Physics – Uspekhi **41** (7) 627 – 638 (1998)

REVIEWS OF TOPICAL PROBLEMS

PACS numbers: 02.10.Sp, **11.25.–w**, **11.55.–m**

Introduction to string dualities†

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Contents

Abstract. Recent progress in superstring theory is reviewed. It is shown how S- ,T- and U-dualities arise in the study of string compactifications, solitons and D-branes to interrelate string theories previously thought to be completely different. Though there are still no proofs for a number of statements, dualities provide an insight not only into string theory itself, but also into geometry and supergravity. Special attention is given to physical aspects which may escape notice in specific problems. The article is intended for a very general reader with little or no background knowledge.

1. Introduction

String theory was founded in the seventies as an attempt to explain strong interactions [1]. Rather than succeed in the initial realm, it led to the emergence of a new branch of theoretical physics. Nowadays string theory has a very broad range of applications: from black hole thermodynamics to the Grand Unification of interaction forces [2]. Indeed, numerous earlier attempts at unification fit nicely into the string theory framework. Being a self-consistent quantum theory it includes gravity. On the other hand, at low energies string dynamics reduces to the Yang – Mills action which is a vital ingredient of all realistic models of particle interactions.

As we already mentioned string theory incorporates gravitons (and higher spin states) that lead to UV divergencies in quantum perturbation theory. String theory avoids this problem by considering one-dimensional strings instead of point particles. Therefore, at large scales (low energy) we recover general relativity, while the string size provides a natural cut-off at small distances (a potential source of divergences). This concept also changes our understanding of space – time geometry. At large scales it resembles classical Euclidean geometry, but attains a completely unusual string geometry when points become close to each other — there are no well-defined separated points any more. One could hope for a generalization of this idea to *membranes*‡ and objects of arbitrary dimension: p-*branes*. We shall use these designations later, taking p to be the spatial dimensionality. In this notation, point particles become 0-branes, strings — 1-branes, and instantons are (-1)-branes as they are localized in all space and time directions, etc. We can write the classical action, similar to the Nambu – Goto action for strings [2 – 4]:

$$S = -T_p \int \mathrm{d}^{p+1}\xi \left[-\det\left(\partial_i X^\mu\, \partial_j X^\nu \eta_{\mu\nu}\right)\right]^{1/2}, \tag{1}$$

$$[T_p] = \left[(\mathrm{mass})^{p+1}\right], \tag{2}$$

so that the brane tension T_p has dimension $p+1$ in mass units. X^μ provide a map from the $(p+1)$-dimensional world-volume $\xi = (\xi_0, \ldots, \xi_p)$ to the D-dimensional Minkowski space with flat metric: $\eta_{\mu\nu} = \mathrm{diag}\,(-,+,\ldots,+)$ and $\mu = 0, \ldots, D-1$.

At least in the string case, this action is equivalent to the action in Polyakov's form:

$$S = T_p \int \mathrm{d}^{p+1}\xi\, \sqrt{\gamma}\, \gamma^{ij} \eta_{\mu\nu}\, \partial_i X^\mu\, \partial_j X^\nu , \tag{3}$$

with $\gamma = \det \gamma_{ij}$. The equations of motion for the metric determine it up to position-dependent normalization factor:

$$\gamma_{ij} \propto \partial_i X^\mu\, \partial_j X_\mu . \tag{4}$$

Pull-back into Eqn (3) leads to the original Nambu – Goto action (1). We must stress here that in the action (3) one has to

† From material of the presentation at the 34th jubilee scientific conference of MIPT "Current problems of fundamental and applied physics and mathematics"..

‡ The italicized special terms are defined in the Glossary.

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Received 4 November 1997
Uspekhi Fizicheskikh Nauk **168** (7) 705 – 717 (1998)
Translated by S G Gukov; edited by L V Semenova

integrate over all distinct metrics and sum over all world-sheet topologies.

For an arbitrary value of p, the theory of p-branes is highly nonlinear. It is a great fortune that in the case $p = 1$ there exists a huge general covariance symmetry. At least on a sphere, it allows us to eliminate the metric from Eqn (3), and therefore to linearize the theory. In this particular case γ^{ij} has exactly the same number of independent components as the number of reparametrizations we can play with: two choices come from the coordinate transformation and the other from the scale invariance. So far so good, but we have considered only the classical action. On the other hand, in quantum theory there is a so-called Weyl anomaly that breaks the scaling invariance. It is proportional to the total *central charge* of the underlying algebra. For the simplest case of a bosonic string, ghost fields contribute 26, so that $c = D - 26$ defines the critical dimension of the bosonic string theory: $D = 26$ [2–5]. If we regarded X's as supersymmetric coordinates, the corresponding supersymmetric (SUSY) *conformal theory* would have a central charge $c = (3/2)D - 15$, i.e. a critical dimension of 10. Similar arguments lead to $D = 2$ for the extended ($N = 2$) supersymmetry, which is not so interesting as the $N = 1$ case.

All the arguments above are supposed to answer the question: "Why strings?", and to show the peculiar role of the one-dimensional (p-)branes. Moreover, a consistent quantum theory may exist in the critical dimension only.

In the case of a free open string the world-sheet integration in the bosonic part of the action (3) runs over all surfaces $\mathcal{M}$ with a boundary. Its variation

$$\delta S = -2T_p \int_{\mathcal{M}} \mathrm{d}^{p+1}\xi\, \delta X_\mu\, \partial_i \partial^i X^\mu + 2T_p \int_{\partial\mathcal{M}} \mathrm{d}^{p+1}\xi\, \delta X_\mu \epsilon_{ij} \partial^j X^\mu \qquad (5)$$

defines classical solutions as *harmonic functions* on the world sheet of a string. The latter are just eigen modes as for a normal violin string. In the case of a closed string we have two sets of left and right modes, correspondingly:

$$X(z,\bar z) = X_{\mathrm{L}}(z) + X_{\mathrm{R}}(\bar z)\,,$$
$$X_{\mathrm{L}}(z) = x_{\mathrm{L}} + \frac{C}{2} - \mathrm{i}\alpha_0 \ln z + \mathrm{i}\sum_{m\neq 0} \frac{\alpha_m}{m z^m}\,, \qquad (6)$$
$$X_{\mathrm{R}}(\bar z) = x_{\mathrm{R}} - \frac{C}{2} - \mathrm{i}\bar\alpha_0 \ln \bar z + \mathrm{i}\sum_{m\neq 0} \frac{\bar\alpha_m}{m \bar z^m}\,,$$

where we introduced the complex coordinates $z = \xi_0 + \mathrm{i}\xi_1$ and $\bar z = \xi_0 - \mathrm{i}\xi_1$†.

Along with the equations of motion, one of the following boundary conditions should also be satisfied. Neumann boundary conditions respect Poincare invariance:

$$\partial_n X_\mu = 0 \leftrightarrow X_{\mathrm{L}}(z) = X_{\mathrm{R}}(\bar z)\,, \qquad (7)$$

and hence momentum is conserved. On the other hand, Dirichlet boundary conditions indicate defects in space – time:

$$\delta X_\mu = 0 \leftrightarrow X_{\mathrm{L}}(z) = -X_{\mathrm{R}}(\bar z)\,. \qquad (8)$$

† The role of the 'unphysical' constant C will become clear later.

The higher excited modes α_m and $\bar\alpha_m$ enjoy harmonic oscillator algebra for each mode:

$$[\alpha_m, \alpha_n] = m\delta_{m+n,0}\,, \qquad (9)$$

and similarly for $\bar\alpha_m$, so that an arbitrary excitation can be represented as

$$\alpha_{-m_1}\ldots\alpha_{-m_r}\bar\alpha_{-m'_1}\ldots\bar\alpha_{-m'_s}|0,k\rangle\,. \qquad (10)$$

The total number of excited oscillators is given by

$$L = m_1 + \ldots + m_r\,, \qquad \bar L = m'_1 + \ldots + m'_s\,. \qquad (11)$$

Five different superstring theories have emerged over the progress of the theory: type I string theory, type IIA and IIB, and two heterotic string theories with gauge groups SO(32) and $\mathrm{E}_8 \times \mathrm{E}_8$. Let us briefly explain how these theories come from the simple bosonic string. The theories differ in the way we supersymmetrize the bosonic string, so that the type refers to space – time supersymmetry. Type I theory is a theory of *open* (and *closed*‡) superstrings with $N = 1$ supersymmetry on the world sheet. The boundary conditions on the ends of a string relate two supercharges of chiral and antichiral modes into single supercharge which has 16 real components. In other words, the type I theory has $N = 1$ supersymmetry in space – time. On the contrary, type II theories have two 16-component supercharges and therefore possess $N = 2$ supersymmetry in space – time. The only difference between type II theories concerns the chirality of left and right modes seen by a ten-dimensional observer. In type IIB both sectors are chiral, while in type IIA they are of opposite chirality.

Because only an odd (even) number of gamma-matrices (Γ_μ) can be placed between two spinors of opposite (the same) chirality, type IIA (IIB) theory contains only odd (even) forms. This affects the dimensionality of (Ramond – Ramond) potentials in the spectrum. Hence type IIA (IIB) strings can end only on p-branes with p even (odd) respectively. A heterotic string is somewhat more sophisticated, but more promising from the phenomenological point of view. A compactification of it onto a certain (*Calabi – Yau*) manifold is a good candidate to describe the real world. It has an $N = 1$ right sector like a type I string, and the usual bosonic left sector. As all the theories have $D = 10$ critical dimension, it seems that we are left with $26 - 10 = 16$ extra bosonic degrees of freedom. They become gauge degrees of freedom, 32 in number after fermionization of bosonic fields with the same central charge. This leads us to the SO(32) gauge group. Detailed analysis indicates that there is only one other anomaly-free way to gauge these extra left fields which gives the $\mathrm{E}_8 \times \mathrm{E}_8$ special group.

Before we delve into the wonderful world of dualities let us digress for a moment to fix the notation and terminology.

Say, if a theory is dual to itself, it is said to be self-dual. We will discuss the following: T-, S- and U-*dualities*. The former relates one theory (A) on a large manifold to another theory (B) on a small manifold. Below we study the simplest example of compactification on a circle. $\mathrm{O}(n,m,Z)$ is the typical symmetry group of T-duality. It is nonperturbative with respect to *string tension*. However one can compare spectra

‡ A theory of only open strings does not exist. It is always possible to find a section of a world sheet, such that it will be represented by a closed curve corresponding to a closed string.

in both theories to any order in *string coupling constant*. Therefore, this is an example of perturbative T-duality. On the contrary, S-duality is nonperturbative, because it 'inverts' the coupling constant like T-duality inverts the size. This duality implements nonperturbative objects that makes it more interesting. Usually S-duality is associated with the $SL(2, Z)$ symmetry group. The latter (U-duality) was first found in compactifications of type II superstrings on $K3 \times T^2$. It possesses the $E_7(Z)$ duality group which unifies T- and S-duality groups, i.e. $SL(2, Z) \times O(6, 6, Z)$.

In spite of extensive citations throughout the text, here we acquaint the reader with a helpful bibliography. Due to the exhaustive literature [2–5] on perturbative string theory, there is no use in reviewing it once again here. Later we assume that the interested reader, if necessary, can use the mentioned references. The most important and heavily used terms and definitions are collected in the Glossary at the end of the paper. There is also a number of lectures [6–11] on nonperturbative aspects of string theory that will be the main subject of our discussion. In the meantime we hope that the present discussion will complement these reviews. It is also instructive to mention some papers on T-duality and D-branes [12–14], and on S-duality [15–17] and p-branes [18, 19]. The Seiberg–Witten exact solution of $N = 2$ supersymmetric Yang–Mills theories in four dimensions [20] is frequently used to verify string dualities. This fact influenced the recital of S-duality which goes along the lines of Seiberg and Witten. To become more familiar with this subject we recommend the beautiful introductory lectures [21–23].

2. Low-energy action in string theory

Due to supersymmetric nonrenormalization theorems, low-energy methods provide the most powerful tools for understanding string theory dynamics. String theory action reduces to the low-energy effective action of massless string excitations in the large tension limit. This description is possible in closed form because higher excitations in Eqn (10) become infinitely massive.

Consider the excitation of the bosonic string next to the tachyon†. It coincides with the first excited state in type II string theory, namely, the massless state (10) with $L = \bar{L} = 1$:

$$\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}|0, k\rangle \,. \tag{12}$$

The trace, symmetric and antisymmetric parts of this state are the dilaton, the graviton and antisymmetric two-form, respectively.

In the same fashion one can construct type II superstring states from the creation and annihilation operators of the Ramond–Ramond sector. Because of the anticommutation property of fermions, these excitations correspond to even (odd) rank forms in type IIB (IIA) theories respectively.

The massless sector of a type I superstring can be obtained by left-right symmetrization of a type IIB theory. More customarily, it is said that type I theory is the Z_2 orbifold of a type IIB supersting. The fact that the type I string theory is a theory of open superstrings is also an important point. Ends of an open string carry so-called Chan–Paton factors which are the generators of some gauge group. It turns out that the theory is anomaly-free only if the gauge group is SO(32) [2]. For this reason the low-energy action of type I string theory also includes Yang–Mills action with SO(32) gauge group, while Eqn (44) is the typical form of the action for closed strings.

† In type II superstrings the tachyon is eliminated by the GSO projection.

3. T-duality

Let us start our tour with the T-duality which is inherent to theories of extended objects like strings. There is no analog of T-duality in the normal field theory describing the dynamics of point particles.

Consider the compactification of the closed bosonic string theory on a circle [11, 12]. This implies the identification of the 25th coordinate:

$$X^{25} \approx X^{25} + 2\pi R \,, \tag{13}$$

so that Kaluza–Klein excitations now carry discrete momenta: $k^{25} = n/R$. This, of course, also takes place in any quantum field theory in compact space. But the following effect is unique for a string theory. A string can wrap the periodic dimension so that

$$X^{25}(2\pi) = X^{25}(0) + 2\pi m R \,. \tag{14}$$

From the mode expansion (6) we see that the eigen values $k^{25}_{\mathrm{L,R}}$ of operators α_0 and $\tilde{\alpha}_0$ are no more equal to each other: $k^{25}_{\mathrm{L}} - k^{25}_{\mathrm{R}} = mR$. Once the net momentum is equal to $(k^{25}_{\mathrm{L}} + k^{25}_{\mathrm{R}})/2$ we conclude that:

$$k^{25}_{\mathrm{L}} = \frac{mR}{2} + \frac{n}{R} \,, \qquad k^{25}_{\mathrm{R}} = -\frac{mR}{2} + \frac{n}{R} \,. \tag{15}$$

The physical state conditions determine the mass spectrum of the theory:

$$M^2 = k_0^2 - \sum_{\mu=1}^{24} k_{\mu}^2 = \frac{m^2 R^2}{4} + \frac{n^2}{R^2} + L + \bar{L} - 2 \,,$$
$$mn + L - \bar{L} = 0 \,.$$

From the above formulas it is clear that two theories with radii R and $2/R$ have the same spectrum so that the Kaluza–Klein states in one theory correspond to the winding modes in the other, and vice versa‡: $m \leftrightarrow n$. This is exactly the action of T-duality which takes

$$k^{25}_{\mathrm{L}} \to k^{25}_{\mathrm{L}} \,, \qquad k^{25}_{\mathrm{R}} \to -k^{25}_{\mathrm{R}} \,.$$

If we also extend its action to the higher excited states:

$$\alpha^{25}_m \to \alpha^{25}_m \,, \qquad \bar{\alpha}^{25}_m \to -\bar{\alpha}^{25}_m \,,$$

so that

$$X^{25}_{\mathrm{L}} \to X^{25}_{\mathrm{L}} \,, \qquad X^{25}_{\mathrm{R}} \to -X^{25}_{\mathrm{R}} \,,$$

it becomes an exact symmetry of the operator product expansion in any order of string interaction. This accounts for the Z_2: $R \leftrightarrow 2/R$ group associated with T-duality. The moduli space, i.e. the space of parameters in the corresponding conformal theory, is a half-line $R > \sqrt{2}$.

‡ An extra $SU(2) \times SU(2)$ massless multiplet emerges at the self-dual value of radius $R = \sqrt{2}$.

An open string cannot wrap the periodic dimension, so we have to consider the action of T-duality on it separately. The usual open string has Neumann boundary conditions $n^i \partial_i X^\mu = 0$. The duality transform $\partial_i X^{25} = \epsilon_i^j \partial_j \tilde{X}^{25}$ with $\tilde{X}^{25} = X_{\mathrm{L}}^{25} - X_{\mathrm{R}}^{25}$, changes these conditions to the following:

$$n^i \epsilon_i^j \partial_j \tilde{X} = t^j \partial_j \tilde{X} . \tag{16}$$

It states that the longitudinal derivative vanishes, therefore $\tilde{X}^{25}$ is constant along the boundary. As follows from the mode expansion (6), the change of sign of the right-movers X only transforms the Neumann boundary conditions (7) into the Dirichlet boundary conditions (8). If we put $R \to 0$ (i.e. $\tilde{R} \to \infty$), we obtain the space where string end-points are allowed to move on the hyperplane of constant $\tilde{X}^{25} = X_{\mathrm{L}}^{25} - X_{\mathrm{R}}^{25} = C$. This means that open strings can terminate only on this hyperplane, while closed strings (which exist in any theory of open strings) can propagate anywhere. The hyperplane should preserve Lorentz invariance, and therefore must be a dynamic object: D-*brane* [11, 12]. Interestingly, by this definition a D-brane turns out to be a nonperturbative state charged under *Ramond – Ramond fields*. We will return to this question in Section 5.

It is worthy to stress one more important feature of T-duality. For the example we have been discussing throughout this section, the parity operator in the spinor space has the following form†: $-\mathrm{i}\Gamma^{25}\Gamma_{27}$. Therefore, because the duality exchanges the even and odd form potentials, it also exchanges type IIA and IIB string vacua.

4. S-duality

Let us now turn to the more remarkable duality which not only led us to the equivalence of various string vacua but also allowed some models to be solved. For instance, it is a vital ingredient for the solution of the $N = 2$ super-Yang – Mills theory [20]. A very good 'stringy' introduction to the Seiberg – Witten theory can be found in Ref. [21].

Electromagnetic duality in ordinary quantum electrodynamics (QED) is the four-dimensional image of strong-weak coupling duality in string theory. Point particles naturally couple to the vector potential A_ν whose field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{17}$$

satisfies the Maxwell equations:

$$\partial_\mu F^{\mu\nu} = 0 , \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0 , \tag{18}$$

The dual‡ field strength in the last formula is defined via contraction with the antisymmetric tensor $\epsilon^{\mu\nu\gamma\delta}$.

Equations (18) are manifestly symmetric with respect to the exchange $F \leftrightarrow \tilde{F}$. The electric source in the right-hand side of the first equation breaks this symmetry unless an analogous magnetic source is also introduced. This restores the symmetry if we require electric and magnetic charges to transform into each other. However the vector potential A_μ can no longer be globally defined, as it identically satisfies the second equation of set (18). In other words, the nonzero source term in the right-hand side is inconsistent with the Bianchi identities in the dual theory. This fact leads to the Dirac string attached to the monopole. It is not dynamic and represents just a topological singularity in space – time. Defining the vector potential separately on the two hemispheres, surrounding the magnetic charge, allows us to remove the singularity in A_μ. According to Stocks theorem, the net flux through the sphere reduces to integration over the equator, or a loop over the Dirac string. The electron wave function in quantum theory acquires a phase during motion in a magnetic field. In order for the string to stay 'invisible' the change in phase over the loop should be multiple of 2π. This is precisely the Dirac quantization condition for the electric Q_{e} and magnetic Q_{m} charges: $Q_{\mathrm{e}} Q_{\mathrm{m}} = 2\pi n$. For dyons, particles carrying both electric and magnetic charges, the analogous condition is known as the Dirac – Schwinger – Zwanziger quantization condition. For two dyons with charges Q_{e}^1, Q_{m}^1 and Q_{e}^2, Q_{m}^2 respectively, the condition has the following form:

$$Q_{\mathrm{e}}^1 Q_{\mathrm{m}}^2 - Q_{\mathrm{e}}^2 Q_{\mathrm{m}}^1 = 2\pi n , \qquad n \in Z . \tag{19}$$

The simplest solution is given by the charge lattice:

$$Q_{\mathrm{e}} = e n_1 , \qquad Q_{\mathrm{m}} = \frac{2\pi}{e} n_2 , \tag{20}$$

where n_1 and n_2 are integers. Therefore, a theory with an electrically charged particle $(e, 0)$ and a magnetic charge $(0, 2\pi/e)$ possesses electric-magnetic duality which acts as $e \leftrightarrow 2\pi/e$, interchanging weak and strong couplings.

Note that we derived the non-trivial condition on charges assuming only existence of magnetic charge. Unfortunately, so far we have been dealing with QED that does not have a monopole in the spectrum. Hence the duality cannot be a symmetry of the theory.

Let us now turn to the more complicated Georgi – Glashow theory which includes a monopole from the very beginning. The model starts with SO(3) gauge field interacting with the isovector Higgs field ϕ via Lagrangian [22]

$$L = -\frac{1}{4} F_i^{\mu\nu} F_{i\mu\nu} + \frac{1}{2} \mathrm{D}^\mu \phi \mathrm{D}_\mu \phi - V(\phi) , \qquad i = 1, 2, 3 , \tag{21}$$

where D_μ is a covariant derivative and $V(\phi) = \lambda(\phi^2 - a^2)^2/4$ is the Higgs potential. Finite energy solutions must have the profile of the Higgs field asymptotically going to the constant a. The remaining gauge symmetry is U(1), the rotation is over $\phi_a = a\delta_{a,3}$. Therefore, only the Abelian part of the gauge boson survives at large scales. Moreover, an element of the second homotopy group is associated with any finite energy solution according to the ϕ map: $S^2 \to S^2$ from spatial infinity to the vacuum space. As was shown by t'Hooft and Polyakov [28], the latter is defined by an integer proportional to the magnetic charge Q_{m} of the solution. The authors of Ref. [28] suggested a topologically stable solitonic solution that at large distances resembles the Dirac monopole in the effective QED theory. It avoids singularities at small scales restoring the whole non-Abelian gauge symmetry.

Because the monopole considered above has a smooth field profile, calculation of its mass makes sense and turns out to be related to its magnetic charge. As was shown by Bogomolny [29], the mass of the solution with magnetic charge Q_{m} satisfies the following inequality: $M \geqslant a|Q_{\mathrm{m}}|$, that, according to Prasad and Sommerfield, can be saturated in the case of so-called BPS *monopoles* [30]. A similar formula

† In terms of gamma-matrices the chirality operator is defined as $\Gamma_{27} = \Gamma_0 \Gamma_1 \ldots \Gamma_{25}$.

‡ From now on tilde refers to the dual variable.

also holds for vector bosons which acquire mass $M = a|Q_e|$ via the Higgs mechanism. Combination of these two results yields the mass formula for dyons, i.e. solutions carrying both electric and magnetic charges:

$$M = \left|ae(n_e + \tau n_m)\right| . \tag{22}$$

We have introduced the complex coupling constant† $\tau = \theta/2\pi + i4\pi/e^2$.

Theories which have states saturating the BPS *condition* may have the $SL(2, Z)$ duality group generated by the transformations

$$\tau \to -\frac{1}{\tau}, \qquad \tau \to \tau + 1 . \tag{23}$$

The former is a generalization of electric-magnetic duality, and the latter shifts the θ-angle by 2π. For example, the energy of a state in a four-dimensional field theory with extended supersymmetry is bound by the absolute value of the central charge of SUSY algebra. For sufficiently large supersymmetry, the mass formula (22) is protected from quantum corrections. In that case, strong-weak coupling duality is an exact symmetry of the spectrum and may be also a symmetry of the whole quantum theory.

One of the simplest examples with this property is a 'kink' solution in $(1+1)$-dimensional space–time. A solution minimizing the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - U(\phi) \tag{24}$$

with the potential term $U(\phi) = \lambda(\phi^2 - m^2/\lambda)^2/4$ and dimensionless coupling constant $g = \lambda/m^2$ carries conserved topological charge:

$$T = \frac{1}{2}\sqrt{g}\left[\phi(+\infty) - \phi(-\infty)\right] , \tag{25}$$

where $T = +1$ $(T = -1)$ for a kink (antikink), respectively, and ϕ varies from $\phi = -1/\sqrt{g}$ at the minimum of U at $x = -\infty$ to another extremal value $\phi = +1/\sqrt{g}$ when $x = +\infty$. The mass (rest energy) of the kink is given by

$$E = \int \mathrm{d}x\,\frac{1}{2}(\phi')^2 + U(\phi) = \frac{2\sqrt{2}}{3}\,\frac{m}{g} , \tag{26}$$

As one might expect for nonperturbative solution, it is inversely proportional to the coupling constant.

In the supersymmetric case the Lagrangian has the form

$$\mathcal{L} = -\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}\,\bar{\psi}i\gamma_\mu\partial^\mu\psi - \frac{1}{2}\,U^2(\phi) - \frac{1}{2}\,U'(\phi)\psi\psi , \tag{27}$$

where ψ is a Majorana fermion and $U(\phi) = \lambda(\phi^2 - a^2)$. The theory contains two supercharges:

$$Q_\pm = \int \mathrm{d}x\,(\dot{\phi} \pm \phi')\psi_\pm \mp U(\phi)\psi_\mp , \tag{28}$$

† In this expression we take into account the topological θ-term. As was shown by Witten [27], with nonzero θ a monopole acquires an effective electric charge $\theta n_m/2\pi$.

$\psi_\pm$ denotes the left and right components of ψ respectively. They form the supersymmetry algebra:

$$Q_+^2 = P_+ , \quad Q_-^2 = P_- , \quad \{Q_+, Q_-\} = T , \tag{29}$$

where $P_\pm = P_0 \pm P_1$, and the central charge T is purely topological. The relation:

$$P_+ + P_- = (Q_+ + Q_-)^2 - T = (Q_+ - Q_-)^2 + T \tag{30}$$

leads to the *Bogomolny bound* $M \geqslant T/2$ for the rest mass M. This bound is saturated for a state $|s\rangle$ which is annihilated by the following combination of supercharges: $(Q_+ \pm Q_-)|s\rangle = 0$. Indeed, the requirement for preserved supersymmetry is equivalent to taking the square root of the equations of motion. And the fact that the kink solution is annihilated by a combination of supercharges implies saturation of the Bogomolny bound. The other combination of supercharges rather creates a fermionic zero-mode in the kink background. This follows from the (nonzero) supersymmetric variation of the bosonic equation of motion which is equivalent to the fermionic equation of motion in the bosonic background.

This property (that half the supercharges annihilating the classical solution leads to saturation of the Bogomolny bound, while the other half produces fermionic zero-modes in the soliton background) takes place for many solitonic solutions in supersymmetric theories. As a result, the search for configurations that preserve some SUSY provides a shortcut to the solution of the equations of motion. Indeed, it is usually easier to find solutions to these first order equations than to solve the equations of motion, generally of the second order. In the next section we demonstrate this by some examples of p-branes — string solitonic solutions.

5. Branes and the duality of dualities

In Section 2 we found that a consistent theory of open strings possessing T-duality requires the ends of strings to move on static objects extended in p spatial dimensions. They are described by the boundary conditions:

$$\partial_\perp X^{0,\ldots,p} = 0 , \qquad X^{p+1,\ldots,D-1} = 0 , \tag{31}$$

which correspond to the motion of the ends of open strings on a $(p+1)$-dimensional hyperplane (brane world-volume). Because there are no independent open strings in type II theories, their existence is intimately related with the presence of D-branes. We have already mentioned that the latter must be dynamic object in order to preserve Poincare invariance.

On the other hand, from the discussion above we learnt about the existence of string solitons, so-called p-branes. These extended BPS solutions can be obtained from the low-energy supergravity theory of the corresponding string theory. Such solutions are parametrized by the tension T_p and the charge density μ_p with respect to the Ramond–Ramond $(p+1)$-form $A^{(p+1)}$ described by the following term [18] in the world-volume effective action:

$$S_{\text{int}} = \mu_p \int \mathrm{d}^{p+1}\xi\, A^{(p+1)} \tag{32}$$

From now on we omit space–time indices, and specify the rank of the corresponding form by a superscript in parentheses: $A^{(i)}$. Like for any BPS states, the mass formula (22) equates brane tension and its charge μ_p: $T_p = \mu_p$. This fact entails an important corollary for D-branes which frequently turn out to be p-branes. One can evaluate the interaction energy between two identical parallel D-branes [12, 13]. It is proportional to $T_p^2 - \mu_p^2$. Hence dilaton and graviton attraction is exactly canceled by repulsion due to Ramond – Ramond fields. The same happens in supersymmetric field theories where two static monopoles (more generally, BPS states) do not exert a mutual force.

Like any BPS states, p-branes usually break half of the supersymmetry. If we consider intersecting branes or special configurations of them (D-brane instantons), only 1/4 (or less) of the original supersymmetry would be preserved: a half due to each brane.

A consistent quantum theory of such objects encounters serious problems (partly outlined in the Introduction). All we can do at this point is to study the low-energy physics.

One can rewrite Eqn (17) for arbitrary p and D in terms of differential forms. In this language the gauge transformations look like:

$$\delta A^{(p+1)} = \mathrm{d}\Lambda^{(p)} . \tag{33}$$

The gauge-invariant field strength†

$$F^{(p+2)} = \mathrm{d}A^{(p+1)} \tag{34}$$

satisfies the Bianchi identity

$$\mathrm{d}F^{(p+2)} = 0 . \tag{35}$$

Neglecting other interactions, the equations of motion for the $(p+1)$-form potential acquire the form

$$\mathrm{d}\tilde{F}^{(p+2)} = \tilde{J}^{(D-p-1)} , \tag{36}$$

where the source term J is a $(p+1)$-form.

A point particle naturally couples to a vector potential, so that the world-line integral of the one-form is well-defined. In the same way a p-brane is naturally coupled to a $(p+1)$-form [since it has $(p+1)$-dimensional world-volume]. This gives a $(p+2)$-form field strength. From the Hodge conjugation (contraction with a totally antisymmetric tensor) one can easily see that it is dual to a $[D-(p+2) = D-p-2 = \tilde{p}+2]$-form. This gives us the dual pairs (p and $\tilde{p}$) in D dimensions:

$$p + \tilde{p} = D - 4 . \tag{37}$$

For example, this useful formula suggests the possible self-duality of point particles in four dimensions (which is realized by electrons and monopoles). Because our main subject is string theory in the critical dimension of 10, it is useful to guess for a dual of a string, i.e. 1-brane. According to Eqn (37) this is the NS5-*brane*. Under the transformation *String* $\leftrightarrow$ NS5-*brane*, S- and T-dualities also exchange. This means that the essentially nonperturbative S-duality in string theory becomes T-duality from an NS5-brane point of view. As already mentioned, the latter can be checked to any order in perturbation theory. Therefore we have reduced the verification of S-duality to the equivalent statement, string – NS5-brane equivalence, described above. This logic is referred to as the duality of dualities.

Like the Maxwell equation (18), formula (36) implies the existence of 'electric' charge, p-brane, but also magnetic charge, i.e. $\tilde{p} = (\mathrm{D}-\mathrm{p}-4)$-brane.

To restore the duality symmetry introducing a $(\mathrm{D}-\mathrm{d}-4)$-brane we must modify Eqn (34) to

$$F^{(p+2)} = \mathrm{d}A^{(p+1)} + \omega^{(p+2)} , \tag{38}$$

so that the Bianchi identity (35) takes the following form:

$$\mathrm{d}F^{(p+2)} = X^{(p+3)} , \tag{39}$$

where

$$X^{(p+3)} = \mathrm{d}\omega^{(p+2)} . \tag{40}$$

For electric and magnetic charges

$$\mu_p = \int_{S^{D-p-2}} \tilde{F}^{(D-p-2)} = \int_{M^{D-p-1}} \tilde{J}^{(D-p-1)} , \tag{41}$$

$$\mu_{D-p-4} = \int_{S^{p+2}} F^{(p+2)} = \int_{M^{p+3}} J^{(p+3)} \tag{42}$$

we have a generalization of the Dirac quantization condition (19). Following Dirac, one can smoothly define the potential everywhere except at a singular (hyper)string impaling S_{D-p-2}. This singularity may become dangerous once the Aharonov – Bohm experiment involving $(\mathrm{D}-\mathrm{p}-4)$-branes can detect it. Indeed, the $(\mathrm{D}-\mathrm{p}-4)$-brane wave function acquires the phase

$$\mu_d \, \mu_{D-d-4} = 2\pi n \tag{43}$$

over a loop around the singularity.

Surprisingly, D-brane charges expressed via their tensions satisfy this relation with $n = 1$. The simple cylinder diagram somehow knows about this nonperturbative consistency check [13]. Moreover we have just realized that D-branes carry the minimal allowed R – R charges. Hence one can assume that these are the only ones.

The only thing left is the explicit form of the p-brane solutions. Consider a d-rank antisymmetric tensor potential $A_{M_1 M_2 \ldots M_d}$ interacting with gravity g_{MN} and dilaton ϕ via the action

$$I_D(d) = \frac{1}{2} \int \mathrm{d}^D x \sqrt{-g} \biggl\{ R - \frac{1}{2} (\partial\phi)^2 - \frac{1}{2(d+1)!} \exp\bigl[-a(d)\phi\bigr] F_{d+1}^2 \biggr\} . \tag{44}$$

The field strength F_{d+1} of rank $(d+1)$ is defined in Eqn (34), and $a(d)$ is (by now) an unknown constant. Even though here we allow arbitrary D and d, the most interesting cases will be the string and NS5-brane‡. A d-dimensional elementary object, '$(\mathrm{d}-1)$-brane' is described by its trajectory in space – time $X^M(\xi^i)$ $[i = 0, 1, \ldots, (d-1)]$, world-volume metric $\gamma_{ij}(\xi)$ and the tension T_d. It interacts with the bulk

† This formula changes a little on inclusion of the interaction.

‡ A self-dual string in six dimensions is another interesting example.

fields via the action

$$S_d = T_d \int \mathrm{d}^d\xi \Big\{ -\frac{1}{2}\sqrt{-\gamma}\,\gamma^{ij}\partial_i X^M \partial_j X^N g_{MN} \exp\Big[a(d)\frac{\phi}{d}\Big] + \frac{(d-2)}{2}\sqrt{-\gamma} - \frac{1}{d!}\,\varepsilon^{i_1 i_2 \ldots i_d}\partial_{i_1}X^{M_1}\partial_{i_2}X^{M_2}\ldots\partial_{i_d}X^{M_d}A_{M_1 M_2 \ldots M_d}\Big\}. \quad (45)$$

The Bianchi identity (35) stays intact, while the equations of motion (36) for the field A become

$$\mathrm{d}\Big\{\exp\big[-a(\tilde{d})\phi\big]F\Big\} = 2(-1)^{d^2}\tilde{J}, \quad (46)$$

with the source term J of rank d:

$$J^{M_1\ldots M_d} = T_d \int \mathrm{d}^d\xi\, \varepsilon^{i_1 i_2 \ldots i_d}\,\partial_{i_1}X^{M_1}\partial_{i_2}X^{M_2}\ldots\partial_{i_d}X^{M_d}\,\frac{\delta^D(x-X)}{\sqrt{-g}}\,. \quad (47)$$

Now we can define two conserved charges: the Neuter 'electric' charge (41) and the topological 'magnetic' charge (42). The latter does not vanish if the action I_D admits $\tilde{d}$-dimensional solitonic solutions — '$(\tilde{\mathrm{d}}-1)$-branes'. These charges satisfy the Dirac quantization condition (43). Of course at this level it is not clear whether the system has elementary or extended solitonic solutions, and even if it has any — what values of electric and magnetic charges μ_d and $\mu_{\tilde{d}}$ are.

Let us first consider the equations of motion that follow from $I_D + S_d$. The Einstein equations:

$$\sqrt{-g}\Big\{R^{MN} - \frac{1}{2}g^{MN}R - \frac{1}{2}\Big[\partial^M\phi\partial^N\phi - \frac{1}{2}g^{MN}(\partial\phi)^2\Big] - \frac{1}{2}\frac{1}{d!}\Big[F^M_{M_1\ldots M_d}F^{NM_1\ldots M_d} - \frac{1}{2(d+1)}g^{MN}F^2\Big] \times \exp\big[-a(d)\phi\big]\Big\} = \sqrt{-g}\,T^{MN}_{d-1}, \quad (48)$$

contain the $(\mathrm{d}-1)$-brane energy-momentum tensor equal to

$$T^{MN}_{d-1} = -T_d \int \mathrm{d}^d\xi\,\sqrt{-\gamma}\,\gamma^{ij}\partial_i X^M\partial_j X^N \exp\Big[\frac{a(d)\phi}{d}\Big]\frac{\delta^D(x-X)}{\sqrt{-g}}\,. \quad (49)$$

The antisymmetric tensor field equation has the following form:

$$\partial_M\Big(\sqrt{-g}\,\exp\big[-a(d)\phi\big]F^{MM_1\ldots M_d}\Big) = 2T_d\int \mathrm{d}^d\xi\,\varepsilon^{i_1\ldots i_d}\partial_{i_1}X^{M_1}\ldots\partial_{i_d}X^{M_d}\delta^D(x-X)\,, \quad (50)$$

the dilaton equation is

$$\partial_M\big(\sqrt{-g}\,g^{MN}\partial_N\phi\big) + \frac{a(d)}{2(d+1)!}\sqrt{-g}\,\exp\big[-a(d)\phi\big]F^2 = \frac{a(d)T_d}{d}\int \mathrm{d}^d\xi\,\sqrt{-\gamma}\,\gamma^{ij}\partial_i X^M\partial_j X^N g_{MN} \times \exp\Big[\frac{a(d)\phi}{d}\Big]\delta^D(x-X)\,, \quad (51)$$

and the position of the $(\mathrm{d}-1)$-brane is given by

$$\partial_i\Big\{\sqrt{-\gamma}\,\gamma^{ij}\partial_j X^N g_{MN}\exp\Big[\frac{a(d)\phi}{d}\Big]\Big\} - \frac{1}{2}\sqrt{-\gamma}\,\gamma^{ij}\partial_i X^N\partial_j X^P\partial_M\Big\{g_{NP}\exp\Big[\frac{a(d)\phi}{d}\Big]\Big\} - \frac{1}{d!}\,\varepsilon^{i_1\ldots i_d}\partial_{i_1}X^{M_1}\ldots\partial_{i_d}X^{M_d}F_{MM_1\ldots M_d} = 0\,, \quad (52)$$

where

$$\gamma_{ij} = \partial_i X^M\partial_j X^N g_{MN}\,\exp\Big[\frac{a(d)\phi}{d}\Big]\,. \quad (53)$$

To solve these equations of interacting $(\mathrm{d}-1)$-brane fields we introduce the ansatz for the D-dimensional metric g_{MN}, d-form $A_{M_1\ldots M_d}$, the dilaton ϕ and coordinates $X^M(\xi)$ corresponding to the most general coordinate split $d/(D-d)$ invariant under $\mathrm{P}_d \times \mathrm{SO}(D-d)$, where P_d refers to the d-dimensional Poincare group [18, 19]. We divide the indices

$$x^M = (x^\mu, y^m) \quad (54)$$

into $\mu = 0, 1, \ldots, (d-1)$ and $m = d, d+1, \ldots, (D-1)$. Then the space – time interval takes the form

$$\mathrm{d}s^2 = \exp(2A)\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + \exp(2B)\delta_{mn}\,\mathrm{d}y^m\,\mathrm{d}y^n\,, \quad (55)$$

and the gauge d-form is

$$A_{\mu_1\ldots\mu_d} = -\frac{1}{{}^d g}\,\varepsilon_{\mu_1\ldots\mu_d}\exp C\,, \quad (56)$$

where ${}^d g$ is the determinant of $g_{\mu\nu}$, $\varepsilon_{\mu_1\ldots\mu_d} \equiv g_{\mu_1\nu_1}\ldots g_{\mu_d\nu_d}\varepsilon^{\nu_1\ldots\nu_d}$ and $\varepsilon^{012\ldots(d-1)} = 1$, i.e. $A_{01\ldots(d-1)} = -\exp C$. All the other components of $A_{M_1\ldots M_d}$ we put to zero. The P_d invariance requires the (so far) arbitrary functions A, B, C to depend on y^m only. Then $\mathrm{SO}(D-d)$ invariance further restricts the dependence to only the parameter $y = \sqrt{\delta_{mn}y^m y^n}$. There is a similar ansatz for the dilaton

$$\phi = \phi(y)\,. \quad (57)$$

In the same fashion we separate coordinates in the $(\mathrm{d}-1)$-brane sector:

$$X^M = (X^\mu, Y^m) \quad (58)$$

and choose the static gauge

$$X^\mu = \xi^\mu\,, \quad (59)$$

assuming

$$Y_m = \mathrm{const}\,. \quad (60)$$

Substituting all this into the $(\mathrm{d}-1)$-brane equations of motion we get five equations of the four functions A, B, C, ϕ and one unknown number $a(d)$.

Assuming that metric at infinity tends to η_{MN} gives the unique solution:

$$A = \frac{\tilde{d}}{2(d+\tilde{d})}(C - C_0)\,,$$

$$B = -\frac{d}{2(d+\tilde{d})}(C - C_0)\,,$$

$$\frac{a(d)}{2}\,\phi = \frac{a^2(d)}{4}(C - C_0) + C_0\,, \tag{61}$$

where $C_0 = a(d)\phi_0/2$ and ϕ_0 is the dilaton vacuum expectation value. C is determined by

$$\exp(-C) = \exp(-C_0) + \frac{k_d}{y^{\tilde{d}}}\,,$$

$$\tilde{d} > 0 = \exp(-C_0) - \frac{T_d}{\pi}\ln y\,, \qquad \tilde{d} = 0\,, \tag{62}$$

and

$$k_d = \frac{2T_d}{\tilde{d}}\,\Omega_{\tilde{d}+1}\,, \tag{63}$$

where $\Omega_{\tilde{d}+1}$ is the volume of $S^{\tilde{d}+1}$. The parameter $a(d)$ is defined via the relation:

$$a^2(d) = 4 - \frac{2d\tilde{d}}{d+\tilde{d}}\,. \tag{64}$$

It is worthwhile noting that in the case at hand the δ-function coefficient vanishes in both the Einstein equations and the dilaton equation at $y = 0$. Therefore we have found solutions to free equations; the source term shows up only in the equations of motion of the antisymmetric tensor field.

It is important that we could determine the number $a(d)$ requiring the theory to have elementary $(d-1)$-brane solutions.

The mass per unit $(d-1)$-dimensional volume of elementary (d $-$ 1)-brane is equal to

$$\mathcal{M}_d = \int \theta_{00}\,\mathrm{d}^{\,D} - d$$

where θ_{MN} is the total pseudotensor of the gravity–matter system. Therefore we find

$$\mathcal{M}_d = T_d \exp C_0\,. \tag{66}$$

In order to calculate the electric charge μ_d of Eqn (41) it is convenient to introduce the spherical coordinates:

$$y^m = (y, \theta^i)\,, \tag{67}$$

where $i = 1, \ldots, (\tilde{d}+1)$, so that

$$\delta_{mn}\,\mathrm{d}y^m\,\mathrm{d}y^n = \,\mathrm{d}y^2 + y^2\,\mathrm{d}\Omega^2_{\tilde{d}+1}\,, \tag{68}$$

and $\mathrm{d}\Omega^2_{\tilde{d}+1}$ is the metric on $(\tilde{d}+1)$-dimensional unit sphere $S^{\tilde{d}+1}$. Then the antisymmetric tensor field equations give

$$F_{y\mu_1\ldots\mu_d} = -\frac{1}{d_g}\,\varepsilon_{\mu_1\ldots\mu_d}\partial_y \exp C\,. \tag{69}$$

The tensor $\tilde{F}$ dual to F has nonvanishing components in the θ^i direction only:

$$\sqrt{-g}\,\tilde{F}^{\theta_1\ldots\theta_{D-d-1}} = -(-1)^{(D-d)(d+1)}\exp(2C)\partial_y\exp(-C)\,. \tag{70}$$

Therefore, using Eqns (61)–(63) we obtain

$$\exp\left[-a(d)\phi\right]\tilde{F}_{\theta_1\ldots\theta_{D-d-1}} = (-1)^{(D-d)(d+1)}2T_d\,\frac{\varepsilon_{\theta_1\ldots\theta_{D-d-1}}}{\Omega_{\tilde{d}+1}}\,. \tag{71}$$

Equation (41) leads to the relation

$$\mu_d = \sqrt{2}\,T_d(-1)^{(D-d)(d+1)}\,, \tag{72}$$

which means that

$$\mathcal{M}_d = \frac{1}{\sqrt{2}}\,|\mu_d|\exp\frac{a(d)\phi_0}{2}\,. \tag{73}$$

Therefore, although we did not imply any supersymmetry, we have obtained the same relation between mass and charge as in the case of the supersymmetric solutions in Section 3. Actually this is a corollary of our assumption that the ratio of coefficients of the kinetic term and Wess–Zumino term in the σ-model p-brane action (45) is consistent with supersymmetry.

A generalization to exact stable multi-(d $-$ 1)-brane configuration is very straightforward. It is given by a superposition of solutions (61):

$$\exp(-C) = \exp(-C_0) + \sum_l \frac{k_d}{|y - y_l|^2}\,, \tag{74}$$

where y_l refers to the position of each (d $-$ 1)-brane. To make the zero-force condition manifest consider, for instance, the multi-(d $-$ 1)-brane configuration (74) with N source (d $-$ 1)-branes. In general there is no transverse $\mathrm{SO}(D-d)$ symmetry, however there is P_d, Poincare-invariance of the configuration (74). Suppose that every (d $-$ 1)-brane with index l satisfies $X^\mu(l) = \xi^\mu$, so that all of them are aligned. The Lagrangian of every (d $-$ 1)-brane with index l in the background source given by Eqns (54)–(57), follows from Eqn (45) via substitution of γ_{ij} from Eqn (53) and g_{MN} from Eqns (55), (61), (74):

$$\mathcal{L}_d = -T_d\left[\left(-\det\left\{\exp\left[2A + \frac{a(d)\phi}{d}\right]\eta_{ij} + \exp\left[2B + \frac{a(d)\phi}{d}\right]\partial_i Y^m(l)\partial_j Y_m(l)\right\}\right)^{1/2} - \exp C\right]. \tag{75}$$

This corresponds to the potential energy

$$V = T_d\left[\exp\frac{dA + a(d)\phi}{2} - \exp C\right], \tag{76}$$

which vanishes on the dilaton field equations. It is the generalization of the zero-force condition discussed above to arbitrary d and D.

The elementary (d $-$ 1)-branes discussed so far correspond to solutions of the interacting system fields–branes with the action $I_D(d) + S_d$. They have δ-function singularities at $y = 0$ and a nonzero Neuter electric charge μ_d. To conclude this section we find solitonic $(\tilde{\mathrm{d}} - 1)$-branes corresponding to solutions of the equations of motion derived from $I_D(d)$ only. Such solutions are regular at $y = 0$ and describe nonzero topological magnetic charge $\mu_{\tilde{d}}$ (recall $\tilde{d} = D - d - 2$).

To carry this out, we use the ansatz invariant under $P_{\tilde{d}} \times SO(D-\tilde{d})$. Then Eqns (54) and (55) still have the same form as before, but with $\mu = 0, 1, \ldots, (\tilde{d}-1)$ and $m = \tilde{d}, \tilde{d}+1, \ldots, (D-1)$. We take an ansatz for the strength of the antisymmetric field, not for the potential itself. Remember from the above discussions that a nonzero electric charge corresponds to

$$\frac{1}{\sqrt{2}} \exp(-a\phi^*) F_{\tilde{d}+1} = \frac{\mu_d \, \varepsilon_{\tilde{d}+1}}{\Omega_{\tilde{d}+1}} , \qquad (77)$$

where $\varepsilon_{\tilde{d}+1}$ is the volume form on $S^{\tilde{d}+1}$. Therefore, to get a nonzero magnetic charge, we put

$$\frac{1}{\sqrt{2}} F_{d+1} = \frac{\mu_{\tilde{d}} \, \varepsilon_{d+1}}{\Omega_{d+1}} \qquad (78)$$

with the volume form ε_{d+1} on S^{d+1}. Since it is a harmonic form, one cannot write F globally as a curl of A any more. However the Bianchi identity is still satisfied. It is easy to see that all the field equations which follow from $I_D(d)$ are satisfied provided that $d \to \tilde{d}$, so that $a(d) \to a(\tilde{d}) = -a(d)$ in the Einstein equations and the dilaton field equation with zero source term. For future reference it is convenient to write down the explicit solution for $\phi_0 = 0$:

$$\mathrm{d}s^2 = \left(1 + \frac{k_{\tilde{d}}}{y^d}\right)^{-d/(d+\tilde{d})} \mathrm{d}x^\mu \, \mathrm{d}x_\mu + \left(1 + \frac{k_{\tilde{d}}}{y^d}\right)^{\tilde{d}/(d+\tilde{d})} \mathrm{d}y_m \, \mathrm{d}y_m ,$$

$$\exp(2\phi) = \left(1 + \frac{k_{\tilde{d}}}{y^d}\right)^{a(d)} ,$$

$$F_{d+1} = \sqrt{2} \, \mu_{\tilde{d}} \, \varepsilon_{d+1} \Omega_{d+1} . \qquad (79)$$

Note that we found solution over all space including $y = 0$, because now there are no δ-functions in the equations of the dilaton and graviton fields.

The mass per unit $(\tilde{d}-1)$-dimensional volume is equal to:

$$\mathcal{M}_{\tilde{d}} = \frac{1}{\sqrt{2}} |\mu_{\tilde{d}}| \exp\left[\frac{a(\tilde{d})\phi_0}{2}\right] = \frac{1}{\sqrt{2}} |\mu_{\tilde{d}}| \exp\left[-\frac{a(d)\phi_0}{2}\right] . \qquad (80)$$

Note that the dependence on ϕ_0 is such that $\mathcal{M}_{\tilde{d}}$ is large when $\mathcal{M}_d$ is small, and vice versa.

The electric charge of the fundamental solution as well as the magnetic charge of the solitonic solution satisfy the Dirac quantization condition and saturate the Bogomolny bound.

Having considered the interaction of solitonic objects (the zero-force condition), we now turn to the analysis of singularities in the dual picture. Consider the radial trajectory of a $(\mathrm{d}-1)$-brane infalling to the center of a dual $(\tilde{\mathrm{d}}-1)$-brane. We assume that $(\mathrm{d}-1)$-brane and $(\tilde{\mathrm{d}}-1)$-brane do not intersect each other, which is usually the case for $D = d + \tilde{d} + 2$. And we choose the following criterion for a $(\tilde{\mathrm{d}}-1)$-brane to be singular: if a probe $(\mathrm{d}-1)$-brane 'can see' the singularity of the $(\tilde{\mathrm{d}}-1)$-brane source (i.e. the probe brane falls in a finite time), then we regard the $(\tilde{\mathrm{d}}-1)$-brane as a singular solution from the point of view of the $(\mathrm{d}-1)$-brane. In the other case the brane solutions are mutually nonsingular which is the case, for example, for strings and five-branes. The complete final answer is that only point particles and strings are nonsingular with respect to their duals. Fortunately, the string and NS5-brane satisfy this criterion.

6. Conclusions

The primary goal of this review is an attempt to provide the elementary introduction to string dualities, one of the most exciting subjects in modern theoretical physics. Throughout the discussion, a number of specific examples were used to explain the basic ideas underlying string dualities, rather than to review the whole network. Now, with these simple examples in mind, we recuperate the complete picture of nonperturbative string theory from the duality point of view.

The present understanding of string theory implies the following picture. On the huge moduli space of the "String Theory" there is a set of special points corresponding to types IIA, IIB, I, and heterotic strings. Previously we regarded all these as different, independent string theories. Now all the points are connected, and moving on the moduli space we come from one theory to another. But this is not yet a duality. Indeed, two-dimensional theories of a scalar boson on a circle also represent a different point on the moduli space $\mathcal{M} = \{\mathcal{R}\}$, so that all such theories are connected by a continuous variation of radius. Duality implies a stronger relation: the theories must be equivalent. In the example above, only the theories living on radii R and $\tilde{R}$ so that $R\tilde{R} = 2$ are equivalent to each other. This is precisely the T-duality that relates compactifications of type IIA and type IIB theories on circles or torii. Such compactifications preserve all the supersymmetries of the ten-dimensional theory. To preserve only part of the original SUSY, one has to compactify on certain Calabi–Yau manifolds. The conjecture that any such manifold has its 'mirror' partner has not yet been completely proven by mathematicians†. String theory nearly 'proves' this conjecture assuming that every compactification of a type IIA theory has the corresponding type IIB compactification. This is an example of T-duality at work. Another pair of theories related by T-duality are heterotic string theories with gauge groups SO(32) and $E_8 \times E_8$ respectively. Unlike S-duality, the easiest way to check such dualities is via a perturbative expansion.

S-duality connecting theories at strong and weak couplings is one of the most powerful tools in string (field) theory. In order to 'confirm' it, one has to know not only all the perturbation series, but also the nonperturbative effects. Unfortunately there are only a few theories where so much is known‡. On the other hand to make use of S-duality, it is usually conjectured from some indirect arguments, e.g. from analysis of the BPS spectrum, low-energy effective action, etc. Then, assuming it is a true quantum symmetry, one can get 'exact' results. However the existence of such a duality remains conjectural. Among the known examples in string theory, type I and SO(32) heterotic strings are related by S-duality. Type IIB string theory turns out to be self-dual. When the coupling constant goes to infinity, nonperturbative degrees of freedom start to behave like fundamental excitations in the original theory. Therefore it remains to be

† Manifolds X and $\tilde{X}$ are related by so-called *mirror symmetry* [31, 9], if the complex structure X of the first becomes the Kahler structure $\tilde{X}$ of the second, and vice versa. Namely, the duality action on a two-torus exchanges its complex structure $\tau = \mathrm{i}R_1/R_2$ and Kahler structure $\rho = \mathrm{i}R_1R_2$.

‡ Kramers–Wannier duality is a good analog in condensed matter physics [32]. It relates the original spin system to the theory on the dual lattice at the inverse temperature $T \leftrightarrow 1/T$ [32]. Having assumed the existence of the only singular point, one can 'guess' for the phase transition temperature $T = 1$ corresponding to the self-duality point.

clarified what the counterparts for type IIA and $E_8 \times E_8$ heterotic string theories in strong coupling limit are. Somewhat miraculously, it turns out that both of these theories follow from compactification of an eleven-dimensional M-theory on a circle and interval respectively. The type IIA coupling constant is related to size of the circle in such a way that the weak coupling limit corresponds to the small radius limit, so that we end up with a ten-dimensional theory. On the contrary, in the strong coupling limit the circle decompactifies to the eleventh dimension.

D – p-branes, p-dimensional hypersurfaces where open strings can end, provide a typical way for duality verification. D-branes are nonperturbative supergravity solutions charged under Ramond – Ramond fields. Let us see how all the branes can be obtained from the corresponding solutions of maximal $d = 11$ supergravity. M-theory has two fundamental objects: a membrane, a dual (in $d = 11$) five-brane and Kaluza – Klein excitations. Compactification of this theory on a circle leads to a type IIA theory with all the kinds of branes. An NS5-brane in type IIA theory is an M5-brane embedded in noncompact space – time. On the other hand, if one of its dimensions wraps the circle, we end up with a D4-brane in type IIA string theory. A fundamental type IIA string and a D2-brane follow from a membrane in a similar fashion. D0-branes (and dual to them D6-branes) are nothing but Kaluza – Klein excitations along the compact dimension. All the IIB type branes can be obtained by T-duality action in one-space direction. This action along an NS5-brane leaves it intact, while the D-branes of even dimensions (in type IIA) are transformed into D-branes of odd dimensions in IIB theory. The relation between type II theories turns out to be even closer. Namely, the $\mathrm{SL}(2, Z)$ S-duality group of type IIB theory has an interpretation as a modular group of a torus composed of a type IIA compactification circle and an M-theory circle. In this way T- and S-dualities are unified into a U-duality.

One of the tempting directions for future investigations is the geometrical interpretation of $\mathrm{SL}(2, Z)$ self-duality of type IIB theory via compactification of an F-theory on a small torus. Like in the previous passage, the S-duality group is associated with the modular group of the torus [9].

Among other interesting directions are a matrix description of the "String Theory" [33] and applications to low-dimensional field theories as low-energy limits of D-brane configurations [34]. String theory turned out to be very fruitful for a microscopic picture of quantum black holes [24 –26, 9]†. These topics make up the subject of future investigations.

Acknowledgements
I would like to acknowledge E Akhmedov, A Morozov and L Okun for valuable comments on the manuscript. I am also grateful to the organizers and participants of the 33rd Winter School of Theoretical Physics "Duality: Strings and Fields" in Poland and the summer school in Cargese.

The work was supported in part by grant RFBR No 96-15-96939.

† An extremal magnetically charged black hole is obtained via dimensional reduction of a certain configuration of interacting D-branes. And its Bekenstein – Hawking entropy comes from the possible string states [24 – 26, 9].

7. Glossary

BPS inequality (bound) for masses of states in a theory with extended supersymmetry arises from the central extension of supersymmetry algebra:

$$M \geqslant |Z| \,. \tag{81}$$

This condition is named after Bogomolny, Prasad and Sommerfield who derived it in Refs [29, 30]. Some examples can be found in Section 3 where we discussed S-duality (22).

If Eqn (81) becomes an equality, the bound is saturated, e.g., Eqn (22). States whose mass is strictly equal to the absolute value of the central charge fall into 'short' multiplets. The latter is so called because it provides a representation of supersymmetry algebra with a lower number of states (in contrast to the 'long' multiplet composed of states with larger mass). This fact has a very important corollary: the number of states cannot jump during continuous variation of the parameters, e.g. coupling constants. Therefore BPS states remain in short multiplets during motion over the moduli space, and their masses receive no perturbative corrections.

S-duality. An equivalent description of theories in different regions of moduli space is called a duality (later S-, T- and U-duality). In practice this means that two equivalent theories that we used to think of as different theories are actually represented by different regions on the moduli space of a corporate theory. The equivalence of theories is a very strong statement and, for example, implies that theories have the same spectrum of physical states (but not only that). Sometimes it allows the 'solution' of a theory by gluing together perturbative expansions in different regions.

For example, S-duality relates one theory at weak coupling to another theory at strong coupling. Of course, it is not easy to make sure that the states in both theories coincide. To do this one has to know the strong coupling dynamics to all orders in the perturbation theory. Considering BPS states can substantially simplify the analysis. Expressed in algebraic terms they cannot receive quantum corrections, and therefore easily follow to the strong coupling region under the renormalization group.

The modular group $\mathrm{SL}(2, Z)$ is a typical S-duality group. It is generated by two elements. The first

$$\tau \to \tau + 1 \,, \tag{82}$$

usually corresponds to a shift of some topological number by unity and does not affect the physics. The second generator,

$$\tau \to -\frac{1}{\tau} \,, \tag{83}$$

inverts the coupling constant. For the sake of clarity we refer to $N = 2$ supersymmetric Yang – Mills theory [20] where τ is nothing but the complexified coupling constant.

T-duality. Theory A is called T-dual to theory B if theory A compactified on a small manifold is equivalent to theory B on a large manifold. Unlike S-duality, it is perturbative in the sense that it can be checked step by step to any order in the string coupling constant.

Because T-duality is a transformation of a d-dimensional compact part of space – time, usually it is associated with the orthogonal symmetry group $\mathrm{O}(d, d, Z)$ [or $\mathrm{O}(d, d + 16, Z)$ in the case of heterotic string theory compactified on a d-dimensional torus].

U-duality is composed of S- and T-duality in roughly the same way as the S-duality symmetry group $\mathrm{SL}(2, Z)$ is generated by two elements. Acting on a certain brane (BPS state) it is easy to see that T and S do not commute with each other. Hence U-duality is naturally named after unification of T- and S-dualities.

We expect the corresponding symmetry group to have $\mathrm{SL}(2, Z)$ and $\mathrm{O}(d, d, Z)$ as subgroups. In general the group $\mathrm{E}_d(Z)$ has this property.

Bogomolny bound (see BPS inequality).

Membrane. A two-dimensional dynamical surface, an elementary (like ten-dimensional superstring) excitation in M-theory. A nonperturbative two-dimensional object (membrane) is a specific example of p-branes.

D-brane. A hyperplane (hypersurface) of Dirichlet boundary conditions where open strings are allowed to end. In order to preserve Lorentz invariance a D-brane must be a dynamic object. Moreover, D-branes turn out to be sources of Ramond – Ramond fields, and p-branes [12]. In order to specify the world-volume dimension it is convenient to write D – p-brane.

p-brane. Usually we think of a p-brane as a nonperturbative object extended in p spatial directions; and even more frequently — as the corresponding supergravity solution. The latter interacts with a $(p+1)$-form, a low-energy state from the Ramond – Ramond sector.

The classical p-brane action is given by the space (world-volume) swept by the brane during its motion.

NS-brane. A solitonic five-brane, S-dual partner of a fundamental string.

Conformal field theory (CFT). A theory symmetric with respect to scaling transformation. The most popular example is the two-dimensional conformal theory. In that case an infinite-dimensional conformal group is generated by local transformations preserving angles on a two-dimensional surface [2, 4].

It was the conformal symmetry which allowed us to cancel out the two-dimensional metric from Eqn (3).

It is possible to write the explicit dependence of conformal generators via Laurent coefficients L_m of the stress-energy tensor:

$$T(z) = \sum_{m=-\infty}^{+\infty} \frac{L_m}{z^{m+2}}\,, \qquad \bar{T}(\bar{z}) = \sum_{m=-\infty}^{+\infty} \frac{\bar{L}_m}{\bar{z}^{m+2}}\,. \tag{84}$$

To make a connection with Eqn (6), we give the typical dependence of eigen modes in a free theory:

$$L_m \approx \frac{1}{2} \sum_{n=-\infty}^{+\infty} \alpha_n \alpha_{m-n}\,. \tag{85}$$

These operators are called Virasoro generators because they satisfy the following Virasoro algebra:

$$\left[L_m, L_n\right] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{86}$$

and the same for $\bar{L}_m$ with the central charge $\bar{c}$.

Central charge, or in other words the central extension of an algebra, is represented by a c-number term in Eqn (86). Here we talk about Virasoro algebra. It is clear that the central charge is a parameter convenient for classification of theories. For example one can check that the central charge in the theory of a free boson is equal to 1. In the more general case, where X^μ is a coordinate on D-dimensional manifold, $c = \mathrm{D}$.

There is another example of an algebra with a central extension that we discuss in the paper. It is the algebra of extended supersymmetry. The presence of the central terms in such a theory is very important since supersymmetry mixes internal and space – time symmetries. Namely, the central charges determine the lower bound (22) for the masses of states in the theory [29, 30].

We give an example of maximally extended eleven-dimensional supergravity [7]:

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^M)_{\alpha\beta} P_M + (C\Gamma_M N)_{\alpha\beta} Z_{(2)}^{MN} + (C\Gamma_M NPQR)_{\alpha\beta} Z_{(5)}^{MNPQR}\,. \tag{87}$$

It is called maximally extended SUGRA, because the total number of possible central terms in Eqn (87) is equal to the net number of independent components of antisymmetric forms $Z_{(i)}$:

$$11 + 55 + 462 = 528\,. \tag{88}$$

We have already encountered one more example in Section 3 in the discussion of the two-dimensional supersymmetric kink. Then the role of the central charge was played by the topological charge of a kink.

Open (closed) string. A string is a 1-brane or, put differently, a one-dimensional extended object with the topology of an interval (a circle). The open string action (3) and the mode expansion (6) coincide with those of a closed string. The only difference is in the boundary conditions (7) or (8) for an open string. These conditions relate the left ($\tilde{\alpha}_m$) and right (α_m) movers, and therefore preserve only 16 out of 32 independent supercharges in a ten-dimensional superstring theory. That is the reason why an open superstring theory is called a type I theory, while the theories of closed strings, type II theories, possess 32 independent supercharges.

A harmonic function satisfies the Laplace equation which is a typical equation of free motion.

***SL*(2, *Z*) group** is a special linear group of rank 2 generated by the elements

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \tag{89}$$

where $ad - bc = 1$. Frequently these transformations are associated with S-duality action, so that τ stands for a complexified coupling constant.

States in the type IIA theory spectrum. One can impose two sorts of boundary conditions on the fermionic superpartners of the coordinates x_μ. There are periodic (Ramond) and antiperiodic (Neveu – Schwarz) boundary conditions. Depending on which conditions are imposed on fermions in the left and right sectors on string world sheet, we distinguish the following four types of excitations: R – R, NS – NS, R – NS and NS – R. The first two are bosons, while the last two are their fermionic superpartners with respect to the target – space supersymmetry.

The NS – NS sector contains a graviton, a dilaton and an antisymmetric two-tensor, and the R – R sector includes various antisymmetric tensor fields called Ramond – Ramond fields.

Calabi – Yau manifolds are Ricci-flat Kahler manifolds. The simplest example of such a manifold is a two-dimensional (of real dimension) torus. A more non-trivial example is a four-dimensional K3-manifold.

***O*(*n*, *m*) group**. The group whose action leaves the line element

$$ds^2 = dx_1^2 + \ldots + dx_n^2 - dy_1^2 - \ldots - dy_m^2 \quad (90)$$

invariant in the space $R^{n,m}$ with the Lorentz signature. $O(n, m, Z)$ is its subgroup composed of matrices with integer entries.

Coupling constants. There are two expansion parameters in string theory. The first is the string tension: a perturbative parameter in the nonlinear sigma-model (the theory living on the world sheet). The other is the string coupling constant — the parameter associated with the genus expansion of the string world sheet.

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