partial x^2}\right) = 0 \right\}.$$

In other words, it is necessary to consider the 'critical surface' (whose dimension is equal to the dimension of the manifold Y of observation) in the direct product $X \times Y$ consisting of the critical points of the function $S(\cdot, y)$, i.e. given by the equation $\partial S/\partial x = 0$, and then to project it on Y along X. A set of *critical values* of this mapping of one k-dimensional manifold on another is called the *caustic* of a given system of waves (or rays).

Case A_2. $S = x_1^3 + x_1 y_1 \pm x_2^2 \pm \ldots \pm x_k^2$. Critical surface: $3x_1^2 + y_1 = 0$, $x_2 = \ldots = x_k = 0$, $\partial^2 S/\partial x_1^2 = 6x_1 = 0$. Caustic: $y_1 = 0$ [for $n > 1$ this is $(n-1)$-dimensional surface]. In the Fresnel case ($n = 1$) the caustic is a single point on a straight line.

Case A_3. $S = x_1^4 + x_1^2 y_1 + x_1 y_2$. Critical surface: $4x_1^3 + 2x_1 y_1 + y_2 = 0$. Caustic [compare with the case (1) in the introduction]: $y_1 = -6x_1^2$, $y_2 = 8x_1^3$ — semicubic parabola $y_1^3 = \text{const}\, y_2^2$.

In the case $n = 2$ there are no other singularities except for the regression point A_3. For example, the singularity A_3 is given by the system of normals to ellipse.

In the case $n > 3$ the singularity A_3 is observed on the 'edge of regression' of co-dimension 2 (curve of regression for a caustic in 3-dimensional space for $n = 3$) (Fig. 9).

This edge of regression is semicubic in the sense that the plane transverse to it intersects the caustic along a curve with a semicubic singularity.

Case D_4. $k = 2$, $n = 3$. In these two cases, D_4^+ and D_4^-, the calculations are similar to those given above, but become more cumbersome, and it is better to carry them out using the reflection group in the way I shall explain later.

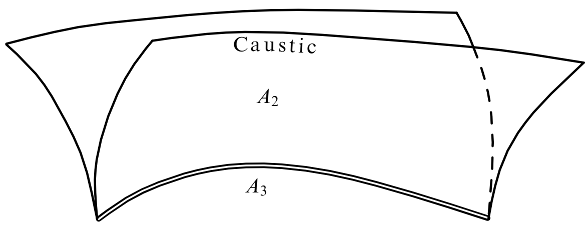


Figure 9. Singularity A_3 as the edge of regression of a caustic.

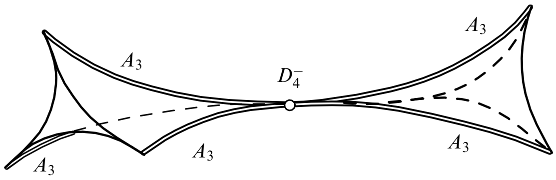


Figure 10. Pyramid D_4^-.

The answers are topologically simple enough: in the case D_4^- (pyramid) the caustic is topologically arranged as a two-sided 3-hedral pyramid whose edges — three smooth parabolas tangent in the vertex — are semicubic edges of regression directed outwards (Fig. 10).

In the case D_4^+ the caustic in the *complex domain* is the same, but in the real domain is quite another (a 'purse') (Fig. 11). Topologically this purse is similar to a pair of smooth surfaces $z = \pm xy$, but actually each of the surfaces is not quite smooth: one surface has a semicubic edge of regression over the diagonal $x = y > 0$, and the second surface — for $x = y < 0$. In both cases the edges are directed to the second part of the surface.

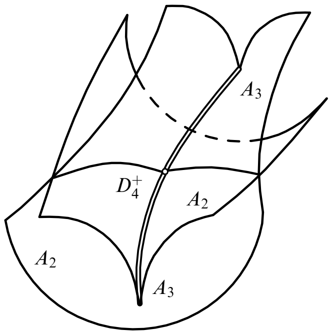


Figure 11. Purse D_4^+.

It is worth mentioning that the holomorphically identical surfaces of the pyramid and the purse have very different groups of real symmetries.

Remark. It is interesting (even from the viewpoint of the number theory where the asymptotics of oscillating integrals are also applied — I M Vinogradov became famous due to this discovery) to obtain as exact an evaluation as possible of the behavior of the oscillating integral in the neighborhood of a caustic.

Outside the caustic the asymptotics are of the form of a sum of Fresnelian contributions of stationary points, with the amplitude coefficients depending on the transformation of variables bringing the phase to the sum of quadrates:

$$I(y) \sim \Sigma (C_s h)^{n/2}, \qquad C_s \sim \frac{1}{\det_s} \left(\frac{\partial^2 S}{\partial x^2} \right). \tag{6}$$

Approaching the caustic, one $\det_s$ goes to zero, so that the evaluation is no more of the form $Ch^{n/2}$. However Colin de Verdier put forward a hypothesis that expression (6) remains nevertheless a *uniform evaluation* outside the caustic even near the bad points of the latter. He proved this for points A_2, A_3, A_4, D_4 and, in general, for all singularities of the phase function S of general position depending on not more than 7 parameters y (as will be shown later, it is this condition that guarantees the so-called *simplicity* of the singularities as well as their link with simple Lie algebra).

The Ukrainian vagrant and philosopher G Skovoroda long ago thanked the Creator who made everything that is necessary simple and everything that is complex unnecessary. Here Skovoroda made use of logic irreproachable from mathematical a viewpoint, which seems to be inaccessible to some physicists (*JETP* once rejected an interesting paper on adiabatic invariants because "the author asserted that A implies B, though in the words of the vice-editor-in-chief of *JETP* every physicist knows that B does not imply A!").

Caustics and reflection groups

In the foregoing we related a reflection group to the manifold of orbits and submanifold (with singularities) of nonregular orbits in it. The manifold of the orbits of the reflection group by itself is always smooth. This is a generalization of the theorem on symmetric functions of the corresponding symmetry group A_k of a k-dimensional simplex.

I give here the classification of the groups of Euclidean reflections. It is clear that the direct product of a reflection group in $\mathbf{R}^m$ by a reflection group in $\mathbf{R}^n$ is a reflection group in $\mathbf{R}^{m+n}$. The action of a reflection group in Euclidean space is called *irreducible*, if there is no nontrivial subspace invariant with respect to the whole group.

Any group of Euclidean reflections is a product of irreducible reflection groups which are the only ones to be described.

Irreducible reflection groups can be divided into two classes: crystallographic groups in $\mathbf{R}^N$, keeping unchanged some lattice $\mathbf{Z}^N$ of linear integer combinations of N linearly independent vectors, and non-crystallographic groups.

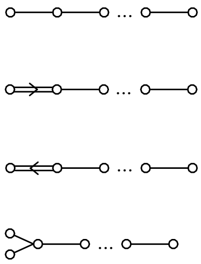
The crystallographic groups make four infinite series

A_k: $k \geqslant 1$

B_k: $k \geqslant 2$

C_k: $k \geqslant 3$

D_k: $k \geqslant 4$



and five exceptional groups:

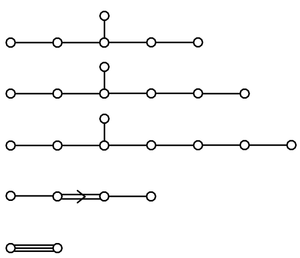
E_6:

E_7:

E_8:

F_4:

G_2:



Here the diagrams indicate the location of generating mirrors. A circle on a diagram means the basis vector orthogonal to a mirror. The line between two circles means an angle of 120°, and the absence of the line — an angle of 90°, two lines — an angle of 135°, three lines — an angle of 150°.

The sign $>$ indicates the length of basis vectors: on one side from it they are $\sqrt{2}$ times longer then on the other. The index k is a dimension of space (and number of circles in the diagram).

Diagrams of this kind were certainly used by Coxeter and Witt, that is why they are usually called Dynkin diagrams.

Example. B_2 and C_2 are symmetry groups of a quadrate, B_3 and C_3 are those of a cube. The difference between B_k and C_k is only in lattices generated by basis vectors: the mirrors by themselves (and, therefore, groups) are the same.

Noncrystallographic groups of Euclidean reflections are the following:

$I_2(p)$ is the symmetry group of a regular p-gon on a plane, $p \neq 2, 3, 4, 6$;

H_3 is the symmetry group of a regular icosahedron in $\mathbf{R}^3$;

H_4 is the symmetry group of a regular hypericosahedron in $\mathbf{R}^4$.

An icosahedron has 20 faces, 12 vertices, a group of motion consisting of 60 elements from $SO(3)$, a symmetry group consisting of 120 elements from $O(3)$.

A hypericosahedron has 120 vertices and 600 faces (tetrahedron). To construct it, let us consider a two-sheet spin covering:

$$\mathbf{R}^4 \supset S^3 = SU(2) = \mathrm{Spin}(3) \to SO(3)\,.$$

The inverse image of the group of 60 rotations of an icosahedron consists of 120 points in S^3. They are vertices of the hypericosahedron.

Remark. It is proved that the manifolds of orbits of all these and only these groups are smooth [6].

Example. For A_k let us start from $\mathbf{R}^{k+1}$ with basis $\alpha_0, \ldots, \alpha_k$. The vectors $e_1 = \alpha_0 - \alpha_1$, $e_2 = \alpha_1 - \alpha_2, \ldots, e_k = \alpha_{k-1} - \alpha_k$ make a frame in $\mathbf{R}^k$ (orthogonal to diagonal) with scalar products given by the diagram A_k. The mirror, orthogonal to e_j, determines the same reflection as the transposition of the coordinates α_{j-1} and α_j. That is why the reflection group A_k acts in $\mathbf{R}^k$ as a symmetric group $S(k+1)$ of the coordinate permutations in $\mathbf{R}^{k+1}$.

The basic invariants $\sigma_2 = \alpha_1\alpha_2 + \ldots + \alpha_{k-1}\alpha_k, \ldots, \sigma_k = \alpha_1\alpha_2 \ldots \alpha_k$ serve as coordinates on the manifold of orbits. By the fundamental theorem on symmetric functions, all polynomials in $\mathbf{R}^k$, invariant in respect to the action of $S(k+1)$, are polynomials of $\sigma_1, \ldots \sigma_k$. It follows from this that $\mathbf{C}^k/S(k+1) \approx \mathbf{C}^k$ in the sense of algebraic geometry. Actually this theorem is valid even for smooth functions, and the smooth $\mathbf{R}$-manifold $\mathbf{C}^k/S(k+1)$ is diffeomorphic to $\mathbf{R}^{2k}$.

The manifold of nonregular orbits of the reflection group is called its *discriminant*. In the complex manifold of the orbits this hypersurface (in general, complex) usually has singularities.

Example. For A_2 the discriminant is a semicubic parabola $\sigma_2^3 = C\sigma_3^2$ in $\mathbf{C}^2$ with a singularity in 0.

Discriminants of the reflection groups immediately give the singularities of the wave fronts of typical wave families. But it turns out caustics also have a natural algebraic description in these terms (found in Ref. [8]).

Example. For A_3 the discriminant is a surface in $\mathbf{C}^3$ called a *swallow tail* and made in the space of polynomials $z^4 + az^2 + bz + c$ (with coordinates a, b, c) by polynomials having multiple roots (Fig. 12).

Really, the orbit of the point (z_0, z_1, z_2, z_3) under the action of the permutation of coordinates consists of 24 points

[6] It would be interesting to study Lie groups with such a property of the manifolds of orbits: the Maxwell theorem on spherical functions gives interesting examples [9].

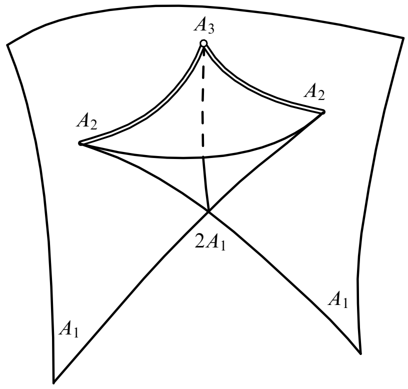


Figure 12. Swallow tail — discriminant for A_3 and the caustic on it for A_4.

and, therefore, is regular, if and only if all four roots z_j are different.

We obtain convenient formulas parametrizing the discriminant: there

$$x^4 + ax^2 + bx + c \equiv (x-u)^2(x^2 + 2ux + v)$$

(we used the fact that $\sigma_1 \equiv 0$). Thus,

$$a = v - 3u^2\,, \qquad b = 2u^3 - 2uv\,, \qquad c = u^2 v\,.$$

To study this surface, it is convenient to intersect it with the plane $a = \mathrm{const}$. The intersection is the curve

$$b = -4u^3 - 2au\,, \qquad c = au^2 + 3u^4\,.$$

For $a = 0$ it is a parabola of power $3/4$. The singular points of the curve are given by the equation $6u^2 - a = 0$.

If $a < 0$, then there are no real singularities and in the real plane the curve is smooth. When a increases to 0, at the point 0 arises a singularity of the power $3/4$ which later, for $a > 0$, decays into two semicubic singularities. In this case the planar curve transforms like a system of ellipse equidistants when the first singularity arises in the focal point of the caustic.

It can be shown that these phenomena are not only similar but diffeomorphic as well: the propagating waves sweep through space-time a surface with a 'swallow tail' singularity when the front passes through the cusp.

The analysis of this example leads to the following general construction which turns out to be admissible in the case of general reflection groups as well.

Let us consider the discriminant of a reflection group (as a complex hypersurface in $\mathbf{C}^n$) with the leading singularity at the point $\sigma = 0$. Let us suppose that the manifold of orbits $\mathbf{C}^n$ is a fibre bundle over $\mathbf{C}^{n-1}$, i.e. that it is given a smooth mapping $\mathbf{C}^n \to \mathbf{C}^{n-1}$ of rank $n-1$ at the point $\sigma = 0$.

It turns out that:

(1) all such mappings in a general position are locally equivalent (can be transformed into each other by local holomorphic diffeomorphism of $\mathbf{C}^n$ in the vicinity of 0 which maps the discriminant into itself and the fibres of the mapping into the fibres);

(2) the projection of the edge of the discriminant regression is the caustic of the corresponding singularity.

Example. Projecting the swallow tail (discriminant of A_3) is equivalent to forgetting the coordinate c. The edge of regression is projected into a semicubic parabola which is

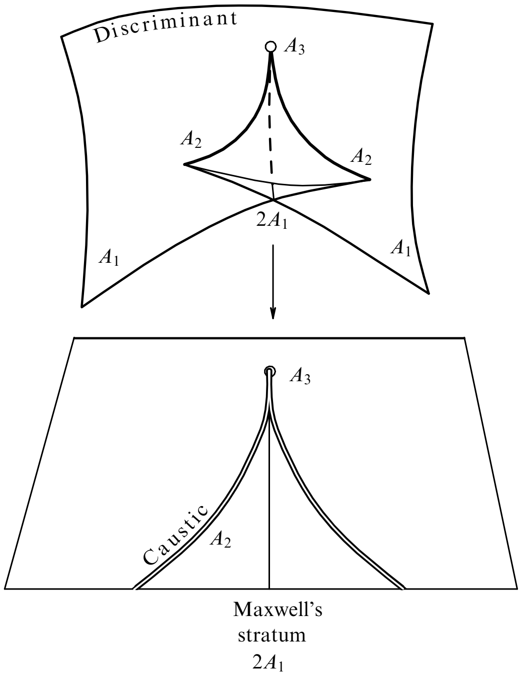


Figure 13. Typical projection of the swallow-tail onto the plane.

the caustic of the singularity A_3. On the edge of the regression $a = v - 3u^2$, $b = 2u^3 - 2uv$, $a = 6u^2$, so on its projection $a = 6u^2, b = -16u^3$ (Fig. 13).

Remark. Besides the edge of regression the swallow tail has a line of self-intersection ($b = c = 0$). The projection of this line onto a plane is a smooth line. If projecting $\mathbf{C}^3 \to \mathbf{C}^2$ is considered as differentiating the polynomial $z^4 + az^2 + bz + c$, then the line of self-intersection consists of a set of polynomials with two double roots. It is projected into the 'Maxwell stratum' consisting of cubic polynomials with zero integral between critical points.

Under such interpretation the caustic consists of cubic polynomials with multiple critical points. All this can be understood especially easy, if one takes into account that the value of the coordinate c (differentiating of polynomials is equivalent to projecting along this axis) on the discriminant is simply the critical value (taken with minus) of the corresponding polynomial $x^4 + ax^2 + bx$.

Rearranging of propagating waves

A system of propagating waves can be described using a set of momentary wave fronts Φ_t in 'physical' space, but instead of them one can consider unified 'big front' Φ in space-time whose intersections with isochrones $t = \text{const}$ give the momentary fronts [10].

It turns out that large fronts propagating in n-dimensional space have the same singularities as the momentary fronts in $(n+1)$-dimensional space.

Example. Momentary fronts in the plane are curves. Their propagation can be described by a big front in 3-dimensional space-time. In a typical situation such front has singularities no more complicated than the swallow tail. Therefore, typical rearrangements of the wave fronts propagating in a plane are typical rearrangements of the cross-sections of swallow tail with surfaces $t = \text{const}$, where t is a function of general position defined on a 3-dimensional space which contains the swallow tail (Fig. 14).

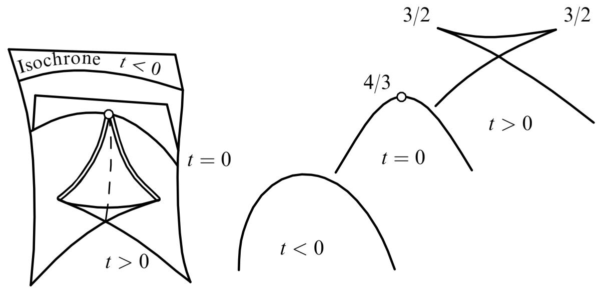


Figure 14. Rebuilding of the cross-sections of the swallow tail.

These rearrangements are exhausted with the following variants:

(1) the birth or death of two semicubic cusps of the front when the swallow tail of big front passes;

(2) the birth or death, or rebuilding of two semicubic points of regression of the front when isochrone touches the line of regression of big front;

(3) the birth or death, or rebuilding of two branches of the front when isochrone touches the smooth part of front;

(4) other points of contact of the isochrone line of self-intersection with the front.

The rebuildings A_4 and D_4 are interesting examples of front rebuildings in $\mathbf{R}^3$. Under rebuilding A_4 two swallow tails of the momentary front merge while the isochrone touches the line of regression of the big front.

Different phenomena can happen, depending on sign, under rearrangement D_4. Maybe the most interesting thing is connected with turning inside out the trifoliate knot in $\mathbf{R}^3$, during which its vertices slide along the edges of regression of a caustic pyramid. The corresponding pictures are shown in Refs [3, 4, 6].

The theories constructed above have found use in the study of shock waves and cosmological bifurcations of the motion of dust-like media (see Refs [4, 6]).

Versal deformations

The analysis given above of the singularities of caustics, wave fronts and their rearrangements was founded on the formulas for the phase function from the section 'Oscillating integrals'. Now I proceed to the description of algebraic technique giving these formula.

The singularity of the oscillating integral is given by itself by the phase function $F(x)$ [for the Airy integral $F(x) = x^3$]. To calculate the multiplicity of a critical point (which we shall consider to be the origin) one has to consider the orbit F under the action of the group of diffeomorphisms. It turns out that the infinite-dimensional space of functions in the neighborhood of a critical point of finite multiplicity k can be represented as the direct product of a k-dimensional space transversal to the orbit and an infinite-dimensional space along which the type of singularity does not change. In the Airy case the finite-dimensional space consists of the functions $x^3 + ax + b$ and the infinite-dimensional space consists of the functions X^3, where $X = h(x)$ is a change of variables.

In general the multiplicity is determined by the following algebraic construction. At first we take the space $\mathcal{E}$ of all functions of x (here it is admissible to take functions of different degree of smoothness; in the case under consideration one can take even formal Taylor series $f(x) = f_0 + f_1 x + f_2 x^2 + \ldots$ making the ring $\mathcal{E} = \mathcal{C}[[x]]$).

In the ring $\mathcal{E}$ of functions we consider the *gradient* ideal spanned by partial derivatives of the initial function F under consideration:

$$I_F = \left\{ h_1 \frac{\partial F}{\partial x_1} + \ldots + h_n \frac{\partial F}{\partial x_n} \right\}$$

(here $h_s \in \mathcal{E}$ are any functions from $\mathcal{E}$). This space I_F actually is the tangent space to the orbit F under the action of the diffeomorphism group because

$$F(x_1 + \varepsilon h_1, \ldots, x_n + \varepsilon h_n) = F(x_1, \ldots, x_n) + \varepsilon \left(h_1 \frac{\partial F}{\partial x_1} + \ldots + h_n \frac{\partial F}{\partial x_n} \right) + o(\varepsilon) \, .$$

Example. For $F = x^3$ one finds

$$I_F = \{hx^2\} = \{h_2 x^2 + h_3 x^3 + \ldots\} \, .$$

This ideal is a subspace of co-dimension 2 in the space $\{h_0 + h_1 x + h_2 x^2 + \ldots\}$ of all power series of x.

Definition. The co-dimension

$$\mu = \dim_{\mathcal{C}} \frac{\mathcal{C}[[x_1, \ldots, x_n]]}{\{\sum h_s \, \partial F / \partial x_s\}}$$

of the gradient ideal is called the *multiplicity* of the critical point 0 of the function F.

Example. For $F = x^3$ we find that

$$\mu = \dim_{\mathcal{C}} \frac{\{h_0 + h_1 x + \ldots\}}{\{h_2 x^2 + \ldots\}} = \dim_{\mathcal{C}} \{ax + b\} = 2 \, .$$

Let us suppose that the critical point 0 of the function F is of finite multiplicity and that the factor-space $\mathcal{E}/I_F$ is generated over $\mathbf{C}$ by functions $g_1, \ldots, g_\mu$.

Then the μ-parametric space

$$\{F + \lambda_1 g_1 + \ldots + \lambda_\mu g_\mu\} \, , \qquad \lambda_s \in \mathbf{C}$$

can be taken as a transversal to the orbit of the function F.

Example. For $F = x^3$ one obtains the transversal $\{x^3 + ax + b\}$.

Definition. The function $S(x, y)$ is called the *deformation* of the function $F(x)$, if $S(x, 0) = F(x)$. The transversal to the orbit constructed above is a two-parameter deformation of the function x^3 with parameters $y_1 = a$ and $y_2 = b$.

Definition. *Initial rates* of deformation S are called functions $g_s(x) = \partial S / \partial y_s \big|_{y=0}$. In our example these are $g_1 = x$ and $g_2 = 1$.

The rule for writing down a transversal to the orbit (which is also called *versal deformation* of function F) is established by the fact that the deformation is versal each time its initial rates generate a basis of the factor-space $\mathcal{E}/I_F$.

The phase functions, used in the section 'Oscillating integrals' for construction of a swallow tail, purse and pyramid, are versal deformations S of the functions $F = x_1^4 \pm x_2^2$ and $F = x_1^2 x_2 \pm x_2^3$.

To study the structure of the orbits of the diffeomorphism group in the vicinity of F, it is sufficient to study only the finite-parametrical family — versal deformation. In the infinite-dimensional space the situation is the same, only everything is multiplied by a smooth infinite-dimensional manifold, so that every orbit has the form of a cylinder with the infinite-dimensional smooth generator, and the genetrix is a manifold of functions from an appropriate class in versal deformation.

Example. Singularity A_2 (orbit of the function x^3) makes the edge of regression in the manifold of functions with critical value 0.

Remark. The algebraic results given above are valid for formal as well as for analytic and even smooth functions, with natural reservations: not all μ bifurcating critical points can be real (for example, there are no real critical points for $x^3 + ax$ when $a > 0$ though the multiplicity μ is equal to 2).

The intersection line of the swallow tail corresponds both to real and complex points with multiple critical values. In the real case only half of this line is a genuine line of self-intersection of a real swallow tail, and the second half is its analytical continuation.

Coxeter numbers

The exponents of singularity for the asymptotics of oscillating integrals can be algebraically described as follows. Let us consider mirrors of corresponding group of Euclidean reflections in real space $\mathbf{R}^\mu$. They divide the space into parts called Weyl cells. Each cell is a cone with a simplicial base. The continuated walls of a cell (there are μ of them) divide the space into 2^μ parts called *Springer cones*. Each Springer cone consists of several Weyl cells. The geometry of Weyl cells and Springer cones made of them decisively influences the geometry of wave fronts and caustics.

Definition. The product of reflections in walls of one Weyl cell is called a *Coxeter transformation*.

Example. Three mirrors A_2 divide the plane into six Weyl cells — angles of 60°. Each Coxeter transformation is a rotation of 120°.

The eigenvalues of a Coxeter transformation are roots of different powers m of 1. The order of the Coxeter transformation is called the *Coxeter number of a reflection group*.

Example. Coxeter numbers m of some crystallographic reflection groups in $\mathbf{R}^n$ are given in the following table:

A_n	B_n, C_n	D_n	E_6	E_7	E_8
$m = n + 1$	$2n$	$2n - 2$	12	18	30

It turns out that these numbers are closely connected with the asymptotics of the corresponding oscillating integrals.

Example. For D_n on the plane x, y

$$I_h = \iint \exp\left[\frac{\mathrm{i}(x^2 y + y^{n-1})}{h}\right] \mathrm{d}x \, \mathrm{d}y \, .$$

Let us make a change of coordinates $x = h^\alpha X$, $y = h^\beta Y$ destroying division by h:

$$I_h = \iint \exp\left[\mathrm{i}(X^2 Y + Y^{n-1})\right] h^{\alpha+\beta} \, \mathrm{d}X \, \mathrm{d}Y \, ,$$

$$2\alpha + \beta = (n-1)\beta = 1 \, .$$

We get $\beta = 1/(n-1)$, $\alpha = (n-2)/(2n-2)$, $\alpha + \beta = n/(2n-2)$. Therefore the integral I_h decreases as $h^{n/(2n-2)}$ at the points D_n instead of usual rate of decrease $h^{2/2}$. For example, at the points D_4 we obtain $h^{2/3}$ instead of h, i.e. the rate becomes $h^{-1/3}$ times worse.

Such calculations give the exponents of singularity of the rest singularities in the list — obviously they repeat the list of Coxeter numbers:

$$\begin{array}{ccccc} A_n & D_n & E_6 & E_7 & E_8 \\ \dfrac{n-1}{2n+2} & \dfrac{n-2}{2n-2} & \dfrac{5}{12} & \dfrac{8}{18} & \dfrac{14}{30} \end{array} .$$

An analogous theory has already been constructed for singularities B, C, F, G and should be constructed for $I_2(p)$, H_3, H_4.

Example. At the usual point A_1 the exponent of the singularity is 0, i.e. the power multiplier for a 2-dimensional oscillating integral is proportional to h. At the double point A_2 (the usual point of the caustic) the exponent of the singularity is $1/6$, i.e. the power multiplier is $h^{1-1/6}$. At a 3-fold point (the cusp of caustic) the light is even brighter, the exponent of singularity for A_3 is $1/4$, and the power multiplier is $h^{1-1/4}$.

At singular points D_4 (corresponding to pyramids and purses of the caustic) the exponent of singularity is $1/3$, the power multiplier is $h^{1-1/3}$. But these points in a 2-dimensional system of general position appear only for distinct values of the parameters (for example, time, if the question is not the light but the noise of a plane flying by).

Springer cones and Bernoulli numbers

Opera has become the realm of boredom because there is too much music in it.
Bomarche, Foreword to "Tarar"

Let us consider the reflection group in $\mathbf{R}^n$ generated by reflections in n walls of a Weyl cell. The continuations of these walls, like coordinate planes, divide the space into 2^n parts — Springer cones each of which consists of several (maybe one) Weyl cells. One of these cones consists of the maximal number of cells — let us call it the *chief* Springer cone.

Example. Group A_2 of the plane reflections in three mirrors making angles 120° in 0, has chief Springer cones consisting of two Weyl cells.

Definition. The number of Weyl cells in the chief Springer cone of the reflection group is called its *Springer number*.

Example. The springer number of the group A_2 is $a_2 = 2$, as we saw before. To calculate the following Springer numbers a_m of the groups A_m is already not so easy (this is the so-called theory of unknown consequences). The first of them are as follows: $a_m = 2, 5, 16, 61, 272, 1385, \ldots$ $(m = 2, 3, \ldots)$.

The appearance of Euler number 61 suggest that this consequence is connected with Euler and Bernoulli numbers.

Let us write the exponential generating function

$$P(t) = \sum_{k=0} \frac{a_k t^k}{k!} .$$

It turns out (the theorem of the French mathematician Andre) that

$$P(t) = \tan t + \sec t .$$

Moreover, the coefficients a_k can be easily calculated using a Pascal type triangle:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 0 & & & \\ & & 0 & & 1 & & 1 & & \\ & 2 & & 2 & & 1 & & 0 & \\ 0 & & 2 & & 4 & & 5 & & 5 \\ . & . & . & . & . & . & . & . & . \end{array}$$

Each element in each row is equal to the sum of numbers of the preceding row standing to the right or to the left from a given element (depending on the parity of the number of the row).

Along the left side of the triangle run (starting from a_1) the coefficients of a Taylor series for a tangent divided by $k!$, and along the right side — for a secant.

There is similar theory for the other reflection groups (see Ref. [11]).

As for the geometry of the Springer cone subdivision into Weyl cells, it is convenient to start from the case A_3 in $\mathbf{R}^3$. In this case the cone is a triangular pyramid divided into 5 cells (Fig. 15). This division (by symmetry planes of the tetrahedron) is conveniently depicted as a division of the triangle PQR of the pyramid base into 5 triangles — bases of Weyl cells. Let us choose points S and T on the sides RQ and RP and draw the intervals PS and QT intersecting each other in U. The bases of Weyl cells are triangles PQU, QUS, UST, TSR, and PUT.

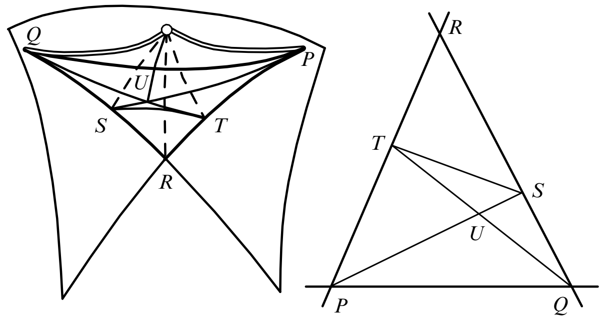


Figure 15. Subdivision of the base of Springer cone of the group A_3 into Weyl cells.

Let us consider now swallow tail as a caustic of the family of functions of x of the following form:

$$x^5 + ax^3 + bx^2 + cx .$$

Points a, b, c, corresponding to the functions with four real critical points, make a 'pyramid' between the edge of regression and the line of self-intersection of the swallow tail.

The Maxwell stratum divides this pyramid into domains inside each of which functions are topologically equivalent (i.e. the orders of critical values on the axis of values are the same). It turns out that there are five such domains, they are bound by algebraic surfaces and all division topologically is the same as the division of Springer cone into cells described above.

This topological equivalence is given by algebraic homeomorphism which even provides *diffeomorphism* of the interior of pyramid on the interior of the Springer cone (as well as the interior of each face on the interior of the corresponding face of the cone).

All these results apply to other reflection groups.

The diffeomorphism, which is under discussion here, is a real form of the remarkable Lyashko–Looijenga mapping (see Refs [12, 13]) putting the polynomial

$$a(z) = z^{k+1} + a_1 z^{k-1} + \ldots + a_k , \qquad a_s \in \mathbf{C}$$

in correspondence with polynomial

$$b(w) = \prod (w - c_s) = w^k + b_1 w^{k-1} + \ldots + b_k , \qquad b_s \in \mathbf{C} ,$$

whose roots c_s are critical values of polynomial a.

3. The frontier between mathematics and physics

The problem of relations between these two sciences has been widely discussed. For example, Hilbert explicitly announced that geometry is a part of physics because both the geometer and the physicist obtain their achievements in quite the same way.

True, I am afraid that Hilbert just did not consider geometry as a part of mathematics because he said that for mathematics it is the same whether its 'points' are glasses of beer and all 'straight lines' are benches. This statement is not quite senseless, for example, in Lobachevsky geometry (in Poincareé model) circles are considered as straight lines, and this is useful.

It is pity that his followers, like Bourbaki, applied these 'harmless' ideas in the teaching of school mathematics substituting juggling of logic symbols for the content rich science of the structure of the world. The hatred of mathematics spread itself all over the world, and we, Russians, are even behind.

Recently one such follower sent me a letter criticising my statement that mathematics is a part of physics, insisting that there is nothing in common between these sciences. It is worth of mentioning that the same outstanding Bourbakist refused to take part in writing a review book for 2000 pleading his non-participation by mentioning that 'mathematical joint ventures are always a failure'. I do not know whether the Bourbaki venture is really finished.

It would take too much time to cite all the remarkable statements (of Pascal, Descartes, Newton, Huygens, Leibniz) on this matter but I can not resist the temptation to mention Dirac, who said that a physicist should never lean on physical intuition which most often is a name for prejudiced judgements. According to Dirac's opinion, the right way consists of taking mathematical theory and consequently developing it considering at the same time its applications to as many important models as possible.

For example, the right electromagnetic theory follows from the Maxwell equations and not from more precise specifications of breeds of cat and types of amber. A question about the color of a meridian is an abuse of 'intuition' of prejudiced opinions.

I hope that I have shown above to what results one can come following Dirac's advice.

Descartes' discussion of the barometric ideas of Pascal are given as an example of the inadmissible influence of prejudiced ideas in the case when one should invent mathematical theory rich in content. Pascal took as his initial point the Torricelli experiment with a mercury column and constructed a corresponding device substituting water and French wine for Italian mercury (it was difficult because the keg should be very strong to resist the pressure of a ten-meter high column of wine or water). But Pascal managed to do everything — first on the St. Jacob tower in Paris and later on the mountain Pui de Dome in Auvergne — and constructed the first water barometers (with vacuum above the water column). He come to Descartes, the greatest scientist of the time (Pascal was very young), and told him about the theory — the Pascal law and so on. Descartes, the forerunner of Bourbaki, the man who banished drawings from geometry, considered all that to be idle theory and wrote to Huygens: "As for me — I do not see vacuum anywhere, besides, maybe, in Pascal's head". Several months later Descartes already asserted that it was he who taught Pascal everything. The axiom of the 'horror of vacuum' was more dear for Descartes than Pascal's theory (later Descartes became dissatisfied with Newton's long-range action, considering planets to be moved by ether vortices).

Besides the vacuum in Pascal's head Descartes discovered much of interest, for example, in the theory of caustics he found the explanation of the rainbow and derived its spread (43°) from the refractive index of water.

Recently a general tendency, observed all over the world, is the offensive on science and education on the part of bureaucrats and managers. Mathematics and physics are endangered among the first. For example, I can mention recent 'Californian wars': the state of California under the guidance of G Seaborg accepted new school requirements disapproved at federal level as contradicting all-American standards. The senate raised an objection.

Here are two examples. A new program provided the addition of common fractions for ten year old school children in the course of mathematics and theory on the three phase states of water in the course of physics. In the federal programme water has only two phase states (changing into one another in the fridge), steam being considered inaccessible and rather abstract for the deprived schoolchildren. Questioning of teachers revealed that, as a rule, they cannot manage even the basics — to divide 111 by 3 requires a computer. Attempts to abolish mathematics (especially proofs) are seen by our educators too as 'humanitisation'.

I underline that proofs in mathematics have always played only a subservient role, somewhat like orthography or calligraphy in poetry. Mathematics as well as physics is an experimental science, and conscious addition of common fractions $1/2$ and $1/3$ is a standard element common to all mankind culture. The attempts to break the habit of time and to stop any kind of progress is a natural, but very dangerous consequence of the universal bureaucratization and world struggle with culture.

Romans tried to retain from Greek science only the part 'of practical use', and the result was the gloomy obscurantism of the Medieval epoch.

This work is carried out under partial financial support of the Russian Foundation for Basic Research (project 99-01-01109) and the Institute Universitaire de France.

References

1. Arnol'd V I "O prepodavanii matematiki" ("On the teaching of mathematics") *Usp. Mat. Nauk* **53** (1) 229 (1998)
2. Whitney H "Singularities of mappings of Euclidean spaces I. Mappings of the plane into the plane" *Ann. Math.* **62** 374 (1955)
3. Arnol'd V I, Varchenko A N, Guseĭn-Zade S M *Osobennosti Differentsiruemykh Otobrazheniĭ* Vol. I, II (Singularities of Differentiable Maps) (Moscow: Nauka, 1982, 1984) [Translated into English (Boston: Birkhäuser, 1985 – 1988)]
4. Arnol'd V I *Teoriya Katastrof* (Catastrophe Theory) 3rd ed. (Moscow: Nauka, 1990) [Translated into English (Berlin: Springer-Verlag, 1992)]
5. *Sovremennye Problemy Matematiki. Fundamental'nye Napravleniya* Vol. 1, 3 – 6, 39 (Modern Problems of Mathematics. Fundamental Directions) (Moscow: VINITI, 1985 – 1989)
6. Arnol'd V I *Osobennosti Kaustik i Volnovykh Frontov* (Singularities of Caustics and Wave Fronts) (Moscow: Fazis, 1998) [Translated into English (Dordrecht: Kluwer Acad. Publ., 1990)]
7. Arnol'd V I "Integraly bystro ostsilliruyushchikh funktsiĭ i osobennosti proektsiiĭ lagranzhevykh mnogoobraziiĭ" ("Integrals of fast oscillating functions and singularities of projections of Lagrangian manifolds") *Funkts. Analiz i Ego Prilozh.* **6** (3) 61 (1972)

8. Arnol'd V I "Indeksy osobykh tochek 1-form na mnogoobrazii s kraem, svorachivanie invariantov grupp porozhdennykh otrazheniyami i osobye proektsii gladkikh poverkhnosteĭ" ("Indices of singular points of 1-forms on a manifold with a boundary, contraction of the invariants of groups generated by reflections and singular projections of smooth surfaces") *Usp. Mat. Nauk* **34** (2) 3 (1979)
9. Arnol'd V I "Topologicheskaya teorema Maksvella o mul'tipol'nom predstavlenii sfericheskikh funktsiĭ" ("Maxwell topological theorem on multipole representation of spherical functions") *Usp. Mat. Nauk* **51** (6) 227 (1996)
10. Arnold V I "Wave front evolution and equivariant Morse lemma" *Comm. Pure Appl. Math.* **29** (6) 537 (1976)
11. Arnol'd V I "Ischislenie zmeĭ i kombinatorika chisel Bernulli, Eĭlera i Springera" ("Snake calculus and combinatorics of Bernoulli, Euler and Springer numbers") *Usp. Mat. Nauk* **47** (1) 3 (1992)
12. Arnol'd V I "Kriticheskie tochki funktsiĭ i klassifikatsiya kaustik" ("Critical points of functions and classification of caustics") *Usp. Mat. Nauk* **29** (3) 243 (1974)
13. Looijenga E "The complement of the bifurcation variety of a simple singularity" *Invent. Math.* **23** (2) 105 (1974)