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The Bénard – Marangoni thermocapillary-instability problem

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Contents

Abstract. Physically, there are two main mechanisms responsible for driving the instability in the coupled buoyancy (Bénard) and thermocapillary (Marangoni) convection problem for a weakly expansible viscous liquid layer bounded from below by a heated solid surface and on the top by a free surface subject to a temperature-dependent surface tension. The first mechanism is density variation generated by the thermal expansion of the liquid; the second results from the surface-tension gradients due to temperature fluctuations along the upper free-surface. In the present paper we consider only the second effect as in the Bénard experiments [the so-called Bénard – Marangoni (BM) problem]. Indeed, for a thin layer we show that it is not consistent to consider both effects simultaneously, and we formulate an alternative concerning the role of buoyancy. In fact, it is necessary to consider two fundamentally distinct problems: the classical shallow-convection problem for a non-deformable upper surface with partial account of the Marangoni effect (the RBM problem), and the full BM problem for a deformable free surface without the buoyancy effect. We shall be mostly concerned with the thermocapillary BM instabilities problem on a free-falling vertical film, since most experiments and theories have focused on this (in fact, wave dynamics on an inclined plane is quite analogous). For a thin film we consider three main situations in relation to the magnitude of the characteristic Reynolds number (Re) and we derive various model equations. These model equations are analyzed from various points of view but the central intent of this paper is to elucidate the role of the Marangoni number on the evolution of the free surface in space and time. Finally, some recent numerical results are presented.

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1. Introduction

In considering a variety of problems in hydrodynamics and in the theory of convective heat and mass transfer in two-phase systems with a deformable interphase boundary (or free surface: liquid – gas) one encounters so-called capillary phenomena, i.e., phenomena caused by the existence of surface tension on the interface boundary. Concerning a physical description of surface-tension-driven phenomena the reader can find a very valuable review in the paper by Levich and Krylov [1]. See also the interesting and very well documented paper by Sarpkaya [2], concerning the discussion of various surface-tension effects, the role of surfactants, the topology of interfacial interactions and the elucidation of the physics of fluid-mechanic phenomena at pure and contaminated interfaces.

In fact, capillary phenomena may occur in two cases:

(a) when the surface of phase separation possesses considerable curvature, and

(b) when the surface tension varies from point to point on the surface.

In both cases, forces arise at the interphase boundary which change the nature of motion in each of the phases or induce motion originally absent. Formally, the effect of surface tension on the hydrodynamics (and via the hydrodynamics also on the heat and mass transfer) appears in a change of the boundary conditions on the interface (see Section 2.2). In a liquid layer, wavy motion of the free surface open to ambient passive air (at constant temperature T° and constant pressure p_a) represents one of the most important cases where capillary forces are displayed. In fact, the surface tension dominates the process of formation of waves whose wavelength λ° is 'short' (large Weber number), but for surface waves, in a thin layer, $\varepsilon = h^\circ/\lambda^\circ \ll 1$, where h° is the thickness of the liquid layer (the long-wave approximation). One the other hand, variation of the surface tension from point to point on a deformable free surface gives rise to a tangential strain on this boundary surface and as a consequence convective currents will inevitably arise in the liquid (as in the Bénard [3] experiments). The motion induced by tangential gradients of surface tension is customarily called the Marangoni effect (after one of the first scientists to give an explanation of the effect). We shall consider some particular examples where varying surface tension affects the hydrodynamics. For a pertinent review concerning Marangoni instability, the reader can consult the paper by Davis [4]. There are a number of excellent sources of information on various features of variable surface-tension effects. Among these are: Sterning and Scriven [5], Levich [6], Kenning [7], Norman et al. [8], Velarde and Chu [9], and Probstein [10].

In the present paper we consider only thermocapillarity effects and pose an equation of state:

$$\sigma = \sigma(T)\,,$$

where the surface tension σ is a function of the temperature T (only). The simplified case we examine involves a simple geometry — *the one-layer system* — in which there is a liquid (weakly expansible) layer, whose lower boundary is a rigid plate and whose upper boundary is an deformable free surface with a passive gas (having negligible viscosity and density).

At the present time there are a large number of studies of the hydrodynamics of a wavy thin falling film in the literature (see, for example, the book edited by Meyer [11]) and a review and further references (up to 1984 – 1985) can be found in Lin and Wang [12]. For a non-expansible, strictly incompressible liquid (without the Marangoni effect) and two-dimensional wave flow, a discussion of the basic results is given in the recent papers by Trifonov and Tsvelodub [13] and Prokopiou, Cheng and Chang [14]. For the vertical film, the recent paper by Chang [15] gives an excellent review concerning mostly various transition regimes on a free-falling vertical film. For an extension of this review, see Chang and Demekhin [16] — but in both these review papers there is no discussion concerning the Marangoni effect!

For investigations where the Marangoni effect is taken into account see the references (up to 1985 – 1986) in [4] and also the recent (1990 – 1996) papers [17] to [46].

In the next Section 2 we present the exact formulation of the full three-dimensional problem for an expansible liquid, in a horizontal layer, and we derive the associated dimensionless dominant problem (for the details of this derivation for a weakly expansible liquid, see our paper [47]). Section 3 is devoted to a consistent formulation of the Bénard – Marangoni problem for a non-deformable free surface (but with buoyancy effect) and a deformable free surface (without the buoyancy effect). For this we elucidate the role of the buoyancy coupled with the free surface effect, according to our recent paper [48]. Concerning the Bénard – Marangoni instability problem with a non-deformable upper surface, when the buoyancy effect is dominant, see [48, § 4] and the recent papers [49] (by Dauby and Lebon) and [50] (by Vince).

In Sections 4 to 6 we consider three limiting regimes for the high, moderate and low Reynolds numbers (Re), in the long-wave approximation $\varepsilon \ll 1$, and we systematically derive various model equations for the film thickness. For $\mathrm{Re} \gg 1$, we generalize the classical Shkadov [51] integral boundary-layer equations and we derive a Stuart – Landau equation. A finite-dimensional dynamical-system approach is also considered (with some recent numerical result obtained by S Godts and M Zghal) and the appearance of bifurcations, chaos and strange attractors are discussed. For $\mathrm{Re} = O(1)$, we derive the Kuramoto – Sivashinsky (KS) equation, where the Marangoni and Biot effects are taken into account. The hierarchy of bifurcations and attractors of the KS equations is also discussed (see also, for instance, the paper [52] by Demekhin, Tokarev and Shkadov). For $\mathrm{Re} \ll 1$, a KS – KdV model equation is derived for the evolution of the thickness of the free surface. Some features of solutions of this KS – KdV equation are discussed and various associated amplitude equations are derived.

2. Governing equations and boundary conditions

2.1 Equations

Consider an expansible viscous layer lying on a horizontal plane held at the uniform temperature $T^\circ + \Delta T^\circ$, with a deformable free surface open to ambient passive air (at temperature T° and pressure p_a, having negligible viscosity and density). The expansible viscous liquid, with the equation of state

$$\rho = \rho(T)\,, \tag{2.1.1}$$

is a Newtonian fluid with viscosities $\lambda(T)$ and $\mu(T)$, density $\rho(T)$, specific heat $C(T)$ and thermal conductivity $k(T)$; $\varkappa = k/\rho C$ is the thermal diffusivity and $\nu = \mu/\rho$ is the kinematic viscosity. The surface tension is assumed to be a linear function of the temperature [and as a consequence the term $\partial\sigma/\partial t$ is absent in the energy balance, see (2.2.3)]:

$$\sigma(T) = \sigma(T^\circ) - \gamma(T - T^\circ)\,, \tag{2.1.2}$$

where $\gamma = -\,\mathrm{d}\sigma(T)/\,\mathrm{d}T$ is a constant which is positive for most liquids.

The exact (Navier – Stokes) equations governing the expansible fluid flow for the velocity vector $\mathbf{u}$, pressure p and temperature T, are:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\mathbf{u} = 0\,, \tag{2.1.3}$$

$$\rho\,\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \nabla p + \rho g\mathbf{k} = \nabla\cdot\mathrm{T}\,, \tag{2.1.4}$$

$$\rho C(T)\,\frac{\mathrm{D}T}{\mathrm{D}t} + p\nabla\cdot\mathbf{u} = \nabla\cdot(k\nabla T) + \Phi(\mathbf{u})\,, \tag{2.1.5}$$

where T is the (symmetric) viscous-stress tensor of the liquid and $\Phi(\mathbf{u})$ is the dissipation function. In equations (2.1.3) –

(2.1.5), the notation is standard:

$$\mathbf{u}(t,\mathbf{x}) = \big(u_i(t,x_i)\big)\,, \quad \frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\,,$$
$$\nabla = \frac{\partial}{\partial x_i}\,, \quad \mathbf{x} = (x_i)\,, \quad i = 1,2,3,$$

and (O, x_i) is a Cartesian coordinate system in which $\mathbf{g} = -g\mathbf{k}$ acts in the negative x_3 direction. For a Newtonian fluid

$$\mathrm{T} = \lambda(T)(\nabla\cdot\mathbf{u})\mathrm{I} + 2\mu(T)\mathrm{D}\,, \tag{2.1.6}$$

$$\Phi(\mathbf{u}) = 2\mu(T)\mathrm{D}:\mathrm{D} + \lambda(T)(\nabla\cdot\mathbf{u})^2\,, \ \mathrm{D}:\mathrm{D} = d_{ij}^2\,, \tag{2.1.7}$$

where D, with the (Cartesian) components $d_{ij} = (1/2)(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$, is the strain-rate tensor. Finally, $\mathrm{I} = \delta_{ij}$ is the Kronecker delta (i and $j = 1,2,3$).

If $e(T)$ is the specific internal energy of the expansible liquid with the equation of state (2.1.1), then

$$\frac{\mathrm{D}e}{\mathrm{D}t} = C(T)\frac{\mathrm{D}T}{\mathrm{D}t}\,, \quad C(T) = \frac{\mathrm{D}e}{\mathrm{D}T}\,.$$

2.2 Boundary conditions

For equations (2.1.3) – (2.1.5), the relevant boundary conditions are

$$\mathbf{u} = 0\,, \quad T = T^\circ + \Delta T^\circ\,, \quad x_3 = 0\,, \tag{2.2.1}$$

$$-(p - p_\mathrm{a})\mathbf{n} + \mathrm{T}\cdot\mathbf{n} = 2\sigma(T)H\mathbf{n} + \nabla_S\sigma(T)\,, \quad x_3 = h(t,x_1,x_2)\,, \tag{2.2.2}$$

$$-k(T)\nabla T\cdot\mathbf{n} = q^\circ(T - T^\circ) + k(T^\circ)\left(\frac{\Delta T^\circ}{h^\circ}\right)\,, \quad x_3 = h(t,x_1,x_2)\,. \tag{2.2.3}$$

The location of the deformable free surface, $x_3 = h(t,x_1,x_2)$, is determined via the kinematic condition

$$u_3 = \frac{\partial h}{\partial t} + u_1\frac{\partial h}{\partial x_1} + u_2\frac{\partial h}{\partial x_2}\,, \quad x_3 = h(t,x_1,x_2)\,. \tag{2.2.4}$$

In the boundary conditions (2.2.2) and (2.2.3), $H = -(1/2)(\nabla_S\cdot\mathbf{n})$ is the mean interface curvature, q° is the heat-transfer constant coefficient, $\mathbf{n}$ is the normal unit vector (directed from liquid to air) and ∇_S is the surface gradient at the free surface. Note that in condition (2.2.2) we do not take into account the surface viscosities.

In the reference state, the fluid is at rest and heat propagates only by conduction: no flow is present; under these conditions we have a quiescent stationary-state temperature:

$$T_\mathrm{s}(x_3) = T^\circ + \Delta T^\circ\left[1 - \left(\frac{x_3}{h^\circ}\right)\right]\,, \tag{2.2.5}$$

and, as a consequence, in (2.2.3) it is necessary to take into account the imposed temperature gradient (see [4, p. 407]): $\mathrm{d}T_\mathrm{s}(x_3)/\mathrm{d}x_3 \equiv -(\Delta T^\circ/h^\circ)$.

Concerning the free surface jump conditions (2.2.2) and (2.2.3), see the book [53, p. 20] by Joseph and Renardy. Afterwards we assume that $\Delta T^\circ > 0$.

2.3 Dimensionless dominant problem

For a weakly expansible liquid, when the dimensionless parameter

$$\alpha = \beta\Delta T^\circ\,, \quad \beta = -\left[\frac{\mathrm{d}(\log\rho)}{\mathrm{d}T}\right]_{T=T^\circ}\,, \tag{2.3.1}$$

is a small parameter, it is possible to derive a simplified dimensionless dominant problem from the full exact problem (2.1.3) – (2.1.5), (2.2.1) – (2.2.4). As in [47] and [48], let the coefficients with the superscript ° be reference values for these coefficients at $T = T^\circ$. If we take into account (2.1.1) and introduce the pressure and temperature perturbations by

$$\theta = \frac{T - T^\circ}{\Delta T^\circ}\,, \tag{2.3.2a}$$

$$\pi = \frac{1}{\mathrm{Fr}^2}\left[\left(\frac{p - p_\mathrm{a}}{\rho_0 g h^\circ}\right) + \left(\frac{x_3}{h^\circ} - 1\right)\right]\,, \tag{2.3.2b}$$

then for $v_i = u_i/(\nu_0/h^\circ)$, θ and π, as a function of the dimensionless variables

$$x_i' = \frac{x_i}{h^\circ}\,, \quad t' = \frac{t}{h^{\circ 2}/\nu_0}\,,$$

we derive the following dominant dimensionless equations:

$$\frac{\partial v_k}{\partial x_k'} = O(\alpha)\,, \tag{2.3.3a}$$

$$\frac{\mathrm{D}v_i}{\mathrm{D}t'} + \frac{\partial\pi}{\partial x_i'} - \frac{\alpha}{\mathrm{Fr}^2}\theta\delta_{i3} - \nabla'^2 v_i = O(\alpha)\,, \tag{2.3.3b}$$

$$\left[1 - \alpha\,\mathrm{Bo}(p_\mathrm{a}' + 1 - x_3')\right]\frac{\mathrm{D}\theta}{\mathrm{D}t'} - \frac{1}{\mathrm{Pr}}\nabla'^2\theta - 2\mathrm{Bo}\,\mathrm{Fr}^2(d_{ij}')^2 = O(\alpha)\,, \tag{2.3.3c}$$

in place of (2.1.3) – (2.1.5). Explicitly,

$$d_{ij}' = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j'} + \frac{\partial v_j}{\partial x_i'}\right)\,, \quad \frac{\mathrm{D}}{\mathrm{D}t'} = \frac{\partial}{\partial t'} + v_i\frac{\partial}{\partial x_i'}\,,$$

$$\nabla'^2 = \frac{\partial^2}{\partial x_1'^2} + \frac{\partial^2}{\partial x_2'^2} + \frac{\partial^2}{\partial x_3'^2}\,, \quad p_\mathrm{a}' = \frac{p_\mathrm{a}}{\rho_0 g h^\circ}\,.$$

In [48, Appendix] the reader can find the expressions for the terms $O(\alpha)$ in the right-hand side of equations (2.3.3). In the above dimensionless relations and equations,

$$\mathrm{Pr} = \frac{\nu_0}{\varkappa_0}\,, \quad \mathrm{Fr} = \frac{\nu_0/h^\circ}{(gh^\circ)^{1/2}}\,, \quad \mathrm{Bo} = \frac{gh^\circ}{C_0\Delta T^\circ} \tag{2.3.4}$$

are the Prandtl, Froude and Boussinesq numbers, respectively.

For these dominant equations (2.3.3) it is necessary to derive a set of dominant dimensionless boundary conditions for v_i, θ and π from (2.2.1) – (2.2.4). For this we write the free surface equation in the form $x_3 = h^\circ\left[1 + \delta\eta(t', x_1', x_2')\right]$, where δ is an amplitude parameter for the free surface deformation $\eta(t', x_1', x_2')$ relative to plane $x_3' = 1$, and in this case we obtain from (2.2.2) and (2.2.3), with (2.1.2), the following four dominant dimensionless boundary conditions at the deformable free surface, $x'_3 = 1 + \delta\eta(t', x'_1, x'_2)$:

$$\pi = \frac{\delta\eta}{\mathrm{Fr}^2} + 2d'_{ij}n'_i n'_j + (\mathrm{We} - \mathrm{Ma}\,\theta)(\nabla'_S \cdot \mathbf{n}') + O(\alpha), \quad (2.3.5a)$$

$$d'_{ij}\,t'^{(s)}_i n'_j + \frac{1}{2}\mathrm{Ma}\,t'^{(s)}_i \frac{\partial\theta}{\partial x'_i} = O(\alpha), \quad s = 1, 2, \quad (2.3.5b)$$

$$\nabla'\theta \cdot \mathbf{n}' + \mathrm{Bi}\,\theta + 1 = O(\alpha). \quad (2.3.5c)$$

In the dimensionless dominant boundary conditions (2.3.5b), $t'^{(1)}_i$ and $t'^{(2)}_i$ are the dimensionless components of two orthonormal tangential vectors to the free surface $x'_3 = h'$ $(= 1 + \delta\eta)$ and n'_i are the dimensionless components of $\mathbf{n}'$, namely

$$\mathbf{t}'^{(1)} = \frac{1}{\sqrt{N'_1}}\left(1, 0, \frac{\partial h'}{\partial x'_1}\right),$$

$$\mathbf{t}'^{(2)} = \frac{1}{\sqrt{N'_1 N'}}\left[-\frac{\partial h'}{\partial x'_1}\frac{\partial h'}{\partial x'_2},\quad 1 + \left(\frac{\partial h'}{\partial x'_1}\right)^2,\quad \frac{\partial h'}{\partial x'_2}\right],$$

$$\mathbf{n}' = \frac{1}{\sqrt{N'}}\left(-\frac{\partial h'}{\partial x'_1},\quad -\frac{\partial h'}{\partial x'_2},\quad 1\right).$$

According to [54, p. 213], the unit vectors tangential and normal to the deformed free surface $x'_3 = h'(t', x'_1, x'_2)$ are written in terms of the (x'_1, x'_2, x'_3) system. For $(\nabla'_S \cdot \mathbf{n}')$ we have the following expression:

$$\nabla'_S \cdot \mathbf{n}' = -\frac{1/h^\circ}{(N')^{3/2}}\left[N'_2\left(\frac{\partial^2 h'}{\partial x'_1 \partial x'_1}\right) + N'_1\left(\frac{\partial^2 h'}{\partial x'_2 \partial x'_2}\right) - 2\left(\frac{\partial^2 h'}{\partial x'_1 \partial x'_2}\right)\left(\frac{\partial h'}{\partial x'_1}\right)\left(\frac{\partial h'}{\partial x'_2}\right)\right],$$

where

$$N'_1 = 1 + \left(\frac{\partial h'}{\partial x'_1}\right)^2, \quad N'_2 = 1 + \left(\frac{\partial h'}{\partial x'_2}\right)^2,$$

$$N' = 1 + \left(\frac{\partial h'}{\partial x'_1}\right)^2 + \left(\frac{\partial h'}{\partial x'_2}\right)^2.$$

The expressions for the terms $O(\alpha)$ in the right-hand side of conditions (2.3.5) are presented in the appendix of [48]. Finally, instead of (2.2.1) we write

$$v_i = 0, \quad \theta = 1, \quad x'_3 = 0, \quad (2.3.6a)$$

and instead of (2.2.4)

$$v_3 = \delta\left(\frac{\partial\eta}{\partial t'} + v_1\frac{\partial\eta}{\partial x'_1} + v_2\frac{\partial\eta}{\partial x'_2}\right), \quad x'_3 = 1 + \delta\eta. \quad (2.3.6b)$$

In equations (2.3.5)

$$\mathrm{We} = \frac{\sigma_0 h^\circ}{\rho_0 v_0^2}, \quad \mathrm{Ma} = \frac{\gamma h^\circ \Delta T^\circ}{\rho_0 v_0^2}, \quad \mathrm{Bi} = \frac{q^\circ h^\circ}{k_0} \quad (2.3.7)$$

are the Weber, Marangoni [when we take into account (2.1.2)] and Biot numbers, respectively.

3. The Bénard – Marangoni problem

3.1 The role of buoyancy — an alternative

It is usual in the literature [55, Chapter 2] to refer to the Rayleigh – Bénard (RB) shallow convection as the instability problem produced by buoyancy. However, Bénard convective cells, [3], are primarily induced by the surface-tension gradients resulting from temperature variations along the free surface (the so-called Marangoni effect). For this, it seems justified to use the term 'Bénard – Marangoni (BM) thermocapillary-instability problem' when, as in Bénard's experiment, the dominant acting driving force is the surface-tension gradient. Naturally, in the full dominant convection problem (2.3.3), (2.3.5), (2.3.6) for a weakly expansible and viscous liquid, both the buoyancy and the surface-tension gradient are operative, so it is important (from our point of view) to ask: how are the two effects coupled when the liquid is weakly expansible ($\alpha \to 0$)?

It is also necessary to note that in the dominant equations (2.3.3b) and (2.3.3c), two terms $(\alpha/\mathrm{Fr}^2)\theta\delta_{i3}$ and $\alpha\mathrm{Bo}(p'_a + 1 - x'_3)(\mathrm{D}\theta/\mathrm{D}t')$ containing α are present! On the other hand, in boundary condition (2.3.5a), we have the term $\delta\eta/\mathrm{Fr}^2$!

As a consequence, if we want to take into account the buoyancy term $(\alpha/\mathrm{Fr}^2)\theta\delta_{i3}$ in (2.3.3b) then it is necessary that $\alpha \to 0$ and $\mathrm{Fr} \to 0$ with $\alpha/\mathrm{Fr}^2 = O(1)$ fixed. But in such a case, in (2.3.5a), the first term on the right-hand side (for $\delta = O(1)$) is unbounded!

Hence, we obtain the following *alternative*:

"Either the buoyancy is taken into account and in this case the free surface deformation effect is negligible and we have the possibility to take into account the Marangoni effect only partially, or the free surface deformation effect is taken into account, in which case the buoyancy does not play a significant role in the Bénard – Marangoni full thermocapillary problem".

3.2 The BM problem with a non-deformable upper surface (RBM problem)

From equation (2.3.3b), when $\alpha \to 0$, it is obvious that if we want to take into account the buoyancy term $\mathrm{Gr}\theta\delta_{i3}$, it is necessary to impose the following 'Boussinesq limiting process' [47]:

$$\alpha \to 0, \quad \mathrm{Fr} \to 0 \quad \left(\frac{\alpha}{\mathrm{Fr}^2} = \mathrm{Gr} = O(1)\right). \quad (3.2.1)$$

In this case, if all the variables t' and x'_i and the parameters Pr, Bo are fixed and the terms $O(\alpha) \to 0$, then we derive the following classical shallow-convection dimensionless Boussinesq equations for v_i, θ and π:

$$\frac{\partial v_k}{\partial x'_k} = 0, \quad (3.2.2a)$$

$$\frac{\mathrm{D}v_i}{\mathrm{D}t'} + \frac{\partial\pi}{\partial x'_i} - \mathrm{Gr}\theta\delta_{i3} = \nabla'^2 v_i, \quad (3.2.2b)$$

$$\frac{\mathrm{D}\theta}{\mathrm{D}t'} = \frac{1}{\mathrm{Pr}}\nabla'^2\theta. \quad (3.2.2c)$$

As $\mathrm{Bo} = O(1)$ and $\mathrm{Fr} \ll 1$, the model equations (3.2.2) are valid for the following physical condition:

$$\left(\frac{v_0^2}{g}\right)^{1/3} \ll h^\circ \leqslant C_0\frac{\Delta T^\circ}{g}. \quad (3.2.3)$$

Now, if we consider the dominant dimensionless boundary conditions (2.3.5), as a consequence of the limit $\mathrm{Fr} \to 0$, we see that in the free surface condition (2.3.5a) the first term on the right-hand side (for $\delta = O(1)$) is unbounded! Therefore, the free surface pressure condition (2.3.5a) for a weakly expansible liquid is asymptotically consistent with the limit model equations (3.2.2) only for a small free surface amplitude, when

$$\delta \to 0\,, \quad \mathrm{Fr} \to 0 \quad \left(\frac{\delta}{\mathrm{Fr}^2} = \delta^* = O(1) \right). \tag{3.2.4}$$

On the other hand, if we want to take account the influence of a (large) Weber number in the BM limit problem with a non-deformable upper surface, it is also necessary that

$$\mathrm{We} \gg 1\,, \quad \delta \ll 1\,, \quad \delta\,\mathrm{We} = \mathrm{W}^* = O(1)\,. \tag{3.2.5}$$

Finally, since $\delta \ll 1$, from (2.3.6b) we also have $v_3 = O(\delta)$ on $x_3 = 1$.

Consequently, for the model Boussinesq equations (3.2.2), when we take into account (3.2.4) and (3.2.5) and if Ma and Bi are both $O(1)$, we can prescribe the following 'non-deformable' free surface boundary conditions (at $x_3' = 1$):

$$\pi = \delta^* \eta - \mathrm{W}^* \left(\frac{\partial^2 \eta}{\partial x_1'^2} + \frac{\partial^2 \eta}{\partial x_2'^2} \right), \tag{3.2.6}$$

$$v_3 = 0\,, \quad \frac{\partial v_1}{\partial x_3'} = -\mathrm{Ma}\frac{\partial \theta}{\partial x_1'}\,, \quad \frac{\partial v_2}{\partial x_3'} = -\mathrm{Ma}\frac{\partial \theta}{\partial x_2'}\,, \tag{3.2.7a}$$

$$\frac{\partial \theta}{\partial x_3'} + \mathrm{Bi}\theta + 1 = 0\,. \tag{3.2.7b}$$

In fact, condition (3.2.6) for π at $x_3' = 1$ is a condition for the determination of the free surface deformation $\eta(t', x_1', x_2')$ relative to the plane $x_3' = 1$.

Finally, for the RBM problem (the BM problem with a non-deformable upper free surface), when we take into account the Biot, Marangoni and Weber effects, we have the equations (3.2.2) for v_i, θ and π, with the conditions (3.2.7a, b) at $x_3' = 1$ and (2.3.6a) at $x_3' = 0$. In this RBM problem we have three similarity parameters (Gr, δ^* and W^*) and three dimensionless parameters (Pr, Ma and Bi). In [49], precisely this RBM model problem was recently considered.

3.3 The BM problem for a thin layer (film) with a deformable free surface

For the derivation of the full BM model problem, when the free surface deformation $\delta = O(1)$ in equation $h' = 1 + \delta\eta$ plays an essential role, it is necessary to assume that the Froude number Fr is $O(1)$. As a consequence, we must consider the following incompressible limiting process:

$$\alpha \to 0\,, \quad \mathrm{Fr} = O(1)\,. \tag{3.3.1}$$

But in this case it is obvious that $\mathrm{Bo} \ll 1$, since $C_0 \Delta T^\circ / g \gg h^\circ \approx (v_0^2/g)^{1/3}$. As a consequence, in place of the Boussinesq equations (3.2.2), we derive the following 'incompressible' equations:

$$\frac{\partial v_k}{\partial x_k'} = 0\,, \tag{3.3.2a}$$

$$\frac{\mathrm{D} v_i}{\mathrm{D} t'} + \frac{\partial \pi}{\partial x_i'} = \nabla'^2 v_i\,, \tag{3.3.2b}$$

$$\frac{\mathrm{D}\theta}{\mathrm{D}t'} = \frac{1}{\mathrm{Pr}} \nabla'^2 \theta\,. \tag{3.3.2c}$$

For these equations (3.3.2), since $\mathrm{Fr} = O(1)$ and $\delta = O(1)$, we must use the deformable-free-surface conditions at $x_3' = h'(t', x_1', x_2')$. According to (2.3.5) we derive the following dimensionless, very complicated conditions (written at $x_3' = h'(t', x_1', x_2')$):

$$\begin{aligned} \pi = {} & \frac{h'-1}{\mathrm{Fr}^2} + \frac{2}{N'} \left[\frac{\partial v_1}{\partial x_1'} \left(\frac{\partial h'}{\partial x_1'} \right)^2 + \frac{\partial v_2}{\partial x_2'} \left(\frac{\partial h'}{\partial x_2'} \right)^2 + \frac{\partial v_3}{\partial x_3'} \right. \\ & + \left(\frac{\partial v_1}{\partial x_2'} + \frac{\partial v_2}{\partial x_1'} \right) \frac{\partial h'}{\partial x_1'} \frac{\partial h'}{\partial x_2'} - \left(\frac{\partial v_1}{\partial x_3'} + \frac{\partial v_3}{\partial x_1'} \right) \frac{\partial h'}{\partial x_1'} \\ & \left. - \left(\frac{\partial v_2}{\partial x_3'} + \frac{\partial v_3}{\partial x_2'} \right) \frac{\partial h'}{\partial x_2'} \right] - \frac{\mathrm{We} - \mathrm{Ma}\theta}{N'^{3/2}} \left[N_2' \frac{\partial^2 h'}{\partial x_1' \partial x_1'} \right. \\ & \left. + N_1' \frac{\partial^2 h'}{\partial x_2' \partial x_2'} - 2 \frac{\partial^2 h'}{\partial x_1' \partial x_2'} \frac{\partial h'}{\partial x_1'} \frac{\partial h'}{\partial x_2'} \right], \end{aligned} \tag{3.3.3a}$$

$$\begin{aligned} & \left(\frac{\partial v_1}{\partial x_1'} - \frac{\partial v_3}{\partial x_3'} \right) \frac{\partial h'}{\partial x_1'} + \frac{1}{2} \left(\frac{\partial v_1}{\partial x_2'} + \frac{\partial v_2}{\partial x_1'} \right) \frac{\partial h'}{\partial x_2'} \\ & + \frac{1}{2} \left(\frac{\partial v_2}{\partial x_3'} + \frac{\partial v_3}{\partial x_2'} \right) \frac{\partial h'}{\partial x_1'} \frac{\partial h'}{\partial x_2'} - \frac{1}{2} \left[1 - \left(\frac{\partial h'}{\partial x_1'} \right)^2 \right] \\ & \times \left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right) = \frac{\mathrm{Ma}}{2} \left(\frac{\partial \theta}{\partial x_1'} + \frac{\partial h'}{\partial x_1'} \frac{\partial \theta}{\partial x_3'} \right) N'^{1/2}, \end{aligned} \tag{3.3.3b}$$

$$\begin{aligned} & \left(\frac{\partial v_1}{\partial x_1'} - \frac{\partial v_2}{\partial x_2'} \right) \frac{\partial h'}{\partial x_2'} \left(\frac{\partial h'}{\partial x_1'} \right)^2 + \left(\frac{\partial v_1}{\partial x_3'} + \frac{\partial v_3}{\partial x_1'} \right) \frac{\partial h'}{\partial x_1'} \frac{\partial h'}{\partial x_2'} \\ & + \left(\frac{\partial v_2}{\partial x_2'} - \frac{\partial v_3}{\partial x_3'} \right) \frac{\partial h'}{\partial x_2'} + \frac{1}{2} \left[1 + \left(\frac{\partial h'}{\partial x_1'} \right)^2 - \left(\frac{\partial h'}{\partial x_2'} \right)^2 \right] \\ & \times \left(\frac{\partial v_1}{\partial x_2'} + \frac{\partial v_2}{\partial x_1'} \right) \frac{\partial h'}{\partial x_1'} - \frac{1}{2} \left[1 + \left(\frac{\partial h'}{\partial x_1'} \right)^2 - \left(\frac{\partial h'}{\partial x_2'} \right)^2 \right] \\ & \times \left(\frac{\partial v_2}{\partial x_3'} + \frac{\partial v_3}{\partial x_2'} \right) = \frac{\mathrm{Ma}}{2} \left\{ - \frac{\partial h'}{\partial x_1'} \frac{\partial h'}{\partial x_2'} \frac{\partial \theta}{\partial x_1'} \right. \\ & \left. + \left[1 + \left(\frac{\partial h'}{\partial x_1'} \right)^2 \right] \frac{\partial \theta}{\partial x_2'} + \frac{\partial h'}{\partial x_2'} \frac{\partial \theta}{\partial x_3'} \right\} N'^{1/2}, \end{aligned} \tag{3.3.3c}$$

$$\frac{\partial \theta}{\partial x_3'} = \frac{\partial h'}{\partial x_1'} \frac{\partial \theta}{\partial x_1'} + \frac{\partial h'}{\partial x_2'} \frac{\partial \theta}{\partial x_2'} - (\mathrm{Bi}\theta + 1) N'^{1/2}\,. \tag{3.3.3d}$$

Finally, in place of (2.3.6b) and (2.3.6a) we write:

$$v_3 = \frac{\partial h'}{\partial t'} + v_1 \frac{\partial h'}{\partial x_1'} + v_2 \frac{\partial h'}{\partial x_2'}\,, \quad x_3' = h'(t', x_1', x_2')\,, \tag{3.3.3e}$$

and

$$v_1 = v_2 = v_3 = 0\,, \quad \theta = 1\,, \quad x_3' = 0\,. \tag{3.3.3f}$$

A BM problem similar to (3.3.2), (3.3.3) was considered recently by Nepomnyashchy and Velarde [33], but our boundary conditions (3.3.3a) – (3.3.3d) are somewhat different [our conditions (3.3.3) are consistent with the exact starting boundary conditions (2.2.2) and (2.2.3), if we take into account the definition of the unit vectors tangential ($\mathbf{t}'^{(1)}$ and $\mathbf{t}'^{(2)}$) and normal ($\mathbf{n}$) to the deformed free surface, at the incompressible limit (3.3.1)].

3.3.1 The BM Boundary-Layer (BMBL) model problem. The full BM model problem, (3.3.2), (3.3.3), is very complicated, but for a thin film flow we can apply the long-wave approximation. In this case, we derive a simplified BMBL model problem for high Reynolds numbers. According to the long-wave approximation, we assume that the characteristic value for the horizontal wavelength $\lambda^\circ \gg h^\circ$. In this case, instead of the dimensionless variables (t', x'_1, x'_2, x'_3), it is judicious to introduce the following new dimensionless variables:

$$x = \varepsilon x'_1\,, \quad y = \varepsilon x'_2\,, \quad z \equiv x'_3\,, \quad \tau = \varepsilon \operatorname{Re} t' \tag{3.3.4}$$

and new functions

$$u = \frac{v_1}{\operatorname{Re}}\,, \quad v = \frac{v_2}{\operatorname{Re}}\,, \quad w = \frac{v_3}{\varepsilon \operatorname{Re}}\,, \quad \Pi = \frac{\pi}{\operatorname{Re}^2}\,, \tag{3.3.5}$$

where

$$\varepsilon = \frac{h^\circ}{\lambda^\circ}\,, \quad \operatorname{Re} = \frac{U^\circ h^\circ}{v_0}\,. \tag{3.3.6}$$

Finally, in place of the dimensionless parameters Fr, We and Ma we introduce corresponding modified Froude, Weber and Marangoni numbers based on the characteristic velocity U°:

$$Fr = \frac{U^\circ}{\sqrt{gh^\circ}}\,, \quad We = \frac{\sigma_0}{\rho_0 h^\circ U^{\circ 2}}\,, \quad Ma = \frac{\gamma \Delta T^\circ}{\rho_0 h^\circ U^{\circ 2}}\,. \tag{3.3.7}$$

When [48, 56]

$$\varepsilon \to 0\,, \quad \operatorname{Re} \to \infty\,, \quad \operatorname{We} \to \infty \tag{3.3.8a}$$

with the following similarity relations

$$\varepsilon \operatorname{Re} = \operatorname{Re}^* = O(1)\,, \quad \varepsilon^2\, We = \mathrm{W}^* = O(1) \tag{3.3.8b}$$

in place of the full BM model problem (3.3.2), (3.3.3), we derive for the functions

$$u(\tau, x, y, z), \quad v(\tau, x, y, z), \quad w(\tau, x, y, z),$$
$$\Pi(\tau, x, y, z), \quad \theta(\tau, x, y, z)\,, \quad h' = H(\tau, x)$$

the following approximate BMBL model problem:

$$\mathbf{D} \cdot \mathbf{V} + \frac{\partial w}{\partial z} = 0\,, \tag{3.3.9a}$$

$$\frac{\mathrm{D}\mathbf{V}}{\mathrm{D}\tau} + \mathbf{D}\Pi = \frac{1}{\operatorname{Re}^*}\,\frac{\partial^2 \mathbf{V}}{\partial z^2}\,, \tag{3.3.9b}$$

$$\frac{\partial \Pi}{\partial z} = 0 \Rightarrow \Pi = \frac{H-1}{Fr^2} - \mathrm{W}^* \mathbf{D}^2 H\,, \tag{3.3.9c}$$

$$\Pr \frac{\mathrm{D}\theta}{\mathrm{D}\tau} = \frac{1}{\operatorname{Re}^*}\,\frac{\partial^2 \theta}{\partial z^2}\,. \tag{3.3.9d}$$

At $z = 0$

$$u = v = w = 0\,, \quad \theta = 1\,, \tag{3.3.10a}$$

$$\frac{\partial \mathbf{V}}{\partial z} = -\operatorname{Re}^* Ma \left[\mathbf{D}\theta + (\mathbf{D}H) \frac{\partial \theta}{\partial z} \right], \tag{3.3.10b}$$

at $z = 0$

$$\frac{\partial \theta}{\partial z} + 1 + \mathrm{Bi}\theta = 0\,, \tag{3.3.10c}$$

$$w = \frac{\partial H}{\partial \tau} + \mathbf{V} \cdot \mathbf{D}H\,. \tag{3.3.10d}$$

In these BMBL model equations we have the operators

$$\frac{\mathrm{D}}{\mathrm{D}\tau} = \frac{\partial}{\partial \tau} + \mathrm{V} \cdot \mathbf{D} + w \frac{\partial}{\partial z}\,, \qquad \mathbf{D} = \frac{\partial}{\partial x}\,, \frac{\partial}{\partial y}\,,$$

$$\mathbf{D}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,.$$

In (3.3.9) and (3.3.10) the parameters Fr, Ma, Pr, Bi, Re* and W* are $O(1)$. The above BMBL problem is a very significant approximate model for the investigation of the instability problem for a thin, slightly viscous film, and this model deserves further consideration.

3.3.2 The limiting case: Pr→ 0. When Pr→ 0, the solution for the temperature perturbation θ is very simple, according to (3.3.9d), (3.3.10a) and (3.3.10c):

$$\theta = 1 - (1 + \mathrm{Bi})\,\frac{z}{1 + \mathrm{Bi}\,H}\,. \tag{3.3.11}$$

On the other hand (for all Pr and Re* fixed), from (3.3.3a) with the conditions:

at $z = 0$

$$w = 0\,,$$

at $z = 0$

$$w = \frac{\partial H}{\partial \tau} + \mathbf{V} \cdot \mathbf{D}H\,,$$

we derive the following averaged (evolution) equation

$$\frac{\partial H}{\partial t} + \mathbf{D} \cdot \int_0^H \mathbf{V}\,\mathrm{d}z = 0\,. \tag{3.3.12}$$

As a consequence, in this case, we derive the following system of two nonlinear evolution equations for the two functions $\mathbf{V}(\tau, x, y, z)$, and $H(\tau, x, y)$: (3.3.12) and

$$\frac{\mathrm{D}\mathbf{V}}{\mathrm{D}\tau} - \frac{1}{\operatorname{Re}^*}\,\frac{\partial^2 \mathbf{V}}{\partial z^2} = -\frac{1}{Fr^2}\,\mathbf{D}H + \mathrm{W}^* \mathbf{D}[\mathbf{D}^2 H]\,, \tag{3.3.13}$$

since

$$w = -\int_0^z (\mathbf{D} \cdot \mathbf{V})\,\mathrm{d}z\,.$$

For these two equations, (3.3.12) and (3.3.13), we have as boundary conditions (in z):

at $z = 0$

$$\mathbf{V} = 0\,,$$

at $z = 0$

$$\frac{\partial \mathbf{V}}{\partial z} = Ma \operatorname{Re}^* (1 + \mathrm{Bi})\,\frac{\mathbf{D}H}{(1 + \mathrm{Bi}H)^2}\,. \tag{3.3.14}$$

If now $\operatorname{Re}^* \ll 1$, in the limit $\operatorname{Re}^* \to 0$, from (3.3.13), we obtain, with (3.3.14), the following limiting solution for the horizontal velocity $\mathbf{V} = \mathbf{V}(H)$:

$$\mathbf{V}(H) = \frac{1}{3}\,\frac{\mathrm{d}}{\mathrm{d}H} \left\{ H^3 \left[a^\circ \mathbf{D}(\mathbf{D}^2 H) - \mathbf{D}H \right] + b^\circ (1 + \mathrm{Bi})\, H^2 \frac{\mathbf{D}H}{(1 + \mathrm{Bi}\, H)^2} \right\} \tag{3.3.15}$$

and, as a consequence, in place of (3.3.12), we derive a single evolution equation for the thickness of the film $H(\tau, x, y)$:

$$\frac{\partial H}{\partial t}+\frac{1}{3}\,\mathbf{D}\cdot\left\{H^3\left[a^\circ\mathbf{D}(\mathbf{D}^2H)-\mathbf{D}H\right]+b^\circ(1+\mathrm{Bi})\,H^2\frac{\mathbf{D}H}{(1+\mathrm{Bi}\,H)^2}\right\}=0\,. \tag{3.3.16}$$

In (3.3.15) and (3.3.16)

$$a^\circ=\frac{\sigma_0}{\rho_0 g\lambda^{\circ 2}}\,,\quad b^\circ=\frac{1}{2}\,\frac{\gamma\Delta T^\circ}{\rho_0 g h^{\circ 2}} \tag{3.3.17}$$

with the following choice for the characteristic velocity: $U^\circ=gh^{\circ 3}/\lambda^\circ\nu_0$.

In the above equation (3.3.16) (very similar to that derived by A Oron and Ph Rosenau [21]), the term proportional to a° has a stabilizing effect while the term proportional to b°, on the contrary, has a destabilizing impact.

It should be noted that gravity [the term $-(1/3)\mathbf{D}\cdot[H^3\mathbf{D}H]$ in (3.3.16)] stabilizes the evolution of the interface when the film is supported from below.

3.4 The BM problem for a free-falling vertical film

It what follows we consider the thermocapillary instabilities on a free-falling vertical (2D) film, since most experiments and theories have focused on the latter — in fact, wave dynamics on an inclined plane is quite analogous.

In dimensionless variables [see (3.4.3)], the governing equations and boundary conditions for the (incompressible but thermally conducting) liquid motion down the vertical plane are as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0\,, \tag{3.4.1a}$$

$$\frac{\mathrm{D}u}{\mathrm{D}t}+\frac{\partial p}{\partial x}-\frac{1}{\varepsilon\mathrm{Fr}^2}=\frac{1}{\varepsilon\,\mathrm{Re}}\left(\frac{\partial^2 u}{\partial z^2}+\varepsilon^2\frac{\partial^2 u}{\partial x^2}\right), \tag{3.4.1b}$$

$$\varepsilon^2\frac{\mathrm{D}w}{\mathrm{D}t}+\frac{\partial p}{\partial z}=\frac{\varepsilon}{\mathrm{Re}}\left(\frac{\partial^2 w}{\partial z^2}+\varepsilon^2\frac{\partial^2 w}{\partial x^2}\right), \tag{3.4.1c}$$

$$\mathrm{Pr}\frac{\mathrm{D}\theta}{\mathrm{D}t}=\frac{1}{\varepsilon\mathrm{Re}}\left(\frac{\partial^2\theta}{\partial z^2}+\varepsilon^2\frac{\partial^2\theta}{\partial x^2}\right), \tag{3.4.1d}$$

where $\mathrm{D}/\mathrm{D}t=\partial/\partial t+u\partial/\partial x+w\partial/\partial z$

At $z=0$

$$u=w=0\,,\quad \theta=1\,, \tag{3.4.2a}$$

at $z=h(t,x)$

$$\frac{\partial u}{\partial z}=-\varepsilon\,\mathrm{Ma}\left(\frac{\partial\theta}{\partial x}+\frac{\partial h}{\partial x}\frac{\partial\theta}{\partial z}\right)-\varepsilon^2\left(\frac{\partial w}{\partial x}+4\frac{\partial h}{\partial x}\frac{\partial w}{\partial z}\right)-\frac{3}{2}\varepsilon^3\,\mathrm{Ma}\left(\frac{\partial h}{\partial x}\right)^2\left(\frac{\partial\theta}{\partial x}+\frac{\partial h}{\partial x}\frac{\partial\theta}{\partial z}\right), \tag{3.4.2b}$$

$$p=p_{\mathrm{a}}+2\frac{\varepsilon}{\mathrm{Re}}\left(\frac{\partial w}{\partial z}-\frac{\partial h}{\partial x}\frac{\partial u}{\partial z}\right)-\varepsilon^2\,\mathrm{We}\frac{\partial^2 h}{\partial x^2}+\varepsilon^2\frac{\mathrm{Ma}}{\mathrm{Re}}\frac{\partial^2 h}{\partial x^2}\,\theta+2\frac{\varepsilon^3}{\mathrm{Re}}\left[\left(\frac{\partial h}{\partial x}\right)^3\frac{\partial u}{\partial z}-2\left(\frac{\partial h}{\partial x}\right)^2\frac{\partial w}{\partial z}-\frac{\partial h}{\partial x}\frac{\partial w}{\partial x}\right], \tag{3.4.2c}$$

$$\frac{\partial\theta}{\partial z}=-(1-\mathrm{Bi}\,\theta)+\varepsilon^2\left[\frac{\partial h}{\partial x}\frac{\partial\theta}{\partial x}-\frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2(1+\mathrm{Bi}\,\theta)\right], \tag{3.4.2d}$$

$$w=\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}\,. \tag{3.4.2e}$$

We note that the free-surface boundary conditions (3.4.2b – d) are written with an error of $O(\varepsilon^4)$. In (3.4.1) and (3.4.2) all the functions and variables are dimensionless, namely (the * marks dimensional quantities),

$$x=\frac{x^*}{\lambda^\circ}\,,\ z=\frac{z^*}{h^\circ}\,,\ t=\frac{t^*}{t^\circ}\,,\ u=\frac{u^*}{U^\circ}\,,\ w=\frac{w^*}{\varepsilon U^\circ}\,,\ p=\frac{p^*}{\rho_0 U^{\circ 2}}\,,$$
$$\theta=\frac{T^*-T^\circ}{\Delta T^\circ}\,,\ t^\circ=\frac{\lambda^\circ}{U^\circ}\,,\ p_{\mathrm{a}}=\frac{p_{\mathrm{a}}^*}{\rho_0 U^{\circ 2}}\,, \tag{3.4.3}$$

and h°, λ°, U°, ρ_0, T°, $\Delta T^\circ>0$ are the characteristic values for the thickness of the film, wavelength, velocity, density, temperature and rate of increase at the lower horizontal boundary $z^*=0$. In the equations (3.4.1) and boundary conditions (3.4.2) we have the following dimensionless parameters:

$\varepsilon=h^\circ/\lambda^\circ$ — long-wave parameter,
$\mathrm{Re}=Uh^\circ/\nu_0$ — Reynolds number,
$\mathrm{Fr}=U^\circ/\sqrt{gh^\circ}$ — Froude number,
$\mathrm{Pr}=\nu_0/\varkappa_0$ — Prandtl number,
$\mathrm{Ma}=\gamma\Delta T^\circ/U^\circ\rho_0\nu_0$ — Marangoni number,
$\mathrm{We}=\sigma_0/h^\circ\rho_0 U^{\circ 2}$ — Weber number,
$\mathrm{Bi}=q^\circ h^\circ/k_0$ — Biot number. (3.4.4)

We note that x^* (along the vertical plate) and z^* (perpendicular to vertical plate) are horizontal and vertical coordinates, respectively; u^* and w^* are the corresponding horizontal and vertical velocity components. For the surface tension $\sigma(T)$ we again have relation (2.1.2). As the characteristic velocity U° we can choose the interface velocity

$$U^\circ=g\frac{h^{\circ 2}}{\nu_0}\,, \tag{3.4.5}$$

and in this case $\mathrm{Re}/\mathrm{Fr}^2=1$. The product

$$\mathrm{Re\,We}=\frac{\sigma_0}{\rho_0 g h^{\circ 2}}=\frac{1}{\mathrm{K}}\,, \tag{3.4.6}$$

where K is the so-called capillary number.

4. High Reynolds number boundary-layer regime: Re ≫ 1

4.1 Generalized integral boundary-layer model equations

From the continuity condition (3.4.1a), the kinematic condition (3.4.2e) and the condition that $w=0$ at $z=0$, we can easily derive the following averaged continuity equation:

$$\frac{\partial h}{\partial t}+\frac{\partial q}{\partial x}=0\,, \tag{4.1.1}$$

where

$$q(t,x)=\int_0^{h(t,x)}u(t,x,z)\,\mathrm{d}z\,. \tag{4.1.2}$$

The function $q(t,x)$ is the velocity component u (parallel to the solid vertical plate $z=0$) averaged over the film thickness. Due to the fact that $q(t,x)$, as a general rule, cannot be expressed in terms of $h(t,x)$ or some of its spatial derivatives, equation (4.1) is not a closed-form evolution equation for the thickness $h(t,x)$. Our main goal in this section is to examine a particular situation arising mainly from averaging process where a closed system of equations can be obtained. Specifically, we consider the following situation (long waves and large Reynolds numbers):

$$\varepsilon \ll 1\,, \quad \mathrm{Re} \gg 1\,, \quad \varepsilon\,\mathrm{Re} = \mathrm{Re}^* = O(1)\,. \tag{4.1.3}$$

With (4.1.3), in place of (3.4.2c) we obtain the limiting equation $\partial p/\partial z = 0$ and in this case, according to (3.4.2c), we derive the following relation between p and h:

$$p = p(h) \equiv p_{\mathrm{a}} - \mathrm{W}^* \frac{\partial^2 h}{\partial x^2}\,, \tag{4.1.4}$$

if (large Weber numbers)

$$\varepsilon\,\mathrm{We} = \mathrm{W}^* = O(1)\,. \tag{4.1.5}$$

As a consequence, the following leading-order limiting system for u, w, θ and h is derived:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0\,, \tag{4.1.6a}$$

$$\mathrm{Re}^* \frac{\mathrm{D}u}{\mathrm{D}t} - \frac{\partial^2 u}{\partial z^2} = 1 + \frac{1}{\mathrm{K}^*}\frac{\partial^3 h}{\partial x^3}\,, \tag{4.1.6b}$$

$$\mathrm{Re}^*\,\mathrm{Pr}\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{\partial^2 \theta}{\partial z^2}\,, \tag{4.1.6c}$$

with $1/\mathrm{K}^* = \mathrm{Re}^*\,\mathrm{W}^* = O(1)$ and if we assume that $\mathrm{Re}/\mathrm{Fr}^2 = 1$.

For equations (4.1.6) we have the following reduced boundary conditions:

at $z=0$

$$u = w = 0\,, \quad \theta = 1\,, \tag{4.1.7a}$$

at $z = h(t,x)$

$$\frac{\partial u}{\partial z} = -\mathrm{M}^*\left(\frac{\partial \theta}{\partial x} + \frac{\partial h}{\partial x}\frac{\partial \theta}{\partial z}\right), \tag{4.1.7b}$$

$$\frac{\partial \theta}{\partial z} = -(1 + \mathrm{Bi}\,\theta)\,, \tag{4.1.7c}$$

$$w = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}\,, \tag{4.1.7d}$$

if we assume that (large Marangoni numbers)

$$\varepsilon\,\mathrm{Ma} = \mathrm{M}^* = O(1)\,. \tag{4.1.8}$$

The above, reduced problem (4.1.6), (4.1.7) remains a complicated boundary-value problem. When $\mathrm{M}^* = 0$ (in this case the thermal field is decoupled from the dynamical field), Shkadov [51], using the integral method, have reduced the problem to a system of two averaged equations for $h(t,x)$ and $q(t,x)$ and applied also the self-similarity assumption for the horizontal velocity component u (see Section 4.1.1 below). For example, using the integral method, we can write the following averaged equation in place of (4.1.6b):

$$\mathrm{Re}^*\left\{\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left[\int_0^{h(t,x)} u^2\,\mathrm{d}z\right]\right\} + \left(\frac{\partial u}{\partial z}\right)_{z=0} = \mathrm{M}^*\left(\frac{\partial \theta}{\partial x} + \frac{\partial h}{\partial x}\frac{\partial \theta}{\partial z}\right) + \frac{1}{\mathrm{K}^*}\,h\frac{\partial^3 h}{\partial x^3} + h\,, \tag{4.1.9}$$

if we use the boundary condition (4.1.7b).

4.1.1 Shkadov averaged classical model (Ma = 0).

When Ma = 0, the term proportional to Ma^* in (4.1.9) disappears and the thermal field is decoupled from the dynamical field. Since (when Ma = 0)

at $z=0$

$$u = 0\,,$$

at $z = h(t,x)$

$$\frac{\partial u}{\partial z} = 0\,,$$

we can use

$$u(t,x,z) = \frac{U(t,x)}{h(t,x)}\left[z - \frac{1}{2h(t,x)}z^2\right] \tag{4.1.10}$$

as a self-similarity assumption for $u(t,x,z)$.

With (4.1.10) we easily derive the following equation relating $h(t,x)$ and $q(t,x)$ in place of (4.1.9):

$$\mathrm{Re}^*\left[\frac{\partial q}{\partial t} + \frac{6}{5}\frac{\partial}{\partial x}\left(\frac{q^2}{h}\right)\right] + \frac{3}{h^2}\,q = \frac{1}{\mathrm{K}^*}\,h\frac{\partial^3 h}{\partial x^3} + h\,, \tag{4.1.11}$$

where we assume that the capillary number $\mathrm{K}^* = O(1)$. We note that with (4.1.2) and (4.1.10) we have the following relation between $q(t,x)$ and $U(t,x)$:

$$U(t,x) = \frac{3}{h}q\,. \tag{4.1.12}$$

The system of two evolution equations, (4.1.1) and (4.1.11), for $h(t,x)$ and $q(t,x)$ forms the Shkadov averaged model.

4.1.2 Shkadov generalized model with the Marangoni effect (Pr = 0).

When Pr = 0, the horizontal velocity u satisfies the following two boundary conditions:

at $z=0$

$$u = 0\,,$$

at $z = h(t,x)$

$$u = \mathrm{M}^*(1+\mathrm{Bi})\frac{\partial h/\partial x}{(1+\mathrm{Bi}\,h)^2}\,,$$

since for Pr = 0, in place of (4.6c), we have the limiting equation $\partial^2\theta/\partial z^2 = 0$, with the boundary conditions: $\theta = 1$ at $z=0$ and $\partial\theta/\partial z = -(1+\mathrm{Bi}\,\theta)$ at $z = h(t,x)$, and the solution for θ is:

$$\theta = 1 - (1+\mathrm{Bi})\frac{z}{1+\mathrm{Bi}\,h}\,. \tag{4.1.13}$$

As a consequence, in place of (4.10) we write:

$$u(t,x,z) = \frac{U(t,x)}{h(t,x)}\left[z - \frac{1}{2h(t,x)}z^2\right] + \mathrm{M}^*(1+\mathrm{Bi})\frac{\partial h/\partial x}{(1+\mathrm{Bi}\,h)^2}\,z\,. \qquad (4.1.14)$$

In this case, thanks to (4.1.14), we obtain [in place of (4.1.12)]

$$U(t,x) = \frac{3}{h}q - \frac{3}{2}\mathrm{M}^*\frac{1+\mathrm{Bi}}{(1+\mathrm{Bi}\,h)^2}\,h\frac{\partial h}{\partial x} \qquad (4.1.15)$$

and from (4.1.9) we derive a second averaged equation relating $h(t,x)$ and $q(t,x)$:

$$\mathrm{Re}^*\left\{\frac{\partial q}{\partial t} + \frac{6}{5}\frac{\partial}{\partial x}\left(\frac{q^2}{h}\right) + \frac{1}{20}\mathrm{M}^*(1+\mathrm{Bi})\frac{\partial}{\partial x}\times\left[h\frac{\partial h}{\partial x}\frac{q}{(1+\mathrm{Bi}\,h)^2}\right]\right\} + \frac{3}{h^2}q = \frac{1}{\mathrm{K}^*}h\frac{\partial^3 h}{\partial x^3} + h + \frac{3}{2}\mathrm{M}^*(1+\mathrm{Bi})\frac{\partial h/\partial x}{(1+\mathrm{Bi}\,h)^2} - \frac{1}{120}\mathrm{Re}^*\mathrm{M}^{*2}(1+\mathrm{Bi})^2\frac{\partial}{\partial x}\left[\left(\frac{\partial h}{\partial x}\right)^2\frac{h^3}{(1+\mathrm{Bi}\,h)^4}\right]. \qquad (4.1.16)$$

The system of two equations (4.1.1) and (4.1.16), for two functions $h(t,x)$ and $q(t,x)$, generalizes the Shkadov classical system for the case when the Marangoni effect is important and the Biot effect is taken into account, but $\mathrm{Pr}=0$.

4.1.3 Shkadov generalized model with Marangoni and Prandtl effects ($\mathrm{Bi}=0$). This case is more complicated, since for $\mathrm{Pr}\neq 0$ it is necessary to derive a third averaged equation from the equation (4.1.6c) for θ. Here we derive such an averaged equation only for the case when $\mathrm{Bi}=0$ (it is not clear if it is possible to derive such an averaged equation for the case $\mathrm{Bi}\neq 0$!). For this, it is necessary to introduce the new function Θ in place of θ such that

$$\Theta = \theta - (1-z)\,.$$

The function $\Theta(t,x,z)$ is the solution of the following problem [according to (4.1.6c), (4.1.7a) and (4.1.7c), with $\mathrm{Bi}=0$]:

$$\mathrm{Re}^*\mathrm{Pr}\left(\frac{\mathrm{D}\Theta}{\mathrm{D}t} - w\right) = \frac{\partial^2\Theta}{\partial z^2}\,, \qquad (4.1.17a)$$

at $z=0$

$$\Theta = 0\,, \qquad (4.1.17b)$$

at $z=0$

$$\frac{\partial\Theta}{\partial z} = 0\,. \qquad (4.1.17c)$$

According to condition (4.1.17c), in place of (4.1.7b) we can write the condition

$$\frac{\partial u}{\partial z} = \mathrm{M}^*\left(\frac{\partial h}{\partial x} - \frac{\partial\Theta}{\partial x}\right), \quad z = h(t,x)\,. \qquad (4.1.18)$$

From (4.1.17) we derive the averaged equation

$$\mathrm{Pr}\,\mathrm{Re}^*\left[\frac{\partial\chi}{\partial t} + \frac{\partial}{\partial x}\left(\int_0^{h(t,x)} u\Theta\,\mathrm{d}z - \int_0^{h(t,x)} w\,\mathrm{d}z\right)\right] + \left(\frac{\partial\Theta}{\partial z}\right)_{z=0} = 0\,, \qquad (4.1.19)$$

where

$$\chi(t,x) = \int_0^{h(t,x)}\Theta(t,x,z)\,\mathrm{d}z\,. \qquad (4.1.20)$$

But thanks to the conditions (4.1.17b, c) we can assume the following self-similarity for the solution $\Theta(t,x,z)$

$$\Theta(t,x,z) = 2\left[1 - \frac{\Sigma(t,x)}{h(t,x)}\right]\left[z - \frac{1}{2h(t,x)}z^2\right]. \qquad (4.1.21)$$

Our problem is now to derive two averaged equations for three unknown functions $h(t,x)$, $q(t,x)$ and $\Sigma(t,x)$. Firstly, from (4.1.21) we easily obtain

at $= h(t,x)$

$$\frac{\partial\Theta}{\partial x} = \frac{\partial h}{\partial x} - \frac{\partial\Sigma}{\partial x}\,,\quad \frac{\partial u}{\partial z} = \mathrm{M}^*\frac{\partial\Sigma}{\partial x}\,. \qquad (4.1.22)$$

As a consequence of (4.1.22) we write for $u(t,x,z)$ [by analogy with (4.1.14), but with $\mathrm{Bi}=0$]:

$$u(t,x,z) = \frac{U(t,x)}{h(t,x)}\left[z - \frac{1}{2h(t,x)}z^2\right] + \mathrm{M}^*\frac{\partial\Sigma}{\partial x}\,z\,. \qquad (4.1.23)$$

With (4.1.23) we obtain

$$q(t,x) = \frac{1}{3}\,Uh + \frac{1}{2}\,\mathrm{M}^*h^2\frac{\partial\Sigma}{\partial x}\,, \qquad (4.1.24a)$$

and (in place of (4.1.15), since $\mathrm{Bi}=0$)

$$U(t,x) = \frac{3}{h}q - \frac{3}{2}\mathrm{M}^*h\frac{\partial\Sigma}{\partial x}\,. \qquad (4.1.24b)$$

By analogy with (4.1.16) we derive the following averaged equation (in the terms proportional to M^* we replace $\partial h/\partial x$ by $\partial\Sigma/\partial x$ and we assume that $\mathrm{Bi}=0$):

$$\mathrm{Re}^*\left[\frac{\partial q}{\partial t} + \frac{6}{5}\frac{\partial}{\partial x}\left(\frac{q^2}{h}\right) + \frac{1}{20}\mathrm{M}^*\frac{\partial}{\partial x}\left(h\frac{\partial\Sigma}{\partial x}q\right)\right] + \frac{3}{h^2}q = \frac{1}{\mathrm{K}^*}h\frac{\partial^3 h}{\partial x^3} + h + \frac{3}{2}\mathrm{M}^*\frac{\partial\Sigma}{\partial x} - \frac{1}{120}\mathrm{Re}^*\mathrm{M}^{*2}\frac{\partial}{\partial x}\left[\left(\frac{\partial\Sigma}{\partial x}\right)^2 h^3\right]. \qquad (4.1.25)$$

Now from (4.1.20), and (4.1.21) we obtain the relation

$$\chi = \frac{2}{3}h(h-\Sigma)\,, \qquad (4.1.26)$$

and in (4.1.9) for the term $\partial\chi/\partial t$ we find: $-(2/3)h\{\partial\Sigma/\partial t + [2-(\Sigma/h)]\partial q/\partial x\}$. Further,

$$\int_0^{h(t,x)} w\,\mathrm{d}z = -\frac{1}{48}\,\mathrm{M}^*\frac{\partial}{\partial x}\left(h^3\frac{\partial\Sigma}{\partial x}\right) + q\frac{\partial h}{\partial x} - \frac{3}{8}\frac{\partial(hq)}{\partial x}\,, \qquad (4.1.27a)$$

and

$$\int_0^{h(t,x)} u\Theta \,\mathrm{d}z = \frac{12}{15} q(h-\Sigma) + \frac{1}{60}\mathrm{M}^*(h-\Sigma)h^2 \frac{\partial \Sigma}{\partial x}\,. \tag{4.1.27b}$$

Since $(\partial\Theta/\partial z)_{z=0} = 2[1-(\Sigma/h)]$, for $\Sigma(t,x)$ we derive the following averaged equation, in place of (4.1.19),

$$\frac{\partial \Sigma}{\partial t} + \left(2-\frac{\Sigma}{h}\right)\frac{\partial q}{\partial x} + \frac{6}{5h}\frac{\partial}{\partial x}\big[q(\Sigma-h)\big] - \frac{9}{16h}\frac{\partial}{\partial x}(qh) + \frac{3}{2}\frac{q}{h}\frac{\partial h}{\partial x} + \frac{\mathrm{M}^*}{40h}\frac{\partial}{\partial x} \times \left[h^2(\Sigma-h)\frac{\partial \Sigma}{\partial x}\right] + \frac{\mathrm{M}^*}{32h}\frac{\partial}{\partial x}\left(h^3\frac{\partial\Sigma}{\partial x}\right) + \frac{3}{\mathrm{Re}^*\mathrm{Pr}}\frac{\Sigma-h}{h^2} = 0\,. \tag{4.1.28}$$

Finally for our three unknown functions, $h(t,x)$, $q(t,x)$ and $\Sigma(t,x)$, we derive an averaged system of three equations, (4.1.1), (4.1.25) and (4.1.28).

4.2 Analysis of the linear problem

A basic (constant) solution of the averaged equations (4.1.1), (4.1.25) and (4.1.28) for $h(t,x)$, $q(t,x)$ and $\Sigma(t,x)$ is

$$h=1\,, \quad q=\frac{1}{3}\,, \quad \Sigma=1\,. \tag{4.2.1}$$

Since $h(t,x) = 1+\delta\eta(t,x)$, we can write

$$q = \frac{1}{3} + \delta\varphi(t,x)\,, \qquad \Sigma = 1+\delta\zeta(t,x)\,. \tag{4.2.2}$$

When $\delta \ll 1$, from (4.1.1), (4.1.25) and (4.1.28) we derive the following system of two linear evolution equations for $\eta(t,x)$ and $\zeta(t,x)$:

$$\frac{\partial^2\eta}{\partial t^2} + \frac{4}{5}\frac{\partial^2\eta}{\partial x\partial t} + \frac{2}{15}\frac{\partial^2\eta}{\partial x^2} + \frac{3}{2}\frac{\mathrm{M}^*}{\mathrm{Re}^*}\frac{\partial^2\zeta}{\partial x^2} - \frac{\mathrm{M}^*}{60}\frac{\partial^3\zeta}{\partial x^3} + \mathrm{W}^*\frac{\partial^4\eta}{\partial x^4} + \frac{3}{\mathrm{Re}^*}\left(\frac{\partial\eta}{\partial x}+\frac{\partial\eta}{\partial t}\right) = 0\,, \tag{4.2.3a}$$

$$\frac{\partial\zeta}{\partial t} - \frac{7}{16}\frac{\partial\eta}{\partial t} - \frac{7}{80}\frac{\partial\eta}{\partial x} + \frac{2}{5}\frac{\partial\zeta}{\partial x} + \frac{\mathrm{M}^*}{32}\frac{\partial^2\zeta}{\partial x^2} + \frac{3}{\mathrm{Pr}\,\mathrm{Re}^*}(\zeta-\eta) = 0\,, \tag{4.2.3b}$$

since $\partial\varphi/\partial x = -\partial\eta/\partial t$.

This closed linear system (4.2.3a, b) can be resolved numerically (with appropriate initial conditions and periodicity relative to x) for the investigation of the stability of the (constant) Nusselt type smooth basic regime (4.2.1). Here we consider only infinitesimal disturbances of the form

$$\eta(t,x) = A^\circ \exp\big[\mathrm{i}k(x-ct)\big]\,, \quad \zeta(t,x) = B^\circ \exp\big[\mathrm{i}k(x-ct)\big] \tag{4.2.4}$$

and in such a case we derive the following dispersion relation

$$\frac{3}{\mathrm{Re}^*}(c-1) - \mathrm{i}k\left(c^2-\frac{4}{5}c+\frac{2}{15}\right) + \mathrm{i}k^3\mathrm{W}^* = \frac{B^\circ}{A^\circ}\mathrm{M}^*\left(\frac{k^2}{60}+\frac{3}{2}\mathrm{i}k\frac{1}{\mathrm{Re}^*}\right), \tag{4.2.5}$$

with

$$B^\circ\left[\frac{3}{\mathrm{Pr}\,\mathrm{Re}^*} - \frac{1}{32}\mathrm{M}^*k^2 - \mathrm{i}k\left(c-\frac{2}{5}\right)\right] - A^\circ\left[\frac{3}{\mathrm{Pr}\,\mathrm{Re}^*} + \frac{7}{16}\mathrm{i}k\left(c-\frac{1}{5}\right)\right] = 0\,. \tag{4.2.6}$$

From (4.2.6) we see that the ratio B°/A° is a complex function of k and c, and as a consequence, when $\mathrm{Pr} \neq 0$, the dispersion relation (4.2.5) is very complicated! Therefore, below we consider only the case when $\mathrm{Pr}=0$ but $\mathrm{M}^* \neq 0$. In this case $A^\circ = B^\circ$ and in place of the characteristic equation (4.2.5) we obtain the following reduced dispersion relation (with the Marangoni effect):

$$\frac{3}{\mathrm{Re}^*}(c-1) - \mathrm{i}k\left(c^2-\frac{4}{5}c+\frac{2}{15}\right) + \mathrm{i}k^3\mathrm{W}^* = \frac{1}{2}\mathrm{M}^*\left(\frac{k^2}{30}+3\mathrm{i}\frac{k}{\mathrm{Re}^*}\right). \tag{4.2.7}$$

Since the stability with respect to disturbances limited throughout the space at any moment of time is of particular interest here, the wavenumber k in (4.2.7) is assumed to be real. Then the values of the complex phase velocity $c = c_\mathrm{r} + \mathrm{i}c_\mathrm{i}$ may be found from the real and imaginary part of (4.2.7), respectively:

$$\frac{3}{\mathrm{Re}^*}(1-c_\mathrm{r}) + kc_\mathrm{i}\left(\frac{4}{5}-2c_\mathrm{r}\right) + \frac{1}{60}\mathrm{M}^*k^2 = 0\,, \tag{4.2.8a}$$

and

$$\frac{3}{\mathrm{Re}^*}c_\mathrm{i} - k\left(c_\mathrm{r}^2 - c_\mathrm{i}^2 - \frac{4}{5}c_\mathrm{r} + \frac{2}{15} + \frac{3}{2}\frac{\mathrm{M}^*}{\mathrm{Re}^*}\right) + k^3\mathrm{W}^* = 0\,. \tag{4.2.8b}$$

If $c_\mathrm{i} > 0$ the disturbance is amplified and if $c_\mathrm{i} < 0$ it disappears. From (4.2.8a), when $c_\mathrm{i} = 0$, we derive the following relation for the phase speed of a neutral disturbance:

$$c_\mathrm{r}^* = c^* = 1 + \frac{\mathrm{M}^*\,\mathrm{Re}^*}{180}k^2\,, \tag{4.2.9}$$

and infinitesimal disturbances thus appear to be dispersive – the coefficient

$$\beta = \frac{\mathrm{M}^*\,\mathrm{Re}^*}{180} \tag{4.2.10}$$

is the rate of the dispersive term and is an explicit function of M^*. The imaginary part, (4.2.8b), when $c_\mathrm{i} = 0$ and according to (4.2.9), gives a biquadratic algebraic equation for the neutral wavenumber k^*

$$\beta(k^{*2})^2 + \left(\frac{6}{5}\beta - \mathrm{W}^*\right)k^{*2} + \frac{1}{3} + \frac{3}{2}\frac{\mathrm{M}^*}{\mathrm{Re}^*} = 0\,. \tag{4.2.11}$$

It is obvious that for

$$\mathrm{W}^* \geqslant \frac{6}{5}\beta + 2\beta\left(\frac{1}{3}+\frac{3}{2}\frac{\mathrm{M}^*}{\mathrm{Re}^*}\right)^{1/2} \tag{4.2.12}$$

(when $\mathrm{M}^* \neq 0$ or $\beta \neq 0$) we obtain one or two values for k^{*2} and these values are always positive.

A particular case corresponds to the relation [in (W^*, M^*, Re^*) space]

$$\frac{30W^*}{M^* Re^*} = \frac{1}{5} + \frac{1}{3}\left(\frac{1}{3} + \frac{3}{2}\frac{M^*}{Re^*}\right)^{1/2} \tag{4.2.13}$$

between the dimensionless parameters W^*, M^*, and Re^*, and in this case a single neutral wavenumber k^* (associated with the neutral curve of stability, corresponding to $c_i = 0$) exists such that:

$$k^{*2} = \frac{180}{M^* Re^*}\left(\frac{1}{3} + \frac{3}{2}\frac{M^*}{Re^*}\right)^{1/2}. \tag{4.2.14}$$

All the disturbances with $k < k^*$ are unstable and those with $k > k^*$ disappear.

When $M^* = 0$ we obtain, from (4.2.9) and (4.2.11), the classical result [13, 57]:

$$c_r^* = c^* = 1\,, \quad k^* = \left(\frac{1}{3W^*}\right)^{1/2} \tag{4.2.15a}$$

or

$$W_c^* = \frac{1}{3\,k^{*2}}\,; \tag{4.2.15b}$$

W_c^* is the 'critical W^* corresponding to the cutoff wavenumber k^*'.

As a consequence of the above analysis, for $M^* \neq 0$, we observe the existence of a cutoff wavenumber which is a function of the three dimensionless parameters: W^*, M^*, and Re^*. In fact, each of these parameters can play the role of a bifurcation parameter (in contrast with (4.2.15a, b), where the bifurcation parameter is only W^*). At the present time a numerical investigation of the full linear stability problem (4.2.3a, b) is in progress by S. Godts and M Zghal in L.M.L. (University of Lille I), but here I do not discuss the details of this investigation.

4.3 Stuart – Landau equation

The linearized stability theory is based upon the main assumption of small (infinitesimal) amplitude disturbances, that is, all terms involving quadratic or higher powers in the disturbances are neglected. Consequently, the solution of the governing partial differential equations for the disturbances is considerably simplified by linearization [see our linear equations (4.2.3a, b)] and Fourier analysis of the disturbances is used with great success to obtain the solutions. In fact, the linearized stability theory determines the conditions of a given steady flow (Nusselt smooth regime, for our case) which allow the growth of a small disturbance. According to the theory, the amplitude of the disturbance is found to grow exponentially in time for values of certain flow parameters above a critical value [see, for example, (4.2.15)]. In reality, such disturbances do not grow exponentially without limit!

So the validity of the process of linearization, even though it is so often used in many physical problems, has been questioned in connection with the instability problem. Landau [58] first described nonlinear instability phenomena of certain classes of flows using the nonlinear amplitude equation

$$\frac{d}{dt}\left(|A|^2\right) = \nu|A|^2 - \lambda|A|^4\,, \tag{4.3.1}$$

where $A = A(t)$ is the amplitude of the leading Fourier mode, ν and λ are constants, the latter being called the Landau constant. In general, ν, λ and A are complex. The parameter ν is the eigenvalue of the linear stability problem and if W^* is the bifurcation parameter, then

$$\mathrm{Real}(\nu) \sim W^* - W_c^*\,, \quad W^* \to W_c^*\,.$$

The case $\lambda = 0$ corresponds to the linear equation given by the linearized theory. The second term on the right-hand side of (4.3.1) is due to nonlinearity and may accelerate or decelerate the exponential growth of the disturbance depending on the signs of ν and λ. The original Stuart equation [59] is

$$\frac{dA}{dt} = \nu A + \mu A|A|^2\,, \tag{4.3.2}$$

and the equivalent form of (4.3.2) is

$$-c\,\frac{dA}{d\xi} = \nu A + \mu A|A|^2\,, \tag{4.3.3}$$

where $\xi = x - ct$ and $A = A(\xi)$.

Below we derive an equation of the form (4.3.3) for the Shkadov averaged nonlinear system (4.1.1), (4.1.11) for $h(t, x)$ and $q(t, x)$. The derivation of the Stuart – Landau type amplitude equation near criticality, using the technique of multiple scales, is now well known and the details can be found in Newell [60] or Stewartson and Stuart [61]. Our main small parameter is $\delta \ll 1$.

First we write

$$W^* = W_c^* + \sigma\delta^2 \tag{4.3.4}$$

and we consider the case when $\sigma > 0$ ($k < k^*$). As $\delta \ll 1$ we consider, in fact, a weakly nonlinear theory. For the phase velocity we write, according to (4.2.15) and (4.3.4),

$$c_r = c_r^* + \delta^2 c_2\,, \quad c_r^* = 1\,. \tag{4.3.5}$$

Next, we introduce slow variables

$$\xi_k = \delta^k \xi\,, \quad k = 1, 2, 3, \ldots, \tag{4.3.6}$$

where $\xi = x - c_r t$. But for a weakly nonlinear case, if we want to derive the associated Landau – Stuart (LS) envelope evolution, it is sufficient to assume that the amplitude of the wave-packet envelope is a function of $\xi_2 \equiv \eta$ only. As a consequence, we can assume that $h(t, x)$ and $q(t, x)$ have the following form:

$$h = h^*(\xi, \eta, \delta) = 1 + \delta h_1 + \delta^2 h_2 + \delta^3 h_3 + \ldots, \tag{4.3.7a}$$

$$q = q^*(\xi, \eta, \delta) = \frac{1}{3} + \delta q_1 + \delta^2 q_2 + \delta^3 q_3 + \ldots \tag{4.3.7b}$$

But, according to linear theory,

$$h_1(\xi, \eta) = A(\eta)E(\xi) + A^*(\eta)E(-\xi)\,, \tag{4.3.8}$$

with $E(\pm\xi) = \exp(\pm i k^* \xi)$, and A^* is the complex conjugate of the amplitude A ($AA^* = |A|^2$).

For the derivation of the LS equation for the amplitude $A(\eta)$ of the wave-packet envelope it is necessary to eliminate the secular terms in the equation for h_3 and q_3! In other words, we assume that the asymptotic expansions (4.3.7a, b) are uniformly valid with respect to the variable ξ. Now, taking into account the relations

$$\frac{\partial}{\partial t} = -c_{\mathrm{r}}\left(\frac{\partial}{\partial \xi} + \delta^2 \frac{\partial}{\partial \eta}\right), \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \delta^2 \frac{\partial}{\partial \eta}, \tag{4.3.9}$$

we substitute the above expansion (4.3.7) and (4.3.4), (4.3.5) for W^* and c into the nonlinear system of two equations (4.1.1), (4.1.11) and identify different orders to derive a sequence of differential equations. To obtain the LS amplitude equation at the lowest order, we only need to consider $n = 1, 2$ and 3 powers of the small parameter δ.

4.3.1 For $h_1(\xi, \eta)$ and $q_1(\xi, \eta)$ we obtain the classical homogeneous linear system:

$$\frac{\partial h_1}{\partial \xi} = \frac{\partial q_1}{\partial \xi}, \quad \Lambda(h_1, q_1) = 0, \tag{4.3.10}$$

where

$$\Lambda(h, q) \equiv -\frac{1}{5}\frac{\partial q}{\partial \xi} - \frac{2}{15}\frac{\partial h}{\partial \xi} + \frac{3}{\mathrm{Re}^*}(q - h) - \frac{1}{3k^{*2}}\frac{\partial^3 h}{\partial \xi^3}. \tag{4.3.11}$$

Since $\Lambda(E(\pm\xi),\ E(\pm\xi)) \equiv 0$, the solution of (4.3.10) is

$$q_1(\xi, \eta) = h_1(\xi, \eta) = A(\eta)E(\xi) + A^*(\eta)E(-\xi). \tag{4.3.12}$$

4.3.2 For $h_2(\xi, \eta)$ and $q_2(\xi, \eta)$ we derive a non-homogeneous system:

$$\frac{\partial h_2}{\partial \xi} = \frac{\partial q_2}{\partial \xi}, \tag{4.3.13a}$$

$$\Lambda(h_2, q_2) = -\frac{8}{15}\frac{\partial (q_1)^2}{\partial \xi} + \frac{3}{\mathrm{Re}^*} q_1^2 + \frac{1}{3k^{*2}} q_1 \frac{\partial^3 q_1}{\partial \xi^3}. \tag{4.3.13b}$$

If we take into account the solution (4.3.12) for $q_1(\xi, \eta)$ and $h_1(\xi, \eta)$, then in place of (4.3.13b) we obtain

$$\Lambda(h_2, q_2) = \frac{6}{\mathrm{Re}^*}|A|^2 + \left(\frac{3}{\mathrm{Re}^*} - \frac{7}{5}\mathrm{i}k^*\right)A^2 E(2\xi) + \left(\frac{3}{\mathrm{Re}^*} + \frac{7}{5}\mathrm{i}k^*\right)A^{*2}E(-2\xi). \tag{4.3.14}$$

The equations (4.3.13a) and (4.3.14) for $q_2(\xi, \eta)$ and $h_2(\xi, \eta)$ support solutions of the following type

$$h_2(\xi, \eta) = q_2(\xi, \eta) - \alpha|A|^2,$$
$$q_2(\xi, \eta) = B(\eta)E(2\xi) + B^*(\eta)E(-2\xi), \tag{4.3.15}$$

with $E(\pm 2\xi) = \exp(\pm 2\mathrm{i}k^*\xi)$

Since [according to (4.3.16)]

$$\Lambda(h_2, q_2) \equiv \frac{3}{\mathrm{Re}^*}\alpha|A|^2 + 2\mathrm{i}k^* B(\eta)E(2\xi) - 2\mathrm{i}k^* B^*(\eta)E(-2\xi),$$

we see that

$$\alpha = 2, \quad B(\eta) = \beta\left[A(\eta)\right]^2, \quad \beta = -\left(\frac{7}{10} + \mathrm{i}\frac{3}{2k^*\mathrm{Re}^*}\right). \tag{4.3.16}$$

Finally, for $h_2(\xi, \eta)$ and $q_2(\xi, \eta)$ we obtain as a solution

$$h_2(\xi, \eta) = q_2(\xi, \eta) - 2|A|^2,$$
$$q_2(\xi, \eta) = \beta\left[A(\eta)\right]^2 E(2\xi) + \beta^*\left[A^*(\eta)\right]^2 E(-2\xi), \tag{4.3.17}$$

where β^* is the complex conjugate of the coefficient β.

4.3.3 For $h_3(\xi, \eta)$ and $q_3(\xi, \eta)$ we first obtain the following equation

$$\frac{\partial h_3}{\partial \xi} - \frac{\partial q_3}{\partial \xi} = c_2 \frac{\partial h_1}{\partial \xi},$$

since $h_1 = q_1$, and we derive the following relation between h_3 and q_3:

$$h_3 = q_3 - c_2 h_1 + H(\eta). \tag{4.3.18a}$$

The second equation for $h_3(\xi, \eta)$ and $q_3(\xi, \eta)$ is

$$\Lambda(h_3, q_3) = \gamma A^3\left[A(\eta)\right]^3 E(3\xi) + S(A)E(\xi) + \text{c.c.} \tag{4.3.18b}$$

when we take into account the solutions (4.3.12) and (4.3.17). In (4.3.18b)

$$S(A) = -\frac{2}{3}\frac{\partial A}{\partial \eta} + \lambda A - \mu A|A|^2. \tag{4.3.19}$$

For the complex coefficients γ, λ and μ we have the following expressions:

$$\gamma = -\frac{55}{2\mathrm{Re}^*} + \mathrm{i}\left[\frac{507}{50}k^* - \frac{9}{k^*}\left(\frac{1}{\mathrm{Re}^*}\right)^2\right], \tag{4.3.20a}$$

$$\lambda = -\mathrm{i}k^*(c_2 - \sigma k^{*2}), \tag{4.3.20b}$$

$$\mu = \frac{93}{10\mathrm{Re}^*} + \mathrm{i}\left[\frac{31}{50}k^* + \frac{9}{k^*}\left(\frac{1}{\mathrm{Re}^*}\right)^2\right]. \tag{4.3.20c}$$

From (4.3.18b) we conclude that, necessarily, $H(\eta) = 0$ and if we utilize (4.3.18a) (with $H(\eta) = 0$) and the expression (4.3.11) for the operator $\Lambda(h_3, q_3)$, we derive for q_3 the following equation

$$\frac{\partial}{\partial \xi}\left(\frac{1}{k^{*2}}\frac{\partial^2 q_3}{\partial \xi^2} + q_3\right) = 3\left[\varkappa c_2 A - S(A)\right]E(\xi) - 3\gamma\left[A(\eta)\right]^3 E(3\xi) + \text{c.c.}, \tag{4.3.21}$$

where

$$\varkappa = \frac{3}{\mathrm{Re}^*} - \frac{1}{5}\mathrm{i}k^*. \tag{4.3.22}$$

The solution of (4.3.21) has the following form

$$q_3 = D(A)E(3\xi) - \frac{3}{2}\left[\varkappa c_2 A - S(A)\right]E(\xi)\xi + \text{c.c.} \tag{4.3.23}$$

and as a consequence the term proportional to $-(3/2)[\varkappa c_2 A - S(A)]E(\xi)$ is a secular term (this term is very large for very large values of ξ and, as a consequence, the term $\delta^3 q_3$ in (4.3.7b) may not be small relative to the term $\delta^2 q_2$!).

Finally, for the amplitude $A(\eta)$ we derive the following LS evolution equation:

$$\frac{2}{3}\frac{\partial A}{\partial \eta} + \nu A + \mu A|A|^2 = 0, \tag{4.3.24}$$

where

$$\nu = \varkappa c_2 - \lambda = \frac{3}{\mathrm{Re}^*} c_2 + \mathrm{i}\left(-\frac{6}{5} k^* c_2 + \delta k^{*3} \right), \qquad (4.3.25)$$

and for the complex Landau coefficient μ we have the formulae (4.3.20c).

It is not difficult to derive an analogous LS equation for the system (4.1.1), (4.1.16), when the Marangoni effect is taken into account, but this derivation is more laborious and the coefficients ν and μ, in this case, are rather awkward!

In fact, in (4.3.25) it is reasonable to choose

$$\left(-\frac{6}{5} k^* c_2 + \sigma k^{*3} \right) = 0 \rightarrow c_2 = \frac{5}{6} \sigma k^{*2}, \qquad (4.3.26)$$

and in this case, the LS evolution is

$$-\frac{\partial A}{\partial \eta} = \alpha \sigma A + \frac{3}{2} \mu A |A|^2 \qquad (4.3.27)$$

with

$$\alpha = \frac{15}{4} \frac{k^{*2}}{\mathrm{Re}^*} > 0, \quad \mu = \frac{93}{10\mathrm{Re}^*} + \mathrm{i}\left[\frac{31}{50} k^* + \frac{9}{k^*} \left(\frac{1}{\mathrm{Re}^*} \right)^2 \right]. \qquad (4.3.28)$$

The Landau – Stuart equation implies that the solution $|A| = 0$ is an equilibrium solution which is stable or unstable if $W^* < W^*_c$ or $W^* > W^*_c$, respectively, and $|A| \rightarrow |A|_e$ is a new equilibrium value as $\eta \rightarrow -\infty$. The branching of the curve of the equilibrium solution $|A| = 0$ at $\mathrm{W}^* = \mathrm{W}^*_c$ is called the Landau bifurcation.

Since Real(μ), when $\mathrm{W}^* > \mathrm{W}^*_c$, then $\sigma > 0$ so that both the terms on the right side of (4.3.27) are positive and hence $|A|$ increases exponentially as $\eta \rightarrow -\infty$. This case corresponds to a rapid transition to turbulence (chaos!).

4.4 Finite-dimensional dynamical-system approach

In many cases, it is allowable to deal with a dynamical system (DS) of finite dimension as a model of a continuous fluid and this is particularly the case at the stage of generation of turbulence, at which only a limited number of degrees of freedom of motion have been excited. In practice the study of so-called 'chaos' is mostly made on the basis of a low-dimensional dynamical system, whereas 'turbulence' is generally dealt with as motion in a fluid of infinite number of degrees of freedom. The approximation of a fluid in terms of a finite-dimensional model system provides us with a very powerful means of analysis and for this reason recent progress in the theory of chaos has enabled us to look straight at the fundamental mechanism of turbulence. On the other hand, it is generally recognized that turbulence, when fully developed, has a singular structure in space and time.

Such singular behavior of a liquid cannot be described correctly by means of a finite-dimensional model system. Thus, in a restricted sense, chaos in fluids covers only a part of turbulent phenomena. Much of chaos as a science is connected with the notion of 'sensitive dependence on initial conditions'. Technically, scientists term as 'chaotic' those non-random complicated motions that exhibit a very rapid growth of errors, which, despite perfect determinism, inhibits any pragmatic ability to render accurate long-term prediction. In fact, temporal chaos is known to appear in certain systems having only a few degrees of freedom. Take for example the Lorenz [62] model which has only three degrees of freedom — it was in this very Lorenz DS that the first strange attractor, in a problem arising from fluid dynamics, was discovered numerically by Lorenz in 1963.

Here, we consider [in fact, the case with Pr = 0: $\Sigma = h$ in (4.1.28)] the averaged system of equations (4.1.1) and (4.1.16) for the two functions $h(t, x)$ and $q(t, x)$.

To initiate a weakly nonlinear finite-dimensional dynamical-system analysis, we first expand these equations about the Nusselt smooth regime [see (4.2.1) and (4.2.2)].

One can show that up to the third order [relative to $\delta \ll 1$ and with an error of $O(\delta^2)$], the desired starting evolution equations are

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \eta}{\partial t} = 0, \qquad (4.4.1)$$

$$\begin{aligned}
&\frac{\partial \eta}{\partial t} + \frac{4}{5}\frac{\partial \varphi}{\partial x} - \frac{2}{15}\frac{\partial \eta}{\partial x} - \frac{3}{2}\frac{\mathrm{M}^*}{\mathrm{Re}^*}\frac{\partial \eta}{\partial x} + \frac{\mathrm{M}^*}{60}\frac{\partial^2 \eta}{\partial x^2} - \mathrm{W}^*\frac{\partial^3 \eta}{\partial x^3} \\
&\quad + \frac{3}{\mathrm{Re}^*}(\varphi - \eta) + \delta\left[\frac{\partial}{\partial x}\left(\frac{6}{5}\varphi^2 - \frac{4}{5}\varphi\eta + \frac{2}{15}\eta^2 \right) \right. \\
&\quad + \frac{3}{\mathrm{Re}^*}(\eta^2 - 2\varphi\eta) - \mathrm{W}^*\eta\frac{\partial^3 \eta}{\partial x^3} + \frac{\mathrm{M}^*}{20} \\
&\quad \left. \times \frac{\partial}{\partial x}\left(\varphi\frac{\partial \eta}{\partial x} + \frac{1}{3}\eta\frac{\partial \eta}{\partial x} \right) + \frac{\mathrm{M}^{*2}}{120}\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial x} \right)^2 \right] = 0. \qquad (4.4.2)
\end{aligned}$$

For a solution periodic in x we write:

$$\eta(t, x) = \sum A_p(t)\cos(pkx) + B_p(t)\sin(pkx), \ p \geqslant 1, \qquad (4.4.3a)$$

$$\varphi(t, x) = \sum C_p(t)\cos(pkx) + D_p(t)\sin(pkx), \ p \geqslant 1. \qquad (4.4.3b)$$

Then, by substituting (4.4.3a, b) into (4.4.1) and (4.4.2), we get an infinite systems of ordinary differential equations. If we introduce the vector of unknown amplitudes

$$\mathbf{X}_p(t) = \left[A_p(t), B_p(t), C_p(t), D_p(t) \right]^T, \qquad (4.4.4)$$

then this DS can be written (for $p \geqslant 1$) as

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{X}_p(t)}{\mathrm{d}t} &= L_0\mathbf{X}_p(t) + \frac{\delta}{2}\sum \left[L_1\mathbf{X}_m(t) \right]\mathbf{X}_{m+p}(t) \\
&\quad + \left[L_2\mathbf{X}_m(t) \right]\mathbf{X}_{m-p}(t), \quad m \geqslant 1, \qquad (4.4.5)
\end{aligned}$$

where L_0, L_1 and L_2 are constant matrix coefficients. These matrices contains the integers p, m and k and also the parameters Re^*, W^*, and M^*. The explicit expressions of these matrices (as functions of $p, m, k, \mathrm{Re}^*, \mathrm{W}^*, \mathrm{M}^*$) are given in Zghal's thesis [63]. We note that

$$A_{-p} = A_p, \quad B_{-p} = -B_p, \quad C_{-p} = C_p, \quad D_{-p} = -D_p.$$

In the process of numerical integration of the dynamical system (4.4.5), performed by Godts and Zghal (see [63]), the series seemed to be finite, containing N terms. In this process of numerical integration the wavenumber k plays the role of the bifurcation parameter and when k decreases the number of active modes in the dynamical system (4.4.5) increases.

Bifurcations of the dynamical system (4.4.5) appear with decreasing k. For a very small value of the parameter $\delta (= 0.025)$ we in fact have a quasiperiodic transition to chaos. It should be pointed out that as rule (for small δ), stochastic (chaotic) behavior was noted after the destruction of invariant tori. More precisely, chaos is the result of the strong instability of the three-frequency quasiperiodic regime.

An interesting quality of the dynamical system (4.4.5), relative to the number of active modes (amplitudes) N, is the rapid convergence of the stochastic properties to a limiting state. In fact, the difference in these stochastic properties for $N = 8$ and $N = 10$ is almost indistinguishable!

In Zghal's thesis, [63], the reader can find various results of the numerical integration of the dynamical system (4.4.5) for $N = 10$ (in this case we have 40 amplitudes equations) and various values of δ. For the characterization of the chaotic behavior of the above dynamical system (4.4.5), Zghal numerically calculated the correlation dimension, the Lyapunov exponents, the Poincaré section and the power spectra. Different types of strange attractors are also presented and the destabilizing Marangoni effect is studied. It is obvious that the dynamical system (4.4.5) is very sensitive to the values of the parameter δ and surprisingly, when this parameter increases, the route to chaos change! It is also necessary to note that the Marangoni effect (the influence of the terms proportional to M^*) accelerates the emergence of strange attractors.

The preliminary numerical results (for $\mathrm{M}^* = 0$) show that the curve that characterizes the variation of the correlation dimension with N, for $\delta = 0.025$, $k = 0.25$, $\mathrm{Re}^* = 7.5$, $\mathrm{W}^* = 1$ and $\mathrm{M}^* = 0$, tends (with increasing N) to an asymptotic constant position, which is between 3 and 4. Concerning the Poincaré section, for these values of δ, k, Re^*, W^*, and $\mathrm{M}^* = 0$ in this section we obtain a limited 'cloud zone' of intersection points. This feature is a very representative property of the temporal chaos. For these same values the power spectra of the amplitude is continuous. Finally, it is interesting to note that the amplitude variations (with time t) and the projection of two strange attractors onto the phase plane (A_1, B_1), for $k = 0.24$, but for two different initial states (amplitude differences being only 2%) are very different: when $t > 200$ long-term prediction is impossible (sensitive dependence of the initial conditions) and even flows that are quasi-identical for $t < 200$ ultimately differ strongly because of the exponential divergence phenomenon.

4.4.1 Competition between subharmonic and sideband secondary instabilities. The secondary instabilities on a falling film which cause a monochromatic wave to evolve into solitary waves have been examined by Cheng and Chang, [64], with a weakly nonlinear theory from a system of equations similar to (4.4.1), (4.4.2) but with $\mathrm{M}^* = 0$. According to [64], the unique phase-speed dispersion relation dictated by inertia and capillarity is found to favor a nonlinear, two-wave subharmonic resonance that causes neighboring crests to coalesce. This occurs when the fundamental wave frequency is below a critical value ω_c which can be approximated by a nonlinear resonant frequency. An entire band of secondary waves are excited by this static subharmonic mechanism such that the coalescence occurs non-uniformly. In the lower half of this band secondary waves travel faster than the fundamental, which induces the coalesced waves to move faster than the slower fundamental. For monochromatic waves with frequencies above ω_c, a three-wave oscillatory sideband instability is triggered which generates two secondary waves that are slower than the fundamental. This sideband instability involves a long-wave modulation that causes several crests to coalesce simultaneously. Both secondary transitions are important intermediate stages of the route to spatio-temporal chaos involving solitary waves. In the review paper [16] by Chang and Demekhin, the reader can find a deep discussion concerning the evolution toward solitary waves (saturation, subharmonic secondary instability, and synchronization) and also the coalescence, transition state, and dynamics on falling films.

M Cheng and H Ch Chang have explained and characterized the relative dominance of the subharmonic and sideband secondary instabilities observed by Liu and Gollub [65]. Both instabilities represent important bifurcations of periodic traveling waves on falling films. In [64], the authors formulated this bifurcation as a time-periodic evolution in space, most pertinent to the periodic-forcing experiment of Liu and Gollub. The subharmonic instability involves a Floquet multiplier of -1 and the sideband instability corresponds to one with a complex Floquet multiplier. They are hence analogous to period doubling and the quasiperiodic transition in dynamical systems. They also correspond to Arnold tongues away from the linear resonant frequencies when a nonlinear oscillator is forced periodically. However, in opinion of the authors of [64], one cannot extend this analogy to the next bifurcation, with the hope of capturing a Feigenbaum cascade, a quasiperiodic transition to chaos, or Devil's staircase-type locking. The one-hump solitary-wave solution branch of the Shkadov averaged equations (4.1.1), (4.1.11) is shown in Fig. 1 below, according to [16, p. 33]. In this figure, the parameter δ is a normalized Reynolds number (see [15, p. 109]).

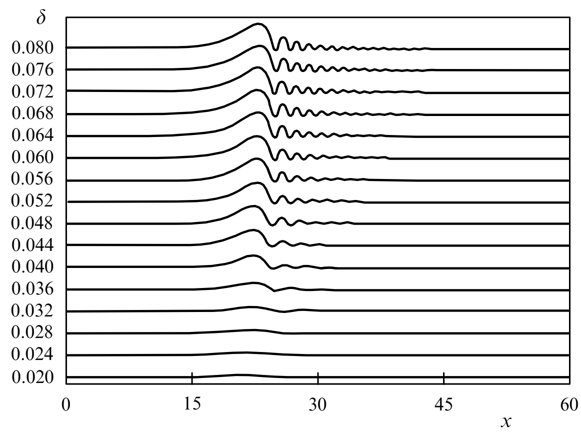

Figure 1. One-hump solitary solution branch of the Shkadov equations.

5. Moderate Reynolds number, Poiseuille regime: Re = $O(1)$

5.1 KS equation

We start again with the equations (3.4.1) and boundary conditions (3.4.2). As $\mathrm{Re} = O(1)$ we get that ε is the main small parameter occurring within equations (3.4.1) and conditions (3.4.2), for the full starting problem. As the characteristic velocity U° we choose the interface velocity

$$U^\circ = g\frac{h^2}{\nu_0}\,,$$

and in this case

$$\frac{\mathrm{Re}}{\mathrm{Fr}^2} = 1\,.$$

We assume that the Weber number We is large and we introduce a Weber similarity parameter

$$\mathrm{W}^* = \varepsilon^2\,\mathrm{We} = O(1)\,.$$

Concerning the Prandtl, Marangoni and Biot numbers we suppose that

$$\mathrm{Pr} = O(1)\,,\quad \mathrm{Ma} = O(1)\,,\quad \mathrm{Bi} = O(1)\,.$$

We are looking for the solution of (3.4.1) with (3.4.2) and the above relations for Re, Fr^2, We, Pr, Ma and Bi, an expansion in the form proposed by Benney [66], namely:

$$U = (u, w, \pi, \theta)^{\mathrm{T}} = U_0 + \varepsilon U_1 + O(\varepsilon^2)\,,\ \varepsilon \to 0\,, \qquad (5.1.1)$$

but we shall not expand the thickness of the film $h(t,x)$ for the moment.

The solution U_0 is

$$u_0 = -z\left(\frac{1}{2}\,z - h\right),\ w_0 = -\frac{1}{2}\frac{\partial h}{\partial x}\,z^2\,,\ p_0 = p_{\mathrm{a}} - \mathrm{W}^*\frac{\partial^2 h}{\partial x^2}\,, \qquad (5.1.2\mathrm{a})$$

$$\theta_0 = 1 - (1+\mathrm{Bi})\frac{z}{1+\mathrm{Bi}\,h}\,. \qquad (5.1.2\mathrm{b})$$

From (4.1.1) we also get

$$\frac{\partial h}{\partial t} + h^2\,\frac{\partial h}{\partial x} = O(\varepsilon)\,. \qquad (5.1.3)$$

Now, writing out the set of equations and boundary conditions for U_1, and again, assuming that $h(t,x)$ is not expanded, it is easy to get an analytic expression for u_1, as a function of z. Using u_0 and u_1, we may compute q_0 and q_1 within the expansion

$$q(t,x) = \int_0^{h(t,x)} u(t,x,z)\,\mathrm{d}z = q_0 + \varepsilon q_1 + O(\varepsilon^2)\,. \qquad (5.1.4)$$

Concerning $q_0 = (1/3)h^3$, this has already been taken into account with (5.1.3), while for q_1 we get the following expression:

$$q_1(t,x) = \frac{1}{3}\,\mathrm{Re}\,\mathrm{W}^* h^3\,\frac{\partial^3 h}{\partial x^3} + \frac{1}{2}\,\mathrm{Ma}(1+\mathrm{Bi})h^2\,\frac{\partial h/\partial x}{(1+\mathrm{Bi}\,h)^2} + \frac{2}{15}\,\mathrm{Re}\,h^6\,\frac{\partial h}{\partial x}\,. \qquad (5.1.5)$$

From equation (4.1.1), we may get an equation for the thickness of the film $h(t,x)$, involving the $O(\varepsilon)$ term occurring in (5.1.3):

$$\frac{\partial h}{\partial t} + h^2\frac{\partial h}{\partial x} + \varepsilon\,\frac{\partial}{\partial x}\left[\frac{1}{3}\,\mathrm{Re}\,\mathrm{W}^* h^3\,\frac{\partial^3 h}{\partial x^3} + \frac{2}{15}\,\mathrm{Re}\,h^6\,\frac{\partial h}{\partial x} + \frac{1}{2}\,\mathrm{Ma}(1+\mathrm{Bi})h^2\,\frac{\partial h/\partial x}{(1+\mathrm{Bi}\,h)^2}\right] = 0\,. \qquad (5.1.6)$$

This evolution equation (5.1.6) of the 'Benney type' contains the small parameter ε, since the fully consistent asymptotic approach based on an expansion in ε is not applied to $h(t,x)$. Of course we may expand $h(t,x)$ in different ways and we shall investigate here only the same kind of phenomenon as that which led to the Kuramoto – Sivashinsky (KS) equation. In order to achieve this, we put in (5.1.6):

$$\tau = \varepsilon t\,,\quad \xi = x - t\,,\quad h(t,x) = 1 + \varepsilon\eta(\tau,\xi) + \ldots\,,\quad \varepsilon = \delta\,. \qquad (5.1.7)$$

Since $\partial h/\partial t = -\varepsilon\,\partial h/\partial\xi + \varepsilon^2\partial\eta/\partial\tau$ and $\partial h/\partial x = \varepsilon\partial\eta/\partial\xi$, if we let $\varepsilon \to 0$ within the transformed version of equation (5.1.6) we derive the following KS equation:

$$\frac{\partial\eta}{\partial\tau} + 2\eta\frac{\partial\eta}{\partial\xi} + \alpha\frac{\partial^2\eta}{\partial\xi^2} + \gamma\frac{\partial^4\eta}{\partial\xi^4} = 0\,, \qquad (5.1.8)$$

where

$$\alpha = \frac{2}{15}\,\mathrm{Re} + \frac{1}{2}\,\frac{\mathrm{Ma}}{1+\mathrm{Bi}}\,,\quad \gamma = \frac{1}{3}\,\mathrm{Re}\,\mathrm{W}^*\,. \qquad (5.1.9)$$

The KS equation (5.1.8) is asymptotically consistent when $\varepsilon = \delta \to 0$ and is an approximate equation valid with an error of the order $O(\varepsilon)$. It is interesting to note that the Benney type equation (5.1.6) is also derived from the averaged 'Shkadov type' equations (4.1.1), (4.1.16), when $\mathrm{Re}^* \to 0$, if we assume that K^* and M^* remain $O(1)$ in this limit. In this last case we again derive a KS equation, similar to (5.1.8), but [see (5.1.19)] the first term in the coefficient α is absent and in place of Ma we have M^*. Respectively, in coefficient γ, in place of $\mathrm{Re}\,\mathrm{W}^*$ we have the coefficient $1/\mathrm{K}^*$.

5.1.1 Analysis of the KS equation. A very naive linear stability analysis shows that a cutoff wavenumber exists for the KS equation (5.1.8). Indeed, if $\eta(\tau,\xi) \sim \exp(\omega\tau + \mathrm{i}k\xi)$, then for ω we derive the following dispersion relation

$$\omega - \alpha k^2 + \gamma k^4 = 0\,. \qquad (5.1.10)$$

The curve $\omega = 0$ determines the neutral curve of the (linear) steady stability (in this case the phase velocity $\omega/k = c = 0$, where the wavenumber k is assumed to be real) and as a consequence we obtain a cutoff wavenumber k^* such that

$$k^{*2} = \frac{\alpha}{\gamma} = \frac{1}{\mathrm{W}^*}\left[\frac{2}{5} + \frac{3}{2}\,\frac{\mathrm{Ma}}{\mathrm{Re}(1+\mathrm{Bi})}\right]. \qquad (5.1.11)$$

Now, if we introduce the amplitude $2\eta(\tau,\xi) = A(\tau,\xi)$, then in place of (5.1.8) we obtain the following KS canonical equation

$$\frac{\partial A}{\partial\tau} + A\frac{\partial A}{\partial\xi} + \alpha\frac{\partial^2 A}{\partial\xi^2} + \gamma\frac{\partial^4 A}{\partial\xi^4} = 0\,. \qquad (5.1.12)$$

When $\alpha = 0$ and $\gamma = 0$, we have the well-known equation (see G B Whitham [67])

$$\frac{\partial A}{\partial\tau} + A\frac{\partial A}{\partial\xi} = 0\,,$$

and along the characteristics [defined by $\partial\xi/\partial\tau = A(\tau,\xi)$] the solution $A(\tau,\xi(\tau))$ is constant.

When $\gamma = 0$ ($W^* = 0$), the surface-tension term is removed and (5.1.12) reduces to Burger's equation. In this case, the Cole – Hopf transformation (see, for example, the paper by Hopf [68]) further reduces it to the heat equation. Since $\alpha > 0$ the Cole – Hopf transformation produces a heat equation backward in time and the initial disturbance will then grow without limit. Hence, we shall include the surface-tension term and discuss equation (5.1.12) when $\alpha > 0$ and $\gamma > 0$.

The full KS equation (5.1.12) is capable of generating solutions in the form of irregularly fluctuating quasiperiodic waves. This KS model equation provides a mechanism for the saturation of an instability, in which the energy in long-wave instabilities is transferred to short-wave modes which are then damped by surface tension.

In the full KS equation (5.1.12), the terms $\partial A/\partial\tau + A\,\partial A/\partial\xi$ lead to steepening and wave breaking in the absence of stabilizing terms. The term $\alpha\,\partial^2 A/\partial\xi^2$ destabilizes shorter-wavelength modes preferentially and therefore aggravates wave steepening (since M^* and Bi are both positive, for small Bi the Marangoni effect aggravates this destabilization). Finally, the term $\gamma\partial^4 A/\partial\xi^4$ is required for saturation, [53, p. 333]. Unfortunately, explicit analytic solutions of the KS equation are not available, but approximate analytic solutions and numerical solutions are discussed in Hooper and Grimshaw [69].

The linear dispersion relation (5.1.10) shows that short waves are stable, and long waves are unstable. The critical wavenumber is $k^* = \sqrt{\alpha/\gamma}$ which ought to be small for the analysis of long waves to make sense. The maximum growth rate is $(\alpha^2/4\gamma)$ and occurs at $k^*/\sqrt{2}$. It is anticipated that the effect of the nonlinear term in (5.1.12) will be to allow energy exchange between a wave with the wavenumber k and its harmonics with the end result being nonlinear saturation. The final state may be either chaotic oscillatory motion (see, below, the Section 5.2) or a state involving only a few harmonics.

The energy equation [53, p. 334] corresponding to (5.1.12) is obtained by multiplying (5.1.12) by A and integrating by parts, assuming A to be periodic with period $2L$:

$$\frac{1}{2}\frac{\partial}{\partial\tau}\int_0^{2L} A^2\,\mathrm{d}\xi = \int_0^{2L}\left[\alpha\left(\frac{\partial A}{\partial\xi}\right)^2 - \gamma\left(\frac{\partial A^2}{\partial\xi^2}\right)^2\right]\mathrm{d}\xi\,. \quad (5.1.13)$$

The minimization of the right-hand side of (5.1.13) over all periodic functions shows that this right-hand side will be negative for $\pi/L > k^*$; and therefore the nonlinear equation (5.1.12) is globally stable for an initial condition with a wavenumber satisfying the linear stability criterion. In other words, if you put in an initial disturbance [e.g. $\sin(k\xi)$] with a wavenumber k greater than k^*, then the nonlinear term in (5.1.12) creates higher harmonics, but it will not create waves with wavenumbers smaller than k, so there will be stability. If you want to generate a component with a wavenumber in the unstable region, you have to put in an initial condition with a wavenumber less than k^*.

Hence, we need to consider only the case $k < k^*$. The periodic boundary conditions allow A to be written as a Fourier series:

$$A = \sum_{-\infty}^{+\infty} A_n(\tau)\exp(\mathrm{i}nk\xi)\,, \quad A_{-n} = A_n^*\,, \quad (5.1.14)$$

where A_n^* is the complex conjugate of A_n. Since A_0 is constant we may put $A_0 = 0$, and the substitution of (5.1.14) into (5.1.12) gives:

$$\frac{\partial A_n}{\partial\tau} - \sigma_n A_n + \mathrm{i}nkB_n = 0\,, \quad (5.1.15)$$

where

$$B_n = \sum_{r=1}^{\infty} A_r^* A_{r+n} + \frac{1}{2}\sum_{r=1}^{n-1} A_r A_{n-r}\,, \quad \sigma_n = \alpha(nk)^2 - \gamma(nk)^4\,. \quad (5.1.16)$$

A significant feature of the system of equations (5.1.15) is that for any given k, only a finite number of Fourier modes, $A_1, A_2, \ldots, A_n$ say, are unstable ($\sigma_n > 0$), and all higher modes are stable. Note that the nth mode has a critical wavenumber k^*/n, and a maximum growth rate of $(\alpha^2/4\gamma)$ – independent of n – at $k^*/(n\sqrt{2})$. This implies that unstable modes will be stabilized by energy transfer to higher harmonics. The simplest case amenable to some analysis is when $k^*/2 < k < k^*$. Only $n = 1$ is unstable and in the following it is assumed that it is sufficient to consider just the interaction between $n = 1$ and $n = 2$. The approximate version of (5.1.15) is then

$$\frac{\partial A_1}{\partial\tau} - \sigma_1 A_1 + \mathrm{i}kA_2A_1^* = 0\,, \quad \frac{\partial A_2}{\partial\tau} - \sigma_2 A_2 + \mathrm{i}\varkappa(A_1)^2 = 0\,. \quad (5.1.17)$$

Note that A_1 is unstable ($\sigma_1 > 0$) but A_2 ($\sigma_2 < 0$) is stable; $\sigma_1|A_1|^2 + \sigma_2|A_2|^2 = 0$ reflecting the required energy balance in the approximate version of

$$\frac{\partial}{\partial\tau}\left(\sum|A_n|^2\right) = 2\sum\sigma_n|A_n|^2\,, \quad \text{where} \quad n = 1, 2, 3, \ldots,$$

as a consequence of (5.1.13) and (5.1.14). Equation (5.1.17) has the steady solution

$$|A_1| = \left(-\frac{1}{k^2}\,\sigma_1\sigma_2\right)^{1/2}, \quad A_2 = \frac{\mathrm{i}k}{\sigma_2}A_1^2\,. \quad (5.1.18)$$

Here, A_1 is growing and A_2 is stabilizing. However, as k is decreased, the hypothesis that only two modes are involved becomes more suspect! Indeed, as k is decreased, the steady solution of (5.1.15) given approximately by (5.1.18) is first modified by the presence of a small correction due to A_3 and then, when $k^*/3 < k < k^*/2$ (i.e. $\sigma_2 > 0$, but $\sigma_3 < 0$), replaced by another 'two-mode equilibrium' in which A_2 and A_4 are the dominant components. A further decrease in k then leads to a succession of states, altering between 'two-mode equilibria' and 'bouncy states'. If the steady solution for A_2 in (5.1.18) is substituted into the first equation in (5.1.17) a LS equation is obtained for A_1,

$$\frac{\partial A_1}{\partial\tau} = \sigma_1 A_1 + \frac{k^2}{\sigma_2}\,|A_1|^2 A_1\,, \quad (5.1.19)$$

and this equation (5.1.19) is, in fact, valid only for k close to k^*. If in (5.1.19) we assume that $A_1 = |A_1|\exp(\mathrm{i}\phi)$, then $\phi = \text{const}$ and for $|A_1|$ we derive a Landau equation

$$\frac{\partial|A_1|}{\partial\tau} = \sigma_1|A_1| + \lambda|A_1|^3\,, \quad (5.1.20)$$

and $\lambda = (k^2/\sigma_2) < 0$, since $\sigma_2 < 0$. The solution of (5.1.20) gives

$$|A_1| \sim A_1^{\circ}\exp(\sigma_1\tau)\,, \quad \tau \to -\infty\,,$$

where according to A_1° is the initial value at $\tau = 0$ and $\sigma_1 > 0$, which decays like according to the linearized theory. However, $|A_1|^2 \to |A_1|^2 e = -(2\sigma_1/\lambda)$, for all values of $\tau \to +\infty$ – this case is called 'supercritical stability'. If we now introduce a small perturbation parameter $\varkappa$ defined by

$$\varkappa^2 \mu = k^2 \left(\frac{a}{b} - k^2 \right) > 0 \,, \qquad (5.1.21)$$

and a slow scale $T = \varkappa^2 t$, then for a slowly varying amplitude of the fundamental wave $H(T)$ such that $|A_1| = \varkappa H$, from (5.1.20), we derive the following Landau equation for $H(T)$:

$$\frac{\partial H}{\partial T} = b\mu H - \lambda H^3 \,, \qquad (5.1.22)$$

where the (positive) Landau constant is

$$\lambda = \frac{1}{16b} \left(k^2 - \frac{a}{4b} \right) > 0 \,.$$

5.2 Hierarchy of bifurcations and attractors of the KS equation

5.2.1 Stationary wave solutions. A more convenient reduced form of the KS equation (5.1.8) is derived if we introduce the following new function $H(x,t)$ and new variables t and x by the relations

$$\eta = 2\alpha \left(\frac{\alpha}{\gamma} \right)^{1/2} H \,, \quad \tau = \frac{\gamma}{\alpha^2}\, t \,, \quad \xi = \frac{\gamma}{\alpha}\, x \,;$$

in this case we obtain the following reduced KS equation for the amplitude $H(x,t)$:

$$\frac{\partial H}{\partial t} + 4H \frac{\partial H}{\partial x} + \frac{\partial^2 H}{\partial x^2} + \frac{\partial^4 H}{\partial x^4} = 0 \,. \qquad (5.2.1)$$

Transforming (5.2.1) to a moving coordinate system with speed v (a derivation speed) and integrating once, one obtains for $H^*(\xi)$

$$\frac{\partial^3 H}{\partial \xi^3} + \frac{\partial H^*}{\partial \xi} - vH^* + 2H^2 = Q \,, \qquad (5.2.2)$$

where $Q = \langle 2H^{*2} \rangle$ is the deviation flux in the moving frame obtained by invoking the constant-thickness condition,

$$\langle H^* \rangle = 0 \,, \qquad (5.2.3)$$

and $\langle\ \rangle$ denotes averaging over one wavelength in the scaled ξ-coordinate.

If, however, the constant-flux condition is imposed ($Q = 0$), (5.2.2) reduces to

$$\frac{\partial^3 H'}{\partial \xi^3} + \frac{\partial H'}{\partial \xi} - vH' + 2H'^2 = 0 \,, \qquad (5.2.4)$$

and (5.2.3) is unnecessary and no longer holds, $\langle H' \rangle \neq 0$.

The constant-flux equation has one less parameter and involves only one equation (5.2.4), whereas two equations (5.2.2) and (5.2.3) must be solved for the constant-thickness approach and two parameters, Q and v are involved (for a detailed discussion of the properties of these equations see [70]). In Figure 2 below, according to [70, p. 446], we depict a more detailed version of the stationary-wave-solution branches reported in [52] (concerning this last paper see the discussion below in Section 5.2.2). There are myriad infinite wave families. Owing to the symmetry of (5.2.2), a mirror image of Fig. 2 exists for negative v. However, only a few families are pertinent to the boundary-layer averaged (Shkadov type) equations.

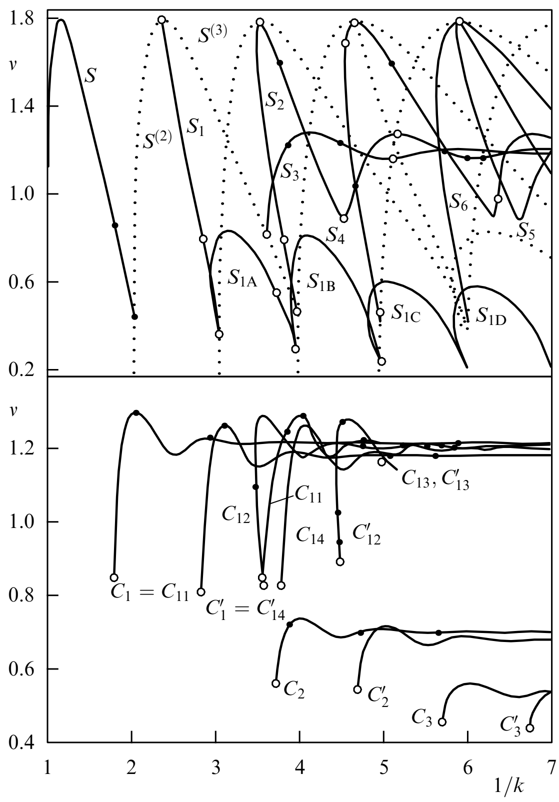

Figure 2. Stationary solution branches of the KS equation (5.2.2).

The primary branch S is a standing-wave solution for (5.2.2) with $v = 0$. We note that k is the wavenumber in the original x-coordinate and not the normalized ξ-coordinate of (5.2.2). At wavenumber $k = 0.49775$, the S branch of (5.2.2) coalesces with the second branch $S^{(2)}$ which bifurcates from $k = 1/2$. Likewise the $S^{(n)}$ branches bifurcate from $k = 1/n$ and they are all identical to S except that n units exist in one wavelength. For $k < 0.50$, one family of waves consists of the S_n branches (see the Fig. 3 below) that come from the $S^{(n)}$ branches. As is evident in Fig. 3, at the bifurcation point on $S^{(n)}$, the waves on S_n are still quite sinusoidal but subsequent ones develop a very rich structure with a broad Fourier content. Some of them (S_2, S_3, S_4, etc) extend to $k = 0$ and terminate as solitary waves. Unlike the $S^{(n)}$ branches, these branches do not resemble each other. A second family of waves consists of C_n and C'_n branches in Fig. 2 and in contrast to the $S^{(n)}$ and S_n families, these are traveling-wave solutions of (5.2.2) with phase speeds in excess of 3. These traveling waves have unique solitary-wave shapes (see, for instance, [71]).

It is important to note that after the solitary-wave regime, the wave breaks into non-stationary three-dimensional patterns. This implies that 3D stationary waves either do not exist or have very short lifetimes, thus being insignificant. This final transition to interfacial 'turbulence' must then be analyzed with an entirely different approach.

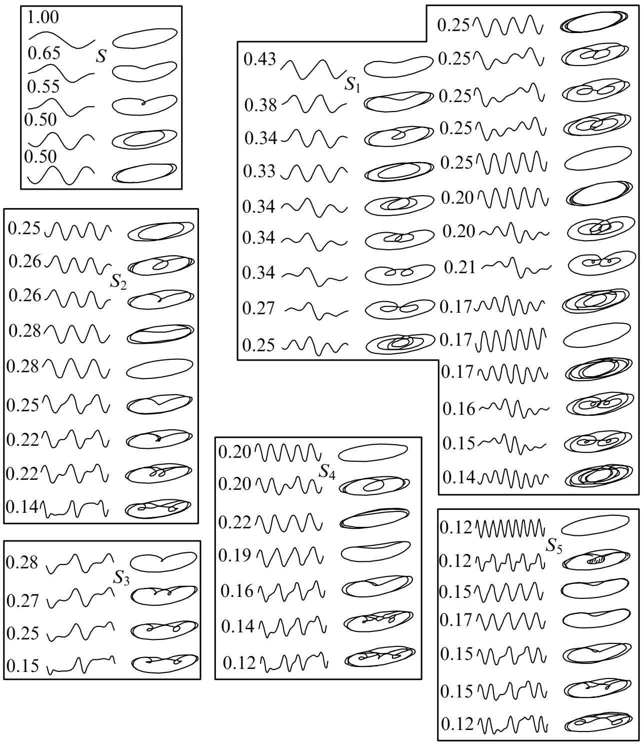

Figure 3. Wave profiles $H^*(\xi)$ of the KS equation (5.2.2) and closed trajectories in the phase space of (H^*, $\mathrm{d}H^*/\mathrm{d}\xi$, $\mathrm{d}^2H^*/\mathrm{d}\xi^2$).

5.2.2 Non-stationary wave solutions. Now, we consider the non-stationary-solution behavior and attractors of the KS equation in the form (5.2.1) and we assume that $(x, t) \in R^1 x R^+$. We impose an initial condition

$$H(x, 0) = H^\circ(x)\,, \quad x \in R^1\,, \tag{5.2.5}$$

and the periodic boundary conditions are

$$H(x, t) = H\left(x + \frac{2\pi}{k}\,,\ t\right), \tag{5.2.6}$$

where k is the wavenumber. Below we mainly present the results of the numerical investigations by Demekhin, Tokarev and Shkadov [52]. A comprehensive review dedicated to the research of the KS equation (up to 1986) is presented in paper [72]. First it is clear that for any (x, t), the solutions $H(x, t)$ of equation (5.2.1) are invariant referring to transformations:

$$H(x, t) \to H(x + x^\circ, t + t^\circ)\,, \quad H(x, t) \to -H(-x, t)\,, \tag{5.2.7a}$$

$$H(x, t) \to H(x - ct) + \frac{1}{4}\,c\,. \tag{5.2.7b}$$

Since $H = \mathrm{const}$ is a trivial solution of the problem (5.2.1), (5.2.5), (5.2.6), as a consequence of (5.2.7b), we assume that $H = 0$ is a trivial solution of this problem. The linear characteristic equation is $c = \mathrm{i}k(1 - k^2) = c_\mathrm{i}$, $c_\mathrm{r} = 0$, where $c = c_\mathrm{r} + \mathrm{i}c_\mathrm{i}$ is the phase velocity. When $k > 1$ the $H = 0$ solution is stable and when $k < 1$ it is not. For $k = 1$ we have a neutral stability. When $k = k_{\max} = (1/2)\sqrt{2}$ the increment of growth $(k\,c_\mathrm{i})$ has its maximum $(kc_\mathrm{i})_{\max} = 1/4$.

For a periodic wave solution

$$H(x, t) = \sum_{n=1}^{\infty} \left[A_n(t)\cos(knx) + B_n(t)\sin(knx)\right]. \tag{5.2.8}$$

By a simple linear transformation $kx \to x$ and $kt \to t$, and by substitution of (5.2.8) into (5.2.1) we get an infinite system of ordinary differential equations for the amplitudes $A_n(t)$ and $B_n(t)$:

$$\frac{\mathrm{d}A_n(t)}{\mathrm{d}t} = k\Big\{\gamma_n A_n(t) - \sum_{m=1}^{\infty} A_m(t)\left[B_{m+n}(t) - B_{m-n}(t)\right] - B_m(t)\left[A_{m+n}(t) - A_{m-n}(t)\right]\Big\}, \tag{5.2.9a}$$

$$\frac{\mathrm{d}B_n(t)}{\mathrm{d}t} = k\Big\{\gamma_n B_n(t) + \sum_{m=1}^{\infty} A_m(t)\left[A_{m+n}(t) + A_{m-n}(t)\right] - B_m(t)\left[B_{m+n}(t) + B_{m-n}(t)\right]\Big\}, \tag{5.2.9b}$$

where

$$\gamma_n = nk\left[1 - (nk)^2\right], \ n = 1, 2, 3, \ldots,$$
$$A_{-n} = A_n\,,\ B_{-n} = -B_n\,,\ A_0 = B_0 = 0\,.$$

But the existence of a class of antisymmetric solutions of the KS equation (5.2.1) follows from (5.2.7a). Such solutions $A_n = 0$, for all n and $B_n(t)$, are described by an infinite chain of o.d. eqs.:

$$\frac{\mathrm{d}B_n(t)}{\mathrm{d}t} = k\Big\{\gamma_n B_n(t) + \sum_{m=1}^{\infty} B_m(t)\left[B_{m+n}(t) + B_{m-n}(t)\right]\Big\}, \quad n = 1, 2, 3, \ldots \tag{5.2.10}$$

In the process of numerical integration of the dynamical system the series (5.2.8) seemed to be finite, containing N terms. In the range of wavenumbers $0.15 < k < 1$, the number of harmonics is usually satisfied the ratio $N = 2\,\mathrm{entier}\,(1/k)$ (that is, half of the modes were in the linear unstable zone $nk < 1$, and half of them in the stable one $nk > 1$, $n = 1, 2, \ldots, N$). In paper [52], many of the calculations were corrected by the given $N = 4\,\mathrm{entier}\,(1/k)$. So, for $0.15 < k < 1$ the number varied from 3 to 20, while the dimensionality of the dynamical system (5.2.9a, b) varied correspondingly from 6 to 40. Bifurcations of the dynamical system (5.2.9a, b) which appear with decreasing k from 1 to 0.15 – 0.2 are shown schematically in Fig. 4 below (for details, see [52, pp. 342 and 343]). In this figure, the attractor identification corresponds to that of the stationary-problem solution families.

At $k > 1$ the origin 0 is the single global attractor in the phase space of the dynamical system (5.2.9a, b). At $0.5547 < k < 1$ the attracting manifold produces S-type solutions. At $k = 0.5547$ the basic family loses stability, which is inherited by the C_1 family branching off from S. At $0.49775 < k < 0.5547$ the S family may be such a surface or, to be more exact, a manifold of codimension 1. For any k value such that $0.4212 < k < 0.49775$ there is a new invariant attracting manifold – a rough heteroclinic contour $L_1^{(2)}$. At $0.4212 < k < 0.49775$ each of the stationary points of the $S^{(2)}$ family has a one-dimensional unstable manifold. The $L_1^{(2)}$-contours are unstable at $0.4859 < k < 0.49775$. At $k \approx 0.4795$ the family C_{11} loses its stability due to the Hopf bifurcation (Fig. 5).

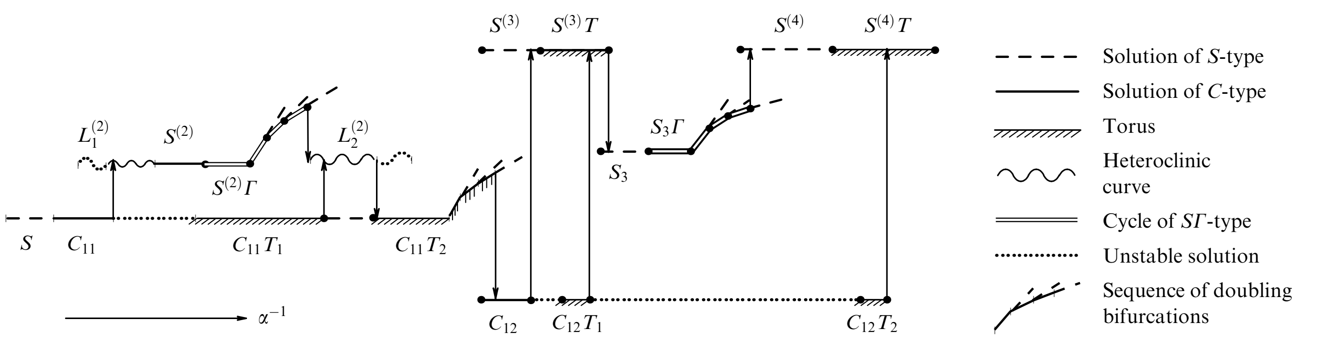

Figure 4. Bifurcations of the dynamical system (5.2.9a, b).

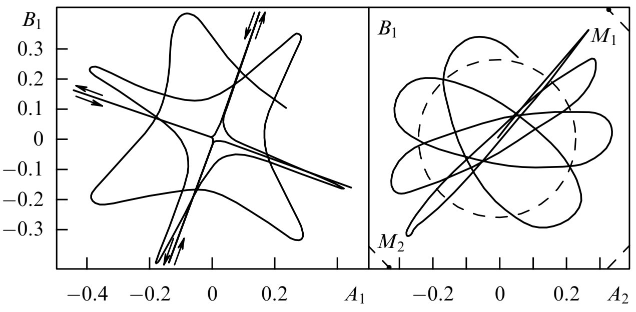

Figure 5. Beginning of the phase-trajectory transition from the unstable cycle C_{11} to the contour $L_1^{(2)}$. Two projections are given: onto the plane A_1B_1 and onto the plane A_2B_2 at $k = 0.47$.

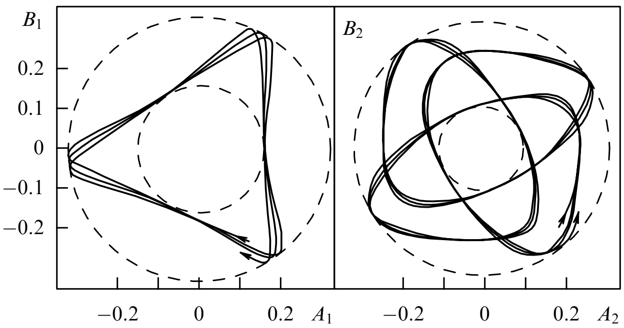

Figure 6. $k = 0.35$.

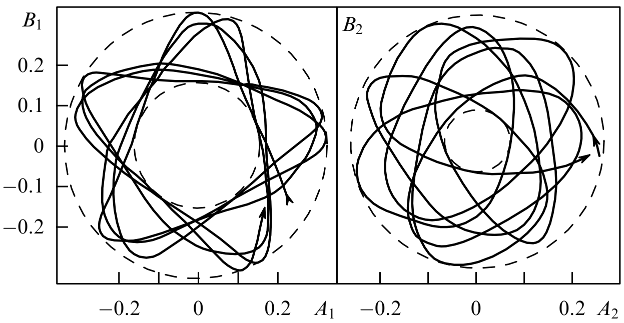

Figure 7. $k = 0.345$.

The authors of paper [52] (see pp. 350, 351), in order to explain the absence of a torus in the whole supercritical region $0.3925 < k < 0.4795$, suggest a hypothesis of the existence of an unstable torus family surrounding the solution family C_{11} at $k > 0.4795$. According to this hypothesis, as k tends to 0.4795, the unstable torus 'sits down' on a stable cycle from C_{11} inheriting the torus instability; that is, a rigid loss of the stability of the C_{11}-type cycle occurs. It seems that bifurcations associated with the absence of tori are connected with the attractors $L_1^{(2)}$ and $S^{(2)}$, which appear to be a trace of the stationary solution family $S^{(2)}$ at $0.3925 < k < 0.4795$. At $k \approx 0.4212$, the S_1 family branches off from the unstable family of solutions $S^{(2)}$ inheriting its instability. As a result of this bifurcation, the stationary solutions from $S^{(2)}$ become stable and the attractors $L_1^{(2)}$ disappear. $S^{(2)}$ ceases to be the only attractor at $k \approx 0.3925$. At $0.338 < k < 0.3925$ the phase trajectory is wound on the stable invariant torus if the initial conditions are set in the proximity of the unstable solutions of family C_{11}. The corresponding attractor is designated as $C_{11}T$ in Fig. 4. The projections $A_1 - B_1$ and $A_2 - B_2$ of the phase trajectory moving along the $C_{11}T_1$ torus at $k = 0.35$ and 0.345 are shown in Figs 6 and 7.

The stationary solutions from $S^{(2)}$ lose their stability at $k \approx 0.3635$ due to the Hopf bifurcation and in the vicinity of each unstable stationary point $S^{(2)}$ a limit cycle appears. The family of these cycles will conditionally be designated as $S^{(2)}\Gamma$. In Figure 8 below the evolution of the $S^{(2)}\Gamma$ cycles with the decrease of k is shown. At $k \approx 0.3482$ the cycle undergoes the first period-doubling bifurcation and at $k \approx 0.3481$ the second. It seems that this cascade of bifurcations is not infinite and is cut short during the third doubling — this is due to the fact that the cycle quickly 'swells' with decreasing k and at $k \approx 0.34799$ the phase trajectory comes onto the separatrix surface dividing the two centrosymmetric points from $S^{(2)}$ which are at the ends of the diameter of the circle $a_l^2 + b_l^2 = \text{const}$, a_l, $b_l \in S^{(2)}$, $l = 1, 2, \ldots$. While doing this, the trajectory passes along the separatrix loop of the origin and the cycle is destroyed. This develops into a situation which is in many respects similar to the Lorenz system!

The phase-trajectory behavior in the projection onto the (B_2, B_4) plane at $k \approx 0.3$ is shown in Fig. 9. Let us point out that the value of $k = 0.3$ is close to the critical one, $k^* \approx 0.2967$.

At $k \approx 0.2988$ the second tori families appeared in a way similar to that of the tori family $C_{11}T_1$; the birth of the second family is connected with the family of stationary running waves C_{11} and is designated as $C_{11}T_2$. In the range $0.283 < k < 0.295$ the tori $C_{11}T_2$ are influenced by the

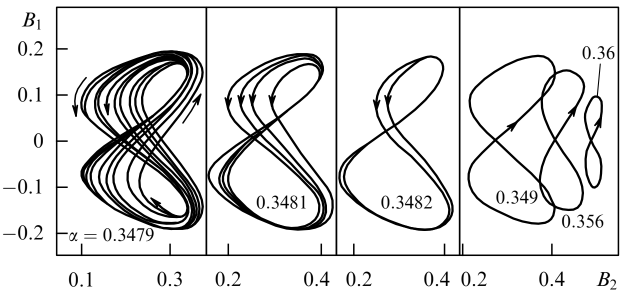

Figure 8. Evolution of the $S^{(2)}\Gamma$ cycles with the decrease of k.

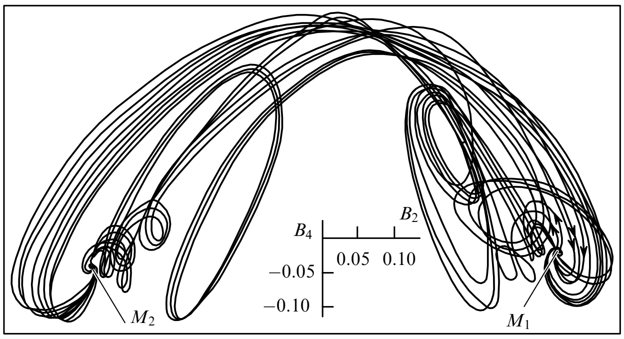

Figure 9. Complex movement observed in the vicinity of $S^{(2)}\Gamma$.

cascade of period-doubling bifurcations of the Feigenbaum type (the reader can find a review concerning the various routes to chaos in our paper [73]). The family of stationary running waves C_{12} becomes stable at $k \approx 0.2831$ and loses its stability due to the Hopf bifurcation at $k \approx 0.2751$. A non-stationary regime of the metastable chaos type (Fig. 10) can be observed and, as a result of this, the phase trajectory is

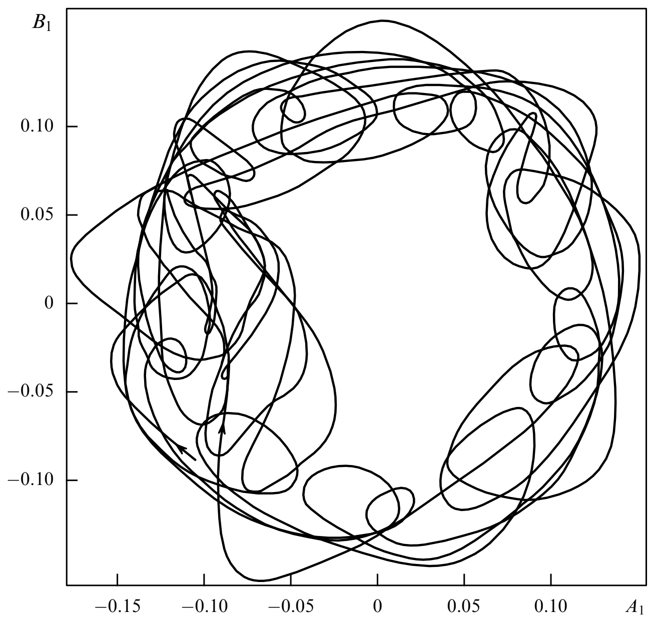

Figure 10. Metastable chaos regime at $k = 0.273$.

attracted to some point from $S^{(3)}$. The trajectory approaches the stationary point from $S^{(3)}$ in jumps fairly closely and leaves the vicinity slowly, again approaching the destroyed torus.

The time of the trajectory's being in the vicinity of $S^{(3)}$ increases in each new cycle and the vicinity itself narrows. At $0.2449 < k < 0.2803$ the stationary points $S^{(3)}$ are stable and the family $S^{(3)}$ acquires stability as a result of the contact of the branches of the families $S^{(3)}$ and S_2 at $k \approx 0.2449$. In a very narrow range of the wavenumber $0.2392 < k < 0.2394$ the phase-trajectory behavior is quite stochastic. At $0.2195 < k < 0.2392$ the stationary points of the family S_3 are attractors and at $k \approx 0.2195$ the solutions of S_3-type lose stability as a result of Hopf bifurcation and a family of the $S_3\Gamma$ cycles forms. At $k \approx 0.2169$ in $S_3\Gamma$ a cascade of the Feigenbaum-type period-doubling bifurcations begins and this results in the formation of a chaotic attractor at $k \approx 0.2150$. The stability loss in the families $S^{(4)}$ and $S^{(5)}$ takes place similarly to that of the family $S^{(3)}$. Finally, we note that the tori $S^{(4)}T$ exist in the range $0.1820 < k < 0.1865$ and $S^{(5)}T$ in the range $0.1500 < k < 0.1506$.

It is interesting to note that for $k > 0.35$ a truncated approximate low-dimensional dynamical system will be considered. This system has only three modes (as does the Lorenz system), B_1, B_2 and B_3:

$$\frac{dB_1}{dt} = \gamma_1 B_1 + 2(B_1 B_2 + B_2 B_3)\,,$$
$$\frac{dB_2}{dt} = \gamma_2 B_2 + 2(2B_1 B_3 - B_1^2)\,,$$
$$\frac{dB_3}{dt} = \gamma_3 B_3 - 6B_1 B_2\,, \qquad (5.2.11)$$

with $\gamma_j = j\sigma_j = j^2 k(1 - j^2 k^2)\,,\ j = 1, 2, 3$

The phase flow of the dynamical system (5.2.11) is dissipative if

$$\gamma_1 + \gamma_2 + \gamma_3 < -B_2\,. \qquad (5.2.12)$$

In this case, because of the dissipation, the attractors have zero phase volume and a dimensionality lower than 3.

6. Low Reynolds number, second Poiseuille regime: Re $\ll$ 1

6.1 KS – KdV equation

6.1.1 The KS – KdV equation for low Marangoni numbers. The problem considered in this Section 6 is interesting when we look at (5.1.6). As a matter of fact this equation (5.1.6) is a singular perturbation of the hyperbolic equation

$$\frac{\partial h}{\partial t} + h^2 \frac{\partial h}{\partial x} = 0\,.$$

Curiously, we again get a singular perturbation of this same equation, but of another type, if we make the assumption (low Reynolds number, see [56]):

$$\mathrm{Re} \ll 1\,, \quad \mathrm{Ma} \ll 1\,,$$

such that

$$\frac{\mathrm{Re}}{\varepsilon} = \mathrm{R}^\circ\,, \quad \frac{\mathrm{Ma}}{\varepsilon} = \mathrm{M}^\circ\,, \qquad (6.1.1)$$

and again assume that $\varepsilon^2\mathrm{We} = \mathrm{W}^*$ and $\mathrm{Re}/\mathrm{Fr}^2 = 1$, in the full starting problem (3.4.1), (3.4.2). In this case, in place of this starting problem, we obtain the following problem:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0\,, \tag{6.1.2a}$$

$$\frac{\partial^2 u}{\partial z^2} + 1 = \varepsilon^2 \mathrm{R}^\circ \left(\frac{\mathrm{D}u}{\mathrm{D}t} + \frac{\partial p}{\partial x} - \frac{1}{\mathrm{R}^\circ}\frac{\partial^2 u}{\partial x^2} \right), \tag{6.1.2b}$$

$$\frac{\partial p}{\partial z} - \frac{1}{\mathrm{R}^\circ}\frac{\partial^2 w}{\partial z^2} = \varepsilon^2 \left(\frac{1}{\mathrm{R}^\circ}\frac{\partial^2 w}{\partial x^2} - \frac{\mathrm{D}w}{\mathrm{D}t} \right), \tag{6.1.2c}$$

$$\frac{\partial^2 \theta}{\partial z^2} = \varepsilon\left(\mathrm{Rr}\,\mathrm{R}^\circ \frac{\mathrm{D}\theta}{\mathrm{D}t} - \frac{\partial^2\theta}{\partial x^2} \right). \tag{6.1.2d}$$

At $z = 0$

$$u = w = 0\,, \quad \theta = 1\,, \tag{6.1.3a}$$

$$\frac{\partial u}{\partial z} = -\varepsilon \mathrm{M}^\circ \left(\frac{\partial \theta}{\partial x} + \frac{\partial h}{\partial x}\frac{\partial \theta}{z} \right) - \varepsilon^2 \left(\frac{\partial w}{\partial x} + 4\frac{\partial h}{\partial x}\frac{\partial w}{\partial z} \right) + O(\varepsilon^4)\,; \tag{6.1.3b}$$

at $z = h(t, x)$

$$p = p_\mathrm{a} - \mathrm{W}^* \frac{\partial^2 h}{\partial x^2} + \frac{2}{\mathrm{R}^\circ}\left(\frac{\partial w}{\partial z} - \frac{\partial h}{\partial x}\frac{\partial u}{\partial z} \right) + O(\varepsilon^2)\,, \tag{6.1.3c}$$

$$\frac{\partial \theta}{\partial z} = -(1 + \mathrm{Bi}\,\theta) + O(\varepsilon^2)\,, \tag{6.1.3d}$$

$$w = \frac{\partial h}{\partial t} + u\,\frac{\partial h}{\partial x}\,. \tag{6.1.3e}$$

Obviously in this case, the formal (Benney [66]) expansion in ε is modified. Specifically,

$$U = (u, w, p, \theta)^T = U_0 + \varepsilon^2 U_2 + \dots\,, \quad \varepsilon \to 0\,. \tag{6.1.4}$$

The solution U_0 is obtained in a straightforward way. In this case, when $\varepsilon \to 0$ in (6.1.2) and (6.1.3) we obtain the following leading-order solution:

$$u_0 = -\frac{1}{2}z^2 + hz\,, \quad w_0 = -\frac{1}{2}\frac{\partial h}{\partial x}z^2\,, \tag{6.1.5a}$$

$$p_0 = p_\mathrm{a} - \mathrm{W}^*\frac{\partial^2 h}{\partial x^2} - \frac{1}{\mathrm{R}^\circ}\frac{\partial h}{\partial x}(h + x)\,, \tag{6.1.5b}$$

$$\theta_0 = 1 - (1 + \mathrm{Bi})\,\frac{z}{1 + \mathrm{Bi}\,h}\,. \tag{6.1.5c}$$

From (4.1.1) and (4.1.2), with (6.1.5a), we get, this time,

$$\frac{\partial h}{\partial t} + h^2\frac{\partial h}{\partial x} = O(\varepsilon^2)\,, \tag{6.1.5г}$$

since $q_0 = (1/3)h^3$.

Writing out the set of equations and boundary conditions at the order ε^2 (from (6.1.2) and (6.1.3), for U_2, and assuming that $h(t, x)$ is not yet expanded, we may get an awkward expression for u_2 that may be integrated with respect to z in order to obtain an explicit expression for q_2 in [as a consequence of (4.1.1) and (6.1.4)]:

$$\frac{\partial h}{\partial t} + h^2\frac{\partial h}{\partial x} + \varepsilon^2\frac{\partial q_2}{\partial x} = O(\varepsilon^4)\,.$$

The final result is analogous to (5.1.6), but with some additional terms, and reads

$$\frac{\partial h}{\partial t} + h^2\frac{\partial h}{\partial x} + \varepsilon^2\frac{\partial}{\partial x}\left\{ \frac{1}{3}h^3\left[\mathrm{R}^\circ\,\mathrm{W}^*\frac{\partial^3 h}{\partial x^3} + 7\left(\frac{\partial h}{\partial x}\right)^2 \right] + h^4\frac{\partial^2 h}{\partial x^2} + \frac{2}{15}h^6\frac{\partial h}{\partial x} + \frac{1}{2}\mathrm{M}^\circ(1 + \mathrm{Bi})\left[h^2\frac{\partial h/\partial x}{(1 + \mathrm{Bi}h)^2} \right] \right\} = 0\,. \tag{6.1.6}$$

The evolution equation (6.1.6) for $h(t, x)$ is valid with an error of $O(\varepsilon^4)$. Now, with this last evolution equation (6.1.6), we intend to play the same game as considered for the reduction of (5.1.6) in the KS equation (5.1.8). We use, specifically:

$$\tau = \delta t\,, \ \xi = x - t\,, \ h = 1 + \frac{1}{\phi}\varepsilon^2\eta(\tau, \xi) + \dots\,, \ \delta = \frac{1}{\phi}\varepsilon^2\,, \tag{6.1.7}$$

where ϕ is the dispersive similarity parameter. Again, carrying out the limiting process $\varepsilon \to 0$, we find, in place of (6.1.6), an equation which combines the features of KdV on the one hand and KS on the other hand:

$$\frac{\partial \eta}{\partial \tau} + 2\eta\frac{\partial \eta}{\partial \xi} + \alpha\frac{\partial^2 \eta}{\partial \xi^2} + \phi\frac{\partial^3 \eta}{\partial \xi^3} + \gamma\frac{\partial^4 \eta}{\partial \xi^4} = 0\,, \tag{6.1.8}$$

where

$$\alpha = \phi\left[\frac{2}{15}\mathrm{R}^\circ + \frac{1}{2}\frac{\mathrm{M}^\circ}{1 + \mathrm{Bi}} \right], \quad \gamma = \frac{1}{3}\phi\,\mathrm{R}^\circ\,\mathrm{W}^*\,. \tag{6.1.9}$$

The evolution KS – KdV equation (6.1.8) is again a significant model equation valid for large time with an error of $O(\delta)$. The coefficients, α, γ and ϕ are all positive constants characterizing instability, dissipation and dispersion. As a consequence of the derivation of the KS – KdV equation (6.1.8) valid for low Reynolds numbers, we conclude that the features of the thin film for a strongly viscous liquid are quite different. The dispersive term, $\phi(\partial^3\eta/\partial\xi^3)$, changes the behavior of the thickness $\eta(\tau, \xi)$ in space and in time.

6.1.2 The KS – KdV equation for high Prandtl numbers. The above theory, which leads to the KS – KdV equation, is valid only for low Marangoni numbers, such that $\mathrm{Ma}/\varepsilon = \mathrm{M}^\circ = O(1)$. Here we want to derive a KS – KdV equation another way, when the Prandtl number is large, such that

$$\varepsilon^2\mathrm{Pr} = \mathrm{Pr}^* = O(1)\,. \tag{6.1.10}$$

For this, firstly, we consider the full problem (3.4.1), (3..2), for $\varepsilon \ll 1$, and assume that

$$U(u, w, p, \theta)^T = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \dots\,, \quad \varepsilon \to 0\,. \tag{6.1.11}$$

In this case, if $\varepsilon^2\mathrm{We} = \mathrm{W}^* = O(1)$ and $\mathrm{Re}/\mathrm{Fr}^2 = 1$, we successively derive the following solutions for q_0, q_1 and q_2, When we assume that Re, Ma, Pr, W^* are fixed, $\varepsilon \to 0$, and $\mathrm{Bi} = 0$:

$$q_0 = \frac{1}{3} h^3 , \qquad (6.1.12a)$$

$$q_1 = \frac{1}{3} \operatorname{Re} W^* + \frac{1}{2} \operatorname{Ma} h^2 \frac{\partial h}{\partial x} + \ldots , \qquad (6.1.12b)$$

$$q_2 = \frac{1}{16} \operatorname{Ma} \operatorname{Re} \operatorname{Pr} \frac{\partial}{\partial x}\left(h^4 \frac{\partial h}{\partial x}\right) h^2 + \ldots , \qquad (6.1.12c)$$

where

$$q_i(t,x) = \int_0^{h(t,x)} u_i(t,x,z)\,\mathrm{d}z , \quad i = 1,2,3 . \qquad (6.1.13)$$

In (6.1.12) we have written only those terms which explicitly appear in the KS – KdV equation (6.1.17) derived below. In fact, for the consistency of this KS – KdV equation it is now necessary to assume that $\operatorname{Re} \ll 1$, $\operatorname{Pr} \gg 1$ and $W^* \gg 1$. From the averaged equation (4.1.1), thanks to (6.1.12), we obtain the following '*à la* Benney' equation:

$$\frac{\partial h}{\partial t} + h^2 \frac{\partial h}{\partial x} + \varepsilon \frac{\partial}{\partial x}\left(\frac{1}{3} h^3 \operatorname{Re} W^* \frac{\partial^3 h}{\partial x^3} + \frac{1}{2} \operatorname{Ma} h^2 \frac{\partial h}{\partial x} + \ldots\right) + \frac{1}{16} \varepsilon^2 \operatorname{Ma} \operatorname{Re} \operatorname{Pr} \left\{ \frac{\partial}{\partial x}\left[h^2 \frac{\partial}{\partial x}\left(h^4 \frac{\partial h}{\partial x}\right)\right] + \ldots \right\} = 0 . \qquad (6.1.14)$$

Finally, if we use the transformation

$$\tau = \delta t, \; \xi = x - t, \; h = 1 + \delta \eta(\tau, \xi) + \ldots , \; \delta = \varepsilon , \qquad (6.1.15)$$

and assume that

$$\operatorname{Re} = \varepsilon, \; \varepsilon W^* = W^{**} = O(1), \; \varepsilon^2 \operatorname{Pr} = \operatorname{Pr}^* = O(1), \qquad (6.1.16)$$

then from (6.1.14) we derive the following KS – KdV equation for $\eta(\tau, \xi)$:

$$\frac{\partial \eta}{\partial \tau} + 2\eta \frac{\partial \eta}{\partial \xi} + \frac{1}{2} \operatorname{Ma} \frac{\partial^2 \eta}{\partial \xi^2} + \frac{1}{16} \operatorname{Ma} \operatorname{Pr}^* \frac{\partial^3 \eta}{\partial \xi^3} + \frac{1}{3} W^{**} \frac{\partial^4 \eta}{\partial \xi^4} = 0 . \qquad (6.1.17)$$

This above equation (6.1.17) is valid for low Reynolds numbers, very large Weber numbers and a large Prandtl number. In (6.1.17) the destabilizing term is controlled by the Marangoni number [$\operatorname{Ma} = O(1)$] and dispersion arises thanks to the effect of the large Prandtl number [$\operatorname{Pr}^* = O(1)$]. The dissipation is present only when the Weber number is very large [of the order $O(1/\varepsilon^3)$].

6.2 Some features of solutions of the KS – KdV equation

The linear dispersion relation of equation (6.1.8) for the wave $\eta(\tau, \xi) \approx \exp(\mathrm{i}k\xi + \sigma\tau)$ is expressed as

$$\sigma = \alpha k^2 - \gamma k^4 + \mathrm{i}\phi k^3 . \qquad (6.2.1)$$

For $\operatorname{Real}(\sigma) > 0$ we have instability and for $\operatorname{Real}(\sigma) < 0$, stability. $\operatorname{Real}(\sigma) = 0$, if

$$k = k_c \left(\frac{\gamma}{\alpha}\right)^{1/2} . \qquad (6.2.2)$$

Consequently, the cutoff wavenumber for equation (6.1.8), satisfies the relation

$$k_c^2 = \frac{2}{5W^*} + \frac{3}{2} \frac{M^\circ}{R^\circ W^*} (1 + \operatorname{Bi}) . \qquad (6.2.3)$$

Thus the waves of small wavenumber are amplified while those of large wavenumbers are damped, and the maximum growth rate occurs at $k_m = (\gamma/2\alpha)^{1/2}$. For $\phi = 0$, equation (6.1.8) is reduced to a self-exciting dissipative KS system which exhibits turbulent (chaotic) behavior (see, for instance, [52]). For $\alpha = \gamma = 0$ (the limiting case when $R^\circ = 0$ and $M^\circ = 0$ – a non-viscous liquid film, without the Marangoni effect), on the other hand, equation (6.1.8) reduces to the KdV equation which is known to admit solutions instead of chaos!

Thus, in the general case of non-zero α, γ and ϕ, the increasing value of ϕ is expected to change the character of the solution of equation (6.1.8) from an irregular wave train to a regular row of solitons (a row of pulses of equal amplitude; see Fig. 11). The trend is more amplified at larger values of ϕ, and the asymptotic state of the solution for large ϕ takes the form of a row of KdV solitons [74, p. 3]. There seems to exist a critical value of about unity for the dimensionless parameter $\mu = \phi/(\gamma\alpha)^{1/2}$, representing the relative importance of dispersion, which corresponds to the transition from an irregular wave train to a regular row of solitons. We note that the complicated evolution of solutions of (6.1.8) is described by the weak interaction of pulses, each of which is a steady solution of (6.1.8). When dispersion is strong, the pulse interactions become repulsive, and the solutions tend, in fact, to form stable lattices of pulses [75].

The laminarizing effects of dispersion in equation (6.1.8) were discussed in depth by H -Ch Chang et al. [71]. These authors show that the linear dispersion term $\phi(\partial^3\eta/\partial\xi^3)$ tends

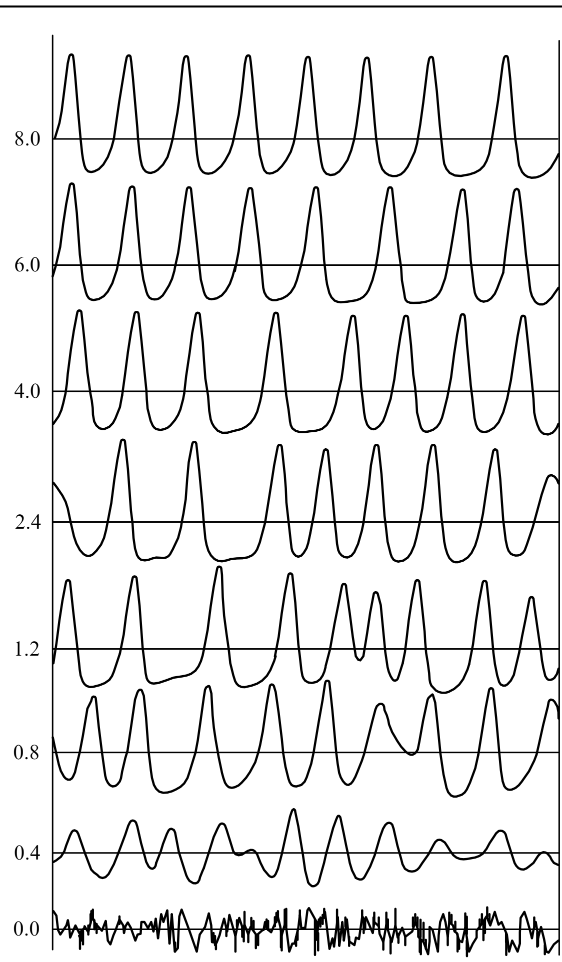

Figure 11. Temporal evolution of $\eta(\tau, \xi)$.

to arrest the irregular behavior (in the KS equation, which exhibits spatial-temporal chaos) in favor of a spatially periodic cellular structure, consistent with prior numerical and experimental observations.

In [71, pp. 301 – 312] the study also includes a normal-form analysis, for stationary waves in phase space. In this case, carrying out a moving-coordinate transformation and stipulating that the waves are stationary in this moving frame, $x = \xi - c^\circ\tau$, we obtain their governing equation from (6.1.8), integrating once,

$$\gamma\frac{d^3H}{dx^3} + \phi\frac{d^2H}{dx^2} + \alpha\frac{dH}{dx} - c^\circ H + 2H^2 = Q \qquad (6.2.4)$$

and

$$\int_0^{2\pi/k} H(x)\,dx = 0\,, \qquad (6.2.5)$$

where $H(x) = (1/2)\eta(\xi - c^\circ\tau)$. Relation (6.2.5) is the condition of zero mean, $2\pi/k$ is the wavelength of the stationary wave and Q is the integration constant. In fact, we can write

$$\gamma\frac{d^3f}{dx^3} + \phi\frac{d^2f}{dx^2} + \alpha\frac{df}{dx} - \lambda f + 2f^2 = 0\,, \qquad (6.2.6)$$

where the new speed is $\lambda = (8Q - c^{\circ 2})^{1/2}$ and the new wave profile

$$f(x) = H - \frac{1}{4}c^\circ + \left(\frac{1}{16}c^{\circ 2} + \frac{1}{2}Q\right)^{1/2}. \qquad (6.2.7)$$

It is important to note that $f(x)$ does not satisfy the zero-mean condition; however, it should be realized that equation (6.2.6) with parameters α, γ, λ and ϕ is equivalent to (6.2.4) with α, γ, λ, ϕ and Q because of the additional constraint (6.2.5). In fact, for the case of a stationary wave with infinite wavelength and vanishing derivatives at the two ends ($x \to \pm\infty$), Q vanishes exactly and (6.2.4) is equivalent to (6.2.6). Equation (6.2.6) is invariant under the transformation $x \to -x$, $f \to -f$, $\lambda \to -\lambda$, and in [71] the authors investigate the case of either $\lambda > 0$ or $\phi > 0$ and write (6.2.6) as a dynamical system (for $\alpha = \gamma = 1$)

$$\frac{d\mathbf{F}}{dx} = L_1\mathbf{F} + L_2\mathbf{F} - 2Cf^2\,, \quad \mathbf{F} = \left(f, \frac{df}{dx}, \frac{d^2f}{dx^2}\right)^T, \qquad (6.2.8)$$

where

$$L_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & -\phi \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \qquad (6.2.9)$$

Two fixed points, $A = (0, 0, 0)$ and $B = (\lambda/2, 0, 0)$, exist for the dynamical system (6.2.8) and the limit cycles of (6.2.8) represent spatially periodic stationary waves and some of them can be shown to bifurcate from A and B via the Hopf bifurcation (B and A undergo a Hopf bifurcation at $\lambda = \pm\delta$, respectively with a frequency of $\sigma_i = 1$). Spatially periodic stationary waves with unit wavelength are hence expected near these points. Stationary solitary waves correspond to the homoclinic trajectory Γ of A, but solitary waves can exist only in the sectors

$$\psi \in \left(0, \frac{3\pi}{4}\right) \cup \left(\pi, \frac{7\pi}{4}\right)$$

in the $\lambda - \phi$ parameter space. In fact, it is necessary to distinguish two types of solitary waves and for large ϕ we obtain Kawahara's asymptotic estimate

$$f(x) \sim \frac{21\phi}{20ch^2}\left[\left(\frac{7x}{20}\right)^{1/2}\right], \quad \lambda = \frac{7}{5}\phi\,. \qquad (6.2.10)$$

In [71], from (6.2.8), a numerical construction of stationary solitary and periodic waves is presented. An interesting result of this numerical investigation is that the stationary solitary waves are unstable in an extended domain and hence cannot be an attractor of the KS – KdV equation (6.1.8)! It is also shown that by $\phi > 1.1$, the infinite families of stationary periodic waves of the KS equation, each ending in a solitary wave, have been annihilated successively such that only a lone family of periodic waves remains, consisting of one-hump KdV pulses for $\phi > 3.7$, as the only periodic waves have much larger domains of attraction than the strange attractors and hence tend to dominate spatial-temporal chaos in an extended domain with significant dispersion. In Figure 12 the wave profiles of the dominant periodic-wave family at $\phi = 0.3$ and $\phi = 5.0$ are presented according to [71, p. 315]. At a given ϕ the amplitude of the periodic solution increases with decreasing wavenumber k. This is consistent with the analytical result of Kawahara and Toh [75].

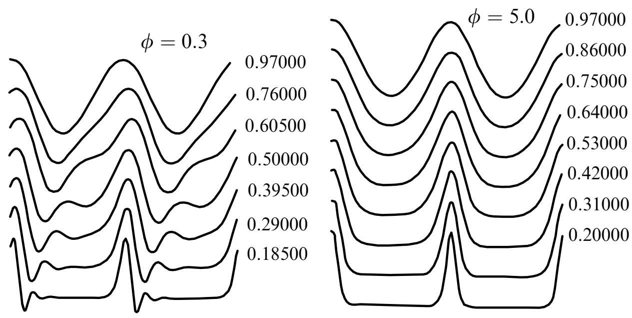

Figure 12. Wave profiles of the dominant periodic wave family at $\phi = 0.3$ and $\phi = 5.0$. Note the profiles for smaller ϕ.

In [76] two different situations are thoroughly investigated. First, the production – dissipation part $(\alpha\partial^2\eta/\partial\xi^2 + \gamma\partial^4\eta/\partial\xi^4)$ of equation (6.1.8) is taken as a small perturbation to the KdV equation $(\partial\eta/\partial\tau + 2\eta\,\partial\eta/\partial\xi + \phi\,\partial^3\eta/\partial\xi^3 = 0)$ proportional to a smallness parameter μ:

$$\frac{\partial\eta}{\partial\tau} + 2\eta\frac{\partial\eta}{\partial\xi} + \phi\frac{\partial^3\eta}{\partial\xi^3} + \mu\left(\frac{\partial^2\eta}{\partial\xi^2} + \frac{\partial^4\eta}{\partial\xi^4}\right) = 0\,, \qquad (6.2.11)$$

when $\alpha = \gamma = \mu$. It is shown that within times limited by $1/\mu$ (and beyond) the KdV hyperbolic secants interact similarly to the Zabusky and Kruskal's findings, retaining the 'aging' they experience (concerning the Z – K paper, see, for instance, our review paper [77], on nonlinear long waves on water and solitons). At longer times the localized solutions adopt the terminal shape and phase velocity, and different humps can form bound states. The increase of the production – dissipation parameter exaggerates the effects through reducing the 'practical infinity' for the time scale. For $\mu > 2$ with the rest of parameters equal to unity, the solution goes chaotic. These result of Christov and Velarde [76] outline the region where the fairly long transients can be approximately considered as solitons, albeit imperfect ones. The second situation is when the fairly KS part of (6.2.11) is predominant. This happens either when μ is not small enough or for very long times ($t \to \infty$, or $t \gg 1/\mu$) when strictly permanent shapes are attained which are in fact short waves and dissipation is dominant. In fact, in [76], the soliton concept (a 'dissipative soliton'!) is extended in two directions: to long transients, practically 'permanent' and solitonic in the time scale $1/\mu$ set by the production – dissipation processes, and to truly permanent wave – particles with, however, inelastic behavior upon collisions.

In paper [78], exact soliton- and cnoidal-wave solutions for equation (6.1.8) are derived and in [79], cnoidal and solitary waves of (6.1.8) are obtained both asymptotically and numerically. The purpose of paper [80] by J Liu and P Gollub is to report an experimental study of the dynamics of 2D solitary waves and their interactions, as part of an effort to understand more complex and disordered film flows. The study of solitary-wave interactions is critical to understanding the dynamics of film flows. Interactions are strongly inelastic in the sense that two interacting pulses merge: a large solitary wave overtakes and absorbs slower ones in front, leaving a long flat interface behind. A cascade of interactions can occur on a sufficiently long film plane. The chaotic pulse trains of equation (6.2.4), for $Q = 0$ and $\alpha = \gamma = 1$, are also studied in [81] and [82]. The authors describe the homoclinic orbits and pulse – train solutions of (6.2.4) in some detail, develop the asymptotic method to the first order and derive the timing map for the pulse separations. In [82], Balmforth presents a discussion of the theory of solitary-wave-solution equilibria and dynamics, for the equation

$$\left(\frac{d^3}{d\xi^3} + \mu\frac{d^2}{d\xi^2} + \frac{d}{d\xi}\right)\Theta - c\,\Theta + \frac{1}{2}\,\Theta^2 = 0\,, \qquad (6.2.12)$$

analogous to (6.2.4), with $Q = 0$, within the framework of asymptotic analysis and dynamical-system theory. In Figure 13, below, we present an illustration of the pulse train

$$\Theta = \sum_k H_k + \varepsilon R + O(\varepsilon^2)\,, \quad H_k = H(\xi - \xi_k)\,, \qquad (6.2.13)$$

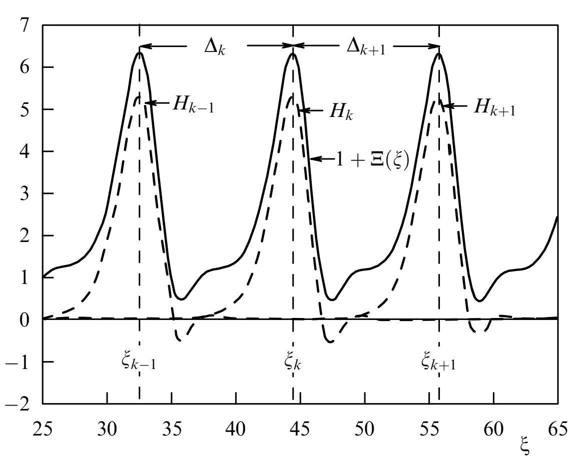

Figure 13. Illustration of the pulse train for equation (6.2.12).

where ξ_k denotes the position of the pulse and εR is the error-correction term.

Finally, a simple sequence of bifurcations is shown in Fig. 14, which shows (according to [82]) the succession of states that are related as c is varied for $\mu = 0.7$, in (6.2.12).

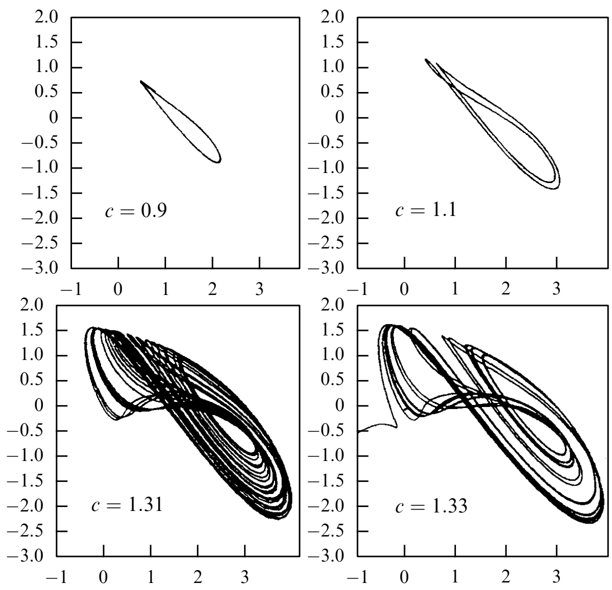

Figure 14. Bifurcation sequence of equation (6.2.12) for $\mu = 0.7$ and four values of c.

6.3 Amplitude equation

6.3.1 To demonstrate the competition between the stationary waves and the non-stationary (possibly chaotic) attractors of the KS – KdV equation (6.1.8), we render (6.1.8), with $\alpha = \gamma = 1$, into a dynamical system by the Galerkin projection in a periodic medium with wavelength $2\pi/k$:

$$\eta(\tau, \xi) = \frac{1}{2}\sum A_p(\tau)\cos(pk\xi) + B_p(\tau)\sin(pk\xi)\,, \quad p \geqslant 1\,. \qquad (6.3.1)$$

For a qualitative analysis of the projections of the phase trajectory onto the plane it is sufficient to consider a dynamical system truncated at three harmonics.

This system can easily be written in an explicit form. Specifically, first we make a simple linear transformation of the coordinate $k\xi \to x$, $k\tau \to t$; the space period of the equation (6.1.8), with the initial $\eta(0, \xi) = \eta^{\circ}(\xi)$ and periodic boundary $\eta(\tau, \xi) = \eta(\tau, \xi + 2\pi/k)$ conditions, $\xi \in [0, 2\pi/k]$, will transit to $x \in [0, 2\pi]$. Next, substituting (6.3.1) into the KS – KdV equation (6.1.8), we derive for the amplitudes $A_1(t)$, $B_1(t)$ and $B_2(t)$, the following reduced dynamical system:

$$\frac{dA_1}{dt} = \sigma_1 A_1 + k^2\phi B_1 - 2A_1B_2\,, \qquad (6.3.2a)$$

$$\frac{dB_1}{dt} = \sigma_1 B_1 - k^2\phi A_1 + 2B_1B_2\,, \qquad (6.3.2b)$$

$$\frac{dB_2}{dt} = 2\sigma_2 B_2 + 2\,(A_1^2 - B_1^2)\,, \qquad (6.3.2c)$$

where $\sigma_1 = k(1 - k^2)$, $\sigma_2 = 2k(1 - 4k^2)$.

The phase flow of the above dynamical system (6.3.2) is dissipative if the following relation is satisfied: $\sigma_1 + \sigma_2 < 0$, and because of this dissipative effect, the attractors have zero phase volume and dimensionality lower than 3 (when t tends to infinity) for a wavenumber k such that

$$0.58 < k < 1\,. \tag{6.3.3}$$

This three-amplitude dynamical system (6.3.2) can be studied qualitatively and numerically.

6.3.2 Another way of deriving a three-amplitude dynamical system for the KS – KdV equation (83) is using the Fourier series. In this case we assume that (see, for instance, [83])

$$\eta(\tau, \xi) = \frac{1}{2}\sum A_n(\tau)\exp\left[\mathrm{i}n(\omega_1^\circ \tau - k_1\xi)\right], \tag{6.3.4}$$

where $A_n(\tau)$ is the complex amplitude of the nth spatial harmonic, and ω_1° is the linear angular frequency (in fact, the angular frequency of the fundamental harmonic, with k_1 as the wavenumber, at the first stage of its growth).

It must be stressed that if the wavenumber $k_n = nk_1$ is the actual wavenumber of the nth harmonic, the frequency $n\omega_1^\circ$ cannot be considered as its actual frequency ω_n (the latter may vary a little, owing to possible small dispersive effects). The slow variation of the phase $\varphi_n(\tau)$ corresponding to this small frequency shift is taken into account in the complex amplitude:

$$A_n(\tau) = |A_n(\tau)|\exp\left[\mathrm{i}\varphi_n(\tau)\right]. \tag{6.3.5}$$

Inserting (6.3.4) into (6.1.8), for the fist three harmonics we derive the following three-amplitude dynamical system (again, with $\alpha = \gamma = 1$):

$$\frac{\mathrm{d}A_1}{\mathrm{d}\tau} = \gamma_1 A_1 + \mathrm{i}k_1 A_1^* A_2\,, \tag{6.3.6a}$$

$$\frac{\mathrm{d}A_2}{\mathrm{d}\tau} = (\gamma_2 - 6\mathrm{i}k_1^3\phi)A_2 + \mathrm{i}k_1 A_1^2\,, \tag{6.3.6b}$$

$$\frac{\mathrm{d}A_3}{\mathrm{d}\tau} = (\gamma_3 - 24\mathrm{i}k_1^3\phi)A_3 + 3\mathrm{i}k_1 A_1 A_2\,, \tag{6.3.6c}$$

where $\gamma_n(nk_1)^2[1-(nk_1)^2]$, $n = 1, 2, 3$.

Near criticality, where the mode A_1 is the only unstable mode, while the others are linearly strongly damped, the dynamics is controlled by the marginally unstable mode A_1, to which the other two modes are slaved. As a consequence, from (6.3.6b), we have that the dynamics of the harmonic A_2 is slaved to the fundamental harmonic A_1 according to

$$A_2 = -\frac{\mathrm{i}k_1}{\gamma_2 - 6\mathrm{i}k_1^3\phi}A_1^2\,. \tag{6.3.7}$$

From (6.3.6a), with (6.3.7), the fundamental harmonic A_1 obeys the following Stuart – Landau equation:

$$\frac{\mathrm{d}A_1}{\mathrm{d}\tau} = \gamma_1 A_1 + \lambda A_1^* A_1^2\,, \tag{6.3.8}$$

where

$$\lambda = \gamma_2\frac{k_1^2}{a^2}\left(1 + \mathrm{i}\frac{6k_1^3\phi}{\gamma_2}\right), \quad a^2 = \gamma_2^2 + 36k_1^6\phi^2\,, \qquad \gamma_2 < 0\,,$$

is the complex Landau constant. Its real part (positive) corresponds to nonlinear dissipation; its imaginary part, to nonlinear frequency correction (due to dispersive effects).

As mentioned in [83], the dispersive character of the waves plays a crucial role [via the parameter ϕ in (6.1.8)] in the occurrence of amplitude collapses and frequency locking. This may be understood within the framework of dynamical system (6.3.6) after separation of the modulus and phase of the complex amplitudes [according to (6.3.5)].

For the simple case of $|A_1(\tau)|$ and $|A_2(\tau)|$ and the phase difference $\Phi(\tau) = \varphi_2 - 2\varphi_1$, we derive the following dynamical system of three equations in place of (6.3.6):

$$\frac{\mathrm{d}|A_1|}{\mathrm{d}\tau} = \gamma_1|A_1| - k_1|A_1||A_2|\sin\Phi\,, \tag{6.3.9a}$$

$$\frac{\mathrm{d}|A_2|}{\mathrm{d}\tau} = \gamma_2|A_2| + k_1|A_1|^2\sin\Phi\,, \tag{6.3.9b}$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\tau} = -6k_1^3\phi + k_1\frac{|A_1|^2 - 2|A_2|^2}{|A_2|}\cos\Phi\,. \tag{6.3.9c}$$

This dynamical system (6.3.9) is a particular case of that considered in [83, p.48] and deserves further careful numerical investigation!

7. Conclusion and comments

For high Reynolds numbers and long waves, when the Weber and Marangoni numbers are large, inertia-induced instability, dispersion, dissipation and Marangoni and Biot effects are taken into account in predicting the thickness $h(t,x)$ of the vertical falling film by the very significant boundary-layer model problem (4.1.6), (4.1.7). From my point of view, precisely this boundary-layer model problem is adequate for a careful numerical investigation! On the other hand, the generalized integral-boundary-layer model (averaged) equations, (4.1.1), (4.1.25) and (4.1.28), are an *ad hoc* but convenient simplification of the full boundary-layer model problem (4.1.6), (4.1.7), when a self-similar profile is arbitrarily assumed for the horizontal velocity and for the perturbation of the temperature beneath the film. Although the integral-boundary-layer model of three averaged equations, (4.1.1), (4.1.25) and (4.1.28), is derived in an *ad hoc* manner, it seems that these averaged equations (according to preliminary numerical results by M Zghal [63]) yield the correct linear and nonlinear behavior for the evolution of $h(t,x)$ for large times. It is hence a good substitute for the 'Benney' type equation (5.1.6), and the KS equation (5.1.8) for moderate Reynolds numbers. The linear equations (4.2.3a, b) are interesting for the investigation of the (linear) stability of the film. Finally, the finite-dynamical-system approach is also very interesting and a good quality of the corresponding dynamical system (4.4.5), relative to number of active modes (amplitudes) N, is the rapid convergence of the stochastic properties to a limiting state.

Concerning low Reynolds numbers, for long waves, when the Weber number is large but the Marangoni number is small, the evolution equation (6.1.6) for $h(t,x)$, valid with an error of $O(\varepsilon^4)$, is very significant and gives a KS – KdV evolution equation (6.1.8) in the limit of small amplitudes of the film thickness $h(t,x)$. As consequence, the features of a thin strongly-viscous-liquid film, when the Marangoni effect is small but the Weber number is large, are quite different because the appearance of a dispersive term. We note that our derivation of the KS – KdV equation (6.1.8) from the full problem (6.1.2), (6.1.3), with the assumptions (6.1.1), is only asymptotically correct!

Of course, we may expand the thickness of the film $h(t,x) = 1 + \delta\eta$ in a number of ways and derive various evolution equations for η! For example, for the complex amplitude A of a monochromatic wave with the wavenumber k_c,

$$\eta = A(T,X)\exp(\mathrm{i}k_c x) + \text{c.c.}$$

and amplitude B of the stream function (for a two-dimensional flow), so that

$$\psi = B(T,X)F(z)\exp(\mathrm{i}k_c x) + \text{c.c.},$$

in a weakly nonlinear multiple-scale analysis we can derive, with application of a third-order solvability condition, the following system of two evolution equation for $A(T,X)$ and $B(T,X)$:

$$\alpha\frac{\partial B}{\partial T} - \mu B + \delta\frac{\partial^2 B}{\partial X^2} + \lambda|B|^2|B| = \gamma AB,$$

$$\frac{\partial A}{\partial T} + a\frac{\partial^2 A}{\partial X^2} + b\frac{\partial^4 A}{\partial X^4} = c\frac{\partial^2(|B|^2)}{\partial X^2},$$

where $X = \varepsilon x$ and $T = \varepsilon^2 t$, when we assume that $\mathrm{Ma} - \mathrm{Ma}_s \sim \varepsilon^2\mu$ and Ma_s relates to $k_c \neq 0$ and indicates the threshold of the short-scale mode caused by surface-tension gradients alone, without surface deformation. But we also have a long-scale mode, which is influenced by gravity and capillary forces, and surface deformation plays a crucial role in its development. These two types of modes, having different scales, can interact with each other in the course of their nonlinear evolution (see, for instance, paper [32]). The coefficient λ is the Stuart – Landau constant and for a rigorous derivation of this constant, in the framework of the derivation (via the center-manifold theory) of the Ginzburg – Landau (GL) equation [when $\gamma = 0$ and $\alpha = 1$, in equation for $B(T,X)$], see paper [84]. We note that the linear part of the equation for $B(T,X)$ must be consistent with the original (linear) dispersion relation for the starting problem. The constants γ and c are the interaction coefficients and when these interaction coefficients are both zero, then we again derive the GL equation for B and a linear KS equation for A. We note that the approach for the resolution of the sideband instability of near-neutral waves is possible from the GL equation, by including the sideband effects in the complex coefficient δ.

In [32], by deriving the amplitude equations for the fundamental and is sidebands, a simple generalized criterion is shown to contain all the classical results. However, it still applies for conditions near the neutral curve but away from critical and contains nonlinear interactions (with low-wave-number modes) that have been omitted by classical theories. Finally, in [85], a numerical study of the above two nonlinear evolution equations for A and B is performed and it is shown that, due to a nonlinear coupling with the deformation of the liquid – gas interface, the primary convection pattern can undergo oscillatory instability generating various kinds of long surface waves which modulate the short-scale convection.

In conclusion, it is important to note that I am conscious that it is very desirable to have worked concrete cases of engineering importance and results available that could be compared with reality or other theories! But, from my point of view, it is no less important to derive conceptually coherent approximate models for the falling film flows. Indeed, without such models it is not 'efficient' to perform 'costly' numerical study — film-flow science seems to be suffering today from an excess of numerical investigations!

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