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# Decoherence and the theory of continuous quantum measurements

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<u>Abstract.</u> The decoherence of a quantum system, i.e., its becoming partly classical, results from its interaction with the environment and is well described in terms of continuous quantum measurement theory. In the present paper, various approaches to CQM theory are reviewed, of which that using effective complex Hamiltonians is discussed in detail. The effective complex Hamiltonian is obtained from the restricted path integral, the latter highlighting the role of information in the dynamics of the measured system, and is applied to the energy measurement of a two-level system. For such a measurement, quantum transition monitoring is shown to be possible and its back-action on the transition probability analyzed. The performance of such a measurement using a long series of soft observations is described.

# 1. Introduction

Continuous or repeated measurements of quantum systems have been actively discussed over the past decade — in the first place because this is where the intrinsic features of quantum theory are manifested to the utmost, and secondly because such measurements are steadily gaining practical importance [1-12]. Twenty years ago it was theoretically demonstrated [13-16], and later confirmed experimentally [17] that repeated measurements of a discrete observable lead

<sup>†</sup>Sections which may be omitted on a first reading are marked with an asterisk.

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Received 21 April 1998 Uspekhi Fizicheskikh Nauk **168** (9) 1017–1035 (1998) Translated by A S Dobroslavskii; edited by A I Yaremchuk to freezing of the system in the original state (the so-called quantum Zeno effect). If, however, the accuracy of each of the repeated measurements is not high, then their effect on the measured system is not as strong, and the continuously measured system is not frozen [2, 9]. Recently it was demonstrated [18, 19] that a soft continuous quantum measurement (CQM) is capable of monitoring a quantum transition, and an experiment was proposed in which the transition between two atomic levels induced by resonant radiation is monitored through observing a series of electron scatterings by the atom. Actually, instead of electron scattering one could use a series of short and weak interactions of the atom with any auxiliary system. Other quantum phenomena can be monitored in a similar way. Thus, soft continuous quantum measurements offer a new tool for experimental study of quantum processes, capable of giving an insight deeper than previously seemed possible.

We shall outline these new possibilities at the end of this review, and start it with a theoretical analysis of quantum measurement in general, and CQM in particular. We are going to give a thorough analysis of the phenomenon of decoherence of a quantum system — the process through which the system acquires classical features, and the environment stores information about the system [12]. The phenomenon of decoherence is associated with any measurement, and plays a decisive role in the dynamics of a system subjected to repeated or continuous quantum measurements.

Using a simple example we shall see how the measurement of a quantum system occurs in the process of interaction between the system and its environment, and why this interaction inevitably leads to decoherence of the measured system. We shall briefly discuss two models of quantum diffusion — that is, the continuous monitoring of the coordinate of the quantum particle.

Then we are going to present different phenomenological approaches to CQM, which can be studied without resorting to particular models of the surrounding (measuring) medium.

In special detail we shall discuss the phenomenological approach based on restricted integrals over paths (quantum corridors), which effectively reduces to the Schrödinger equation with a complex Hamiltonian. Subsequently we shall use this approach for analyzing the continuous measurement of the energy of a multilevel system and — in greater detail — a two-level one. We shall prove that such a measurement is capable of monitoring the quantum transition (the Rabi oscillations).

The relationship between quantum and classical descriptions of physical processes has been actively discussed since the early days of quantum mechanics, and still is. This issue is highly complicated because of the fundamental differences between the quantum and classical representations of physical systems and phenomena. The main distinction is apparently the quantum theoretical principle of superposition for the states of 'corpuscular' systems, for which in classical physics superposition is not possible. Because of this, it is not quite true that classical theory gives an approximate description of something that is more precisely rendered in terms of quantum theory. Niels Bohr believed that a complete description of physical phenomena ought to include classical elements in addition to a quantum representation.

Quantum mechanics assumes that, along with any two states of a quantum system  $|\psi_1\rangle$ ,  $|\psi_2\rangle$ , there exists also their quantum mechanical (also known as coherent) superposition  $|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle$ . If, however, the states  $|\psi_1\rangle$ ,  $|\psi_2\rangle$  differ considerably — are said to be classically distinctive — for example, relate to the states of an elementary particle localized at points far from each other, then their superposition is not observed under ordinary circumstances. Such states may be referred to as non-coherent.

To be more precise, a superposition of macroscopically distinctive states can be realized; but to have it survive for some time the system must be completely isolated. Even a very slight interaction with the environment will very quickly reduce the superposition  $|\psi\rangle$  to one of the stable states,  $|\psi_1\rangle$  or  $|\psi_2\rangle$ , with only the respective probabilities  $|c_1|^2$  and  $|c_2|^2$  known in advance. Such a conversion is the simplest example of *decoherence*. The fathers of quantum mechanics referred to this process as reduction or collapse. Today its nature is well understood, and the dynamics of this process are studied, the development in time.

A couple of decades ago the process of decoherence was mostly of academic interest. Currently, however, the experimental techniques in the field of, for example, quantum optics [20], have become so much refined that it has become possible to observe the formation of a superposition of macroscopically distinctive states and the subsequent process of decoherence [21, 22]. Moreover, the processes of decoherence must necessarily be included into a correct description of quantum systems interacting with the environment (open systems).

In particular, the processes of decoherence are important for the theory and practice of quantum computers [23, 24]. A quantum computer is a device capable of performing parallel computations by operating with a quantum superposition involving an enormous number of terms.

In quantum computers (whose elements have already been realized in practice) decoherence plays a dual role. In the course of calculations, decoherence is a harmful process, since the superposition has to be prevented from falling apart. However, when the computation is complete, its result must be retrieved from the computer and represented in the classical (that is, stable and steady) form. This is accomplished by an appropriate measurement of the state of computer as a quantum system. In other words, the computer is made to interact with a special device which acts as a measuring system. This device measures certain parameters of the state of the computer, thus causing the decoherence of this state. In this case the decoherence is brought about deliberately.

Hypothetical quantum computers are devices in which decoherence plays a crucial role. It is of no less importance, however, for many other quantum devices. The phenomenon of decoherence occurs whenever the system interacts or is made to interact with its environment, and the state of the system has some impact on the state of the environment. By observing the state of the environment one can then gain some information regarding the state of the system. Accordingly, the interaction of the system with its environment may be interpreted as the measurement of the system. The information about the system is recorded in the environment. Then we say that a measurement of the quantum system has taken place. The environment that performs this measurement can be created on purpose (a measuring device or measuring medium), although in many cases it exists beyond the experimenter's discretion, and often plays an undesirable role, leading to a special kind of dissipation.

It is important that such an interaction with the environment inevitably modifies the state of the quantum system, causing decoherence. It turns out, however, that the behavior of the quantum system being measured can be described with due account for its decoherence, without the need for explicit description of the measuring medium. The back effects of the environment are taken into account implicitly. This means that the system in question is treated as an open system, and its evolution is described phenomenologically.

The phenomenological theory of continuously measured quantum systems is an extension of conventional quantum mechanics and much augments its capabilities. This theory is closed and intrinsically elegant.

In this way, the answers to the questions "How does a quantum measurement occur?", and "How does a continuously measured system behave?" bring us to the theory of open continuously measured quantum systems.

From the general course of quantum mechanics we know that a measurement of a quantum system obeys von Neumann's reduction postulate, which in the simplest case is represented by the scheme

$$c_1|1\rangle + c_2|2\rangle \rightarrow \begin{cases} |1\rangle, & p_1 = |c_1|^2, \\ |2\rangle, & p_2 = |c_2|^2. \end{cases}$$

In this scheme,  $|1\rangle$  and  $|2\rangle$  are states corresponding each to a certain outcome of the measurement. According to the reduction postulate, one or the other measurement output are random quantities with respective probabilities  $p_1$  and  $p_2$ , and the system assumes the corresponding state. This instantaneous transition, which cannot be described by the Schrödinger equation, is known as reduction or collapse of the state of the system. Mathematically, the reduction can be described as the projection of the initial state vector on to the subspace of vectors proportional to either  $|1\rangle$  or  $|2\rangle$ .

In the more general case, the measurement is described by a set of projectors  $P_i$ , where the subscript *i* numbers the alternative outcomes of the measurement. If the measurement results in the *i*th alternative, the initial state  $|\psi\rangle$  after the measurement will go into state  $|\psi_i\rangle = P_i |\psi\rangle$  — that is, the reduction of state is described by the corresponding projector. The probability that the measurement will result in the *i*th alternative is $\dagger$ 

$$p_i = \|\psi_i\|^2 = \langle \psi_i | \psi_i \rangle = \langle \psi | P_i | \psi \rangle.$$
(1)

This most simple scheme of description of measurement raises a number of questions, the first of which is how and why the collapse takes place. The answer to this question has been sought by many authors (see, for example, Refs [25, 26, 10, 11] and an excellent review of the topical literature in the book [12]). A very clear analysis of the mechanism leading to collapse was done by Zurek [27, 28], where it was termed environment-induced superselection. We shall briefly touch upon this issue in Section 2.1.

Another question arises when we consider a series of successive quantum measurements each of which is described by von Neumann's reduction. How does a system behave when subjected to a series of such instantaneous measurements? How does this behavior change if the intervals between successive measurements tend to zero, so that the measurement becomes continuous? This paper for the most part is devoted to the discussion of these questions.

The analysis of these problems reveals that repeated von Neumann measurements of an observable with a discrete spectrum (for example, measurement of the energy of a multilevel system) lead to suppression of quantum transitions. As a result of continuous measurement, the system is completely frozen at one point of the spectrum. This phenomenon has attracted much interest and became known as the Zeno quantum paradox (or effect) [13–16].

The Zeno paradox indicates that continuous measurement may lead to trivialization, to the disappearance of dynamics. But is it always the case? The answer is negative. In the first place, the system is not frozen if the measured observable has a continuous spectrum. Secondly, the dynamics remain nontrivial even in the case of a discrete spectrum as long as the measurement is soft (not too accurate). From this standpoint, a too accurate measurement of the quantum system is not advantageous, and this is a manifestation of the paradoxical nature of quantum mechanics.

The dynamics of a quantum system subjected to continuous measurement is a new type of dynamics which is more general than that described by the conventional Schrödinger equation. It includes dissipation due to the effects of the environment. It is the dynamics of open quantum systems.

There are different approaches to the description of open (continuously measured) quantum systems. Later on we are going to discuss these in greater detail; at this point we shall just enumerate some of them.

— The model of measurement which includes the principal quantum system S, its environment (or measuring device)  $\mathcal{M}$ , and the interaction between them.

— The equation for the density matrix S of the system (master equation), obtained after summation over the degrees of freedom of the environment M (a special case of the Lindblad equation).

† Usually the state after the measurement is described in terms of a normalized vector. We, however, prefer using the vector  $|\psi_i\rangle$ , since the reduction then is represented in a very straightforward manner, and the norm of the resulting vector is equal to the probability of the respective outcome of the measurement. This approach is especially convenient for describing repeated measurements.

- Restricted path integrals (quantum corridors), which can be reduced to the Schrödinger equation with a complex Hamiltonian.

— The stochastic Schrödinger equation.

Finally, let us make two conceptual remarks. The theory of open continuously measured quantum systems throws new light on the old question whether or not quantum mechanics is a closed theory. The answer is affirmative if we are considering the Feynman formulation of quantum mechanics which is extremely rich in ideas. This ideological diversity allows us not to introduce the theory of quantum measurements as a special independent postulate, but rather derive it from the Feynman formulation of quantum mechanics.

The phenomenology of continuously measured quantum systems leads to the conclusion of the dynamic role of information in the following sense. Of course, the dynamics of the measured system are determined by the nature of the measuring medium and its interaction with the system. All the essential features of the dynamics, however, can be reconstructed using only the information about the system that is recorded in the environment. It is information that determines the dynamics.

# 2. Measurement of a quantum system by its environment

In this section we shall consider the physically more obvious 'straightforward' descriptions of the measurement of a quantum system, leaving for the next section the more abstract phenomenological approaches which have the advantage of being universal and independent of the model used.

# 2.1 Environment-induced superselection

As already mentioned, the behavior of a quantum system subjected to measurement (idealized, of course) is described by von Neumann's reduction postulate [29]. Let us consider very schematically the physical nature of the reduction (collapse) of state. To illustrate the main idea it will suffice to analyze the simplest measurement amounting to the choice between two alternatives. Assume, for example, that the measured observable A may take on one of the two values a', a''. Then, as a result of the measurement, the system, with the appropriate probability, will go over into one of the eigenstates of the observable:

$$\psi\rangle = c'|a'\rangle + c''|a''\rangle \rightarrow \begin{cases} |a'\rangle, & p_1 = |c'|^2, \\ |a''\rangle, & p_2 = |c''|^2. \end{cases}$$
(2)

The same change in terms of the density matrix is expressed as the transition of the density matrix  $\rho_0 = |\psi\rangle\langle\psi|$  of the pure state into the density matrix of the mixed state:

$$\rho_0 = \begin{pmatrix} |c'|^2 & c'c''^* \\ c'^*c'' & |c''|^2 \end{pmatrix} \to \begin{pmatrix} |c'|^2 & 0 \\ 0 & |c''|^2 \end{pmatrix} = \rho.$$
(3)

The resulting density matrix  $\rho$  contains the same information as the right-hand side of Eqn (2), including information about the transition probabilities.

The physical process leading to the transition (3) is called decoherence. It results in the conversion of a superposition of a set of states into a mixture of the same states. The mark of decoherence is the disappearance of the nondiagonal elements

$$\langle a'|\rho|a''\rangle = \langle a''|\rho|a'\rangle = 0.$$
<sup>(4)</sup>

After the measurement, the superposition of states  $|a'\rangle$  and  $|a''\rangle$  is no longer possible, and only one of them may exist with no admixture of the other. We speak then of *superselection* — prohibition of the superposition of states from a given set of subspaces (in our example it is two one-dimensional subspaces).

Now let us consider the physical mechanism which leads to decoherence and superselection. This mechanism consists in the interaction of the measured system with its environment (measuring device) according to the scheme:

Interaction leads to entanglement, to quantum correlation of the two systems, so that the state of one of them contains information about the state of the other. This is how this occurs.

As already mentioned, a situation equivalent to the measurement of a quantum system is often encountered even in case when the measurement was not aimed by the experimenter. Nevertheless, for the sake of simplicity we shall speak of a device  $\mathcal{M}$  that measures the system  $\mathcal{S}$ . Assume that the device prior to the interaction is in the state  $|\phi_0\rangle$ . Interaction between the two systems is referred to as measurement when it results in a specific correlation between these two systems, so that the information about one system is recorded in the state of the other. In our case, the interaction between the two subsystems must translate the state  $|a'\rangle|\phi_0\rangle$  of the compound system into  $|a'\rangle|\phi'\rangle$ , and the state  $|a''\rangle|\phi_0\rangle$  into  $|a''\rangle|\phi''\rangle$ . Then the state of the device after the measurement will tell us about the state of the measured system.

Assume that prior to the measurement the composite system, comprising S and M as subsystems, occurs in the state  $|\Psi_0\rangle = |\psi\rangle |\phi_0\rangle$ , where  $|\psi\rangle$  is the state of the measured system discussed above. Then the interaction between the two subsystems will result in the following change of state of the composite system:

$$\begin{split} |\Psi_{0}\rangle &= |\psi\rangle|\phi_{0}\rangle = \left(c'|a'\rangle + c''|a''\rangle\right)|\phi_{0}\rangle \\ &\to c'|a'\rangle|\phi'\rangle + c''|a''\rangle|\phi''\rangle = |\Psi\rangle \,. \end{split}$$
(5)

Observe that, by contrast to the reduction of states (2) or (3), the transition (5) occurs by way of conventional quantum mechanical evolution, and can be described by the Schrödinger equation. This evolution results in the state  $|\Psi\rangle$  of the composite system, in which the subsystems S and M are entangled or, in other words, a quantum correlation is established between them.

Constructing the density matrix  $|\Psi\rangle\langle\Psi|$  of the composite system after the measurement and calculating its trace with respect to all degrees of freedom of the measuring device, we obtain the density matrix of the measured system:

$$\rho = \operatorname{Tr}_{\phi} |\Psi\rangle \langle \Psi| = |c'|^2 |a'\rangle \langle a'| + |c''|^2 |a''\rangle \langle a''| + \langle \phi'' | \phi'\rangle c' c''^* |a'\rangle \langle a''| + \langle \phi' | \phi''\rangle c'^* c'' |a''\rangle \langle a'|.$$
(6)

In this expression the off-diagonal matrix elements are nonzero. The following analysis indicates, however, that in reality they are negligibly small, and condition (4) is satisfied to a high degree of accuracy.

A large (macroscopic) number of degrees of freedom is a prerequisite for any measuring device, as is the fact that its states corresponding to different outcomes of the measurement (in our case  $|\phi'\rangle$  and  $|\phi''\rangle$ ) are 'macroscopically distinctive'. This means that the corresponding wave functions depend on very many variables, and exhibit different functional dependences on the large number of these variables. The scalar product of such wave functions is practically equal to zero (to be more precise, it is exponentially small with an exponent of the order of  $10^{-23}$ ). The reason is that the scalar product is an integral with respect to an enormous (macroscopic) number of variables. Even if the integral with respect to each separate variable is a little less than one, the total multiple integral will be close to zero. Hence, to a high degree of accuracy we have

$$\langle \phi' | \phi'' \rangle = \langle \phi'' | \phi' \rangle = 0.$$
<sup>(7)</sup>

As a result, the off-diagonal terms of the density matrix vanish, and it becomes

$$\rho = \operatorname{Tr}_{\phi} |\Psi\rangle \langle \Psi| = |c'|^2 |a'\rangle \langle a'| + |c''|^2 |a''\rangle \langle a''|$$
(8)

in accordance with Eqn (4). In this way, the measurement leads to decoherence or superselection. The nature of this phenomenon has rather long been understood (see, for example, the excellent papers [30-32]). Zurek furthered the analysis of this phenomenon and aptly christened it 'environment-induced superselection' [28].

We did not go into the details of the interaction between the measured system and its environment leading to transition (5). The analysis of relevant models (see book [12] and references therein) reveals that decoherence arises (that is, the off-diagonal terms vanish) exponentially fast in accordance with the expression

$$\left| \langle a'' | \rho(t) | a' \rangle \right|^2 \sim \exp\left[ -\kappa (a'' - a')^2 t \right].$$
(9)

This occurs as more and more degrees of freedom of the environment get entangled with the measured system. As follows from Eqn (9), the characteristic time  $t_d$  of decoherence is inversely proportional to the squared difference between the measured values of the observable,

$$t_{\rm d} \sim (a'' - a')^{-2}$$
. (10)

By the way, this explains why it is practically impossible to realize a superposition of two states of a particle which are localized at points far from each other. Even if such a superposition were to arise, it would very quickly suffer decoherence owing to the interaction with the environment from which it cannot be completely isolated.

# 2.2 Models of continuous measurements

The physically most intelligible approach to the description of quantum measurements, including continuous measurements, relies on some model of the process of measurement in accordance with the general scheme outlined in Section 2.1. The model must include the principal quantum system S, its environment  $\mathcal{M}$ , and the interaction between them. If the

macroscopic number of degrees of freedom of the environment is explicitly taken into account, then the model becomes rather complicated.

A large variety of models of quantum measurements have been proposed (see, for example, Refs [33, 34, 30, 31, 35, 28, 36-38, 4, 3, 39, 7, 40, 12]). Without going into the details, we are going to describe and compare two models of one and the same continuous measurement which consists in monitoring the coordinate of a pointlike particle. The first of these models was proposed as part of a theory of quantum diffusion [36]. Consider a pointlike quantum particle interacting with the atoms in crystal lattice as shown in the following diagram of decoherence by crystal modes:

#### Decoherence by crystal modes

(Caldeira & Leggett, 1983)



The 'particle' in our context is the measured system S, and the 'crystal' is its environment M. Such interaction results in a correlation between the coordinate of the particle and the state of the crystal (in other words, the state of the phonons that represent the motion of crystal atoms). The particle then suffers decoherence, its state is described by the density matrix, and the time evolution is described by equation<sup>†</sup>

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] - \frac{1}{2}\kappa[\mathbf{r},[\mathbf{r},\rho]], \qquad (11)$$

where the coefficient

$$\kappa = \frac{2\eta k_{\rm B}T}{\hbar^2} \tag{12}$$

depends on the temperature of the crystal T and the damping coefficient  $\eta$ , the same as enters the classic equation of motion of particle in a medium

$$m\ddot{\mathbf{r}} + \eta\dot{\mathbf{r}} + V'(\mathbf{r}) = F(t).$$
(13)

Another model of motion of a particle through a medium was specially designed for describing the continuous measurement of the particle's coordinate [40, 41]. This model assumes that the interaction with the particle excites the internal degrees of freedom of the atom, and the decoherence occurs through interaction with these degrees of freedom rather than with the modes corresponding to the displacement of atoms. This model can be schematically represented by the following diagram of decoherence by the internal structure of atoms:

**Decoherence by the internal structure of atoms** (Konetchnyi, Mensky & Namiot, 1993)

			·		-			·						
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	2	0	$\stackrel{\circ}{\sim}$	%	6	0
0	0	%	$^{\circ}$	6	م	0	ৎ	0	/0	$\checkmark$	0	~o	0	0
0	Ŷ	0	0	0	0	0	0	Y	0	0	0	0	0	0
ó	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	ο	0	ο	ο	0	ο	ο	0	ο	0	0	0	0	0

This model also leads to an equation of the form of Eqn (11), but the coefficient

$$\kappa = \frac{2}{\lambda^2 \tau} \tag{14}$$

in this case will depend on the distance  $\lambda$  at which the atom reacts to the particle, and the relaxation time  $\tau$  of the atom excited by the passage of the particle. Under certain conditions, this second mechanism of decoherence will prevail.

For our future discussion it is important that both models lead to Eqn (11) for the density matrix, in which the effect of decoherence is represented by the double commutator of the density matrix with the measured observable (which in our case is the position of the particle). As will be demonstrated in Section 3, such an equation, which represents the phenomenological description of a continuous quantum measurement, can be derived without using the model of interaction.

# **3.** Phenomenology of continuously measured systems

As demonstrated in Section 2, the behavior of a quantum system subjected to continuous measurement can be derived by considering the model of the environment. It is possible, however, to do without such a model, taking the effects of the environment into account implicitly in accordance with the scheme



We shall discuss different approaches to this phenomenological description of continuous quantum measurements (CQM), paying special attention to the method of restricted path integrals.

#### 3.1 Approaches to describing continuous measurements

A continuous quantum measurement can be approximated by a series of repeated instantaneous measurements, each of which is described by the von Neumann projector. Evolution of the system between two measurements is described by the Schrödinger equation, or (which is equivalent) by the unitary evolution operator. This gives us the following law of evolution between the times  $t_0$  and  $t = t_N$ :

$$|\psi(t)\rangle = U(t_N, t_{N-1})P_{i_{N-1}}U(t_{N-1}, t_{N-2})\dots$$
  
...  $U(t_3, t_2)P_{i_2}U(t_2, t_1)P_{i_1}U(t_1, t_0)|\psi(t_0)\rangle.$  (15)

<sup>&</sup>lt;sup>†</sup>A more general equation can be found in Ref. [36]. We give a simplified expression whose validity is subject to certain restrictions on the parameters of the system.

The result of a series of measurements is represented by a sequence of numbers  $\{i_1, i_2, \ldots, i_{N-1}\}$ . We see that the evolution of the measured system depends on these numbers — that is, on the result of the series of measurements. The description can be made more realistic if we represent the instantaneous measurement not by the von Neumann projectors, but rather by their generalizations — positive operators. The spectrum of possible measurement outputs can then be made continuous.

Passing to the limit from this formalism for repeated measurements, one can obtain a tool for studying continuous quantum measurements (see, for example, Refs [42, 2, 43]). The resulting approach is phenomenological and does not require an explicit model of the measuring medium. The continuously measured system is then assumed to be open, and the effects of the environment are taken into account in an implicit way.

There are quite a few phenomenological approaches that lead straight to the continuous measurement, without first representing it as a repeated measurement with subsequent passage to the limit. Let us list the main ones.

1. The equation for the density matrix (master equation) of a continuously measured system S can be represented in the form

$$\dot{\rho} = -\frac{\mathrm{i}}{\hbar} [H,\rho] - \frac{1}{2} \kappa [A,[A,\rho]], \qquad (16)$$

where *H* is the Hamiltonian of the measured system, *A* is the continuously measured observable of the system — that is, the observable whose information is 'recorded' in the state of the environment, and the constant  $\kappa$  characterizes the strength of interaction between the measured system and its environment. Equation (16) is a special case of the Lindblad equation [44] which describes a more general class of open systems. In Section 2.2 we saw that an equation of this form can be derived from the model of interaction between the system and its environment. In Ref. [44], however, the Lindblad equation was derived without using any particular model, under the assumption of the Markovian nature of the process. Interpretation of the special case (16) of the Lindblad equation as describing a continuous measurement can only be corroborated either by the models of measurement, or with the aid of restricted path integrals (see the next paragraph).

2. Restricted path integrals or quantum corridors [9] reduce the description of continuously measured system S to the Schrödinger equation with a complex Hamiltonian,

$$\left|\dot{\psi}\right\rangle = \left[-\frac{\mathrm{i}}{\hbar}H - \kappa \left(A - a(t)\right)^{2}\right]\left|\psi\right\rangle.$$
(17)

By a(t) here we denote the value of observable A at the time t found in the course of continuous measurement. Thus, unlike the Lindblad equation, this description of continuous measurement is selective: it takes into account the result of the measurement — that is, the information recorded in the environment (despite the fact that this approach does not rely on any model of the environment). This is the information approach, which assumes that the dynamics of the measured system are determined not by the particulars of the environment, but rather by the information recorded in the environment. Equation (16) can be derived from Eqn (17) by carrying out a summation over all possible curves a(t) in a certain way. This approach will be discussed in detail in Section 3.2.

**3.** The stochastic Schrödinger equation for a continuously measured system

$$d|\psi\rangle = \left[-\frac{\mathrm{i}}{\hbar} H - \kappa \left(A - \langle A \rangle\right)^2\right] |\psi\rangle \,\mathrm{d}t + \sqrt{\kappa} \left(A - \langle A \rangle\right) |\psi\rangle \,\mathrm{d}w$$
(18)

can be derived [45] from the equation with a complex Hamiltonian (17). In the stochastic equation, w is a random variable of the type of a Brownian walk (white noise), which characterizes the effects of the environment. The differential of this variable, which enters this equation, satisfies the condition  $dw^2 = dt$ , which reflects the properties of Brownian motion (the mean deviation is proportional to the square root of the time interval). Other stochastic equations have also been proposed [46-48] leading to the Lindblad equation (16). The problem is that the stochastic equation cannot be unambiguously derived from the Lindblad equation. The advantage of Eqn (18) is that it follows from the equation with a complex Hamiltonian (17), which can be derived from first principles. In Ref. [49] and subsequent publications of the same authors, the noise in the stochastic equation was interpreted in a different fashion - not as an effect of the environment, but rather as an independent fundamental physical process called spontaneous localization.

The relationship between different phenomenological approaches to continuous quantum measurements can be illustrated by the following table:



In this table we emphasize the distinction between selective and nonselective descriptions of continuous quantum measurements. The selective description is more detailed. It portrays the evolution of the measured system for only one out of many alternative results of measurement. In this description the state of the measured system remains pure (if, of course, it was pure prior to the measurement). In the equation with a complex Hamiltonian, the alternative is given by the function a(t), which has a straightforward physical meaning — it is the result of monitoring the observable A. In the stochastic equation, the alternative is given by the random function w.

#### Quantum mechanics of open measured systems

Nonselective description represents evolution of the measured system irrespective of the measurement readout. This description takes into account all possible measurement readouts, and the actual measurement readout is assumed not to be known. Accordingly, the nonselective description (equation for the density matrix) can be derived from the selective description by carrying out the summation with respect to the alternatives (see, for example, Ref. [50]). The transition back from a nonselective (less detailed) to a selective (more detailed) description is ambiguous and requires additional assumptions for removing the uncertainty. The logical links between the approaches, and the possibilities for deriving one approach from another, are shown on the diagram by arrows (see also Section 3.3).

# 3.2 Restricted path integrals (quantum corridors)

Let us consider in greater detail the method of restricted path integrals (RPI) for describing continuous quantum measurements. In this approach, an open continuously measured system is described (as opposed to a closed system) not by a single unitary evolution operator, but rather by a whole family of partial evolution operators (propagators), in accordance with the numerous alternative measurement readouts. Each result of measurement defines one channel of quantum evolution with the aid of the relevant partial propagator. And it is all these channels taken together that give a complete description of the dynamics of the open continuously measured system.

**3.2.1 Main principles.** The starting point for constructing the method of restricted path integrals is the Feynman formulation of quantum mechanics based on the formalism of path integrals. The Feynman path integrals are convenient for developing the main principles of the approach, being an excellent tool for conceptual analysis. Having formulated the basics of the theory, however, it will be easy to switch to the mathematically more simple formalism of Schrödinger equation with a complex Hamiltonian. The Schrödinger equation can then be used for practical calculations, without using the path integrals.

In Feynman's formulation of quantum mechanics, the amplitude of transition of the system from one point of the configuration space to another (the propagator of the system) is represented by the path integral

$$U_T(q'',q') = \int d[q] \exp\left(\frac{1}{\hbar} S[q]\right)$$
$$= \int d[p] d[q] \exp\left[\frac{i}{\hbar} \int_0^T (p\dot{q} - H(p,q,t))\right].$$
(19)

Here S[q] is the classical action of the system in question, which may be expressed as the integral of the Lagrangian along the path,

$$S[q] = \int_0^T dt \, L(q, \dot{q}, t) \,, \tag{20}$$

and H is its Hamiltonian. The first integral in Eqn (19) is the path integral in the configuration space (which may be multidimensional), and the second is the path integral in the corresponding phase space. The integrals are equal to each other, and any of them can be used for describing a closed system. For describing a continuously measured system, however, one generally needs the path integral in the phase space.

The operator  $U_t$  with the kernel (19) — that is, one whose matrix elements are

$$\langle q^{\prime\prime}|U_t|q^{\prime}\rangle = U_t(q^{\prime\prime},q^{\prime})\,,\tag{21}$$

is the evolution operator, and describes the evolution of the system in accordance with the equations

$$|\psi_t\rangle = U_t |\psi_0\rangle, \qquad \rho' = U_t \rho_0 U_t^{\dagger}. \tag{22}$$

We know that the vector  $|\psi_t\rangle$  can be found by solving the Schrödinger equation with the initial condition  $|\psi_0\rangle$ . The classical Hamiltonian *H* of the system is real, and the corresponding quantum operator is Hermitian. Therefore, the evolution operator  $U_t$  is unitary,

$$U_t^{\dagger} U_t = \mathbf{1} \,, \tag{23}$$

and the vector  $|\psi_t\rangle$  has a unit norm (provided, of course, that the initial vector  $|\psi_0\rangle$  is normalized to unity).

To go over to the description of continuous measurement, we must first recall the conceptual basis of Feynman's representation (19) of the propagator. According to Feynman's ideology, the exponential under the path integral is the amplitude of probability of transition of the system from the starting point to the end point along the given path (which may be a path in the configuration space or phase space). Since the path the transition takes is not known, the total amplitude of the transition probability is found by carrying out a summation (integration) over all possible paths, which leads to expression (19).

The latter argument only holds, however, if it is really not possible in principle to find out along which path the system propagates. Such is indeed the case when the system is closed. The situation is different if the system is open — that is, if the system interacts with the environment in some way or other. In this case the state of the environment is modified by the interaction, and this change will depend on the state of the system. Monitoring the change of the state of the environment, one can gain certain information about the evolution of the system that has caused this change. In particular, some knowledge can be obtained about the path of propagation of the system. In such a case the path integral must be restricted to those paths which comply with the information obtained.

This is how the restricted path integral (RPI) arises. The idea of its application to continuous measurements was briefly formulated by Feynman in his original paper [51]. This approach was developed technically and conceptually in the author's papers [52, 53, 2, 9] (see also Refs [54, 55, 42, 56]). In two directions it has been possible to advance Feynman's ideas much further: (1) the RPI formalism has been extended to arbitrary continuous measurements, not limited to the continuous monitoring of the coordinates of the system; and (2) it has been proved that the calculation of RPI not only gives the probability distribution of different results of measurement, but also describes the evolution of the measured system (which in turn allows the refinement of the first conclusions concerning the probability distribution).

Now the easiest way to proceed is to use the most simple example of continuous measurement. Assume that the measurement consists in monitoring the position of the system in the configuration space (for definiteness, we may speak of monitoring the coordinate of a moving particle). Then at any time *t* the measurement gives an estimate of the position — that is, point a(t) in the configuration space. In total, the result of the measurement is represented by the curve  $[a] = \{a(t) | 0 \le t \le T\}$ . Since the accuracy of measurement cannot be infinite, the resulting curve [a] only gives limited information about the position of the system at any time. Namely, it tells us that at the time *t* the system was in a certain vicinity of point a(t). The size of this vicinity depends on the error  $\Delta a$  of measurement. Accordingly, the result of continuous measurement as a whole defines a corridor in the configuration space, centered around the curve [a] and having a width  $\Delta a$  (Fig. 1). In view of this, the path integral in the calculation of propagator ought to be restricted to this corridor.



Figure 1. Corridor restricting the path integration in the case of continuous monitoring of coordinate.

In more general terms, if the readout of continuous measurement  $\alpha$  implies that path [q] belongs to the family of paths  $I_{\alpha}$ , then the propagator of the system must be calculated by the formula

$$U_T^{\alpha}(q'',q') = \int_{I_{\alpha}} \mathbf{d}[q] \, \exp\left(\frac{\mathbf{i}}{\hbar} \, S[q]\right). \tag{24}$$

Similarly, if the readout of continuous measurement  $\alpha$  implies that path [p,q] in the phase space belongs to the family  $J_{\alpha}$  of paths in the phase space, then the expression for the propagator of the system is

$$U_T^{\alpha}(q'',q') = \int_{J_{\alpha}} \mathbf{d}[p] \, \mathbf{d}[q] \, \exp\left[\frac{\mathrm{i}}{\hbar} \int_0^T (p\dot{q} - H(p,q,t))\right]. \quad (25)$$

Thus, we find the propagator as the integral over the corridor of paths. By analogy with Carmichael's *quantum trajectory* [6], we may refer to  $I_{\alpha}$  or  $J_{\alpha}$  as the *quantum corridor*.

The corridor  $I_{\alpha}$  in the configuration space may be regarded as a particular case of the corridor  $J_{\alpha}$  in the phase space. This special case is realized when the information gained from the measurement is adequately expressed in terms of coordinates and does not require the involvement of momentum. Such is the case, for example, when the measurement consists in monitoring the coordinate<sup>†</sup>. Usually the information supplied by the measurement does not allow the corridor of paths to be defined with clearcut limits. A more adequate description of the measurement is based on the weight functional

$$U_T^{\alpha}(q'',q') = \int \mathbf{d}[q] \, w_{\alpha}[q] \exp\left(\frac{\mathbf{i}}{\hbar} \, S[q]\right) \tag{26}$$

or, in the general case,

$$U_T^{\alpha}(q^{\prime\prime},q^{\prime}) = \int \mathrm{d}[p] \,\mathrm{d}[q] \,w_{\alpha}[p,q] \exp\left[\frac{\mathrm{i}}{\hbar} \int_0^T (p\dot{q} - H(p,q,t))\right].$$
(27)

In such a situation we may also speak of a quantum corridor, but the boundaries of this corridor are blurred. If, for example, the coordinate q is measured continuously, and the measurement readout is represented by curve  $\alpha = [a]$ , then the functional  $w_{\alpha}[q]$  must be equal to one for paths [q] lying close to curve [a], and become zero for paths straying far from this curve.

**Remark 1.** A still more realistic description of the measurement requires using a complex-valued functional  $w_{\alpha}[p,q]$  in place of the real-valued one. This means that the measurement involves not only projecting the system on to the subspace corresponding to the measurement readout, but also a change of phase of the wave function depending on the measurement readout. Such measurement is not minimal because the same information can be obtained without distortion of phase (while the projection is absolutely necessary). Real devices, however, may realize such nonminimal measurement may be advantageous.

Having expressed in one way or another the propagator  $U_T^{\alpha}(q'',q')$  and passing to the relevant evolution operator  $U_T^{\alpha}$  using a formula similar to Eqn (21), we get the law of evolution in the form

$$|\psi_T^{\alpha}\rangle = U_T^{\alpha}|\psi_0\rangle, \qquad \rho_T^{\alpha} = U_T^{\alpha}\rho_0(U_T^{\alpha})^{\dagger}$$
(28)

(provided that the continuous measurement in question is performed and yields the desired result). Now we have a whole family of *partial evolution operators*  $U_T^{\alpha}$ , and these operators are not unitary. Vector  $|\psi_T^{\alpha}\rangle$  produced by the application of such an operator has a norm of less than one (even if the initial vector is normalized). The trace of the density matrix  $\rho_T^{\alpha}$  is less than one, even if the initial density matrix has a unit trace. This circumstance is not accidental, because the new norms give us the probability distribution of different measurement readouts. Namely, the quantity

$$P(\alpha) = \operatorname{Tr} \rho_T^{\alpha} = \operatorname{Tr} \left( U_T^{\alpha} \rho_0 (U_T^{\alpha})^{\dagger} \right)$$
(29)

is the density of the probability that the measurement will yield the result  $\alpha$ . The distribution is normalized with respect to a certain measure  $d\alpha$ , so that

$$\int \mathrm{d}\alpha \, P(\alpha) = 1 \,. \tag{30}$$

Carrying out summation (using this measure) over all possible outcomes of measurement  $\alpha$ , we may go over to the nonselective description of continuous measurement:

$$\rho_T = \int \mathrm{d}\alpha \, \rho_T^{\alpha} = \int \mathrm{d}\alpha \, U_T^{\alpha} \, \rho_0 (U_T^{\alpha})^{\dagger} \,. \tag{31}$$

<sup>&</sup>lt;sup>†</sup>Note that the information about the path [q] obtained from the measurement allows certain conclusions to be made concerning the velocity  $[\dot{q}]$ , but this does not yet automatically give the momentum, since the classical relation  $p = m\dot{q}$  does not hold in the quantum regime (see Refs [57, 9]).

Then the measurement readout is assumed to be unknown, and the evolution of the measured system is described by the density matrix. The trace of the density matrix  $\rho_T$  is unity for any initial state  $\rho_0$  of the system, as long as the *condition of generalized unitarity* 

$$d\alpha \left(U_T^{\alpha}\right)^{\dagger} U_T^{\alpha} = \mathbf{1}$$
(32)

is satisfied. This condition ensures preservation of probability (30), and therefore is mandatory for the family of partial evolution operators.

**3.2.2 Monitoring of an arbitrary observable.** Embarking on the construction of the RPI formalism, we selected the monitoring of coordinate as our working example of continuous measurement. Let us now consider a more general case of monitoring of an arbitrary observable which may be a function of coordinates, momenta and time, A = A(p, q, t). The result of monitoring this observable is represented by the function

$$[a] = \{a(t) | 0 \le t \le T\}.$$
(33)

This result implies that the observable A at any time t is close to a(t). In other words, curve [A], as determined by its values

$$A(t) = A(p(t), q(t), t)$$
(34)

is generally close to curve [a]. The word 'close' is rather loose and ought to be given a definition. For the criterion of closeness of two curves we shall select the mean square distance between them,

$$\langle (A-a)^2 \rangle_T = \frac{1}{T} \int_0^T [A(t) - a(t)]^2 dt.$$
 (35)

Now we define the weight functional in Eqn (27) as

$$w_{[a]}[p,q] = \exp\left(-\kappa \langle (A-a)^2 \rangle_T\right)$$
$$= \exp\left\{-\kappa \int_0^T [A(t) - a(t)]^2 \,\mathrm{d}t\right\}.$$
(36)

Thus we have in a certain way concretized the concept of the quantum corridor which describes our measurement (monitoring of the observable *A*). We may say that for the description of monitoring we are using a *Gaussian corridor*.

The measure of closeness of the two curves and the weight functional can be chosen in different ways, and the resulting description of the measurement will to a certain extent depend on this choice. Having decided in favor of a particular choice, we solidify the theory. Selecting different options, we shall be getting a somewhat different description for the measurement which we have called the monitoring of observable *A*. This uncertainty has a straightforward physical interpretation—it is related to the freedom of physical realization of the monitoring.

Mathematically, our definition of the weight functional (36) is the simplest. At the same time, it can be proved that this functional correctly describes the behavior of the system when the monitoring is realized in the form of frequent and short (almost instantaneous) observations of the system — that is, as a series of weak interactions with the measuring

system. In principle, however, one can select a different weight functional, thus describing a different class of measuring devices (see Remark 1 above).

Coefficient  $\kappa$  in Eqn (36) characterizes the 'force of measurement', its accuracy. To better understand the physical meaning of this parameter, we may represent it in the form

$$\kappa = \frac{1}{T\Delta a_T^2} \,. \tag{37}$$

Substituting this expression for  $\kappa$  into Eqn (36), we see that  $\Delta a_T$  is the error of the continuous measurement that lasts for a time *T*. Indeed, the weight functional (36) is designed so that a substantial contribution to the RPI comes only from those paths which correspond to functions A(t) whose mean square deviation from a(t) is not greater than  $\Delta a_T$ .

If the parameter  $\kappa$  remains constant with time, then  $\Delta a_T$  falls,

$$\Delta a_T^2 \sim \frac{1}{T} \,, \tag{38}$$

that is, the resolution of continuous measurement improves as its duration increases. We shall return to this effect later on, in connection with the continuous measurement of energy. At this point we just note that it leads to exponentially fast decoherence of the type of Eqn (9).

Using the Gaussian definition (36) of a quantum corridor, we may rewrite the restricted path integral (27) in the form

$$U_{T}^{[a]}(q'',q') = \int d[p] d[q] \exp\left\{\frac{i}{\hbar} \int_{0}^{T} (p\dot{q} - H(p,q,t)) dt - \kappa \int_{0}^{T} (A(p,q,t) - a(t))^{2} dt\right\}.$$
 (39)

This integral will coincide with the conventional (nonrestricted) Feynman integral (19) if we replace the Hamiltonian *H* by the *effective complex Hamiltonian* 

$$H_{[a]}(p,q,t) = H(p,q,t) - i\kappa\hbar (A(p,q,t) - a(t))^2.$$
(40)

This means that the evolution described by the partial evolution operator (28) can equivalently be described by the Schrödinger equation with the Hamiltonian (40):

$$|\dot{\psi}\rangle = -\frac{\mathrm{i}}{\hbar} H_{[a]}|\psi\rangle = \left[-\frac{\mathrm{i}}{\hbar} H - \kappa \left(A - a(t)\right)^2\right]|\psi\rangle.$$
(41)

This equation describes the evolution of the measured system in a selective way — that is, with due account for the readout of measurement [a]. If we go over to nonselective description, carrying out integration over all [a] in accordance with Eqn (31), then the resulting density matrix will satisfy the Lindblad equation (16) [50].

**Remark 2.** The arguments developed above hold if the time in the course of monitoring of the observable A is measured with absolute precision (that is, if the finite accuracy of timing can be neglected). Of course, in reality the quantity a(t), derived in the course of monitoring, characterizes the value of the observable over a certain time interval, whose length characterizes the accuracy of time measuring or, looking from a different angle, the inertial properties of the measuring device. The general scheme outlined in Section 3.2.1 was extended to this case in Ref.

[41]. It was demonstrated that, in the case of finite time resolution, the evolution of the system cannot be described by a Schrödinger-type differential equation, and the measurement is 'time-integral'. The RPI technique of representation of propagators remains workable in this case as well.

**3.2.3** \*Uncertainties in continuous measurements. A continuous measurement is completely described by the set of partial propagators (27), each of which can be found by calculating the RPI or by solving the Schrödinger equation with a complex Hamiltonian. This, however, is not so easy to do. A natural question is whether it might be possible to use simpler procedures (like the uncertainty relations) for solving at least some of the problems related to continuous measurement. This can indeed be done. Namely, there are rather simple methods that give an approximate answer to the question as to which the measurement readouts [a] occur with a high enough probability.

This answer is expressed by an inequality which is called the *principle of uncertainty of action* [58–60], since it shows how the classical action change when the path is varied within one and the same quantum corridor [a].

The condition that the probability of measurement readout [a] be high is formulated in terms of one of the paths [p,q]reconciled with this result — that is, satisfying the condition

$$A(t) = a(t) \,,$$

where<sup>†</sup>

$$A(t) = A(p(t), q(t), t)$$

Apart from this perfectly matching path, we consider a path  $[p + \Delta p, q + \Delta q]$  which differs from the latter by the increment  $[\Delta p, \Delta q]$  (where  $\Delta p = \Delta p(t), \Delta q = \Delta q(t)$  are functions of time). Then the function A(t) has an increment of  $\Delta A(t)$ . We only consider increments of the path such that the increment  $\Delta A(t)$  remains less than the error  $\Delta a(t)$  of the measurement. This means that the new path  $[p + \Delta p, q + \Delta q]$  within the accuracy  $\Delta a$  of measurement still agrees with the measurement readout [a]. To simplify the terminology, we shall say that the new path lies within the corridor [a] — that is, the increment of the path does not lead beyond this corridor.

Now we find the classical action for the path [p,q] by the formula

$$S[p,q] = \int_0^T (p\dot{q} - H(p,q,t)) \,\mathrm{d}t$$
(42)

and look at the increment  $\Delta S$  upon transition to the new path  $[p + \Delta p, q + \Delta q]$ . The magnitude of the increments of action found in this way is the criterion of how probable is the measurement readout [a]. It turns out that the corridor [a] is probable if the increment  $\Delta S$  of action does not exceed the quantum  $\hbar$  of action until the path lies within the corridor [a]<sup>‡</sup>.

This condition ensures that the measurement readout [a] occurs with a high enough probability. It can be written in the following compact form:

$$\max_{|\Delta A(t)| \leqslant \Delta a(t)} \left| \sum_{i} \int_{0}^{T} \mathrm{d}t \left[ \Delta p_{i} \left( \dot{q}_{i} - \frac{\partial H}{\partial p_{i}} \right) - \Delta q_{i} \left( \dot{p}_{i} + \frac{\partial H}{\partial q_{i}} \right) \right] \right| \lesssim \hbar.$$
(43)

The parentheses under the integral enclose expressions that enter the Hamilton equations; accordingly, this inequality can be interpreted in terms of a 'fictitious force'. Namely, the measurement readout [a] is probable if the corresponding path [p,q] satisfies the modified Hamilton equations

$$\dot{q}_i - \frac{\partial H}{\partial p_i} = -\delta F(t) \frac{\partial A(p,q,t)}{\partial p_i} , \qquad (44)$$

$$\dot{p}_i + \frac{\partial H}{\partial q_i} = \delta F(t) \frac{\partial A(p,q,t)}{\partial q_i}$$
(45)

with the 'fictitious force'  $\delta F(t)$  subject to the restriction

$$\int_{0}^{T} \mathrm{d}t \left| \delta F(t) \right| \Delta a(t) \lesssim \hbar \,. \tag{46}$$

Alternatively, condition (43) can be written as the restriction on the area in the phase space defined by the two paths [p,q] and  $[p + \Delta p, q + \Delta q]$  (Fig. 2):

$$\max_{|\Delta A(t)| \leqslant \Delta a(t)} \left| \sum_{i} \int_{0}^{T} \mathrm{d}t \, \delta \sigma_{i} \right| \lesssim \hbar \,, \tag{47}$$

where

$$\delta\sigma_i(t) = \Delta p_i(t)\delta q_i(t) - \Delta q_i(t)\delta p_i(t),$$
  

$$\delta q_i = \mathrm{d}q_i - \frac{1}{m}p_i\,\mathrm{d}t, \qquad \delta p_i = \mathrm{d}p_i - F_i\,\mathrm{d}t.$$
(48)

Inequality (47) must hold for all paths that belong to this corridor. This implies that for some of the paths the equality (by order of magnitude) will be attained. For a one-dimensional system we then have

$$\left| \int (\Delta p \, \delta q - \Delta q \, \delta p) \right| \sim \hbar \,. \tag{49}$$



**Figure 2.** (a) Area of a parallelogram in a phase space built on two vectors. (b) Area in a phase space defined by two paths [p,q] and  $[p + \Delta p, q + \Delta q]$ , lying within the corridor [a]. If this area is less than  $\hbar$  for each such pair of paths, then [a] will occur as the measurement readout with a sufficiently high probability.

<sup>†</sup> It can be demonstrated that the proof does not depend on which path [p,q] reconciled with the measurement readout [a] is selected.

 $<sup>\</sup>ddagger$  The background of this criterion is obvious. If variation of the path within the corridor results in large variations of action, the imaginary exponential of action in integral (39) exhibits fast oscillations, and the integral is small. Since this integral defines the probability density for the measurement readout [*a*], the probability is also small.

If the measurement is very short, we may write

$$\left| \left( \Delta p \, \delta q - \Delta q \, \delta p \right) \right| \sim \hbar \,. \tag{50}$$

Consider a measurement of momentum with accuracy  $\Delta p$  over a time interval  $\Delta t$ . If the coordinate is not measured, then  $\Delta q$  is very large, and at first sight it may seem that the second term in Eqn (50) dominates. In this case, however,  $\delta p$  must be exactly equal to zero, since otherwise inequality (47) would not be satisfied. Thus, the second term vanishes, and we have

$$\Delta p |\delta q| = \Delta p \left| (q'' - q') - \frac{p}{m} \Delta t \right| \sim \hbar, \qquad (51)$$

where q', q'' are the coordinates before and after the measurement. We see that the effect of measurement may consist in that the velocity  $(q'' - q')/\Delta t$  differs from the classical expression p/m by an amount of the order of  $\hbar/(\Delta p\Delta t)$ . This is an alternative formulation of the well-known uncertainty relation [61]

$$\Delta p | v'' - v' | \Delta t \sim \hbar \,, \tag{52}$$

which is deduced from a thought experiment. Relation (47), however, is much more general.

**3.2.4 Features of the RPI-based approach.** Let us discuss some important features of the approach based on a restricted path integral. We saw that this approach is deduced from first principles — namely, from quantum mechanics in Feynman's formulation. At the same time, it can be given an independent substantiation based on concrete models of measurement (see Section 4.3 below). Validation of this phenomenological approach with models makes it more reliable, while the derivability from first principles points to the fundamentality of the theory.

According to the RPI approach, the continuously measured system is described as an open system, but the description is selective. Evolution of the measured (open) system is described by the state vector rather than by the density matrix. Such a description is applicable to an individual system, and not only to the statistical ensemble of systems corresponding to all possible states of the measuring medium. The RPI approach reveals a very important feature of the interaction between the measured system and the measuring system: the reciprocal effect of the measurement on the measured system depends only on the information obtained from the measurement (and recorded in the state of the environment). In this sense the RPI approach to continuous measurements may be called the information approach [62].

The dynamics of the measured system, described by the set of partial propagators (39) or by the Schrödinger equation with an effective complex Hamiltonian (41), depends only on the information recorded by the measuring medium, but not on the particulars of the interaction of the system with the environment. This is a manifestation of the *dynamic role of information*. A complete description of the open (measured) system does not require a complete model of the environment. The information model is quite sufficient.

As a matter of fact, the dynamic role of information is already clear from von Neumann's reduction postulate, which holds that the final state of the measured system only depends on the outcome of the measurement. In the reduction postulate, however, the change of the system caused by the measurement and the evolution of the system owing to its own dynamic properties are time-separated [cf. Eqn (15)].

By contrast, the two dynamic aspects are inseparable in the evolution of the continuously measured system. Such systems exhibit a new type of dynamics which involves both classical and quantum features at the same time. The dynamics are completely determined by fixing (1) the Hamiltonian of the system, and (2) the information about the system that flows (dissipates) into the environment.

The RPI approach proves that the theory of measurements can be incorporated into quantum mechanics, contrary to the common opinion that it ought to be postulated independently of quantum mechanics. One only has to understand quantum mechanics the way it was formulated by Feynman. This indicates that Feynman's formulation of quantum mechanics is essentially broader and deeper than its conventional operator form. The added depth arises from the fact that a physical interpretation is given not only to the full amplitude of propagation of the system, but also to the amplitude of propagation along the given path. This formulation of quantum mechanics is closed because it naturally incorporates the quantum theory of measurements.

In Section 4 we shall discuss in detail an important application of the RPI method to the continuous measurement of the energy of a multilevel system. At this point we shall just mention some other applications of this method in nonrelativistic and relativistic quantum theories.

In the nonrelativistic theory this method has been applied to the analysis of measurement of the coordinates of oscillator and a system of oscillators [52, 53, 2], and to quantum nondemolishing measurements [63, 60, 64]. The relativistic applications include:

— quantum restrictions on the measurability of an electromagnetic field [66],

— quantum restrictions on the measurability of a gravitational field [66],

— the appearance of classical geometry in quantum gravitation [67],

— the measurement of the position of relativistic particle [68],

— the analysis of Unruh and Hawking thermal effects [69].

#### 3.3 \*Derivation of the stochastic equation

The Schrödinger equation with a complex Hamiltonian (41) can be written as the stochastic Schrödinger equation [45]. To do this, in place of the function a(t) and the state vector  $\psi(t)$  we introduce new variables

$$a = c + \frac{\xi}{\sqrt{2\kappa}}, \qquad \Psi(t) = \exp\left(\frac{1}{2}\int_0^t \mathrm{d}t\,\xi^2\right)\psi(t)\,, \tag{53}$$

where c(t) will be defined later. Then Eqn (41) can be rewritten as

$$\left|\dot{\Psi}\right\rangle = \left[-\frac{\mathrm{i}}{\hbar}H - \kappa(A-c)^2 + \sqrt{2\kappa}\left(A-c\right)\xi\right]\left|\Psi\right\rangle \tag{54}$$

or, if we define a new variable *w* as  $dw = \xi dt$ ,

$$d|\Psi\rangle = \left[-\frac{\mathrm{i}}{\hbar} H - \kappa (A-c)^2\right]|\Psi\rangle \,\mathrm{d}t + \sqrt{2\kappa} \,(A-c)|\Psi\rangle \,\mathrm{d}w\,.$$
(55)

In order to have the norm of vector  $|\Psi\rangle$  conserved,

$$\left(\langle \Psi | + \langle d\Psi | \right) \left( |\Psi\rangle + |d\Psi\rangle \right) = \langle \Psi | \Psi\rangle, \qquad (56)$$

it is sufficient to require that

$$\mathrm{d}w^2 = \mathrm{d}t\,, \qquad c = \langle \Psi | A | \Psi \rangle\,. \tag{57}$$

Equation (55) with the additional conditions (57) is known as the stochastic Schrödinger equation. The quantity w here is a random function (white noise), which describes the impact of the measuring medium on the measured system. The statistics of this noise is determined by the probability distribution

$$P[w] = \langle \psi | \psi \rangle = \exp\left(-\frac{1}{2} \int_0^t \dot{w}^2(t) \,\mathrm{d}t\right). \tag{58}$$

This is the so-called *white noise* or Brownian motion. In the case of such motion, the change of w over a small time interval  $\Delta t$  is described by the distribution

$$P(\Delta w) = \frac{1}{\sqrt{2\pi\Delta t}} \exp\left(-\frac{1}{2}\frac{\Delta w^2}{\Delta t}\right),\tag{59}$$

and, as a consequence, the mean square displacement is proportional to the time:

$$\overline{\Delta w^2} = \int P(\Delta w) \Delta w^2 \, \mathrm{d}\Delta w = \Delta t \,. \tag{60}$$

This last formula offers a physical interpretation of the unusual relation  $dw^2 = dt$ , which is adopted in the stochastic theory in the framework of the so-called *Ito formalism* [70].

In this way, the stochastic equation (55) is derived from the theory of continuous measurements based on a restricted path integral or on the Schrödinger equation with a complex Hamiltonian. Other stochastic equations have also been proposed for describing continuous quantum measurements [46, 71, 47]. The necessary condition is that the density matrix, which describes the same system in a nonselective manner, satisfies the Lindblad equation (16). This holds for Eqn (55) because the Lindblad equation is an implication of the RPI approach [50].

### 3.4 \*Consistent histories

A new direction of research in quantum mechanics, started a few years ago, is based on the concept of *consistent histories* [72-75]. This research is concerned with the emergence of classical features in quantum systems — that is, the effect of decoherence.

The approach based on consistent histories bears some technical semblance to the RPI method, although it is radically different. The difference is that consistent histories are used for describing a closed quantum system (with the purpose of revealing classical features), whereas the RPI method deals from the start with an open system, while the effects of the environment are taken into account implicitly. Let us briefly discuss the method of consistent histories, following the paper by Gell-Mann and Hartle [74].

A history  $\alpha = \{i_1, i_2, \dots, i_{N-1}\}$  is defined in Ref. [74] as a chain of projectors that specify the state of the system at successive times. These projectors define the evolution operator

$$U_{\alpha} = U(t_N, t_{N-1}) P_{i_{N-1}} U(t_{N-1}, t_{N-2}) \dots$$
  
$$\dots U(t_3, t_2) P_{i_2} U(t_2, t_1) P_{i_1} U(t_1, t_0), \qquad (61)$$

which is similar to that used in Eqn (15) for describing a series of instantaneous measurements. In the method of consistent histories, however, the projectors are used not for describing real measurements, but rather for analyzing the free evolution of the system without any external influence. The operator  $U_{\alpha}$ is just one quantum alternative out of many that contribute to the evolution of the quantum system. The total evolution operator is the sum

$$U = \sum_{\alpha} U_{\alpha} \tag{62}$$

over all possible histories — that is, over all possible selections of the projector for each moment of time<sup>†</sup>.

Histories in the sense of Gell-Mann and Hartle are straightforward analogs of Feynman's paths, which determine the evolution of a closed system only when taken together. By contrast to individual paths, histories are coarser alternatives, each of which includes many paths.

According to Gell-Mann and Hartle, each history is associated with the 'probability'

$$P_{\alpha} = \operatorname{Tr}\left(U_{\alpha}\,\rho \,U_{\alpha}^{\dagger}\right).\tag{63}$$

The interpretation of this quantity as a probability is not necessarily correct, and the further analysis is concerned with the question of when exactly such an interpretation is justified.

Along with a certain selected set of histories  $\{\alpha\}$ , the authors also consider sets of histories which follow from the selected set by applying the procedure of *coarse-graining* — when the projections are made onto more extensive subspaces, or are less frequent (not at every selected moment of time). Each history from the coarser set can be represented as a sum of histories from the finer set (a coarse history includes fine histories as alternatives). The corresponding evolution operators are bound by the summation

$$\beta = \sum_{\alpha \in \beta} \alpha , \qquad U_{\beta} = \sum_{\alpha \in \beta} U_{\alpha} . \tag{64}$$

At the same time, the corresponding 'probabilities' of the coarser histories

$$P_{\beta} = \operatorname{Tr}\left(U_{\beta}\,\rho\,U_{\beta}^{\dagger}\right) \tag{65}$$

are not necessarily representable as sums of 'probabilities' of the finer histories.

The key to the entire approach is found in the following argument which leads to the 'condition of consistency' as the necessary criterion that the description of the system in terms of alternatives  $\{\alpha\}$  is classical. A sufficiently coarse-grained description of a quantum system loses its quantum features and becomes purely classical. The alternatives  $\{\alpha\}$  give a coarse-grained description of the system, and if the description is coarse enough, these alternatives may be regarded as classical. In this case, the quantities (63) indeed ought to be

<sup>†</sup> By contrast, if operator (61) describes a real measurement, as has been assumed in Eqn (15), it corresponds to the classical alternative, and the summation of such operators is meaningless.

considered as probabilities complying with the conventional rule of summation.

Now if the alternatives from the set  $\{\alpha\}$  can be regarded as classical alternatives, then any coarser set  $\{\beta\}$  of alternatives is also classical, and therefore the *rule of summation of probabilities* holds:

$$P_{\beta} = \sum_{\alpha \in \beta} P_{\alpha} \,. \tag{66}$$

Thus, if the alternatives  $\{\alpha\}$  are classical, then the transition to the coarser alternatives  $\{\beta\}$  complies not only with the rule of summation of amplitudes (64), but also with the rule of summation of probabilities (66). A straightforward mathematical analysis proves that condition (66) is satisfied if the functional of decoherence

$$C_{\alpha\alpha'} = \operatorname{Tr}\left(U_{\alpha}\,\rho\,U_{\alpha'}^{\dagger}\right) \tag{67}$$

satisfies the condition of consistency

$$C_{\alpha\alpha'} + C^*_{\alpha\alpha'} = 0 \quad \text{for} \quad \alpha \neq \alpha' \,.$$
 (68)

We see that the condition of consistency (68) is necessary for the set of histories  $\{\alpha\}$  to be interpreted as a set of classical alternatives.

It was demonstrated that the condition of consistency holds approximately for the 'situation of measurement' that is, when the system under consideration consists of two parts which interact like the measuring device and the measured subsystem (see Section 2.1). At the same time, the condition of consistency of histories is not sufficient for ensuring the predictability of the quantum system — a criterion that the given degree of coarsening ensures that the description is indeed classical [76]. We see that the theory of consistent histories, certainly being a step forward in the study of decoherence, does not yet give a complete solution of the problem.

Drawing a comparison once again between the two approaches, the methods of RPI and consistent histories, we duly note that the condition of consistency is not a necessary condition in the RPI approach. This is because the RPI approach is concerned with an open system that interacts with its environment (although there is no explicit model of this environment). The RPI-based analysis of the system involves only one family  $\{\alpha\}$  of histories — the family that describes the effects of the environment. There is no need to use the coarser sets of histories, and to provide consistency of descriptions at different levels of coarse-graining.

# 4. Continuous measurement of discrete energy

An important achievement of recent years is the analysis of continuous energy measurement in a system with discrete energy levels. Firstly, the measurement of the energy of an atom is of much practical importance. Secondly, all important features of continuous quantum measurement in this case can be transferred to the continuous measurement of any discrete observable. We are going to consider the continuous energy measurement in a multilevel (especially a two-level) system, and prove that this process allows the monitoring of a quantum transition from one level to another. First we shall analyze the measurement using the Schrödinger equation with a complex Hamiltonian, and then consider a concrete scheme of such measurement using conventional quantum mechanical methods. We shall also show the way of implementing this measurement.

The approach based on RPI and complex Hamiltonians was first applied to the measurement of energy in a two-level system in Refs [77, 78]. It was demonstrated that if the energy is measured with a high enough accuracy, then the system becomes frozen, and transitions between the levels are no longer possible (the Zeno effect). Alternative regimes of measurement have not been (and could not have been) studied in these works because of a serious methodological error. The authors assumed that the result of a continuous measurement is expressed by the function E(t), which does not change and coincides with one of the energy levels of the system.

At first sight, such an assumption seems natural for a system with a discrete spectrum. However, it is not correct, since the accuracy of the measurement is finite. The function [E] which represents the measurement readout can be arbitrary in the approach based on RPI or on the Schrödinger equation with a complex Hamiltonian. By solving the Schrödinger equation [similar to Eqn (41)] with the [E]dependent Hamiltonian, one can find the probability density P[E] of the given measurement readout. It is only then (and by no means *a priori*) that one can decide which functions [E]may arise as the measurement readout. It turns out that in the regime of a very accurate measurement, only those results that correspond to functions [E] which are constant and coincide with the energy levels are probable. This is not true, however, when the accuracy of the measurement is relatively poor. But it is exactly those imprecise or soft measurements that are of special interest, since they do not modify the measured system too much.

This error was corrected in Ref. [18], which has made it possible to carry out a detailed analysis of a moderately accurate continuous quantum measurement of the energy, and propose an entirely new type of measurement monitoring of a quantum transition. The results that could be anticipated from such measurement were thoroughly analyzed in Ref. [19], and a concrete scheme of continuous quantum measurement of the energy of the atom was proposed.

## 4.1 Energy measurement in a multilevel system

Consider a system with a Hamiltonian  $H = H_0 + V$ , where  $H_0$  is the 'free' Hamiltonian of the multilevel system, and V is the perturbation that may induce a transition between the levels. Assume that the observable  $H_0$  is continuously measured in such a system. In accordance with the scheme outlined in Section 3.2, we shall describe such a measurement with the functional

$$w_{[E]}[p,q] = \exp\left\{-\kappa \int_0^T \left[H_0(p(t),q(t),t) - E(t)\right]^2 dt\right\},$$
(69)

that is, with a Gaussian corridor centered around curve [E]. Then the effective Hamiltonian is

$$H_{[E]} = H_0 + V - i\kappa\hbar (H_0 - E(t))^2, \qquad (70)$$

and the effective Schrödinger equation with the complex Hamiltonian is

$$\frac{\partial}{\partial t} |\psi_t\rangle = \left(-\frac{\mathrm{i}}{\hbar} H - \kappa \left(H_0 - E(t)\right)^2\right) |\psi_t\rangle.$$
(71)

Solving this equation, we can find the vector  $|\psi_t\rangle$  at any time. In accordance with the general formula (29), the norm of this vector at the end time of the continuous measurement gives the density of the probability that the measurement readout [E] will be realized:

$$P[E] = \|\psi_T\|^2.$$
(72)

Using the expansion of the state vector with respect to the basis

$$|\varphi_n(t)\rangle = \exp\left(-\frac{\mathrm{i}E_n t}{\hbar}\right)|n\rangle,$$
(73)

we get the following set of equations for the expansion coefficients,

$$\dot{C}_n = -\kappa \left( E_n - E(t) \right)^2 C_n - \frac{\mathrm{i}}{\hbar} \sum_{n'} \langle \varphi_n | V | \varphi_{n'} \rangle C_{n'} , \qquad (74)$$

and the formula for the probability distribution of the measurement readout:

$$P[E] = \sum_{n} |C_{n}(T)|^{2}.$$
(75)

For a free multilevel system, V = 0, the solution has a simple form:

$$C_n(T) = C_n(0) \exp\left[-\kappa \int_0^T dt (E_n - E(t))^2\right]$$
$$= C_n(0) \exp\left[-\frac{T}{T_{\rm lr}} \frac{\langle (E_n - E)^2 \rangle_T}{\Delta E^2}\right], \tag{76}$$

where

$$T_{\rm lr} = \frac{1}{\kappa \Delta E^2} \tag{77}$$

is the *timelike parameter* which characterizes the accuracy of measurement. It is expressed in terms of the 'typical energy difference'  $\Delta E$  in the part of the spectrum we are concerned with.

From Eqn (76) we see that if  $T \ge T_{\rm lr}$  (that is, the duration of measurement is large enough compared to the characteristic parameter  $T_{\rm lr}$ ), then the regime of measurement is realized that resolves (separates) the energy levels. Its characteristic features are:

- [E] is a function which is close to one of the energy levels,  $E(t) \simeq E_n$ ;

— the system after the measurement occurs at the level *n*; — the probability of [E] close to  $E_n$  is  $|C_n(0)|^2$ .

Indeed, if the function [E] is close [in the mean square sense, cf. Eqn (35)] to one of the levels  $E_n$ , then the coefficient  $C_n$  with the relevant number remains the same after the measurement (at t = T) as it was before the measurement, while all the other coefficients are exponentially small. According to Eqn (75), the probability density of each such function is  $|C_n(0)|^2$ .

If function [E], however, does not satisfy this condition (does not remain close to one and the same level in the course of the entire measurement), then all the coefficients  $C_n$  at the time T are exponentially small. The probability density of each of these functions is negligibly small.

So we have to conclude that only those results of measurement with a reasonable probability are realized

which are represented by functions close to one of the levels. The probability that the measurement readout is close to  $E_n$  is proportional to  $|C_n(0)|^2$ . From considerations of normalization we deduce that this probability is equal to  $|C_n(0)|^2$  (which can be proved more rigorously). It is easy to see that these results are in perfect agreement with the way the von Neumann energy measurement is described.

If the measurement is not long enough,  $T \ll T_{lr}$ , then the regime of measurement is not capable of resolving the energy levels:

- Variation  $E_{\text{max}} - E_{\text{min}}$  of curve [E] is less than  $\Delta E \sqrt{T_{\text{lr}}/T}$ , but can be much greater than  $\Delta E$ ;

 $-C_n(T) \simeq C_n(0)$  for levels between  $E_{\min}$  and  $E_{\max}$ ;

—  $C_n(T)$  are exponentially small outside of  $[E_{\min}, E_{\max}]$ .

Thus, by the time T all energy levels vanish outside the interval of width  $\Delta E \sqrt{T_{\text{lr}}/T}$ . As T increases, this interval narrows, and by the time  $T = T_{\text{lr}}$  there is only one energy level — that is, the system occurs in the state with definite energy  $E_n$ . Curve [E] by this time is very close to the level  $E_n^{\dagger}$ . The probability that level number n will survive is  $|C_n(0)|^2$ .

We see that when the time of measurement is not yet long enough,  $T \ll T_{lr}$ , the energy cannot be measured with sufficient precision, and the system still occurs in a state that is a superposition of a number of levels. When the duration of continuous measurement becomes greater than  $T_{lr}$ , the result of the continuous measurement fixes on one of the levels, and the system goes into this level. Accordingly, the time parameter  $T_{lr}$  may be interpreted as the level resolution time.

From the arguments developed above it is clear that the continuous measurement of energy as realized in the method of quantum corridors is a model of the process of decoherence for the von Neumann 'instantaneous' measurement [18]. By this we mean the following. The von Neumann scheme assumes that the measurement of energy in a multilevel system occurs instantaneously, yielding (with the relevant probability  $|C_n(0)|^2$ ) one of the values of  $E_n$ , and the system after the measurement occurs in the *n*th level. In reality, all this takes some finite time which is neglected when the measurement is described in accordance with the von Neumann scheme. The continuous measurement reveals the time structure of this transition — that is, describes what we call the process of decoherence.

Let us now consider a two-level system under a resonant force. Let  $H_0$  be the Hamiltonian of the two-level system, and the potential V defined by its matrix elements

$$\langle \varphi_1 | V | \varphi_2 \rangle = \langle \varphi_2 | V | \varphi_1 \rangle^* = V_0 \,. \tag{78}$$

If we go over from basis (73) to basis  $|n\rangle$ , then the matrix elements of operator V will be harmonic functions of frequency  $\omega = \Delta E/\hbar$  — that is, the potential V will describe the resonant action. The set of equations (74) becomes

$$\dot{C}_{1} = -ivC_{2} - \kappa (E_{1} - E(t))^{2}C_{1},$$
  
$$\dot{C}_{2} = -ivC_{1} - \kappa (E_{2} - E(t))^{2}C_{2},$$
(79)

where  $v = V_0/\hbar$ .

If the measurement does not take place ( $\kappa = 0$ ), these equations describe the *Rabi oscillations*,  $C_1(t) = R_1(t)$ ,

† Closeness is understood in the sense of mean square deviation, so the deviation can actually be very large for a very short time.

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 $C_2(t) = R_2(t)$ , where

$$R_1(t) = C_1(0)\cos vt - iC_2(0)\sin vt, \qquad (80)$$

$$R_2(t) = C_2(0)\cos vt - iC_1(0)\sin vt.$$
(81)

When the measurement does take place, the nature of the process will depend on the relative values of three parameters of the dimension of time: the duration *T* of measurement, the level resolution time  $T_{\rm lr}$ , and the Rabi period  $T_{\rm R} = \pi/v$ . One can distinguish three characteristic regimes of measurement [18]:

- (1) **Zeno regime**  $(T_{lr} \ll T_R \ll T)$ .
- The measurement readout is  $E(t) \simeq E_1$  or  $E(t) \simeq E_2$ .
- The probability that E(t) is close to  $E_n$  is  $|C_n(0)|^2$ .

— If [E] is close to  $E_n$ , the system after the measurement is found at level n.

— The Rabi oscillations are completely suppressed.

- (2) **Rabi regime**  $(T_R \ll T \ll T_{lr})$ .
- The Rabi oscillations remain unchanged.
- [E] is arbitrary in the band of width  $\Delta E_T = \Delta E \sqrt{T_{\rm lr}/T}$ .
- (3) Intermediate regime  $(T_{\rm R} \sim T_{\rm lr} \sim T)$ .
- The period of oscillations is slightly increased.
- -[E] occurs in the band oscillating between the levels.

— Oscillations of [E] correspond to Rabi oscillations.

The last regime is the most interesting, since it offers the possibility (with a certain degree of accuracy) of monitoring the quantum transition. We shall consider this regime in greater detail in Section 4.2.

#### 4.2 Monitoring a quantum transition

The set of equations (79) was used in Ref. [19] for analyzing the continuous energy measurement in a two-level system occurring under the resonant force during the time interval  $[T_1, T_2] T_R/2 = \pi/(2v)$  ( $\pi$ -pulse). Outside this interval the potential (78), as well as the coefficient v in Eqn (79), were assumed to be zero. In the absence of measurement, the resonant action would have led to the transition of the system from level 1 to level 2 with probability 1. One could expect that the measurement somewhat reduces the probability of transition, but allows continuous monitoring of the state of the system in the course of transition.

The set of equations (79) has been solved numerically for many functions E(t) selected at random. For each function E(t) the relevant functions  $C_1(t)$  and  $C_2(t)$  were calculated, and also the probability density  $P[E] = |C_1(T)|^2 + |C_2(T)|^2$ of the measurement readout [E]. The probability distribution of different measurement readouts was obtained, and the behavior of the system depending on the measurement readout analyzed. The behavior of the system is graphically represented by the function

$$\mathcal{P}_{2}(t) = |c_{2}(t)|^{2} = \frac{|C_{2}(t)|^{2}}{|C_{1}(t)|^{2} + |C_{2}(t)|^{2}}$$

In the absence of measurement this curve would go from zero to one over the duration of the  $\pi$ -pulse. In the presence of measurement the curve may behave differently depending on the measurement readout [E].

Figure 3 shows the results of a numerical calculation based on the set of equations (79) for a moderately hard measurement. The density diagrams for curves [E] (upper row) and  $[\mathcal{P}_2]$  (lower row) indicate clearly which of these curves are more probable. It is very important to remember that the curves [E] have been smoothed before putting them



Figure 3. Monitoring of a quantum transition with a moderately hard continuous measurement.

on the diagram. Smoothing was performed in the time scale slightly smaller than the transition time  $T_{\rm R}/2$ . It is because of such smoothing that the curves correctly represent the process of transition: curves without smoothing exhibit fast oscillations, and the information they carry is concealed.

The left-hand pair of diagrams in Fig. 3 represents all possible measurement readouts. We see that they are clearly split into two classes, in one of which the measurement readouts point to the presence of a transition, while in the other the results indicate that the transition does not take place. The middle and right-hand pairs of diagrams represent these two classes of results separately. From these diagrams we see that there is a correlation between the curves [E] and  $[\mathcal{P}_2]$  — that is, the behavior of the system with a reasonable probability corresponds to the result obtained, although there is some probability of error (the measurement noise).

The general conclusions can be formulated as follows:

— The smoothed measurement readout [E] gives a correct description of the evolution of the system with a probability of 80%.

— In the presence of measurement, the probability of transition decreases, and is close to 1/2 in the intermediate regime of measurement.

# 4.3 Realization of continuous measurement of energy

A practical implementation of continuous energy measurement in a two-level system was proposed in Ref. [19]. We take an isolated polarized atom, apply a  $\pi$ -pulse of resonant radiation to induce a transition between the levels, and measure the energy by shooting electrons one by one and examining whether the electron is deflected or continues in the initial direction. Scattering is caused by the dipole moment of the atom, which in turn depends on which level the atom occupies at the time. If the state of the atom is a superposition of states with a definite energy, the probability of scattering depends on the coefficients in the superposition, and thus on the mean energy of the atom in the given state. This in principle allows an estimate of the energy of the atom, whereas the back effect of the electrons on the atom results in the evolution of the atom described by the complex Hamiltonian (see Section 4.2).

This scheme of continuous measurement can be considerably generalized (this will be done in our subsequent paper [79]). In place of scattering of electrons by the atom, it is sufficient to arrange a long series of observations of a twolevel system with an auxiliary measuring system (Fig. 4). The interaction involved in the observation must be weak, so as not to modify the state of the system to any considerable extent. Then the information gained through each observation in the series is also small, but a long series of observations will give information which is adequately represented with the aid of an RPI or the Schrödinger equation with a complex Hamiltonian (79). The function E(t), which represents the measurement readout in this equation, is constructed from experimental data in the following fashion.

Each observation may give either of two possible results: positive, if the state of the measuring device after the interaction differs considerably from its initial state, or negative, if the end state of the measuring device is close to the initial state. A long series of observations is split into shorter (but still long enough) chains of N observations each. For each of these N-series we find the ratio  $n = N_+/N$  of positive outcomes of scattering (number of deflected electrons) to the total number of electrons directed at the atom.



**Figure 4.** Short soft observation, repeated to implement the continuous energy measurement in a two-level system:  $\Phi_0$  is the state of the device before measurement,  $\Phi_+$  ( $\Phi_-$ ) is the state of the device after a positive (negative) result of observation, and D<sub>1</sub>, D<sub>2</sub> are detectors.

This ratio is the experimental evaluation of the probability of scattering for the *N*-series in question. Since the probability depends on the mean energy, a simple calculation will also give an estimate for the mean energy *E*. As a result, we get one point on the curve [E]. Performing the same procedure for each of the *N*-series, we obtain a curve E(t) which represents the result of continuous measurement and enters the equation with the complex Hamiltonian (79).

This train of argument more or less supports the conclusion that a series of electron scatterings by the atom leads to a pattern predicted by the theory of continuous measurements — that is, to the behavior represented by the solution of Eqn (79). As a matter of fact, however, one can give a complete quantum mechanical treatment of this measuring configuration, and prove that there is full agreement between the predictions of the conventional quantum mechanical analysis and the results of the phenomenological RPI approach. This will be done in the forthcoming publication [79].

It is very interesting that a broad class of realizations results in behavior of the system described by the same effective Hamiltonian with the additional quadratic imaginary term. The behavior of the measured system does not depend on the particulars of the measuring procedure — it only depends on the single constant  $\kappa$  (or  $T_{\rm lr}$ , which is equivalent) that appears as a certain combination of the parameters of the measuring system. This proves that the simple equation with the effective complex Hamiltonian, obtained through the RPI approach, is not a bad approximation. The behavior described by this equation is indeed characteristic of real measuring systems.

# 5. Conclusion

In this paper we have considered the continuous measurement of a quantum system, which is a special dissipative process. The continuous measurement can be intentional (as in the case of repeated soft observations, Section 4.3), or spontaneous (as in the case of the diffusion of particle, Section 2.2). In any case, the continuous measurement is accompanied by gradual decoherence — destruction of quantum superpositions of those states that are distinguished (resolved) by the measuring environment.

In addition, the continuous measurement may serve as a model of the process of decoherence associated with the measurement of a discrete variable (Section 4.1). If the description of such measurement disregards its finite duration, the process is adequately described by the von Neumann reduction postulate: instantaneous decoherence. When it is desirable to monitor the time evolution of decoherence, however, we come to deal with continuous measurement and gradual decoherence.

The continuous measurement of a quantum system gives specific features to its evolution, primarily the appearance of classical elements in addition to the quantum ones. Mathematically, such evolution can be described in different ways: using the Lindblad master equation, the stochastic Schrödinger equation, or the equation with the complex Hamiltonian for the state vector (see Section 3). In the last case, the imaginary constituent of the Hamiltonian, which accounts for the effects of the measuring medium, depends on the information obtained in the course of continuous measurement. This is of utmost importance, being a demonstration of the dynamic role of information [62].

The continuous measurement can be softer or harder, depending on the strength of interaction with the measuring medium (mathematically, this is determined by the magnitude of the imaginary term in the Hamiltonian). Soft measurement has little effect on the dynamics of the measuring system, but the information about the system is small. Hard measurement gives more information but considerably modifies the dynamics. In the limit of very hard measurement of the discrete observable we are dealing with the Zeno effect: the system is frozen, the transitions between the states with different values of the observable become unlikely (Section 4.1).

The most interesting is the intermediate regime of measurement, when the measurement is not hard enough to freeze the dynamics of the system completely, but is sufficiently hard to gain a reasonable amount of information. In this regime it becomes possible to monitor the quantum transition, although such information is gained at the expense of reducing the probability of transition (Section 4.2). The continuous measurement permitting the monitoring of a quantum transition can be realized as a series of soft short interactions of the two-level system with some auxiliary system (Section 4.3). One possible realization employs a series of electron scatterings by the atom.

Disputes on conceptual matters have never ceased since the early days of quantum mechanics. The root of these problems is the incompatibility of the classical and quantum descriptions of physical systems and processes. This incompatibility is most dramatically manifested in the measurements aimed at obtaining classical information about a quantum system. Today we are able to study in detail the behavior of the measured system (and the measuring system if necessary) in the course of measurement, and this is the subject of this paper (see also the excellent review [12]). The conceptual problems of the theory of measurement, however, remain essentially unresolved.

Simplifying the situation to the utmost, we may say the following. The measurement is associated with the selection of one out of many alternatives. We know well what happens when a particular alternative is selected, and can calculate the probability of each alternative. This gives answers to all questions that may arise in practice. How and why the selection is made, however, remains obscure. This question is hard to answer because this requires reconciling the quantum and the classical visions of the world. The attempts at solving this problem lead to very unusual constructions, the many-worlds interpretation of quantum mechanics being one example [80]. This scope of problems that may be regarded as the conceptual aspect of decoherence has been left out completely from the present paper.

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