Physics – Uspekhi **41** (4) 411 – 416 (1998)

BIBLIOGRAPHY

PACS numbers: **02.50.+r**, **05.45.+b**

The Fokker – Planck equation

A I Olemskoĭ

The Fokker – Planck Equation. Risken H
(Berlin, Heidelberg: Springer-Verlag, 1989)

Beginning from the early works of Boltzmann, Gibbs, and Maxwell, the theory of stochastic systems has attracted considerable attention but recently it has experienced an explosive-like growth of interest. Let alone the numerous conferences, several dedicated journals have sprung into life (*Physical Review*, for example, has introduced the E series) and a lot of monographs devoted to various aspects of the theory of stochastic systems have been published (see Refs [1 – 5] and the bibliographies therein). Among them the Springer series in synergetics, edited by H Haken, is one of the best. The monograph written by his collaborator H Risken was included in the 18th volume of this series and was first published in 1984 and then, with a few corrections and additions, in 1989. It is frequently cited in the English scientific literature but, unfortunately, is almost unknown in the CIS. Therefore it is advantageous to become familiar with its content – the more so as the book written by H Risken is in my opinion one of the best monographs on the theory of stochastic systems.

First and foremost, its value is in the absence of any spells like 'it is easy to see that...', to which the Russian audience is accustomed due to the Course of Theoretical Physics by Landau and Lifshitz†. Risken set forth all the aspects of the topic in succession, without going into unnecessary details, whether for the well-known theory of Brownian movement of particles or special approaches to the Fokker – Planck equation. In addition the author never 'pours water' and the book is written in splendid scientific language — it is a sheer pleasure to read!

The monograph is quite voluminous: it has over 470 pages, 441 references, twelve main chapters, appendices, additions (to the first edition), and an extensive subject index. Somewhat arbitrarily the material can be grouped into three parts. The introductory Chapters 1 – 3 introduce the fundamental notions of the theory of stochastic systems, present the required data from the theory of probability, and consider the Langevin equation. The central Chapters 4 – 10 are dedicated to the major subject of the monograph, to the Fokker – Planck equation (FPE). Chapter 4 is the key chapter of the book. Here various presentations of the FPE are given.

† According to K P Belov, the reviewer of the Course: 'Landau guessed the result while Lifshitz did not necessarily present its proof'.

Uspekhi Fizicheskikh Nauk **168** (4) 475 – 480 (1998)
Translated by D Kh Gan'ga; edited by A Radzig

Then Chapters 5 and 6 describe how the FPE can be solved in the case of one or more variables. Chapter 7 stands somewhat apart and is devoted to a presentation of the fundamental data on correlative functions and susceptibilities. Chapter 8 considers how the number of determining variables in the system can be decreased. In Chapter 9 the FPE is solved by the recursive method, the use of which results in the introduction of continued fractions. In Chapter 10 this method is applied to the Kramers equation, the special case of the FPE for the diffusion of particles in an arbitrary potential. More than a third of the book is filled by the presentation of various applications: Chapter 11 considers the Brownian movement of particles in a periodic potential, and Chapter 12 sets forth the statistical properties of laser radiation. It is somewhat wearing to read through these last chapters because of the abundance of specific results but it seems inevitable for Chapters 11, 12, where Risken set forth the material close to his field of scientific interests.

Here we shall dwell in more detail on the cited topics. We shall present as many details as required for the reader to get the overall impression about the issues covered and the essence of methods used, even if he or she has not touched this book.

The introductory Chapter 1 includes the following sections: 1.1 Brownian motion; 1.2 Fokker-Planck equation; 1.3 Boltzmann equation; 1.4 Master equation. Section 1.1 shows how the Langevin stochastic equation can be derived from the equation of motion of a ball in a liquid. White and colour noises are defined and the white noise intensity is found based on the equipartition law. It is shown that the $x(t)$ solution to the stochastic equation at t, due to its random nature, can be associated with the distribution function $W(x, t)$, the form of which is found from the FPE. Section 1.2 lists the principal types of the FPE. In its general form this is the Chapman – Kolmogorov equation

$$\frac{\partial W(x,t)}{\partial t} = \int M(x,t;x',t')W(x',t')\,\mathrm{d}x'(t')\,, \qquad (1)$$

where the memory function $M(x, t; x', t')$ accounts for the distribution function $W(x', t')$ in the previous instances of time $t' < t$. Expansion of the kernel M in terms of the difference $x - x'$ yields a generalized (in the sense of the memory effects) Kramers – Moyal equation

$$\frac{\partial W(x,t)}{\partial t} = \sum_{n=1}^{\infty}(-\nabla)^n \int_{-\infty}^{t} D^{(n)}(x,t-t')W(x,t')\,\mathrm{d}t'\,, \quad (2)$$

where $\nabla \equiv \partial/\partial x$ and the coefficients $D^{(n)}$ are the moments of the memory function M divided by $n!$. In the case of Markovian processes they are proportional to $\delta(t - t')$ and

the integration in (2) vanishes. In addition, in the case of the δ-correlated (white) noise we have $D^{(n)} \equiv 0$ for $n > 2$, and Eqn (2) brings about the conventional form of the FPE:

$$\frac{\partial W(\mathbf{x}, t)}{\partial t} = L_{\mathrm{FP}} W(\mathbf{x}, t)\,, \quad L_{\mathrm{FP}} \equiv -\nabla_i D_i^{(1)}(\mathbf{x}) + \nabla_i \nabla_j D_{ij}^{(2)}(\mathbf{x})\,, \tag{3}$$

where the equation is extended to the case of several variables so that $\{x_i\} \equiv \mathbf{x}$ and $\nabla_i \equiv \partial/\partial x_i$, the summation is taken over repeating indices, and the quantities $D_i^{(1)}$, $D_{ij}^{(2)} = D_{ji}^{(2)} > 0$ determine the intensities of drift and diffusion processes. In the Kramers equation the set of variables $\{x_i\}$ is reduced to a coordinate and a velocity of one-dimensional Brownian motion. For a large viscosity, the inertial terms in the stochastic equation can be neglected and the velocity of a Brownian particle can be factored out. As a result the set $\{x_i\}$ is reduced to a coordinate and the FPE takes the simplest form of the Smoluchowski equation.

The list of cases when an analytical solution of the FPE can be found is very interesting (Section 1.2.4). Both steady and unsteady analytical solutions exist when the drift factor $D^{(1)}(\mathbf{x})$ depends linearly on the diffusion coefficient $D^{(2)}$ and the latter is constant. The steady distribution can be reduced to quadratures provided that a detailed balance exists and the flow of probability is zero. In addition, in Section 1.2.4 various methods for solving the FPE are cited, including computer simulation, numerical integration, reduction to the Schrödinger equation, analytical solution (especially for the one-dimensional case) when the dependences $D_i^{(1)}(\mathbf{x})$, $D_{ij}^{(2)}(\mathbf{x})$ have a special form, an unsteady solution for a weak external action (theory of linear response), and the method of continued fractions. Unfortunately, this list misses the important case of the analytical solution to the FPE in the unsteady self-modelling regime (this solution was first obtained in 1958 by I M Lifshits and V V Slezov when they considered the phase decomposition ensemble in the course of coalescence). With the use of the self-similarity condition for such a system, the dependence $W(x, t)$ on the two variables x and t can be reduced to one on a single argument $y = x/a(t)$, where $a(t)$ represents the efficient scale of x. It turns out that as in the aforementioned simple cases the $W(y)$ dependence can be presented as an exponential Gibbsian distribution.

In Section 1.3, the kinetic Boltzmann equation is compared with the FPE. They differ in that the first equation describes the ensemble evolution when all the particles are similar, while the second one governs the stochastic behaviour of a probe (large) particle. With linearization of the collision integral, the Boltzmann equation takes the form of the Kramers equation.

Section 1.4 gives the groundwork for the master equation governing the microscopic behaviour of a stochastic system. For example, it presents an expression for the δ-shaped microstate transition rate when the master equation takes the form of the FPE.

Chapter 2 under the title 'Probability Theory' starts with the definitions of probability and probability density through the Heaviside and Dirac functions, respectively. It is shown how the features of the latter can be used to go over to other stochastic variables. Primary emphasis is placed on the characteristic (generating) function $C(u)$, which is the average of the exponent $\exp(\mathrm{i}u\zeta)$, where ζ is a random variable and u is an arbitrary parameter. If follows from the definition that, firstly, the characteristic function is the Fourier image of the probability density and, secondly, for $u = 0$ it is reduced to a partition function. If the dependence $C(u)$ is expanded into a Maclaurin series in terms of u, then the factors are reduced to the moments M_n for a random variable ζ. However, since it is the free energy $-T\ln C(0)$ (where T is the temperature) that is physically meaningful rather than the statistical sum $C(0)$, the expansion $\ln C(u)$ should be used (the expansion coefficients are referred to as cumulants K_n and in diagrams they are presented as irreducible graphs). The author presents explicit expressions for K_n via M_n (and vice versa) for $n = 1, \ldots, 4$ and, moreover, he finds determinant expressions to go over from $M_n(K_n)$ to $K_n(M_n)$ for any n. The consideration is conducted first for a single stochastic variable and then generalized to several variables. A relationship between the conditional and total probabilities is found. The Gaussian distribution is considered separately (in my opinion it is not quite adequate; in any case there is a better and fuller description in Appendix 2.1 to the book [6]). Sections 2.4 and 2.5 are devoted to the study of stochastic processes, in which the stochastic variable is a function of time. It is shown that the distribution function of several variables can be reduced to a product of appropriate functions of a single variable for purely random processes. In going to Markovian processes we obtain the product of pair functions of conditional probability. This yields the integral Chapman – Kolmogorov equation, from which the desired probability can be found (see Section 2.4.2). In conclusion to Chapter 2, the Wiener – Khinchin formula is given for the Fourier image of the correlation function of random variables through the Fourier images of these variables.

Chapter 3 is devoted to the Langevin equation, on which the entire theory of stochastic systems is based (in the absence of a regular force it describes a Wiener process, and with a linear dependence of the force on a stochastic variable it describes the Ornstein – Uhlenbeck process). The consideration starts from the simplest case of Brownian motion (Section 3.1). Here the feature of the book that we noted at the very beginning of the review is most conspicuous as the author goes into details of calculation of the velocity and drift correlators of Brownian particles at an arbitrary instant of time (usually only the $t \to \infty$ asymptotics are given).

The derivation of the Maxwellian distribution through the calculation of moments and the definition of a generating function are very instructive. The Ornstein – Uhlenbeck process is considered based on the Green function method, the use of which enables the time dependences to be found for the two first moments as $t \to 0$. Due to the linear nature of the Ornstein – Uhlenbeck equation, the exact solution can be obtained in going over to the Fourier images with respect to time. The analysis of the Langevin equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = h(x) + g(x)\zeta(t)\,, \quad \langle\zeta\rangle = 0\,, \quad \langle\zeta(t)\zeta(t')\rangle = 2\delta(t - t') \tag{4}$$

in Section 3.3, where $h(x)$ is the deterministic force, and $g(x)$ is the amplitude of a random force $\zeta(t)$ (a multiplicative function), plays a leading part not only in Chapter 3 but in the entire theory of stochastic systems as well. Firstly (Section 3.3.1) the author determines the time dependences of moments for a Wiener process (the force is $h = 0$) with the noise amplitude $g = x$ (the Verhulst model). It turns out that although the deterministic force h is absent, the drift coefficient $D^{(1)} \propto x$ is not zero. In the general case of several

variables x_i, if the noise amplitude is a function $g_{ij}(\mathbf{x})$ of a random variable x_i, then the drift and diffusion coefficients are

$$D_i^{(1)} = h_i + g_{kj}\nabla_k g_{ij}\,, \quad D_{ij}^{(2)} = g_{ik}g_{jk}\,. \tag{5}$$

Thus, the multiplicative nature of noise brings about a fictitious force in addition to a real force h_i. This addition appears when the Stratonovich calculus is used and vanishes when we apply the Ito calculus (Section 3.3.3). They differ in the choice of the point $\tilde{t}_n = t_n + \lambda\Delta t_n$, $\lambda \in [0,1]$ for an infinitesimal time interval $\Delta t_n = t_{n+1} - t_n \to 0$, on which the stochastic variable $x_i(t)$ is defined. In the Ito calculus this point is taken at the left end of the interval ($\lambda = 0$), whereas in the Stratonovich calculus it is taken in the middle ($\lambda = 1/2$). Unfortunately, the author prefers the Stratonovich calculus and does not mention that a continuum set of calculuses is possible, depending on the choice of the parameter $\lambda \in [0,1]$. However, in this way the reader is saved from the probable vague speculations as in Ref. [4], the reading of which gives the idea that the authors do not understand what they are writing about. At the end of Chapter 3 the relationships are found between the coefficients $D_i^{(1)}$ and $D_{ij}^{(2)}$ depending on the choice of the stochastic variables x_i (Section 3.4.2). It is shown how a non-Markovian process can be presented as a Markovian process through introduction of an auxiliary variable (Section 3.5), and how the Langevin equation can be simulated on a computer (Section 3.6).

Chapter 4 presents an introduction to the principal subject of investigation, which is the FPE. In Sections 4.1 and 4.2 the author considers three different methods of how the forward and backward Kolmogorov equations can be obtained for opposite directions of the time arrow (the FPE is reduced to the forward equation). These are formal methods and in essence they consist in expanding either the generating function, or δ-function, or conditional probability in terms of the difference $x - x'$. In my opinion the common derivation via the master equation as in Ref. [7] is more physically meaningful, while the formal procedure is more straightforward. The Pawula theorem is very important and its simple formulation is based on the Schwartz inequality (Section 4.3). This theorem shows that the Kramers–Moyal series (2) can be truncated either at the first term or at the second term, and that the truncation procedure becomes contradictory when an arbitrary number of terms $n > 2$ is retained. In Section 4.4, the FPE is studied for the case of a single variable. It is shown that it can be presented in the form of the equation of continuity and that in this form it can be reduced to the Maxwellian distribution by setting a steady flow for the probability density equal to zero. For small times the conditional probability takes the form of the functional Gaussian distribution throughout the velocity $\dot{x}$, which is shifted by an expectation $D^{(1)}$ with a variance $2D^{(2)}$. After the consideration of some examples in Sections 4.5 and 4.6, the author extends the results obtained to an FPE of several variables (Section 4.7). In Section 4.8, some examples of an FPE of several variables are given, including three-dimensional Brownian motion in a velocity field, one- and three-dimensional Brownian motion in an external field, and, finally, the Brownian movement of two interacting particles. Section 4.9 deals with going over to new variables in the FPE. The parameters $D_i^{(1)}$ and $D_{ij}^{(2)}$ are shown to be transformed by means of the same formulae as those found in Section 3.4.2 on consideration of the Langevin equation. And, finally, Section 4.10 shows that using a common construction of differential geometry the FPE can be presented in a covariant form, where the diffusion coefficient $D_{ij}^{(2)}$ matrix plays the role of a contravariant metric tensor and the inverse diffusion matrix is a covariant tensor.

It is a real pleasure to read Chapter 5, where the single-variable FPE is solved. The introductory Section 5.1 shows that using the transformation of variable from Sections 3.4, 4.9 the diffusion coefficient $D^{(2)}(x)$ can be reduced to a constant $D > 0$. As a result the steady solution of the FPE takes an exponential form with index $-\Phi \equiv -V/D$, $V \equiv -\int D^{(1)}\,\mathrm{d}x$ (Section 5.2). In the simple case of the Ornstein–Uhlenbeck process ($D^{(1)} \propto x$), the steady distribution takes a Gaussian form (Section 5.3). Expansion of the Fokker–Planck operator $L_{\mathrm{FP}} \equiv -\nabla D^{(1)} + \nabla^2 D^{(2)} = \nabla D^{(2)} \exp(-\Phi)\nabla \exp\Phi$ in terms of the eigenfunctions φ_n is one of the primary methods for solving the FPE. In Section 5.4, the operator is shown to have Hermitian properties under the transformation $\exp(\Phi/2)$. The relevant eigenfunction $\psi = \exp(\Phi/2)\varphi$ has a steady form $\psi_0 \propto \exp(-\Phi/2)$ (and the eigenvalue is $\lambda = 0$). The eigenvalue problem is related to the FPE solution by the following equality for the pair distribution function

$$W_2(x,t;x',t') = \psi_0(x)\psi_0(x')\sum_n \psi_n(x)\psi_n(x') \times \exp\left(-\lambda_n|t-t'|\right). \tag{6}$$

As is known, the principal problem of quantum mechanics is how to determine eigenvalues λ_n and eigenfunctions $\psi_n(x)$. It turns out that the Hermitian operator $L = \exp(\Phi/2)L_{\mathrm{FP}}\exp(-\Phi/2)$ can be written as a common Hamiltonian in the Schrödinger equation:

$$L = D\nabla^2 - U(x)\,, \quad U \equiv \frac{D}{4}(\nabla\Phi)^2 - \frac{1}{2}\nabla D\nabla\Phi\,, \tag{7}$$

where the quantity $\sqrt{2D}$ plays the part of the Planck constant, and the drift coefficient $D^{(1)} \equiv -\nabla V$ determines the effective potential energy $U(x)$. Moreover, expression (7) can be presented in a quadratic form $L = a^+a$, where the operators a^+, a are obtained from the momentum operators $p^+ = -\mathrm{i}\sqrt{D}\nabla$, $p = \mathrm{i}\sqrt{D}\nabla$ as a result of the transforms $\exp(\Phi/2)$, $\exp(-\Phi/2)$ and obey the commutation relation $[a, a^+] = -\nabla^2 V$. Hence, besides an analogy with the Schrödinger equation, powerful methods of supersymmetry from quantum theory can be used to solve the FPE [8]. Unfortunately, these methods are totally ignored by the author. Instead, the analogy with the Schrödinger equation receives primary attention. For example, in Sections 5.5–5.10 it is used to study the parabolic, inverse parabolic, V-shaped ($V = Dk|x|$), bistable, metastable (both discontinuous and smooth) types of potential $V(x)$ and that associated with a rectangular well-like dependence $U(x)$ in Eqn (7). Section 5.8 shows how the solution for the inverse potential $-V(x)$ can be reconstructed when the solution for the direct potential $V(x)$ is known. Section 5.9 is concerned with numerical methods for calculation of eigenfunctions and eigenvalues. In the final section 5.10, the rate of escaping the potential barrier is determined for the system in a metastable state. It is shown, within the scope of the quasi-classical approach and according to the Kramers formula, to be proportional to $\exp(-\Delta/D)$, where Δ is the barrier height. On the other hand, this rate is reduced to the least eigenvalue λ_0, the reciprocal of which is the time it takes to escape the potential barrier.

Chapter 6 differs from the preceding one in that it considers the case of several variables. However, since in this case the transformation of variables, by which the diffusion coefficient matrix is reduced to constant quantities, is not always possible, then the conditions should be studied under which the solution of the FPE can be reduced to the eigenvalue problem. To this end the drift coefficient $D_i = D_i^{(s)} + D_i^{(a)}$ should be divided into symmetric $D_i^{(s)} = \nabla_j D_{ij} - D_{ij}\nabla_j \Phi$ and antisymmetric $D_i^{(a)}$ components. Accordingly, the operator $L = \exp(\Phi/2) L_{\text{FP}} \exp(-\Phi/2)$ is divided into the Hermitian $L_{\text{H}} = \exp(\Phi/2)\nabla_i D_{ij} \exp(-\Phi)\nabla_j \exp(\Phi/2) \equiv L_{\text{H}}^+$ and anti-Hermitian $L_{\text{A}} = -\exp(\Phi/2)\nabla_i D_i^{(a)} \exp(-\Phi/2) \equiv -L_{\text{A}}^+$ components (the latter satisfies the condition $L_{\text{A}} W_0^{1/2} = 0$, where W_0 is a steady solution). Given $D_i^{(s)}$, the quantities $A_i \equiv \nabla_i \Phi = D_{ij}^{-1}(\nabla_k D_{jk} - D_i^{(s)})$ can be found such that the potentiality condition $\nabla_j A_i = \nabla_i A_j$ is satisfied and the exponent index $-\Phi$ for the steady distribution is defined by the equality $\Phi = \int A_i \, \mathrm{d}x_i$. Then the Hermitian component L_{H} takes the form of Eqn (7), where D, ∇ should be replaced by D_{ij}, ∇_i and the eigenfunctions φ_n, φ_n^+ of the FPE can be expressed via the eigenfunctions ψ_n, ψ_n^+ for the operators L_{H}, L_{A}^+ as $\varphi_n = \exp(-\Phi/2)\psi_n$, $\varphi_n^+ = \exp(\Phi/2)\psi_n^+$. Moreover, the Hermitian component $\psi_0^{\text{H}} \propto \exp(-\Phi/2)$ corresponding to the eigenvalue $\lambda = 0$ is reduced to the square root of the steady distribution function. In Section 6.4, the potentiality condition (and hence the splitting of the drift coefficients into symmetric and antisymmetric components) is shown to follow from the detailed balance condition. The last condition implies that microscopic transitions to and from an arbitrary level occur at the same rate at any instant of time. It can be expressed formally by the equality $L_{\text{FP}}(\mathbf{x}) W_0(\mathbf{x}) = W_0(\varepsilon\mathbf{x}) L_{\text{FP}}^+(\varepsilon\mathbf{x})$, where the time reversal operator is $\varepsilon_i = 1$ for even coordinate-like variables, and $\varepsilon_i = -1$ for odd velocity-like variables. It turns out that with this operator the quantities $D_i(\mathbf{x})$, $L_{\text{FP}}(\mathbf{x})$, $L(\mathbf{x})$ can be divided in the aforementioned way according to the definitions $D_i^{(s)}(\mathbf{x}) = \varepsilon_i D_i^{(s)}(\varepsilon\mathbf{x})$, $D_i^{(a)}(\mathbf{x}) = -\varepsilon_i D_i^{(a)}(\varepsilon\mathbf{x})$; $L_{\text{FP}}^{(s)}(\mathbf{x}) = L_{\text{FP}}^+(\varepsilon\mathbf{x})$, $L_{\text{FP}}^{(a)}(\mathbf{x}) = -L_{\text{FP}}^{(a)}(\varepsilon\mathbf{x})$, and $L(\mathbf{x}) = L^+(\varepsilon\mathbf{x})$ (the last equation for eigenfunctions means that $\psi_n(\mathbf{x}) = \psi_n^+(\varepsilon\mathbf{x})$). Thus, under the detailed balance conditions the solution to the FPE of several variables is obtained principally in the same manner as in the case of a single variable for the Ornstein – Uhlenbeck equation (see Section 6.5). The final Section 6.6 lists the methods by which the FPE can be solved when the detailed balance condition is broken: the elimination of rapidly changing variables (adiabatic approximation); the transformation of variables under which the FPE is reduced to the integrable Schrödinger equation; the variational method of fitting eigenfunctions to minimize the eigenvalue; reduction of the problem to the Hermitian operator $L_{\text{FP}}^+ L_{\text{FP}}$; numerical integration; expansion in terms of basis functions; the method of continued fractions, and, finally, the quasi-classical approximation.

Chapter 7 'Linear Response and Correlation Functions' seems, in my opinion, not quite adequate. The author considers three different proofs of the fluctuation-dissipation theorem, and according to the proverb 'not seeing the wood for the trees' the content of this chapter seems somewhat arbitrary. Certainly this is not the case and the consideration of this subject by D Forster in Ref. [9] is more appropriate.

Chapter 8 is concerned with three cases when the number of variables used to describe a stochastic system can be reduced. For example, Section 8.1 studies the T-time problem of the first passage out of a given region $[x_1, x_2]$ when the time derivative vanishes. The quantity T is distributed according to the probability density $w(x', T) = -\int_{x_1}^{x_2} \dot{P}(x, T|x', 0) \, \mathrm{d}x$, where $P(x, T|x', 0)$ is the probability of transition from the state x' at $t = 0$ into the state x at $t = T$. The solution of this problem amounts to determining the moments $T_n(x') \equiv \int_0^\infty T^n w(x', T) \, \mathrm{d}T \equiv \int_{x_1}^{x_2} p_n(x, x') \, \mathrm{d}x$ or the relevant densities $p_n(x, x')$. The equations for the moments $L_{\text{FP}}^+(x') T_n(x') = -n T_{n-1}(x')$, $n \geqslant 1$ follow from the backward Kolmogorov equation, and the recurrent relations $L_{\text{FP}} p_n(x, x') = -n p_{n-1}(x, x')$, $n \geqslant 1$ follow from the FPE. Section 8.2 deals with a particular case of the FPE when the coefficients $D_i^{(1)}(\mathbf{x})$, $D_{ij}^{(2)}(\mathbf{x})$ are independent of n arbitrary components x_i from the full set of variables $\{x_i\}^N$, $N > n$. In this case the Fourier images of these components should be used. Then a component can be separated out from the Fokker – Planck operator such that it is independent of the derivatives ∇_i, $i = 1, \ldots, n$, and thus the number of variables in the FPE is reduced to $N - n$. This procedure is most efficient when the dependence on x_i is periodic (for example, a variable x_i represents an angle). Fourier images are also used when one wants to find the distribution of the integral $I(t) = \int_{t_0}^t f(\mathbf{x}(t')) \, \mathrm{d}t'$, where $f(\mathbf{x})$ is a given function, rather than the distribution of the stochastic variables $x_i(t)$ themselves. In this case, if the equation $\dot{I}(t) = f(\mathbf{x})$ is interpreted as an addition to the Langevin system of equations for initial stochastic variables x_i, then the Fokker – Planck operator takes an elongated form $L = L_{\text{FP}}(\mathbf{x}) - f(\mathbf{x})(\partial/\partial I)$. In going over to Fourier images with respect to I, the derivative $\partial/\partial I$ is changed for $\mathrm{i}k$ and the Fourier image of the desired average quantity $M(k, t) \equiv \int G(k, \mathbf{x}, t) \, \mathrm{d}\mathbf{x}$ is given by the Green function G of the elongated FPE: $\dot{W} = LW$. As a result, if the dependence $G(\mathbf{x})$ is known, then the Langevin system of N equations for the initial variables x_i is reduced to a single equation $\dot{M}(k) = -\mathrm{i}k \int f(\mathbf{x}) G(k, \mathbf{x}) \, \mathrm{d}\mathbf{x}$ for the moment M. Section 8.3 is devoted to the adiabatic approximation, under which fast-varying stochastic quantities can be eliminated. Let, for example, the characteristic time it takes for the variable y to change be $\gamma \gg 1$ times less than that for the variable x. Then the solution of the FPE $\dot{W} = (L_x + \gamma L_y) W$ can be conveniently represented as an expansion $W(x, y, t) = \sum_n C_n(x, t) \varphi_n(y, x)$ in terms of the eigenfunctions φ_n of the operator L_y: $L_y(y, x)\varphi_n(y, x) = -\lambda_n(x)\varphi_n(y, x)$ (here the slow-varying variable x is set to be fixed). As a result the system of equations $\dot{C}_n + \gamma\lambda_n C_n = \sum_m L_{nm} C_m$, $L_{mn} \equiv \int \varphi_n^+ L_x \varphi_m \, \mathrm{d}y$ are obtained for the coefficients C_n. In the adiabatic approximation, the time derivative is retained only for $n = 0$ since it is negligible for $n \neq 0$ according to the condition $\gamma \gg 1$. Consequently, the coefficient $C_0 \sim \gamma^0$ significantly exceeds all the others $C_n \sim \gamma^{-1}$ and we have $C_n \simeq L_{n0} C_0 / \gamma\lambda_n$, $n \neq 0$ correct to $\gamma^{-1} \ll 1$. As a result the system can be described by the single equation $\dot{C}_0 = (L_{00} + \gamma^{-1} \sum_n L_{0n} \lambda_n^{-1} L_{n0}) C_0$, $n \neq 0$ for the quantity $C_0(x, t) \equiv \int W(x, y, t) \, \mathrm{d}y = W(x, t)$. In the final Section 8.3.2, the Zwanzig – Mori projection technique is briefly considered. By this technique, the equations can be reduced to a chain of equalities such that the correlator of the quantity with the largest relaxation time is expressed via the correlator of a faster variable; in a similar manner the latter is expressed via the correlator of an even faster variable, and so on. I would recommend the reader to look over book [9] if he (or she) wants to get a deeper knowledge of this powerful method.

Chapter 9 is dedicated to the solution of the FPE with the use of the continued fraction

$$\frac{a_1}{b_1+\left(\dfrac{a_2}{b_2+\ldots}\right)} \equiv [a_1, b_1; a_2, b_2; \ldots], \tag{8}$$

where the pairs a_n, b_n can be numbers, functions, operators, etc. The principal advantage of using continued fractions is that in this way the coefficients $C_n(t)$ for the expansion of the distribution function $W(x,t) = \sum_n C_n(t)\psi_n(x)$ in terms of the eigenfunctions $\psi_n(x)$ of the Fokker–Planck operator L_{FP} can be approximated most efficiently. The time dependences of these coefficients $C_n(t) = G_{nm}(t)C_m(0)$ are given by the Green matrix $G_{nm}(t)$ reporting the solution of the equation

$$\dot{G}_{nm} = Q_n^- G_{n-1m} + Q_n G_{nm} + Q_n^+ G_{n+1m}, \tag{9}$$

to which the FPE is reduced upon substitution of the cited expansion ($Q_n^{\pm}$, Q_n are given constants; they depend on the type of the L_{FP} operator). The distinguishing feature of Eqn (9) is that it establishes a relationship between three neighbouring matrix elements G_{nm}, and because of this the author calls it a 'tridiagonal recurrence relationship'. It turns out that it is a fundamental relationship for problems where the solution can be reduced to a consideration of (either ordinary or partial) differential equations of Bloch, Mathieu, etc. types. For example, if a recurrent relation can be found for $2L+1$, $L>1$ matrix elements, rather than for 3 matrix elements, then it can be reduced to (9) by introducing the L-component vectors $\mathbf{G}_{nm}$ and L-rank matrices $\widehat{Q}_n^{\pm}$, $\widehat{Q}_n$. Equation (9) can be solved on elimination of the derivatives in t by going over to Laplace images in s and by introducing the variables $S_n^+ = G_{n+1m}/G_{nm}$, $S_n^- = G_{n-1m}/G_{nm}$. The use of these variables yields expressions for G_{n+1m}, G_{n-1m} via G_{nm}, i.e. the index of a matrix element G_{nm} can be raised/lowered. Limiting the index n in (9) by the upper N_+ and lower N_- values and transferring from these limiting values to $m \in [N_-, N_+]$ we obtain, with the use of the S_n^+ and S_n^- operators, the following expressions via continued fractions:

$$\begin{aligned} S_m^+(s) &= [-Q_{m+1}^-, Q_{m+1} - s; -Q_{m+1}^+ Q_{m+2}^-, Q_{m+2} - s; \ldots], \\ S_m^-(s) &= [-Q_{m-1}^+, Q_{m-1} - s; -Q_{m-1}^- Q_{m-2}^+, Q_{m-2} - s; \ldots]. \end{aligned} \tag{10}$$

They can be used to determine the memory function $K_m \equiv Q_m^- S_m^- + Q_m^+ S_m^+$ and the diagonal element $G_{mm}(s) = [s - Q_m - K_m(s)]^{-1}$ for the Green function. For $n \neq m$, we have $G_{nm}(s) = U_{nm}(s)G_{mm}(s)$, where $U_{nm} = \prod_{l=1}^{|n-m|} S_{n\pm l}^{\pm}$ and the plus signs refer to the case $n > m$ while the minus signs refer to the case $n < m$. The above method seems very similar to the Zwanzig–Mori projection technique, at least formally, since both of them use continued fractions [9, 10].

Chapter 10 is devoted to the study of the Kramers equation for Brownian motion in an external field as one of the most popular forms of the FPE. The distinguishing feature of the Kramers equation is that the drift coefficient $D^{(1)}$ depends linearly on velocity. It is shown that the operator L_{FP} is divided into irreversible L_{ir} and L_{rev} reversible components. Under the transformation $\exp\{(v/2)^2 + V(x)/2T\}$ the former $L_{\mathrm{ir}} \propto -b^+b^-$ is expressed via the Bose operators $b^{\pm} = \mp\partial/\partial v + v/2$, and the latter has the form $L_{\mathrm{rev}} = -(b^-a^- + b^+a^+)$, where $a^{\pm} = \nabla[1 \pm (1/2)V(x)]$, $\nabla \equiv \partial/\partial x$ (here the coordinate x and the velocity v are measured in units of $(T/m)^{1/2}$). In the case of a harmonic potential $V(x)$ (Section 10.2), linear combinations of the operators a^+, b^+ form creation operators c_1^+, c_2^+, n_1-fold and n_2-fold applications of which to the function $\psi_{00} \propto \exp\{-(v/2)^2 - (\omega x/2)^2\}$, where ω is the eigenfrequency, yield the eigenfunctions $\psi_{n_1n_2} = (n_1!n_2!)^{-1/2} \times (c_1^+)^{n_1}(c_2^+)^{n_2}\psi_{00}$ and eigenvalues $\lambda_{n_1n_2} = (\gamma/2)(n_1+n_2) + [(\gamma/2)^2 - \omega^2]^{1/2}(n_1 - n_2)$, where γ is a kinetic coefficient. Expansion in terms of eigenfunctions leads to recurrence relations, the solutions of which are expressed via continued fractions (Section 10.3). The final Section 10.4 deals with the case of a large damping coefficient γ, when expansion in terms of the parameter $\gamma^{-1} \ll 1$ is possible. In this chapter the distinguishing feature of this monograph that we have noted at the very beginning of the review, namely, its comprehensive and accurate consideration of the subject, is most pronounced.

Chapter 11 'Brownian Motion in Periodic Potentials' is highly voluminous. It consists of nine extensive sections, occupies 97 pages, includes 94 figures, and refers to 57 papers and monographs. Firstly (Section 11.1), the author reviews problems that can be reduced to the problem of diffusion in a periodic potential: a pendulum, superionic conductor, Josephson contact, the rotation of a dipole in a constant field, phase stabilization, and the relation to the sin-Gordon equation. He distinguishes three different regimes depending on a small, intermediate, or large coefficient of friction. In the first case the particle coordinate plays a role of the hydrodynamic mode, so that the velocity drops out of the FPE and the latter reduces to the Smoluchowski equation (Section 11.3). For weak friction the velocity as well as the coordinate exhibit strongly fluctuating behaviour and the energy of particles plays the part of a hydrodynamic mode (Section 11.4). The intermediate case can be studied only with the use of continued fractions. If a homogeneous field is applied in addition to a periodic one, then the particle undergoes a drift with the mobility depending on the given field strength and on the noise intensity and being determined in Sections 11.3–11.5. The graphical display of steady distributions of coordinate and velocity for a particle for various homogeneous field strengths, coefficients of friction, and noise intensity in Section 11.5 is very impressive. Section 11.6 is devoted to the study of the transition from the localization regime to the travelling particle regime. This transition is shown to be possible only in the absence of noise as the homogeneous field strength increases. Unsteady solutions are studied in Section 11.7 with the use of Floquet's theorem to determine the eigenfunctions. The diffusion coefficient and mobility are shown to be related by the Einstein–Smoluchowski equality and the Fourier images of the structural factor is shown to take the diffusive form in the hydrodynamic limit. Section 11.8 deals with the determination of the frequency dependence of susceptibility. The use of the method of continued fractions shows that both the real and imaginary components of the susceptibility have peaks at the zero frequency ω and $\omega \neq 0$; these peaks smooth out as friction increases. And, finally, in Section 11.9 eigenvalues and eigenfunctions are studied for different relations between homogeneous and harmonic components, the coefficient of friction and the noise intensity. The plots therein seem very similar to the 'web' of electronic energy spectra of a crystal.

The final Chapter 12 is concerned with the study of statistical properties of laser radiation. Firstly, within the framework of the microscopic approach, the operators

corresponding to the electromagnetic wave amplitude on the one side and to the electron polarization and population difference in two energy levels on the other side are shown to satisfy to the well-known Lorenz synergistic system. Then within the framework of the adiabatic approximation this system is shown to reduce to the familiar van der Pol equation. With the introduction of stochastic sources, this last equation yields the Langevin equation for laser radiation with different intensities and phases. As a result the FPE and the recurrent relation for the intensity distribution moments are found. In Section 12.2, the steady intensity distribution is shown to have the form of the Gibbsian distribution with the Landau potential as an exponent. It is significant that the moments of the distribution increase in magnitude infinitely near the critical value of the controlling parameter as their order increases. Section 12.3 is concerned with the problem of presentation of eigenvalues and eigenfunctions. As is shown in the next Section 12.4, the relevant expansion coefficients are determined via the continued fractions, the use of which makes it possible to find the probability that the laser will emit spontaneously (Section 12.5). And, finally, in Section 12.6 it is established how the distribution of spontaneous emission intensity relates to that of photoelectrons emitted due to the photon detector exposure to this radiation.

The book includes six appendices more or less related to the use of the method of continued fractions and eleven additions in the second edition. Most of the additions are very concise and include references to other works, which have been published since the first edition. Among them the last Addition S.11 seems the most interesting since it considers the FPE with negative elements in the diffusion coefficient matrix and with third-order derivatives. Appendix A.1 and Addition S.10 consider colour noise, whose intensity has an exponential nature. It is shown that this case can be reduced to additive noise using the colour noise intensity as a new variable. In turn this enables the results of Chapters 3, 4 to be reproduced based on the continued fraction method. Appendix A.2 displays that the collision integral in the Boltzmann equation can be transformed into the Fokker – Planck operator and, thus, a continued fraction can be applied to it. Appendix A.3 presents calculations of matrix elements for Green's function of a harmonic oscillator with the aid of a continued fraction. In this way the results of Section 5.5 are reproduced. In Appendix A.4, the quantum-mechanical FPE is found using the characteristic function of a damped harmonic oscillator and then the steady distribution is derived from it. Appendix A.5 describes a very elegant method (the fourth one!) to derive the FPE by averaging the equation for the microcanonical distribution $\delta(t) = \delta(\zeta(t) - x)$, where $\zeta(t)$ complies with the stochastic equation of motion. And, finally, Appendix A.6 is devoted to the van der Pol equation with a fluctuating controlling parameter. The amplitude noise of the stochastic variable is shown now to be proportional to the amplitude value and, hence, the steady distribution can take a δ-like form. Going over to a logarithmic variable, the noise can be transformed into an additive function and the FPE can be reduced to the Schrödinger equation with a Toda potential.

References

1. Risken H *The Fokker – Planck Equation* (Springer Series in Synergetics, Vol. 18) (Berlin, New York: Springer-Verlag, 1989)
2. Haken H *Synergetics* (Berlin: Springer-Verlag, 1978) [Translated into Russian (Moscow: Mir, 1980)]
3. Gardiner C W *Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences*) (Springer Series in Synergetics, Vol. 13) (Berlin, New York: Springer-Verlag, 1983) [Translated into Russian (Moscow: Mir, 1986)]
4. Horsthemke W, Lefever R *Noise-Induced Transitions. Theory and Applications in Physics, Chemistry and Biology* (Springer Series in Synergetics, Vol. 15, Ed. H Haken) (Berlin: Springer-Verlag, 1984) [Translated into Russian (Moscow: Mir, 1987)]
5. Van Kampen N G *Stochastic Processes in Physics and Chemistry* (Amsterdam: North-Holland Physics Pub., 1984) [Translated into Russian (Moscow: Vysshaya Shkola, 1990)]
6. Doi M, Edwards S F *The Theory of Polymer Dynamics* (Oxford: Clarendon Press, 1987)
7. Lifshitz E M, Pitaevskiĭ L P *Fizicheskaya Kinetika* (Physical Kinetics) (Moscow: Nauka, 1979) [Translated into English (Oxford: Pergamon Press, 1981)]
8. Cooper F, Khare A, Sukhatme U *Phys. Rep.* **251** 267 (1995)
9. Forster D *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (London: W.A. Benjamin, Inc., 1975) [Translated into Russian (Moscow: Atomizdat, 1980)]
10. Olemskoi A I, in *Physics Reviews* (Ed. I M Khalatnikov) **18** Part I 1 (1996)