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# **Bose – Einstein condensation in magnetic traps. Introduction to the theory**

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Abstract. The recent realization of Bose-Einstein condensation in atomic gases opens new possibilities for the observation of macroscopic quantum phenomena. There are two important features of these systems - weak interaction and significant spatial inhomogeneity. Because of this a non-trivial 'zerothorder' theory exists, compared to the 'first-order' Bogolubov theory. The zeroth-order theory is based on the mean-field Gross – Pitaevskii equation for the condensate  $\psi$ -function. The equation is classical in its essence but contains the constant  $\hbar$ explicitly. Phenomena such as collective modes, interference, tunneling, Josephson-like current and quantized vortex lines can be described using this equation. Elementary excitations define the thermodynamic behavior of the system and result in a Landau-type damping of collective modes. Fluctuations of the phase of the condensate wave function restrict the monochromaticity of the Josephson current. Fluctuations of the numbers of quanta result in quantum collapse-revival of the collective oscillations.

# 1. Introduction

In 1924–1925, Albert Einstein published two articles [1] in which Bose's quantum statistics of photons was extended to the case of a perfect gas of a given number of atoms. In the

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Received 5 January 1998 Uspekhi Fizicheskikh Nauk 168 (6) 641–653 (1998) Translated by E G Strel'chenko; edited by A Yaremchuk second of the articles, a spectacular phenomenon, the condensation of atoms in their lowest quantum state, was predicted. It was discovered, namely, that at high enough temperature the momentum distribution of atoms is given by†

$$n_p = \left\{ \exp\left[\frac{\epsilon(p) - \mu}{T}\right] - 1 \right\}^{-1}, \quad T > T_c^0, \tag{1}$$

where  $\mu$  is the chemical potential of the gas and  $\epsilon(p) = p^2/2m$ . The situation, however, is different at the phase transition temperature (we take the degeneration factor g = 1)

$$T_{\rm c}^0 = 3.31 \frac{\hbar^2}{m} (n)^{2/3} \,, \tag{2}$$

where *n* is the gas density. Below this temperature the number  $N_0$  of atoms in the state with  $\mathbf{p} = 0$  is macroscopically large, i.e. proportional to the total number *N* of atoms,

$$N_0 = N \left[ 1 - \left( \frac{T}{T_c} \right)^{3/2} \right], \quad T < T_c^0,$$
 (3)

whereas the rest of the atoms are distributed according to Eqn (1) but with the chemical potential  $\mu = 0$ , so that

$$n_p = \left\{ \exp\left[\frac{\epsilon(p)}{T}\right] - 1 \right\}^{-1}, \quad T < T_c^0.$$
(4)

This 'Bose-Einstein condensation' (BEC) plays an important role in the theory of physical systems such as superfluid helium, electrons in superconductors, and excitons in insulators. In all these cases, however, BEC is masked

<sup>†</sup> Throughout this paper, the convention  $k_{\rm B} = 1$  is used.

either by a strong interaction or by the complexities of a system. For example, although hardly anybody can doubt the existence of Bose condensate in superfluid <sup>4</sup>He, complicated theoretical calculations are needed when using neutron scattering data to reliably determine the number of atoms in the condensate.

As recently as 1995, a sophisticated experimental technique enabled BEC to be observed directly in alkaline-earth metal vapors in a work which, in my view, crowns the development of 20th century experimental physics [2-4].

Due to their magnetic moments, alkaline-earth atoms have the ability to be confined in an appropriately arranged 'magnetic trap.' Once trapped, the atoms must be cooled, first by 'laser cooling' and then by evaporation from the trap. Since high-energy atoms evaporate more rapidly, the latter process makes the gas cooler (in full analogy to what happens to tea in a cup). The resulting temperatures of a few hundred nK have allowed condensation to be observed. At present, several experimental groups have already been able to achieve BEC.

These experiments, while confirming the famous prediction by Einstein, present scientists with a new form of substance in which strong spatial inhomogeneity makes quantum effects very important on a macroscopic scale.

The present paper reviews the main theoretical methods currently used to treat a magnetic-field-trapped Bose-condensed gas and, in contrast to — and hopefully complementing — our previous *Uspekhi* publication [5], is mostly concerned with the quantitative aspects of the problem.

As already mentioned, magnetically trapped atoms are kept near the potential minimum of the trap. Their interaction with the trap can be described by an external potential  $V_{\text{ext}}(\mathbf{r})$  which, under typical experimental conditions, we will assume to be harmonic,

$$V_{\text{ext}}(\mathbf{r}) = \frac{m}{2} (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2).$$
 (5)

For a perfect Bose gas in a trap, the transition temperature and the gas density at the center of the trap are related by Eqn (2), which, for a harmonic trap, gives [6]

$$T_{\rm c}^0 = \hbar \omega_{\rm H} \left(\frac{N}{\zeta(3)}\right)^{1/3}, \quad \omega_{\rm H} = \left(\omega_x \omega_y \omega_z\right)^{1/3}.$$
 (6)

For a phase transition to be sharp, the 'thermodynamic limit' should be taken. In our particular case, however, we should simultaneously let  $N \to \infty$  keeping constant the product  $\omega_{\rm H} N^{1/3}$  (i.e.,  $T_{\rm c}^0$ ) in doing so [7]. Note also that while in a homogeneous perfect Bose gas a phase transition is of third order, so that at  $T_{\rm c}^0$  there is a jump in the temperature derivative of the specific heat; in a trap we are dealing with a second-order transition already in a perfect gas.

If the rarefaction is great enough, the transition temperature expression (6) can also be applied to good accuracy to an interacting gas. To describe the spectral distribution of condensed atoms, however, the interatomic interaction and the energy of vacuum quantum oscillations are to be taken into account. Indeed, in the absence of these effects the condensate, which has no thermal pressure, would contract itself into a point at the minimum of the potential energy. Although the effects mentioned above prevent this collapse, the condensate still forms a sharp density peak in the neighborhood of this point. Clearly, a quantitative description of such phenomena requires that a theory of inhomogeneous Bose-condensed gas be developed.

#### 2. Condensate at zero temperature

# 2.1 Basic equations

The first theoretical treatment of a rarefied Bose gas was given in 1947 when Bogolyubov [8] separated the classical part (condensate) from the second-quantized operator  $\psi$  of atom annihilation,  $\hat{\psi}(\mathbf{r}, t) \approx \sqrt{n_0} + \hat{\theta}(\mathbf{r}, t)$ , where  $\sqrt{n_0} = c$ -number = const. For a rarefied gas,  $\theta \ll \sqrt{n_0}$ . Here the condensate density  $n_0 = N_0/V$ , where  $N_0$  is the number of atoms in the condensate. Treating the quantum part of the  $\psi$  operator as a perturbation, Bogolyubov developed a 'first-order' (in quantum effects) theory of a spatially homogeneous Bose gas.

The same idea applies to an inhomogeneous gas. The difference is that for this case there exists a non-trivial 'zeroorder' theory in which the operator properties of  $\hat{\psi}$  can be neglected. The point is that now the interaction is not just a small correction, but plays a key role in the behavior of the condensed atoms. Extending Bogolyubov's ansatz for the  $\hat{\psi}$  operator to a spatially inhomogeneous gas yields

$$\hat{\psi}(\mathbf{r},t) \approx \psi_0(\mathbf{r},t) + \hat{\theta}(\mathbf{r},t)$$
 (7)

The task now is to derive an equation for  $\psi_0$ , the classical part of the 'condensate wavefunction.' The starting point is the exact operator equation for the operator  $\psi$  in the Heisenberg representation,

$$i\hbar \frac{\partial}{\partial t} \hat{\psi}(\mathbf{r}, t) = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) + \int \hat{\psi}^{\dagger}(\mathbf{r}', t) V(\mathbf{r}' - \mathbf{r}) \hat{\psi}(\mathbf{r}', t) \, \mathrm{d}\mathbf{r}' \right] \hat{\psi}(\mathbf{r}, t) \,, \quad (8)$$

where  $V_{\text{ext}}(\mathbf{r})$  is the confining potential of the trap and  $V(\mathbf{r}' - \mathbf{r})$  is the atom-atom interaction potential. The difficulty is that neglecting  $\hat{\theta}$  in Eqn (8) is equivalent to neglecting all interatomic correlations, which is unacceptable on the scale of  $r_0$ , the range of V. This difficulty does not exist, however, if V satisfies the validity conditions of the Born approximation. We then can safely replace  $\hat{\psi}$  by  $\psi_0$  and take the function  $\psi_0$  (varying slowly on the scale of  $r_0$ ) out of the integral. The result is

$$i\hbar\frac{\partial}{\partial t}\psi_0(\mathbf{r},t) = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{\rm ext}(\mathbf{r}) + g|\psi_0(\mathbf{r},t)|^2\right]\psi_0(\mathbf{r},t),$$
(9)

where

$$g = \int V(r) \,\mathrm{d}\mathbf{r} \,. \tag{10}$$

Now Eqn (10) can be rewritten as

$$g = \frac{4\pi\hbar^2 a}{m} \,, \tag{11}$$

where *a* is the Born amplitude of the *s*-scattering of lowenergy atoms on each other (the Born approximation is applicable if  $a \ll r_0$ ). Now let us take advantage of the fact that, from general arguments, the low-temperature properties of a rarefied ( $na^3 \ll 1$ ) gas are dominated by the *s*-scattering amplitude whatever the potential V. This means that Eqn (9) is valid independently of the applicability of the Born approximation if the quantity a in Eqn (11) is identified as the exact s-scattering amplitude. (Equation (10) is generally invalid in this case).

Note here that neglecting the correlations in deriving (9) is often referred to as the 'mean field approximation.'

Equation (9) was derived by E P Gross and L P Pitaevskiĭ [10] independently in 1961. Note that it can be obtained by varying a suitably chosen action S,

$$\delta S = \delta \left( -i\hbar \int \psi_0^* \frac{\partial}{\partial t} \psi_0 \, \mathrm{d}\mathbf{r} \, \mathrm{d}t + \int E \, \mathrm{d}t \right) = 0 \,, \tag{12}$$

where the energy functional E is

$$E = \int \left[ \frac{\hbar^2}{2m} |\nabla \psi_0|^2 + V_{\text{ext}}(\mathbf{r}) |\psi_0|^2 + \frac{g}{2} |\psi_0|^4 \right] d\mathbf{r} \,. \tag{13}$$

Equation (9) describes the dynamics of an inhomogeneous rarefied Bose gas at zero temperature, T = 0, and is classical in the sense that the function  $\psi_0$  is not an operator and determines the real spatial distribution of condensate atoms rather than the probability density. The modulus and phase gradient of  $\psi_0$  have a clear classical meaning:

$$\psi_0 = |\psi_0| \exp(\mathbf{i}\Phi), n(\mathbf{r}, t) \approx n_0(\mathbf{r}, t) = |\psi_0|^2, \mathbf{v}(\mathbf{r}, t) = \frac{\hbar}{m} \nabla \Phi,$$
(14)

where n and  $\mathbf{v}$  are the number of atoms per unit volume and the atomic velocity, respectively.

The condensate wavefunction can be said to represent the classical limit of the atomic de Broglie wave, when corpuscular properties do not play a role. Describing the condensate in terms of  $\psi_0$  is similar to representing a classical electromagnetic field in terms of electric and magnetic field strengths obeying Maxwell's equations. However, in contrast to Maxwell's equations, Eqn (9), which is their analogue in the present context, contains the Planck's constant  $\hbar$  explicitly.

Equation (9) can be rewritten as a system of equations for n and  $\Phi$ , one having the familiar hydrodynamic continuity form,

$$\frac{\partial}{\partial t}n + \nabla \cdot (\mathbf{v}n) = 0, \qquad (15)$$

and the other being analogous to the Josephson's equation known from the theory of superconductivity,

$$\hbar \frac{\partial}{\partial t} \Phi + \left(\frac{1}{2} m \mathbf{v}^2 + V_{\text{ext}} + gn - \frac{\hbar^2}{2m\sqrt{n}} \nabla^2 \sqrt{n}\right) = 0. \quad (16)$$

If the gas is in its ground state, the wavefunction  $\psi_0$  varies with time as  $\psi_0 \sim \exp(-i\mu t/\hbar)$ , where  $\mu$  is the chemical potential. Thus, the ground state density distribution is determined by the equation

$$\left(-\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) - \mu + g |\psi_0(\mathbf{r}, t)|^2\right) \psi_0(\mathbf{r}, t) = 0. \quad (17)$$

<sup>†</sup> Henceforth the subscript '0' is omitted when there is no possible confusion.

Equation (17) can also be obtained directly by minimizing Eqn (13) for the energy with the number of particles kept fixed. (A similar equation was previously analyzed by V L Ginzburg and the present author in connection with the superfluidity of liquid helium near the  $\lambda$  point [11]. Note, however, that the coefficients of the equation have distinctly different meanings in these two problems).

#### 2.2 Hydrodynamics

Equation (9) contains an 'internal' parameter with a dimension of length, and the correlation radius  $\xi = \hbar / \sqrt{mgn}$ , whose ratio to the size of the condensate 'cloud' *R* determines the general behavior of the solutions involved. Since both *n* and *R* increase with the number of condensed atoms *N*, for *N* large enough the inequality  $R \ge \xi$  holds. It is readily verified that this condition can be rewritten as

$$N \gg \frac{a_{\rm H}}{a}, \quad a_{\rm H} = \left(\frac{\hbar}{m\omega_H}\right)^{1/2}$$
 (18)

(in fact this inequality is generally well satisfied for all experiments of interest). In this case the 'quantum pressure'  $[\hbar^2/(2m\sqrt{n})]\nabla^2\sqrt{n}$  can be neglected as small compared to the remaining terms in Eqn (16). Taking gradients on both sides of the equation now yields

$$m \ \frac{\partial}{\partial t} \mathbf{v} + \nabla \left( \frac{1}{2} m \mathbf{v}^2 + V_{\text{ext}} + gn \right) = 0.$$
 (19)

Eqns (15) and (19) no longer contain  $\hbar$  and are just classical hydrodynamical equations for the potential motion of a medium with the pressure depending on density as  $P = gn^2/2$ . Note a special symmetry property resulting from this equation of state: it turns out that the governing equations are invariant under the transformations  $\mathbf{v} \to C\mathbf{v}, n \to C^2n, \mathbf{r} \to C\mathbf{r}, C$  being a constant.

Equation (19) shows that in the above large N limit, the ground state density distribution has the simple 'Thomas – Fermi' form [12, 14]

$$n_{\rm TF}(\mathbf{r}) = \frac{\mu - V_{\rm ext}(\mathbf{r})}{g} \,. \tag{20}$$

Recalling the quasiclassical requirement of constancy for the chemical potential of a system in a field,

$$\mu_{\rm l}[n(\mathbf{r})] + V_{\rm ext}(\mathbf{r}) = \mu = \text{const}, \qquad (21)$$

one recognizes that Eqn (20) specifies this requirement for a gas with local chemical potential  $\mu_1(n) = gn$  (the term local meaning the value for the homogeneous gas of the same density *n*).

In the same approximation the chemical potential  $\mu$  is related to the number of atoms *N* by

$$\mu_{\rm TF} = \frac{\hbar\omega_{\rm H}}{2} \left(15N\frac{a}{a_{\rm H}}\right)^{2/5}.$$
(22)

Note that Eqn (20) is meaningful only if the scattering amplitude a, and hence the coupling constant g, are positive, i.e., when atoms effectively repel each other. Indeed, the solution of Eqn (20) for g < 0 is readily verified to correspond to a maximum (rather than minimum) of energy and is therefore completely unstable. The gas in this case has a

negative compressibility, dn/dP < 0. In the opposite limiting case  $N \ll a_{\rm H}/|a|$  the coupling term in Eqn (17) may be neglected, which enables a steady-state solution to be constructed. This latter is metastable, however, in that a strong enough compression will bring the gas into an interaction-dominated state, after which it will compress itself infinitely — in so far, at least, as Eqn (17) holds. The reasoning above implies that a metastable solution may exist only for N below a certain  $N_{\rm c} \sim a_{\rm H}/|a|$  [13, 14]. Although atoms with a < 0 have been studied experimentally [4], this subject need not be pursued here.

One of the most interesting applications of Eqs (15) and (19) is the problem of condensate oscillations, which have been studied experimentally [16, 17] and represent a very effective tool for diagnosing systems of interest here. To find condensate eigenfrequencies, a system of linearized equations (15), (19) must be solved. This has been done by Stringari [18].

Linearizing Eqs (15) and (19) and eliminating the velocity one easily obtains the following equation for the small deviation  $\delta n$  of the density from its equilibrium value (20):

$$m\omega^2 \delta n = -\nabla \cdot \left\{ \left[ \mu - V_{\text{ext}}(\mathbf{r}) \right] \nabla \delta n \right\}.$$
<sup>(23)</sup>

The reason that this equation does not contain the coupling constant g explicitly is that the hydrodynamical limit corresponds, in a sense, to the limit  $g \to \infty$ . Equation (23) can be solved analytically for an isotropic trap  $\omega_x = \omega_y = \omega_z = \omega_H$  when the eigenfrequencies are of the form

$$\omega = \omega_{\rm H} (2n_r^2 + 2n_r l + 3n_r + l)^{1/2}, \qquad (24)$$

where l and  $n_r$  are integers, l determining the angular momentum of the oscillations.

In most experiments thus far, traps have been axisymmetric:  $\omega_x = \omega_y = \omega_{\perp}$ ,  $\omega_z = \lambda \omega_{\perp}$ , implying that the *z* component of the angular momentum is conserved and that the eigenfrequencies may be characterized by specifying the azimuthal quantum number *m*.

Note, first of all, that the motion of the mass center in a harmonic trap is totally separable and that the frequencies of interest therefore have a free-particle form,  $\omega = \omega_{\perp}$ ,  $m = \pm 1$  and  $\omega = \omega_z$ , m = 0. The lowest three 'non-trivial' frequencies are

$$\omega^2 = 2\omega_\perp^2 , \qquad m = \pm 2 , \tag{25}$$

and

$$\omega^{2} = \omega_{\perp}^{2} \left( 2 + \frac{3}{2} \lambda^{2} \mp \frac{1}{2} \sqrt{9\lambda^{4} - 16\lambda^{2} + 16} \right), \quad m = 0.$$
 (26)

Although frequency measurements have now reached an accuracy level where the above hydrodynamic approximation is no longer valid, direct (and very involved) numerical frequency computations using the complete Eqn (9) [18] have showed a very good agreement with experiment.

The hydrodynamic approximation can also be applied to the nonlinear dynamics of the condensate, in which case the additional symmetry property noted above is important. Owing to this symmetry, the hydrodynamical equations have exact solutions in which the density n and velocity v are a second-order polynomial in coordinates and a linear function of coordinates, respectively. Let us next consider the example problem of spherically symmetrical gas pulses in an isotropic trap. Taking the solution in the form  $\mathbf{v} = u(t)\mathbf{r}$ ,  $n = c_1(t) + c_2(t)r^2$ , substituting into Eqns (15) and (19), eliminating  $c_1$  and  $c_2$ , and introducing an unknown function w defined by  $u = \dot{w}/w$ , we obtain [20-22]

$$\ddot{w} + \omega_{\rm H}^2 \frac{w - 1}{w^4} = 0.$$
<sup>(27)</sup>

Equation (27) is readily integrated in quadratures. Linearizing Eqn (27) around the equilibrium value w = 1 yields harmonic oscillations of frequency  $\omega = \sqrt{5}\omega_{\rm H}$  in consistence with Eqn (24) with l = 0,  $n_r = 1$ .

Similar solutions for an anisotropic trap are more complex in nature, indicating in particular the development of chaotic motion at large amplitudes [20].

#### 2.3 Interference and vortices

The phenomena discussed in the preceding section cannot strictly be termed a 'macroscopic quantum effect' since the Plank constant disappeared in proceeding from Eqn (16) to (19). However, the complete equation (9) — and hence the condensate wavefunction concept — can be verified experimentally in a variety of interesting ways.

Unfortunately, the numerical integration of Eqn (9), especially for non-steady multi-dimensional problems, is extremely difficult even for state-of-the-art computation facilities [23-25], and under these circumstances the variational method of Ref. [26] may be of use.

The starting point for this method is to choose a trial wavefunction  $\psi(\mathbf{r}, a_i)$  plausibly approximating the desired solution and containing a number of parameters  $a_i$ . Taking the  $a_i$  to be functions of time t, one substitutes  $\psi[\mathbf{r}, a_i(t)]$  into the expression (12) for S and integrates over dr. The function  $S(\dot{a}_i, a_i)$  so obtained is then varied with respect to the  $a_i$  thus giving a relatively simple system of ordinary first-order differential equations in the  $a_i(t)$ .

Recently, spectacular experiments on condensate interference have been conducted [27]. The authors had a cigarlike trap at their disposal and employed a laser beam at a frequency below the atomic absorption line, as their 'working tool'. Because the dielectric permittivity of the gas satisfies inequality  $\epsilon(\omega) < 1$  in this region, atoms are pushed out of the beam thus dividing the condensate into two halves spatially. Turning off the confining potential then causes the two halves to expand and interfere with each other. Figure 1 shows the experimental results along with those obtained theoretically by integrating Eqn (9) numerically [28]. Theory and experiment agree quite well if the computed curves are roughened enough to account for the finite resolution of the experimental apparatus.

Another interesting quantum effect described by Eqn (9) is the Josephson effect. Suppose the magnetic trap consists of two potential wells with a potential barrier in-between. If the atomic chemical potentials in the wells are different, an oscillating current of atoms will flow between the wells. Let us consider the simplest possible example of a one-dimensional symmetrical trap shown schematically in Fig. 2.

If the two wells contain different numbers of atoms, their chemical potentials,  $\mu_1$  and  $\mu_2$ , are also different. It will be assumed that the barrier between the wells is high enough. There are two types of solutions to Eqn (17). The first type, which we shall denote  $\psi_1(x)$ , corresponds to the chemical potential  $\mu_1$  and is localized in well 1. The second,  $\psi_2(x)$ , with



**Figure 1.** Density distribution for interference between two condensates: (a) theory, (b) experiment, (c) finite resolution theory.



Figure 2. Schematic diagram of the double trap for the Josephson experiment.

chemical potential  $\mu_2$ , is localized in well 2. These two functions overlap only in the classically forbidden portion of the barrier, where they are small in value so that the linear term in the equation is negligible. As a result, the linear combination

$$\psi(x,t) = \psi_1(x) \exp\left(-i\frac{\mu_1 t}{\hbar}\right) + \psi_2(x) \exp\left(-i\frac{\mu_2 t}{\hbar}\right)$$
(28)

represents a time-dependent solution of Eqn (9). The atomic flux corresponding to wavefunction (28) is:

$$I = \frac{i\hbar}{2m} \left[ \psi(x,t) \frac{\partial}{\partial x} \psi^*(x,t) - \psi^*(x,t) \frac{\partial}{\partial x} \psi(x,t) \right].$$
(29)

This has the typical Josephson form

$$I = I_0 \sin \left[ \Phi_2(t) - \Phi_1(t) \right] = I_0 \sin \frac{(\mu_1 - \mu_2)t}{\hbar} , \qquad (30)$$

where  $I_0 = (\hbar/m)(\psi_1\psi'_2 - \psi_2\psi'_1)$  and  $I_0$  is calculated by solving a rather peculiar nonlinear tunneling problem [29]. (The reader is referred to Ref. [30] for a more realistic experimental set-up).

The major feature of superfluid systems is the presence of Onsager-Feynman quantum vortices, i.e., of lines with the property that the phase  $\Phi$  of the condensate wavefunction changes by  $2\pi\varkappa$  ( $\varkappa$  being an integer) in bypassing the line. Clearly, the velocity circulation around a vortex is  $2\pi\hbar\varkappa/m$ . One can employ Eqn (17) to construct the condensate wavefunction in the trap in the presence of such vortices. (Incidentally, it was the problem of vortices in a rarefied Bose gas which was originally addressed in Refs [9, 10]).

Letting the vortex line be directed along the trap symmetry axis z, the desired wavefunction may be taken in the form

$$\psi(\mathbf{r}) = \sqrt{n(r_{\perp}, z)} \exp(i\varkappa\phi), \qquad (31)$$

(where  $r_{\perp}^2 = x^2 + y^2$  and  $\phi$  is an angle in the *x*, *y* plane), which corresponds to a vertex with a tangential velocity  $v = \hbar \varkappa / m r_{\perp}$ . The solution of Eqn (17) for this form is given elsewhere [31]. Figure 3 shows the results for the vortex with  $\varkappa = 1$  in a trap with the parameters from Ref. [2]. It is seen



**Figure 3.** Absolute value of the wavefunction for 5000 <sup>87</sup>Rb atoms for the  $\varkappa = 1$  vortex. Distances in units of  $a_{\perp} = (\hbar/m\omega_{\perp})^{1/2}$ .

that the vortex acts to push atoms away from the axis thus producing a toroidal density distribution.

The problem of creating and observing vortex filaments in a magnetic trap is one of the most challenging problems in the subject of Bose–Einstein condensation.

# 3. Elementary excitations

#### 3.1 Thermodynamical similarity

The structure of a Bose-Einstein condensate is different for traps at finite temperatures than for traps at T = 0. At finite temperatures some of the atoms are outside the condensate,

occupying excited levels produced by the trap potential. These atoms form an 'atmosphere' around the higherdensity condensate cloud and, together with the excitations inside the condensate, determine the thermodynamic functions of the systems. The calculation of these functions therefore requires a detailed study of the excitations. Although such a calculation depends crucially on the assumption of a finite number of atoms in the trap, in this section we will be concerned with the thermodynamic limit  $N \rightarrow \infty$  (with  $N^{1/3} \omega_{\rm H}$  kept finite) which we mentioned in the introduction. We shall see that in this limit the properties of a gas in a trap can be expressed in terms of the properties of a homogeneous gas and that the magnetic field only appears in the combination with N[7] as we mentioned earlier. Since the trap potential varies slowly in the limit  $\omega_{\rm H} \rightarrow 0$ , the quasiclassical condition (21) is satisfied, but this time for the gas as a whole rather than the condensate alone. Then, if  $n(\mu, T)$  is the density of a homogeneous gas as a function of  $\mu$ and T, the gas density at point  $\mathbf{r}$  in the trap is found to be  $n[\mu - V_{\text{ext}}(\mathbf{r}), T]$ . The total number N of atoms is obtained by integrating over the volume. Changing the integration variable to  $\xi \equiv V_{\text{ext}}$  gives

$$N = 2\pi \left(\frac{2}{m\omega_{\rm H}^2}\right)^{3/2} \int_0^\infty n(\mu - \xi, T) \sqrt{\xi} \, \mathrm{d}\xi \,, \tag{32}$$

indicating that trap parameters do not indeed appear unless in the combination  $N^{1/3}\omega_{\rm H}$ . Note that Eqn (32) determines the chemical potential  $\mu$  of the system.

The above formulae hold for any system in a trap in the thermodynamic limit. For a rarefied gas, where interaction is characterized by a single parameter *a*, by simple dimensional arguments  $\mu(N, T) = T_c^0 v(\zeta, T/T_c^0)$ , where  $T_c^0$  is given by Eqn (6),  $\zeta = (N^{1/3}\omega_H ma^2)/\hbar$ , and *v* is a dimensionless function [32]. Note that in the limit considered the condensate has a sharp boundary determined by  $n[\mu(N, T) - V_{\text{ext}}(\mathbf{r}), T] = n_c(T)$ , where  $n_c(T)$  is the homogeneous gas density for which *T* is the transition temperature.

A similar kind of analysis applies to other thermodynamic functions. For the thermodynamic potential  $\Omega(\mu, T)$ , for example,

$$\Omega(\mu, T) = 2\pi \left(\frac{2}{m\omega_{\rm H}^2}\right)^{3/2} \int_0^\infty \Theta(\mu - \xi, T) \sqrt{\xi} \,\mathrm{d}\xi\,,\qquad(33)$$

where  $\Theta(\mu, T)$  is the potential per unit volume of a homogeneous gas. From this it is easily shown that the entropy per particle  $S/N = \sigma(\zeta, T/T_c^0)$ , implying that in two experiments with the same similarity parameter  $\zeta$  we shall have equal entropies provided the values of  $T/T_c^0$  are equal. Calculations show [32] that the thermodynamic functions are actually dominated by the parameter  $\zeta^{1/5}$  and depend only slightly on *N*.

We emphasize that the results above are valid in the thermodynamic limit over the entire temperature range, including the critical region near the transition. One cannot expect, however, that these limiting relations will be very accurate for the values of *N* currently achievable.

#### **3.2 Condensation kinetics**

Despite intense interest in the subject, few reliable results are available on the time evolution of the condensation process. The condensation of a homogeneous perfect gas was treated in considerable detail in Ref. [33] in which Boltzmann's equation is solved by assuming that the distribution function f depends on the atomic energy alone. Boltzmann's equation then becomes

$$\frac{\mathrm{d}f(\epsilon_1)}{\mathrm{d}t} = C \int F[f] \frac{P}{p_1} d\epsilon_1' \,\mathrm{d}\epsilon_2' \,, \tag{34}$$

where the constant *C* is proportional to  $g^2$ , *P* equals the least of the quantities  $p_1$ ,  $p_2$ ,  $p'_1$ ,  $p'_2$ , the region of integration is  $\epsilon_1 < \epsilon'_1 < \infty$ ,  $\epsilon_1 - \epsilon'_1 < \epsilon'_2 < \infty$ ,

$$F[f] = (1+f_1)(1+f_2)f_1'f_2' - (1+f_1')(1+f_2')f_1f_2, \quad (35)$$

and  $\epsilon_2 = \epsilon'_1 + \epsilon'_2 - \epsilon_1$ . By integrating Eqn (34) numerically it is found that if the distribution function is initially a smooth function of the energy, then in some finite time it becomes infinity at  $\epsilon = 0$ , thus signaling that the condensate starts to form. References [34, 35] come to different conclusions, however, so that the question remains open at present. What is going on at the next stage of the process, when the condensate has already formed, is even less known. Some plausible considerations may be found in Ref. [36].

## 3.3 Linearized equations

Turning now to elementary excitations we note that considerable simplifications are achieved by using the 'classical' equation (9). It suffices to find the classical frequencies  $\omega_{cl}$ of the oscillations described by the linearized version of this equation, after which the energy levels of the elementary excitations are written as  $\epsilon = \hbar \omega_{cl}$ .† (Clearly, this procedure is equivalent to the diagonalization of the Hamiltonian in the Bogolyubov approximation).

Let  $\psi = \psi_g(\mathbf{r}) + \theta(\mathbf{r}, t)$ , where  $\psi_g$  is the ground state wavefunction obeying Eqn (17), and  $\theta = u(\mathbf{r}) \exp(-i\omega t) + v(\mathbf{r}) \exp(i\omega t)$ . Substituting this into Eqn (9) and linearizing with respect to *u* and *v* we obtain after a little manipulation [10]

$$\left[-\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) - \mu + 2gn(\mathbf{r})\right] u(\mathbf{r}) + gn(\mathbf{r})v(\mathbf{r}) = \epsilon u(\mathbf{r}),$$
$$\left[-\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) - \mu + 2gn(\mathbf{r})\right]v(\mathbf{r}) + gn(\mathbf{r})u(\mathbf{r}) = -\epsilon v(\mathbf{r}),$$
(36)

where  $n = |\psi_{\rm g}|^2$ .

We are now in a position to perform the 'secondary quantization' of  $\theta$  by introducing the excitation creation,  $\hat{\alpha}$ , and annihilation,  $\hat{\alpha}^{\dagger}$ , operators. The operator  $\hat{\theta}$  can then be expressed as

$$\hat{\theta} = \sum_{j} \left[ u_j(\mathbf{r}) \hat{\alpha}_j + v_j(\mathbf{r})^* \hat{\alpha}_j^{\dagger} \right], \qquad (37)$$

where  $u_j$  and  $v_j$  are solutions of Eqn (36) normalized so that [38]

† In this review we are mainly concerned with the case of low temperatures, when the properties of elementary excitations are temperature independent. Under realistic experimental conditions, however, these properties do exhibit a temperature dependence, and this is rather difficult to take into account theoretically [37, 38]. Note that the very concept of excitations with a temperature dependent spectrum relies on the low-density assumption.

$$\int d\mathbf{r} \left[ u_i^*(\mathbf{r}) u_j(\mathbf{r}) - v_i^*(\mathbf{r}) v_j(\mathbf{r}) \right] = \delta_{ij} , \qquad (38)$$

with  $|u_j(\mathbf{r})|^2 - |v_j(\mathbf{r})|^2$  interpreted as the probability density for the excited state *j*.

#### 3.4 Quasiclassical approximation

Although Eqn (36) is not amenable to an analytical general solution, fortunately, in many cases a quasiclassical approximation can be employed [38, 40, 41]. In this spirit, let

$$u(\mathbf{r}) = \bar{u}(\mathbf{r}) \exp\left[i\varphi(\mathbf{r})\right], \quad v(\mathbf{r}) = \bar{v}(\mathbf{r}) \exp\left[i\varphi(\mathbf{r})\right], \quad (39)$$

where the  $\varphi$  phase is related to the excitation momentum **p** by

$$\mathbf{p} = \hbar \nabla \varphi \,. \tag{40}$$

Now let  $p \ge \hbar/a_{\rm H}$ , where  $a_{\rm H} = (\hbar/m\omega_{\rm H})^{1/2}$  is the magnetic length defined in Eqn (18). [For the sake of simplicity we omit the state-labeling subscript *j* in Eqns (39)–(49)]. Also assume that the amplitudes  $\bar{u}(\mathbf{r})$  and  $\bar{v}(\mathbf{r})$  vary slowly on the scale of  $a_{\rm H}$ . Since the cloud outside of condensate is about  $(2T/m\omega_{\rm H}^2)^{1/2}$  in size, the quasiclassical approximation is valid only for  $T \ge \hbar\omega_{\rm H}$ . Substituting Eqn (39) into (36) and neglecting the derivatives of the amplitudes  $\bar{u}$  and  $\bar{v}$  and the second derivatives of the phase  $\varphi$ , we obtain

$$\begin{bmatrix} \frac{p^2}{2m} + V_{\text{ext}}(\mathbf{r}) - \mu + 2gn(\mathbf{r}) \end{bmatrix} u(\mathbf{r}) + gn(\mathbf{r})v(\mathbf{r}) = \epsilon(\mathbf{p}, \mathbf{r})u(\mathbf{r}),$$
$$\begin{bmatrix} \frac{p^2}{2m} + V_{\text{ext}}(\mathbf{r}) - \mu + 2gn(\mathbf{r}) \end{bmatrix} v(\mathbf{r}) + gn(\mathbf{r})u(\mathbf{r})$$
$$= -\epsilon(\mathbf{p}, \mathbf{r})v(\mathbf{r}). \quad (41)$$

In these equations the momentum  $\mathbf{p}$  is a function of  $\mathbf{r}$  defined by Eqn (40). Writing the compatibility condition for these equations determines the quasiclassical spectrum of elementary excitations [12, 40]:

$$\epsilon(\mathbf{p}, \mathbf{r}) = \left\{ \left[ \frac{p^2}{2m} + V_{\text{ext}}(\mathbf{r}) - \mu + 2gn(\mathbf{r}) \right]^2 - g^2 n^2(\mathbf{r}) \right\}^{1/2}.$$
(42)

There are several interesting points to note about Eqn (42). Inside the condensate cloud, where the Thomas–Fermi expression (20) for the equilibrium condensate density is applicable, Eqn (42) can be rewritten in the form

$$\epsilon(\mathbf{p}, \mathbf{r}) = \left\{ \left[ \frac{p^2}{2m} + gn(\mathbf{r}) \right]^2 - g^2 n^2(\mathbf{r}) \right\}^{1/2}, \tag{43}$$

which is exactly Bogolyubov's homogeneous gas spectrum. In particular, in the  $p \rightarrow 0$  limit the dispersion relation takes the phonon form  $\epsilon = c(\mathbf{r})p$ , with  $c = (gn/m)^{1/2}$  the sound velocity. Importantly, Eqn (43) does not depend explicitly on the external potential but only on the condensate density. This is consistent with Landau's superfluidity theory in that the energy of an elementary excitation depends on  $\mathbf{r}$  only through the coordinate dependence of the liquid density.

In the low-density region outside the condensate cloud, Eqn (42) takes the free-atom form

$$\epsilon(\mathbf{p}, \mathbf{r}) = \frac{p^2}{2m} + V_{\text{ext}}(\mathbf{r}) - \mu$$
(44)

(note that this energy is measured from the chemical potential  $\mu$ , however).

The fact that in the intermediate region the energy  $\epsilon$  depends on both *n* and  $V_{\text{ext}}$  makes it rather difficult to construct a full system of equations for a Bose-gas condensate in a trap at finite temperature. Such a system, an analog of the Landau-Khalatnikov system of equations for superfluid helium (see, e.g., Ref. [42], § 77), should contain a kinetic equation for the momentum and coordinate distributions of elementary excitations, and provide a finite-temperature extension of Eqn (9) for the function  $\psi$ . Such a system is still lacking, however. (See Refs [43] and [44] for a discussion of the relevant problems).

The phase  $\varphi$  satisfies the Hamilton – Jacobi equation

$$\epsilon(\hbar \nabla \varphi, \mathbf{r}) = \epsilon \,, \tag{45}$$

where  $\epsilon$  is the prescribed excitation energy value. Solving Eqn (45) with appropriate boundary conditions at the boundary of the classically allowed region yields quasiclassical conditions for the quantization of the excitation energy in the trap.

Equations (41) give only one relation between v and u. To obtain the second relation necessary to find these functions, a next-order quasiclassical equation is needed. A suitable choice for this purpose is the continuity equation, whose steady-state form is

$$\nabla \cdot \left[ \mathbf{v}_g(|u|^2 - |v|^2) \right] = 0, \qquad (46)$$

where  $\mathbf{v}_g = \partial \epsilon / \partial \mathbf{p}$  is the group velocity of the excitation. This equation is satisfied by setting

$$u = U\Psi(\mathbf{r}), v = V\Psi(\mathbf{r}), \qquad (47)$$

where the function  $\Psi = A(\mathbf{r}) \exp[i\varphi(\mathbf{r})]$  obeys the continuity equation in its conventional form,

$$\nabla \cdot (\mathbf{v}_{g} | \boldsymbol{\Psi} |^{2}) = 0, \qquad (48)$$

with  $|U|^2 - |V|^2 = 1$ . From Eqn (38), the function  $\Psi$  is normalized so that  $\int |\Psi|^2 d\mathbf{r} = 1$ . We thus see that the function  $\Psi$  is a quasiclassical wavefunction of an elementary excitation in the usual sense of the word, with energy (42) playing the role of the Hamiltonian; u and v being auxiliary functions from this viewpoint.

The quantities U and V are now given by the well-known Bogolyubov expressions, which are conveniently rewritten as

$$U, V = \pm \left\{ \frac{\left[\epsilon^2 + g^2 n^2(\mathbf{r})\right]^{1/2} \pm \epsilon}{2\epsilon} \right\}^{1/2}.$$
 (49)

The energy  $\epsilon$  here is different for different excited states. Although we can simply set  $\epsilon = \epsilon_j$ , it is more convenient, in some cases, to treat  $\epsilon$  as the function (42) with  $\hat{\mathbf{p}}$  viewed as an operator acting on  $\Psi_j$ . Then in the averaging procedure, Eqn (50) described below, the quantity  $\epsilon$  will play the role of the classical function  $\epsilon(\mathbf{p}, \mathbf{r})$ .

The function  $\Psi$  can be expressed analytically only in some special cases, for example, in the variable-separating case of an isotropic trap. Very often, however, it suffices to take a quasiclassical limit of expressions like  $\operatorname{tr}(Q) = \sum_j \int \Psi_j^* Q(\hat{\mathbf{p}}, \mathbf{r}) \Psi_j \, d\mathbf{r}$ , where  $Q(\hat{\mathbf{p}}, \mathbf{r})$  is a certain function of the operators  $\hat{\mathbf{p}}$  and  $\mathbf{r}$ . To calculate  $\operatorname{tr}(Q)$ , note that the trace of an operator can generally be calculated with respect to any full basis set. In particular, we can take plane waves  $\Psi_{\mathbf{p}} = \exp(i\mathbf{p}\mathbf{r})/\sqrt{V}$ , where V is the volume of a certain large 'box' containing our system. The sum over j can be replaced by an integration with respect to  $V \, d\mathbf{p}/(2\pi\hbar)^3$ . In the quasiclassical limit we can consider the operators  $\hat{\mathbf{p}}$  and  $\mathbf{r}$  as commutative, and all  $\hat{\mathbf{p}}$  in Q as standing to the right of  $\mathbf{r}$  and hence acting on the function  $\Psi_{\mathbf{p}}$  alone. One then has

$$\operatorname{tr}(Q) = \int \Psi_{\mathbf{p}}^{*}Q(\hat{\mathbf{p}}, \mathbf{r})\Psi_{\mathbf{p}} \frac{V \,\mathrm{d}\mathbf{p} \,\mathrm{d}\mathbf{r}}{(2\pi\hbar)^{3}}$$
$$= \int |\Psi_{\mathbf{p}}|^{2}Q(\mathbf{p}, \mathbf{r}) \frac{V \,\mathrm{d}\mathbf{p} \,\mathrm{d}\mathbf{r}}{(2\pi\hbar)^{3}} = \int Q(\mathbf{p}, \mathbf{r}) \frac{\mathrm{d}\mathbf{p} \,\mathrm{d}\mathbf{r}}{(2\pi\hbar)^{3}}, \quad (50)$$

thus reducing the calculation of tr(Q) to integrating the classical function  $Q(\mathbf{p}, \mathbf{r})$  over the phase space.

The considerations above provide the basis for the calculation of the thermodynamic functions of a Bose-condensed gas in a trap (see Ref. [38] and references therein).

As an example, we next estimate the number of atoms excited from the condensate at finite temperatures. By definition, this number can be written as

$$N_T \equiv \int n_T(\mathbf{r}) \, \mathrm{d}\mathbf{r} = \int \left\langle \theta^{\dagger}(\mathbf{r}) \theta(\mathbf{r}) \right\rangle \mathrm{d}\mathbf{r} \,. \tag{51}$$

The task now is to apply transformation (37) using the quasiclassical expressions (47) and (49). Noting that the expectation value  $\langle \alpha_i^{\dagger} \alpha_i \rangle$  is the Bose distribution function

$$\langle \alpha_j^{\dagger} \alpha_j \rangle \to f(\mathbf{p}, \mathbf{r}) = \frac{1}{\exp\left[\epsilon(\mathbf{p}, \mathbf{r})/T\right] - 1},$$
(52)

and using Eqn (50) to sum over the states yields

$$N_T = \int n_T(\mathbf{r}) \, \mathrm{d}\mathbf{r} = \int F(\mathbf{p}, \mathbf{r}) \frac{\mathrm{d}\mathbf{p} \, \mathrm{d}\mathbf{r}}{\left(2\pi\hbar\right)^3} \,, \tag{53}$$

where the atomic distribution function turns out to be

$$F(\mathbf{p}, \mathbf{r}) = \left[ U^2(\mathbf{p}, \mathbf{r}) + V^2(\mathbf{p}, \mathbf{r}) \right] f(\mathbf{p}, \mathbf{r})$$
$$= -\left[ \frac{\partial \epsilon(\mathbf{p}, \mathbf{r})}{\partial \mu} \right]_n f(\mathbf{p}, \mathbf{r})$$
(54)

with

$$\left[\frac{\partial\epsilon(\mathbf{p},\mathbf{r})}{\partial\mu}\right]_{n} = -\frac{p^{2}/2m + V_{\text{ext}}(\mathbf{r}) - \mu + 2gn(\mathbf{r})}{\epsilon(\mathbf{p},\mathbf{r})}.$$
 (55)

It should be noted that the distribution functions of atoms and excitations differ only if the excitation energy is low enough. For high energies  $\partial \epsilon / \partial \mu \rightarrow -1$ , so that  $F(\mathbf{p}, \mathbf{r}) = f(\mathbf{p}, \mathbf{r})$ . The same is true for the region outside the condensate [cf. Eqn (44)], where there is no line of separation between the excitation and atom concepts. In the opposite limit of small  $\rho$ , i.e., for phonons,  $F(\mathbf{p}, \mathbf{r}) = mc(\mathbf{r})f(\mathbf{p}, \mathbf{r})/p$ , so that  $F \ge f$ .

The chemical potential  $\mu$  is defined by specifying the total number of atoms,

$$N = N_0(T) + N_T = \int n_0(\mathbf{r}) \, \mathrm{d}\mathbf{r} + \int n_T(\mathbf{r}) \, \mathrm{d}\mathbf{r} \,.$$
(56)

To conclude this section, let consider a limiting case opposite to the quasiclassical example above, namely the hydrodynamical limit of low-energy excitations. In this case the frequencies can be found by solving Eqn (23) and the functions u and v have the form [44]

$$u, v = \left[\pm \sqrt{\frac{gn(\mathbf{r})}{2\omega}} + \frac{1}{2}\sqrt{\frac{\omega}{2gn(\mathbf{r})}}\right]\chi(\mathbf{r}), \qquad (57)$$

where  $\chi$  is the eigenfunction of Eqn (23) obeying the normalization condition  $\int |\chi|^2 d\mathbf{r} = 1$ .

### 3.5 Landau damping

We mentioned earlier that measuring condensate oscillation frequencies gives values that agree well with theoretical predictions. Another oscillation property measured experimentally is the damping, which also poses a challenging theoretical problem. The most probable damping mechanism is Landau damping, in which the vibration energy is absorbed by elementary excitations in a non-collisional process. (This mechanism was first suggested in Ref. [46]).

The standard approach to Landau damping is to use the kinetic equation. For a gas in an external potential, the problem is more conveniently treated by direct perturbation theory, as was first done by Landau and Rumer in their work on ultrasound attenuation in insulators (see, e.g., Ref. [42], § 73). Let us denote by *E* the energy of a classical oscillation of frequency  $\omega$  (we assume the number of quanta to be large when speaking of classical oscillations). The elementary excitations existing in the system may both absorb and emit oscillation quanta. The energy loss rate is given by

$$\dot{E} = -\hbar\omega[W^{(a)} - W^{(e)}], \qquad (58)$$

where  $W^{(a)}$  and  $W^{(e)}$  is the probability for a quantum of frequency  $\hbar\omega$  to be absorbed (emitted). In perturbation theory the transition probability is given by

$$W = \pi \sum_{i,k} \left| \langle k | V_{\text{int}} | i \rangle \right|^2, \tag{59}$$

where the matrix element is for the transition of the excitation from state *i* to state *k*. Note that  $\epsilon_k = \epsilon_i + \hbar \omega$  when the quantum is absorbed (a) and  $\epsilon_k = \epsilon_i - \hbar \omega$  when it is emitted (e).

The interaction operator here is given by the fourth-order term in  $\psi$  which is involved in the energy expression (13) and should in this context be treated as an 'effective Hamiltonian':

$$\hat{V}_{\text{int}} = \frac{g}{2} \int \hat{\psi}^{\dagger} \hat{\psi}^{\dagger} \hat{\psi} \hat{\psi} \, \mathrm{d}\mathbf{r} \,. \tag{60}$$

The operator  $\hat{\psi}$  in Eqn (60) should be represented in the form  $\hat{\psi} = \psi_0 + \hat{\theta}$ , with  $\hat{\theta}$  given by Eqn (37). In the sum (37) we should distinguish between operators for collective oscillations, whose damping we are seeking and for which we

<sup>†</sup> The same approach can be used to retrieve Belyaev's [47] result on the probability of one phonon decaying into two, when both the *i* and *k* excitations are emitted and  $\epsilon_k + \epsilon_i = \hbar\omega$  (see Ref. [48]).

introduce the quantities  $u_{osc}$ ,  $v_{osc}$ ,  $\hat{\alpha}_{osc}$ ,  $\hat{\alpha}_{osc}^{\dagger}$ , and those for 'thermal' excitations, which will be referenced by employing the subscripts *i*, *k* as in Eqn (59). Retaining only the terms linear in  $\hat{\alpha}_{osc}$  and  $\hat{\alpha}_{osc}^{\dagger}$ , and those with products  $\hat{\alpha}_{k}^{\dagger}\hat{\alpha}_{i}$  or  $\hat{\alpha}_{k}\hat{\alpha}_{i}^{\dagger}$  in Eqn (60), we now turn to the process of interest here, in which a quantum  $\hbar\omega$  is absorbed or emitted with the *i* excitation going over to the *k* excitation.† This yields the following result for the energy loss rate

$$\dot{E} = -\omega 2\pi \sum_{ik} |A_{ki}|^2 \delta(\epsilon_k - \epsilon_i - \hbar\omega) (f_k - f_i) , \qquad (61)$$

where  $E = \hbar \omega n_{\text{osc}}$  is the energy of the classical oscillation  $(n_{\text{osc}} \ge 1)$  and the matrix element is [49]

$$A_{ki} = 2g \int d\mathbf{r} \,\psi_0 \left[ (u_k^* v_i + v_k^* v_i + u_k^* u_i) u_{\text{osc}} + (v_k^* u_i + v_k^* v_i + u_k^* u_i) v_{\text{osc}} \right].$$
(62)

In deriving Eqn (61) it has been assumed that the occupation probabilities of states *i*, *k* are given by the Bose function  $f_j = [\exp(\epsilon_j/T) - 1]^{-1}$ . Taking the oscillation  $\gamma$  to be damped according to  $\dot{E} = -2\gamma E$  we finally obtain

$$\gamma = -\omega\pi \sum_{ik} |A_{ki}|^2 \delta(\epsilon_k - \epsilon_i - \hbar\omega) \frac{\partial f(\epsilon_i)}{\partial \epsilon}, \qquad (63)$$

where the additional assumption  $\hbar \omega \ll T$  has been made.

Equations (62) and (63) can be further simplified by employing the hydrodynamic expressions (57) for the functions  $u_{osc}$ ,  $v_{osc}$ . Equation (63) can be used in calculating the damping numerically. For a sufficiently large number of atoms, the excitation spectrum of the system is virtually continuous so that the summations may be replaced by integrations, one of which eliminates the  $\delta$ -functions in Eqn (63).

Analytical damping formulae can be derived for the case of a homogeneous gas, when u and v are plane waves with amplitudes of the form (49). We find

$$\frac{\gamma}{\omega} = \left[a^3 n_0(T)\right]^{1/2} F(\tau) , \qquad (64)$$

where  $\tau = T/mc^2$  and the function *F* is given by

$$F(\tau) = \frac{4\sqrt{\pi}}{\tau} \int_0^\infty dx \left(1 - \frac{1}{2u} - \frac{1}{2u^2}\right)^2 \times \left[\exp\left(\frac{x}{2\tau}\right) - \exp\left(-\frac{x}{2\tau}\right)\right]^{-2}, \quad (65)$$

with  $u(x) = (1 + x^2)^{1/2}$ .

At low temperatures  $(T \ll \mu$ , where  $\mu = gn_0(0)$  is the chemical potential for T = 0), Eqn (64) becomes

$$\gamma = \frac{2\pi^2 T^4}{45\hbar^3 c^5} = \frac{27\pi}{16} \frac{\omega \rho_n}{\rho} \,, \tag{66}$$

where  $\rho_n$  is the density of the normal component of the phonon gas. Equation (66) was first obtained by Hohenberg and Martin [50].

Of much importance is the case of high temperatures  $(T \ge \mu)$ , but, of course,  $T < T_c$ ). While thermodynamic functions in this case are dominated by excitations with energies  $\epsilon \sim T$ , in function (65), which determines the damping, the main contribution comes from energies  $\epsilon \sim \mu \sim gn_0 \ll T$ , which makes it possible to take the limiting 'Rayleigh-Jeans' expression  $f(\epsilon) \approx T/\epsilon$  as the distribution

function. The resulting  $\gamma$  is linear in temperature, and it is worthwhile noting that at 'high temperatures' this dependence is also expected for a gas in a trap.

Integration within this limit yields [51, 52]

$$\gamma = \frac{3\pi}{8} \frac{Taq}{\hbar^2} \,. \tag{67}$$

This high-temperature regime has recently been used to obtain a quantitative estimate for trap damping [46]. For q taken to be of order  $\hbar\omega/c$ , where  $c = (\hbar/m)[4\pi a n_0(r=0)]^{1/2}$  is the sound velocity at the center of the trap, the value of  $\gamma$  is found to be in semiquantitative agreement with experimental data [17], thus supporting the view that the attenuation is due to Landau damping. It has been suggested [53] that Eqn (67) may also be applied to a gas in an anisotropic trap if the average over the density distribution is taken properly.

# 4. Quantum fluctuations

# 4.1 Corrections to density distribution

By following the same procedure used in deriving Eqns (53)–(54) it is readily shown that for T = 0, in the quasiclassical approximation, the atomic number density is given by the integral

$$n_{\text{out}}(\mathbf{r}) = \int V^2(\mathbf{p}, \mathbf{r}) \frac{\mathrm{d}\mathbf{p}}{\left(2\pi\hbar\right)^3} \,, \tag{68}$$

where  $V^2(\mathbf{p}, \mathbf{r})$  is defined by Eqn (49). Integration yields the same results as in the homogeneous case,

$$n_{\text{out}}(\mathbf{r}) = \frac{8}{3\sqrt{\pi}} \left[ n_{\text{TF}}(\mathbf{r}) a \right]^{3/2},\tag{69}$$

but this quantity is difficult to measure experimentally.

Of considerable interest are the corrections to Eqn (20) for the gas density  $n_{\text{TF}}$ , which are of two types. Firstly, there is a quantum pressure correction, omitted in Eqn (20), which is easily obtained from Eqn (9) by treating the pressure as a perturbation. Our concern here is with the correlation which results from the quantum fluctuations of  $\psi$  and which was already omitted in Eqn (9). This correction is obtained most easily by using the chemical potential expression for a homogeneous gas,

$$\mu(n) = g \left[ n_{\rm TF} + \frac{32}{3\sqrt{\pi}} (n_{\rm TF}a)^{3/2} \right], \tag{70}$$

which was obtained by Lee and Yang [53] taking the quantum correction into account. We are now in a position to employ the equilibrium condition (21), with  $\mu_{l}(n)$  given by Eqn (70). Equation (21) can be solved by iteration bearing in mind, however, that  $\mu_{TF}$  should itself be corrected in order to have the same total number of atoms *N*. For an isotropic trap we find

$$n(r) = \frac{1}{g} \left[ \mu_{\rm TF} \left( 1 + \frac{\sqrt{m}}{2\hbar} a \mu_{\rm TF}^{1/2} \right) - V_{\rm ext}(r) \right] - \frac{32}{3\sqrt{\pi}} (n_{\rm TF} a)^{3/2}, \qquad (71)$$

where  $\mu_{\text{TF}}$  and  $n_{\text{TF}}$  are the density and the chemical potential as given by the Thomas–Fermi expressions (20) and (22), respectively. The difference between Eqns (71) and (69) yields the condensate density,  $n_0(r) = n(r) - n_{\text{out}}(r)$ .

## 4.2 Phase fluctuations

Although Eqn (30) predicts the Josephson current to be fully monochromatic, this prediction is invalidated by fluctuations in the phase of the condensate wavefunction  $\Phi$ . The spectral distribution of the current may be characterized by the Fourier components of its correlation function

$$C(\tau) = \left\langle I(t+\tau)I(t)\right\rangle. \tag{72}$$

Since the calculation of  $C(\tau)$  is difficult and depends heavily on the particular experimental setup, we limit ourselves to an illustrative problem to demonstrate the general nature of the relations involved.

Representing the phase in the form  $\Phi(t) = -\mu t + \delta \mu t$  and assuming the fluctuations of  $\delta \mu$  to be independent of t and Gaussian, after taking the average and regrouping terms in Eqn (30) we obtain (see Refs [55, 56])

$$C(\tau) = I_0^2 \operatorname{Re}\left[\exp(-i\omega_J \tau) \exp\left(-\frac{\langle \delta \mu \rangle^2 \tau^2}{\hbar^2}\right)\right]$$
  
$$\equiv I_0^2 \operatorname{Re}\left[\exp(-i\omega_J \tau) \exp\left[-\left(\frac{\tau}{\tau_0}\right)^2\right],\tag{73}$$

where  $\omega_{\rm J} = (\mu_2 - \mu_1)/\hbar$ . In the temperature range  $T \gg \hbar \omega_{\rm H}$ 

$$\langle \delta \mu \rangle^2 = T \frac{\partial \mu}{\partial N} \,.$$
 (74)

The damping of correlations described by the above equation is usually called the 'phase diffusion' effect<sup>†</sup>. If, furthermore,  $T \ll T_c$ , one can employ expression (22) for the chemical potential, giving

$$\tau_0^2 = \frac{\hbar^2}{T} \frac{\partial N}{\partial \mu} = \frac{5\hbar^2 N}{2\mu T}$$
(75)

for the correlation function decay time  $\tau_0$ . Thus the spectral distribution of the current is found to be Gaussian with a width of order  $1/\tau_0$ .

We will not discuss this important problem in more detail, though.

#### 4.3 Quantum collapse of condensate oscillations

Note that quantum phase diffusion, discussed in the preceding section can be observed because the number of atoms in the trap is relatively low, which implies that our system may be considered mesoscopic. Another mesoscopic phenomenon with no classical analogy is the collapse and subsequent revival of condensate oscillations, an effect which is due to the amplitude dependence of the natural oscillation frequencies. The observation of this phenomenon in classical wave packets was predicted by Averbukh and Perelman [57]. Until now, it has been observed for highly excited states of atomic electrons, for molecular vibrations, and for atoms in interaction with the resonator field. Bose – Einstein condensation allows this effect to be calculated on a macroscopic scale as a quantum collapse of condensate oscillations (see Refs [58 – 60] and references therein).

The basic idea is that, quantum-mechanically, the oscillation of a certain collective mode is a coherent superposition of the stationary states of the oscillator, the scale of the phenomenon depending on the shift in the oscillation frequency as a function of its amplitude.

Let us write the frequency of a given mode in the form

$$\omega = \omega_0 + \delta \omega = \omega_0 (1 + \varkappa E), \qquad (76)$$

where *E* is the oscillation energy. Assuming the frequency shift to be small,  $|\varkappa| E \ll 1$ , and using the quasiclassical relation  $\hbar \omega = (\partial E_n / \partial n)$ , this is rewritten as  $\omega_n = E_n / \hbar = \omega_0 n + bn^2/2$ , where *n* is the number of quanta at a given level  $E_n$  ( $n \ge 1$ ), and  $b = \hbar \omega_0^2 \varkappa$ . Experimentally [15, 16], oscillations were excited by varying the magnetic trap frequency in a sinusoidal fashion, thus bringing the oscillator to a coherent state. The wavefunction of this state can be expanded in terms of stationary state functions,

$$\psi = \sum_{n} c_n \psi_n \exp(-\mathrm{i}\omega_n t) \,, \tag{77}$$

with

$$|c_n|^2 = \frac{\bar{n}^n}{n!} \exp(-\bar{n}) \approx \frac{1}{\sqrt{2\pi\bar{n}}} \exp\left[-\frac{(n-\bar{n})^2}{2\bar{n}}\right], \quad (78)$$

where  $\bar{n}$  is the average number of excited quanta ( $\bar{n} \ge 1$ ), and the energy  $E = \hbar \omega_0 \bar{n}$ . We next ascribe a certain 'coordinate'  $\xi(t)$  to our oscillation and calculate the average of this coordinate over the state (77), (78). For small nonlinearities only the transitions  $n \to n \pm 1$  must be considered, giving

$$\langle \xi(t) \rangle \propto \sum_{n} |c_{n}|^{2} \cos\left[(\omega_{0} + bn)t\right].$$
 (79)

For times t long enough, the summation over n may be replaced by integration, with the result that the average value of  $\langle \xi(t) \rangle$  undergoes an attenuation, or 'collapse,'  $\langle \xi \rangle \sim \exp(-\bar{n}b^2t^2/2) \equiv \exp[-(t/\tau_c)^2]$ , with a characteristic time

$$\tau_{\rm c}^{-1} = \left(\frac{\bar{n}}{2}\right)^{1/2} |b| = \omega_0 \left(\frac{E\hbar\omega_0}{2}\right)^{1/2} |\varkappa| \,. \tag{80}$$

Because expression (79) is periodic, however, the oscillation periodically 'comes alive' with a period of  $\tau_r = 2\pi/(\hbar\omega_0^2|\varkappa|)$ , which is consistent with Ref. [57]. Note that  $\tau_c \approx \sqrt{(1/\bar{n})} \tau_r \ll \tau_r$ .

A general picture of the effect and the meaning of the parameters  $\tau_c$  and  $\tau_r$  are shown schematically in Fig. 4 using arbitrary units for time and the oscillator coordinate.

It should be emphasized that in quantum mechanics measuring the coordinate  $\xi$  of the oscillator changes the value of its momentum, so that the collapse and subsequent revival of oscillations can only be observed by repeating the experiment many times for various oscillation phases. This, however, is exactly the approach used in Refs [16, 17].

<sup>&</sup>lt;sup>†</sup> One might argue (and some people do) that the existence of such diffusion compromises the concept of the condensate wavefunction  $\psi_0$  obeying Eqn (9). It seems to me there is a misconception behind this view. Although, as already pointed out, fluctuations are neglected in Eqn (9), they can be included in the next-order approximation. In doing this, one should calculate physical observables which are always gauge-invariant, i.e., independent of the general phase of the function  $\psi_0$ . Averaging the gauge-invariant  $\psi_0$  itself over the phase fluctuation is meaningless.

5	7	9

Table											
Reference	Atoms	$a, a_0$	$\omega_x/2\pi$ , Hz	$\omega_y/2\pi$ , Hz	$\omega_z/2\pi$ , Hz	T <sub>c</sub>	$N_{\rm c}$	$N_0$			
[2]	<sup>87</sup> Rb	100	78	78	220	100 nK	10 <sup>4</sup>	4500			
[15]	<sup>87</sup> Rb	100	132	132	373	100 NK	$10^{4}$	4500			
[3]	<sup>23</sup> Na	70	745	235	410	2 μΚ	$2  imes 10^6$	$1.5  imes 10^5$			
[16]	<sup>23</sup> Na	70	320	320	18	2 mK	$15  imes 10^6$	$5  imes 10^6$			
[4]	<sup>7</sup> Li	-14.5	151	153	132	300 nK	$1.4  imes 10^5$	1400			



Time (arb. units)

Figure 4. Schematic diagram of the collapse-revival process.

Turning now to the factor  $\varkappa$ , simple dimensional arguments show that in the Thomas-Fermi approximation

$$\varkappa = \frac{\nu}{\mu_{\rm TF}N}, \quad E = \epsilon \mu_{\rm TF} N A^2 \,, \tag{81}$$

where A is the relative oscillation amplitude and v and  $\epsilon$  are dimensionless factors of order unity. Thus

$$\tau_{\rm c} = \frac{\sqrt{2}}{|\nu|\omega_0} \frac{\mu N}{\sqrt{E\hbar\omega_0}} = \frac{\sqrt{2}}{|\nu|\sqrt{\epsilon}\omega_0} \sqrt{\frac{\mu N}{\hbar\omega_0}} \frac{1}{A} \,. \tag{82}$$

The experimental study of Ref. [15] was carried out for the following values of the parameters involved (see also Table 1):  $N = 4500, \omega_{\perp}/2\pi = 132$  Hz,  $\lambda = \sqrt{8}$  and  $a = 100a_0$ , where  $a_0$ is the Bohr radius. For these values,  $\sqrt{\mu_{\rm TF}N/\hbar\omega_{\perp}} \approx 195$ .

The frequency of the m = 2 mode in the linear limit is, from Eqn (25),  $\omega_0 = \sqrt{2\omega_{\perp}}$ . A calculation using Eqn (82) yields v = 0.26,  $\epsilon = 0.57$  [60] for this mode. For an amplitude of order A = 0.2 the collapse time is approximately  $\tau_c = 4.98$ s, and for the lower m = 0 mode, Eqn (26), we similarly have  $\tau_{\rm c} = 1.4$  s. Reference [15] reports oscillation decay with a characteristic time of order 100 ms, which means that under the experimental conditions used, Landau damping masks the quantum effects of interest here. Since, however, this attenuation must rapidly decrease with temperature [cf. Eqn (66)], it is hoped that lowering the temperature will allow quantum collapse to be observed on our macroscopic object.

Among other mesoscopic problems one should mention the possibility [61] that at T = 0 a gas in a trap is described by a linear combination of wavefunctions for various numbers of particles.

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# Appendix

For purposes of reference, Table 1 lists the major parameters used in the experimental studies cited in the text. One expects, however, that owing to the spectacular development of experimental technique - and considering the ever increasing number of laboratories where Bose condensation has been achieved — much more impressive results will undoubtedly have been obtained by the time of publication of this review.

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