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What are stochastic filtering and stochastic resonance?

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Contents

Abstract. The concept of stochastic resonance (SR) was introduced in 1981 in the study of ice-age periodicity in the northern hemisphere. To describe this phenomenon, a relaxation model — an overdamped bistable oscillator — is used. SR is caused by the simultaneous action of a periodic signal and noise and appears as a nonmonotone response to noise intensity variations. Since the subject of the study is actually the filter passband width as a function of noise intensity, 'stochastic filtering' (SF) seems to be a more appropriate term to describe the phenomenon. It is shown that when driven by a signal and noise, a low-attenuation bistable oscillator also displays ordinary SR when the signal frequency coincides with the effective noise-intensity-dependent frequency of the oscillator. Thus the possibility of the resonance being controlled by varying the noise intensity arises.

1. Introduction

The new term 'stochastic resonance' (SR) was introduced in 1981 – 1982 in Refs [1 – 3] dealing with the periodic advance of glaciers on Earth. This effect consists in that when periodic signal and noise are applied simultaneously to an input, the system can be tuned into 'resonance' owing to the nonmonotone dependence of the response of the system on the noise intensity (for instance, the temperature).

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Theoretical analysis of SR is usually based on the relaxation model — an overdamped bistable oscillator. 'Overdamped' means that the coefficient of friction is much greater than the characteristic eigenfrequency of the oscillator. The meaning of 'resonance', however, is not quite conventional. Here we mean the nonmonotone ('resonance') response of the bistable element to an external force as function of noise intensity.

An overdamped oscillator is a relaxation system, and therefore there can be no resonance in the common physical sense. The effect in question actually consists in the nonmonotone dependence of the filter passband width on noise intensity, and so would be more appropriately referred to as 'stochastic filtering' (SF).

The imprecision of the term 'resonance' with reference to the process in an overdamped oscillator has been noted more than once (see, for example, Refs [4, 5]). Still, the term SR in the above sense continues to be used in the scientific literature.

The first two reviews of theoretical and experimental studies of stochastic resonance [6, 7] came out in 1994; a more detailed review appeared in January 1998 [8]. A most up-to-date review on stochastic resonance is published in this issue of *Physics Uspekhi* [9]. By no means does it duplicate article [8]: new problems are discussed by the authors who themselves have contributed much to their investigation and solution.

Now what are the grounds for using the term 'stochastic resonance' in the context of an input periodic signal and noise in an overdamped bistable oscillator?

Starting with the earliest studies (see Refs [8, 9]), the concept of resonance has been framed as follows. The 'resonance' is assumed to occur in the overdamped bistable oscillator when the frequency of the periodic signal coincides with the frequency of switching between the two states of bistable system — the Kramers frequency. As we shall see, however, it is a different phenomenon that takes place in the overdamped oscillator. Here we are dealing with a relaxation process. Using an electromechanical analogy, we may say that we are dealing with an electric circuit consisting of a resistance and capacitor (RC circuit), where the capacitance is nonlinear. This circuit may be regarded as an electric filter characterized by its response to an external periodic input. The filter passband width depends on the noise intensity. It is important that generally this dependence is nonmonotone, which makes it possible to control the passband of the filter by varying the noise intensity. It would be natural to call this process 'stochastic filtering' rather than 'stochastic resonance'.

It is also interesting to analyze the effects of a periodic signal and noise on the bistable oscillator in the other limiting case, when the coefficient of friction is much less than the characteristic eigenfrequency (the 'effective frequency') of a bistable oscillator. Resonance occurs when the frequency of the signal coincides with the effective frequency. Since the effective frequency depends on the intensity of noise, the latter can be used for controlling the resonance. In such a case the term 'stochastic resonance' is appropriate.

In this paper we shall discuss the phenomena of both stochastic filtering and stochastic resonance. The description of SF is based on the Einstein – Smoluchowski equation for the distribution function of the values of generalized coordinate and time [10 – 13]. Stochastic resonance is described with a more general equation — the Fokker – Planck equation for a distribution function of generalized coordinate and velocity, and time as well [10 – 13].

We also note that a unified description of SF and SR is possible using the generalized Fokker – Planck equation [12, 13].

Both SF and SR are described in two alternative ways. One is based on the self-consistent equations in first and second moments of the coordinate and velocity of the oscillator. This approximation was used [12 – 14] for calculating the fluctuations in autooscillatory systems (the van der Pol oscillator and lasers), and in the theory of second-order phase transitions. The other is close to the traditional approach, when the Kramers transitions between the two states of a bistable oscillator are important.

The conditions under which the two alternative descriptions yield the same results up to a constant factor are identified. This is possible when the height of the barrier that separates the states of the bistable oscillator is low. When the barrier is high, the two methods give different results. The suitability of one or the other much depends on the circumstances of the experiment.

Stochastic filtering phenomenon (or stochastic resonance in the traditional sense) has been discovered and studied in many physical, chemical and biological systems. The results of these studies were discussed in detail in the two recent reviews [8, 9] on stochastic resonance (in its traditional sense). It is worth noting that review [9] is entitled "Stochastic resonance: noise-enhanced order". The discussion of this most interesting aspect of the theory of stochastic resonance calls for a separate treatment.

2. Generalized Fokker – Planck equation for a bistable oscillator

2.1 General case

Let us write the generalized Fokker – Planck equation (see Eqn (17.1.4) in Ref. [13]) for a one-dimensional bistable oscillator (a Duffing oscillator) placed in a thermostat at the temperature T:

$$\frac{\partial f}{\partial t} + v\,\frac{\partial f}{\partial x} + \frac{F(x,t)}{m}\,\frac{\partial f}{\partial v} = \frac{\partial}{\partial v}\left(D_{(v)}\,\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial v}(\gamma v f) + \frac{\partial}{\partial x}\left(D_{(x)}\,\frac{\partial f}{\partial x} - \frac{F(x,t)}{\gamma m}\,f\right), \qquad \int f(x,v,t)\,\mathrm{d}x\,\mathrm{d}v = 1\,. \quad (1)$$

Here we assume that the mean velocity of the medium which hosts the Brownian motion of the bistable oscillator is zero.

Let us explain the notation used in this equation.

The force is represented as a sum of two parts:

$$F(x,t) = -\frac{\partial U(x)}{\partial x} + F(t)\,, \quad (2)$$

where $U(x)$ is the potential energy of the oscillator with nonlinear rigidity:

$$U(x) = \frac{m\omega_0^2 x^2}{2}\left(1 - a + \frac{ab}{4}\,x^2\right). \quad (3)$$

Here ω_0 is the eigenfrequency of a nonlinear oscillator; b is the coefficient of nonlinearity, and a is the control parameter.

2.2 Selection of the control parameter

We distinguish two cases.

1. Switching system. Assume that the control parameter is defined by the switching system: as a increases from 0 to 2, the coefficient of linear rigidity decreases from its maximum positive value $m\omega_0^2$ to its largest negative value $-m\omega_0^2$. Under this condition, the values of the control parameter lie between

$$0 \leqslant a \leqslant 2\,. \quad (4)$$

Below we distinguish three characteristic values of the control parameter: $a = 0$, $a = 1$, $a = 2$.

2. Temperature control. Rigidity depends on temperature. At a certain 'critical temperature' it becomes zero, and then negative as the temperature further decreases. This behaviour is possible if the bistability is caused, for example, by a phase transition of the second order in the dielectric of the capacitor in the electric circuit. For the purposes of a qualitative treatment, the temperature dependence can be expressed as

$$1 - a = \tanh\frac{T - T_{\mathrm{c}}}{\Delta T}\,. \quad (5)$$

The quantity ΔT characterizes the 'width' of the critical region in terms of temperature.

We see that in this case there are two possibilities of controlling the bistable element: by varying the noise intensity (temperature), and by varying the control parameter a.

2.3 Equilibrium solution and diffusion coefficients

Now let us return to the generalized kinetic equation (1). At $F(t) = 0$, it has the equilibrium solution in the form of a canonical Gibbs distribution (or combined Maxwell – Boltzmann distribution) with the appropriate Hamilton function:

$$f(x,v) = C\exp\left(-\frac{H(x,v)}{k_{\mathrm{B}}T}\right), \qquad H(x,v) = \frac{mv^2}{2} + U(x)\,. \quad (6)$$

The potential energy is given by Eqn (3).

The dissipation effect in Eqn (1) is caused by diffusion with respect to both velocity and coordinate. In case of a linear (at $a = 0$) oscillator, the appropriate diffusion coefficients are given by the Einstein relations

$$D_{(v)} = \gamma \frac{k_B T}{m}, \qquad D_{(x)} = \frac{k_B T}{\gamma m}. \tag{7}$$

At $F(t) = 0$, the two diffusion terms describe evolution towards equilibrium with respect to both velocity and coordinate.

3. Diffusion approximation

3.1 Diffusion equation

For $a = 0$, the relaxation times $\tau_{(v)}$ and $\tau_{(x)}$ are expressed by

$$\tau_{(v)} \approx \frac{1}{\gamma}, \qquad \tau_{(x)} \approx \frac{\gamma}{\omega_0^2} \equiv \frac{1}{\Gamma}, \qquad \Gamma = \frac{\omega_0^2}{\gamma}. \tag{8}$$

Hence it follows that, depending on the value of dimensionless parameter γ/ω_0, we may identify two limiting cases.

The first corresponds to the inequality

$$\frac{\tau_{(x)}}{\tau_{(v)}} \approx \frac{\gamma^2}{\omega_0^2} \gg 1 \quad \text{for} \quad \gamma \gg \omega_0. \tag{9}$$

In this case, the volume diffusion is much slower than the diffusion with respect to velocities.

This allows going over from the generalized Fokker–Planck equation to the Einstein–Smoluchowski equation for the distribution function $f(x, t)$:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left(D_{(x)} \frac{\partial f}{\partial x} - \frac{F(x,t)}{\gamma m} f\right), \qquad \int f(x,t)\,\mathrm{d}x = 1. \tag{10}$$

We shall call this limit 'the diffusion approximation'. The equilibrium solution of Eqn (10) is the Boltzmann distribution function.

3.2 Three characteristic values of the control parameter

Three characteristic states may be identified depending on the value of the control parameter:

(1) linear oscillator ($a = 0$);

(2) critical point ($a = 1$). At this value of the control parameter the linear rigidity of an oscillator becomes zero, and the potential energy is given by

$$U(x) = \frac{m\omega_0^2}{8} b x^4; \tag{11}$$

(3) bistable oscillator ($1 \leqslant a \leqslant 2$). When the negative value of the linear rigidity reaches its maximum — that is, when $a = 2$, the nonlinear potential takes the form

$$U(x) = \frac{m\omega_0^2 x^2}{2}\left(-1 + \frac{b}{2} x^2\right) \tag{12}$$

and the barrier separating two potential wells is the highest. The height of the barrier depends on two parameters, ω_0 and b. At $F(t) = 0$, the height of the barrier is expressed by

$$\Delta U = U(x_{\max}) - U(x_{\min}), \qquad x_{\max} = 0, \qquad x_{\min} = \pm\sqrt{\frac{1}{b}}. \tag{13}$$

In addition to the characteristic times (9), in the region of bistability there appears another characteristic time — the time taken by the transition between the two stable states. This time is commonly referred to as the 'Kramers time'. When the noise intensity is not high (low temperatures), this time depends exponentially on the height of the barrier, and is much greater than the relaxation times for the linear oscillator.

In measuring the response to a small external force (it is only such perturbations that we are going to consider), the identification of the stationary state requires that the observation time be greater than any relaxation times, including the Kramers time. Under this condition it is possible to regard the transitions over the barrier in the presence of noise (temperature) as a slow diffusion process.

Let us quote a few more characteristic parameters. They are the amplitude of thermal displacements, and the dimensionless parameter ε:

$$x_T^2 = \frac{k_B T}{m\omega_0^2}, \qquad \varepsilon \equiv x_T^2 b < 1. \tag{14}$$

We can also evaluate the time of space diffusion at small values of the control parameter:

$$\tau_{D_{(x)}} \approx \frac{x_T^2}{D_{(x)}} \approx \frac{\gamma}{\omega_0^2} = \frac{1}{\Gamma} \approx \tau_{(x)}. \tag{15}$$

As the control parameter a increases, we get yet another characteristic time — the Kramers time, which characterizes the process of switching between the states of the bistable oscillator.

3.3 Calculation of variance. Self-consistent approximation in the second moment

The exact value of the variance at equilibrium is given by

$$\langle x^2 \rangle = \int x^2 f_0(x)\,\mathrm{d}x, \tag{16}$$

where $f_0(x)$ is the Boltzmann distribution with a bistable potential (9). For the purposes of physical analysis, we once again select three values of the control parameter and consider the self-consistent approximation in the second moment. This approximation was used for calculating the fluctuations in a van der Pol oscillator and in lasers, as well as in the theory of phase transitions of the second order [12–14].

Assume that at $F(t) = 0$ we have an ensemble of non-interacting bistable oscillators. Then the occupancy of the two potential wells is equiprobable, and therefore the first moment is zero, $\langle x \rangle = 0$. Accordingly, we must turn to the equation in the second moment.

We denote the square of displacement by $x^2 = E$, and use the self-consistent approximation in the second moment of variable x:

$$\langle x^4 \rangle \equiv \langle E^2 \rangle \to \langle E \rangle \langle E \rangle. \tag{17}$$

Then from the diffusion equation we get a closed equation in $\langle E \rangle$:

$$\frac{\mathrm{d}\langle E \rangle}{\mathrm{d}t} = 2\left[D_{(x)} - \Gamma\left(1 - a + \frac{ab}{2} b\langle E \rangle\right)\langle E \rangle\right]. \tag{18}$$

Observe that relaxation of the second moment to the steady state at $a = 0$ occurs over a time characteristic of space diffusion [see Eqn (15)]. Let us demonstrate that in the presence of an external force, when the mean displacement $\langle x\rangle$ is no longer equal to zero, its relaxation time can be much greater than that for the second moment. This allows us to neglect the time derivative in the equation for $\langle E\rangle$ and rewrite it in the form

$$\langle E\rangle\left[(1-a)+\frac{ab}{2}\,\langle E\rangle\right]=\frac{D_{(x)}}{\Gamma}=x_T^2\,,\qquad \langle E\rangle=\left\langle x^2\right\rangle. \quad (19)$$

The values of variance from this equation for three selected values of the control parameter are close to those given by the exact expression (16).

3.4 Effective Gaussian approximation

An analytical solution of the diffusion equation (10) for a bistable oscillator for arbitrary values of the control parameter is not feasible. The simplified model equations are therefore useful.

We shall treat the transitions over the potential barrier in the presence of noise as a diffusion process with the Einstein relation $\langle x^2\rangle = D_{(x)}\tau_{(x)}$. Then in place of Eqn (10) we get a simpler model equation

$$\frac{\partial f}{\partial t}=\frac{\partial}{\partial x}\left[D_{(x)}\,\frac{\partial f}{\partial x}+\left(\frac{1}{\tau_{(x)}}\,x-\frac{F(t)}{\gamma m}\right)f\right],\qquad \left\langle x^2\right\rangle=D_{(x)}\tau_{(x)}\,. \quad (20)$$

Its equilibrium solution for $F(t)=0$ is the Gaussian distribution function

$$f(x)=\sqrt{\frac{1}{2\pi\langle x^2\rangle}}\,\exp\left(-\frac{x^2}{2\langle x^2\rangle}\right). \quad (21)$$

The variance $\langle x^2\rangle$ in the self-consistent approximation in the second moment is given by Eqn (19) for arbitrary values of the control parameter. Another parameter is the temperature (noise intensity). The model diffusion equation (20) is valid for arbitrary values of these two parameters. It will be convenient to represent the Gaussian distribution (21) in the more customary form of a Boltzmann distribution for the effective linear oscillator

$$f(x)=\sqrt{\frac{m\omega_{\mathrm{eff}}^2}{2\pi k_{\mathrm B}T}}\,\exp\left(-\frac{m\omega_{\mathrm{eff}}^2x^2}{2k_{\mathrm B}T}\right). \quad (22)$$

The effects of nonlinearity are taken into account by introducing the effective frequency ω_{eff}. Comparing the distributions (21), (22), we get the following relation between the effective frequency squared and variance:

$$\omega_{\mathrm{eff}}^2=\omega_0^2\,\frac{x_T^2}{\langle x^2\rangle}\,,\qquad x_T^2=\frac{k_{\mathrm B}T}{m\omega_0^2}\,. \quad (23)$$

Using the above formulae, we find ω_{eff}^2 for the three selected values of the control parameter.

4. Stochastic filtering

4.1 Relaxation of the first moment in the self-consistent approximation in the second moment

In order to obtain the equation for describing the evolution of the first moment under a small external force, we shall use the approximate diffusion equation (20). As a result, for a bistable oscillator at the largest arbitrary value of the control parameter and in the presence of a weak external field, we get the following equation in $\langle x\rangle$:

$$\frac{\mathrm d\langle x\rangle}{\mathrm dt}+\frac{1}{\tau_{(x)}}\,\langle x\rangle=\frac{F(t)}{m\gamma}\equiv f(t)\,,\qquad \frac{1}{\tau_{(x)}}\equiv\Delta\omega_{(x)}=\frac{D_{(x)}}{\langle x^2\rangle}\,. \quad (24)$$

The variance $\langle x^2\rangle$ (for low strengths of the external force) at all values of the control parameter is determined by the solution of Eqn (19).

In this way, we have a closed system of equations for the first and the second moments. Recall that the time dependence is neglected in Eqn (19). This is justified when the control parameter is in the range $1\leqslant a\leqslant 2$.

4.2 Response to external force. Stochastic filtering

Equation for the first moment accounts for the nonlinearity of the initial equation for an overdamped oscillator through the definition of relaxation time.

Let us write the expression for the response to the external force:

$$\langle x\rangle_\omega=\chi(\omega)\,\frac{F(\omega)}{\gamma m}\,,\qquad \chi(\omega)=\frac{1}{-\mathrm i\omega+1/\tau_{(x)}}\,. \quad (25)$$

The real part of the susceptibility is an example of the Rayleigh spectral line

$$\operatorname{Re}\chi(\omega)=\frac{1/\tau_{(x)}}{\omega^2+\left(1/\tau_{(x)}\right)^2}\,,\qquad \frac{1}{\tau_{(x)}}=\frac{D_{(x)}}{\langle x^2\rangle}\,. \quad (26)$$

The above formulae define the linewidth when the control parameter is in the range $0\leqslant a\leqslant 2$.

Now we define the input signal as

$$F(t)=A\cos\Omega t\,,\qquad F(\omega)=\pi A\left[\delta(\omega+\Omega)+\delta(\omega-\Omega)\right]. \quad (27)$$

Then the response $\langle x(t)\rangle$ at frequency Ω is given by

$$\left\langle x(t)\right\rangle_\Omega=A\operatorname{Re}\chi(\Omega)\cos\Omega t\,. \quad (28)$$

Upon averaging over the period $2\pi/\Omega$, we arrive at

$$\sqrt{\left\langle\left\langle x(t)\right\rangle_\Omega^2\right\rangle^{2\pi/\Omega}}=\frac{A}{\sqrt2}\operatorname{Re}\chi(\Omega)\,. \quad (29)$$

The linewidth $\operatorname{Re}\chi(\Omega)$ (filter passband $\Delta\omega_{\langle x\rangle}=D_{(x)}/\langle x^2\rangle$) through the values of $\langle x^2\rangle$ depends both on the control parameter a and on the noise intensity (temperature). The mean value $\langle E\rangle=\langle x^2\rangle$ is found by solving Eqn (19). Let us consider three characteristic values of the control parameter.

1. Linear oscillator ($a=0$). Then

$$\langle E\rangle=x_T^2\,,\qquad \frac{1}{\tau_{\langle x\rangle}}=\frac{D_{(x)}}{x_T^2}=\Gamma\,. \quad (30)$$

Naturally, in this case the relaxation time for the first moment coincides with the corresponding time for a linear oscillator. Important changes occur at the critical point.

2. Critical point ($a=1$). Then the solution of Eqn (19) is

$$\langle E\rangle=x_T^2\sqrt{\frac{2}{x_T^2b}}\,,\qquad \frac{1}{\tau_{\langle x\rangle}}=\sqrt{\frac{x_T^2b}{2}}\,\frac{D_{(x)}}{x_T^2}<\frac{D_{(x)}}{x_T^2}\,. \quad (31)$$

We see that at the critical point the relaxation time of the first moment becomes a function of the noise intensity (temperature). As the temperature decreases, the filter passband width narrows, and the relaxation time increases.

3. Bistable oscillator at the largest value of the control parameter ($a = 2$). The solution of Eqn (19) is

$$\langle E \rangle = \frac{1}{b}, \quad \frac{1}{\tau_{\langle x \rangle}} \equiv \Delta\omega_{\langle x \rangle} = \Gamma x_T^2 b \ll \Gamma . \tag{32}$$

We see that in our self-consistent approximation in the second moment the filter passband width decreases linearly with falling temperature.

Calculations of this section used the Einstein relation $\langle x^2 \rangle = D_{(x)} \tau_{\langle x \rangle}$ for the bistable oscillator as well. This is natural for a critical region, where the barrier separating the two possible states is low. To allow for high barriers, we shall also use another approximation based on the definition of relaxation time as the time of overcoming the barrier. This will bring us to the known Kramers formula. First, however, we want to discuss the 'resonance' response to a periodic disturbance and features of the spectral density of fluctuations as functions of noise intensity.

4.3 'Resonance' in the case of stochastic filtering

Let us return to Eqn (25) which defines the response to a periodic external force. The same function also defines the spectral density of fluctuations δx at $F(t) = 0$:

$$(\delta x \delta x)_\omega = \frac{2/\tau_{(x)}}{\omega^2 + (1/\tau_{(x)})^2} \langle x^2 \rangle \equiv 2 \operatorname{Re} \chi(\omega) \langle x^2 \rangle . \tag{33}$$

Let us consider by way of example the dependence of $\operatorname{Re}\chi(\omega)$ on noise intensity in the case of bistable behaviour at $a = 2$. Using Eqn (19), we find

$$\langle x^2 \rangle = \frac{1}{b} \quad \text{and} \quad \operatorname{Re}\chi(\omega) = \frac{\Gamma x_T^2 b}{\omega^2 + (\Gamma x_T^2 b)^2}, \quad x_T^2 = \frac{k_B T}{m\omega_0^2} . \tag{34}$$

We see that with fixed frequency ω the temperature dependence (dependence on the noise intensity) is nonmonotone. The maximum value is observed when

$$\frac{k_B T_{\text{res}}}{m\omega_0^2} = \omega^2 . \tag{35}$$

Since the response to a periodic perturbation is expressed via the function $\operatorname{Re}\chi(\omega)$ too, it also exhibits a 'resonance' dependence on the noise intensity. This effect could be aptly termed 'stochastic filtering', since we are actually dealing with the dependence of the filter passband width on the intensity of noise. The term 'stochastic resonance' in this case is rather conventional, and ought to be reserved for the description of oscillation rather than relaxation processes in a bistable oscillator placed in a 'thermostat'.

Observe that the dependence of the above characteristics on the noise intensity in the neighborhood of 'resonance' is asymmetrical. With fixed frequency $\omega = \Omega$ the mean response is proportional to T for low noise intensities, and to $1/T^2$ when the intensity of noise is high. This agrees qualitatively with the results of numerical experiments quoted in the review [8].

So far the phenomenon of stochastic filtering has been treated using the self-consistent approximation in the second moment. Let us now consider the approximation based on the Kramers theory — it is this approximation that is used in the conventional theory of 'stochastic resonance'.

5. Kramers theory and stochastic filtering

5.1 Introductory remarks

The Kramers problem is the subject of many papers and several reviews. A comprehensive review [15] was published to mark the 50th anniversary of Kramers' classic paper [16]. For the sake of justice we ought to mention here that the seminal work in this field was co-authored by L Pontryagin, A Andronov and A Vitt in 1933 [17], long before the paper by Kramers.

The problem of the time of transition over the barrier can be solved in two ways: by solving the time-domain equation (10) for a bistable oscillator, or by solving the stationary diffusion equation with an appropriate 'source' and 'sink' that give a constant flow of Brownian particles. A relatively simple technique for solving the stationary problem was used in Section 17.6 of the book [13]. Let us quote the pertinent results.

We use Eqn (10) for the stationary state. In this equation we carry out one integration and define the constant flux of particles by $j_0 \equiv 1/\tau_{\text{tr}}$. The flux has the dimension of inverse time; τ_{tr} denotes the transition time. As a result, we get a first-order equation

$$D_{(x)} \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{1}{\gamma m} \frac{\mathrm{d}U(x)}{\mathrm{d}x} f = \frac{1}{\tau_{\text{tr}}}, \tag{36}$$

where, as before, $D_{(x)} = k_B T / m\gamma$ is the coefficient of space diffusion, and τ_{tr} is the time of passing over the barrier. The solution of this equation brings us to the following definition of the transition time

$$\tau_{\text{tr}} = \frac{1}{D_{(x)}} \int_{-\infty}^{0} \exp\left(-\frac{U(x)}{k_B T}\right) \left(\int_x^0 \exp \frac{U(x')}{k_B T} \, \mathrm{d}x'\right) \mathrm{d}x . \tag{37}$$

Let us consider again three characteristic cases.

1. Linear oscillator ($a = 0$)

$$\frac{U(x)}{k_B T} = \frac{x^2}{2x_T^2}, \quad \tau_{\text{tr}} \approx \frac{x_T^2}{D_{(x)}} . \tag{38}$$

Comparing this result with Eqn (37), we get the following limit for the transition time

$$\tau_{\text{tr}} \approx \frac{x_T^2}{D_{(x)}} = \frac{\gamma}{\omega_0^2} = \frac{1}{\Gamma} = \tau_{D_{(x)}} = \tau_{(x)} . \tag{39}$$

We see that in the limit of monostable oscillator, the transition time coincides with the above-defined diffusion time [see Eqn (15)].

2. Critical point ($a = 1$)

$$U(x) = \frac{m\omega_0^2}{8} b x^4, \quad \tau_{\text{tr}} \approx \sqrt{\frac{1}{x_T^2 b}} \frac{x_T^2}{D_{(x)}} . \tag{40}$$

We see that the transition time at the critical point is of the same order as the diffusion time (31) in the self-consistent approximation with respect to the second moment. As we go deeper into the bistable region (as $a \to 2$), however, the discrepancy between the two differently calculated relaxation times increases exponentially.

3. Bistable oscillator at the maximum value of the control parameter ($a = 2$). Then Eqn (37) can be reduced to

$$\tau_{\rm tr} \approx \sqrt{2}\,\pi\,\frac{x_T^2}{D_{(x)}}\exp\frac{1}{4x_T^2 b}\,. \tag{41}$$

The corresponding expression for the transition time is written as

$$\tau_{\rm tr} \approx \frac{\sqrt{2}\,\pi}{\Gamma}\exp\frac{1}{4x_T^2 b} \geqslant \frac{1}{\Gamma}\,. \tag{42}$$

Let us apply these results to the formulation of the model equation of spatial diffusion.

5.2 Response to an external force. Stochastic filtering

The equation in the first moment in the Kramers theory is

$$\frac{{\rm d}\langle x\rangle}{{\rm d}t} + \frac{1}{\tau_{\rm tr}}\langle x\rangle = \frac{F(t)}{\gamma m}\,. \tag{43}$$

It differs from Eqn (24) by the replacement

$$\frac{1}{\tau_{(x)}} = \frac{D_{(x)}}{\langle x^2\rangle} \to \frac{1}{\tau_{\rm tr}}\,. \tag{44}$$

Recall that, in the self-consistent approximation with respect to the second moment, the quantity $\langle x^2\rangle$ for all values of the control parameter is given by the solution of Eqn (19).

Equation (43) takes into account the nonlinearity of the initial equation for an overdamped oscillator through the Kramers time, the calculation of which depends essentially on the nonlinearity of the bistable oscillator.

Now we apply substitution (44) to Eqns (25), (26), getting as a result the desired formulae for describing stochastic filtering in the Kramers theory language:

$$\langle x\rangle_\omega = \chi(\omega)\,\frac{F(\omega)}{\gamma m}\,, \qquad \chi(\omega) = \frac{1}{-{\rm i}\omega + 1/\tau_{\rm tr}}\,. \tag{45}$$

Hence follows the expression for the real part of the susceptibility, which is an example of the Rayleigh spectral line:

$$\operatorname{Re}\chi(\omega) = \frac{1/\tau_{\rm tr}}{\omega^2 + (1/\tau_{\rm tr})^2}\,. \tag{46}$$

Formulae (38) – (41) define the filter passband and the shape of the 'resonance curve' of the overdamped oscillator for all values of the control parameter in the range $0 \leqslant a \leqslant 2$.

The filter passband and the shape of the 'resonance curve' depend on both the control parameter a and the noise intensity (temperature). With a fixed noise intensity, the filter passband narrows down as the control parameter increases. Alternatively, if the control parameter is fixed, the filter passband narrows down as the temperature decreases.

In the region of bistability at the largest value of the control parameter $a = 2$, the difference between the results derived by the above two distinctive approaches depends only on the value of one dimensionless parameter $x_T^2 b$ that defines the height of the barrier in the Kramers formula

$$\frac{U(x_{\max}) - U(x_{\min})}{D_{(x)}} = \frac{1}{4x_T^2 b}\,. \tag{47}$$

When this dimensionless parameter is $4x_T^2 b \geqslant 1$ ('high temperatures'), both methods of describing stochastic filtering yield close results. When the temperatures are 'low', $4x_T^2 b \ll 1$, we get into the range of frequencies so low that flicker noise becomes an important factor.

From the arguments developed above it follows that the widely held belief in the concept of 'stochastic resonance' in an overdamped bistable oscillator in the presence of noise is not well grounded. The traditional concept of 'resonance' as typical of oscillatory systems is used here for describing the nonmonotone dependence of the response on the intensity of noise in a relaxing system. This does not mean of course that there is no stochastic resonance as such. It is only that stochastic resonance occurs under conditions opposite to those described above. Namely, we ought to consider a low-friction oscillator rather than the overdamped bistable oscillator. With this purpose, in place of the diffusion equation we shall use the Fokker – Planck equation for the distribution function $f(x, v, t)$.

6. Stochastic resonance

For describing the actual stochastic resonance we shall use the Fokker – Planck equation

$$\frac{\partial f}{\partial t} + v\,\frac{\partial f}{\partial x} + \frac{F(x,t)}{m}\,\frac{\partial f}{\partial v} = \frac{\partial}{\partial v}\left[D_{(v)}\,\frac{\partial f}{\partial v}\right] + \frac{\partial}{\partial v}\,[\gamma v f]\,. \tag{48}$$

Under the condition $F(t) = 0$, the first moments are zero; in the weak field the equations for the first moments are

$$\frac{{\rm d}\langle x\rangle}{{\rm d}t} = \langle v\rangle\,, \qquad \frac{{\rm d}\langle v\rangle}{{\rm d}t} + \gamma\langle v\rangle + \omega_{\rm eff}^2\langle x\rangle = \frac{F(t)}{m}\,. \tag{49}$$

Here we again use the notation for the effective frequency (or effective rigidity). In the self-consistent approximation with respect to the second moment for the arbitrary value of the control parameter the quantity $\omega_{\rm eff}^2$ is given by

$$\omega_{\rm eff}^2 = \omega_0^2\left(1 - a + \frac{ab}{2}\langle E\rangle\right). \tag{50}$$

The set of equations in $\langle x\rangle$, $\langle v\rangle$ is not closed, since the effective frequency depends on $\langle E\rangle \equiv \langle x^2\rangle$. Let us show how these equations can be closed in the weak-field approximation, when the products of the first moments $\langle x\rangle$, $\langle v\rangle$ may be regarded as negligibly small.

Using the Fokker – Planck equation (48) we write a set of two equations

$$\frac{{\rm d}\langle x^2\rangle}{{\rm d}t} = 2\langle x\rangle\langle v\rangle\,, \qquad \langle x^2\rangle = \langle E\rangle\,, \tag{51}$$

$$\frac{{\rm d}\langle x\rangle\langle v\rangle}{{\rm d}t} - \langle v^2\rangle + \omega_0^2\left(1 - a + \frac{ab}{2}\langle E\rangle\right)\langle E\rangle - \gamma\langle x\rangle\langle v\rangle = \frac{\langle x\rangle F(t)}{m}\,. \tag{52}$$

The right-hand side of the former equation is small in the second order, which is negligible in our current approximation. Accordingly, the derivative of the second moment $\langle x^2\rangle$ is also negligibly small. In the same approximation, in the second equation we may drop all terms containing the first moments $\langle x\rangle$, $\langle v\rangle$. Then, given that $\langle v^2\rangle = k_{\rm B}T/m$, we get the following equation

$$\left(1 - a + \frac{ab}{2}\langle E\rangle\right)\langle E\rangle = x_T^2\,. \tag{53}$$

We have arrived at an equation which coincides with Eqn (19).

Now equations (49), (50), (53) comprise a closed set of equations. Let us consider again the solution of Eqn (53), as we did earlier with Eqn (19), for three characteristic values of the control parameter, and find the corresponding values for the square of effective frequency ω_{eff}^2.

1. Linear oscillator ($a = 0$). Then

$$\langle E\rangle = x_T^2\,, \qquad \omega_{\text{eff}}^2 = \omega_0^2\,. \tag{54}$$

2. Critical point ($a = 1$). One has

$$\langle E\rangle = x_T^2\sqrt{\frac{2}{x_T^2 b}}\,, \qquad \omega_{\text{eff}}^2 = \omega_0^2\sqrt{\frac{x_T^2 b}{2}}\,. \tag{55}$$

3. Bistable oscillator at the maximum value of the control parameter ($a = 2$). Then the solution is

$$\omega_{\text{eff}}^2 = \omega_0^2 x_T^2 b\,, \qquad x_T^2 = \frac{k_{\text{B}}T}{m\omega_0^2}\,. \tag{56}$$

As a result, we get the following equation for the bistable oscillator:

$$\frac{\mathrm{d}^2\langle x\rangle}{\mathrm{d}x^2} + \gamma\,\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} + \omega_0^2 x_T^2 b\langle x\rangle = \frac{F(t)}{m}\,. \tag{57}$$

Now we find the expression for the corresponding complex susceptibility, which we define as

$$\langle x\rangle_\omega = \chi(\omega)\,\frac{F(\omega)}{m}\,. \tag{58}$$

As a result we find

$$\chi(\omega) = \frac{1}{\omega_0^2 x_T^2 b - \omega^2 - \mathrm{i}\omega\gamma}\,. \tag{59}$$

From the above formulae it follows that tuning-in to resonance may be effected by adjusting the temperature (noise intensity). By T_{res} we denote the temperature corresponding to the point of resonance:

$$\omega^2 = \omega_{\text{eff}}^2(T_{\text{res}})\,. \tag{60}$$

In particular, at $a = 2$ the condition of resonance is expressed as

$$\omega^2 = \omega_0^2 x_{T_{\text{res}}}^2 b\,, \qquad x_{T_{\text{res}}}^2 = \frac{k_{\text{B}}T_{\text{res}}}{m\omega_0^2}\,. \tag{61}$$

At temperatures above or below T_{res}, the condition of resonance is not satisfied. This allows us to speak of the condition of stochastic resonance.

Observe that formally the effects of Kramers diffusion on the stochastic resonance may be taken into account by making the substitution

$$x_T^2 b \Leftrightarrow \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{4x_T^2 b}\right) = \frac{1}{\Gamma\tau_{\text{tr}}} \tag{62}$$

in the last of the above formulae. Then the condition of stochastic resonance at the largest value of the control parameter ($a = 2$) may be written as

$$\omega^2 = \frac{1}{\Gamma\tau_{\text{tr}}}\,. \tag{63}$$

Notice also that the condition of stochastic resonance for the model of an overdamped oscillator is set (without sufficient grounds) as

$$\omega \approx \frac{1}{\tau_{\text{tr}}}\,. \tag{64}$$

Usually it is assumed that stochastic resonance occurs when the signal frequency is of the order of the frequency of Kramers transitions. In the theory of stochastic resonance under consideration [when the Kramers theory is formally included through substitution (62)], the phenomenon of stochastic resonance takes place when

$$\omega \approx \omega_0\sqrt{\frac{1}{\Gamma\tau_{\text{tr}}}} \approx \sqrt{\frac{\gamma}{\tau_{\text{tr}}}} \gg \frac{1}{\tau_{\text{tr}}}\,. \tag{65}$$

This condition is different from Eqn (64). Stochastic resonance occurs at frequencies much higher than the Kramers frequency.

Although the reviews [8, 9] draw a comparison between theory and experiment for numerous physical, chemical and biological systems, it would be useful to extend this comparison with due account for the alternative descriptions of stochastic filtering and stochastic resonance as described above.

7. Conclusions

More than two decades have passed since the introduction of the concept of 'stochastic resonance' for describing the periodic motion of glaciers in the northern hemisphere of the Earth. The meaning of this term, as we have seen, is that the response of the system to the combined action of harmonic signal and noise exhibits a nonmonotone dependence on the noise intensity. The 'resonance' response to the noise gave rise to the term 'stochastic resonance'. The example of an overdamped bistable oscillator was used for formulating the 'oscillatory' interpretation of such 'resonance' — this takes place when the signal frequency coincides with the Kramers frequency (the frequency of transitions over the barrier separating the two minima of potential energy of the bistable oscillator).

Such a theory, however, has more to do with the imagination than with physical reality. As a matter of fact, an overdamped oscillator is a pure relaxation system that has no natural frequencies and, as a result, cannot exhibit resonance in the traditional sense.

Since, as we have seen, the electromechanical analog of the overdamped bistable oscillator is an RC circuit with a nonlinear capacitance — that is, a nonlinear filter, the phenomenon that has become known as stochastic resonance would be more appropriately termed 'stochastic filtering'. This term implies that the response of an overdamped bistable oscillator in the presence of signal and noise depends nonmonotonically on the intensity of noise. Then the stochastic filtering can be controlled by varying two independent parameters — the intensity of external noise and the temperature (for example, of the ferroelectric capacitor). The latter control is especially efficient in the temperature range adjacent to the point of phase transition in the dielectric substance filling the capacitor.

By contrast, stochastic resonance is only possible in a situation opposite to that of an overdamped bistable oscillator — that is, when the coefficient of friction is much less than the effective frequency of the oscillator. It was shown that the effective frequency is noise-dependent. This opens the possibility of tuning to resonance by varying the intensity of noise. This is the phenomenon that is referred to here as 'stochastic resonance'.

This commentary touches upon just one — even though important — aspect of the theory of controlling nonlinear systems by varying the noise intensity. In the reviews [8, 9], the reader will find a lot of interesting experimental and theoretical material illustrating the wealth of phenomena related to the simultaneous action of signal and noise in nonlinear systems of diverse nature. One of the most intriguing certainly is the phenomenon of noise-enhanced order at stochastic synchronization.

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Note added in proof

The special issue of *Chaos* magazine (September 1998) is to a large extent devoted to the problem of stochastic resonance. The original papers are preceded with a brief review entitled "The Constructive Role of Noise in Fluctuation Driven Transport and Stochastic Resonance" by R Dean Astumian and Frank Moss. Frank Moss is one of the authors of the review published in this issue of *Physics – Uspekhi*.

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