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Time, tunneling and turbulence

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Contents

Abstract. **This paper considers tunneling traversal time; tunneling through an alternating potential, its enhancement and its fractal (Hofstadter butterfly) and chaotic resonances; and Hartree liquid resonance tunneling and its (specifically quantum) instability and turbulence.**

1. Introduction

One of the main manifestations of quantum mechanics is tunneling. Tunneling is involved in the decay of heavy nuclei, field emission from an atom or a solid surface, paraelectric defects in solid, metal – insulator – metal, and Josephson junctions, p – n diodes, superconducting quantum interference device (SQUID) rings, transport in superlattices, quantum diffusion, absorption, and desorption. Quantum transport (e.g., variable-range-hopping conductivity), as well as wave propagation beyond the geometrical-optics region, is related to tunneling.

The knowledge of the tunneling traversal time (TTT) is important in a wide range of physical problems, from tunneling chemical reactions to stationary many-body transport [1]. That is why it is almost as old as quantum mechanics itself [2]. It was studied by Wigner [3], was brought into sharp focus by Büttiker and Landauer [4], and has drawn much attention recently [5]. Bosanac [6] and Band [7] discussed anomalously large traversal (electromagnetic waves) velocities. It was demonstrated [8] that the velocity of the probability density maximum may be infinite or even negative (cf. Ref. [9]). Recently Enders and Nimtz and others rediscovered [10] infinite and superluminal velocities for the evanescent mode packet phase, whose propagation is equivalent to quantum tunneling. These experiments were discussed by Landauer [11] (see also Ref. [1b]). The possibility of time-scale invariance in tunneling (which implies no characteristic TTT) was also established [12]. The situation clearly calls for a physically meaningful answer to the question: what is TTT, what it is not, and how to reconcile presumably infinite and superluminal velocities with the uncertainty principle, relativity and causality [13].

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Tunneling and activation in an alternating potential are important in a variety of physical problems [11 – 15]: interband breakdown, charge exchange between deep-lying impurity centers in semiconductors, resonance tunneling [16], Coulomb blockades [17], and the destruction of adiabatic invariants [18a]. The study of the alternating potential may also be useful for stationary many-body tunneling and evaporation, if the latter is reduced to an approximate single-particle problem. Then some of the degrees of freedom adjust to the progress in particle escape and yield an effective time-dependent potential. Also, the characteristic time T of an alternating potential may be related to the effective temperature $\theta \sim \hbar/T$. Hence, an alternating-potential study may be helpful for the quantum transport problem (e.g., variable-range hopping conductivity).

An extensive and accurate study of particle transmission is also important in view of the experimental rates, often being, by dozens of orders of magnitude, above the theoretical values [19, 20], and even the upper bound [21].

A wave function collapse (WFC), when a probability density disappears in one place and emerges in another, is one of the fundamental concepts in quantum mechanics. It is usually thought of as the implication of a measurement by a classical device [18]. Meanwhile, effective WFC occurs in each Mott hop [22] which determines electron transport in disordered semiconductors [23]. It may also be important for the Mott hopping frequency dependence, tunneling and relaxation [1], quantum electronics [24], transport in disordered systems in general [25], time-dependent Zener tunneling and breakdown, as well as for other problems in postmodern quantum mechanics (to use the term coined by R Harris), discussed in four papers in *Physics Today* in 1993 alone [26]. The improvements in the technology in the last two decades, the success in the fabrication of quantum dots and mesoscopic systems [26, 27], and the experiments on single electron transport [26] allow for a microscopic study of a WFC in a single Mott-type hop.

The resonant tunneling first observed by Chang, Esaki and Tsu [28] is rich in physics. It exhibits structures related to phonon [29] and plasmon [30] assisted tunneling, Landau level matching [31], intrinsic dynamic bistability, hysteresis [32], oscillations† [16], quantum chaos [33] and Coulomb blockades [17]. Recently, Heilblum et al. studied resonance tunneling and its phase evolution through a single quantum dot [34]. Resonant tunneling and activation in an alternating potential are important in the Mott hopping frequency dependence [35]. An alternating potential leads to resonant activation [36], oscillatory wave function collapse [37], evanescent modes [38], photoassisted tunneling [39], and left-right symmetry breaking [40]. The nonlinear Hartree-type Bogolyubov – Pitaevskiĭ – Gross equation [42] in an external potential [43] numerically demonstrated dynamic evolution to chaos [33].

This paper considers three problems specified above: the tunneling traversal time; the tunneling through an alternating potential; and the Hartree liquid tunneling and its intrinsic turbulence.

2. Tunneling traversal time

Start with a stationary flow of free quantum particles. To measure its traversal time, from point A to point B, measure at B the linear response to a time dependent perturbation potential at A. The perturbation Fourier transformation yields arbitrarily high (albeit of correspondingly low probability) frequencies. They allow for an arbitrarily high energy increase, zero minimal traversal time in non-relativistic mechanics and light speed traversal in the relativistic case. Any singularity in the perturbation time dependence (which may accurately determine the moment when it is switched on) only increases the probability density of such a response. An analytical time dependence with the characteristic switching time δt implies a characteristic energy ε change $\delta\varepsilon \sim h/\delta t$, a relative velocity v change $\delta v/v \sim \delta\varepsilon/\varepsilon$ and a relative traversal time change $(v\delta t/\Delta x) + (\delta\varepsilon/\varepsilon) \sim (hv/\delta\varepsilon\Delta x) + (\delta\varepsilon/\varepsilon)$, where Δx is the distance from A to B. The minimal inaccuracy, in agreement with the Heisenberg uncertainty, is $\delta t/\Delta t \sim (\lambda_{\mathrm{dB}}/\Delta x)^{1/2}$, where $\lambda_{\mathrm{dB}} \sim h/p$ is the de Broglie wave length for the momentum p and $\Delta t = \Delta x/v$. Then the traversal time corresponds to the maximal response (which is followed by an infinitely long and correspondingly weak retarded response to vanishing frequencies in the perturbation spectrum).

Now we apply the same approach to the tunneling of a stationary flow through an opaque barrier. Then δt, which is small compared to the Büttiker – Landauer time [4] $t_{\mathrm{BL}} = \int \mathrm{d}x/|\mathbf{v}|$, implies an exponentially higher probability of activation above the barrier than the probability of tunneling. Then the response may be exponentially stronger for a much weaker, but significantly shorter, perturbation. In particular, the response may reach its maximum prior to the maximum of the stronger perturbation. Not surprisingly — the maxima belong to unrelated phenomena [44]. (This is the origin of the pseudo-superluminal velocities in Ref. [10].) But even a rapid perturbation yields arbitrarily low frequencies in the general case, and thus an arbitrarily delayed time response, and an arbitrarily long traversal time. A slow perturbation ($\delta t \gg t_{\mathrm{BL}}$) implies a correspondingly low probability for the particle energy increase. However, it also implies a correspondingly slow response, and the corresponding inaccuracy in the traversal time determination. (Thus, traversal time is not universal for a given barrier and incoming energy. Rather, it crucially depends on the way it is determined.)

The Büttiker – Landauer time determines the transition from tunneling to activation and thus provides an estimate of the tunneling response time. Clearly, since the characteristic tunneling de Broglie wave length is approximately the barrier width, there can be no accurate definition of the tunneling traversal time. This is specially explicit in resonant tunneling. If $h/\delta t$ exceeds its (exponentially narrow) width, it exponentially reduces the transmittance, and implies a characteristic time of resonance tunneling of approximately the dwell time at the quasi-eigenstate in the well.

To quantitatively study time dependent tunneling we introduce a model. Consider two remote potential wells, separated by an opaque potential barrier. Initially the particle density is localized in one of the wells (A), with an exponentially small tail in the other well (B). A fluctuation is modeled by a time-dependent potential, which exists over a finite time. I demonstrate that only under special conditions the particle density practically completely shifts to the other well. This happens periodically with the potential strength change in the general case, and at certain values in the case which models Mott hopping.

The Schrödinger equation in the potential $V(t, r)$ for a particle with mass $1/2$ and $\hbar = 1$ reads

$$\mathrm{i}\dot{\psi} + \Delta\psi - V(t, r)\psi = 0\,, \qquad V(t, \infty) = 0\,, \tag{1}$$

where $\dot{\psi} = \partial\psi/\partial t$. The physics is elucidated by two extreme cases.

(i) If V changes adiabatically with time t, then the particle eigenstate follows its instantaneous value. Until the eigenstate is not too close to the degenerate one, the particle is localized in well A. When the eigenenergies of A and B come to the closest distance ω allowed by the level repulsion ('quasidegeneracy'), one may easily prove (see later) that the particle moves to the well B. Then the level repulsion shifts the levels back to their initial positions. As a result, when V returns to its initial value, the particle returns to well A — see Fig. 1, curve *1* (where ζ is related to the maximal change in V and η to its characteristic time).

(ii) If V at $t = t_1$ jumps to a new value, the continuity of the wave function at $t = t_1$ distributes the particle between the states which correspond to $V(t_1 - 0)$. At $t = t_1$ the wave function continuity preserves it in A. At $t_1 < t < t_2$ the states interfere, and the probability density oscillates between the wells. So, the result of the second jump of V at $t = t_2$ to the initial value $V(t_1 - 0)$ depends on and oscillates with a time lag $\Delta t = t_2 - t_1$. If at $t = t_2 - 0$ the particle is approximately in well B, the jump at $t = t_2$ fixes it there forever (if there are no further jumps). A similar situation occurs when V quickly (compared to ω^{-1}) reaches quasidegeneracy and slowly (compared to the eigenperiod) passes it — see Fig. 2. Characteristically, case (ii) has very quick (quasijumps) and very slow (quasistationary) potential changes. The latter (see later) must be slow compared to the quasidegenerate instantaneous level splitting (which is related to the exponentially weak tunneling through the opaque interwell barrier) to significantly shift the particle to B.

Typically a particle is finally redistributed between the wells. The 'ultimate' wave function collapse (WFC) corre-

† Ref. [41] demonstrated magnetic field induced oscillations in an asymmetric double barrier structure.

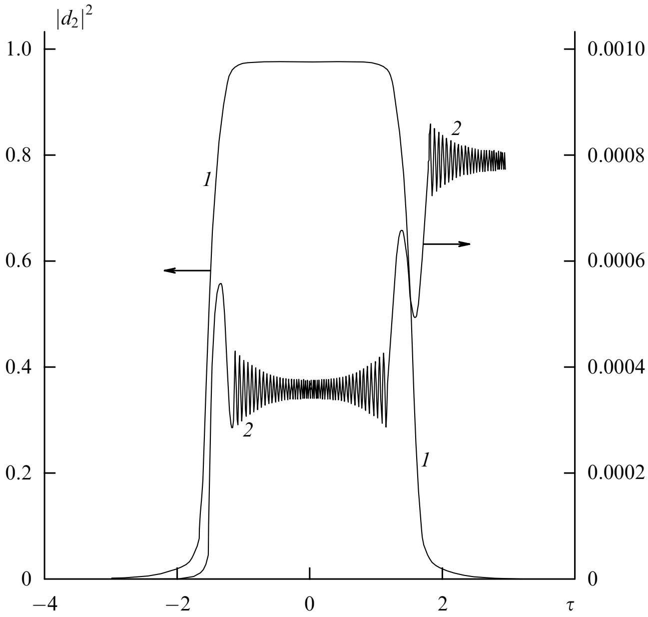

Figure 1. Time dependence of the probability $|d_2(\tau)|^2$ of finding the particle in well B for $f(\tau) = 2/\cosh\tau - 1$ and $\zeta = 500$, $\eta = 50$ (curve *1*); $\zeta = 100$, $\eta = 0.1$ (curve *2*).

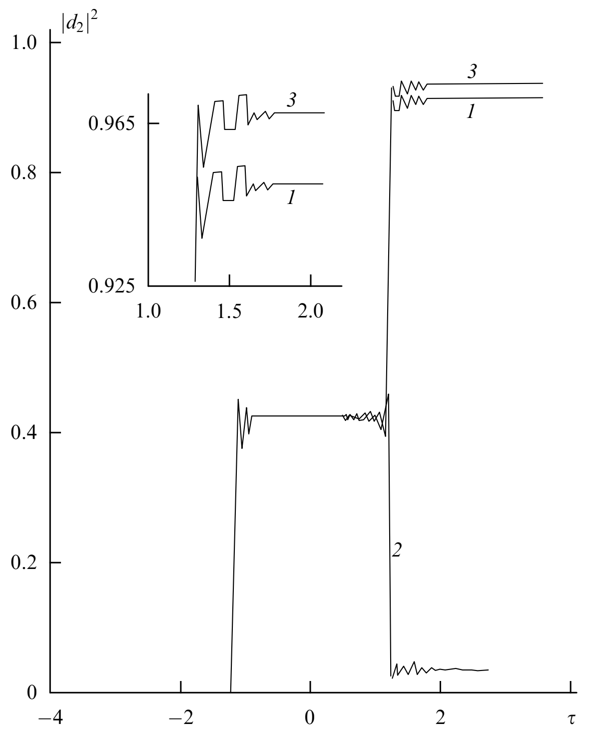

Figure 2. As Fig. 1, but $\eta = 50$ and $\zeta = 15000$ (curve *1*); $\zeta = 15001$ (curve *2*), i.e., ζ changed by approximately half a period (11); and $\zeta = 15002$ (curve *3*).

sponds to an exponentially small ($\propto \omega^2$) probability density in well A. The calculation specifies extremely rare fluctuations which do it — cf. the role of 'special' states in quantum measurements and WFC in Ref. [45]. Such a fluctuation is probable only in a macroscopic 'classical' system (where ω^2 may also be vanishingly small), with its vast range of fluctuations.

To describe a WFC analytically, consider the instantaneous eigenenergies. Suppose they are closest when $t = 0$ (if this happens only once) or when $t = t_1, t_2, \ldots$ (if it happens several times). Denote the corresponding instantaneous value of V by $V_0(r)$ and present Eqn (1) as

$$\mathrm{i}\dot{\psi} + \Delta\psi - V_0\psi = \psi\psi\,, \qquad V_0(r) = V(t_0, r)\,, \qquad v = V - V_0\,. \tag{2}$$

Choose a complete orthogonal and normalized set of functions ϕ, which yield

$$\Delta\phi_n + (\omega_n - V_0)\phi_n = 0 \tag{3}$$

and present ψ in Eqn (2) in the form

$$\psi = \sum_n b_n(t)\phi_n(r) = \mathbf{b}\cdot\boldsymbol{\phi}\,. \tag{4}$$

Then

$$\mathrm{i}\dot{b} = (\hat{\omega} + \hat{v})b\,, \qquad \omega_{nn} = \delta_{mn}\,, \qquad v_{mn} = \int v\phi_m^*\phi_n\,\mathrm{d}r \tag{5}$$

(an asterisk denotes complex conjugation). Suppose $V_0 = V_1 + V_2$, where V_1 and V_2 correspond to wells A and B. I am interested in the transitions between the wells. When the characteristic frequency v is sufficiently low, their probability exponentially decreases with $|\omega_m - \omega_n|$. Suppose the closest level corresponds to ω_1, ω_2 and keep in Eqn (4) only $n = 1, 2$, thus reducing the problem to a two-level one. To monitor the localization positions, we introduce (real localization) eigenstates $\tilde{\phi}$ in each of the wells and decompose ψ with respect to $\tilde{\phi}$ according to

$$\psi = \exp\left[-\mathrm{i}\overline{\omega}t - \mathrm{i}\int^t \overline{v}(t')\,\mathrm{d}t'\right]\tilde{\phi}d\,. \tag{6a}$$

Then the main equation of the problem reads

$$\mathrm{i}\dot{d} = \begin{pmatrix} w & \omega \\ \omega & -w \end{pmatrix} d\,, \tag{6}$$

where

$$\overline{\omega} = \frac{\omega_1 + \omega_2}{2}\,, \qquad \omega = 0.5(\omega_2 - \omega_1)\sin(2\gamma)\,,$$
$$w(t) = \int \frac{v(\tilde{\phi}_1^2 - \tilde{\phi}_2^2)\,\mathrm{d}r}{2}\,, \qquad \overline{v} = \int \frac{v(\tilde{\phi}_1^2 + \tilde{\phi}_2^2)\,\mathrm{d}r}{2}\,. \tag{7}$$

Here γ is the angle between ϕ and $\tilde{\phi}$ (for identical wells $\gamma = \pi/4$); the interwell (time dependent) interaction $\omega \propto \sqrt{P_{\mathrm{T}}}$, P_{T} is the exponentially small tunneling transmission coefficient, and the proximity of levels implies $|\overline{v}| \ll |\overline{\omega}|$. By Eqn (6), $|d|^2$ is conserved and equal to 1 in the case considered, while $\left|d_m(t)\right|^2$ is the probability of the state localized in the mth well at moment t ($m = 1, 2$).

When w is time independent, the solution to Eqn (6) is obvious. When $w(t)$ jumps at $t = t_1$ from one constant value of w to another, then the continuity $d(t_1 - 0) = d(t_1 + 0)$ provides the transfer matrix for $d(t)$ from $t < t_1$ to $t > t_1$. The adiabatic solution is also straightforward.

Consider a general case. To model a large fluctuation (like the Mott one), assume $\max_t\left|w(t)\right| \gg \omega$. When $|w| \gg \omega$, Eqn (6) yields

$$d \cong \hat{B}D\,, \tag{8a}$$

where

$$\hat{B}(F) = \begin{pmatrix} \exp(-\mathrm{i}F) & 0 \\ 0 & \exp(\mathrm{i}F) \end{pmatrix}, \qquad F = \int_0^t w(t')\,\mathrm{d}t', \quad (8)$$

and D is time independent. (Further approximations in ω/w lead to an accurate solution.) In the vicinity of $t = 0$, where $w(t) \cong \dot{w}(0)t$, the substitution $d = \exp[-\mathrm{i}w(0)t^2/2]f$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{f}_1 + 2\mathrm{i}\dot{w}(0)tf_1\right] + \omega^2 f_1 = 0\,,$$

$$f_2 = C - \frac{2\dot{w}(0)tf_1}{\omega}\,. \quad (9)$$

This equation is linear in t and may be solved by the Laplace method. The solution to Eqn (9) allows for the matching of d from Eqn (8a) for $t < 0$ and $t > 0$, and yields the corresponding transfer matrix for D from $t < 0$ to $t > 0$.

The change $t = \tau\omega/\dot{w}(0)$ in Eqn (6) demonstrates that $|M| \gg 1$, $M = \omega^2/\dot{w}(0)$ implies an adiabatic case, while $|M| \ll 1$ corresponds to a rapidly changing potential. Assume

$$w = Wf\left(\frac{t}{\theta}\right), \quad f(\pm\infty) = 1\,, \quad f(0) = 0\,,$$

$$W \gg \omega, \frac{1}{\theta}\,. \quad (10)$$

Introduce $\eta = \omega\theta$, $\zeta = W\theta$ (and thus $M \sim \eta^2/\zeta$). Then one arrives at the following results, which are readily verified with numerical experiments using Eqn (6). Start with $\dot{w}(0) \neq 0$ and the numerical example of $f(\tau) = 2/\cosh\tau - 1$.

(1) $\zeta \gg \eta \gg \sqrt{\zeta}$ corresponds to the adiabatic situation — see Fig. 1, curve *1*. A particle moves from A to B and then returns to A.

(2) $\sqrt{\zeta} \gg 1 \gg \eta$ is a rapidly changing ($M \ll 1$) potential, which quickly ($\eta \ll 1$, i.e., $\theta \ll 1/\omega$) passes the transition interval $|w| \lesssim \omega$ — see Fig. 1, curve *2*. A particle always stays in the same well, without ever noticeably moving to well B.

(3) $\sqrt{\zeta} \gg \eta \gg 1$ is a rapidly changing potential, which slowly ($\eta \gg 1$, i.e., $\theta \gg 1/\omega$) moves through the transition interval — see Fig. 2. This is the only possibility for a high probability of a hop. Note that its conditions are rather special. The characteristic frequency $1/\theta$ of the potential change must be low compared to an exponentially small frequency ω (which is related to the tunneling transmission). The change $2W$ in the potential strength must be large compared to $\omega^2\theta$. In virtue of Eqn (8), it oscillates with W with approximately the period $\Delta W = \Delta\zeta/\theta$,

$$\Delta\zeta = \pi\left[\int_{\tau_1}^{\tau_2} f(\tau)\,\mathrm{d}\tau\right]^{-1}, \qquad f(\tau_1) = f(\tau_2) = 0\,. \quad (11)$$

The physical origin of the periodicity is the slow wave function phase change when $\tau_1 < \tau < \tau_2$. The function $f(t)$ yields

$$\Delta\zeta = \frac{3\pi}{2}\left[2\pi - 3\ln(2 + \sqrt{3})\right] \cong 2.0205\,.$$

Certain W's change $|d_2|^2$ from 0 to 1 (WFC).

3. Tunneling in an alternating potential

The previous section's approach may be generalized to study the impact of an alternating potential on tunneling [46]. It implies, in particular, that transmission through a one-dimensional (1D) barrier in an alternating potential has a resonant nature, if part of the time the total potential has a potential well. A 1D potential well always has an eigenstate. An instantaneous eigenenergy of a time-dependent well moves with time and disappears together with the well. When a particle is activated to the lowest instantaneous ground-state energy $\tilde{\omega}$ (to be more specific, to the corresponding instantaneous tunneling resonance, which has a finite width), then it is trapped there and follows this 'elevator' free of activation energy ('Elevator Resonance Activation' — ERA). When $\tilde{\omega}$ is less than the incoming particle energy Ω, the activation energy is zero, although transmitted particles have the instantaneous eigenenergy, significantly higher than Ω. ERA also occurs in higher dimensionalities, when the deepest instantaneous well has an eigenstate, and is a possibility for all types of waves in their penetration into a region forbidden by geometrical optics. Of course, all this is true only for a sufficiently slow alternating potential.

Thus, a time-dependent opaque barrier may be a model for exponentially enhanced space-time fluctuations and transmission (and thus diffusion) rates. This may suggest their common origin. The results are very general. They are valid for the penetration of any waves (quantum, electromagnetic, sonic, hydrodynamic, etc.) into a classically forbidden region.

To study resonance transmittance in an alternating potential in more detail, consider non-linear activation and tunneling through an arbitrary set of potential barriers and wells in the presence of an alternating harmonic point potential. The problem accurately reduces to a single one-dimensional finite difference equation. Activation exhibits fractal (Hofstadter butterfly-type) and chaotic resonances and their cut-offs, which are related to the alternating potential strength. At low frequencies activation yields a strong dispersion and generates high frequencies. There are evanescent modes below the vacuum (which do not exist in the stationary case). The transmittance may be linear with the alternating potential at certain (medium) frequencies only. Activation is sensitive to the spatial heterogeneity of the alternating potential. These predictions may be directly tested in quantum dot experiments with an alternating gate voltage at low frequencies and with an external microwave field (modulating static potential) at higher frequencies (cf. experiments in Ref. [32]).

The rich physics of activation is related to the superposition of several factors. An alternating potential may activate the incoming particle to an energy in the proximity of a stationary resonance. (At low frequencies this happens only in a sufficiently strong alternating potential.) Thereafter the transmittance strongly depends on the proximity to and the relative timespan at this energy. The timespan is affected by transitions to and from the energy, especially by those to other stationary resonances (which are quasi-equidistant in semiclassics and equidistant in a parabolic well). Transitions are sensitive to the commensurability of the alternating and inter-resonance frequencies; this may lead to a fractal Hofstadter butterfly picture of resonance frequencies. They strongly depend on the heterogeneity of the alternating potential, and the resonances are more pronounced, the higher heterogeneity is.

We start with a point alternating potential and consider an arbitrary 1D static potential $V_0(x)$, $V_0(\pm\infty) = 0$, which may imply any number of barriers and wells. The wave function ψ yields the Schrödinger equation

$$\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial x^2} + V_0\psi - 2f\psi\delta(x)\sin(\omega t)\,. \quad (12)$$

The boundary conditions imply that at $x \to -\infty$ there is only the incoming wave with energy Ω and that as $x \to \infty$ there are no reflected waves. The solution to Eqn (12) may be presented as

$$\psi = \sum_{n=-\infty}^{\infty} \psi_n(x)\exp(-\mathrm{i}\Omega_n t)\,; \qquad \Omega_n = \Omega + n\omega\,, \quad (13)$$

where, by Eqns (12, 13),

$$\psi_n'' + (\Omega_n - V_0)\psi_n = 0\,, \quad (14)$$

$$\delta\psi_n(0) = 0\,; \qquad \delta\psi_n'(0) = \mathrm{i}f\big[\psi_{n+1}(0) - \psi_{n-1}(0)\big]\,. \quad (15)$$

According to the boundary conditions,

$$\psi_n = \delta_{n0}\tilde{\varphi}_0^+ + v_n^-\varphi_n^- \quad \text{if} \quad x < 0\,,$$

$$\psi_n = v_n^+\varphi_n^+ \quad \text{if} \quad x > 0\,, \quad (16)$$

where φ_n^-, φ_n^+, $\tilde{\varphi}_0^+$ are generated by the reflected, transmitted and the (only) incoming waves correspondingly, i.e.,

$$\tilde{\varphi}_0^+ = \exp(\mathrm{i}k_0 x)\,, \quad \varphi_n^- = \exp(-\mathrm{i}k_n x) \quad \text{at} \quad x \to -\infty\,,$$

$$\varphi_n^+ = \exp(\mathrm{i}k_n x) \quad \text{at} \quad x \to \infty\,. \quad (17)$$

Here $k_n = \Omega_n^{1/2}$ if $\Omega_n > 0$, $k_n = |\Omega_n|^{1/2}$ (a.c. evanescent modes below vacuum [19], which are non-existent in the stationary case). Equations (13), (15), (16) accurately reduce, e.g., $\psi(x)$ at $x > 0$, to the solution w_n of a finite difference equation (which is very convenient in numerical calculations):

$$\psi(t,x) = \frac{\tilde{\chi}\tilde{\varphi}_0(0)}{\chi_0}\exp(-\mathrm{i}\Omega t)\sum_{n=-\infty}^{\infty}\frac{\varphi_n^+(x)}{\varphi_n^+(0)}\,w_n\exp(-\mathrm{i}n\omega t), \quad (18)$$

$$\chi_n w_n + \mathrm{i}f(w_{n+1} - w_{n-1}) = \chi_0\delta_{n0}\,, \qquad w_{\pm\infty} \to 0\,, \quad (19)$$

$$\chi_n = \left[\ln\frac{\varphi_n^-(0)}{\varphi_n^+(0)}\right]'\,, \qquad \tilde{\chi} = \left[\ln\frac{\varphi_0^-(0)}{\tilde{\varphi}_0^+(0)}\right]'\,, \quad (20)$$

where the coefficients χ_n are related to stationary solutions of the Schrödinger equation. Consider a semiclassical $U(x)$, which reduces to two opaque barriers, separated by the well. Then the known transfer matrices [47] explicitly relate χ_n to the phase areas in the well ($d_n = \oint K_n\,\mathrm{d}x$) and through the left ($S_1$) and right ($S_2$) barriers ($S = \int|K_n|\,\mathrm{d}x$); the wave vector $K_n = \sqrt{\Omega_n - V_0}$. With an accuracy $\propto \exp[-2(S_1 + S_2)]$, χ_n is real, and Eqn (19) is Hermitian. The approximate Eqns (18)–(20) read:

$$\chi_n^* w_n + \frac{\mathrm{i}f^2}{2}(w_{n+1} - w_{n-1}) = \chi_0^*\delta_{n0}\,, \quad (21)$$

$$\chi_n^* = \cos(d_n) + \mathrm{i}\exp\big[-2S^*(\Omega_n)\big]\sin(d_n)\,, \quad (22)$$

$$\psi(t,0) = \frac{q}{\chi_0^*}\exp\big[-S_1(\Omega) - \mathrm{i}\Omega t\big]G(\omega t)\,,$$

$$G(\varphi) = \sum_{n=-\infty}^{\infty} w_n\exp(-\mathrm{i}n\varphi)\,, \quad (23)$$

where $S^* = \min(S_1, S_2)$; $|q| \sim 1$, $f^2 \sim 2f/k_0$. Beyond the barriers,

$$\psi(t,x) \sim \frac{1}{\chi_0^*}\exp\big[-S_1(\Omega) - S_2(\Omega) + \mathrm{i}k_0 x - \mathrm{i}\Omega t\big]$$

$$\times \sum w_n\exp\left[\mathrm{i}\omega x\left(\frac{\mathrm{d}\omega_0}{\mathrm{d}k_0}\right)^{-1} - \mathrm{i}\omega t - \omega n\,\frac{\mathrm{d}S_2(\omega_0)}{\mathrm{d}\omega_0}\right]\,.$$

If one disregards the exponentially small second term in Eqn (22), then the homogeneous Eqn (21) becomes Hermitian. It has a solution at resonance frequencies ω, when the solution of the inhomogeneous Eqn (21) diverges. Consider $n\omega \ll \Omega^*$. Then $d_n \cong d_0 + \pi n\omega/\Omega^*$, where Ω^* is the eigenstate distance ($\cong$ constant in the semiclassical case). With such an accuracy, the homogeneous Eqn (21) formally reduces to the Harper equation at the band center in the fictitious magnetic flux ω/Ω^* per site [48]. This equation has solutions for fractal values of the flux and phase d_0 in the vicinity of $f^* \cong 1$. (For a solution see Ref. [48].) A more accurate calculation yields χ_n [in Eqn (19)] oscillating with (in general, incommensurate) phase areas in the well at $x < 0$ and $x > 0$. This 'chaotizes' the spectrum. (A set of more than two barriers also yields irregular oscillations and leads to a chaotic spectrum.) Thus, the transmittance may be chaotic and be a fractal function of the alternating current frequency. When in Eqn (21) $|f^*| \ll 1$ (this excludes fractality and chaos), we may derive an analytical formula.

Consider $|\cos(d_n)| \ll |f^*|$. Then in the leading approximation one may neglect w_{n-1} in Eqn (21) when $n > 0$ and w_{n+1} when $n < 0$, and obtain w_n exponentially vanishing with $|n|$:

$$w_n \propto |f^*|^{-|n|}\prod\cos(d_n)\,.$$

If $|\cos(d_n)| \ll f^* \ll 1$, this implies a narrow vicinity of the resonance $\Omega_{\mathrm{res}} = \Omega + (v + \tilde{n})\omega$, where $\cos[d(\Omega_{\mathrm{res}})] = 0$, v is an integer, and $|\tilde{n}| < 1/2$. There,

$$\chi_n^* = \frac{\mathrm{i}\Delta + (n - v - \tilde{n})\omega}{\tilde{\Omega}}\,, \quad (24)$$

where $\Delta \sim \mathrm{Im}\,(\chi_n^*)$ is the natural resonance width, and $\tilde{\Omega}$ the interresonance distance. The periodic $G(\varphi)$ from Eqn (23), by Eqns (21), (24), is presented by an explicit analytical formula:

$$G(\varphi) = \sigma^*\big[\exp(2\pi\sigma^*) - 1\big]^{-1}\exp(-\sigma^*\varphi + \mathrm{i}g\cos\varphi)$$

$$\times \int_{\varphi}^{\varphi+2\pi}\exp(\sigma^*\varphi' - \mathrm{i}g\cos\varphi')\,\mathrm{d}\varphi'\,, \quad (25)$$

$$\sigma^* = \sigma + \mathrm{i}(\tilde{n} + v)\,, \qquad \sigma = \frac{\Delta}{\omega}\,, \qquad g = \frac{f^*\tilde{\Omega}}{\omega} \equiv \frac{\tilde{f}}{\omega}\,.$$

Clearly, $G(\varphi)$ is non-linear in the alternating potential strength g. By Eqns (18), (23), the transmission amplification $p_n = |w_n|^2$ (due to the activation by $n\omega$) is related to the Fourier component w_n of $G(\varphi)$. Equation (25) presents an accurate analytical solution for approximation (24). It yields several frequency regions.

(i) Very low frequencies: $\omega \ll \tilde{f}$, $\Delta^2/\tilde{f}$, i.e., $1 < |g| < \sigma^2$. This is an adiabatic regime, when transmission is the same as in the instantaneous stationary potential. If $\tilde{f} \gg \Delta$, $v \ll \sigma$, then G is very non-linear and non-monochromatic. It is $\cong 1$ when $t \lesssim \Delta/\omega\tilde{f} \ll 1/\Delta$ (higher frequency $\sim \omega\tilde{f}/\Delta$ generation; when

$\tilde{f} \sim \Delta^2/\Omega$, then $G \cong 1$ when $t \sim 1/\Delta$). Thereafter $G \sim \Delta/\tilde{f}\sin(\omega t)$ decreases to $\Delta/\tilde{f}$. The energies which dominate the transmission are $\sim \omega f/\Delta < \Delta$ in the vicinity of the resonance.

(ii) Low frequencies: $\Delta^2/\tilde{f} < \omega < \Delta$, i.e., $1 < \sigma^2 < |g|$. Then $G \sim 0.5(1+\mathrm{i})(\pi/g)^{1/2}$, $|G| < 1$, when $t \lesssim 1/\omega\sqrt{g}$; thereafter $G \sim \Delta/\left[\tilde{f}\sin(\omega t)\right]$. Note that the characteristic time scale is Δ^{-1}.

(iii) Medium frequencies: $\tilde{f} < \omega < \Delta^2/\tilde{f}$, i.e., $|g| < 1, \sigma^2$. This is the linear response case.

(iv) High frequencies: $\omega > \Delta$, $\Delta^2/\tilde{f}$, i.e., $\sigma^2 < 1, |g|$. When $\sigma, |\tilde{n}| \ll 1$, then

$$G(\varphi) = (-1)^{\nu/2} \frac{\sigma^*}{\mathrm{i}\tilde{n} + \sigma} J_\nu(g) \exp(-\mathrm{i}\nu\varphi + \mathrm{i}g\cos\varphi) \,. \quad (26)$$

Thus, $\left|G(\omega t)\right|^2$ is approximately stationary, non-linear and oscillates with the alternating potential strength and ω. If there is a resonance at $|\tilde{n}| \sim \sigma$; when $\tilde{f} \gg \omega$, then $|G|^2 \propto \omega/\tilde{f}$. The energy significantly increases, by $\sim \hbar\tilde{f}$. By Eqn (25),

$$w_n \sim \frac{\sigma^*}{|\tilde{n}| + \sigma} J_{n-\nu}(g) J_\nu(g) \,.$$

It is exponentially cut off when $|\nu| > g$ or $|n-\nu| > g$ (Fig. 3). Sufficiently large $\tilde{f}$ allows for a significant energy rise to a resonance. In the leading approximation the current is also approximately stationary (naturally, due to the superposition of different frequencies) and significantly non-linear.

The total transmission at the moment t is proportional to $\left|G(\omega t)\right|^2$. It depends on the incoming frequency Ω, the alternating potential strength $\tilde{f}$, and the frequency ω. Suppose $|\Omega - \Omega_{\mathrm{res}}| \equiv \Omega' \gg \omega$. Then in the adiabatic case $\tilde{f} > |\Omega'|$ yields a transmission resonance at $\tilde{f}\sin(\omega t) = \Omega'$ with width $\omega\delta t \sim \Delta/\Omega'$ and $G_{\max} \sim \Omega'/\Delta$. High frequency transmission, by Eqn (26), is time independent and has a resonance at $\tilde{f} \cong \Omega'$ with a relative width $\delta\tilde{f}/\Omega' \sim |\omega/\Omega'|^{2/3}$ and $G_{\max} \sim (\Omega'/\omega)^{2/3}/(|\tilde{n}| + \sigma) < (\Omega'/\Delta)^{2/3}$. In both cases the resonant activation is by $\tilde{f}$.

4. Quantum turbulence and resonant tunneling

The Coulomb interaction in resonant tunneling leads to turbulence of a specifically quantum nature. Indeed, suppose the incoming energy slightly exceeds the well's eigenenergy. If a random fluctuation increases the well's charge, and thus the Coulomb and eigenstate energy, then the tunneling approaches the resonance. This increases the transmission and thus the well's charge. So, the eigenstate energy further approaches the incoming energy, this further increases the well's charge, and so on. Thus, the fluctuation progressively grows. If, however, another random fluctuation meanwhile decreases the well's charge, and thus its eigenenergy, then the tunneling moves away from the resonance and decreases. And thus further decreases the well's charge and its eigenenergy, and so on. Therefore, such a fluctuation also progressively grows. The fluctuation increases may be exponentially high, since the charge density in the well is exponentially higher than in the incoming current at resonance and is exponentially lower outside it, beyond the resonance width [47]. Even when the incoming current is very low, such random fluctuations specifically imply quantum turbulence, which is intrinsically related to resonant tunneling. Resonant tunneling and thus turbulence, are most pronounced in 1D, and in a double barrier structure in a strong magnetic field, which effectively reduces it to 1D (see later). The characteristic turbulence timescale is related to the eigenstate width. On a shorter timescale the tunneling current oscillates [16] and may have a Coulomb blockade [17] type shape; in the general case its charge transfer per period is fractional (see later). Similar reasoning is applicable to resonant tunneling of the Aharonov – Bohm persistent current through a quantum dot, but with three amendments. The energy fluctuations must exceed the interlevel spacing in the ring (whose perimeter must be correspondingly larger, the lower its temperature). Dissipation implies that an alternating current may survive only over a time scale small compared to the dissipation time. The

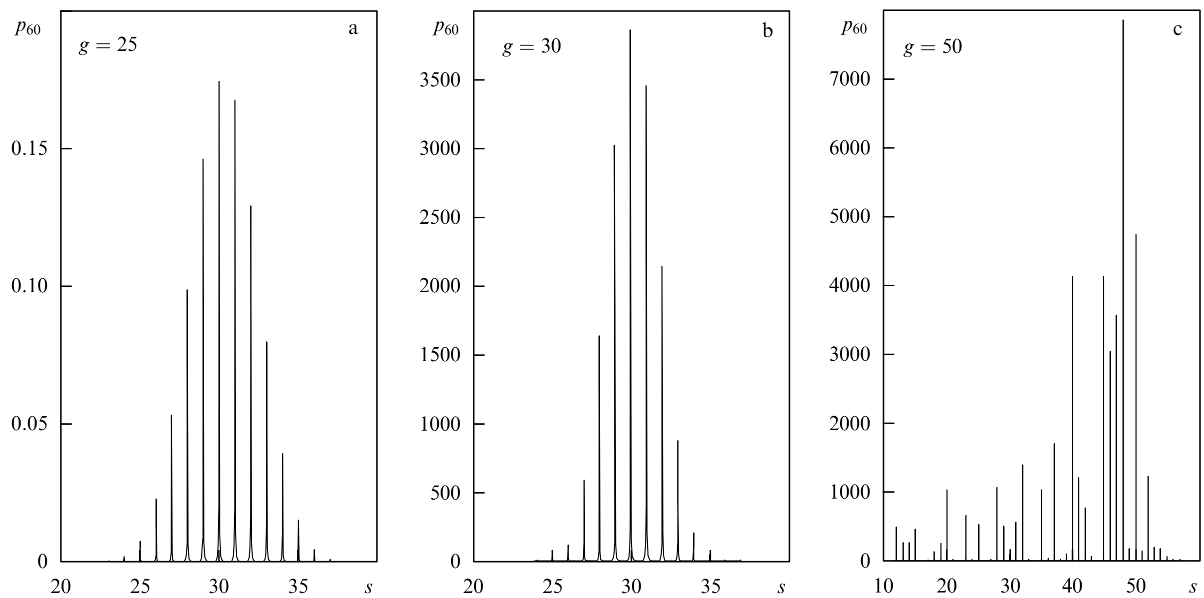

Figure 3. Transmission amplification p_{60} (due to the activation by 60ω) vs. the reduced incoming energy $s = (\Omega_{\mathrm{res}} - \Omega)/\omega$ for $\sigma = 0.01$, $g = 25$ (a), 30 (b), 50 (c). Note the multiple resonances and their cut-offs (in s and g), which are determined by the alternating potential strength rather than by its frequency.

magnetic field, which alternates with the corresponding frequency, may allow one to observe intrinsic oscillations, instability and turbulence of such a quasi-persistent current.

Consider the relevant time scales. The shortest one is related to the period of classical oscillations in a potential well. This scale is usually a fraction of a nanosecond, and determines [16] the oscillation period. Instability and turbulence develop during the particle dwell time in the well. The latter is inversely proportional to the barrier transmittance, and may reach the microsecond range. The dissipation time is related to electron-phonon scattering and radiation, at 10 mK it is ~ 0.1 s. Alternating quasipersistent currents may be observed when the ring perimeter (in μm) is $\gtrsim 10^{-7} v_F/T$, where the temperature T is in degrees of Kelvin, and v_F is the Fermi velocity (in cm s^{-1}).

We also consider a Hartree liquid. The latter reduces to a non-linear Schrödinger equation, which is very rich in physics and applications and has been extensively studied [49]. However, a conventional (in particular, solution) approach is hardly applicable to a system in an external potential, and especially to resonant tunneling, where the linear response charge density, which is exponentially higher inside the potential well than outside it, implies extreme space heterogeneity. Earlier this very heterogeneity elucidated the physics of resonant turbulence.

Consider a geometry reduced to one dimension. Then the non-linear Schrödinger equation in the static potential $U(x)$ with the interaction potential $V(x,x') = V(x',x)$ reads [42]:

$$\mathrm{i}\dot{\psi} = \hat{H}\psi \equiv -\psi'' + U\psi + \psi \int V(x,x') \left|\psi(x',t)\right|^2 \mathrm{d}x' , \quad (27)$$

where $\dot{\psi} \equiv \partial\psi/\partial t$, $\psi' \equiv \partial\psi/\partial x$ and $V(x,x') = V(x',x)$ may also account for the heterogeneous dielectric constant ε_d. [If ε_d is homogeneous, then $V(x,x') = V(|x-x'|)$]. The conserved energy ε is

$$\varepsilon = \mathrm{i}\int \psi^* \dot{\psi}\, \mathrm{d}V - \frac{1}{2}\int V(x,x')\left|\psi(x)\right|^2 \left|\psi(x')\right|^2 \mathrm{d}x\, \mathrm{d}x' \quad (28)$$

(a star denotes complex conjugation). We consider a potential U which consists of two barriers with a well between them, start with a stationary solution $\psi = \psi_0(x)\exp(-\mathrm{i}\omega_0 t)$, then prove its instability, find time-dependent solutions and study their stability. By Eqn (27), $\psi_0(x)$ yields

$$\psi_0'' + \left[\psi_0 - U - \int V(x,x')\left|\psi_0(x')\right|^2 \mathrm{d}x'\right]\psi_0 = 0 . \quad (29)$$

Suppose $U(-x) = U(x)$, each barrier is opaque, and the incoming current is sufficiently weak to disregard the interaction term outside the well. If the incoming energy ω_0 is within the well eigenenergy width $\delta\omega$, then in the well $|\psi_0|$ is exponentially higher than outside it. In the semiclassical case $|\psi_0|$ rapidly oscillates in the well, and in the leading approximation one may replace $|\psi_0|^2$ there by its average value $\langle|\psi_0|^2\rangle$ in the well. If ω_0 is outside $\delta\omega$, then $|\psi_0|$ nowhere exceeds its value $|a|$ in the incoming wave, and the interaction energy may be disregarded everywhere. So, in a general semiclassical case the effective potential in Eqn (29) is

$$U_{\mathrm{eff}} \cong U + \left\langle|\psi_0|^2\right\rangle \int V(x,x')\, \mathrm{d}x' ,$$

and the transmittance T is [47]

$$T \cong \frac{1}{1 + \exp(-4S)\sin^2\alpha} ;$$

$$\alpha = 2\int_0^{x_1} k(x)\, \mathrm{d}x , \qquad S = \int_{x_1}^{x_2} \left|k(x)\right| \mathrm{d}x , \quad (30)$$

$$k = (\omega_0 - U_{\mathrm{eff}})^{1/2} , \qquad U_{\mathrm{eff}}(x_1) = U_{\mathrm{eff}}(x_2) = 0 , \qquad x_1 < x_2 . \quad (31)$$

The transmittance is related [47] to $\langle|\psi_0|^2\rangle/|a|^2$:

$$T \sim \frac{\langle|\psi_0|^2\rangle}{|a|^2}\exp(-2S) . \quad (32)$$

So, Eqns (30) and (31) determine T — see Fig. 4 [where $\lambda = 2x_1$, $\Lambda = x_2 - x_1$, $U_{\max} = \max(U)$]. The average interaction energy in the well is $\sim e^2/\lambda$. So, when $|a|^2 \gtrsim (U_{\max}/e^2)\exp(-2S)$, the incoming energy ω_0 yields non-resonant tunneling and multiple resonances (with the resulting non-monotonic current — voltage dependence — cf. experiments in Ref. [50]). The latter are related to self-organized charge build-ups in the well, which lift different eigenenergies to resonance with the incoming energy.

We now consider the stability of stationary tunneling. According to Eqn (27), an infinitesimally weak perturbation to $\psi_0\exp(-\mathrm{i}\omega_0 t)$ may be presented as

$$\psi = \psi_0\exp(-\mathrm{i}\omega_0 t) + \psi_+\exp(-\mathrm{i}\omega_+ t) + \psi_-\exp(-\mathrm{i}\omega_- t) ,$$
$$\omega_\pm = \omega_0 \pm \omega . \quad (33)$$

We introduce the vector $\boldsymbol{\psi}$ with components ψ_+ and ψ_-; the Hermitian matrix $\tilde{W}(x,x')$ with components

$$\left[\tilde{W}(x,x')\right]_{++} = V(x,x')\psi_0(x)\psi_0^*(x') ,$$
$$\left[\tilde{W}(x,x')\right]_{+-} = V(x,x')\psi_0(x)\psi_0(x')$$

and the operator

$$\hat{W}\boldsymbol{\psi} \equiv \int \tilde{W}(x,x')\boldsymbol{\psi}(x')\, \mathrm{d}x' .$$

In the linear approximation Eqns (27), (33) yield:

$$(\hat{H}_0 + \hat{W} - \omega\hat{\sigma}_z)\boldsymbol{\psi} = \omega_0\boldsymbol{\psi} ,$$
$$\hat{H}_0 = -\frac{\partial}{\partial x^2} + U + \int V(x,x')\left|\psi_0(x')\right|^2 \mathrm{d}x' , \quad (34)$$

where $\hat{\sigma}_z$ is (formally, of course) the z-projection of the spin operator. If the incoming current is fixed, $\boldsymbol{\psi}$ must yield outgoing waves only, i.e. the stationary leakage from the well. This is obviously impossible for any real eigenvalue ω_0 of the Hermitian operator, implying [18] $\mathrm{Im}(\omega) \sim \delta\omega$, and thus, by Eqn (33), the instability of any stationary tunneling in the Hartree model. [In the approximation $\delta\omega = 0$, the eigenvalues ω in Eqn (34) imply the possibility of the corresponding resonance.] Similar reasoning demonstrates the instability of an alternating solution to Eqn (27).

An explicit analytical solution may be found in a shallow well, i.e. in a model $U = U_1(x) - U_0\delta(x)$, $V(x,x') = V(x)\delta(x)\delta(x')$. Then outside $x \neq 0$ Eqn (27) is leaner, and ψ_s (where $s = +1$ corresponds to $x > 0$ and $s = -1$ to $x < 0$)

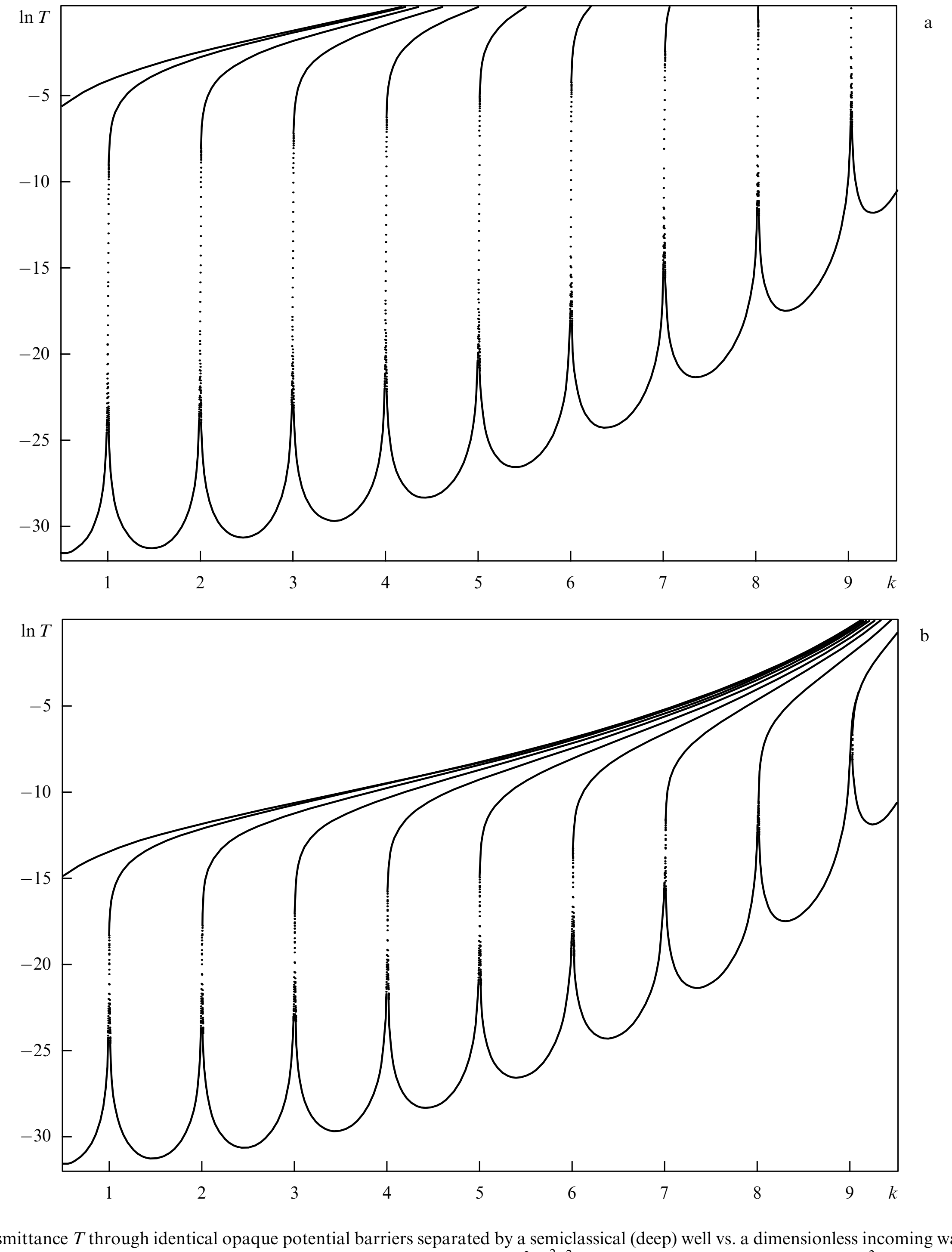

Figure 4. Transmittance T through identical opaque potential barriers separated by a semiclassical (deep) well vs. a dimensionless incoming wave vector $\tilde{k} = \lambda\sqrt{\omega_0}$ for different values of the dimensionless incoming particle density $\tilde{\rho} = e^2|a|^2\lambda^2$. The dimensionless potential height $\Lambda^2 U_{\max} = 1000$; λ and $\Lambda = \lambda/3$ are the well and the barrier widths; $|a|$ is the incoming wave amplitude; $\tilde{\rho} = 10^{-4}$ (a) and $\tilde{\rho} = 1$ (b). Note that $\tilde{\rho} = 1$ yields 20 resonant branches at $\tilde{k}/\pi = 9$ (of course, each resonance has a narrow width which cannot be seen on this scale).

may be presented as

$$\psi_s = \left[\frac{a}{2}(1-s)\psi_{\mathrm{i}}(x) + \int a_{\omega s}\psi_{\omega s}(x)\exp(-\mathrm{i}\omega t)\,\mathrm{d}\omega\right]\exp(-\mathrm{i}\omega_0 t)\,. \qquad (35)$$

Outside the barrier the incoming $\psi_{\mathrm{i}}(x) = \exp\left(\mathrm{i}x\sqrt{\omega_0}\right)$; the reflected $(s=-1)$ and transmitted $(s=+1)$ $\psi_{\omega s}(x) = \exp\left(\mathrm{i}|x|\sqrt{\omega_0+\omega}\right)$. Matching conditions for $\psi(\pm 0)$ and $\psi'(\pm 0)$, by Eqn (27), reduce ψ to $\varphi(t) = \psi(+0,t) = \psi(-0,t)$ and yield the equation

$$k(\omega)\varphi_\omega = b\delta(\omega) + V_0\left(|\varphi|^2\varphi\right)_\omega\,, \qquad (36)$$

$$k(\omega) = \left[\ln\frac{\psi_{\omega+}(0)}{\psi_{\omega-}(0)}\right]'; \qquad b = a\psi_{\mathrm{i}}(0)\left[\ln\frac{\psi_{\mathrm{i}}(0)}{\psi_{0^-}(0)}\right]'. \qquad (37)$$

The subscript in Eqn (36) denotes the Fourier component. When $U_1(x)$ is opaque, the coefficients $k(\omega)$ and b are

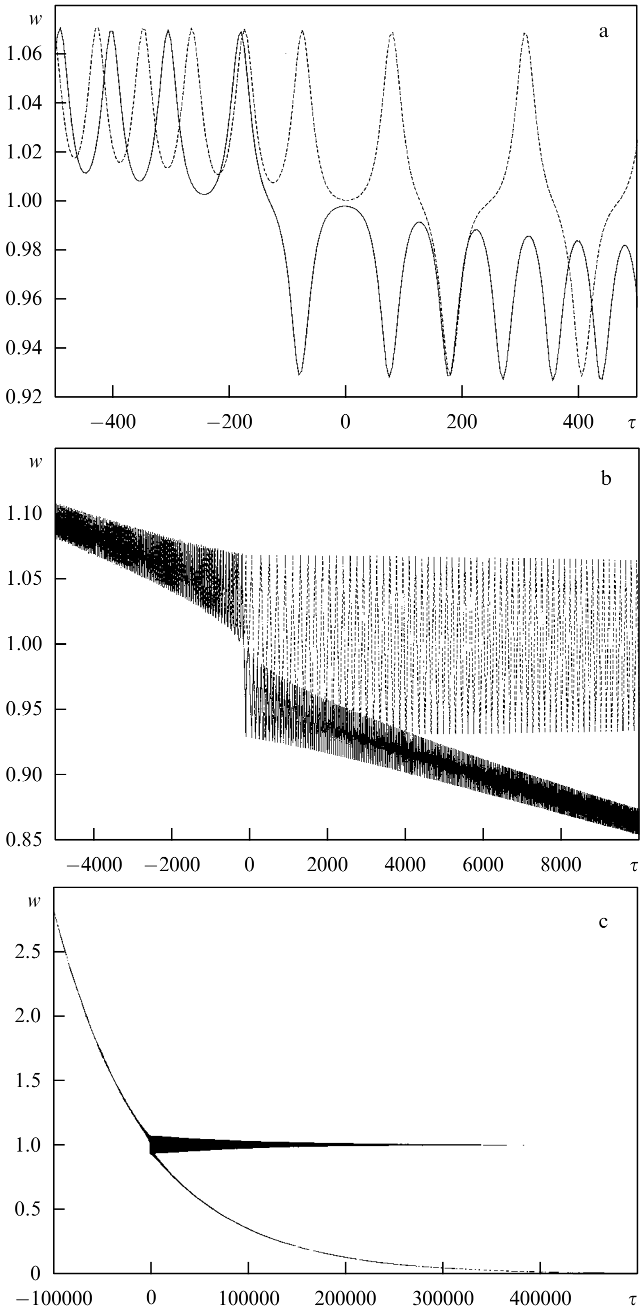

Figure 5. Dimensionless amplitude w of the wave function in the well's center as function of dimensionless time τ for $\beta = 0.0025$, $r = 10^{-5}$; $S(0) = 0$ and $w(0) = 0.997494$ (solid line) and $w(0) = 1$ (dashed line) on a large (a), medium (b) and small (c) scale. Note the divergence of the solutions.

readily calculated in the WKB approximation of the stationary equation. Then $\mathrm{Re}\left[k(\omega)\right] \cong 2\sqrt{q^2 - \omega}$, $q = \left[\max(U_1) - \omega_0\right]^{1/2}$; $\mathrm{Im}\left[k(\omega)\right] \sim \exp(-2S)$, $|b| \sim a\exp(-2S)$. Simple algebra ascertains the instability of a stationary solution to Eqn (36).

To study the general case, consider a quasi-stationary φ, which is reduced to low frequencies $\omega \ll q^2$, and thus yields $k(\omega) \cong k_1 + (\omega/q) + \mathrm{i}k_2$, $k_1 = \mathrm{Re}\left[k(0)\right]$, $k_2 = \mathrm{Im}\left[k(0)\right]$, and Eqn (36) may be rewritten as

$$(k_1 + \mathrm{i}k_2)\varphi + \frac{\mathrm{i}\dot{\varphi}}{q} = b + V_0|\varphi|^2\varphi \,. \tag{38}$$

We introduce

$$b = \frac{\beta k_1^{3/2}}{V_0^{1/2}}\exp(\mathrm{i}B)\,, \qquad \varphi = \left(\frac{k_1}{V_0}\right)^{1/2} w\exp(\mathrm{i}B - \mathrm{i}S)\,;$$
$$\frac{k_2}{k_1} = r\,, \qquad t - t_0 = \frac{\tau}{k_1 q}\,, \tag{39}$$

where w and β are positive and t_0 is an arbitrary constant. Then Eqn (38) reduces to

$$\frac{\mathrm{d}w}{\mathrm{d}\tau} = \beta\sin S - rw\,, \qquad \frac{\mathrm{d}S}{\mathrm{d}\tau} = w^2 + \beta w^{-1}\cos S - 1\,. \tag{40}$$

When $c_1 > 2\beta$, then $\varphi(t)$ is periodic in t with the period $\Delta t = (2/\sqrt{c_1})K(\sqrt{2\beta/c_1})$, where K is the complete elliptic integral. If $c_1 \cong 2\beta$, then $|\varphi|$ has a Coulomb blockade shape. The charge transfer per period depends on $\varphi(t_0)$ and is, in general, fractional. The value of r is proportional to the barrier transmittance $\exp(-2S)$. It is exponentially small in an opaque barrier, and allows for perturbations in rw in Eqn (17). At $|\tau| \gg 1/r$, depending on the initial conditions, the solution either diverges as $\tau \to -\infty$ and approaches a stationary solution (which was earlier demonstrated to be unstable) at $\tau \to \infty$, or vice versa. The numerical calculations for Eqn (40) verify the above analytical results (Fig. 5).

Equations (40) explicitly demonstrate the origin of the instability. They yield solutions, whose proximity allows for fluctuation induced transitions [Fig. 5a; in particular, their increasing proximity to the branching point $w = 1$ makes the transition moment from $w > 1$ to $w < 1$ unstable, but they are remote elsewhere and finally diverge (Fig. 5b, c)].

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