

REVIEWS OF TOPICAL PROBLEMS PACS numbers: **05.40.+j**, 05.70.Fh, **64.60.–i**, 82.20.Fd

Theory of stochastic systems with singular multiplicative noise

A I Olemskoĭ

Contents

Abstract. Noisy, interacting, stochastic systems are analyzed for the case in which their noise intensity varies with the hydrodynamic mode amplitude x according to the power law x^{2a}, $x \in [0, 1]$. It is shown that the phase space domain of definition of the stochastic variable x forms a self-affine set of fractal dimensionality $D = 2(1 - a)$. Using the gauge procedure, a system of calculus is chosen which is not reducible either to the Ito case or the Stratonovich case. By generalizing the microscopic picture of phase transitions it is demonstrated that the system may reduce its symmetry (for $1 < D \leqslant 2$) or lose ergodicity (for $0 < D \leqslant 1$). Over the entire interval $D \in [0, 2]$, a noise-induced transition is shown to be possible.

1. Introduction

The theory of stochastic systems has been attracting special attention ever since the works of J Maxwell, L Boltzmann and J Gibbs, and the study of such systems today occupies a leading position in theoretical physics. This is apparently due to the universality of the stochasticity concept stating that the behaviour of a system with an infinite number of degrees of freedom cannot be described in an unambiguous (deterministic) way. This circumstance has already been understood by Gibbs, who proposed a scheme of statistical physics based on the ergodic hypothesis [1]. The basic assumption of the latter is that the behaviour of a macroscopic system at thermodynamic equilibrium, characterized by time averages, can be represented by the distribution with respect to the statistical ensemble which imitates the evolution of the system. As a result, averaging over the Gibbs ensemble allows a description, up to the fluctuation corrections, of the time-averaged observables. Boltzmann's kinetic description of nonequilibrium systems is based essentially on the same ideas [2]. The solution of Boltzmann kinetic equation gives not only the distribution over the statistical ensemble, but also its time evolution. Of fundamental importance therewith is the theorem of entropy growth. It states that even though the motion of an individual particle is reversible, the ensemble behaves in such a way that the disorder in the distribution of these particles cannot abate over the time. In this way, Boltzmann for the first time encountered the problem of the linkage between mechanical reversibility and statistical irreversibility. According to the reciprocal theorem of Poincaré and Zermelo, this linkage is due to the fact that any point in the phase trajectory of the ergodic system of many particles after a lapse of fairly extended time returns arbitrarily close to its original position [3]. Because of this, averaging over a finite volume, which is always required for the transition to macroscopic variables, results in a coarser representation of the stochastic systems. The loss of the fine structure of the phase trajectory, attendant on averaging, therewith leads to irreversibility. From the geometrical standpoint, as discovered recently, this fact is related to the fractal properties of the phase trajectory, which is a self-affine geometrical pattern of a fractional dimension [4]. It was also found that this property is observed in a phase space whose dimension is not less than three [5].

A distinguishing feature of the systems under consideration is that, by virtue of their being conservative, the

A I Olemskoĭ Sumy State University,
ul. Rimskogo-Korsakova 2, 244007 Sumy, Ukraine
Tel. (7-0542) 33 34 65
E-mail: alexander@olem.sumy.ua

Received 23 September 1997
Uspekhi Fizicheskikh Nauk **168** (3) 287 – 321 (1998)
Translated by A S Dobroslavskiĭ; edited by A Radzig

irreversible behaviour is caused by the deterministic chaos [6, 7]. This implies that the exact equations of motion of the original degrees of freedom are perfectly deterministic, but the transition to averaged variables gives rise to stochastic terms. In recent years, in connection with the advent of the science of synergetics, such systems were found to possess a rather curious property: the increasing intensity of the stochastic terms may not only increase the disorder, but may also result in the formation of ordered structures, thus leading to self-organization of the system and a decrease in the entropy [5, 8–11]. The latter is apparently associated with the nonconservative nature of the phase space volume over which averaging is carried out, since the entropy cannot decrease in a completely closed system. Far from equilibrium, the behaviour of open systems displays qualitatively new features, such as self-induced oscillations of the type appeared in the Belousov–Zhabotinskiĭ reaction [12], the formation of hierarchically subordinated structures [13], etc.

The main idea in the construction of the theory of statistical systems consists in the transition from the exact equation of mechanical motion of individual particles to the approximate stochastic equations describing the behaviour of the hydrodynamic degrees of freedom, characterized by the dynamic variables averaged over a finite volume of the phase space. In addition to the interaction between these modes, which is represented by the deterministic terms, one must also take into account the effects of the environment which simulates the nonhydrodynamic degrees of freedom, not included explicitly. The environment may give rise both to the deterministic terms and to fluctuations. It is of basic importance that these fluctuations (external with respect to the selected degrees of freedom) are distinct from the internal fluctuations, which owe their existence to the random deviations of the hydrodynamic variables from their mean values. The relative intensity of internal fluctuations varies as $N^{-1/2}$, where N is the number of particles within the volume of averaging, and so they usually can be neglected when $N \to \infty$. This does not apply in the least to the external fluctuations: since they are caused by the influence of environment, their intensity cannot depend on the number of particles N which constitutes a characteristics of the separated subsystem rather than the environment. Accordingly, unlike the internal fluctuations, the external fluctuations may considerably change the behaviour of the system. The most popular example of this kind is given by the stochastic systems which exhibit noise-induced phase transitions [14]. The increasing intensity of external fluctuations therewith leads to a drastic change in the form of the stationary distribution function of microscopic states.

The first example of a stochastic system in which external fluctuations play a decisive role is the problem of Brownian movement of a particle under the action of collisions with the molecules of the surrounding liquid [15]. The solution of this problem reveals that the mean square of the particle displacement is equal to twice the time multiplied by the diffusion coefficient. Accordingly, the stationary distribution of the possible values of the coordinate is Gaussian. If the diffusion coefficient does not depend on the coordinate, the noise in the system is referred to as additive [14–18]. It turns out, however, that not only the external fluctuations affect the stochastic system, but the latter also has a reciprocal effect on their intensity (in case of Brownian movement this implies that the diffusion coefficient depends on the coordinate of the particle). Then the external fluctuations are said to be of multiplicative nature. The existence of this effect was pointed out in Refs [19–21]. The concept of multiplicative noise was first used by R Kubo [22] in the description of the stochastic shape of the spectral line. Studied in Refs [23, 24] was the effect of the external Gaussian noise on the chaotic behaviour of the system, which was simulated by a logistic sequence. It was found that in the neighbourhood of attractors which determine the limiting behaviour of the stochastic system, such fluctuations will either generate bifurcations, or cause a transition from the chaotic regime to self-induced oscillations.

One of the problems in the description of a stochastic system with multiplicative noise consists in the choice of the calculus. The fact is that the solution of the stochastic equation of motion is not unique but a continual set of such solutions is realized. The probability of each realization is found from the solution of the Fokker–Planck equation [15]. Definition of the force that enters this equation is not unambiguous. In Ito's calculus [25] the force is interpreted as the real force acting on the selected degree of freedom. In Stratonovich's calculus [26] there appears an additional term proportional to the noise intensity and the derivative of the effective diffusion coefficient. In Klimontovich's kinetic formulation [27] this additional term is doubled. It can be demonstrated that the set of all possible calculuses is continual, which means that the magnitude of this additional term is arbitrary. At the same time, the additional term strongly affects the behaviour of the stochastic system, and it is therefore important to understand the physical meaning of this term and select the right calculus.

This paper is devoted to the theoretical analysis of stochastic systems with singular white noise, whose intensity depends on the stochastic variable x and vanishes at $x = 0$. In Section 2 we develop the mathematical tools used for the description of the stochastic system. In Section 2.1 we follow the standard scheme for constructing the stochastic equation of motion and study the nature of multiplicative noise in homogeneous and spatially distributed systems. Analyzed in Section 2.2 is the approach which leads to different calculuses in the solution to the stochastic equation of motion. We find that the calculus choice depends on the selection of the point used for taking the derivative of the time-dependent stochastic variable. In Ito's calculus this point corresponds to the left-hand border of the infinitesimal time interval giving rise to the time differential. In Stratonovich's and Klimontovich's formulations this point corresponds, respectively, to the centre and the right-hand border of this time interval. In general, the calculus choice is determined by the parameter $\lambda \in [0, 1]$, which in the above three cases takes on the values 0, 1/2, and 1. The solution to the stochastic equation of motion involves a fictitious force, whose magnitude is proportional to the parameter λ and the derivative of the multiplicative function. Section 2.3 deals with the application of field theory methods to the study of stochastic systems. We demonstrate that the standard field scheme based on expressing the generating functional in terms of the generalized action is only valid for systems with additive noise. In this connection for systems with multiplicative noise we propose the transfer to a new stochastic variable whose noise becomes additive. Our study is based on the Fokker–Planck equation whose solution describes the probability distribution for the realizations of values of the stochastic variable. This equation is derived in Section 2.4, and its solution methods are discussed in Section 2.5. The simplest way of constructing the Fokker – Planck equation (see Section 2.4.1) consists in considering the equation of motion for the mean value of an arbitrary function of the original stochastic variable. This method is rather formal and is based on the fact that the differential of the drift term in the Ito stochastic equation of motion has the order of the diffusion component differential to the second power. The method described in Section 2.4.2, which is similar to the derivation of the kinetic equation, seems to be more physical. It is based on the averaged equation of continuity for the one-particle distribution function. We show that its correlator with the random force component equals the sum of the fictitious force, which depends on the selected calculus, and a component which somewhat resembles the Onsager diffusion flow but is not reducible to the latter. Unlike these, the derivation of the Fokker – Planck equation in Section 2.4.3 is microscopic in character rather than phenomenological. It can be used for Markovian processes described by the master equation, which allows the force and the multiplicative function to be represented in terms of the moments of the microscopic transition intensity. It is worth noting that the resulting value of the fictitious force, which depends on the choice of calculus, is twice the value obtained in Section 2.2 from the solution of the stochastic equation. The final Section 2.5 deals with those cases when the Fokker – Planck equation admits analytical solutions. This situation is shown to occur either in the stationary case, or in the self-modelling regime when the time dependence is only contained in the characteristic value of the stochastic variable, selected as the scale. In both cases the distribution function is quasi-Gibbsian: the role of the temperature is played by the noise intensity, and the effective Hamiltonian is reduced to the bare one only for additive noise.

Section 3 is devoted to the study of the effects of singular multiplicative noise on the behaviour of the stochastic system. Since our treatment is based on the stationary solution of the Fokker – Planck equation, in Section 3.1 we analyze the issue of the existence of the force ensuing from the arbitrariness in the choice of calculus. We show that this force can be cancelled out if the bare probability density is multiplied by an exponential whose index also depends on the selected calculus. Such probability gauge removes the problem of choice of calculus. At the same time, the exponent determines the behaviour of the stochastic system over the entire range of its evolution. This treatment is carried out both for the forward (Section 3.1.1) and the inverse (Section 3.1.2) Kolmogorov equations. The difference in the gauge schemes for these equations consists in the different signs ahead of the exponent. The central place is occupied by Section 3.2, which describes the behaviour of the stochastic system depending on the nature of the multiplicative noise. In Section 3.2.1 we study the noise-induced transition which qualitatively changes the form of the density of probability but does not give rise to any singularities. We demonstrate that this transition is continuous only in the limit of additive noise, and when the multiplicative function is linear. Section 3.2.2 deals with the analysis of the character of the distribution function divergence, which occurs when the growth of the multiplicative function is fast and the density of probability is nonintegrable. We show that this singularity reflects the presence of the deterministic condensate, in which the stochastic variable reduces to a time-independent constant. As it turns out, this regime may involve a finite proportion of the degrees of freedom which form the deterministic condensate. The density of condensate depends on the nature of the multiplicative function and noise intensity. The discussion in Section 3.3 reveals that the arbitrariness in the choice of calculus and the pattern of phase transitions presented in Section 3.2 are associated with the fractal nature of the domain of definition of the stochastic variable in the phase space. The fractal dimension depends on the character of the multiplicative function. In consideration of similarity we demonstrate that the probability of transitions between the microscopic states is nonanalytical. This gives rise to a singular force similar to the force associated with the calculus choice. This time, however, the force is not arbitrary, and its magnitude is proportional to the amplitude of the noise rather than its intensity.

A stochastic system is usually analyzed under the assumption that there is no interaction in the many-particle ensemble simulated by the system. This is the reason why until recently no phase transitions in the common sense have been discovered [28]. Section 4 deals with the effects of interaction between the particles on the behaviour of the stochastic system. In Section 4.1 we obtain the stochastic equation of motion which contains the force of interaction between particles in the mean field approximation, which is used for finding the stationary distribution of probability. It turns out that the interparticle interaction leads to renormalization of the parameters in the Landau expansion and gives rise to a contribution to the effective potential which has the least power in the stochastic variable. Section 4.2 deals with the symmetry breaking in the stochastic system. The distribution function therewith becomes asymmetrical with respect to the inversion of sign of the stochastic variable. The condition of self-consistency for finding the long-range order parameter is defined. We show that symmetry breaking occurs when the dimension of the phase space is between 1 and 2. The long-range order parameter is plotted as a function of noise intensity, and the relevant phase diagram is constructed. The resulting functions are nonmonotone, and we prove that this feature is associated with the fractal nature of the domain of definition of the stochastic system in the phase space. Section 4.3 is devoted to the loss of ergodicity by the stochastic system with interaction between particles. The effect of the latter on the pattern of the loss of ergodicity is found to be not as strong as that on the symmetry breaking. A similar situation is encountered in case of a noise-induced transition. As compared with systems without interaction between particles, the difference consists in the renormalization of the critical temperature which enters the Landau expansion. In the final Section 4.4 we analyze the linkage between the fractal nature of the phase space and the behaviour of the stochastic system, taking into account not only the force of interaction between particles, but also the singular force introduced in Section 3.3. Its effect is found to be equivalent to the effective increase of noise intensity. For example, an increase in the singular force leads to a decrease in the density of deterministic condensate, the long-range order parameter, and the abscissa of the maximum of the distribution function.

Section 5 deals with the effects of noise on the behaviour of a synergetic system represented by the standard Lorenz scheme. Such a scheme has been analyzed in detail in the deterministic regime, when the order parameter, the conjugate field and the controlling parameter do not contain fluctuating components [5, 10]. By contrast, our main task in Section 5 is to find out how the behaviour of synergetic system will change if all three degrees of freedom exhibit additive noise. In Section 5.1 we show that if the changes in the controlling parameter and the conjugate field are subordinated to the magnitude of the order parameter, the additive noise of the former two degrees of freedom will transform to multiplicative one. As this takes place, the multiplicative functions of the conjugate field and the controlling parameter do not coincide. At the end of Section 5.1 we find the stationary solution of the Fokker – Planck equation and the location of the maximum of the probability density as a function of noise intensity. In Section 5.2 we demonstrate that additive noise in the order parameter does not produce any qualitative changes in the behaviour of a synergetic system. The time dependence of the order parameter is found in the absence of noise. The central place belongs to Section 5.3, devoted to the effects of multiplicative noise on the pattern of self-organization. In Section 5.3.1 we demonstrate that the multiplicative noise of conjugate field only leads to the displacement of the point of synergetic transformation. Much more important is the effect of the stochastic controlling parameter studied in Section 5.3.2. It develops that the increasing noise intensity suppresses the disordered state. A phase diagram is constructed which shows the domains of existence of ordered and disordered phases. We demonstrate that, similar to the systems considered in Section 3.2.2, the synergetic system at low noise intensities may produce a deterministic condensate. In the final Section 5.3.3 we generalize the results obtained in Sections 5.3.1 and 5.3.2 to the case of combined inclusion of noise arisen from the conjugate field and the controlling parameter. Then the phase diagram displays regions of metastable ordered and disordered phases.

In Section 6 we summarize the results and show how this formalism can be used for describing the effects of memory and nonergodicity in the presence of multiplicative noise.

2. Methods of description of stochastic systems

2.1 Stochastic equation of motion

Let us consider the simplest example of a stochastic system with a single hydrodynamic degree of freedom $x = x(t)$, which is a random function of time t. As indicated in the Introduction, the quantity x for a many-body system is the amplitude of a hydrodynamic mode, such as the concentration of (quasi-) particles, the flow of particles, etc. [29]. If the random variable $x(t)$ has a nonzero average $\langle x(t)\rangle \equiv \eta$, then the system under consideration exhibits a long-range order defined by the parameter $\eta \neq 0$. For visualization of a stochastic system, it is convenient to consider Brownian movement of a macroscopic particle experiencing the action of external force f and random collisions with the molecules of the surrounding liquid. As this takes place, the stochastic variable $x(t)$ is reduced to the coordinate of the particle, and the order parameter η defines its mean location.

In order to develop the stochastic equation of motion which defines the function $x(t)$, consider a deterministic system characterized by the nonconservative quantity x and the force f. The corresponding equation of motion is [30]

$$\rho\ddot{x} + \gamma^{-1}\dot{x} = f, \tag{2.1}$$

where the dot overhead denotes differentiation with respect to time. For a mechanical system, the inertial term $\rho\ddot{x}$ is determined by the effective density ρ, and the dissipative term $\gamma^{-1}\dot{x}$ by the kinetic coefficient γ. If the system behaves in a self-consistent way, the force $f = f(x)$ depends on the coordinate x. In the linear approximation we have $f = -x/\chi$, where χ is the generalized susceptibility, and the solution of Eqn (2.1) gives $\exp(\pm i\omega_0 t - t/\tau)$, where $\omega_0 = (\chi\rho)^{1/2}$ is the fundamental frequency of oscillations, and $\tau = \chi/\gamma$ is the relaxation time. In this way, the inertial term provides for the existence of oscillations with frequency ω_0 (reactive regime), and the dissipative term ensures damping characterized by the relaxation time τ. In the stochastic case, the reactive regime is only realized in systems which exhibit behaviour like that in the Belousov – Zhabotinskiĭ reaction [31]. As a rule, however, $\omega_0\tau \equiv \chi^{3/2}\gamma^{-1}\rho^{1/2} \ll 1$, and the inertial term in Eqn (2.1) can be neglected. Then, given that the force f in the potential systems is determined by the synergetic potential $V(x)$ according to

$$f = -\frac{\partial V}{\partial x}, \tag{2.2}$$

we come to the regression equation

$$\dot{x} = -\gamma\,\frac{\partial V}{\partial x}, \tag{2.3}$$

which describes the dissipative regime of evolution of a nonequilibrium system to the stationary state. This equation was initially proposed by L Landau and I Khalatnikov for thermodynamic systems, for which the synergetic potential reduces to the Landau free energy $F(\eta)$. The latter differs from the conventional thermodynamic potential in that it depends, apart from the parameters of state like the temperature and the volume, on the order parameter η. This means that a nonequilibrium quasi-static state is fixed, corresponding to the given value of η. The state of equilibrium ($\dot{x} = 0$) corresponds to $\partial F/\partial\eta = 0$, which determines the stationary value of the order parameter [28].

The Landau – Khalatnikov equation (2.3), which describes evolution of the thermodynamic system towards equilibrium, is deterministic. Accordingly, the variable x in Eqn (2.3) should be read as the order parameter η, and $V(x)$ must be replaced with $F(\eta)$. Obviously, for describing the fluctuations of the system near the steady state one must include the stochastic component $\zeta = \zeta(t)$ of the velocity $\dot{x}$, which brings us to the Langevin equation

$$\dot{x} = \gamma f + \zeta\,. \tag{2.4}$$

Here the deterministic force f is given by Eqn (2.2), while the stochastic velocity $\zeta(t)$ is expressed in terms of its moments. The first moment is zero by definition. In order to find the second moment we go over to the Fourier transforms $x(\omega)$, $\zeta(\omega)$ with respect to time (ω is the frequency) and use the fluctuation-dissipation theorem [28]

$$\left\langle\left|\delta x(\omega)\right|^2\right\rangle = \frac{2T}{\omega}\,\mathrm{Im}\,\chi(\omega)\,, \tag{2.5}$$

where $\delta x = x - \langle x\rangle$, and $\chi(\omega)$ is the generalized susceptibility, T is the temperature expressed in units of energy. In the limit $\omega \to 0$ we have $\omega^{-1}\,\mathrm{Im}\,\chi(\omega) \to \gamma\tau_0^2$, where τ_0 is the bare time of relaxation [32]. Then, taking into account that $\zeta \to \delta x/\tau_0$, from Eqn (2.5) we get

$$\lim_{\omega\to 0}\left\langle\left|\zeta(\omega)\right|^2\right\rangle = 2\gamma T\,. \tag{2.6}$$

In the case of white noise, all spectral components $\zeta(\omega) = \text{const}$ are represented in the same way, and the limit symbol in Eqn (2.6) can be dropped. Going back to the time-domain representation, we get

$$\langle \zeta(t)\zeta(t') \rangle = 2\gamma T \delta(t - t') . \tag{2.7}$$

We see that in the white noise approximation the correlation of the stochastic component ζ/γ of the force is only observed at the coinciding times $t = t'$. Obviously, such a situation is only possible when the macroscopic relaxation time $\tau = \chi/\gamma$ is much greater than the microscopic time τ_0. If these times are commensurate, the δ-function in Eqn (2.7) spreads out into a bell-shaped characteristic of width $\sim \tau_0$. The Fourier transform $\zeta(\omega)$ of the stochastic force therewith becomes frequency-dependent, and the white noise becomes coloured. Its intensity is determined by the variable T, which for thermodynamic systems is the temperature. Observe that the coefficient 2γ in front of T does not depend on x. In this situation the noise is referred to as additive.

Our treatment is mainly concerned with stochastic systems with multiplicative noise. Following Ref. [14], we shall show that the numerical factor 2 in Eqns (2.6), (2.7) is replaced by $g^2(x)$, which is determined by the form of the multiplicative function $g(x)$. We note that the force $f = f_\alpha(x)$ in Eqn (2.4) depends not only on the variable x defining the stochastic system, but also on the controlling parameter α which characterizes the environment of the system under consideration (for a Brownian particle this parameter is represented by the rate of collisions with the surrounding molecules). In the linear in $\alpha \ll 1$ approximation, we may write

$$f_\alpha(x) \approx f_0(x) + \alpha g(x) , \qquad g(x) \equiv \left. \frac{\partial f_\alpha(x)}{\partial \alpha} \right|_{\alpha=0} . \tag{2.8}$$

In Landau's theory such an approximation evidently corresponds to the expansion of the coefficient in the quadratic term of the free energy in terms of the temperature [28]. As indicated above, the inclusion of the stochastic addition $\zeta(t)$ to the component $f_0(x)$, which is independent of the environment influence, accomplishes the transition from the deterministic Landau–Khalatnikov equation (2.3) to the stochastic Langevin equation (2.4). Obviously, by doing this we only included the internal fluctuations of the system. To allow for the external fluctuations in Eqn (2.8), the stochastic nature must be assigned not only to the internal force $f_0(x)$, but also to the controlling parameter α, which is assumed to depend on the time in the following way

$$\alpha(t) = \alpha + \frac{\sigma}{\gamma}\, \xi(t) , \tag{2.9}$$

where α describes the mean influence of the environment, and σ describes its dispersion. The latter is defined in such a way as to make the stochastic component $\xi(t)$ delta-correlated:

$$\langle \xi(t) \rangle = 0 , \qquad \langle \xi(t)\xi(t') \rangle = \delta(t - t') . \tag{2.10}$$

Substituting Eqn (2.9) into Eqn (2.8), and the result deduced into Eqn (2.4), we find the general form of the stochastic equation of motion for a system allowed to external ζ_e and internal ζ noise:

$$\dot{x} = \gamma f_\alpha(x) + \zeta_e(x, t) + \zeta(t) ; \tag{2.11}$$

$$f_\alpha(x) \equiv f_0(x) + \alpha g(x) , \tag{2.12}$$

$$\zeta_e(x, t) \equiv \sigma g(x) \xi(t) . \tag{2.13}$$

In accordance with condition (2.7), the internal noise is always additive, whereas for the external noise with due account for Eqn (2.10) we have

$$\langle \zeta_e(x, t)\zeta_e(x, t') \rangle = \gamma T g^2(x) \delta(t - t') , \tag{2.14}$$

where the intensity is defined as $T \equiv \sigma^2$ †. Comparing equations (2.7) and (2.14), we see that as long as the external noise is additive, the multiplicative function is constant, viz.

$$g^2(x) = 2 . \tag{2.15}$$

In accordance with definition (2.8), this implies that the response of the system to the influence of the environment does not depend on the state of the system as determined by the stochastic variable x. In other words, there is no feedback between the state of the stochastic system and that of the environment.

Such is the scheme of constructing the stochastic equation for the nonconservative variable x. This scheme is based on the analogy with the mechanical problem of finding the coordinate x of point mass occurring in a medium under the action of the force $f_\alpha + \gamma^{-1}(\zeta_e + \zeta)$. Observe that the stochastic variable x does not depend on the coordinate $\mathbf{r}$, which implies that the system is spatially homogeneous. The situation is entirely different if the quantity x is conserved (like, for example, the concentration of particles). Then we must start from the equation of continuity $\dot{x} + \nabla \mathbf{j} = 0$, where $\nabla = \partial/\partial \mathbf{r}$, $\mathbf{j}$ is the generalized flux defined by the Onsager relation $\mathbf{j} = -M \nabla \mu$, where $M = D/T$ is the mobility of particles expressed in terms of the diffusion coefficient D and the temperature T. In the Ginzburg–Landau theory, the chemical potential $\mu = \partial V/\partial x - \beta \nabla^2 x - h$ is expressed via the synergetic potential $V = V(x)$, the external field $h = h(\mathbf{r}, t)$, and the parameter of inhomogeneity β. Assuming that the field h is purely stochastic, and including the corresponding contributions of $\tilde{\mathbf{j}}$, ζ to the flux $\mathbf{j}$ and velocity $\dot{x}$, we arrive at the general form of the stochastic equation for a spatially distributed system

$$\dot{x} = \frac{D}{T} \nabla^2 \left(\frac{\partial V}{\partial x} - \beta \nabla^2 x - h \right) - \nabla \tilde{\mathbf{j}} + \zeta . \tag{2.16}$$

By reasoning similar to that used for deriving Eqn (2.14), we get the normalization condition for the stochastic component of the velocity

$$\langle \zeta(\mathbf{r}, t)\zeta(\mathbf{r}, t') \rangle = \gamma V T g^2(x) \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') . \tag{2.17}$$

Here V is the volume of the system; the kinetic coefficient $\gamma = D/\beta$ is determined by the temperature T and the gradient parameter β, and the additional δ-function of $\mathbf{r} - \mathbf{r}'$ appears because of inhomogeneity of the system. In the absence of

† Observe that the conventional definition is $T = \sigma^2/2$ [14], and an additional multiplier 2 occurs in Eqn (2.14). We find it more convenient, however, to include this numerical coefficient into the multiplicative factor $g^2(x)$.

feedback between the quantities x and $\mathbf{j}$, the stochastic component of the flux $\tilde{\mathbf{j}}$ satisfies a condition similar to Eqn (2.7), where the noise intensity is represented not by the temperature $T\gamma$, but by the diffusion coefficient D (factor γ is required from considerations of dimensionality). If, however, the change in the stochastic variable $x(\mathbf{r}, t)$ affects the flux $\mathbf{j}(\mathbf{r}, t)$, then the fluctuations of the latter become multiplicative, and we get an equation similar to Eqn (2.14):

$$\left\langle \tilde{\mathbf{j}}(\mathbf{r}, t)\tilde{\mathbf{j}}(\mathbf{r}, t')\right\rangle = DVg_j^2(x)\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \qquad (2.18)$$

where the multiplicative factor $g_j^2(x)$ goes over into a constant multiplier 2 upon transition to additive noise. Fluctuations $h(\mathbf{r}, t)$ of the field conjugate to the stochastic variable $x(\mathbf{r}, t)$ are always multiplicative, and so

$$\left\langle h(\mathbf{r}, t)h(\mathbf{r}, t')\right\rangle = SVg_h^2(x)\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \qquad (2.19)$$

where we have introduced the intensity S and the multiplicative function $g_h(x)$, which determine the nature of the noise of the conjugate field. A salient feature of spatially distributed systems is that upon transition from the fluctuations of the flux $\tilde{\mathbf{j}}$ to the corresponding contribution $\zeta_j = -\nabla\tilde{\mathbf{j}}$ to the noise of velocity $\dot{x}$, the operator $-\nabla^2$ appears on the right-hand side of Eqn (2.18) [31]. Accordingly, for the field component $\zeta_h = -(D/T)\nabla^2 h$ in Eqn (2.19) we get a factor of $(D/T)^2\nabla^4$. This implies that with the increasing scale l on which the fluctuations ζ, ζ_j, ζ_h are observed, the components vary as $\zeta \propto l^0$, $\zeta_j \propto l^{-1}$, $\zeta_h \propto l^{-2}$.

It is often convenient to consider stochastic equations (2.11), (2.16) in a dimensionless form. For nonconservative variable x this is accomplished by expressing the time t in terms of $\tau = \chi/\gamma$, the force $f_\alpha(x)$ in terms of χ^{-1}, and the stochastic components ζ_e, ζ in terms of τ^{-1} (as regards the stochastic variable x, it is dimensionless from the start on the scale of unity). Then the factor γ in Eqn (2.11) disappears, and factor τ occurs on the right-hand sides of equations (2.7), (2.14), which leads to a dimensionless noise intensity $\Theta = \chi T$.

In the case of spatially distributed systems, it is convenient to measure the quantities $t, \mathbf{r}, V(x), h, j$, and ζ in Eqn (2.16) in units of $\beta/(DT)$, $(\beta/T)^{1/2}$, T, T, $D(T/\beta)^{1/2}$, and DT/β, respectively. Then, upon transition to the spatial Fourier transforms, the stochastic equation (2.16) assumes the form

$$\dot{x} = -\mathbf{k}^2\left(\frac{\partial V}{\partial x} + \mathbf{k}^2 x\right) + \mathbf{k}^2 h - \mathrm{i}\mathbf{k}\tilde{\mathbf{j}} + \zeta. \qquad (2.20)$$

The delta-function of $\mathbf{r}-\mathbf{r}'$ disappears in conditions (2.17)–(2.19), and correlated are only the terms corresponding to the wave vectors $\mathbf{k}$ and $-\mathbf{k}'$. With the above units of measurement, the noise intensities of the velocity ζ in Eqn (2.17) and flux $\tilde{\mathbf{j}}$ in Eqn (2.18) are equal to unity, and for the conjugated field h in Eqn (2.19) we have $\Theta_h = SD/(\beta T)$.

On the strength of the arguments developed above, the dimensionless stochastic equation of motion assumes the canonical form

$$\dot{x} = f + \zeta, \qquad (2.21)$$

where the deterministic force f is given by Eqn (2.2) for a homogeneous system, and

$$f = -\mathbf{k}^2\left(\frac{\partial V}{\partial x} + \mathbf{k}^2 x\right) \qquad (2.22)$$

for a spatially distributed system. The stochastic component ζ constitutes the total contributions of the external fluctuations $\zeta_e = \zeta_h + \zeta_j$ of the conjugate field $\zeta_h = \mathbf{k}^2 h$ and the flux $\zeta_j = -\mathrm{i}\mathbf{k}\tilde{\mathbf{j}}$, and the internal noise ζ of the stochastic variable x. In accordance with conditions (2.14), (2.7), (2.17)–(2.19) in dimensionless form, these components are expressed as follows:

$$\zeta_e = \Theta^{1/2} g(x)\xi, \qquad \zeta = (2\Theta)^{1/2}\xi; \qquad (2.23)$$

$$\zeta_h = \mathbf{k}^2\Theta_h^{1/2} g_h(x)\xi, \qquad \zeta_j = -\mathrm{i}k g_j(x)\xi; \qquad \zeta = 2^{1/2}\xi, \qquad (2.24)$$

where Eqn (2.23) relates to the homogeneous system, and Eqn (2.24) to the spatially distributed system (the dimensionless noise intensity $\Theta = \chi T$ occurs in Eqn (2.23) because the scale of measurement is based on the inverse susceptibility χ^{-1}, whereas in Eqn (2.24) the scale is the noise intensity itself: D for ζ_j, and T for ζ). In the absence of feedback applied to the system and the environment, the multiplicative functions $g(x)$, $g_h(x)$, and $g_j(x)$ reduce to a constant multiplier $2^{1/2}$, and the stochastic variable $\xi = \xi_{\mathbf{k}}(t)$ is delta-correlated:

$$\left\langle \xi_{\mathbf{k}}(t)\right\rangle = 0, \qquad \left\langle \xi_{\mathbf{k}}^*(t)\xi_{\mathbf{k}'}(t')\right\rangle = \delta_{\mathbf{k}\mathbf{k}'}\delta(t-t'). \qquad (2.25)$$

For homogeneous systems, the wave vector $\mathbf{k}$ must be dropped [see Eqn (2.10)].

2.2 Solution of the stochastic equation of motion

The study of Brownian movement revealed for the first time that the law of motion $x(t)$ is nonanalytical. In particular, the function $x(t)$, although being everywhere continuous, is nondifferentiable [4, 33]. Because of this, the stochastic equation of motion (2.21) is highly conventional, since its left-hand side contains the time derivative $\dot{x}$ of this function.

Let us show that this singularity makes the solution of the stochastic equation fundamentally ambiguous. This ambiguity consists in the possibility of selecting the continual set of calculuses like those proposed by Ito [25], Stratonovich [26], Klimontovich [27], etc. Each of these calculuses corresponds to a particular solution $x(t)$.

We start from the differential representation of the stochastic equation of motion [14–17]:

$$\mathrm{d}x = f(x)\,\mathrm{d}t + \sigma g(x)\,\mathrm{d}w(t), \qquad (2.26)$$

where the first term describes the drift in the field $f(x)$, and the second term describes the diffusion with the coefficient $D = (\sigma^2/2)g^2(x)$. The stochastic variable $w = w(t)$ is introduced in such a way that its differential replaces the ill-defined quantity $\xi(t)\,\mathrm{d}t$ in Eqns (2.21)–(2.24). The characteristic property of the stochastic equation (2.26) lies in the fact that the time differential $\mathrm{d}t$ is a 2nd-order infinitesimal with respect to the differential $\mathrm{d}w$ of the stochastic variable. Indeed, keeping only the diffusion term in Eqn (2.26), we get $(\mathrm{d}x)^2 = 2D(\mathrm{d}w)^2$, whereas $(\mathrm{d}x)^2 = 2D\,\mathrm{d}t$ by definition. As a result, we get

$$(\mathrm{d}w)^2 = \mathrm{d}t. \qquad (2.27)$$

The formal solution of Eqn (2.26) takes the form

$$x(t) = \int_0^t f\big(x(t')\big)\,\mathrm{d}t' + \sigma\int_{w(0)}^{w(t)} g\big(x(t')\big)\,\mathrm{d}w(t'), \qquad (2.28)$$

where we have assumed that $x(0) = 0$. Since the function $x(t')$ is continuous, evaluation of the first integral on the right-hand side of Eqn (2.28) does not meet with any singularities. However, in the calculation of the diffusion component

$$I(t) \equiv \int_{w(0)}^{w(t)} g\big(x(t')\big)\,\mathrm{d}w(t') \tag{2.29}$$

the integration is not over the time, but with respect to the stochastic variable $w(t')$, which displays singularities. To clarify the situation, we express Eqns (2.26), (2.29) in terms of finite differences, splitting the time interval $[0, t]$ into $N \to \infty$ equal increments $\Delta t_i \equiv t_{i+1} - t_i \to 0$, where $i = 0, 1, \ldots, N$, and $N\Delta t_i = t$:

$$\Delta x(t_i) = f\big(x(t_i)\big)\Delta t_i + \sigma g\big(x(\widetilde{t_i})\big)\Delta w_i\,, \tag{2.30}$$

$$I(t) \equiv \lim_{\Delta t_i \to 0} \sum_{i=0}^{N-1} g\big(x(\widetilde{t_i})\big)\Delta w_i\,, \tag{2.31}$$

where $\Delta w_i \equiv w(t_{i+1}) - w(t_i)$. The salient feature of these expressions is that, because of the singular nature of $w(t_i)$, the result crucially depends on the selection of the point

$$\widetilde{t_i} = t_i + \lambda \Delta t_i\,, \qquad \lambda \in [0, 1]\,, \tag{2.32}$$

used for defining the value of the multiplicative function $g\big(x(\widetilde{t_i})\big)$. For $\lambda = 0$, this point takes up the position at the left-hand border of the interval Δt_i, which corresponds to Ito's calculus. With $\lambda = 1/2$, the time $\widetilde{t_i}$ is fixed in the middle of Δt_i, and we come to Stratonovich's calculus. Finally, with $\lambda = 1$, the time $\widetilde{t_i} = t_{i+1}$ occupies the position at the right-hand border of Δt_i, and this is what was proposed by Klimontovich. There is, however, no physical criterion for fixing any specific value from $\lambda \in [0, 1]$. One may conclude therefore that there exists a continual set of calculuses depending on the selection of the point (2.32).

To evaluate the integral $I(t)$, we express in Eqn (2.31) the magnitude $g\big(x(\widetilde{t_i})\big)$ of the multiplicative function within the interval Δt_i in terms of the value of $g\big(x(t_i)\big)$ at the border of the interval. With this purpose we construct the expansion

$$x(\widetilde{t_i}) \approx x(t_i) + \dot{x}(t_i)\lambda\Delta t_i = x(t_i) + \lambda \Delta x(t_i)\,, \tag{2.33}$$

which follows from Eqn (2.32). Similarly, we get

$$g\big(x(\widetilde{t_i})\big) \approx g\big(x(t_i)\big) + \lambda \nabla g\big(x(t_i)\big)\Delta x(t_i)\,, \tag{2.34}$$

where $\nabla g\big(x(t_i)\big) \equiv \mathrm{d}g/\mathrm{d}x|_{x=x(t_i)}$. Substituting here $\Delta x(t_i)$ from Eqn (2.30), we come to the desired expression

$$\begin{aligned} g\big(x(\widetilde{t_i})\big) &\approx g\big(x(t_i)\big) + \lambda\sigma g\big(x(t_i)\big)\nabla g\big(x(t_i)\big)\Delta w_i \\ &\quad + \lambda f\big(x(t_i)\big)\nabla g\big(x(t_i)\big)\Delta t_i\,. \end{aligned} \tag{2.35}$$

By virtue of Eqn (2.27), the last term is of second order in Δw_i, and can be dropped out. Substituting the retained terms from Eqn (2.35) into Eqn (2.31), and going back to the continual representation (2.29), (2.28), we get the expression

$$x(t) = \int_0^t \Big[f\big(x(t')\big) + h\big(x(t')\big)\Big]\,\mathrm{d}t' + \sigma \int_{w(0)}^{w(t)} g\big(x(t')\big)\,\mathrm{d}w(t')\,, \tag{2.36}$$

which, unlike Eqn (2.28), contains the force

$$h(x) = \frac{\lambda}{2}\,\Theta \nabla g^2(x)\,, \qquad \Theta \equiv \sigma^2\,, \tag{2.37}$$

which depends on the choice of the calculus. Observe that for systems with additive noise, where the function $g(x)$ reduces to a constant, this force is identically equal to zero, and such systems are insensitive to the calculus choice.

Differentiating and then averaging equality (2.36), from condition $\langle \dot{x} \rangle = f$ and definition $\dot{w} = \xi$ we get

$$\big\langle g\big(x(t)\big)\xi(t)\big\rangle = -\lambda\sigma\big\langle g\big(x(t)\big)\nabla g\big(x(t)\big)\big\rangle\,. \tag{2.38}$$

Thus, we see that in the general case of $\lambda \neq 0$ the stochastic velocity $\xi(t)$ correlates with the multiplicative function $g(x)$. Such a correlation is not observed only in Ito's calculus ($\lambda = 0$), which in this respect stands alone. In particular, it is only at $\lambda = 0$ that our condition $\langle \dot{x} \rangle = f$ coincides with the result of averaging Eqn (2.21).

2.3 Field representation of a stochastic system

The use of field-theoretical methods for the description of stochastic systems is one of the most promising directions of research. As we know, field approach is useful for describing systems possessing a continual set of degrees of freedom [34, 35]. In stochastic systems, the degrees of freedom are the space-time fluctuations represented by the function $x(\mathbf{r}, t)$.

The construction of a field-theoretical scheme is based on the stochastic generating functional [35, 36]

$$Z\{u(t)\} = \int Z\{x(t)\}\exp\left\{\int u(t)x(t)\,\mathrm{d}t\right\}\mathrm{D}x\,, \tag{2.39}$$

$$Z\{x(t)\} = \left\langle \prod_t \delta\{\dot{x} - f - \zeta(t)\}\det\left|\frac{\delta\zeta}{\delta x}\right|\right\rangle_\zeta\,, \tag{2.40}$$

whose variation with respect to the test field $u(t)$ yields the correlators of the stochastic variable $x(t)$. Equality (2.39) expresses the functional Laplace transform of $Z\{x\}$, represented by Eqn (2.40), where the presence of a δ-function reflects the stochastic equation of motion (2.21), the determinant ensures the transition from continual integration over ζ to x, and the angle brackets indicate averaging with respect to the random force ζ. The determinant can be eliminated by introducing a pair of Grassmann-conjugate fields, whose condensate determines the density of the antiphase boundaries [31]. In spatially homogeneous systems there are no such boundaries, and the determinant is expressed in terms of the original functions $x(t)$, $f(x)$, and $\zeta(t)$. According to Ref. [37], in Ito's calculus its value reduces to the noise intensity Θ, while in Stratonovich's calculus we get a more complicated expression

$$\det\left|\frac{\delta\zeta}{\delta x}\right| = \exp\left(-\frac{1}{2}\int\frac{\partial f}{\partial x}\,\mathrm{d}t\right). \tag{2.41}$$

To perform averaging with respect to the field $\zeta(t)$, consider the approximation of additive noise described by the Gaussian distribution

$$P\{\zeta(t)\} = (2\pi\Theta)^{-1/2}\exp\left(-\int\frac{|\zeta(t)|^2}{2\Theta}\,\mathrm{d}t\right), \tag{2.42}$$

which corresponds to the normalization condition (2.7) for the complex quantity ζ. Using the integral representation

$$\delta\{v(t)\} = \int_{-i\infty}^{i\infty} \exp\left(-\int v(t)\varphi(t)\,dt\right) \frac{D\varphi}{2\pi i} \qquad (2.43)$$

and averaging over the distribution (2.42), we bring the functional (2.40) to the standard form

$$Z\{x(t)\} = \int \exp\left[-S\{x(t), \varphi(t)\}\right] \frac{D\varphi}{2\pi i} \qquad (2.44)$$

expressed in dimensionless quantities (in particular, $\Theta = 1$). The action $S = \int \mathcal{L}\,dt$ is determined by the Lagrangian $\mathcal{L}$, which in Ito's calculus takes the form

$$\mathcal{L} = \varphi(\dot{x} - f) - \frac{\varphi^2}{2}\,. \qquad (2.45)$$

By virtue of Eqn (2.41), the transition to Stratonovich's calculus gives rise to the additional term $(1/2)\partial f/\partial x$, which can be included into the index of the exponential in Eqn (2.39).

In order to bring the Lagrangian (2.45) to the canonical form, we must introduce a new field $\phi(t)$ defined by the equation

$$\dot{x} = \phi + \varphi\,. \qquad (2.46)$$

As a result, we get

$$\mathcal{L} = \frac{\dot{x}^2}{2} + \left(\phi f - \frac{\phi^2}{2}\right) - f\dot{x}\,. \qquad (2.47)$$

With the aid of Eqn (2.2) we find that the last term here is the total time derivative of $V(x(t))$ which leads to the conventional expression for the partition function [28]. Other terms in Eqn (2.47) are the kinetic energy $\dot{x}^2/2$ and the potential energy $\phi f - \phi^2/2$. The latter assumes the canonical form $\phi^2/2$ if $\phi = f$, which implies that the field ϕ is reduced to the force f conjugated with the order parameter x. Comparing the equation of motion (2.21) with the Euler equation (2.46), corresponding to the least action S in Eqn (2.44), we see that φ defines the most probable value of the amplitude of fluctuations of the conjugate field (its mean value is $\langle\zeta\rangle = 0$). Hence it follows that the distribution in the integrand in Eqn (2.44) is bimodal, unlike the Gaussian distribution introduced by Eqn (2.42).

Obviously, such a situation only occurs in the case of additive noise $\zeta(t)$. When passing to multiplicative noise, we must replace 2Θ with $\Theta g^2(x)$ in distribution (2.42), because of which the averaging with respect to the noise ζ in Eqn (2.40) becomes non-Gaussian. To restore the Gaussian distribution, we go over to the new variable

$$y(x) = \int \frac{dx}{g(x)}\,. \qquad (2.48)$$

According to Eqn (2.27), the terms in the differential dx in Eqn (2.26) are variable in order of infinitesimal, and so for the differential of the new stochastic variable $y(x)$ we get

$$dy = \frac{dy}{dx}\,dx + \frac{1}{2}\frac{d^2y}{dx^2}(dx)^2\,, \qquad (2.49)$$

where in the diffusion approximation we must set

$$(dx)^2 = \sigma^2 g^2(x)(dw)^2 = \Theta g^2(x)\,dt\,. \qquad (2.50)$$

Finding from Eqn (2.48) the derivatives of $y(x)$, and substituting the differentials (2.26), (2.50), we find from Eqn (2.49) the stochastic equation of motion in the new variable:

$$dy = p(y)\,dt + \sigma\,dw\,, \qquad (2.51)$$

$$p(y) \equiv \frac{f(x(y))}{g(x(y))} + \Theta\left(\lambda - \frac{1}{2}\right)\frac{dg}{dx}\bigg|_{x=x(y)}\,. \qquad (2.52)$$

According to Eqn (2.51), its noise is additive, and we may use the field-theoretical scheme described above. The effective force is given in this case by Eqn (2.52), which includes the component (2.37) associated with the arbitrariness in the choice of calculus. Observe that the addition related to the multiplicative nature of the noise [the last term in Eqn (2.52)], disappears in Stratonovich's calculus ($\lambda = 1/2$).

2.4 Fokker – Planck equation

In addition to the features discussed above, the stochastic nature of the system is manifested in that, although the initial conditions are fixed and the calculus is chosen, the equation of motion admits a continual set of solutions $\{x(t)\}$ distributed in a random fashion. This circumstance is a trivial consequence of the presence of a random force ζ in the equation of motion (2.21). Because of this, it is important to arrive at the function

$$P(x,t) = \int \delta\{x - x(t)\}P\{x(t)\}\,Dx(t)\,, \qquad (2.53)$$

which, given the functional $P\{x(t)\}$ of the distribution of solutions of the stochastic equation, allows us to find the density of the probability of realization of a value of x at a given time t. The distribution function $P(x,t)$ is linked to the initial distribution $P(x_0, 0)$ by the following relationship [14 – 18]:

$$P(x,t) = \int P(x,t|x_0,0)P(x_0,0)\,dx_0\,, \qquad (2.54)$$

which includes the transition probability $P(x,t|x_0,0)$ under the integral. In this way, the problem reduces to finding the conditional probability $P(x,t|x_0,0)$, subject to the initial condition $P(x,0|x_0,0) = \delta(x - x_0)$. Formal methods of derivation of the Fokker – Planck equation for the transition probability $P(x,t|x_0,0)$ can be found in the book [15]. Considering that relation (2.54) does not involve the dependence on x, t, these methods can also be used for finding the probability density $P(x,t)$. Since, however, we are mostly interested in the physical content of the problem, we are going to study directly the dependence $P(x,t)$. Of course, we shall arrive at the same results as those obtained by other methods [14 – 18, 38].

2.4.1 Derivation of the Fokker – Planck equation from the equation of motion. Consider an arbitrary analytical function $y(x)$, whose differential can be represented in the form of expression (2.49). As already indicated, it is necessary to take into account the square of the differential (2.26) because its drift and diffusion components are variable in order of infinitesimal, as reflected by Eqn (2.27). Accordingly, the diffusion contribution (2.50) of second order in dx is commensurate with the drift component $f\,dt$ in Eqn (2.26), which is of first order in dt.

Substituting the differentials (2.26), (2.50) into Eqn (2.49), and averaging with respect to the distribution (2.54), we get

$$d\langle y(x)\rangle = \left\langle \frac{dy}{dx} f(x) \right\rangle dt + \sigma \left\langle \frac{dy}{dx} g(x)\,dw \right\rangle + \frac{\Theta}{2} \left\langle \frac{d^2 y}{dx^2} g^2(x) \right\rangle dt\,. \tag{2.55}$$

Here on the left-hand side we interchanged the order of averaging and differentiation, and took Eqn (2.27) into account. According to Eqn (2.38), in Ito's approximation ($\lambda = 0$) the quantities $g(x)$ and $dw = \xi\,dt$ do not correlate, which allows us to factor out $\langle dw \rangle = 0$ in the second term in Eqn (2.55) [see also Eqn (2.10)]. Then, going over to the time derivative and using the definition of the mean

$$\langle y \rangle = \int y(x) P(x,t)\,dx\,, \tag{2.56}$$

we arrive at

$$\int y(x)\dot{P}(x,t)\,dx = \int f(x)P(x,t)\,\frac{dy}{dx}\,dx + \frac{\Theta}{2}\int g^2(x)P(x,t)\,\frac{d^2y}{dx^2}\,dx\,. \tag{2.57}$$

Carrying out integration by parts on the right-hand side, we factor out $y(x)$ in the integrands in Eqn (2.57) [as this takes place, the differentiation operator $\nabla \equiv \partial/\partial x$ is switched to the distribution function $P(x,t)$]. Since the function $y(x)$ is chosen arbitrarily, the integrals are equal only when the integrands are. Eliminating $y(x)$, we come to the Fokker–Planck equation

$$\dot{P} = \nabla\left[-fP + \frac{\Theta}{2}\nabla(g^2 P)\right], \qquad \nabla \equiv \frac{\partial}{\partial x}\,. \tag{2.58}$$

It can be represented in the form of the equation of continuity

$$\dot{P} + \nabla J = 0 \tag{2.59}$$

in the space of the stochastic variable x, where the probability flux $J = J_{dr} + J_{dif}$ is composed of the drift J_{dr} and diffusion J_{dif} components

$$J_{dr} = fP\,, \qquad J_{dif} = -\frac{\Theta}{2}\nabla(g^2 P)\,. \tag{2.60}$$

Here, in accordance with the equation of motion (2.21), the force f ensures the drift velocity $\langle \dot{x} \rangle = f$, and the quantity

$$D(x) = \frac{\Theta}{2} g^2(x) \tag{2.61}$$

is the generalized diffusion coefficient defined by Eqn (2.50). In the case of additive noise $[g(x) = 2^{1/2}]$, this coefficient reduces to the intensity Θ, as ought to be expected.

The above scheme is limited to Ito's calculus. In the general case of $\lambda \neq 0$, the real force $f(x)$ in Eqn (2.55) must be supplemented by the fictitious force $h(x)$ as defined by Eqn (2.37). If, however, we take into account the correlation between $g(x)$ and dw, then, in accordance with Eqn (2.38), we have

$$\left\langle \frac{dy}{dx} h(x) \right\rangle + \sigma \left\langle \frac{dy}{dx} g(x)\xi(t) \right\rangle = 0 \tag{2.62}$$

and the Fokker–Planck equation (2.58) remains invariable.

2.4.2 Construction of the Fokker–Planck equation in the framework of the kinetic approach. The method described above, like those presented in Ref. [15], is rather formal. Therefore, following Klimontovich [27], we shall outline here the kinetic approach which goes back to the Boltzmann equation. Its main advantage consists in its perspicuity, for the mechanical problem of diffusion of the probe particle in the momentum space is considered. In addition, this method demonstrates the linkage between two fundamental equations of statistical physics: the Fokker–Planck equation, which governs the stochastic behaviour of the selected degree of freedom, and the Boltzmann equation describing the statistical ensemble of particles.

The equation of motion of the probe particle in a medium comprised of parent particles is similar to Eqn (2.21):

$$\dot{\mathbf{p}} = -\mu\mathbf{p} + (2D)^{1/2}\xi\,. \tag{2.63}$$

Here, $-\mu\mathbf{p}$ is the force of friction in the medium of viscosity $\mu = \mu(\mathbf{p})$, $D = D(\mathbf{p})$ is the effective diffusion coefficient, linked with the multiplicative function $g(\mathbf{p})$ by a relation similar to Eqn (2.61), and $\xi = \xi(t)$ is the delta-correlated noise. To construct the kinetic equation, we introduce the one-particle distribution function

$$P_1(\mathbf{p}, t) = \delta\big(\mathbf{p} - \mathbf{p}_1(t)\big) \tag{2.64}$$

and average the equation of continuity with respect to the sought-for distribution $P \equiv \langle P_1 \rangle$:

$$\dot{P}_1 + \nabla\mathbf{J}_1 = 0\,, \qquad \nabla \equiv \frac{\partial}{\partial \mathbf{p}}\,, \tag{2.65}$$

where the generalized flux has the form $\mathbf{J}_1 = \dot{\mathbf{p}}P_1$, like the first expression in Eqn (2.60). As a result, we get the equation

$$\dot{P} = \nabla\left[\mu\mathbf{p}P - (2D)^{1/2}\langle \xi P_1 \rangle\right], \tag{2.66}$$

which contains the unknown correlator $\langle \xi P_1 \rangle$.

This correlator is most easily found within the framework of the phenomenological approach based on the equation of continuity (2.59), which unlike Eqn (2.65) involves not the one-particle distribution (2.64), but the total distribution $p = \langle P_1 \rangle$. The flux $\mathbf{J} = \mathbf{J}_{dr} + \mathbf{J}_{dif}$ contains both the drift and the diffusion components

$$\mathbf{J}_{dr} = \mathbf{f}_{ef}P\,, \qquad \mathbf{J}_{dif} = -D\nabla P\,. \tag{2.67}$$

Unlike Eqn (2.60), here the diffusion coefficient D is factored out of the operator $\nabla = \partial/\partial\mathbf{p}$. This implies that, similar to the method described at the end of Section 2.3, in the definition of fluxes (2.67) we have accomplished the transition from momentum $\mathbf{p}$ to a new variable $\mathbf{q} = \int (2D)^{-1/2} d\mathbf{p}$, for which the noise is additive. Then, according to Eqn (2.52), the effective force $\mathbf{f}_{ef}$, which determines the drift component of the flux $\mathbf{J}_{dr}$, has the form

$$\mathbf{f}_{ef} = -\mu\mathbf{p} + \left(\lambda - \frac{1}{2}\right)\nabla D\,. \tag{2.68}$$

Here, the first term contains the force of friction, the parameter λ in the second term takes care of the lack of uniqueness when choosing the calculus, and the term $-(1/2)\nabla D$ accounts for the diffusive contribution $(\mathrm{d}\mathbf{p})^2 = 2D\,\mathrm{d}t$ into the change of the new variable $\mathbf{q}(\mathbf{p})$ [see Eqns (2.49), (2.50)]. Substituting Eqn (2.68) into (2.67), from Eqn (2.59) we get the Fokker – Planck equation in the kinetic form [27]:

$$\dot{P} = \nabla\left\{\left[\mu\mathbf{p} + \left(\frac{1}{2} - \lambda\right)\nabla D\right]P + D\nabla P\right\}. \tag{2.69}$$

Comparing this equation with Eqn (2.66), we find the sought-for correlator

$$\langle \xi P_1 \rangle = \lambda \Theta^{1/2} P \nabla g - \frac{1}{2}\,\Theta^{1/2}\nabla(gP)\,, \tag{2.70}$$

where we have introduced the multiplicative function $g(\mathbf{p}) \equiv \left[(2/\Theta)D(\mathbf{p})\right]^{1/2}$. For additive noise, when $D(\mathbf{p}) = \mathrm{const} \equiv D$, the correlator is $\langle \xi P_1 \rangle = -(D/2)^{1/2}\nabla P$, and equation (2.69) assumes a simplest form

$$\dot{P} = \nabla(-\mathbf{f}P + D\nabla P)\,, \qquad \mathbf{f} = -\mu\mathbf{p}\,. \tag{2.71}$$

A similar situation occurs for the multiplicative noise in Stratonovich's calculus ($\lambda = 1/2$), which gives it an advantage over other calculuses [14].

At equilibrium, we have $\dot{P} = 0$ and the distribution function $P(\mathbf{p})$ is Maxwellian [28]:

$$P \propto \exp\left(-\frac{\mathbf{p}^2}{2mT}\right), \tag{2.72}$$

where m is the mass of the particle. Substituting this expression into Eqn (2.69), we find the equation for the effective diffusion coefficient

$$D(\mathbf{p}) = mT\left[\mu(\mathbf{p}) + \left(\frac{1}{2} - \lambda\right)\frac{\mathbf{p}}{p^2}\,\frac{\mathrm{d}D(\mathbf{p})}{\mathrm{d}\mathbf{p}}\right]. \tag{2.73}$$

The Einstein relation

$$D(\mathbf{p}) = mT\mu(\mathbf{p}) \tag{2.74}$$

holds either for additive noise, or for the Stratonovich's calculus.

2.4.3 Derivation of the Fokker – Planck equation from the master equation. The approaches described above are essentially phenomenological, and are incapable of disclosing the microscopic content of such parameters as the generalized diffusion coefficient. This becomes possible when the treatment is based on the so-called master equation, which expresses the rate of change of the distribution function (2.54) in terms of the intensities of transitions between the microscopic states [38].

Let us first describe the scheme of derivation of the master equation. This equation only holds for Markovian processes, for which the probability of transition from one microscopic state to another does not depend on the way in which the system was brought into the initial state (in other words, the system does not feature microscopic memory). Then we may avail ourselves of the Kolmogorov equation [39]

$$P(x + \mathrm{d}x, t + \mathrm{d}t) = W(x, x + \mathrm{d}x)P(x, t)\,\mathrm{d}x\,\mathrm{d}t\,, \tag{2.75}$$

which describes the evolution of the distribution function in terms of the transition probability $W(x, x + \mathrm{d}x)\mathrm{d}x$ from microscopic state x to state $x + \mathrm{d}x$ per unit time. Hence, the master equation follows directly

$$\dot{P}(x, t) = \int \left[W(x + y, x)P(x + y, t) - W(x, x - y)P(x, t)\right] \mathrm{d}y\,, \tag{2.76}$$

where the first term corresponds to the forward transitions $x + y \to x$, and the second to the back transitions $x \to x - y$. It is easy to see that the latter governs the relaxation process whose effective time is given by

$$\tau^{-1}(x) = \int W(x, x - y)\,\mathrm{d}y\,. \tag{2.77}$$

Formally, this means that the integral intensity of transitions is equal to the inverse relaxation time.

In order to solve Eqn (2.76), one has to find the intensity of transitions $W(x, y)$, which is a separate problem [40]. As will be shown below, however, for our purposes we only need to know the first two moments of the complete distribution function $W(x, y)$, not the function itself. A vital role in their definition is played by the principle of detailed balance [15, 40]

$$P(x)\,W(x, y) = P(y)\,W(y, x)\,, \tag{2.78}$$

which implies that the intensities of forward and back transitions between any two microscopic states x, y are the same. Then the integrand in Eqn (2.76) is zero, and $\dot{P} = 0$, which means that the distribution function (2.54) does not depend on the time. The corresponding microscopic state is referred to as stationary†. Observe that the principle of detailed balance (2.78) does not hold for an arbitrary stationary state.

Embarking on the derivation of the Fokker – Planck equation, we must note that the transfer to a latter is only possible for those processes for which $W(x, y)$ is a continuous function. In the opposite case of discrete processes the transfer is not feasible, and the system is described by the master equation (2.76).

For continuous processes, the first term in brackets in Eqn (2.76) can be represented as the expansion

$$\begin{aligned} W(x + y, x)P(x + y, t) &= W(x, x - y)P(x, t) \\ &\quad + \frac{\partial\left[W(x, x - y)P(x, t)\right]}{\partial x}\,y \\ &\quad + \frac{1}{2}\,\frac{\partial^2\left[W(x, x - y)P(x, t)\right]}{\partial x^2}\,y^2 + \ldots \end{aligned} \tag{2.79}$$

As a result, we get a differential equation containing the moments

$$\left\langle y^n(x) \right\rangle = \int W(x, x - y)y^n\,\mathrm{d}y\,. \tag{2.80}$$

According to Pawula's theorem [41], in the case of Markovian processes for which the master equation just holds, all the

† In should be emphasized that a stationary state is not always equivalent to the equilibrium state. For example, a piped flow is stationary when its velocity is constant, and equilibrium when its velocity is zero.

moments are equal to zero at $n > 2$, the odd-numbered moments identically so. Then Eqn (2.76) becomes

$$\dot{P} = \nabla\left[-fP + \frac{\Theta}{2}\,\nabla(g^2 P)\right], \qquad \nabla \equiv \frac{\partial}{\partial x}\,. \tag{2.81}$$

The force $f \equiv -\langle y\rangle$ and the multiplicative function $g \equiv \left(\Theta^{-1}\langle y^2\rangle\right)^{1/2}$ are given here by the expressions

$$f(x) \equiv -\int_{-\infty}^{\infty} yW(x, x-y)\,\mathrm{d}y\,, \tag{2.82}$$

$$\Theta g^2(x) = \int_{-\infty}^{\infty} y^2 W(x, x-y)\,\mathrm{d}y\,, \tag{2.83}$$

where Θ is the noise intensity.

Let us now demonstrate that, as for the solution of the stochastic equation (2.26), a more careful treatment of the master equation (2.76) gives rise to a fictitious force which depends on the choice of calculus. With this purpose we represent the expansion (2.79) in terms of finite differences

$$W(x_i+\Delta x_i, x_i)P(x_i+\Delta x_i, t) = W(\tilde{x}_i, x_i-\Delta x_i)P(\tilde{x}_i, t) + \frac{\Delta\left[W(\tilde{x}_i, x_i-\Delta x_i)P(\tilde{x}_i, t)\right]}{\Delta x_i}\,\Delta x_i + \ldots \tag{2.84}$$

Like in Section 2.2, here we have taken due account of the fact that, owing to function $W(x, y)$ being nonanalytical, the result of expansion in finite differences Δx_i will depend upon the point fixing [cf. Eqn (2.32)]:

$$\tilde{x}_i = x_i + \lambda\Delta x_i\,, \qquad \lambda \in [0, 1]\,, \tag{2.85}$$

within the interval Δx_i. Like on the time axis, the location of the point is described by the parameter $\lambda \in [0, 1]$, which corresponds to a given calculus. For a smooth distribution function $P(x)$, the arbitrariness in the selection of λ does not matter, and one may replace $\tilde{x}_i$ with x_i in the argument. Conversely, for the intensity of stochastic transitions, as for Eqn (2.84), we have

$$W(\tilde{x}_i, x_i-\Delta x_i) = W(x_i, x_i-\Delta x_i) + \frac{\Delta\left[W(x_i, x_i-\Delta x_i)\right]}{\Delta x_i}\,\lambda\Delta x_i + \ldots \tag{2.86}$$

Substituting this expansion into Eqn (2.84), and the result into the master equation written in terms of finite differences, we pass to the continual limit and get

$$\begin{aligned}\dot{P}(x, t) = \int\Bigg\{ & P(x, t)\left[\frac{\partial W(x, x-y)}{\partial x}\,\lambda y + \frac{1}{2}\,\frac{\partial^2 W(x, x-y)}{\partial x^2}\,(\lambda y)^2\right] \\ & + \frac{\partial}{\partial x}\left[P(x, t)\left(W(x, x-y) + \frac{\partial W(x, x-y)}{\partial x}\,\lambda y\right)\right] y \\ & + \frac{1}{2}\,\frac{\partial^2\left[P(x, t)W(x, x-y)\right]}{\partial x^2}\,y^2\Bigg\}\,\mathrm{d}y\,,\end{aligned} \tag{2.87}$$

where we have dropped out the terms producing the zero moments [see Eqn (2.80)] of order $n > 2$. Taking into account that $\partial W(x, x-y)/\partial x = -\partial W(x+y, x)/\partial y$, and integrating by parts with respect to y, we see that the first term is zero. The second term is eliminated in a similar way. Then, with due account for the definitions (2.82), (2.83), we get the equation

$$\dot{P} = \nabla\left[-(f+h)P + \frac{\Theta}{2}\,\nabla(g^2 P)\right], \qquad \nabla \equiv \frac{\partial}{\partial x}\,. \tag{2.88}$$

As compared with Eqn (2.81), here we have the force

$$h(x) = \lambda\Theta\nabla g^2(x)\,, \tag{2.89}$$

which depends on the choice of calculus. This force vanishes in the case of additive noise and in Ito's calculus ($\lambda = 0$).

We see that both in the solution of the stochastic equation of motion (see Section 2.2) and in the proper derivation of the Fokker–Planck equation, which defines the probability of distribution of such solutions, the real force f is supplemented by the fictitious force h depending on the calculus choice. However, comparing expressions (2.37) and (2.89) we find that the latter gives twice the value of h given by the former, and the two values only coincide in the trivial case of $h = 0$, which corresponds to Ito's calculus (or to additive noise).

2.5 Solution of the Fokker–Planck equation

Expression (2.88) is a differential equation in partial derivatives with varying coefficients, and its solution in the general case is not possible. A rather comprehensive review of methods for solving the Fokker–Planck equation can be found in Ref. [15]. In our case of one variable, the most popular of these methods is based on the transformation of the variable (2.48), which allows the reduction of the multiplicative noise to additive noise, bringing the Fokker–Planck equation to the form of the Schrödinger equation (see Section 3.1.1). This opens the possibility of using the powerful methods of solution of the problem concerned with the analysis of eigenfunctions and eigenvalues of the relevant Sturm–Liouville operator [15]; the methods of supersymmetry can also be used to great advantage [42]. Here we are not going to employ these methods, which are capable of dealing with the most general case of several variables in conditions when the principle of detailed balance does not hold. As it turns out, the coefficients of expansion in eigenfunctions then obey the triangular recurrent relations, which lead to continued fractions [15].

Our treatment will be based on the analysis of the stationary solution to the Fokker–Planck equation. Aside from the fact that it leads to the Gibbs distribution, the cornerstone of statistical physics [28], we shall demonstrate that a natural extension of this method allows determination of a new class of solutions for nonstationary systems in the self-modelling regime.

In stationary systems, the probability distribution (2.54) does not depend on the time, so $\dot{P} = 0$ in Eqn (2.88), and the generalized flux assumes a constant value

$$(f+h)P - \frac{\Theta}{2}\,\nabla(g^2 P) = \mathrm{const} \equiv J\,, \qquad \nabla \equiv \frac{\partial}{\partial x}\,. \tag{2.90}$$

For equilibrium systems we have $J = 0$, and the solution of Eqn (2.90) can be written as

$$P = Z^{-1} g^{-2(1-2\lambda)} \exp\left(\frac{2}{\Theta}\int\frac{f+h}{g^2}\,\mathrm{d}x\right), \tag{2.91}$$

where the constant Z is determined by the normalization condition

$$\int P(x)\,\mathrm{d}x = 1\,. \tag{2.92}$$

In the case of additive noise ($g(x) = 2^{1/2}$, $h = 0$), the distribution (2.91), with due account for definition (2.2), assumes the Gibbsian form

$$P(x) = Z^{-1}\exp\left(-\frac{V(x)}{\Theta}\right). \tag{2.93}$$

Here, the synergetic potential $V = V(x)$ reduces to the effective Hamiltonian of the system [28], Θ is the temperature expressed in energy units, and the normalization constant

$$Z = \int_{-\infty}^{\infty}\exp\left(-\frac{V(x)}{\Theta}\right)\mathrm{d}x \tag{2.94}$$

is the partition function whose magnitude determines the free energy

$$F = -\Theta\ln Z\,. \tag{2.95}$$

The equilibrium distribution can be also written in the quasi-Gibbsian form

$$P(x) = Z^{-1}\exp\left(-\frac{U_{\mathrm{ef}}(x)}{\Theta}\right) \tag{2.96}$$

in the case of multiplicative noise. Then, however, the effective potential $U_{\mathrm{ef}}(x)$ is not reduced to the initial one, $V(x)$:

$$U_{\mathrm{ef}}(x) = 2(1-2\lambda)\Theta\ln g(x) + U(x)\,; \tag{2.97}$$

$$U(x) = -2\int\frac{f(x)}{g^2(x)}\,\mathrm{d}x\,, \qquad f \equiv -\frac{\partial V}{\partial x}\,. \tag{2.98}$$

Expression (2.97) is characterized by the dependence of the logarithmic term on the parameter λ, which determines the choice of calculus.

Now let us consider a nonequilibrium stationary state, in which the flux J, while constant, is not equal to zero (such a state can obviously be realized only in an open system with restricted volume, with $J \neq 0$ on the boundaries). Then the probability distribution retains the quasi-Gibbsian form of Eqn (2.96), but the normalization constant Z^{-1} is replaced by the function

$$Z_{\mathrm{ef}}^{-1}(x) = Z^{-1} - \frac{2J}{\Theta}\int\big(g(x)\big)^{-4\lambda}\exp\left(\frac{U(x)}{\Theta}\right)\mathrm{d}x\,, \tag{2.99}$$

where the dependence on x is contained in the subtrahend proportional to J/Θ. If the flow of probability is outwardly directed, then $J > 0$, and, according to Eqn (2.99), its growth leads to a decrease in the free energy (2.95). Otherwise ($J < 0$) the free energy exceeds the equilibrium value, which means that energy is being pumped into the system.

The study of a nonstationary distribution is only possible in the self-modelling regime, when the dependence on two arguments x, t is expressed in terms of a single variable $y = x/a(t)$:

$$P(x,t) = a^{\alpha}\varphi(y)\,, \tag{2.100}$$

where the functions $a(t)$, $\varphi(y)$ and the exponent α are to be defined. Mathematically, Eqn (2.100) implies that the probability distribution is a homogeneous function of order α. Physically, the transition to the new variable $y = x/a$ corresponds to a scaling of the stochastic quantity x, the scale $a(t)$ being a function of time. The property of homogeneity (2.100) reflects the self-similarity of the phase space of the stochastic system, which allows measurement of the quantity x on an arbitrary scale $a(t)$. We know that such a feature is displayed by fractal objects [4, 33]. Accordingly, the assumption (2.100) implies that the domain of definition of the phase space for the stochastic system is a fractal set, whose dimension D lies between 2 (the conventional phase plane) and 0 (the point of equilibrium).

In order to find the exponent α, we substitute the function (2.100) into the normalization condition (2.92), getting as a result

$$\big(a(t)\big)^{-(1+\alpha)} = \int_{-\infty}^{\infty}\varphi(y)\,\mathrm{d}y\,, \tag{2.101}$$

The left-hand side of this equation depends on the time, whereas the right-hand side does not. Hence it follows that

$$\alpha = -1\,, \tag{2.102}$$

which brings us to the conventional normalization condition

$$\int_{-\infty}^{\infty}\varphi(y)\,\mathrm{d}y = 1\,. \tag{2.103}$$

The form of the function $\varphi(x)$ can only be found at given scaling properties of the force $f(x)$ and the multiplicative function $g(x)$. By analogy with Eqn (2.100), we write

$$f(x) = a^{\beta}F(y)\,, \tag{2.104}$$

$$g(x) = a^{\gamma}G(y)\,, \qquad y = \frac{x}{a}\,, \tag{2.105}$$

where the functions $F(y)$, $G(y)$ and the exponents β, γ are assumed to be known (in Section 3.3 we shall demonstrate that the latter are expressed in terms of the fractal dimension D as $\beta = 1 - D$, $\gamma = 1 - D/2$). Substituting the relations (2.100), (2.104), (2.105) into the Fokker – Planck equation (2.88), and making use of expressions $\dot{P} = (\alpha\varphi - y\varphi')a^{\alpha-1}\dot{a}$, $\nabla \equiv \partial/\partial x = a^{-1}\partial/\partial y$, we get

$$(a^{-\beta}\dot{a})(\alpha\varphi - y\varphi') = -\big[(F+H)\varphi\big]' + \frac{\Theta}{2}\,a^{2\gamma-\beta-1}(G^2\varphi)''\,. \tag{2.106}$$

The prime here denotes differentiation with respect to y. We also assume that the fictitious force (2.89) obeys the same scaling condition (2.104) as does the real force:

$$h(x) = a^{\beta}H(y)\,. \tag{2.107}$$

Equation (2.106) becomes an ordinary differential equation

$$\frac{\Theta}{2}(G^2\varphi)'' - \big[(F+H)\varphi\big]' + \mu(y\varphi' - \alpha\varphi) = 0 \tag{2.108}$$

provided that its coefficients do not depend on the time. With this purpose we must set $a^{-\beta}\dot{a} = \mathrm{const} \equiv \mu$, $2\gamma - \beta - 1 = 0$, whence

$$a = \left[\mu(1-\beta)\right]^{1/(1-\beta)} t^{1/(1-\beta)}\,, \tag{2.109}$$

$$\gamma = \frac{1+\beta}{2}\,. \tag{2.110}$$

Thus, the linkage (2.110) between β and γ verifies our assumption (2.107).

In this way, the considerations of similarity allow us to find the time dependence (2.109) of the characteristic value $a(t)$ of the stochastic variable $x(t)$, and to define the exponent (2.102) of the distribution function (2.100). The definitive function $\varphi(y)$ obeys the ordinary differential equation (2.108). It is easy to see that this equation can be represented as

$$\Theta(G^2\varphi)'' = (\widetilde{F}_{\rm ef}G^2\varphi)'\,, \tag{2.111}$$

where we have introduced the effective force [cf. Eqn (2.2)]

$$\widetilde{F}_{\rm ef}(y) = -\frac{\partial \widetilde{U}_{\rm ef}(y)}{\partial y}\,, \tag{2.112}$$

whose magnitude is determined by the potential

$$\widetilde{U}_{\rm ef}(y) = -4\lambda\Theta\ln G(y) + U_\mu(y) + U(y)\,; \tag{2.113}$$

$$U_\mu(y) = \mu\int G^{-2}(y)\,{\rm d}y^2\,, \tag{2.114}$$

$$U(y) = -2\int\frac{F(y)}{G^2(y)}\,{\rm d}y\,. \tag{2.115}$$

Lowering the order of the differential equation (2.111), and using the boundary conditions

$$\varphi'(y) = 0\,,\qquad \varphi(y) = 0\qquad\text{at}\qquad y = \pm\infty \tag{2.116}$$

we find the general solution of the Fokker–Planck equation in the self-modelling regime:

$$\varphi(y) = Z^{-1}\exp\left(-\frac{U_{\rm ef}(y)}{\Theta}\right). \tag{2.117}$$

Here the effective synergetic potential $U_{\rm ef}\equiv\widetilde{U}_{\rm ef}+2\Theta\ln G$, renormalized on account of the factor G^2 in Eqn (2.111), has the form [cf. Eqn (2.113)]

$$U_{\rm ef}(y) = U_\Theta(y) + U_\mu(y) + U(y)\,, \tag{2.118}$$

where

$$U_\Theta(y) = 2(1-2\lambda)\Theta\ln G(y)\,, \tag{2.119}$$

and the remaining terms are given by Eqns (2.114), (2.115). The normalization constant Z is given by Eqn (2.103).

This analysis reveals that the description of nonstationary self-modelling regime can be performed by analogy with the stationary system, if from the original stochastic variable x and functions $P(x,t)$, $f(x)$, $h(x)$, and $g(x)$ we go over to the variables $y = x/a$, $\varphi = P/a^\alpha$, $F = f/a^\beta$, $H = h/a^\beta$, and $G = g/a^\gamma$, whose magnitudes are determined by the scale $a = a(t)$, which gives the characteristic value of variable x. Comparison between expressions (2.97), (2.118) indicates that the transition to the nonstationary regime also gives rise to an additional term (2.114), resulting from the change of scale $a(t)$ in accordance with Eqn (2.109).

To conclude this section, let us note that the first description of a nonstationary stochastic system in a self-modelling regime was apparently given by I M Lifshitz and V V Slezov [43] for the problem of coalescence of emanation of a new phase. Today the considerations of similarity are widely employed for describing the evolution of spatial structures arising in the course of phase transformations [31, 44].

3. Description of a stochastic system with singular multiplicative noise

3.1 Gauging the probability distribution of a stochastic system

As follows from our discussion in Sections 2.2, 2.4, the non-analytical character of the dependence $x(t)$, which determines the time variation of the stochastic variable, gives rise to the force h depending on the calculus choice [see Eqns (2.37), (2.89)]. Unlike the situation in the field theory, where the arbitrariness in the selection of potential does not affect the experimental field strength [45], the force h has great influence on the distribution of probability of realization of the stochastic variable.

From a mathematical viewpoint, the appearance of the force h is associated with the arbitrariness in the selection of the point (2.85) on the x axis. The singular nature of the function $\xi(t)$, which determines the δ-correlated character of the white noise [see Eqn (2.10)], ensures the fixation of this point. Apparently, as the δ-correlator smears out, which always happens in reality, the force h vanishes automatically [14]. In other words, the force which depends on the calculus choice and therefore has no physical meaning is an artifact of the white noise approximation. In this connection one must either reject this approximation, which will greatly complicate the formalism [14], or use a gauge scheme which would eliminate this ambiguity for δ-correlated noise. Such a scheme was first proposed in Ref. [46] and is reproduced in this section; we shall consider separately the forward and the backward Kolmogorov equations (Sections 3.1.1 and 3.1.2, respectively).

For the sake of simplicity we shall analyze the spatially homogeneous case, with the time measured in units of $\tau = \chi/\gamma$ (χ is the susceptibility, γ the kinetic coefficient), the force on a scale of χ^{-1}, the stochastic velocity component $\dot{x}$ in units of τ^{-1}. Then the stochastic equation of motion assumes the form [see Eqns (2.21), (2.23)]

$$\dot{x} = f(x) + T^{1/2}g(x)\xi(t)\,. \tag{3.1}$$

Here we have explicitly singled out the dimensionless noise intensity T (which was denoted by Θ in Section 2.1) and the multiplicative function $g(x)$; the force $f(x)$ and the function $\xi(t)$ are given by Eqns (2.2), (2.10). According to Eqn (2.88), it is convenient to write the Fokker–Planck equation, which defines the distribution function (2.54), in the form of an equation of continuity in the space of the stochastic variable:

$$\dot{P}_\lambda + \nabla J_\lambda = 0\,,\qquad \nabla\equiv\frac{\partial}{\partial x}\,. \tag{3.2}$$

Here the probability flux is given by the expression

$$J_\lambda = (f + h_\lambda)P_\lambda - \frac{T}{2}\,\nabla(g^2 P_\lambda)\,, \tag{3.3}$$

in which the force [see Eqn (2.89)]

$$h_\lambda(x) = \lambda T \nabla g^2(x) \tag{3.4}$$

depends on the parameter $\lambda \in [0, 1]$, related to the choice of calculus.

3.1.1 Forward Kolmogorov equation. As we know, the evolution of the system in probability space is described by the forward or backward Kolmogorov equations, depending on the time direction [14–18]. Let us consider first the former of these, which is reducible to the Fokker–Planck equation.

By way of an intuitive guideline, we shall sketch out the known transfer from the Fokker–Planck equation to the Schrödinger equation with imaginary time for a system with additive noise, where the fictitious force is $h_\lambda \equiv 0$ by virtue of $g(x) = 2^{1/2}$, and the value of λ does not matter [47]. This transfer is accomplished through replacing the probability P_λ in Eqns (3.2), (3.3) with the 'wave function' $\Psi = P_\lambda \exp(-\alpha)$. As a result, the Fokker–Planck equation with additive noise becomes

$$\dot{\Psi} = \Psi\left[\nabla^2 V + (\nabla V)\nabla\alpha + T(\nabla\alpha)^2 + T\nabla^2\alpha\right] + (\nabla V + 2T\nabla\alpha)\nabla\Psi + T\nabla^2\Psi\,. \tag{3.5}$$

This equality reduces to the Schrödinger-type equation in the absence of terms containing $\nabla\Psi$, which is achieved by selecting the imaginary phase α in accordance with the condition

$$\nabla\alpha = -\frac{\nabla V}{2T}\,. \tag{3.6}$$

This condition implies that the displacement of the origin of the synergetic potential V by δV shifts the phase by $\delta\alpha = -\delta V/(2T)$ without modifying the Schrödinger equation itself, i.e.

$$-T\,\frac{\partial}{\partial t}\,\Psi = -T^2\nabla^2\Psi + U\Psi \tag{3.7}$$

with the imaginary time $-\mathrm{i}t$, the potential energy

$$U \equiv \frac{1}{4}(\nabla V)^2 - \frac{T}{2}\,\nabla^2 V \tag{3.8}$$

and the Planck constant T (the mass of the effective particle is 1/2).

The fact that the origin of phase α does not depend on the 'coordinate' x corresponds to the condition of global gauge invariance in the standard field-theoretical scheme [48], which is obviously satisfied owing to the additive nature of the noise. When the noise is multiplicative, its intensity becomes a function of x, which causes the phase to change as $\alpha(x)$. Because of this, the gauge condition must be made local, and the scheme becomes much more complicated.

In order to analyze the local gauge invariance, we shall express the initial distribution $P_\lambda(x, t)$ in Eqns (3.2), (3.3) in terms of the renormalized distribution function

$$P(x, t) = P_\lambda(x, t)\exp\left[-\alpha(x)\right]\,, \tag{3.9}$$

satisfying the canonical Fokker–Planck equation

$$\dot{P} = \nabla\left[(\nabla V)P + \frac{T}{2}\,\nabla(g^2 P)\right], \tag{3.10}$$

in which the renormalized potential $V = V(x)$ is not reducible to the initial potential $V_0(x) \equiv -\int f(x)\,\mathrm{d}x$. As a result, we get

$$\dot{P} = \nabla\left[(\nabla V_0)P + \frac{T}{2}\,\nabla(g^2 P) - \lambda T(\nabla g^2)P + T(\nabla\alpha)g^2 P\right] + (\nabla V_0)(\nabla\alpha)P - \lambda T(\nabla g^2)(\nabla\alpha)P - \frac{T}{2}\,g^2(\nabla^2\alpha)P + \frac{T}{2}\,g^2(\nabla\alpha)^2 P\,, \tag{3.11}$$

where on the right-hand side we have added and subtracted the term $(T/2)g^2 P\nabla^2\alpha$. Since Eqn (3.10) does not contain terms proportional to P, we must set

$$(\nabla V_0 - \lambda T\nabla g^2)\nabla\alpha = \frac{T}{2}\,g^2\left[\nabla^2\alpha - (\nabla\alpha)^2\right]. \tag{3.12}$$

In addition, the expressions for flows which occur in Eqns (3.10), (3.11) under the operator ∇ must be equal:

$$\nabla V_0 - \lambda T\nabla g^2 + T(\nabla\alpha)g^2 = \nabla V\,. \tag{3.13}$$

Now it is convenient to rewrite the result in the form of a set of equations

$$\nabla\alpha - \nabla\ln\nabla\alpha = -\frac{2\nabla V_0}{Tg^2} + 2\lambda\nabla\ln g^2\,, \tag{3.14}$$

$$\nabla\alpha = \frac{\nabla(V - V_0)}{Tg^2} + \lambda\nabla\ln g^2\,, \tag{3.15}$$

which defines the phase distribution $\alpha(x)$ and the renormalized potential $V(x)$ via the bare potential $V_0(x)$ and the parameter of calculus λ. Eliminating $\lambda\nabla\ln g^2$, we get the equation

$$\nabla\alpha + \nabla\ln\nabla\alpha = \frac{2\nabla V}{Tg^2}\,,$$

the first integral of which is

$$\nabla\exp\alpha = \exp\left(\frac{2}{T}\int\frac{\nabla V}{g^2(x)}\,\mathrm{d}x\right). \tag{3.16}$$

Hence we find the final expression for the phase factor

$$\exp\left[\alpha(x)\right] = \int\exp\left[\frac{U(x)}{T}\right]\mathrm{d}x\,, \tag{3.17}$$

which is characterized by the shape of the effective potential [cf. Eqn (2.98)]

$$U(x) = 2\int\frac{\mathrm{d}V(x)/\mathrm{d}x}{g^2(x)}\,\mathrm{d}x\,. \tag{3.18}$$

As will be shown below, the physical meaning of the phase as defined by Eqns (3.17), (3.18) consists in that the shape of the function $\alpha(x)$ determines the type of the boundary $x = 0$ and the pattern of transition of the stochastic system to the deterministic regime $x(t) = 0$.

As follows from the arguments developed above, the fictitious force h, which occurs in the Fokker–Planck equation in the approximation of δ-correlated noise, can be compensated on account of the transfer from the initial synergetic potential $V_0(x)$ to the renormalized potential $V(x)$. Then the initial probability distribution $P_\lambda(x,t)$ acquires the coefficient $\exp[-\alpha(x)]$, where the phase $\alpha(x)$ is also determined by the renormalized potential $V(x)$. The physical meaning of such a renormalization is that in the white noise approximation one ought to use not the original Fokker–Planck equation (3.2)–(3.4), whose solution depends on the calculus choice (parameter λ), but rather the renormalized equation (3.10), obtained through gauging the probability distribution $P(x,t)$ and using Eqn (3.9). Similarly to the renormalization techniques used in quantum electrodynamics [49], the synergetic potential $V(x)$ should be regarded as a function with an innate physical meaning, whereas $V_0(x)$ is the bare potential without such a meaning.

To find the linkage between $V(x)$ and $V_0(x)$, we rewrite the left-hand side of Eqn (3.14) as $-\nabla(\ln\nabla\exp(-\alpha))$, which brings us to the solution

$$\alpha(x) + \alpha_\lambda(x) = \text{const}. \tag{3.19}$$

This solution links the phase α, defined by the renormalized potential (3.18), and the bare phase α_λ, which is defined by

$$\exp[\alpha_\lambda(x)] = \int (g(x))^{-4\lambda} \exp\left[\frac{U_0(x)}{T}\right] \mathrm{d}x\,, \tag{3.17a}$$

$$U_0(x) \equiv 2\int \frac{\mathrm{d}V_0(x)/\mathrm{d}x}{g^2(x)}\,\mathrm{d}x\,. \tag{3.18a}$$

Differently from α, phase α_λ is determined not only by the shape of the bare potential $V_0(x)$, but also by the parameter λ [in Ito's calculus ($\lambda = 0$) the definitions (3.17), (3.17a) are the same up to the replacement of V with V_0]. The physical meaning of condition (3.19) is that the renormalization of potential leads (up to a constant) to a change of sign of the bare phase α_λ.

3.1.2 Backward Kolmogorov equation. In order to employ the methods developed in the quantum theory [40], it will be convenient to bring the Fokker–Planck equation into the form of the Liouville equation

$$\dot{P}_\lambda(x,t) = L_\lambda(x)P_\lambda(x,t)\,, \tag{3.20}$$

where the operator L_λ is written as follows

$$L_\lambda = \nabla\big[(\nabla V_0) - \lambda T(\nabla g^2)\big] + \frac{T}{2}\nabla^2 g^2\,. \tag{3.21}$$

In the context of this approach, the gauge transformation (3.9) is expressed by

$$P(x,t) = U(x)P_\lambda(x,t)\,, \tag{3.22}$$

which implies passage to a new basis, which is given by the operator

$$U(x) = \exp[-\alpha(x)]\,, \tag{3.23}$$

whose action is determined by the phase distribution $\alpha(x)$. On the basis P, the Liouville equation assumes the form of Eqn (3.20), if from Eqn (3.21) we go to the operator

$$L = UL_\lambda U^{-1}\,. \tag{3.24}$$

The form of the latter follows from the condition that the Liouville equation on the new basis is reduced to Eqn (3.10), whence it follows that

$$L = \nabla(\nabla V) + \frac{T}{2}\nabla^2 g^2\,. \tag{3.25}$$

Substituting Eqns (3.21), (3.23) into Eqn (3.24), and comparing the result with Eqn (3.25), we come to equations (3.12), (3.13), which are separated because the latter occurs under the operator ∇.

Let us now go over to the backward Kolmogorov equation, which corresponds to the inverted time direction [39]. We know that time reversal in quantum theory is achieved by the transition to conjugate quantities [40]. In the case of the Liouville equation (3.20) we must replace the distribution $P_\lambda(x,t)$ by $P_\lambda^+(x,t)$, and the operator $L_\lambda(x)$ by $L_\lambda^+(x)$. Upon transition to the new basis [cf. Eqn (3.22)]

$$P^+(x,t) = U^+(x)P_\lambda^+(x,t) \tag{3.26}$$

we ought to expect that the operator

$$U^+(x) = \exp[\alpha(x)] \tag{3.27}$$

should contain phase $\alpha(x)$ with opposite sign to that in Eqn (3.23) for the forward time flow.

To prove this point, into the Liouville equation [cf. Eqn (3.20)]

$$\dot{P}_\lambda^+(x,t) = L_\lambda^+(x)P_\lambda^+(x,t)\,, \tag{3.28}$$

corresponding to the inverted time flow, we substitute the inverted expression (3.26) and the operator (cf. Eqn (3.21))

$$L_\lambda^+ = \big[(\nabla V_0) - \lambda T(\nabla g^2)\big]\nabla - \frac{T}{2}g^2\nabla^2\,, \tag{3.29}$$

whose form follows from the conventional representation of the back Kolmogorov equation [15]. Then, on the basis P^+, the Liouville equation assumes the form (3.28), where L_λ^+ is replaced by the operator

$$L^+ = (\nabla V)\nabla - \frac{T}{2}g^2\nabla^2\,. \tag{3.30}$$

The latter does not involve the fictitious force $h = \lambda T\nabla g^2$ if the same conditions (3.12), (3.13) are satisfied as for the forward equation.

Thus, assumption (3.27) is true, and comparison of it with Eqn (3.23) brings us to the conclusion that operator $U(x)$ is unitary. This can also be proved by applying the operation of conjugation to Eqn (3.24) followed by the substitution of expressions (3.27), (3.29), (3.30). As a result, we get the same conditions (3.12), (3.13).

This train of thought brings us to an important conclusion: the group of gauge transformations of the Fokker–Planck equation, which eliminates the arbitrariness in the choice of calculus, is unitary. As far as the Liouville operator is concerned, a comparison between expressions (3.25) and

(3.30) indicates that it is non-Hermitian. However, using the $\exp\left[U_{\text{ef}}(x)/(2T)\right]$ transform, where the effective potential is given by Eqn (2.97), the operator can be reduced to the Hermitian form [15].

3.2 Phase transitions in a stochastic system

By contrast to conventional phase transitions, when the qualitative transformation of the system is related to the appearance of integral singularities in the distribution function, the noise-induced transitions cause less dramatic changes of the system — for example, additional maxima may arise as the noise intensity decreases. In case of multiplicative noise, however, whose amplitude depends on the stochastic variable x, these maxima may occur alongside the infinite growth of the probability density in the neighbourhood of the point $x = 0$ [14]. If such a singularity is nonintegrable, the normalization condition is lost, and the deterministic condensate may precipitate in the stochastic system: a finite share of the total number of degrees of freedom, represented by all possible time dependences $x(t)$, assumes the form $x(t) = 0$. The question we are facing in this connection is the following: what will be the distribution of the system states between the maxima corresponding to the deterministic regime and the ordered state?

From a formal standpoint, we have to classify the possible types of boundary $x = 0$ with respect to the form of the multiplicative function. To this end we may use the recipe proposed in Ref. [14], which prescribes finding certain parameters L_1, L_2, L_3 and classifying the type of the boundary in accordance with their divergence. Nothing is said, however, about the physical meaning of parameters L_1, L_2, L_3; what is more, for their determination one needs to know the distribution function which, as we know, depends on the calculus choice. In the preceding section we described the gauge scheme which allowed us to bring the Fokker – Planck equation to the canonical Ito's form without an arbitrary force by introducing the phase factor into the distribution function. In addition to eliminating the ambiguity in the selection of the distribution function, such gauge of the stochastic system allows us to give physical meaning to the parameters L_1, L_2, L_3 — they are defined by the phase factor (3.17) used for gauging the Fokker – Planck equation.

Our task now is to study the singularities of the equilibrium distribution [see Eqn (2.96)]

$$P(x) = Z^{-1} \exp\left(-\frac{U_{\text{ef}}(x)}{T}\right). \tag{3.31}$$

Here and below we are dealing with the gauge distribution (3.22), which implies that in the effective potential (2.97) we set $\lambda = 0$. The bare potential will be approximated by the Landau expansion [28]

$$V(x) = \frac{A}{2}x^2 + \frac{B}{4}x^4, \tag{3.32}$$

$$A = \alpha(T - T_{\text{c}}), \tag{3.33}$$

where α, T_{c}, B are the positive constants, and T is the noise intensity (temperature). With a view to studying the singularities near the point $x = 0$, for the multiplicative function we set

$$g(x) = 2^{1/2}x^a, \tag{3.34}$$

where the exponent a is arbitrary [in Section 3.3 we shall supply evidence to the effect that the power law (3.34) is universal for self-similar systems]. Then the effective synergetic potential (2.97), (2.98) takes the form

$$U_{\text{ef}}(x) = 2Ta\ln x + U(x), \tag{3.35}$$

$$U(x) = \frac{A}{2(1-a)}x^{2(1-a)} + \frac{B}{2(2-a)}x^{2(2-a)}. \tag{3.36}$$

A salient feature of systems with multiplicative noise lies in the nonanalytical form of the renormalized potential $U(x)$ (3.36) in spite of the fact that the bare potential $V(x)$ is given by a very simple expression (3.32). At $a = 0$, the noise is additive, and the potentials $V(x)$ and $U(x)$ coincide. The cases of $a = 1$, $a = 2$, for which Eqn (3.32) does not hold, will be treated separately. Finally, in accordance with Eqns (2.113), (2.114), upon transition to the nonstationary self-modelling regime the effective potential (3.35) acquires the term U_μ, which is of the same nature as the first term in Eqn (3.36). This implies that this term may be taken into account by the appropriate renormalization of the critical temperature T_{c}, which enters Eqn (3.33).

3.2.1 Noise-induced phase transition. As indicated above, this transition consists in that, as the noise intensity T decreases, distribution (3.31) exhibits maxima at points $\pm x_0 \neq 0$ [14]. The location of these points is determined by the condition

$$\frac{\partial U_{\text{ef}}(x)}{\partial x} = 0, \tag{3.37}$$

which defines the minima of the effective potential. Substituting here Eqns (3.33), (3.35), (3.36), for the location of maximum x_0 measured in units of $\alpha T_{\text{c}}/B$, one gets the equation

$$x_0^2 + \frac{2a}{\alpha}\Theta x_0^{-2(1-a)} = 1 - \Theta, \tag{3.38}$$

where we have introduced the dimensionless temperature

$$\Theta \equiv \frac{T}{T_{\text{c}}}. \tag{3.39}$$

This equation admits solutions only for values of Θ limited from above by a certain Θ_0. To find the latter, we note that at the point $\Theta = \Theta_0$ the first and second derivatives of $U_{\text{ef}}(x)$ are zero. This gives us an additional equation

$$x_0^{2(2-a)} = \frac{2a}{\alpha}(1-a)\Theta_0. \tag{3.40}$$

Eliminating x_0 from Eqns (3.38), (3.40), for the temperature of transition we find

$$\frac{(1-\Theta_0)^{2-a}}{\Theta_0} = \frac{2a}{\alpha}\frac{(2-a)^{2-a}}{(1-a)^{1-a}}, \qquad \Theta_0 \equiv \frac{T_0}{T_{\text{c}}}. \tag{3.41}$$

The critical intensity Θ_0 as a function of the exponent a is plotted in Fig. 1 for different values of α. In the limit of $a = 0$, which corresponds to additive noise, the transition point T_0 coincides with the critical value T_{c}. As the exponent increases in the lower range, the quantity T_0 decreases, the rate of decline being the faster, the smaller the parameter α, the motive power of the phase transition. As a increases further, T_0 passes through a minimum and then increases to the value given by Eqn (3.44) and corresponding to $a = 1$. Then the

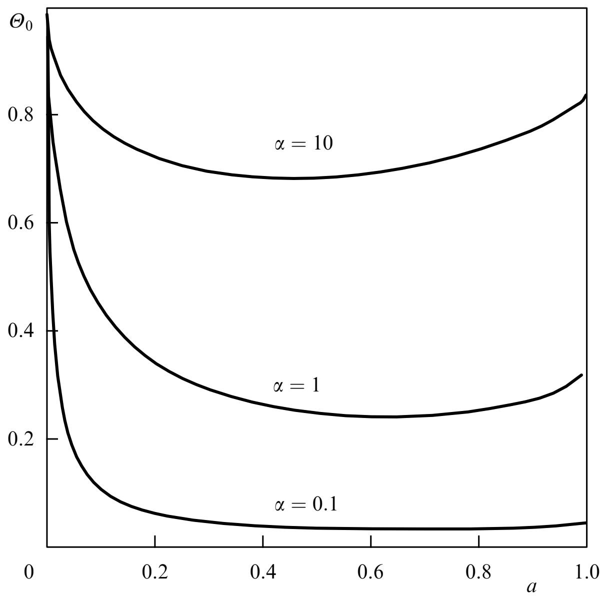

Figure 1. Limiting value $\Theta_0 = T_0/T_c$ of noise intensity as a function of exponent a of the multiplicative function.

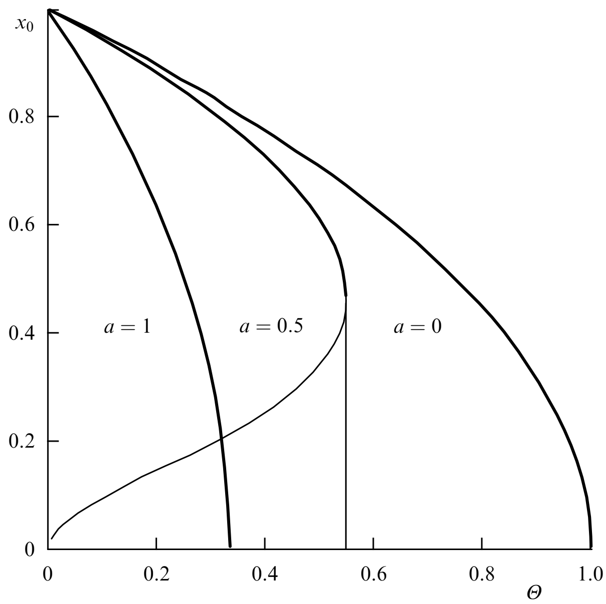

Figure 2. Most probable value x_0 of the stochastic variable as a function of the noise intensity $\Theta = T/T_c$.

most probable value $x_0(T)$ of the stochastic variable decreases as the noise intensity grows, as shown in Fig. 2. Observe that in the limiting cases $a = 0$, $a = 1$ the phase transition is continuous, whereas between these values the quantity x_0 exhibits a jump at the transition point T_0.

For $a = 1$, in place of Eqn (3.35) we get

$$U_{\text{ef}} = 2T \ln x + U(x) , \qquad U = A \ln x + \frac{B}{2} x^2 . \tag{3.42}$$

Then distribution (3.31) has a maximum at the point

$$x_0 = \left[1 - \left(1 + \frac{2}{\alpha} \right) \Theta \right]^{1/2} , \qquad \Theta \equiv \frac{T}{T_c} , \tag{3.43}$$

and the noise intensity, which corresponds to this maximum, takes the form

$$T_0 = \left(1 + \frac{2}{\alpha} \right)^{-1} T_c . \tag{3.44}$$

3.2.2 Transition of a stochastic system to the deterministic regime. The maximum of the distribution $P(x)$, described above, is not the only one possible. Indeed, substituting Eqn (3.35) into (3.31), we see that for positive a the distribution function has a singularity $P \simeq Z^{-1} x^{-2a}$ at $x \to 0$. This means that the normalization constant Z at the lower limit of integration $b \to 0$ has the form $Z \sim b^{1-2a}$. Consequently, if the exponent of the multiplicative function a lies in the interval 0 to 1/2, this singularity of the distribution (3.31) is integrable, and the system behaves in an ordinary way. For $a > 1/2$, however, we get $Z = \infty$, and the distribution $P(x)$ in the limit $x = 0$ becomes nonnormalizable. This implies that a condensate is formed, which corresponds to the deterministic behaviour of the system at $x = 0$. With due account for Eqn (3.35), it is convenient to split the distribution function (3.31) into two factors, viz.

$$P(x) = P_0(x) P'(x) ; \tag{3.45}$$

$$P_0(x) = Z_0^{-1} x^{-2a} , \qquad P'(x) = \frac{1}{Z'} \exp\left[-\frac{U(x)}{T} \right] , \tag{3.46}$$

where the potential $U(x)$ is given by Eqn (3.36). In the first term, which corresponds to the condensate, the constant Z_0 is infinite, whereas in the second $Z' = Z/Z_0 < \infty$. For $a = 1$, the terms in the distribution (3.45) become

$$P_0(x) = Z_0^{-1} x^{-2} , \qquad P'(x) = \frac{1}{Z'} x^{-A/T} \exp\left(-\frac{B}{2T} x^2 \right) . \tag{3.47}$$

In this way, in systems with multiplicative noise (3.34) the point $x = 0$ appears as an attractor, the presence of which has a considerable effect on the entire axis of x values. Representing the stochastic system as the process of diffusion of a particle with coordinate x, we may regard this attractor as an attractive boundary. Following Ref. [14], we shall give the recipe which allows different types of diffusion processes to be classified depending on the existence of the boundary $b \to 0$.

Let us introduce the function

$$\phi(x) = \exp\left[-\int \frac{2f(x)}{Tg^2(x)} \, dx \right] \tag{3.48}$$

and analyze the convergence of the following integrals

$$L_1 = \int_b^\beta \phi(x) \, dx , \tag{3.49}$$

$$L_2 = \int_b^\beta \frac{1}{Tg^2(y)} \int_b^y \exp\left[-\int_\beta^x \frac{2f(z)}{Tg^2(z)} \, dz \right] dx \times \exp\left[\int_\beta^y \frac{2f(z)}{Tg^2(z)} \, dz \right] dy , \tag{3.50}$$

$$L_3 = \int_b^\beta \left(g^2(x) \phi(x) \right)^{-1} dy , \tag{3.51}$$

where b and β are the lower and upper limits of the diffusion interval, respectively. If the first of the integrals is infinite

($L_1 = \infty$), then the boundary b is referred to as natural, and cannot be reached even over the infinite time $t = \infty$. If $L_1 < \infty$, $L_2 = \infty$, the boundary is attractive, and can be reached by the particle only at $t \to \infty$. If L_1, $L_2 < \infty$, $L_3 = \infty$, the boundary b can be reached within a finite time, and one has to define the boundary conditions. Finally, when all three integrals L_1, L_2, L_3 are finite, the boundary is regular, and the particle may at any time occur at the point $x = b$.

A disadvantage of this classification is that expressions (3.49)–(3.51) are quite cumbersome, and the physical content of the parameters L_1, L_2, L_3 is not clear. However, from a comparison of Eqns (3.17), (3.18) with (3.48) one may see that the function $\phi(x)$, which determines the parameters L_1, L_2, L_3, is linked with phase $\alpha(x)$, which sets the gauge of the Fokker–Planck equation, by the following relation

$$\exp\left[\alpha(x)\right] = \int \phi(x)\,\mathrm{d}x\,. \tag{3.52}$$

Then the definitions (3.49)–(3.51) become

$$L_1(b,\beta) = \exp\left[\alpha(\beta)\right] - \exp\left[\alpha(b)\right]\,, \tag{3.53}$$

$$L_2(b,\beta) = \frac{Z}{T}\int_b^\beta \exp\left[\alpha(x)\right]P(x)\,\mathrm{d}x\,, \tag{3.54}$$

$$L_3(b,\beta) = Z\int_b^\beta P(x)\,\mathrm{d}x\,, \tag{3.55}$$

where β is an arbitrary parameter. In this way, the use of phase $\alpha(x)$ allows a transparent meaning to be given to the parameters L_1, L_2, L_3. Since the phase factor $\exp\alpha$ constitutes, according to Eqn (3.17), the integral effect of the exponential action of the effective potential $U(x)$, reduced to the noise intensity T, then equation (3.53) defines the difference of these effects at the limits β and b. Function (3.54) is accordingly the phase factor averaged over distribution (3.31), which in turn is reduced to the normalization constant Z. Finally, Eqn (3.55) defines, up to the same reduction, the integral effect of accumulation of probability (3.31) over the interval $[b,\beta]$.

According to the recipe of Ref. [14], the deterministic regime $x = 0$ is not reached even over an infinite time, if $L_1 = \infty$ in the limit $b \to 0$. If $L_1 < \infty$, and L_2 diverges, the deterministic behaviour is only feasible in the limit of $t \to \infty$. In other words, even though such behaviour is manifested at $x \to 0$ in the form of an infinite maximum of the singular term $P_0(x)$, in reality it is not feasible. Because of this, in the normalization of distribution (3.45) we must perform a cutoff at the lower limit $b \to 0$, whose magnitude defines the constant

$$Z_0 = 2(2a-1)^{-1}b^{1-2a}\,. \tag{3.56}$$

If the values of L_1, L_2 are finite, but $L_3 = \infty$, then the deterministic regime is established within a finite time, since, according to (3.55), the system with infinite probability is confined in the range $[b,\beta]$. This implies that in the normalization of the distribution function we must account explicitly for the presence of condensate, which sets apart the δ-shaped singularity:

$$P(x) = C\delta(x) + P_0(x)P'(x)\,. \tag{3.57}$$

Here, the intensity C of the deterministic condensate is determined by the above normalization condition. Finally, if all the parameters L_1, L_2, L_3 are finite, then at any time the deterministic regime, like the stochastic one, is realized in accordance with its distribution $P_0(x)$.

Since the neighbourhood of point $x = 0$ is most important, where the divergence of $P(x)$ is concentrated, one may find the asymptotic dependence of $\alpha(x)$ in the limit $x \to 0$. Setting $B = 0$ in Landau's expansion, from Eqns (3.17), (3.36) for $a \neq 1$ we get

$$\exp(\alpha) = \int \mathrm{d}x\,\exp\left[\frac{A}{2T(1-a)}\,x^{2(1-a)}\right]. \tag{3.58}$$

Introducing a new variable $y \equiv x^{2(1-a)}$, it is convenient to rewrite this expression in the form

$$\exp(\alpha) = \left[2(1-a)\right]^{-1}\int y^{-\gamma}\exp(\lambda y)\,\mathrm{d}y\,, \tag{3.59}$$

where we have introduced the constants

$$\gamma = \frac{1-2a}{2(1-a)}\,, \qquad \lambda = \frac{A}{2T(1-a)}\,. \tag{3.60}$$

Since $\exp(\alpha) > 0$ always, from Eqn (3.59) it follows first of all that the positive parameter a is limited from above by the value of 1, and therefore $\lambda > 0$. At $x \ll 1$, the main contribution to the integral of Eqn (3.59) comes from the power function, and for the phase factor we get $\exp(\alpha) \simeq x$. Therefore, the parameter L_1, which is defined by Eqn (3.53), assumes finite values. The distribution function for $x \ll 1$ is defined by

$$2ZP(x) \simeq x^{-2a}\exp\left[-\lambda x^{2(1-a)}\right]. \tag{3.61}$$

Substituting this into Eqn (3.54), we arrive at

$$L_2 = \frac{1}{2T}\int_b^\beta x^{-2a+1}\exp\left[-\lambda x^{2(1-a)}\right]\mathrm{d}x\,. \tag{3.62}$$

Upon transfer to the variable $y = x^{2(1-a)}$, this integral is easily computed:

$$L_2(b,\beta) = (2A)^{-1}\left\{\exp\left[\lambda\beta^{2(1-a)}\right] - \exp\left[\lambda b^{2(1-a)}\right]\right\}. \tag{3.63}$$

From this we see that in the limit $b \to 0$ the quantity L_2 is always finite. Similarly, for L_3 we get

$$L_3(b,\beta) = \left[2(2a-1)\right]^{-1}\left(\beta^{1-2a} - b^{1-2a}\right). \tag{3.64}$$

The parameter L_3 thus assumes finite values at $a < 1/2$, and becomes infinite at $a > 1/2$.

This analysis reveals that in the range $0 < a \leqslant 1/2$ the deterministic regime is realized on a level with the stochastic regime at any time, and at $1/2 < a < 1$ the system reaches the position $x = 0$ in a finite time. The characteristic time τ_{ef} of evolution of the stochastic subsystem at $x \ll 1$ is given by the expression $\tau_{\mathrm{ef}}^{-1} = (T/2)\nabla^2 g^2$, which follows from the Fokker–Planck equation (3.10) in which the terms exhibiting the most singular behaviour in the limit $x \to 0$ are retained. As a result, we get the equality

$$\tau_{\mathrm{ef}} = \left[2a(2a-1)T\right]^{-1}x^{2(1-a)}\,, \tag{3.65}$$

from which we see that $\tau_{\mathrm{ef}} < 0$ at $a < 1/2$, and $\tau_{\mathrm{ef}} > 0$ at $a > 1/2$. Thus, the time taken to reach the deterministic regime $x = 0$ increases with increasing $x \ll 1$.

The self-consistent behaviour of the condensate, determined by the constant C, is ensured by the normalization condition for distribution (3.57). Making use of Eqn (3.46) and the normalization constant (3.56), we find

$$C = 1 - \frac{2a-1}{Z'}\, b^{2a-1} \int_b^{\infty} x^{-2a} \exp\left[-\frac{U(x)}{T}\right] \mathrm{d}x \,. \qquad (3.66)$$

Now we go over to the integration variable $y = x/b$ normalized to the quantity $b \to 0$, which cuts off the lower limit of integration. Then

$$C = 1 - \frac{2a-1}{Z'} \int_1^{\infty} y^{-2a} \exp\left[-\frac{U(by)}{T}\right] \mathrm{d}x \,. \qquad (3.67)$$

As follows from Eqn (3.36), at $a \neq 1$ the index of the exponential in the limit $b \to 0$ takes on the value of 0, and as a result

$$C = 1 - \frac{1}{Z'} \,, \qquad (3.68)$$

$$Z' = \int_0^{\infty} \exp\left[-\lambda x^{2(1-a)} - \mu x^{2(2-a)}\right] \mathrm{d}x \,, \qquad (3.69)$$

where the expression for the normalization constant Z' follows from Eqns (3.31), (3.35), (3.36), and also $\mu = B/[2T(2-a)]$, $1/2 < a < 1$.

Numerical integration in Eqn (3.68) yields the curves $C(\Theta)$ shown for different a, α in Fig. 3. We see that, as the noise intensity $\Theta = T/T_{\mathrm{c}}$ increases, the density of the deterministic condensate gradually decreases from $C = 1$ at $\Theta = 0$ to $C = 0$ at the critical value of Θ_{c}. For large values of the constant α in Landau's expansion (Fig. 3a), the fall-off of the condensate density C occurs at greater values of the noise intensity Θ, if the value of a is higher. For $\alpha \simeq 1$, the function $C(\Theta)$ becomes more complicated (Fig. 3b): the increase in a causes the density of condensate to increase as before when Θ is small, and to decrease when Θ is large.

Figure 4 shows the curves of the critical noise intensity Θ_{c} as a function of a for different values of α. We see that with small values of α the function monotonically decreases, and increases with large values. Observe that at $\alpha < 1$ the critical value Θ_{c} becomes infinite when a decreases. This means that when the multiplicative noise (3.34) increases slowly as the stochastic variable x grows, the thermodynamic stimulus for ordering is low, and the deterministic condensate is realized at all values of the noise intensity Θ.

For $a = 1$ and $x \ll 1$ we have

$$\exp(\alpha) \simeq \left(\frac{A}{T} + 1\right)^{-1} x^{A/T+1} \,, \qquad P(x) \simeq Z^{-1} x^{-(A/T+2)} \,. \qquad (3.70)$$

Accordingly, in the range restricted by the noise intensity

$$T^0 = \left(1 + \alpha^{-1}\right)^{-1} T_{\mathrm{c}} \,, \qquad (3.71)$$

we get $L_1 = \infty$, while at $T > T^0$ the quantity L_1 is finite. According to Eqn (3.54), one obtains

$$L_2(b, \beta) \simeq (A + T)^{-1} \ln \frac{\beta}{b} \,, \qquad (3.72)$$

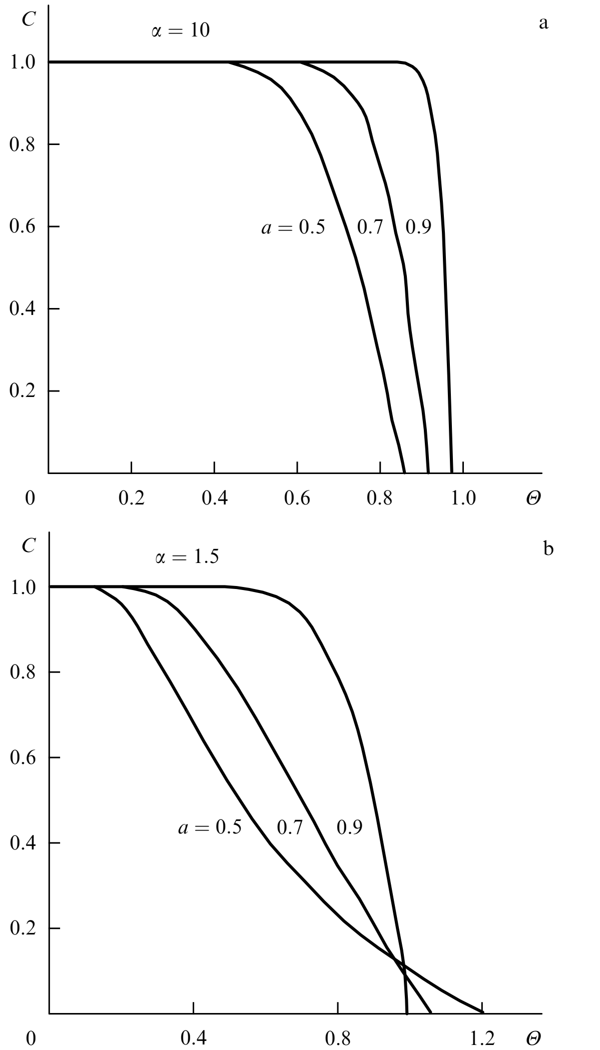

Figure 3. Density of deterministic condensate as a function of the noise intensity Θ for different values of the exponent a and parameter α.

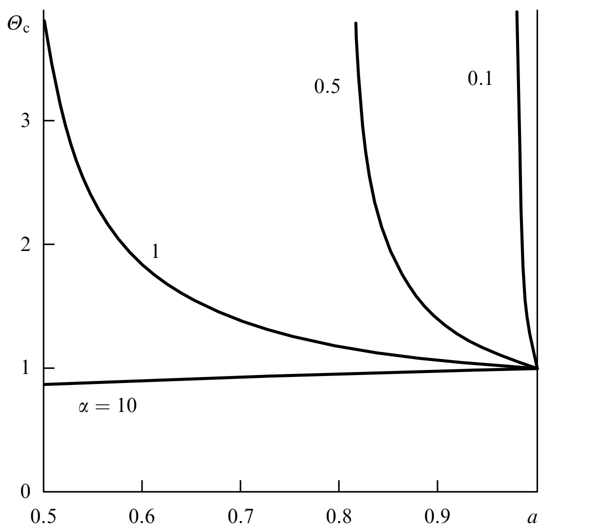

Figure 4. Critical noise intensity $\Theta_{\mathrm{c}} = T/T_{\mathrm{c}}$ as a function of exponent a.

so that for $T > T^0$ we have $L_2 = \infty$. Thus, in the limit of $a = 1$ for the multiplicative function exponent the deterministic regime is not manifested at all. However, while for $T < T^0$ this regime is not attainable even in an infinite time, for $T > T^0$ the condensate will appear at $t \to \infty$ in the limit of $x = 0$.

As follows from Fig. 3, as the exponent of the multiplicative function (3.34) grows up to its limiting value of $a = 1$, the density of condensate $C(\Theta)$ as a function of the noise intensity $\Theta = T/T_c$ follows a stepwise pattern: below the critical value T_c we have $C = 1$, and for overcritical values $(T > T_c)$ we obtain $C = 0$. Such behaviour corresponds to the well-known Verhulst model [14]. The stepwise form of $C(\Theta)$ is ensured by the unlimited growth of the factor $x^{2\delta}/(2\delta)$, $\delta \equiv 1 - a > 0$, which occurs in the first term of the potential (3.36), in the limit of $\delta \to 0$. At $T < T_c$, we have $A < 0$, and the relevant first term in the exponential in (3.69) is positive. As a result, we get $Z' \gg 1$, whence it follows that, in accordance with Eqn (3.68), $C = 1$. In the opposite case of $T > T_c$ both terms in the exponential in Eqn (3.69) are negative, so $Z' \ll 1$ and, hence, $C = 0$. Observe that when $a = 1$ exactly, and the potential $U(x)$ has the logarithmic form (3.42), the step in $C(\Theta)$ is reversed: $C = 0$ at $\Theta < 1$, and $C = 1$ at $\Theta > 1$. This is evidently associated with the change of sign upon transfer from the limiting dependence $x^{2\delta}/(2\delta) > 0$, $\delta \to +0$ to the logarithmic dependence $\ln x < 0$ at $\delta = 0$ $(x \ll 1)$. As the exponent $a > 1$ grows, the inverted step in $C(\Theta)$ is gradually smeared.

3.3 Fractal nature of phase space

Let us represent the resulting pattern by the function $x(t)$, which describes the time dependence of the particle coordinate in the course of generalized diffusion [4]. Since, according to Eqn (3.3), the diffusion coefficient reduces to $(T/2)g^2$, we obtain $x^2 = T(g(x))^2 t$, whence for $a(t) \equiv \langle x^2(t) \rangle^{1/2}$ we get $a(t) = (2Tt)^{1/2(1-a)}$. At the same time, the process of generalized diffusion is described by the Hurst relation $a(t) \propto t^H$, where H is reduced to the Hölder index, which gives the highest order of derivative of a nonanalytical function $x(t)$ [33]. In this way, we find the linkage†

$$H^{-1} = 2(1 - a) \tag{3.73}$$

between the Hölder index H and the parameter a in our theory. According to Eqn (3.73), the process $a(t) \propto t^{1/2}$ is realized in the case of additive noise $(a = 0)$ and it corresponds to common diffusion. The function $x(t)$ is not even once differentiable. Although the Hölder index H grows with increasing a, this situation continues up to $a = 1/2$. In the range $1/2 < a < 3/4$ we have $1 < H < 2$. The function $x(t)$ itself is then smooth, but its first derivative is already nonanalytical. It is easy to see that for an arbitrary interval $(2n - 1)/2n < a < (4n - 1)/2n$ all derivatives up to and including the nth-order ones are smooth functions. Since the deterministic condensate occurs at $1/2 < a < 1$, this implies that here the stochastic function $x(t)$ is smooth to the extent that a finite number of degrees of freedom degenerates into the constant $x(t) = 0$. Conversely, at $0 < a < 1/2$ the nonanalytical function $x(t)$ is so complicated that the stochastic process gets 'entangled' with it, and the deterministic condensate is not formed.

These features of the function $x(t)$ indicate that its graph has a fractal appearance, like the graph of the Weierstrass function [4]. As we know, such functions are characterized by their fractal dimension [4, 33]. Observe, however, that since the stochastic variable x and the time t are measured on different scales, the graph of $x(t)$ represents a self-affine rather than self-similar set. Unlike the latter, a self-affine set is characterized by three fractal dimensions rather than one: global $D = 1$, local $D = 2 - H$, and inner $D = H^{-1}$ [33]. To visualize these numbers, consider a flat filament aligned with the graph of $x(t)$. Then the global dimension features the dimension of the domain of a new phase precipitated on the filament. The local dimension characterizes the process of adsorption of charged particles, and the inner dimension defines the length of the filament. Obviously, it is the latter that reflects the nature of the stochastic process. Using Eqn (3.73), we get the following relationship for this dimension:

$$D = 2(1 - a)\,. \tag{3.74}$$

At $0 < a < 1/2$ we have $2 > D > 1$, and the graph of $x(t)$ is a geometrical object intermediate between a line and a plane. This implies that the domain of definition of the distribution function $P(x, t)$ in Eqn (3.57) is extended to such an extent that the condensate component $C\delta(x)$ does not emerge [$C = 0$ in Eqn (3.68)]. At $1/2 < a < 1$, equation (3.74) gives us $1 > D > 0$, and the domain of definition of $P(x, t)$ is intermediate between a line and a point. Such depletion of the stochastic process causes freezing of the system in the deterministic condensate $x(t) = 0$.

It is obvious that the primal reason for such behaviour of the stochastic system is the fractal nature of the function $W(x, y)$ entering the master equation (2.76). The properties of self-affinity allow us to find the form of this function in the limits of $x, y \to 0$. Indeed, taking advantage of the fact that the sought-for function is homogeneous for the fractal domain of definition [33], we may write

$$W(x, y) = x^b \varphi(u)\,, \qquad u \equiv \frac{y}{x}\,; \tag{3.75}$$

$$W(x, y) = y^c \psi(v)\,, \qquad v \equiv \frac{x}{y}\,, \tag{3.76}$$

where b, c are the exponents to be found. Since these functions tend to the asymptotes $W(x, y) \sim x^b$ at $x \to 0$, and $W(x, y) \sim y^c$ at $y \to 0$, we may conclude that the functions $\varphi(u)$, $\psi(v)$ tend to constant values at infinity. In this respect, expression (3.75) corresponds to the domain $x \ll y$ $(u \gg 1)$, and (3.76) to $y \ll x$ $(v \gg 1)$.

In order to find the exponents b, c, we use the asymptotic approximation $P(x) \propto x^{D-2}$ in the conditions of the detailed balance (2.78). Then for $x \ll y \ll 1$, $y \ll x \ll 1$ the latter brings us to the functional equations

$$x^{D-2} x^b \varphi(u) = y^{D-2} x^c \psi(u)\,, \qquad u \gg 1\,; \tag{3.77}$$

$$x^{D-2} y^c \psi(v) = y^{D-2} y^b \varphi(v)\,, \qquad v \gg 1\,. \tag{3.78}$$

† Because of the importance of Eqn (3.73), we shall quote a different way of getting this result, based on transfer (2.48) to variable y which exhibits additive noise. By definition, $y \propto t^{1/2}$ and the substitution of Eqn (3.34) into (2.48) yields $y \propto x^{1-a}$. Thus, $x \propto t^{1/2(1-a)}$ and a comparison with the definition $x \sim t^H$ brings us to Eqn (3.73).

Hence, in the limits of $x, y \to 0$, we get

$$c = b + D - 2\,, \tag{3.79}$$

$$\varphi(u) = y^{D-2}\psi(u)\,, \tag{3.80}$$

$$\varphi(v) = x^{D-2}\psi(v)\,. \tag{3.81}$$

In order to find b, we substitute Eqn (3.75) into the definition of the multiplicative function (2.83), getting

$$b = -(1 + D) \tag{3.82}$$

and the normalization condition

$$\int_{-\infty}^{\infty} u^2 \varphi(1 - u)\,\mathrm{d}u = 2T\,. \tag{3.83}$$

As a result, the relations of similarity (3.75), (3.76) assume the final form

$$W(x, y) = x^{-(1+D)}\varphi\left(\frac{y}{x}\right) \qquad \text{at} \qquad x \ll y\,, \tag{3.84}$$

$$W(x, y) = y^{-3}\psi\left(\frac{x}{y}\right) \qquad \text{at} \qquad y \ll x\,. \tag{3.85}$$

Obviously, both for $x \ll y$, when $W \sim x^{-(1+D)}$, and for $x \gg y$, when $W \sim y^{-3}$, the function $W(x, y)$ is singular. The drift and diffusion components are asymptotically represented by $f_{\mathrm{s}} \propto x^{1-D}$ and $g \propto x^{1-D/2}$, respectively. As ought to be expected, with increasing parameter a (the fractal dimension D decreases) these singularities are weakened, which causes the appearance of the deterministic condensate at $a > 1/2$ ($D < 1$).

The foregoing fractal analysis reveals that the power-law approximation (3.34) of the multiplicative function $g(x, t)$ reflects its homogeneity as expressed by the equation [cf. Eqn (2.105)]

$$g(x, t) = a^{1-D/2} G(y)\,, \qquad y = \frac{x}{a(t)}\,, \tag{3.86}$$

where $a(t)$ is the time-dependent characteristic scale of the stochastic variable, and $G(y) \to 2^{1/2}$ at $y \to \infty$. In this way, the applicability of the results obtained in Section 3.2 is restricted to those systems for which the domain of definition of the stochastic variable represents a self-similar set characterized by the inner fractal dimension D [33].

It would be interesting to note that in the limit of $x \to 0$ the singular force $f_{\mathrm{s}} \propto x^{1-D}$ is of the same nature as the fictitious force $h = 2\lambda(2 - D)Tx^{1-D}$ in the Fokker–Planck equation (3.2)–(3.4). Since the primal cause of both these forces is the fractal nature of the phase space [for $h(x)$ this can be seen from the derivation of the Fokker–Planck equation in Section 2.4.3], one may assume that these forces coincide. Such a coincidence, however, only relates to the form of the dependence on the stochastic variable x. As far as the dependence on the noise intensity T is concerned, from definitions (2.82), (2.83) it immediately follows that, being the moment of the first order, the force $f \propto T^{1/2}$ is proportional to the noise amplitude, while the force $h \propto T$ is proportional to the noise intensity. In addition, the singular force f_{s} does not involve the arbitrary parameter λ, which determines the fictitious force $h \propto \lambda$ and thus the choice of calculus.

The singular force deduced is governed by the dependence of the multiplicative function $g(x)$ on the stochastic variable. An even higher degree of singularity is displayed by the force $f = -\partial U/\partial x \propto x^{D-1}$, which is conditioned by the functional form

$$U(x) = \frac{A}{D}\,x^D + \frac{B}{2 + D}\,x^{D+2} \tag{3.87}$$

of the synergetic potential (3.36). However, whereas the force $f_{\mathrm{s}}(x)$ always points towards $x = 0$, the same applies to the component $f(x)$ only when the noise intensity exceeds the critical value of T_{c}. Then $A > 0$ and the function $U(x)$ has a minimum at $x = 0$, where the system is locked. Such situation corresponds to the formation of a deterministic condensate, whose density is given by Eqn (3.68). At $T < T_{\mathrm{c}}$, we get $A < 0$, and the force $f(x)$ changes its direction towards the point $x^0 = (-A/B)^{1/2}$. In this case the appearance of deterministic condensate is wholly caused by the action of the force $f_{\mathrm{s}}(x)$, which assumes the most singular nature in the neighbourhood of the point $x = 0$.

4. Symmetry and ergodicity breaking in stochastic systems with interparticle interaction

We know that the principal characteristic of a phase transition in the common sense is the order parameter $\eta = \langle x \rangle$, which is the mean value of the fluctuating quantity $x(t)$ (for example, the spin variable in the case of magnetics) [28]. The nonzero moment η implies that the distribution function $P(x)$ is asymmetrical with respect to the replacement $-x \to x$, which leads to the symmetry breaking inherent in phase transitions. The phase space then displays a region of most preferred values of the microscopic variable x. Obviously, a more radical restructuring is associated with the loss of ergodicity [50]: the phase space then exhibits forbidden regions, where $P(x) = 0$, or condensate regions, where the probability density $P(x)$ is much greater than at other points x (see Section 3.2.2). A most vivid example of such points is the discrete set of atomic coordinates in a vitrified liquid. A salient feature of nonergodic systems is that the mean over the distribution $P(x)$ is not equivalent to the average over the time t [50]. As follows from Section 3.2.1, systems with noise-induced transitions do not exhibit symmetry or ergodicity breaking, because the distribution $P(x)$ is modified more slightly in the course of transformation: its maxima appears at $x \neq 0$.

As a rule, the studies of stochastic systems disregard the interaction between particles, which is the cause of the conventional phase transition. Accordingly, the question is whether its inclusion could lead to a symmetry breaking? The authors of Ref. [51] gave a positive answer to this question. This study, however, was concerned with nonsingular multiplicative noise, whose intensity is finite for all values of x. In Section 5 we shall demonstrate that for a stochastic system simulated with the Lorenz equations, in which each of the three degrees of freedom originally exhibits additive noise, the hierarchy of relaxation times leads not only to the known effect of subordination of modes [5], but also to the transformation of noise inherent in some of these degrees of freedom from additive to multiplicative [52]. In particular, the multiplicative function of the controlling parameter falls off linearly in the limit of $x \to 0$, and the model of Ref. [51] does not work.

In this way, we face the problem of studying a stochastic system with interaction between particles and singular multiplicative noise (3.34). Then the effective diffusion coefficient

$D \propto x^{2a}$ has the form shown in Fig. 5. At $a < 1/2$, the value of D increases very rapidly with x, and the existence of the singular point x where $D = 0$ does not much change the behaviour of the stochastic system as compared with the case of additive noise — as the noise intensity T decreases, the system may exhibit symmetry breaking but will always preserve ergodicity. The situation becomes entirely different for $a > 1/2$, when the effective diffusion coefficient D assumes small values in the neighbourhood of $x = 0$, and grows dramatically at $x \gg 1$. As it turns out, the stationary distribution then always has a symmetrical form with a delta-shaped singularity at $x = 0$. In other words, as the exponent of the multiplicative function grows up to the values $a > 1/2$, the symmetry of the stochastic system is completely restored, but the deterministic condensate is produced. This implies that in spite of the existence of noise, a finite proportion of particles are 'frozen' to their original positions. In other words, the stochastic system acquires memory, inherent to the vitreous state, and also loses ergodicity.

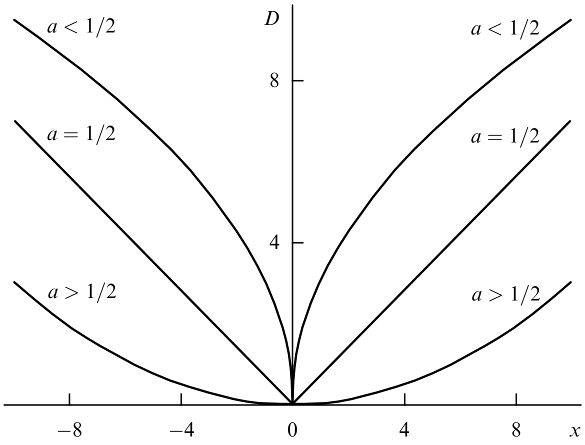

Figure 5. Effective diffusion coefficient D as a function of the stochastic variable x for different values of the exponent a of the multiplicative function.

4.1 Inclusion of the interparticle interaction in the description of a stochastic system

As follows from Section 2, the formalism used in the description of stochastic systems is phenomenological. Because of this, the inclusion of the interaction between particles, which is microscopic by nature, is a nontrivial problem. Our approach will be centred around the most simple model of harmonic oscillations of atoms in a one-dimensional periodic chain [53]. The potential energy of interaction of nearest neighbours with the atom at the site x is given by

$$V_{\text{int}}(x) = \frac{c}{2}\sum_i (x - x_i)^2 \,, \tag{4.1}$$

where c is the stiffness of the effective spring, and the summation is carried out over the locations x_i of nearest neighbours. In the mean field approximation, we replace x_i by the average value $\langle x_i \rangle \equiv \eta$, and the force of interaction $f_{\text{int}} = -\partial V_{\text{int}}/\partial x$ then becomes

$$f_{\text{int}} = -w(x - \eta) \,, \tag{4.2}$$

where we have introduced the characteristic value of the interaction energy $w = cz$, z being the number of nearest neighbours. The model considerations used in deriving Eqn (4.2) are intuitive. In form, Eqn (4.2) coincides with the expression proposed in Ref. [51]; we shall use it as the simplest possible approximation, regarding w as a parameter of the theory.

This force of interaction f_{int} must be taken into account along with the force of self-action f in Eqns (2.96)–(2.98), which determine the stationary distribution of the stochastic variable. As a result, this distribution is given by the following expressions:

$$P = Z^{-1} \exp\left[-\frac{U_{\text{ef}}(x)}{T}\right] , \tag{4.3}$$

$$U_{\text{ef}}(x) = 2aT \ln x + \widetilde{U}(x) \,, \tag{4.4}$$

$$\widetilde{U} = U(x) + U_{\text{int}}(x) \,, \tag{4.5}$$

$$U(x) = \frac{A}{2(1-a)}\, x^{2(1-a)} + \frac{B}{2(2-a)}\, x^{2(2-a)} , \qquad A = \alpha(T - T_{c0}) \,, \tag{4.6}$$

$$U_{\text{int}}(x) = w\left(\frac{x^{2(1-a)}}{2(1-a)} - \eta\,\frac{x^{1-2a}}{1-2a}\right) . \tag{4.7}$$

Here Z is the normalization constant as given by the condition (2.92); the multiplicative function has the same form as Eqn (3.34) in Section 3. The logarithmic term in the effective potential (4.4) is due to the multiplicative nature of the noise, component (4.6) is the renormalized Landau potential, and contribution of (4.7) is due to interaction. A comparison between Eqns (4.6) and (4.7) reveals that the interaction w renormalizes the critical temperature $T_c = T_{c0} - w/\alpha$ in the Landau expansion. If the order parameter is $\eta \neq 0$, the interaction also contributes a term of the lowest order in x.

According to Eqns (4.4)–(4.7), the shape of the distribution (4.3) mainly depends on the value of a, which determines the magnitude of a singular contribution to the effective potential (4.4): at $x \to 0$, the multiplicative function leads to the singularity $P \propto g^{-2}(x) \propto x^{-2a}$. In the range $0 < a < 1/2$, this divergence is integrable, and does not have any dramatic effect. In addition, the last term in Eqn (4.7) is decreasing, and the potential $\widetilde{U}(x)$ is not singular. At $a > 1/2$, the divergence of the distribution $P(x)$ becomes nonintegrable, and the last term in $\widetilde{U}(x)$ diverges. Because of this, the range of large values of a calls for separate treatment.

4.2 Theory of a stochastic system with broken symmetry

Let us first consider the case of $a < 1/2$. In accordance with the normalization condition (2.92), the partition function has the form

$$Z = \int_{-\infty}^{\infty} x^{-2a}\widetilde{P}(x) P_\eta(x)\,\mathrm{d}x \,, \tag{4.8}$$

$$\widetilde{P}(x) \equiv \exp\left\{-\lambda x^{2(1-a)} - \mu x^{2(2-a)}\right\} , \tag{4.9}$$

$$P_\eta(x) \equiv \exp\left(v\eta x^{1-2a}\right) , \tag{4.10}$$

where $\lambda = \alpha(1 - \Theta^{-1})/\left[2(1 - \dot{a})\right]$, $\mu = B/\left[2(2-a)\Theta T_c\right]$, $v = w/\left[(1-2a)\Theta T_c\right]$, and the temperature $\Theta = T/T_c$ is measured in the units of renormalized critical temperature $T_c = T_{c0} - w/\alpha$. Observe that the first two terms in the integrand in Eqn (4.8) are symmetrical with respect to the replacement $-x \to x$, while the term $P_\eta(x)$, associated with the long-range order $\eta \neq 0$, displays asymmetry, which is responsible for the symmetry breaking in the course of ordering. The quantity $\eta = \langle x \rangle$ is given by the condition of self-consistency

$$\eta = Z^{-1} \int_{-\infty}^{\infty} x^{1-2a} P_\eta(x) \widetilde{P}(x)\, \mathrm{d}x\,. \tag{4.11}$$

Since at $\eta = 0$ we have $P_\eta(x) = 1$, the integrand in Eqn (4.11) is antisymmetrical, and the integral is identically equal to zero. Because of this, condition (4.11) always admits the root $\eta = 0$, which corresponds to the disordered phase. As for the ordinary phase transitions, the nonzero solution arises when the graphs of the right-hand and left-hand sides of Eqn (4.11) touch. Evidently, this condition defines the point of phase transition, or the so-called phase diagram of the system.

To construct the latter, we differentiate both sides of Eqn (4.11) with respect to η, and let η tend to zero:

$$1 = \frac{\partial Z^{-1}}{\partial \eta} \int_{-\infty}^{+\infty} x^{1-2a} P_\eta(x) \widetilde{P}(x)\, \mathrm{d}x \bigg|_{\eta=0} + Z^{-1} \frac{\partial}{\partial \eta} \int_{-\infty}^{+\infty} x^{1-2a} P_\eta(x) \widetilde{P}(x)\, \mathrm{d}x \bigg|_{\eta=0}. \tag{4.12}$$

According to Eqns (4.8), (4.10), in the first term we have

$$\frac{\partial Z^{-1}}{\partial \eta}\bigg|_{\eta=0} = -vZ^{-2} \int_{-\infty}^{+\infty} x^{1-4a} \widetilde{P}(x)\, \mathrm{d}x\,. \tag{4.13}$$

This expression reduces to zero, because the first multiplier in the integrand is antisymmetrical, and the second, according to Eqn (4.9), symmetrical with respect to replacement of x with $-x$. As a result, condition (4.12) becomes

$$1 = \frac{v}{Z(0)} \int_{-\infty}^{\infty} x^{2(1-2a)} \widetilde{P}(x)\, \mathrm{d}x\,, \tag{4.14}$$

where $Z(0) \equiv Z(\eta = 0)$. Since the integral on the right-hand side is positive, we have $v > 0$, and the parameter of interaction $w = (1 - 2a)\Theta T_c v$ is also positive.

Numerical analysis of Eqns (4.8), (4.11), (4.14) is facilitated by reducing the integration over the entire axis $-\infty < x < \infty$ to integration over the positive semiaxis $0 \leqslant x < \infty$. With this purpose we represent the integral I on the right-hand side of Eqn (4.11) as a sum of

$$I_+ \equiv \int_0^{\infty} x^{1-2a} \widetilde{P}(x) P_\eta(x)\, \mathrm{d}x\,, \tag{4.15}$$

$$I_- \equiv \int_{-\infty}^{0} x^{1-2a} \widetilde{P}(x) P_\eta(x)\, \mathrm{d}x\,. \tag{4.16}$$

Replacing x by $-x$ in Eqn (4.16), and taking into account the evenness of the function $\widetilde{P}(x)$, we find

$$I_- = -\int_0^{\infty} x^{1-2a} \widetilde{P}(x) P_\eta(-x)\, \mathrm{d}x\,. \tag{4.17}$$

Since, according to Eqn (4.10), $P_\eta(-x) = \big(P_\eta(x)\big)^{-1}$, then for the total integral $I = I_+ + I_-$ one obtains

$$I = \int_0^{\infty} x^{1-2a} \widetilde{P}(x) \left[P_\eta(x) - \frac{1}{P_\eta(x)} \right] \mathrm{d}x\,. \tag{4.18}$$

Similarly, for the normalization constant Z in Eqn (4.8) we have

$$Z = \int_0^{\infty} x^{-2a} \widetilde{P}(x) \left[P_\eta(x) + \frac{1}{P_\eta(x)} \right] \mathrm{d}x\,. \tag{4.19}$$

With due account for Eqns (4.10), (4.18), (4.19), equations (4.8), (4.11), (4.14) take on the form

$$Z = 2 \int_0^{\infty} x^{-2a} \widetilde{P}(x) \cosh\big(v\eta x^{1-2a}\big)\, \mathrm{d}x\,, \tag{4.20}$$

$$\eta = 2Z^{-1} \int_0^{\infty} x^{1-2a} \widetilde{P}(x) \sinh\big(v\eta x^{1-2a}\big)\, \mathrm{d}x\,, \tag{4.21}$$

$$Z(0) = 2v \int_0^{\infty} x^{2(1-2a)} \widetilde{P}(x)\, \mathrm{d}x\,. \tag{4.22}$$

It would be interesting to compare these expressions with the standard Ising model, in which the variable x can only assume the values of ± 1, and the noise is additive [28]. In the context of our approach this means that the distribution is $\widetilde{P}(x) = \delta(x-1) + \delta(x+1)$, and the multiplicative exponent $a = 0$ (therefore, the parameter $v = w/T$). Then expressions (4.20), (4.21) assume the standard form [28]

$$Z = 2\cosh\left(\frac{w\eta}{T}\right), \qquad \eta = \tanh\left(\frac{w\eta}{T}\right). \tag{4.23}$$

Thus, our model of phase transition is an extension of the Ising model. The extension consists in that the δ-shaped distribution function $\widetilde{P}(x)$ is smeared over the entire x-axis, and the noise becomes multiplicative.

Numerical solutions of Eqn (4.21) for different values of a are plotted in Fig. 6. Observe that in the case of additive noise ($a = 0$) the temperature dependence of the order parameter $\eta(\Theta)$ has the conventional monotonically decreasing form [28]. At $a > 0$ it exhibits irregular kinks, which increase in number as a increases. Near the limiting value of $a = 1/2$, on approaching the phase transition temperature, the curve exhibits jumps of the order parameter, followed by rather extensive tails of low values of η. These irregularities of $\eta(\Theta)$ are not related to the accuracy of numerical solution, but are

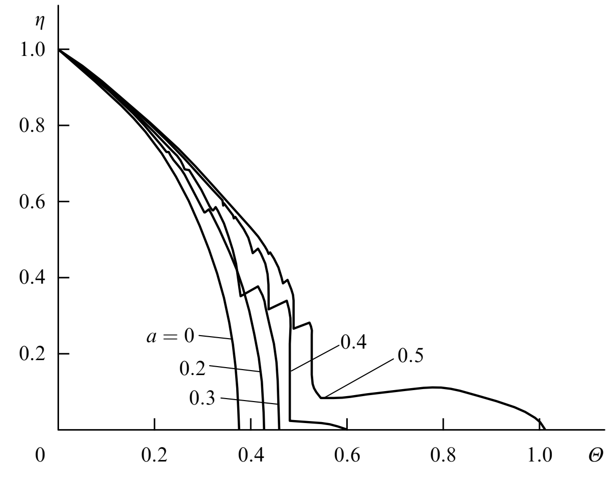

Figure 6. Temperature dependence of the long-range order parameter for different values of the exponent a.

due to the multiplicative nature of the noise. Its physical background will be discussed at the end of this section.

Figure 7 shows the phase diagram of the system based on Eqn (4.22). In this diagram, the phase transition temperature Θ_0 is plotted as a function of the exponent a and the parameter of interatomic interaction w. As follows from the function $\Theta_0(a)$ (curve *1*), with increasing a the quantity Θ_0 gradually increases from a finite value of Θ_0 at $a = 0$ to infinity at $a = 1/2$. At small a, the function $\Theta_0(w)$ (curve *2*) also grows monotonically. However, the tails of the function $\eta(\Theta)$, which appear as a increases, give rise to flat (curve *3*) and even nonmonotone (curves *4*, *5*) stretches on the phase diagram $\Theta_0(w)$.

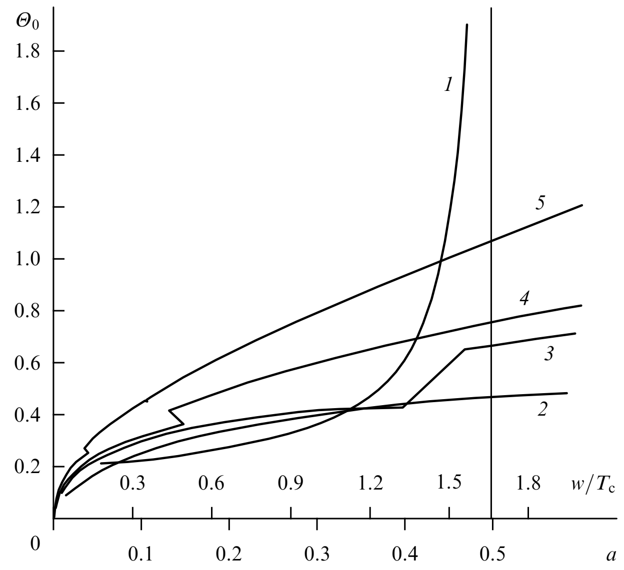

Figure 7. Temperature of symmetry breaking Θ_0 as a function of the exponent a for $w = 0.3$ (curve *1*), and as a function of the interatomic interaction parameter w for $a = 0.3$, 0.4, 0.42, and 0.45 (curves *2*–*5*).

Let us now consider the conditions of symmetry breaking in accordance with the value of the fractal dimension D. From the analysis carried out above, the long-range order $\eta \neq 0$ is observed when the exponent of the multiplicative function is $0 < a < 1/2$. In Section 4.3 we shall prove that in the range $1/2 < a < 1$ the long-range order disappears ($\eta = 0$). Taking into account the linkage (3.74) between the fractal dimension D and the exponent a, we see that the long-range order is realized when $1 < D < 2$, whereas for $0 < D < 1$ we have $\eta = 0$. Obviously, such behaviour corresponds to the well-known breakdown of long-range order by fluctuations in systems with $D < 1$ [28].

The fractional dimension $1 < D < 2$ also explains the irregularities (see Figs 6 and 7) of the temperature dependence of the order parameter $\eta(\Theta)$ and the phase diagram $\Theta_0(w)$. As indicated above, these irregularities are manifested themselves when the exponent a approaches the value of $1/2$. According to the definition of fractal dimension, this means that as the dimensionality decreases to the critical value of $D = 1$, the phase space narrows down to such an extent that, in spite of the retention of order $\eta \neq 0$, the effect of irregularities becomes large enough to cause such peculiarities. In particular, the high-temperature tails of low values of η in Fig. 6 and the corresponding features on the phase diagram (Fig. 7) are due to the fact that, as the temperature increases, the clusterization of the fractal phase space leads to an abrupt rather than gradual contraction of the region corresponding to the ordered phase.

4.3 Theory of a nonergodic stochastic system

Now let us consider the range of $a > 1/2$, corresponding to a faster growth of the multiplicative noise. Then the factor x^{-2a} leads to divergence of the integral in Eqn (4.20). First let us show that such divergence is responsible for the absence of order over the entire temperature range. With this purpose we introduce the lower limit of integration, $b \to 0$. Then, taking into account that the main contribution comes from the low values of x, for which $\widetilde{P}(x) \simeq 1$, from Eqn (4.20) for $\eta = 0$ we get $Z(0) \sim b^{1-2a}$. A similar estimate for the integral I on the right-hand side of Eqn (4.22) yields $I \sim b^{3-4a}$. As a result, from Eqn (4.22) it follows that $v^{-1} \sim b^{2(1-a)}$, and for $a < 1$ in the limit of $b \to 0$ we find that $v = \infty$. Thus, the graphs of the left-hand and right-hand sides of the self-consistency condition (4.21) will only touch at $\Theta = 0$. When the temperature is finite, the graph of the right-hand side of Eqn (4.21) leaves the point $\eta = 0$ at an angle of $\alpha \sim vb^{2(1-a)}$. Aside from the fact that the condition $\alpha > 0$ would require a negative value of the interaction parameter $w \propto (1 - 2a)v$, in the limit $b \to 0$ this angle is infinitesimally small, and condition (4.21) only holds at the point $\eta = 0$.

We see that, in spite of the existence of interaction (4.2), when the exponent of the multiplicative function (3.34) exceeds the value of $1/2$, the self-consistency condition (4.21) admits only the trivial solution $\eta = 0$, which points to the recovery of symmetry over the entire temperature range. As a result, the picture of noise-induced transition and loss of ergodicity (see Section 3.2) is the same as in the absence of interaction. The only difference consists in the renormalization of the critical temperature

$$T_c = T_{c0} - \frac{w}{\alpha}\,, \tag{4.24}$$

where T_{c0} is its bare value in the synergetic potential (4.6), w is the intensity of interaction, and α is the parameter in the Landau expansion (3.32), (3.33).

This pattern of symmetry restoration at $a > 1/2$ can be derived not only from the analysis of equations (4.20)–(4.22), but also directly from the shape of the effective potential $U_{ef}(x)$ as a function of the stochastic variable. According to Eqns (4.4)–(4.7), the first term in Eqn (4.4) defines such a deep minimum at $x = 0$ that the asymmetry caused by the contribution of $U_{int}(x)$ is of no consequence, and the self-consistency condition (4.21) leads to $\eta = 0$. At the same time, the minimum at $x = 0$ causes the appearance of deterministic condensate. One could say therefore that the latter is responsible for the restoration of symmetry. Since the freezing of one-Cth fraction of the number of degrees of freedom at the point $x = 0$ implies that the stochastic system loses its ergodicity, we may state that, as the exponent a increases, the symmetry breaking at $a < 1/2$ is succeeded by the loss of ergodicity at $a > 1/2$.

This phenomenon can be interpreted in plain terms if we go over from the exponent a to the fractal dimension D as defined by equality (3.74). Indeed, at $0 < a < 1/2$ we have $2 > D > 1$, and the graph of $x(t)$ represents a geometrical object intermediate between a line and a plane. Then the domain of definition of the integrand in Eqn (4.21) is so much broadened that the self-consistency condition is satisfied for $\eta \neq 0$. At the same time, the contribution from the δ-shaped term to distribution (3.57) is negligibly small, and the deterministic condensate is not manifested ($C = 0$). With $1/2 < a < 1$ we get $1 > D > 0$, and the domain of definition of the distribution function (4.3) is intermediate between a line and a point. Such depletion of the stochastic process reinforces the contribution from the δ-shaped term to distribution (3.57), which means that the system is frozen in the deterministic condensate $x(t) = 0$. The integral on the right-hand side of Eqn (4.21) is then mainly determined by the range of $x \ll 1$, where its value is negligibly small, and therefore $\eta = 0$.

4.4 Linkage of the fractal nature of the phase space with the behaviour of a stochastic system

Arguments developed above indicate that the representation of multiplicative noise by a power law (3.34), or by a more general expression (3.86) is equivalent to the assumption that the domain of definition of the stochastic system in the phase space is a fractal object whose dimension D is related to the exponent of the multiplicative function a by equality (3.74). In the simplest case, the self-consistent behaviour of the stochastic system is determined by the force of self-action, corresponding to the renormalized Landau potential [see Eqn (4.6)]

$$U(x) = \frac{A}{D}\, x^{D} + \frac{B}{(D+2)}\, x^{D+2}\,, \qquad A = \alpha(T - T_{c0})\,. \quad (4.25)$$

Then over the entire range of $D \in [0, 2]$ only the noise-induced transition is possible, when the stationary distribution function (4.9) exhibits maxima at $x \neq 0$. In the case of low fractal dimension ($D < 1$) there is a nonintegrable singularity at $x = 0$, which causes the appearance of the deterministic condensate leading to the loss of ergodicity. The inclusion of interaction between particles, characterized by the potential [see Eqn (4.7)]

$$U_{\mathrm{int}}(x) = w\left(\frac{x^{D}}{D} - \eta\, \frac{x^{D-1}}{D-1}\right) \qquad (4.26)$$

(here $\eta \equiv \langle x \rangle$ is the long-range order parameter, and w is the intensity of interaction), singles out the range of $D \in (1, 2]$, in which the stochastic system breaks its symmetry with respect to reversal of the sign of variable x. This gives rise to a phase transition similar to that observed in thermodynamic systems [28].

This picture, however, is incomplete because we have not taken into account the singular force due to the fractal nature of the stochastic system. As shown in Section 3.3, this force $f_{\mathrm{s}}(x) \propto T^{1/2} x^{1-D}$ as a function of the stochastic variable is similar to the fictitious force (3.4) associated with the calculus choice, and yet it does not involve the arbitrary parameter $\lambda \in [0, 1]$ and is proportional not to the noise intensity T but rather to its amplitude $T^{1/2}$.

This section is concerned with the analysis of a stochastic system, which includes not only the self-action $U(x)$ and interaction $U_{\mathrm{int}}(x)$ as defined by Eqns (4.25), (4.26) but also the contribution $U_{\mathrm{s}}(x)$ due to the singular force $f_{\mathrm{s}}(x)$ [54]. In order to find this contribution, we rewrite, in dimensional form, equations (2.82), (2.83) expressing the macroscopic characteristics $f(x)$, $g(x)$ in terms of the intensity $W(x, x-y)$ of microscopic transitions. Comparing the dimensional (2.11) and dimensionless (2.21) representations of the stochastic equation, we see that the transfer to dimensional quantities in the Fokker–Planck equation (2.81) gives rise to the kinetic coefficient γ measured in $\mathrm{erg}^{-1}\ \mathrm{s}^{-1}$†, which occurs before the force $f(x)$ and the multiplicative factor $g^2(x)$. Then it is easy to see that definitions (2.82), (2.83) in the dimensional representation may be written as

$$f(x) = -\gamma^{-1} \int_{-\infty}^{\infty} y W(x, x-y)\, \mathrm{d}y\,, \qquad (4.27)$$

$$T g^2(x) = \gamma^{-1} \int_{-\infty}^{\infty} y^2 W(x, x-y)\, \mathrm{d}y\,. \qquad (4.28)$$

Here variables x, y are dimensionless by definition, and the intensity of transitions $W(x, x-y)$ is measured in s^{-1}. Therefore, given the microscopic nature of these transitions, in order to convert the dimensionless similarity relations (3.84), (3.85) into the dimensional form we must single out the factor τ_0^{-1} related to the microscopic time τ_0. As a result, the intensity of transitions in definitions (4.27), (4.28) becomes

$$W(x, x-y) = \tau_0^{-1} x^{-(1+D)} \varphi(1-u)\,, \qquad (4.29)$$

where $u = y/x$ is the new variable, and $\varphi(1-u)$ is the unknown function which decreases with increasing u. The construction of this function is a separate problem, and here we shall use the simplest Gaussian approximation

$$\varphi(1-u) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{u^2}{2\sigma^2}\right). \qquad (4.30)$$

Substituting this expression into Eqn (4.29), and the result into Eqn (4.28), we find the variance

$$\sigma = (2\tau_0 \gamma T)^{1/2}\,. \qquad (4.31)$$

Then Eqns (4.27), (4.29), (4.30) bring us to the following expression for the singular force

$$f_{\mathrm{s}}(x) = -T\left(\frac{T_{\mathrm{s}}}{T}\right)^{1/2} x^{1-D}\,, \qquad (4.32)$$

where we have introduced the characteristic temperature

$$T_{\mathrm{s}} \equiv \frac{4}{\pi \tau_0 \gamma}\,. \qquad (4.33)$$

If, like in Section 2.1, this temperature is measured in units of reciprocal susceptibility χ^{-1}, we get

$$T_{\mathrm{s}}\chi = \frac{4}{\pi}\,\frac{\tau}{\tau_0}\,, \qquad \tau \equiv \frac{\chi}{\gamma}\,. \qquad (4.34)$$

Thus, the temperature T_{s} defines the ratio of the macroscopic relaxation time τ to the microscopic time τ_0.

In order to obtain the expression for the singular contribution $U_{\mathrm{s}}(x)$ to the effective potential [cf. Eqn (4.4)]

$$U_{\mathrm{ef}}(x) = (2-D) T \ln x + U(x) + U_{\mathrm{int}}(x) + U_{\mathrm{s}}(x) \quad (4.35)$$

† Observe that the dimension of the parameter γ^{-1} coincides with the dimension of action S, and so the quantity γ^{-1} may be regarded as the unit of measurement in Eqn (2.44).

we must choose between relations (2.2) and (2.98), which link the force $f(x)$ with the relevant synergetic potential $U(x)$. The decision must be based on the fact that the latter differs from the former by taking into account the multiplicative nature of the noise. It is this circumstance that is responsible for the fact that the exponents in Eqns (4.25), (4.26) for the renormalized potentials of self-action and interparticle interaction involve the fractal dimension D, which in the bare potentials (3.32), (4.1) reduces to $D = 2$. However, expression (4.32) for the singular force takes the multiplicative nature of the noise into account from the outset, since it has been derived on the basis of the similarity relation (4.29) reflecting the fractal nature of the phase space, and therefore the multiplicative behaviour of the system [see Eqn (3.74)]. Because of this, for finding the singular potential one should make use of Eqn (2.2), whence it follows that

$$U_s(x) = \frac{(T_s T)^{1/2}}{2-D}\, x^{2-D}\,. \tag{4.36}$$

Unlike potential components (4.25), (4.26), the growth of fractal dimension here reduces the index of the stochastic variable x. In the case of additive noise ($D = 2$), as ought to be expected, the singular term vanishes.

In the range of $D \leqslant 1$, the stochastic system features an appearance of deterministic condensate whose density is given by the expressions

$$C = 1 - \frac{1}{Z'}\,, \tag{4.37}$$

$$Z' = \int_0^\infty \widetilde{P}(x)\,\mathrm{d}x\,, \tag{4.38}$$

$$\widetilde{P}(x) = \exp\left(-\varkappa x^{2-D} - \lambda x^D - \mu x^{2+D}\right), \tag{4.39}$$

$$\varkappa = \frac{(T_s/T_c)^{1/2}}{2-D}\,\Theta^{1/2}\,,$$
$$\lambda = \frac{\alpha}{D}\left(1-\Theta^{-1}\right), \qquad \mu = \frac{\alpha}{2+D}\,\Theta^{-1}\,, \tag{4.40}$$

which are a generalization of Eqns (3.68), (3.69). The corresponding temperature dependence $C(\Theta)$ is shown in Fig. 8. We see that the increasing ratio T_s/T_c of the characteristic temperatures, which implies a growing contribution of the singular force, leads to gradual reduction of the density of condensate. Observe that distribution (4.39) does not exhibit long-range order ($\eta = 0$). The long-range order appears when the fractal dimension reaches the range of $D \in [1, 2]$, but then the deterministic condensate vanishes (this follows from the fact that at $D > 1$ the distribution function (3.57) becomes normalizable even without the δ-shaped contribution). The self-consistency condition, which determines the long-range order parameter, has the form [cf. Eqns (4.20), (4.21)]

$$\eta = 2Z^{-1}\int_0^\infty x^{D-1}\sinh\left(\nu\eta x^{D-1}\right)\widetilde{P}(x)\,\mathrm{d}x\,, \tag{4.41}$$

$$Z = 2\int_0^\infty x^{D-2}\cosh\left(\nu\eta x^{D-1}\right)\widetilde{P}(x)\,\mathrm{d}x\,, \tag{4.42}$$

$$\nu = \frac{w/T_c}{D-1}\,\Theta^{-1}\,, \tag{4.43}$$

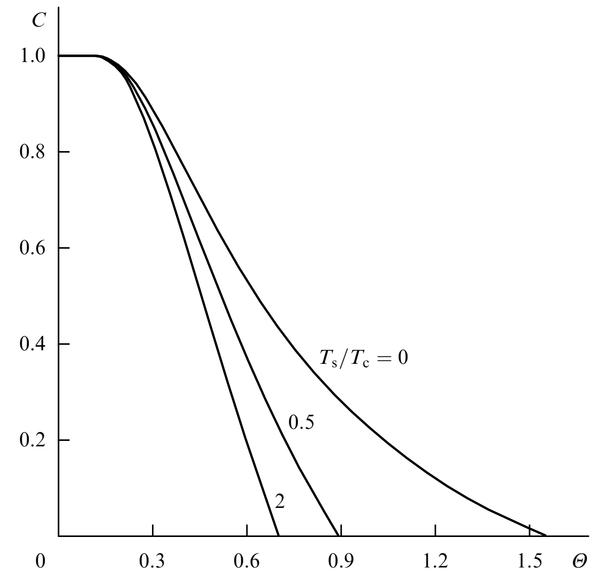

Figure 8. Density of deterministic condensate C as a function of the noise intensity Θ at $a = 0.7$.

where the distribution $\widetilde{P}(x)$ is given by equalities (4.37), (4.38). The phase transition point is given by [cf. Eqn (4.22)]

$$Z(0) = 2\nu\int_0^\infty x^{2(D-1)}\widetilde{P}(x)\,\mathrm{d}x\,, \tag{4.44}$$

where $Z(0) \equiv Z(\eta = 0)$. The temperature dependence of the long-range order parameter and the phase diagram are shown in Figs 9 and 10.

The temperature-dependent functions $C(\Theta)$, $\eta(\Theta)$ and the phase diagram (see Figs 8 – 10) show that the singular force (4.32), whose action is determined by the ratio T_s/T_c, manifests itself in the following manner: the increasing contribution from the singular component lowers the

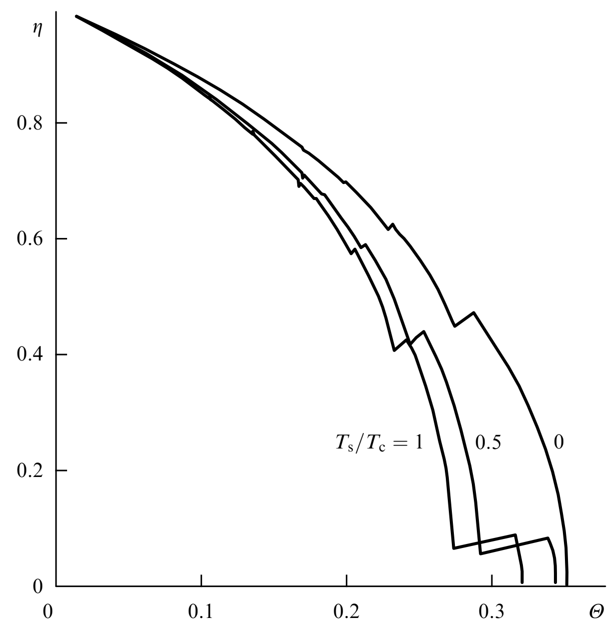

Figure 9. The long-range order parameter η as a function of the noise intensity Θ for different values of T_s/T_c.

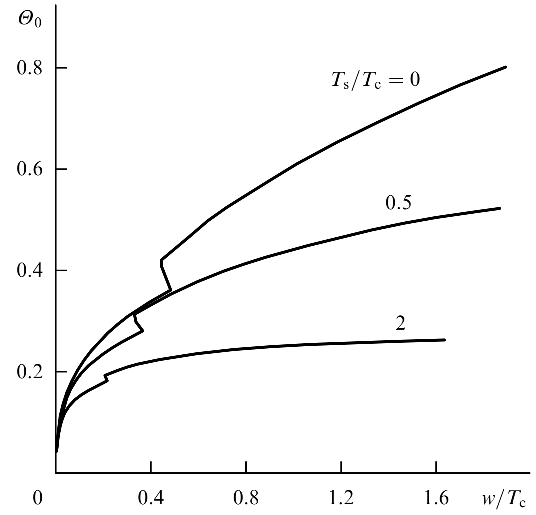

Figure 10. Temperature of symmetry breaking as a function of the parameter of interatomic interaction w for $D = 1.16$.

temperature of formation of deterministic condensate and the threshold of formation of the nonsymmetrical phase ($\eta \neq 0$). Since both these quantities and the singular force (4.32) increase as the fractal dimension D decreases, we may state that this force acts in accordance with the generalized Le Chatelier principle, working to suppress the effects of the fractal structure of the phase space which caused this force in the first place.

5. Effects of noise on the behaviour of a synergetic system

In the preceding sections we have studied those stochastic systems which are characterized by a single hydrodynamic mode parametrically represented by the variable x. We showed that, when the influence of the environment representing the nonhydrodynamic degrees of freedom depends on the state of the stochastic system x, the noise becomes multiplicative, which gives rise to highly nontrivial changes in the behaviour of the system. In particular, the domain of definition of stochastic system in the phase space assumes a fractal nature, because of which the synergetic potential on the one hand is renormalized and on the other acquires a singular term. Recall that the multiplicative noise was represented by a power-law function (3.34).

Now the question that naturally arises is just how realistic our model of multiplicative noise is? Obviously, to answer this question we must initially select several hydrodynamic degrees of freedom rather than one, and consider their self-consistent behaviour with due account for the noise in each of them. If we later single out one of these selected degrees of freedom, we should automatically get the form of the multiplicative function, which will supply the answer to our question [52, 55].

In Section 5.1 we consider the Lorenz stochastic system which differs from the standard synergetic scheme [5] only in that each hydrodynamic mode involves stochastic contributions in addition to the deterministic terms — in a manner similar to the transfer from the Landau–Khalatnikov equation (2.3) to the Langevin equation (2.4). The analysis of the synergetic scheme so modified reveals that even when all degrees of freedom feature additive noise from the start, their hierarchical subordination (giving preference to a single mode) transforms the additive noise of the preferred mode into multiplicative noise. In addition, the use of the synergetic scheme offers an independent method for reproducing the results obtained for the one-parameter system (see Section 3.2).

5.1 Lorenz stochastic system

As we know, synergetics is the extension of thermodynamics of phase transitions [28] to open systems, which may exhibit self-organization as the growing external forces push the system away from equilibrium [5, 10, 11, 14, 56]. According to the Ruelle–Takens theorem, a nontrivial pattern of self-organization, which involves strange attractors, is observed when the number of selected degrees of freedom is not less than three [57]. The most popular three-parameter scheme is the Lorenz system [58]. Initially developed for simulation of atmospheric phenomena, it was later applied in physics, chemistry, biology, sociology, etc. [5, 10, 11, 14–18]. Recently the concepts of synergetics have been used for explaining the effects of restructuring of condensed matter [59, 60].

Embarking on the construction of the Lorenz scheme, we introduce the quantities η, h, S, which are commonly known as the order parameter, the conjugate field, and the controlling parameter, respectively. For the sake of simplicity we consider a spatially homogeneous system, for which our task is reduced to finding the time dependences $\eta(t)$, $h(t)$, and $S(t)$. With this purpose we use the phenomenological approach, in which the equations of motion determine the rates $\dot{\eta}$, $\dot{h}$, $\dot{S}$ of change of the quantities η, h, S depending on their values. In writing these equations one must first of all bear in mind that in the autonomous regime the change of all hydrodynamic modes is dissipative. Also important is the Le Chatelier principle: since self-organization is caused by the growth of the controlling parameter S, then the order parameter η and the conjugate field h must vary in such a way as to resist the growth of S. Formally, this circumstance can be regarded as the existence of negative feedback between η and h. Finally, of primary importance is the positive feedback between the order parameter η and the controlling parameter S, which works to increase the conjugate field h. It is this feedback that is the motive power of self-organization.

The Lorenz system takes all these circumstances into account in the most simple way. With fluctuation terms included, it has the form

$$\dot{\eta} = -\frac{\eta}{\tau_\eta} + \gamma h + 2^{1/2}\sigma_\eta \xi\,, \tag{5.1}$$

$$\dot{h} = -\frac{h}{\tau_h} + g_h \eta S + 2^{1/2}\sigma_h \xi\,, \tag{5.2}$$

$$\dot{S} = \frac{S_0 - S}{\tau_S} - g_S \eta h + 2^{1/2}\sigma_S \xi\,. \tag{5.3}$$

Here the first terms on the right-hand sides describe the autonomous relaxation of the quantities η, h, S to their respective stationary values $\eta = 0$, $h = 0$, $S = S_0$ with relaxation times τ_η, τ_h, τ_S; γ is the kinetic coefficient; the positive constants g_h, g_S are the measures of feedback applied to the system; ξ is the delta-correlated stochastic component [see Eqn (2.10)], and σ_η^2, σ_h^2, σ_S^2 are the noise intensities of the relevant variables.

If $\tau_h, \tau_S \ll \tau_\eta$, the subordination principle allows setting $\tau_h \dot{h} = \tau_S \dot{S} = 0$ in Eqns (5.2), (5.3), and yet retaining the stochastic terms. Then the last two equations express the conjugate field and the controlling parameter in terms of the order parameter:

$$h = \left(1 + \frac{\eta^2}{\eta_m^2}\right)^{-1} \left[A_h \eta (S_0 + 2^{1/2}\sigma_S \tau_S \xi) + 2^{1/2}\sigma_h \tau_h \xi\right], \quad (5.4)$$

$$S = S_0 - A_S \eta h + 2^{1/2}\sigma_S \tau_S \xi, \quad (5.5)$$

where we inserted the quantities

$$A_h \equiv g_h \tau_h, \qquad A_S \equiv g_S \tau_S, \qquad \eta_m^{-2} \equiv A_S A_h. \quad (5.6)$$

Substituting Eqn (5.4) into (5.1), we get the stochastic differential equation in Ito's form

$$\tau_\eta \dot{\eta} = -\eta + \frac{S_c^{-1}\eta(S_0 + 2^{1/2}\sigma_S\tau_S\xi) + 2^{1/2}\sigma_h\tau_h\gamma\tau_h\xi}{1 + \eta^2/\eta_m^2}, \quad (5.7)$$

where

$$S_c^{-1} = \gamma\tau_\eta g_h \tau_h. \quad (5.8)$$

Separating the deterministic and the stochastic components, we bring Eqn (5.7) to the canonical form [cf. Eqn (2.11)]

$$\dot{\eta} = -\frac{\partial V}{\partial \eta} + \left[\sigma_S g_S(\eta) + \sigma_h g_h(\eta)\right]\xi + 2^{1/2}\sigma_\eta \xi, \quad (5.9)$$

where the time t is measured in units of τ_η, the order parameter on the scale of η_m, the noise intensities σ_η, σ_h and σ_S in τ_η^{-1}, $(\tau_\eta\tau_h\gamma)^{-1}$ and S_c/τ_S, respectively. The synergetic potential $V(\eta)$ takes the form

$$V = \frac{1}{2}\left[\eta^2 - \Theta \ln(1+\eta^2)\right], \qquad \Theta \equiv \frac{S_0}{S_c}. \quad (5.10)$$

Expanding this expression in powers of $\eta^2 \ll 1$, we see that it reduces to the Landau expansion (3.32) with the parameters $A = 1 - \Theta \ll 1$, $B = \Theta \approx 1$. The multiplicative functions $g_S(\eta)$, $g_h(\eta)$ have the form

$$g_S(\eta) = \eta g_h(\eta) = 2^{1/2}\eta\left(1+\eta^2\right)^{-1}, \quad (5.11)$$

$$g_h(\eta) = 2^{1/2}\left(1+\eta^2\right)^{-1}. \quad (5.12)$$

From the above treatment we see that in the context of the adiabatic approximation (τ_h, $\tau_S \ll \tau_\eta$), the synergetic system with additive noise reduces to a one-parameter stochastic system with multiplicative noise. It can be studied using the methods described in Section 2, if by the stochastic variable x we understand the order parameter η, whose relaxation time is the largest. The synergetic potential (5.10) exhibits a minimum at the point

$$\eta_0 = \pm(\Theta - 1)^{1/2}, \quad (5.13)$$

which, as distinct from the scheme used above, pertains to large rather than small stationary values $\Theta = S_0/S_c$ of the controlling parameter.

The stochastic component in Eqn (5.9)

$$\zeta(t) = \left[\sigma_S g_S(\eta) + \sigma_h g_h(\eta)\right]\xi(t) + 2^{1/2}\sigma_\eta \xi(t) \quad (5.14)$$

comprises the multiplicative noises of the controlling parameter and conjugate field, and the additive component of the order parameter. Observe that the noise increases with η only for the controlling parameter, whose multiplicative function at $\eta^2 \ll 1$ assumes the form

$$g_S(\eta) \simeq 2^{1/2}\eta. \quad (5.15)$$

Comparing this with Eqn (3.34), we see that the exponent $a = 1$ and Eqn (3.74) gives the minimum value $D = 0$ of the fractal dimension. Thus, the domain of definition of the synergetic system in the phase space reduces to an ensemble of isolated points. Obviously, these can be either the points of maxima of the stationary distribution (2.96), or the point $\eta = 0$ corresponding to the deterministic condensate (see Section 3.2). The domain of definition of the synergetic system can be expanded to dimensions $D > 0$ if we assume that the factor η is raised in Eqn (5.2) to the power $1 - D/2$, and in Eqn (5.3) to the power $1 + D$. This implies that the positive feedback therewith is enhanced ($\eta^{1-D/2} > \eta$), and the negative feedback is reduced ($\eta^{1+D} < \eta$). Observe that the exponent of the order parameter η assumes fractional values only in the nonlinear terms of the set of equations (5.1) – (5.3), which are responsible for the feedback. This is quite natural if we recall that the multiplicative nature of the noise is due to the existence of feedback between the medium and the stochastic system (see Section 2.1).

Finally, let us reproduce the main expressions which define the extreme points of the stationary distribution [cf. Eqn (2.96)]

$$P(\eta) = Z^{-1}\exp\left[-U_{ef}(\eta)\right] \quad (5.16)$$

of the synergetic system (5.1) – (5.3). Here, the effective potential [cf. (2.97), (2.98)]

$$U_{ef}(\eta) = \ln g^2(\eta) + 2\int \frac{\partial V/\partial\eta}{g^2(\eta)}\,d\eta \quad (5.17)$$

is determined by the bare synergetic potential (5.10) and the square of the effective multiplicative function

$$g^2(\eta) = 2\sigma_\eta^2 + \sigma_h^2 g_h^2(\eta) + \sigma_S^2 g_S^2(\eta). \quad (5.18)$$

This expression follows from the known property of additivity [39] of squares of variances of independent Gaussian random quantities [see Eqn (2.10)]. Combining expressions (5.10) – (5.12), (5.17), (5.18), we find the explicit form of $U_{ef}(\eta)$, which is too cumbersome to be reproduced here. Much simpler is the equation

$$x^3 - \Theta x^2 - 2\sigma_S^2 x + 4(\sigma_S^2 - \sigma_h^2) = 0, \qquad x \equiv 1 + \eta^2, \quad (5.19)$$

which defines the locations of the maxima of distribution (5.16). According to Eqn (5.19), they are insensitive to changes in the intensity of noise σ_η^2 of the order parameter, and are determined by the stationary value Θ of the controlling parameter and the relative intensities σ_S^2, σ_h^2 of the multiplicative noises.

5.2 Synergetic transition in the case of additive noise

In the simplest case of $\sigma_\eta = \sigma_h = \sigma_S = 0$, equations (5.1) – (5.3) reduce to the classical Lorenz system [58], and the stochastic equation (5.9) assumes the deterministic

Landau – Khalatnikov form

$$\dot{\eta} = -\eta\left(1 - \frac{\Theta}{1+\eta^2}\right). \tag{5.20}$$

Passing to the new variable $x = 1 + \eta^2$, we transform this into

$$\frac{x\,\mathrm{d}x}{(x-1)(x-\Theta)} = -2\,\mathrm{d}t\,, \tag{5.21}$$

whence it immediately follows that

$$\eta^{-2}\left|\eta^2 - (\Theta - 1)\right|^{\Theta} \propto \exp\left[-2(\Theta-1)t\right]. \tag{5.22}$$

Thus, in the subcritical region $\Theta < 1$ the system relaxes into a disordered state $\eta = 0$, and into an ordered state (5.13) in the supercritical region $\Theta > 1$. Such phases are commonly referred to as symmetrical and asymmetrical [28]. The distribution function (5.16) has one central maximum in the former case, and two maxima at points (5.13) in the latter. Observe that, owing to the absence of noise, all these maxima are delta-shaped:

$$P(\eta) \propto \delta\left(\frac{\mathrm{d}V}{\mathrm{d}\eta}\right). \tag{5.23}$$

As the additive noise of the order parameter becomes available ($\sigma_\eta \neq 0$, $\sigma_h = \sigma_S = 0$), according to Eqn (5.19) the stationary states of the system do not change. However, as follows from Eqns (5.16) – (5.18), the delta-shaped peaks in the distribution function are smeared and they become

$$P(\eta) = Z^{-1}\exp\left[-\frac{V(\eta)}{\sigma_\eta^2}\right]. \tag{5.24}$$

Their width depends on the noise intensity of the order parameter.

5.3 Synergetic transition in the case of multiplicative noise

Let us calculate the integral contribution $U(\eta)$ to Eqn (5.17) for the effective potential when $\sigma_\eta = 0$. According to Eqns (5.11), (5.12), (5.18) we have

$$U(\eta) \equiv \int \frac{\eta(1+\eta^2)(1-\Theta+\eta^2)}{\sigma_h^2 + \sigma_S^2\eta^2}\,\mathrm{d}\eta\,. \tag{5.25}$$

Now we introduce the ratio of noise intensities $\alpha \equiv \sigma_h/\sigma_S$ and the new variable $y \equiv \alpha^2 + \eta^2$. Equation (5.25) then becomes

$$\sigma_S^2 U(y) = \frac{y^2}{4} + \left(\frac{1}{2} - \alpha^2\right)y + \frac{1}{2}(1-\alpha^2)(1-\alpha^2-\Theta)\ln y - \frac{\alpha^2}{2}\,. \tag{5.26}$$

Returning to the old variables and taking advantage of the fact that the potential is defined up to an arbitrary constant, which may be incorporated into the normalization constant of distribution (5.16), for the effective synergetic potential we finally get

$$U_{\mathrm{ef}}(\eta) = \frac{1}{2}\left[\frac{\eta^4}{2} + (2-\Theta-\alpha^2)\eta^2 + (1-\alpha^2)(1-\Theta-\alpha^2)\ln(\alpha^2+\eta^2)\right] + \sigma_S^2\ln\left[g_S^2(\eta) + \alpha^2 g_h^2(\eta)\right], \quad \alpha \equiv \frac{\sigma_h}{\sigma_S}\,. \tag{5.27}$$

5.3.1 Inclusion of the stochasticity of the conjugate field.

Expanding Eqn (5.27) in σ_S, we find that

$$\sigma_h^2 U_{\mathrm{ef}}(\eta) = 2^{1/2}3^{-1}\left(g_h(\eta)\right)^{-3} - 2\Theta\left(g_h(\eta)\right)^{-2} + 2\sigma_h^2\ln g_h(\eta) \tag{5.28}$$

up to an insignificant constant. The corresponding distribution function (5.16) has a minimum at $\eta = 0$ if the stationary value Θ of the controlling parameter does not exceed the critical level

$$\Theta_{\mathrm{c}}^h = 1 - 4\sigma_h^2\,, \tag{5.29}$$

whose value decreases with increasing intensity of noise of the conjugate field. In this case the system is in the symmetric state. At $\Theta > \Theta_{\mathrm{c}}^h$, the solution of Eqn (5.19) yields the locations $\eta_+ = -\eta_-$ of the maxima of distribution (5.16) in the asymmetrical phase. The function $\eta_+ = \eta(\Theta, \sigma_h)$ is plotted in Fig. 11. For small values of Θ and σ_h we have

$$\eta_\pm^2 \approx \begin{cases} (4\sigma_h^2)^{1/3} - 1 + \dfrac{\Theta}{3}(1+3^{-1/2})\,, & \Theta \to 0\,; \\ \Theta - 1 + \dfrac{4\sigma_h^2}{\Theta^2}\,, & \sigma_h^2 \to 0\,. \end{cases} \tag{5.30}$$

Equation (5.30) and Fig. 11 point to the occurrence of a type II transition at the critical value $\Theta = \Theta_{\mathrm{c}}^h$. Observe that at $\Theta = 0$ the synergetic transition into the asymmetric state may also take place, when $\sigma_h > \sigma_{\mathrm{c}} = 1/2$. Obviously, this phenomenon belongs to the class of noise-induced transitions.

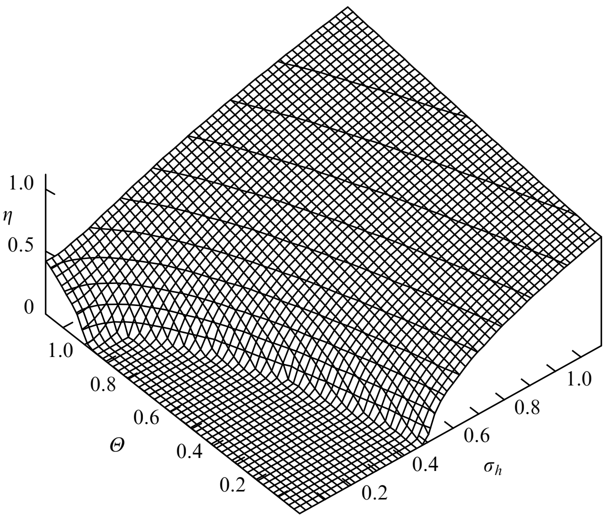

Figure 11. Order parameter η as a function of the controlling parameter Θ and the noise intensity of the conjugate field σ_h.

5.3.2 Inclusion of the stochasticity of the controlling parameter.

Now let us analyze the stationary states of a synergetic system in the presence of the noise of the controlling parameter. As in our previous case, one may demonstrate that the stationary distribution function (5.16) is determined by the effective potential

$$U_{\mathrm{ef}}(\eta) = \frac{\eta^4}{4} + \left(1 - \frac{\Theta}{2}\right)\eta^2 + \left(1 - \Theta + 2\sigma_S^2\right)\ln\eta - 2\sigma_S^2\ln\left(1+\eta^2\right). \tag{5.31}$$

The expression for the minimum of function (5.31) is found from Eqn (5.19) with $\sigma_h = 0$. Let us analyze the solutions of this equation depending on the values of parameters Θ, σ_S. Similarly to Eqn (5.29), the critical value

$$\Theta_c^S = 1 + 2\sigma_S^2 \tag{5.32}$$

limits from above the domain of existence of the zero solution to Eqn (5.19). Other solutions $\eta_\pm$ correspond to the asymmetric phase. Eliminating the root $\eta^2 = 0$, we get the biquadratic equation

$$\eta^4 + (3-\Theta)\eta^2 - (2\Theta + 2\sigma_S^2 - 3) = 0\,, \tag{5.33}$$

which has the roots

$$\eta_\pm^2 = \frac{1}{2}\left[\Theta - 3 + \sqrt{(3-\Theta)^2 + 4(2\Theta - 3 + 2\sigma_S^2)}\right]. \tag{5.34}$$

The magnitude of this solution has its minimum

$$\eta_c^2 = \frac{1}{2}\left[\Theta - 3 + \sqrt{(\Theta+7)(\Theta-1)}\right] \tag{5.35}$$

on the line defined by expression (5.32). At $\Theta < 4/3$, the roots $\pm\eta_c$ are complex, at $\Theta = 4/3$ they become zero, at $\Theta > 4/3$ they are real, and $\eta_+ = -\eta_-$. In this way, the point

$$\Theta = \frac{4}{3}\,, \qquad \sigma_S^2 = \frac{1}{6} \tag{5.36}$$

corresponds to the appearance of roots $\eta_\pm \neq 0$ of Eqn (5.19), corresponding to the asymmetric phase. If condition (5.32) is satisfied, the root $\eta = 0$ corresponds to the minimum of the potential (5.31) at $\Theta < 4/3$, whereas at $\Theta > 4/3$ this root corresponds to the maximum, and the roots $\eta_\pm$ to symmetric minima.

Let us now find the condition of existence of roots $\eta_\pm$. Setting the discriminant of Eqn (5.19) equal to zero, we get the equations

$$\sigma_S^2 = 0\,, \qquad \sigma_S^4 - \sigma_S^2\left[\frac{27}{2}\left(1 - \frac{\Theta}{3}\right) - \frac{\Theta^2}{8}\right] + \frac{\Theta^3}{2} = 0\,, \tag{5.37}$$

the second of which gives

$$2\sigma_S^2 = \left[\frac{27}{2}\left(1 - \frac{\Theta}{3}\right) - \frac{\Theta^2}{8}\right] \pm \left\{\left[\frac{27}{2}\left(1 - \frac{\Theta}{3}\right) - \frac{\Theta^2}{8}\right]^2 - 2\Theta^3\right\}^{1/2}. \tag{5.38}$$

This equation defines a bell-shaped curve $\Theta(\sigma_S)$, which intersects with the horizontal axis at the points $\sigma_S = 0$ and $\sigma_S = (27/2)^{1/2}$, and has a maximum at

$$\Theta = 2\,, \qquad \sigma_S = \sqrt{2}\,. \tag{5.39}$$

It is easy to see that this line touches the curve (5.32) at point (5.36).

The analysis carried out above allows us to construct the phase diagram of the system at hand in the presence of noise of the controlling parameter (Fig. 12). Here, the region S below curve *2* defined by Eqn (5.38) corresponds to the stable symmetric phase ($\eta = 0$), whereas region N above curve *1* defined by Eqn (5.32) corresponds to the asymmetric phase

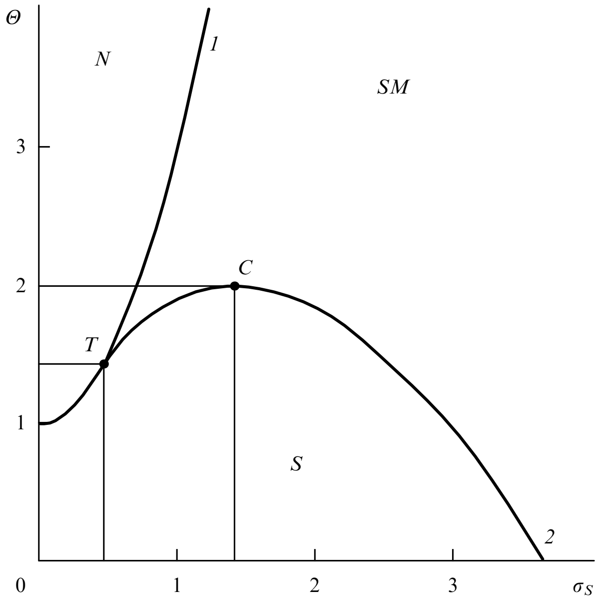

Figure 12. Phase diagram of a system with the noise of the controlling parameter σ_S. Curves *1* and *2* correspond to Eqns (5.32) and (5.38); S denotes the stable symmetric phase, N — asymmetric, SM — stable symmetric and metastable asymmetric, T is the tricritical point, C is the critical point.

($\eta_\pm \neq 0$). The region SM between curves *1* and *2* corresponds to the coexisting stable symmetric and metastable asymmetric phases. Point T where curve *1* intersects with curve *2*, defined by Eqn (5.36), is tricritical, and the point C with the coordinates given by Eqn (5.39) is critical.

Bifurcation diagrams in Fig. 13 depict the behaviour of stationary states of the system depending on the noise intensity σ_S at fixed values of the controlling parameter Θ.

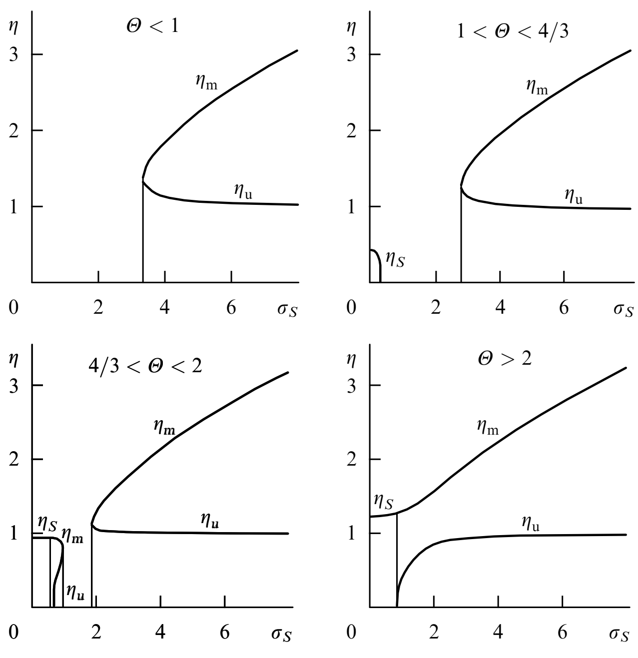

Figure 13. Bifurcation diagrams for stationary values of the order parameter η depending on the noise intensity σ_S and the mean value Θ of the controlling parameter (η_S denotes stable solution, η_m — metastable, η_u — unstable).

On the border of region S at $\sigma_S^2 > 1/6$ we observe a sudden occurrence of two extremes of the potential (5.31), one of which corresponds to the unstable state η_u, and the other to the metastable state η_m.

It is worth noting that the effective synergetic potential (5.31) exhibits a logarithmic singularity at $\eta \to 0$, and so $U_{ef}(\eta) \to -\infty$ below curve (5.32), and $U_{ef}(\eta) \to +\infty$ above it. Let us analyze the behaviour of the system near the singular point $\eta = 0$ according to the scheme described in Section 3.2.2. Similarly to the factorization (3.45), we can represent the distribution function (5.16) as

$$P(\eta) = P_0(\eta)P'(\eta)\,, \tag{5.40}$$

where

$$P_0(\eta) = \frac{Z_0^{-1}\left(1+\eta^2\right)^2}{\eta^2}\,, \qquad P'(\eta) = Z_1^{-1}\exp\left[-\frac{U(\eta)}{\sigma_S^2}\right], \tag{5.41}$$

and the synergetic potential is

$$U(\eta) = \left(1 - \frac{\Theta}{2}\right)\eta^2 + \frac{\eta^4}{4} + (1-\Theta)\ln\eta\,. \tag{5.42}$$

When $\eta \ll 1$, we may use asymptotic approximation and arrive at

$$U(\eta) \simeq (1-\Theta)\ln\eta\,, \qquad P \simeq Z^{-1}\eta^{-(2+(1-\Theta)/\sigma_S^2)}\,. \tag{5.43}$$

Hence for the phase factor (3.17) we get

$$\exp(\alpha) \simeq \frac{\eta^{(1-\Theta)/\sigma_S^2+1}}{(1-\Theta)/\sigma_S^2+1}\,. \tag{5.44}$$

Then the parameter (3.53) takes the form

$$L_1(b,\beta) = \left(\frac{1-\Theta}{\sigma_S^2}+1\right)^{-1}\left[\beta^{(1-\Theta)/\sigma_S^2+1} - b^{(1-\Theta)/\sigma_S^2+1}\right]. \tag{5.45}$$

In the limit $b \to 0$ we have $L_1 = \infty$ in the region restricted from above by the noise intensity

$$\sigma_c^2 = \Theta - 1\,, \tag{5.46}$$

and $L_1 < \infty$ at $\sigma_S > \sigma_c$. For parameter (3.54) we get

$$L_2(b,\beta) = \left(1-\Theta+\sigma_S^2\right)^{-1}\ln\frac{\beta}{b}\,. \tag{5.47}$$

At $\sigma_S > \sigma_c$ we have $L_2 = \infty$. Consequently, in the range $\sigma_S < \sigma_c$ the deterministic condensate does not form even in an infinite time, and for $\sigma_S > \sigma_c$ its appearance is only possible in the limit $t \to \infty$. At $\sigma_h \neq 0$ the condensate disappears, because the multiplicative function at $\eta = 0$ takes on a finite value.

5.3.3 Simultaneous inclusion of the stochasticity of the conjugate field and the controlling parameter. Let us now consider the more general case of two multiplicative noises σ_h and σ_S. The stationary distribution function has the form (5.16), where the effective potential is in the general form (5.27). Unlike Eqn (5.31), it has a finite limit at $\eta \to 0$. Introducing the parameter $a = 1 - \alpha^2$ and the renormalized variables $\widetilde{\sigma} \equiv \sigma_S/a$, $\widetilde{\Theta} \equiv \Theta/a$, $\widetilde{\eta}^2 = (1+\underset{\sim}{\eta}^2)/a - 1$, at $\underset{\sim}{\alpha} < 1$ we may represent Eqn (5.27) in the form $\widetilde{U}_{ef}/\widetilde{\sigma}^2$, where $\widetilde{U}_{ef}$ is found from Eqn (5.31) by replacing σ_S, Θ, η with the renormalized quantities $\widetilde{\sigma}$, $\widetilde{\Theta}$, $\widetilde{\eta}$. Then the action of the noise of the conjugate field is reduced to the renormalization of the minimum value of the order parameter by the quantity $(a^{-1}-1)^{1/2}$, so that the region of divergence $\widetilde{\eta} \approx 0$ becomes inaccessible.

The condition of extremum of the potential (5.27) splits into two equations, one of which is simply $\eta = 0$, and the other is given by Eqn (5.19). The analysis of the latter indicates that the line of existence of the zero solution is defined by an expression which differs from Eqn (5.32) by the added term $-4\sigma_h^2$ on the right-hand side. The tricritical point has the coordinates

$$\Theta = \frac{4}{3}\left(1-\sigma_h^2\right), \qquad \sigma_S^2 = \frac{1}{6}\left(1+8\sigma_h^2\right). \tag{5.48}$$

The phase diagram for fixed values of σ_h is shown in Fig. 14. We see that for $\sigma_h^2 < 1$ it is generally the same as that shown in Fig. 13. At $\sigma_h^2 = 1$, the tricritical point (5.48) occurs on the σ_S axis (Fig. 14b), and at $\sigma_h^2 > 2$ the region of symmetric phase disappears. The main distinction of this situation is that, because the potential (5.27) is finite, the phase diagram exhibits curve *3* corresponding to the coexistence of symmetric and asymmetric phases (binodal). Below this line, the symmetric phase is stable, and the asymmetric phase is

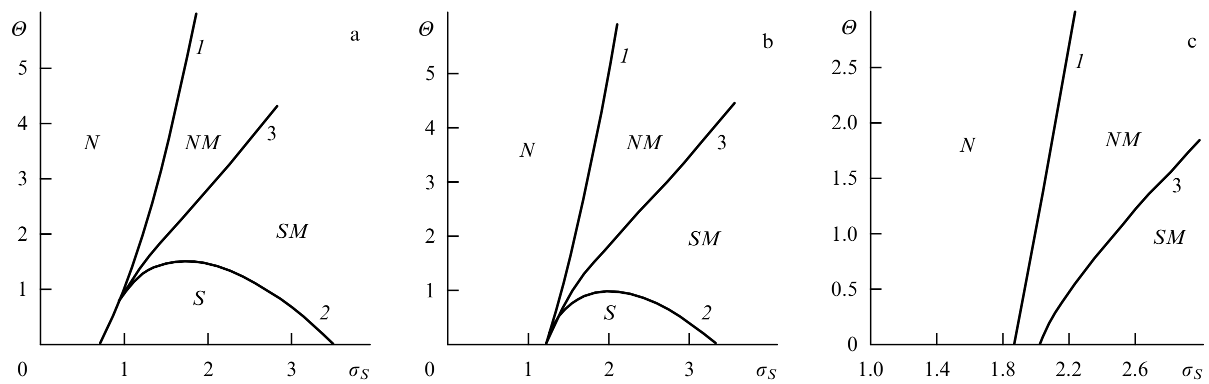

Figure 14. Phase diagrams for fixed values of the noise intensity of the conjugate field: (a) $\sigma_h = 0.5$; (b) $\sigma_h = 1$; (c) $\sigma_h = 2$. Curves *1* and *2* define the boundary of stability of nonsymmetric and symmetric phases, curve *3* corresponds to coexistence of two phases; *N* — nonsymmetric phase, *NM* — nonsymmetric metastable, *S* — symmetric, *M* — metastable, and *SM* — symmetric metastable.

metastable (above this line the situation is reversed). Curve *1* (spinodal) defines the threshold of absolute loss of stability of the symmetric phase. Above this line the system finds itself in a stable asymmetric state.

6. Conclusions

This review is concerned with the study of stochastic systems with singular multiplicative noise and interaction between particles. The system with one degree of freedom is considered along with the synergetic system parametrically represented by three hydrodynamic modes. By assumption, all values of the stochastic variable x can be realized in the initial state (in other words, the initial distribution function is not local). Notwithstanding the fact that we confined our analysis to the stationary distribution and did not consider the time evolution of the system, this circumstance is of fundamental importance, because the diffusion coefficient in the case of singular noise goes to zero at $x = 0$, and the positive and negative semiaxes of x values are disconnected. For example, a Brownian particle with a δ-shaped distribution cannot cross the border $x = 0$.

The main purpose of the review consists in extending the microscopic theory of phase transitions to systems with multiplicative noise. This required analyzing the structure of the Fokker – Planck equation itself. We found that the arbitrary force, which comes about in deciding on the choice of calculus, can be counterbalanced by multiplying the initial probability density by the exponential whose index determines the behaviour of the system over the entire time interval. The synergetic potential must also be renormalized. Such gauge, however, does not imply that Ito's calculus is preferential. In Section 4.4 we show that the nonanalytical behaviour of the rate of microscopic transitions gives rise to a singular force in the Fokker – Planck equation, which depends on the stochastic variable similarly to the fictitious force mentioned above, but does not involve an arbitrary coefficient. This force is proportional to the amplitude (rather than intensity) of the noise, and the square root of the ratio of the macroscopic to microscopic times. This force is directed so as to oppose the factors which brought it into existence.

The main result of this paper is that the existence of multiplicative noise transforms the domain of definition of the stochastic system in the phase space into a self-affine fractal set whose dimension lies between 0 and 2. It is the fractal nature of the phase space that gives rise to the singular force, whose magnitude is defined by the derivative of the multiplicative function with respect to the stochastic variable. Our power-law approximation (3.34) of the multiplicative noise reflects the self-affine nature of the phase space, and equation (3.74) links the geometrical and dynamic characteristics of the stochastic system.

Our treatment reveals that for the fractal dimension $0 < D \leqslant 1$ the deterministic regime is realized, in which the stochastic variable is time-independent. As the noise intensity decreases, a finite share of the degrees of freedom form a deterministic condensate, and the system loses ergodicity in a way similar to the vitrification of liquid. Interaction between particles at $1 < D \leqslant 2$ leads to the symmetry breaking: the distribution function becomes antisymmetrical with respect to sign reversal of the stochastic variable. The temperature dependence of the long-range order parameter is nonmonotonic, which is due to the clusterization of the fractal phase space.

In Section 5 we consider the example of Lorenz system, a popular object in the theory of self-organization. The inclusion of additive noises in all of the Lorenz equations leads in the adiabatic approximation to the transformation of these noises into multiplicative noises of the conjugate field and the controlling parameter. This gives rise to nontrivial restructuring of the synergetic system depending on the noise intensity of the controlling parameter.

As regards our approximations, our treatment was based on a spatially homogeneous system and a nonconservative stochastic variable. In addition, in connection with nonergodic systems, we only mentioned the memory effects, without specifying the parameters of nonergodicity and memory (the latter is apparently associated with the density of deterministic condensate defined in Section 3.2.2). They can be found using the field-theoretical scheme based on correlation techniques [31]. In the case of multiplicative noise (3.34), however, the correlators of the field variables are raised to fractional powers, and the standard field-theoretical scheme does not work. This difficulty can be avoided by passing to a new variable (2.48) in the initial Langevin equation, and basing the action in accordance with the scheme developed in Section 2.3. Since the new variable exhibits additive noise, one may now use the standard scheme, and go over to the initial stochastic variable in the final results. To the author's knowledge, such a program has not yet been realized.

The author is grateful to V G Bar'yakhtar and S V Peletminskiĭ for their support and fruitful discussions. The original results reproduced here were obtained together with D O Kharchenko and E A Toropov, whose cooperation is highly appreciated.

This work was supported by the State Committee for Science and Technology of Ukraine, Grant 2.4/4, and in part by the International Soros Science and Education Program (ISSEP), Grant SPU072044.

References

1. Gibbs J W *Termodinamika. Statisticheskaya Mekhanika* (Thermodynamics. Statistical Mechanics) (Moscow: Nauka, 1982)
2. Boltzmann L *Vorlesungen über Gastheorie* (Leipzig: J.A. Barth, 1896 – 1898) [Translated into Russian (Moscow: Gostekhteorizdat, 1956)]
3. Poincaré H *Izbrannye Trudy* Vol. III (Selected Works) (Moscow: Nauka, 1984)
4. Feder J *Fractals* (New York: Plenum Press, 1988) [Translated into Russian (Moscow: Mir, 1991)]
5. Haken H *Synergetics* (Springer Series in Synergetics, Vol. 1) (Heidelberg: Springer, 1978) [Translated into Russian (Moscow: Mir, 1980)]
6. Loskutov A Yu, Mikhaĭlov A S *Vvedenie v Sinergetiku* (Introduction to Synergetics) (Moscow: Mir, 1990)
7. Schuster H G *Deterministic Chaos: An Introduction* (Weinheim, Federal Republic of Germany: Physik-Verlag; Deerfield Beach, FL: VCH Publishers, 1984) [Translated into Russian (Moscow: Mir, 1988)]
8. Prigogine I *Termodinamicheskaya Teoriya Struktury, Ustoĭchivosti i Fluktuatsiĭ* (Thermodynamic Theory of Structure, Stability and Fluctuations) (Moscow: Mir, 1973)
9. Nicolis G, Prigogine I *Self-Organization in Nonequilibrium Systems: From Dissipative Structures to Order Through Fluctuations* (New York: Wiley, 1977) [Translated into Russian (Moscow: Mir, 1979)]
10. Haken H *Synergetics: An Introduction: Nonequilibrium Phase Transitions and Self-Organization in Physics, Chemistry, and Biology* (Springer Series in Synergetics, Vol. 1) (Berlin, New York: Springer, 1983) [Translated into Russian (Moscow: Mir, 1985)]

11. Haken H *Information and Self-Organization* (Springer Series in Synergetics, Vol. 40) (Berlin, New York: Springer-Verlag, 1988) [Translated into Russian (Moscow: Mir, 1991)]
12. Tyson J J *The Belousov–Zhabotinskiĭ Reaction* (Lecture Notes in Biomath. **10**) (Berlin, New York: Springer-Verlag, 1976)
13. Nicolis J *Dynamics of Hierarchical Systems: An Evolutionary Approach* (Springer Series in Synergetics, Vol. 25) (Berlin, New York: Springer-Verlag, 1986) [Translated into Russian (Moscow: Mir, 1989)]
14. Horsthemke W, Lefever R *Noise-Induced Transitions* (Springer Series in Synergetics, Vol. 15) (Berlin, New York: Springer-Verlag, 1984) [Translated into Russian (Moscow: Mir, 1987)]
15. Risken H *The Fokker–Planck Equation* (Springer Series in Synergetics, Vol. 18) (Berlin, New York: Springer-Verlag, 1989)
16. Gardiner C V *Stokhasticheskie Metody v Estestvennykh Naukakh* (Stochastic Methods in Natural Sciences) (Moscow: Mir, 1986)
17. Van Kampen N G *Stochastic Processes in Physics and Chemistry* (Amsterdam: North-Holland, 1981) [Translated into Russian (Moscow: Vysshaya Shkola, 1990)]
18. Stratonovich R L *Nelineĭnaya Neravnovesnaya Termodinamika* (Nonlinear Nonequilibrium Thermodynamics) (Moscow: Nauka, 1985) [Translated into English (Berlin, New York: Springer-Verlag, 1992)]
19. Kuznetsov P I, Stratonovich R L, Tikhonov V I *Zh. Eksp. Teor. Fiz.* **28** 509 (1955)
20. Stratonovich R L, Landa P S *Izv. Vyssh. Uchebn. Zaved. Radiofiz.* **2** (1) 37 (1959)
21. Stratonovich R L *Izbrannye Voprosy Teorii Fluktuatsiĭ v Radiotekhnike* (Selected Problems of the Theory of Fluctuations in Radio Engineering) (Moscow: Sovetskoe Radio, 1961)
22. Kubo R A, in *Fluctuation, Relaxation and Resonance in Magnetic Systems* (Ed. D Ter Haar) (Edinburgh: Oliver and Boyd, 1962)
23. Mayer-Kress G, Haken H *J. Stat. Phys.* **21** 149 (1981)
24. Crutchfield J P, Farmer J D, Huberman B A *Phys. Rep.* **92** 45 (1982)
25. Ito K *Nagoya Math. J.* **3** 55 (1951)
26. Stratonovich R L *SIAM J. Control* **4** 362 (1966)
27. Klimontovich Yu L *Usp. Fiz. Nauk* **164** 811 (1994) [*Phys. Usp.* **37** (8) 737 (1994)]
28. Landau L D, Lifshitz E M *Statisticheskaya Fizika* Chast' 1 (Statistical Physics, Vol. 1) (Moscow: Nauka, 1976) [Translated into English (Oxford: Pergamon Press, 1980)]
29. Forster D *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Frontiers in Physics, 47) (Reading, Mass.: W.A. Benjamin, Advanced Book Program, 1975) [Translated into Russian (Moscow: Atomizdat, 1980)]
30. Landau L D, Lifshitz E M *Mekhanika* (Mechanics) (Moscow: Nauka, 1988) [Translated into English (Oxford, New York: Pergamon Press, 1976)]
31. Olemskoĭ A I, Koplyk I V *Usp. Fiz. Nauk* **165** 1105 (1995) [*Phys. Usp.* **38** (10) 1061 (1995)]
32. Katsnel'son A A, Olemskoĭ A I *Mikroskopicheskaya Teoriya Neodnorodnykh Struktur* (Microscopic Theory of Inhomogeneous Structures) (Moscow: Izdatel'stvo MGU, 1987)
33. Olemskoi A I, in *Physics Reviews* (Ed. I M Khalatnikov) **18** Part 1 1 (1996)
34. Bogolyubov N N, Shirkov D V *Vvedenie v Teoriyu Kvantovannykh Poleĭ* (Introduction to the Theory of Quantized Fields) (Moscow: Nauka, 1976) [Translated into English (New York: John Wiley, 1980)]
35. Itzykson C, Zuber J-B *Quantum Field Theory* (New York: McGraw-Hill, 1980) [Translated into Russian (Moscow: Mir, 1984)]
36. Parisi G, Sourlas N *Phys. Rev. Lett.* **43** 744 (1979)
37. Zinn-Justin J *Quantum Field Theory and Critical Phenomena* (Oxford: Clarendon Press; New York: Oxford Univ. Press, 1989)
38. Lifshitz E M, Pitaevskiĭ L P *Fizicheskaya Kinetika* (Physical Kinetics) (Moscow: Nauka, 1979) [Translated into English (Oxford: Pergamon Press, 1981)]
39. Gnedenko B V *Kurs Teorii Veroyatnosteĭ* (A Course in Theory of Probabilities) (Moscow: Nauka, 1967)
40. Landau L D, Lifshitz E M *Kvantovaya Mekhanika. Nerelyativistskaya Teoriya* (Quantum Mechanics: Non-Relativistic Theory) (Moscow: Nauka, 1984) [Translated into English (Oxford, New York: Pergamon Press, 1977)]
41. Pawula R F *IEEE Trans. Inform. Theory* **IT–13** 33 (1967)
42. Cooper F, Khare A, Sukhatme U *Phys. Rep.* **251** 267 (1995)
43. Lifshits I M, Slezov V V *Zh. Eksp. Teor. Fiz.* **35** 479 (1958)
44. Bray A J *Adv. Phys.* **43** 357 (1994)
45. Landau L D, Lifshitz E M *Teoriya Polya* (Field Theory) (Moscow: Nauka, 1973) [Translated into English (Oxford: Pergamon Press, 1975)]
46. Olemskoĭ A I, Kharchenko D O *Izv. Vyssh. Uchebn. Zaved. Ser. Fiz.* (2) 121 (1996)
47. Feĭgel'man M V, Tsvelik A M *Zh. Eksp. Teor. Fiz.* **83** 1430 (1982) [*Sov. Phys, JETP* **56** 823 (1982)]
48. Akhiezer A I, Peletminskiĭ S V *Polya i Fundamental'noye Vzaimodeĭstvie* (Fields and Fundamental Interaction) (Kiev: Naukova Dumka, 1986)
49. Berestetskiĭ V B, Lifshitz E M, Pitaevskiĭ L P *Kvantovaya Élektrodinamika* (Quantum Electrodynamics) (Moscow, Nauka, 1989) [Translated into English (Oxford: Pergamon Press, 1982)]
50. Palmer R G *Adv. Phys.* **31** 669 (1982)
51. Van den Broeck C, Parrondo J M R, Toral R *Phys. Rev. Lett.* **73** 3395 (1994)
52. Toropov E A, Kharchenko D O *Izv. Vyssh. Uchebn. Zaved. Ser. Fiz.* (4) 75 (1996)
53. Kosevich A M *Fizicheskaya Mekhanika Real'nykh Kristallov* (Physical Mechanics of Real Crystals) (Kiev: Naukova Dumka, 1981)
54. Olemskoĭ A I, Kharchenko D O *Metallofiz.* (8) 3 (1996)
55. Olemskoĭ A I, Flat A Ya *Fiz. Tverd. Tela* **35** 542 (1993)
56. Olemskoĭ A I, Khomenko A V *Zh. Eksp. Teor. Fiz.* **110** 2144 (1996) [*JETP* **83** 1180 (1996)]
57. Ruelle D, Takens F, in *Strannye Attraktory* (Strange Attractors) (Moscow: Mir, 1981)
58. Lorenz E J *Atmos. Sci.* **20** 1675 (1963)
59. Olemskoĭ A I, Petrunin V A *Izv. Vyssh. Uchebn. Zaved. Ser. Fiz.* **30** (1) 82 (1987) [*Sov. Phys. J.* **30** (1) 61 (1987)]
60. Olemskoĭ A I, Sklyar I A *Usp. Fiz. Nauk* **162** (6) 29 (1992) [*Sov. Phys. Usp.* **35** 455 (1992)]