Physics – Uspekhi **41** (3) 231 – 239 (1998)

REVIEWS OF TOPICAL PROBLEMS PACS numbers: 02.40.Vh, 72.15.Gd

Topological phenomena in normal metals

S P Novikov, A Ya Mal'tsev

Contents

Abstract. Galvanomagnetic phenomena in metals in strong magnetic fields, associated with the Fermi surface geometry are considered. Using three-dimensional topology theorems, a full classification of all possibilities is made. For non-closed general-position electron orbits, special topological characteristics are introduced for the conductivity tensor at $B \to 0$.

1. Introduction. Historical remarks

It is known that the electrical conductivity of metals in strong magnetic fields reveals a wide variety of effects. The present paper is concerned with galvanomagnetic effects related to quasiclassical electron orbits in a single crystal in the presence of a uniform magnetic field **B**. As was shown by I M Lifshitz, M Ya Azbel, and M I Kaganov around 1956, the geometry of such orbits, based on the single-particle Bloch dispersion relation $\epsilon_n(\mathbf{p})$, completely determines the asymptotic behaviour of the electrical conductivity tensor in this case [1].

I M Lifshitz and V G Peschanskiĭ [2, 3] considered, in particular, the important role of non-closed orbits lying in, and passing through, a strip of finite width (for details see below). The experimental study of various relevant situations was carried out in Refs [4 – 7] (literature references are limited to those quoted later on in this paper). Theoretical and experimental developments in this field have been discussed thoroughly both in review articles [8, 9] and a monograph [10]. Since then there have been a large number of studies to examine specific materials based on a general theory of such phenomena [8 – 11].

Around 1982, one of us (S.P.N) observed that the physical picture that emerges from these studies generates some elegant problems in the topology of low-dimensional manifolds ([12], see also Refs [13 – 15]), and a number of topological investigations by his students A V Zorich, I A Dynnikov, and S P Tsarev followed. As a result, fundamental theorems on the topology of open general-position orbitals were proved and some non-trivial ('ergodic') non-general-position orbit geometries found that require special conditions to be met for their observation [16 – 23].

It will be observed that this latter case had remained untreated in the theory of galvanomagnetic effects simply because no such orbits had been known to exist. As regards the general-position orbits, the present authors, using the topological results just mentioned, were able to find universal topological characteristics for metals with a complex Fermi surface, amenable to observation in strong-magnetic-field conductivity studies on single crystals [24] (see below).

Note that complex Fermi-surface shapes occur, for example, in such metals as Au, Pb, Pt, and Ag, to name but a few. The Fermi surface of copper, determined by Pippard [25], is the earliest example of this type.

The purpose of this review is, based on available topological theorems, to present a general description of the asymptotic strong-magnetic-field behaviour of the conductivity tensor. We will consider 'general position' situations in detail, observable with probability unity for an arbitrary direction of the magnetic field and an arbitrarily complex Fermi surface; and will also point out some special cases possible for specially directed **B** (in particular, the 'ergodic case,' initially discovered in Ref. [23] and in Ref. [20], where the more general examples were constructed, and analysed in terms of conductivity in Ref. [27]). It will be shown, in particular, that for $B \to \infty$ all possible conductivity features related to non-closed, open, and general-position orbits may be classified into groups corresponding to certain regions on the unit sphere (parametrising **B** directions), each region being specified by an experimentally observable integer-valued plane Γ which presents a topological characteristic of

S P Novikov L D Landau Institute for Theoretical Physics, Russian Academy of Sciences,
ul. Kosygina 2, 117940 Moscow, Russia
Tel. (7-095) 137 32 44; (7-095) 702 93 17
Fax (7-095) 938 20 77
E-mail: novikov@itp.ac.ru
Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742-2431,
Tel. 301-779-74-72. Fax 301-314-93-63
E-mail: novikov@ipst.umd.edu
A Ya Mal'tsev L D Landau Institute for Theoretical Physics, Russian Academy of Sciences,
ul. Kosygina 2, 117940 Moscow, Russia
Tel. (7-095) 137 32 44; (7-095) 702-93 17
Fax (7-095) 938 20 77
E-mail: maltsev@itp.ac.ru

Received 9 October 1997
Uspekhi Fizicheskikh Nauk **168** (3) 249 – 258 (1998)
Translated by E G Strel'chenko; edited by M S Aksent'eva

the given stable group of open orbits. These integer-plane stability regions, together with the unit-sphere regions containing no open orbits at all, form a set of full measure on the unit sphere and thus embrace all general-position cases possible. In general, we present, for the first time, the classification, obtained in three-dimensional topological studies, of all possible types of open orbits which occur for directions of **B** of irrationality 3; and also summarise all orbit types which may arise when the magnetic field is directed such that the plane $\Pi(\mathbf{B})$ orthogonal to it contains reciprocal lattice vectors. The paper also present an analysis of previous theoretical and experimental studies in the field as well as new theoretical results related to earlier unknown, non-trivial topological aspects. The present work thus gives a full picture of strong-magnetic-field galvanomagnetic phenomena related to the complex topology of the Fermi surface of the metal.

We will concentrate here on the results achieved with the theory of normal metals [24]. Based on the recent topological results of Refs [20–22], it is also shown that these ideas are equally applicable to the theory of semiconductors (see Ref. [26]).

2. Observable quantities. The general position case

The most important case to consider is, in our view, the 'general position' case, which is observed with probability 1 for an arbitrarily directed magnetic field **B**. As already mentioned, the analysis of open orbits in this case brings about non-trivial topological characteristics of the Fermi surface — namely integer-valued (i. e., generated by two reciprocal lattice vectors) planes and the corresponding unit sphere zones — which determine the properties of all general-position orbits occurring for various directions of **B**.

Consider a single crystal of a normal metal with lattice L generated by the vectors $\mathbf{l}_1$, $\mathbf{l}_2$, and $\mathbf{l}_3$. As is well known, in the absence of a magnetic field, $\mathbf{B} = 0$, single-particle electronic states can be described in terms of energy bands and quasimomenta $\mathbf{p} = (p_1, p_2, p_3)$ defined modulo a reciprocal lattice vector, i.e.,

$$\mathbf{p} \text{ is physically equivalent to } \mathbf{p} + \mathbf{l}^*$$

for any vector $\mathbf{l}^*$ such that $\langle \mathbf{l}^*, \mathbf{l}_j \rangle = 2\pi\hbar n_j$, where the n_j are integers. The reciprocal lattice L^* is generated by the vectors $(\mathbf{l}_1^*, \mathbf{l}_2^*, \mathbf{l}_3^*)$ such that $\langle \mathbf{l}_i^*, \mathbf{l}_j \rangle = 2\pi\hbar\delta_{ij}$.

In this approximation we have a set of 'dispersion relations'

$$\epsilon_n(\mathbf{p}) = \epsilon_n(\mathbf{p} + \mathbf{l}^*)\,, \qquad n = 0, 1, 2, \ldots, \tag{1}$$

which describe the quasimomentum dependence of the electron energy.

Electrons in the ground state occupy all levels below the Fermi energy ϵ_{F}, $\epsilon_j(\mathbf{p}) \leqslant \epsilon_{\mathrm{F}}$, leaving all levels above unfilled. The theory of electrical conductivity in normal metals deals with small perturbations of this picture, so that all effects of interest depend on the behavior of the electronic distribution function in a small neighbourhood of the Fermi surface in quasimomentum space $\epsilon(\mathbf{p}) = \epsilon_{\mathrm{F}}$.

For most normal metals, the following conditions are satisfied:

(a) The dispersion relation has no critical points at the Fermi surface, i.e.,

$$\nabla\epsilon_j(\mathbf{p}) \neq 0 \quad \text{for} \quad \epsilon = \epsilon_{\mathrm{F}}\,.$$

(b) The Fermi surfaces of different allowed energy bands $\epsilon_j(\mathbf{p})$ and $\epsilon_i(\mathbf{p})$ do not intersect†, i.e., at the Fermi level $\epsilon = \epsilon_{\mathrm{F}}$

$$\epsilon_j(\mathbf{p}) \neq \epsilon_i(\mathbf{p})\,, \qquad i \neq j\,.$$

For future convenience, we will describe here the experimental aspects of the unit probability general-position situations and introduce integer-valued, observable, topological strong magnetic-field conductivity characteristics whose existence was established in Ref. [24]. Let us apply a strong magnetic field **B** ($B \simeq 10$ T)‡ and a weak electric field **E** orthogonal to **B**. As shown by means of topological theorems [24], in the general-position case there are two possibilities for the asymptotic behaviour of the conductivity tensor in the plane orthogonal to **B**.

Case 1. Compact orbits

The two-dimensional part of the conductivity tensor $\sigma_{\mathbf{B}}^{\alpha\beta}$ tends to zero as $B \to \infty$ for $\mathbf{B}/B$ fixed: $\sigma_{\mathbf{B}}^{\alpha\beta} \to 0$, $\alpha, \beta = 1, 2$ in the plane orthogonal to **B** (this conductivity component is already vanishingly small for $B \simeq 10$ T). The asymptotic form of the conductivity tensor is (see Ref. [1])

$$\sigma_{\mathbf{B}}^{ij} \simeq \frac{ne^2\tau}{m^*} \begin{pmatrix} (\omega_B\tau)^{-2} & (\omega_B\tau)^{-1} & (\omega_B\tau)^{-1} \\ (\omega_B\tau)^{-1} & (\omega_B\tau)^{-2} & (\omega_B\tau)^{-1} \\ (\omega_B\tau)^{-1} & (\omega_B\tau)^{-1} & 1 \end{pmatrix}, \tag{2}$$

where ω_B is the cyclotron frequency, and τ is the electron transit time. The z axis lies along **B**.

Case 2. Open general-position orbits

For a certain direction $\mathbf{B}/B = \mathbf{n}$, as $B \to \infty$, the two-dimensional part of the conductivity tensor $\sigma_{\mathbf{B}}^{\alpha\beta}$, $\alpha, \beta = 1, 2$, tends to the nonzero constant tensor $\sigma_{\infty}^{\alpha\beta}$ dependent on the direction of the unit vector **n**. In this case the (2×2) tensor $\sigma_{\infty}^{\alpha\beta}$ is always of rank unity since one of its eigenvalues is zero. To describe the total, 3×3, conductivity tensor $\sigma_{\mathbf{B}}^{ij}$, $i, j = 1, 2, 3$, we introduce an orthonormal basis in which $\mathbf{e}_1$ is directed along the vector lying in the core of the (2×2) tensor $\sigma_{\mathbf{B}}^{\alpha\beta}$ in the plane orthogonal to **B**; $\mathbf{e}_2$ lies in the same plane $\mathbf{e}_2 \perp \mathbf{B}$, $\mathbf{e}_2 \perp \mathbf{e}_1$; and $\mathbf{e}_3 = \mathbf{B}/B$ (Fig. 1).

Referring to this coordinate system, the three-dimensional conductivity tensor is

$$\sigma_{\mathbf{B}}^{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} + O(B^{-1})\,, \tag{3}$$

where $(*)$ stands for certain nonzero constants. Note that $\sigma_{\mathbf{B}}^{ij} = \sigma_{-\mathbf{B}}^{ji}$ and $\sigma(\mathbf{e}_1) = O(B^{-1})$.

The picture so described is stable in the sense that for magnetic fields with directions $\mathbf{e}_3' = \mathbf{B}'/B'$ close to the original $\mathbf{e}_3 = \mathbf{B}/B$ the conductivity tensor will have the same

† This property may be destroyed by magnetic breakdown.

‡ It can be shown from general arguments relevant to this type of effect (see, e.g., Refs [8–11]) that in order for our 'geometric limit' to be observable, the only condition on the magnitude of the magnetic field is $\omega_B\tau \gg 1$ (ω_B is the cyclotron frequency and τ is the electron transit time). This yields ~ 1 T for pure gold samples at temperatures of $\simeq 4$ K, which are precisely the conditions used in Ref. [7]. The restriction securing the quasiclassical nature of electron motion, $\hbar\omega_B \ll \epsilon_{\mathrm{F}}$, is satisfied for all realistic values of B (the upper bound being $10^3 - 10^4$ T).

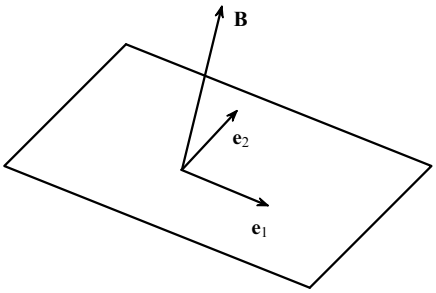

Figure 1. Special basis, corresponding to case 2.

form (3) in a new orthonormal basis $(\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3')$, where $\sigma_{\mathbf{B}'}(\mathbf{e}_1') = 0$, $\sigma_\infty' \neq 0$ in a plane orthogonal to $\mathbf{B}'$.

Our most important statement here is that the plane generated by the 0-vectors $\mathbf{e}_1$ and $\mathbf{e}_1'$ is integral-valued and that it is the same for all small rotations $\mathbf{B}'$ of the direction of the magnetic field $\mathbf{B}$ (Fig. 2).

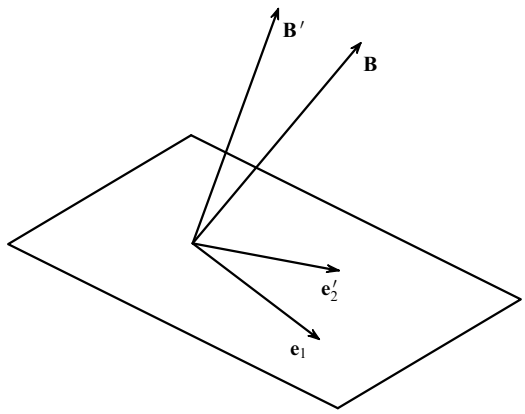

Figure 2. Integer-valued plane, generated by the vectors $\mathbf{e}_1$ and $\mathbf{e}_1'$, for small rotations of the magnetic field.

The term integer-valued is here understood to mean that the plane under consideration is generated by two reciprocal lattice vectors $(\bar{\mathbf{l}}^*, \bar{\bar{\mathbf{l}}}^*)$,

$$\bar{\mathbf{l}}^* = n_1\mathbf{l}_1^* + n_2\mathbf{l}_2^* + n_3\mathbf{l}_3^*\,,$$
$$\bar{\bar{\mathbf{l}}}^* = m_1\mathbf{l}_1^* + m_2\mathbf{l}_2^* + m_3\mathbf{l}_3^*$$

and

$$\mathbf{e}_1 = \alpha\bar{\mathbf{l}}^* + \beta\bar{\bar{\mathbf{l}}}^*\,, \qquad \mathbf{e}_1' = \alpha'\bar{\mathbf{l}}^* + \beta'\bar{\bar{\mathbf{l}}}^*\,.$$

Here the n_j and m_j are integers. The components of the vector $\bar{\mathbf{l}}^* \times \bar{\bar{\mathbf{l}}}^*$ thus specify this plane uniquely, and we arrive therefore at a set of three integers $n_1m_2 - m_1n_2 = M_3$, $n_2m_3 - m_2n_3 = M_1$, $n_3m_1 - m_3n_1 = M_2$ defined up to a common factor (so that in reality only their ratios are meaningful).

We shall call the set of numbers (M_1, M_2, M_3) defined up to a common factor the 'topological type' of the conductivity tensor in a strong magnetic field $\mathbf{B}$ for a given local 'stability region' for Case 2. A topological type is thus specified by two (or more) close magnetic field directions, $\mathbf{B}$ and $\mathbf{B}'$, lying within one and the same (field-direction-specifying) stability region on the unit sphere and represents a locally stable topological characteristic of the Fermi surface. For small field direction variations, the topological type (M_1, M_2, M_3) remains unchanged, i. e., remains constant on a certain open unit sphere region which we call a (M_1, M_2, M_3) topological type 'stability region.'

We will denote the area (or measure) of the (M_1, M_2, M_3) topological type stability regions by $\mu(M_1, M_2, M_3)$, and the measure of the unit sphere set corresponding to Case 1, by μ_0. We argue that

$$\mu_0 + \sum_{(M_1, M_2, M_3)} \mu(M_1, M_2, M_3) = 4\pi \tag{4}$$

where the sum is over all topological types. In fact $\mu(M_1, M_2, M_3)$ for many topological types and in any case topological types with large numbers $|M_j|$ produce very small values of μ.

From what has been said it follows generally speaking, that in a real experiment a finite (and not very large) number of topological types and of their 'stability ranges' can be observed.

Mathematically, Eqn (4) implies that all non-general-position possibilities other than Cases 1 and 2 correspond to the $\mathbf{B}$ directions covering a set of zero measure on a unit sphere. In the following treatment, some of the most interesting examples of non-general position will be considered.

For a comparison with earlier experimental data we also present here the asymptotic form of the resistivity tensor inverse to σ, $R = \sigma^{-1}$, using the same basis as for σ in Eqn (3) (see Refs [10, 11]).

Case 1. The components of $\hat{R}$ are of the order

$$\hat{R} \simeq \frac{m^*}{ne^2\tau}\begin{pmatrix} 1 & \omega_B\tau & 1 \\ \omega_B\tau & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{5}$$

(the part of the matrix proportional to B is skew-symmetric).

Case 2. The components of $\hat{R}$ are of the order

$$\hat{R} \simeq \frac{m^*}{ne^2\tau}\begin{pmatrix} (\omega_B\tau)^2 & \omega_B\tau & \omega_B\tau \\ \omega_B\tau & 1 & 1 \\ \omega_B\tau & 1 & 1 \end{pmatrix}, \tag{6}$$

where $\omega_B = eB/m^*c$ is the cyclotron frequency and τ, the electron free transit time.

We now present some of the experimental data obtained by Gaidukov [7] for Au. As we see from (6), the observed B dependence of the resistance must be of the form $\rho \sim (B^2\cos^2\alpha)\rho_0$ in the plane orthogonal to $\mathbf{B}$, where $\rho_0 = m^*/ne^2\tau$. The factor $(\cos^2\alpha)$ is unity for the electric field along the vector $\mathbf{e}_1$ (see above), which is an eigenvector of the conductivity tensor (3) with an eigenvalue 0 in the plane normal to $\mathbf{B}$.

In Figure 3 (see Fig. 11 from Ref. [7]) the dashed regions are those where the B behaviour was observed (numbers 1,0,0 etc. designating the directions of $\mathbf{B}$). It is interesting to note, according to Ref. [7], that the resistivity in the centres of these regions 'has very deep minima' and behaves as in Case 1. Within the dashed regions but outside the indicated central points, the resistivity should behave like B^2 as in Case 2. Experiment, however, showed B^α for $\alpha \leqslant 2$ ('somewhat less,' to quote Ref. [7]). Probably the magnetic fields $B \approx 2$ T were not sufficient to observe our asymptotic behaviour, so it is

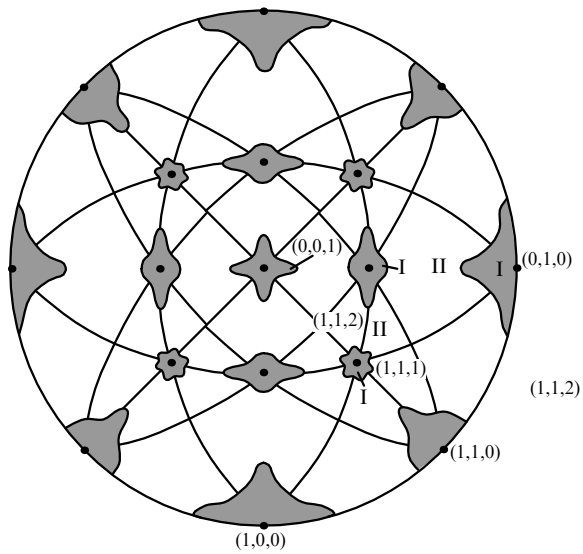

Figure 3. Experimental data obtained by Yu P Gaĭdukov for Au. The dark regions correspond to case 2.

perhaps worthwhile repeating the experiments at $B \geqslant 10$ T. In the non-dashed regions we have Case 1. There are also some interesting features (perhaps, non-general position cases?) seen along the solid curves in these regions, which we will discuss later.

Returning now to the interior of the dashed regions (with remote central points), it is expected that the B^2 behaviour will be experimentally confirmed there. Also, we argue that the dashed regions in this case are in fact the stability regions whose topological types correspond to integer-valued planes orthogonal to unit vectors directed toward the remote central points, so that in Fig. 11 from Ref. [7] (Fig. 3 of ours) the following topological types can be observed:

$$(M_1, M_2, M_3) = (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1), (\pm 1, \pm 1, 0), (\pm 1, 0, \pm 1), (0, \pm 1, \pm 1), (\pm 1, \pm 1, \pm 1). \quad (7)$$

This statement must be verified experimentally, however.

Here we have considered the results of Ref. [7] as direct experimental evidence that both the dashed regions and some points on the solid lines have no relation to Case 1. Our concept of integer-valued planes was not used — nor indeed known — at the time.

Anyway, the disappearance of conductivity in the centres of the dashed regions (to which, in this case, also corresponds a minimum in resistivity) is consistent with our understanding of the situation in Au. The point is that if at a certain **B** the integer-valued plane we described becomes parallel to the field and so open general position orbits exist for all field directions close to that given, these orbits may still disappear for this special direction of the field **B** (see below).

We now turn to a rather interesting case of ergodic open orbits which may occur for special (irrationality 3) directions of the magnetic field **B** [$\Pi(\mathbf{B})$ does not contain reciprocal lattice vectors] and were discovered in Refs [20–22]. Although the magnetic field directions mentioned above form a set of measure 0 on the unit sphere for a given general-type Fermi surface, still special directions of **B** exist for which such situations may even be observed for rather complex Fermi surfaces.

Case 3. Non-general position situation (ergodic orbits)

The experimental data of Ref. [7] indicate an unusual behaviour for the resistivity tensor along the solid lines shown in Fig. 4 (see Fig. 5 of Ref. [7]). It is found, specifically, that at many points along these lines the resistance shows very deep minima where its asymptotic behaviour is B^α, with the values of α lying in the range $1 < \alpha < 1.8$. In our view, it would be interesting to repeat these measurements at say, $B \geqslant 10$ T, instead of $B \approx 2$ T as used in Ref. [7]. In this case, much shallower stability regions may be expected, corresponding to the general-position Case 2 with topological types (M_1, M_2, M_3) of more complexity than that in Eqn (7). There is an alternative possibility, though.

One may expect, that in this region more complex general-position orbits of the type observed in Refs [20–22] ('ergodic' orbits) may exist. This type of orbit is illustrated in Fig. 5. This situation, as one of the present authors (A Ya M) suggests, must 'generally' lead to 'scaling' resistivity behavior $R \sim B^\alpha$, $1 < \alpha < 2$ (see Ref. [27]).

Also, the average value of the group velocity v_z along each such orbit is zero, and these ergodic orbits therefore contribute to the three-dimensional conductivity tensor in

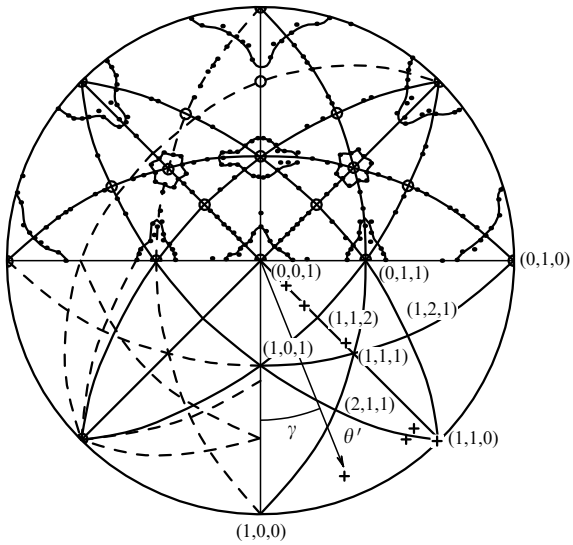

Figure 4. Experimental data obtained by Yu P Gaĭdukov for Au.

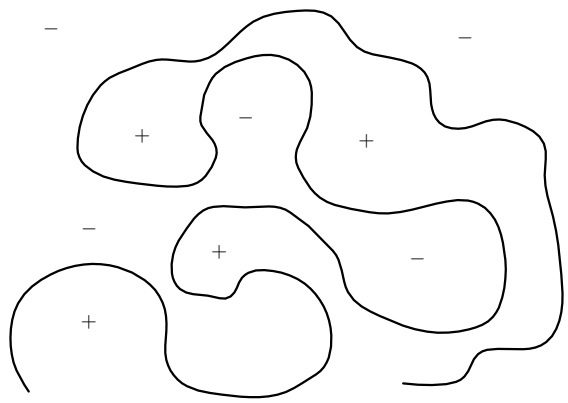

Figure 5. Example view of an ergodic open trajectory constructed by I A Dynnikov. The signs '+' and '−' indicate regions of greater and lesser energy.

such a way that

$$\sigma^{ij}_{\mathbf{B}}(\gamma) \to 0 \tag{8}$$

for $B \to \infty$ for all $i, j = 1, 2, 3$.

For the proper choice of axes in the plane $\Pi(\mathbf{B})$ normal to the magnetic field, the general form of these contributions can be written in the form [27]

$$\sigma^{ij}_{\mathbf{B}} \simeq \frac{ne^2\tau}{m^*} \begin{pmatrix} (\omega_B\tau)^{2\beta-2} & (\omega_B\tau)^{-1} & (\omega_B\tau)^{\beta+\gamma-2} \\ (\omega_B\tau)^{-1} & (\omega_B\tau)^{2\alpha-2} & (\omega_B\tau)^{\alpha+\gamma-2} \\ (\omega_B\tau)^{\beta+\gamma-2} & (\omega_B\tau)^{\alpha+\gamma-2} & \dfrac{T^2}{\epsilon_{\mathrm{F}}^2} + (\omega_B\tau)^{2\gamma-2} \end{pmatrix}, \tag{9}$$

where $0 < \alpha, \beta, \gamma < 1$, $\alpha + \beta = 1$.

To this contribution we must, however, add those from all closed (compact) orbits. These, as can be shown from general arguments, must always exist in Au whatever the direction of the magnetic field applied, so that in the limit of strong magnetic fields we will generally have a nonzero longitudinal (along **B**) conductivity,

$$\sigma^{zz}_{\mathbf{B}} \to \sigma^{zz}_{\infty} \neq 0\,.$$

We expect, however, that for the unity-sphere **B** directions containing ergodic orbits, σ^{zz}_{∞} will be reduced compared with similar general-position directions, thus implying the existence of local minima at such points on the unit sphere. This property can presumably be used to experimentally distinguish between ergodic situations and very small stability regions, where in order to observe the B^2 resistance behaviour in the $\Pi(\mathbf{B})$ plane, with the size of the region decreasing and its topological type growing in complexity, increasingly larger boundary values of **B** are needed.

For the ergodic orbits, the two-dimensional resistivity tensor in the plane orthogonal to **B** has the asymptotic form [27], as $B \to \infty$,

$$\rho_{\eta\xi}(B) = \frac{m^*}{ne^2\tau} \begin{pmatrix} (\omega_B\tau)^{2\alpha} & \omega_B\tau \\ \omega_B\tau & (\omega_B\tau)^{2\beta} \end{pmatrix} \tag{10}$$

if one employs the same coordinate system used in Eqn (9).

A more detailed discussion of ergodic orbits can be found in Ref. [27]. We now turn to a topological interpretation of the results presented above.

3. Topological analysis of general position cases

As is well known and was mentioned earlier, quasi-momenta are represented by vectors $\mathbf{p} = (p_1, p_2, p_3)$ defined modulo reciprocal lattice vectors. Topologically, these equivalency classes can be treated as points of a three-dimensional torus T^3 which we call the 'Brillouin zone'. The entire **p** space R^3 will be referred to as the 'extended Brillouin zone.' In topological terms, we are dealing here with a 'universal covering' over a three-dimensional torus.

In the standard quasiclassical approach to galvanomagnetic effects in the presence of a magnetic field **B**, the 'electronic orbits' for the adiabatic evolution of the Bloch waves can be obtained from the following dynamic systems in $(\mathbf{x}, \mathbf{p})$ space:

$$\dot{\mathbf{x}} = \{\mathbf{x}, \epsilon(\mathbf{p})\}\,, \tag{11}$$

$$\dot{\mathbf{p}} = \{\mathbf{p}, \epsilon(\mathbf{p})\}\,, \tag{12}$$

where $\epsilon(\mathbf{p})$ is the dispersion relation in the absence of **B**, and the Poisson brackets are of the form

$$\{p_i, x_j\} = \delta_{ij}\,, \qquad \{x_i, x_j\} = 0\,,$$
$$\{p_i, p_j\} = \frac{e}{c}\,\epsilon_{ijk}\,B_k\,. \tag{13}$$

For a uniform magnetic field **B** our equations (12) for (p_1, p_2, p_3) are closed (because $B_k = \text{const}$), thus yielding a Hamiltonian system in a three-dimensional torus (Brillouin zone), with Poisson brackets

$$\{p_i, p_j\} = \frac{e}{c}\,\epsilon_{ijk}B_k$$

and Hamiltonian $\epsilon(\mathbf{p})$. The system has two integrals of motion, $\epsilon(\mathbf{p})$ and $\sum B_k p_k$, the latter being the 'Casimir' for the bracket in **p** space. Since $\epsilon_{jqk} = -\epsilon_{jkq}$,

$$\left\{p_j, \sum B_q\,p_q\right\} = \frac{e}{c}\sum_{q,k}\epsilon_{iqk}B_qB_k = 0\,.$$

Thus, electronic orbits are defined in R^3 by the equations

$$\epsilon(\mathbf{p}) = \epsilon_{\mathrm{F}}\,, \qquad \sum B_k\,p_k = \text{const}\,. \tag{14}$$

Geometrically, these are cross sections of the Fermi surface by planes orthogonal to the magnetic field, each section being a union of orbits.

We will call an electronic orbit compact if it is closed in space R^3 (i. e., in the extended Brillouin zone). A curve in R^3 will be called periodic with a period T (non-compact) if $\mathbf{p}(t+T) = \mathbf{p}(t) + \mathbf{l}^*$, where $\mathbf{l}^*$ is a certain reciprocal lattice vector. Strictly speaking, such a curve is closed in the three-dimensional torus T^3 (Brillouin zone) but, in topology parlance, is non-homotopic to zero in T^3. Compact orbits have the property $\mathbf{l}^* = 0$ and are, in topological terms, homotopic to zero in the three-dimensional torus T^3.

It is easily seen that periodic non-compact orbits can only occur if the magnetic field **B** is directed such that the plane $\Pi(\mathbf{B})$ orthogonal to it contains at least one reciprocal lattice vector $\mathbf{l}^* \neq 0$.

Consider now the magnetic fields **B** whose 'irrational' general-position directions satisfy the following conditions:

(1) Plane $\Pi(\mathbf{B})$ does not contain reciprocal lattice vectors.

(2) All the points where planes orthogonal to **B** touch the Fermi surface are non-degenerate [note that these are critical points of the dynamic system (12) on the Fermi surface].

(3) A separatrix orbit emanating from one saddle does not enter any other saddle, i.e., it either does not have the second end or returns to the same saddle it started from. (In the general position case any plane parallel to $\Pi(\mathbf{B})$, i.e., orthogonal to **B**, contains no more than one saddle).

Let us now define the 'topological rank' of the Fermi surface. The relation

$$\epsilon_n(\mathbf{p}) = \epsilon_{\mathrm{F}}$$

in quasimomentum space R^3 (extended Brillouin zone) is specified by the periodic function $\epsilon_n(\mathbf{p})$. The surface it defines in R^3 is a union of 'connectivity components' on each of which two different points can be connected by a path lying at the Fermi surface. We will call the Fermi surface 'topologically complex' if there is at least one connectivity component which does not lie between two parallel planes in R^3. In this case we

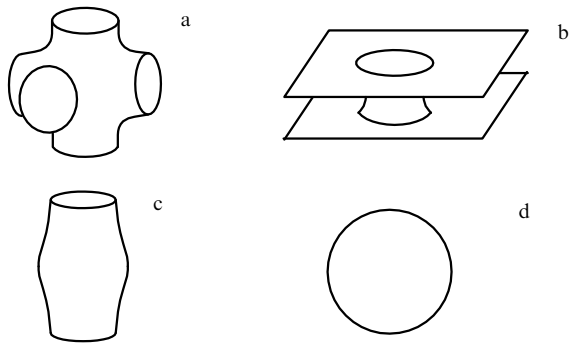

Figure 6. Examples of rank 3, 2, 1 and 0 Fermi surfaces.

will say that this Fermi surface component as well as the Fermi surface itself are of rank 3 topologically (Fig. 6a).

A Fermi surface is said to be of topological rank 2 if any one of its connectivity components may be confined between a certain two parallel planes and if there exists at least one component which cannot be confined in a cylinder ('warped plane' components, see Fig. 6b). In particular, there may exist two (or more) connectivity components of rank 2 confined between a pair of parallel planes with different general directions (Fig. 7).

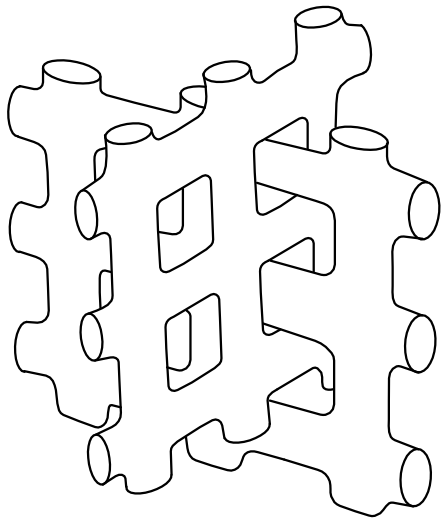

Figure 7. Example of a toppological rank 2 Fermi surface, containing two components with different integral directions.

A Fermi surface has a topological rank 1 if any one of its connectivity components may be confined within a certain cylinder and if there exists at least one component which cannot be confined within a sphere of finite radius ('warped cylinder' components, Fig. 6c).

A Fermi surface has a topological rank 0 if any one of its connectivity components may be confined within a sphere of finite radius (Fig. 6d).

Upon application of a magnetic field we obtain electronic orbits specified by the intersection of the planes $\Pi(\mathbf{B})$ with the Fermi surface. The following possibilities then arise.

(1) The Fermi surface is of topological rank 0. All electronic orbits are closed.

(2) The Fermi surfaces are of topological rank 1. Both closed and open electronic orbits are possible. The latter can only occur if the magnetic field is orthogonal to the axis of the cylinder for one of the components. However, even in this case orbits may all be compact, as exemplified by the 'helix' case of Fig. 8.

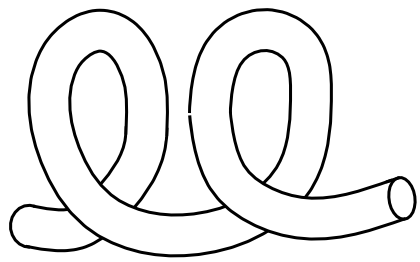

Figure 8. Connection of a 'spiral' type component. Open orbits are absent for any direction of **B**.

The open orbits (if they exist) are periodic, with the period vector directed along the axis of the corresponding cylinder. This is obviously a non-general position situation because open orbits correspond only to the one-parametric family of **B** directions on the unit sphere.

(3) The Fermi surface is of topological rank 2. Closed and open orbits are possible for any direction of **B** and, as already mentioned, there may generally exist second-rank connectivity components with different general directions of the corresponding pairs of parallel planes. It is readily seen that any open orbit lies in a straight strip of finite width, obtained by the intersection with the $\Pi(\mathbf{B})$ plane of a pair of integer-valued planes bounding the corresponding connectivity component. (It is assumed that $\Pi(\mathbf{B})$ has a different direction from all such $\Pi^{(j)}(\epsilon_{\mathrm{F}})$ planes for connected components on the Fermi surface). We assert that for general-position, magnetic field directions of irrationality 3, open orbits may exist only on connectivity components corresponding to one and the same direction of the $\Pi^{(j)}(\epsilon_{\mathrm{F}})$ planes, so that all open orbits have one and the same average direction specified by the intersection of $\Pi(\mathbf{B})$ with the appropriate, uniformly directed $\Pi^{(j)}(\epsilon_{\mathrm{F}})$. We note also that this picture is locally stable to small rotations of the magnetic field **B**, and that the topological type of the conductivity tensor is determined by the integer-valued plane parallel to all corresponding $\Pi^{(j)}(\epsilon_{\mathrm{F}})$. For large departures of the magnetic field directions, open orbits may disappear on the components we consider so as to emerge on those with integer-valued planes directed differently (if such planes exist), implying a transition to another stability region or to Case 1 (if no new open orbits appear at all). Note here that if $\Pi(\mathbf{B})$ coincides with one of $\Pi^{(j)}(\epsilon_{\mathrm{F}})$, then all open orbits are periodic because of the **B** direction being integer-valued. The above picture is enough to demonstrate all basic conductivity tensor features predicted by our rigorous topological analysis [16–23].

(4) The Fermi surface of topological rank 3 is the most interesting to consider. Generally speaking, each time we consider electronic orbits we are dealing with only one of the Fermi-surface connectivity components. Identifying equivalent points in quasimomentum space by

$$\mathbf{p} \equiv \mathbf{p} + \mathbf{l}^{*} ,$$

we obtain a closed two-dimensional surface in a three-dimensional torus T^3 (Brillouin zone). Let the magnetic field $\mathbf{B} = (B_1, B_2, B_3)$ be in totally irrational direction so that $\Pi(\mathbf{B})$ does not contain reciprocal lattice vectors $\mathbf{l}^*$ and the corresponding family of electronic orbits satisfies the non-degeneracy conditions 1, 2, and 3 (see above).

Now let us remove all nonsingular open orbits (which are all closed curves in quasimomentum space R^3) from the Fermi surface. The remaining part is, obviously, a union of surfaces bounded by closed singular (i.e., terminating in a singular point) orbits. Thus, the compact non-singular orbits having been removed, the Fermi surface is a union of components S_i (if open orbits do at all exist). The boundaries of the S_i are sets of closed singular (separatrix) orbits $\gamma_{i\alpha}$ (see Fig. 9).

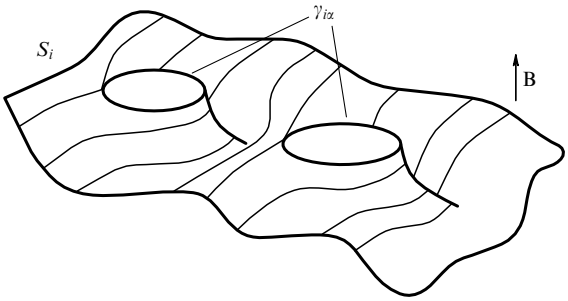

Figure 9. Components S_i, consisting of open orbits. The closed singular $\gamma_{i\alpha}$ and critical points form the boundary of the S_i components.

Each of the separatrix orbits is a planar curve in $\Pi(\mathbf{B})$, with its interior being, in topological terms, a two-dimensional disk $D_{i\alpha}$ lying in the plane $\Pi(\mathbf{B})$. Let us now fill all the bounds of $\gamma_{i\alpha}$ by planar two-dimensional disks $D_{i\alpha}$ by adding these latter to the S_i (i.e., partial Fermi surfaces). We will then have a two-dimensional, piecewise smooth surface in R^3 and hence (following a quasimomentum identification procedure) also their images $\bar{S}_i$ in the three-dimensional torus T^3. By definition, all open orbits lie on surfaces constructed in this way.

We shall call the genus of surface $\bar{S}_i$ in a three-dimensional torus a 'genus of corresponding open orbits' lying in it (for magnetic field directions of irrationality 3).

The result which we consider to be the most important in the work of A V Zorich and I A Dynnikov [16–22] is that in the general-position case all surfaces $\bar{S}_i$ constructed in this way are of genus 1, implying that they are topologically equivalent to two-dimensional tori placed in the Brillouin zone (i.e., a three-dimensional torus T^3). The general proof of this theorem is highly involved [19] and need not be pursued here.

The term 'in the general-position case' is understood here to mean that if for a certain energy level $\epsilon(\mathbf{p}) = \epsilon_0$ this statement does not hold then such a situation is destroyed by an arbitrarily small variation of the level (with the direction of $\mathbf{B}$ fixed [20]); note that the magnetic field directions for which this statement may not hold is a set of measure zero [20–22]. It is readily shown that the $\bar{S}_i$ surfaces are in fact immersed in a three-dimensional torus T^3 which has no self-intersections. Nor do the surfaces intersect one another. It follows from the above that once the two-dimensional disks $D_{i\alpha}$ are filled in the way described, in the general-position case each S_i surface in quasimomentum space R^3 appears as a 'warped plane' and hence its sections by planes parallel to $\Pi(\mathbf{B})$ lie in finite-width strips in those planes.

That the above topological picture is locally stable follows from the fact that the compact nonsingular orbits employed in constructing the 'reduced Fermi surface' (see above) are locally stable towards small magnetic field rotations. The two conditions we have formulated, the 'nondegenerate' Fermi surface and the 'general-position' condition on field $\mathbf{B}$ directions, are both significant in deriving the statements above. It is easily seen that the above arguments provide justification for Case 2 for the general-position conductivity tensor.

The earliest example of such a topologically stable open orbit was given by I M Lifshitz and V G Peschanskiĭ [2] for the Fermi surface of copper (or for the 'fine' space net to use the language of Pippard's work on this metal). For this net (see Fig. 10, or Figs 2, 3 of Ref. [2]), in the dashed stability regions we have open general-position orbits corresponding to Case 2. As noted by I M Lifshitz and V G Peschanskiĭ, the directions of the open orbits are, on average, specified by the intersection of planes orthogonal to the magnetic field with the coordinate planes (xy), (yz), and (xz) (for the corresponding stability regions). Thus, in this case the topological types will be given by the numbers

$$(M_1, M_2, M_3) = (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\,.$$

The stability regions in this case are small regions in the vicinity of the corresponding magnetic field directions on the unit sphere. The remaining part of the unit sphere represents (for a fine enough net) Case 1.

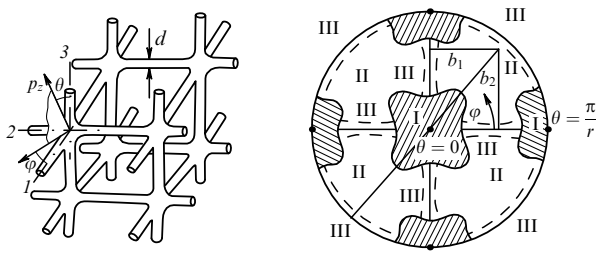

Figure 10. So-called 'fine space net' and its corresponding stability zones. Its shown by I M Lifshitz and V G Peschanskiĭ, the average directions of open orbit in these zones is given by the intersections of $\Pi(\mathbf{B}$ with the coordinate planes (xy), (yz), and (xz).

In a subsequent work by I M Lifshitz and V G Peschanskiĭ [3] the case of a unit sphere region with open orbits is discussed for a more complex Fermi surface defined by

$$\alpha\left(\cos\frac{ap_x}{\hbar} + \cos\frac{ap_y}{\hbar} + \cos\frac{ap_x}{\hbar}\right) + \beta\left(\cos\frac{ap_x}{\hbar}\cos\frac{ap_y}{\hbar} + \cos\frac{ap_y}{\hbar}\cos\frac{ap_z}{\hbar} + \cos\frac{ap_z}{\hbar}\cos\frac{ap_x}{\hbar}\right) + \delta\cos\frac{ap_x}{\hbar}\cos\frac{ap_y}{\hbar}\cos\frac{ap_z}{\hbar} = \zeta_0\,, \tag{15}$$

for various values of the parameters (corresponding integer-valued planes are not mentioned in [3])†. The authors of Ref.

† Our notation α, β, δ, and ζ_0, although different from that adopted in Ref. [3], is more convenient for the present discussion.

[3] are, however, incorrect in claiming the presence of unit sphere regions in which open orbits with different average directions may coexist (see Fig. 4 in Ref. [3]). This result is in conflict with our general arguments and cannot therefore be correct. The situation described in Ref. [3] cannot occur for open unit sphere regions, so that the scenario shown last in Fig. 11 (Fig. 1 in Ref. [3]) is wrong. This statement is a mathematically rigorous consequence of the theorems proved in Ref. [19].

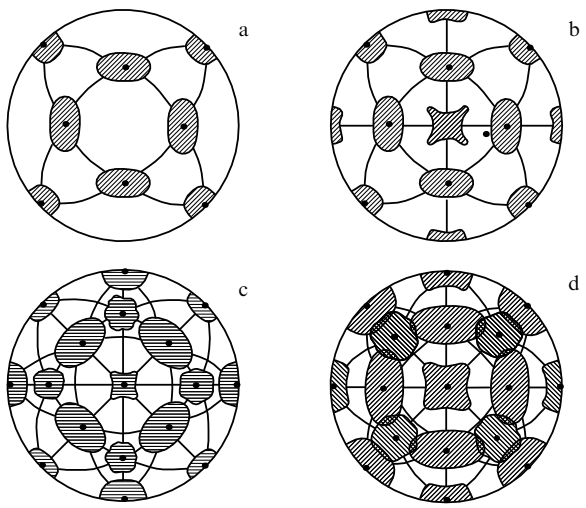

Figure 11. Stability zones given by I M Lifshitz and V G Peschanskiĭ for different analytical examples of Fermi surfaces corresponding to a cubic lattice. Integer planes, corresponding to these zones, are not discussed in the literature. According to our results, the last diagram is erroneous, since the stability zones may not intersect one another for a whole region of the unit sphere, and open general-position orbits with different average directions may not coexist over whole region.

There is one further important point to be made. In Ref. [3] it pointed out that depending on the values of α, β, δ, and ζ_0 there are a finite number of topologically different Fermi surface shapes of the type defined by Eqn (15). All of these are described in some detail and for each of them the diagram of existence of open orbits is given (Fig. 11), it being assumed that the existence regions (on the unit sphere) of open orbits refer to the Fermi surface type as a whole. In our view, this is not an entirely justified approach, however, because as the parameters α, β, δ, and ζ_0 are varied, both the regions of existence of open orbits and the related integer-valued planes undergo changes even if both the surface itself and its topology remain unchanged.

Following Refs [20–22] and [16] we now describe an alternative result, namely:

(1) The union of stability regions (i.e., of those with general-position open orbits) for all surfaces described by Eqn (15) for α, β, and δ fixed and ζ_0 arbitrary, is a dense set everywhere on the unit sphere. Stability regions corresponding to different integer-valued planes, although may have a common point at their boundary, do not intersect. The regions that exist here are those corresponding to topological types with arbitrarily large numbers M_j and representing topological characteristics of the global dispersion relation for α, β, and δ fixed. For ζ_0 fixed, only part of these regions will be seen, the interiors of the remaining regions corresponding to Case 1. If ζ_0 is varied, both the observed topological types and their corresponding regions will change even if the surface topology remains unchanged.

(2) The variation of the parameters α, β, and δ changes the boundaries of large regions and causes a complete rearrangement of the band picture for small regions corresponding to integer-valued planes with large (M_1, M_2, M_3). (As in 1, here too a set of bands for all possible ζ_0 is considered). The corresponding boundary value of $|M_j|$ is the greater the smaller variations in α, β, and δ.

Thus, it follows from the above argument that the general stability pattern of surfaces (15) must actually be more complex. However, the regions (and solid lines) of existence of open orbits, plotted by I M Lifshitz and V G Peschanskiĭ in Figs 1 through 3 in Ref. [3] (Fig. 11 of ours) correctly show the position of the largest stability regions, which always exist for surfaces (15) of a fixed topological type and correspond, in this sense, to the topological types of surfaces (15). We also predict here the existence of smaller zones with conductivity tensors of complex topological types.

The same authors [2] derive the contribution (3) from the open orbits of the type described above (i.e., of those lying in and passing throughout a strip of finite width). This result holds for the general-position open orbits we discussed earlier [24] and consider in the present paper. Our results, however, also rely on a topology related property, namely that on average, stable open general-position orbits are all unidirectional.

The last property, as already mentioned, only applies to stable general-position orbits. Thus, for example, if $\mathbf{B}$ is directed such that the plane $\Pi(\mathbf{B})$ is integer-valued (i.e., contains two non-collinear reciprocal lattice vectors) then we may have open orbits with different integer-numbered average directions of $\mathbf{l}^*$. The classification of open orbits in this case is quite simple, namely each of them is periodic and its contribution to the conductivity is as described in Ref. [2]. The sum of such 'partial conductivity tensors' $\sigma^{\alpha\beta}_{i\infty}$, however, will be more complex in structure than in Case 2; in particular, there will be no eigenvector with zero eigenvalue in $\Pi(\mathbf{B})$ in the $B \to \infty$ limit. Such a pattern is destroyed by an arbitrarily small rotation of the magnetic field. If $\mathbf{B}$ is directed such that the plane $\Pi(\mathbf{B})$ contains only one integer-valued vector $\mathbf{l}^*$ (up to a constant factor), then the above situation is not possible.

As regards open orbits for the case of $\Pi(\mathbf{B})$ with one integer-valued vector ($\mathbf{B}$ direction of irrationality 2), it was shown by I A Dynnikov [20] that open orbits have an asymptotic direction here. It does not necessarily follow, however, that an open orbit lies in and passes through a straight strip of finite width and, along with general-position orbits possessing this property, there may occur for such directions of $\mathbf{B}$ — and an example of this kind was given by S P Tsarev — open orbits obeying the following, comparatively weak restriction: there exist orthonormal basis (e_1, e_2) in $\Pi(\mathbf{B})$ such that

$$\lim_{t\to\infty} \frac{p_1(t) - p_1(0)}{t} = \text{const} \neq 0\,,$$

$$\lim_{t\to\infty} \frac{p_2(t) - p_2(0)}{t} = 0\,.$$

As is the case with ergodic orbits observed for $\mathbf{B}$ directions of irrationality 3, this type of orbit is also unstable towards any arbitrarily small rotations of the $\mathbf{B}$ direction.

4. Conclusions

An important feature of the present analysis should be noted. Up to this point it has been assumed throughout that different portions of the Fermi surface (or Fermi surfaces of various energy bands, as is often the case in metals) do not intersect. A material may or may not have this property, however. In particular, in some metallic lattices various Fermi surfaces (or, as we call them, various components) may, as a consequence of symmetry properties, come very close together and, due to a magnetic breakdown in strong magnetic fields [28], may undergo strong changes with the result that an electron moves in one of the components without noticing the second at their intersection (see Ref. [10] for the physical conditions on the value of **B**). In this case the reasoning above can only be applied to open orbits lying on each individual non-self-intersecting component, with stability regions separated for each such component; the stability regions for different intersecting components being generally independent of each other. Here we can have intersecting stability regions and, consequently, unit sphere regions with general-position orbits differing in their average direction. Such a situation occurs in strong magnetic fields in the 'strong magnetic breakdown' region. (The problem of magnetic breakdown in this context was brought to the authors' attention by M I Kaganov).

Helpful discussions with M I Kaganov, V G Peschanskiĭ, L A Falkovsky and M E Fisher are gratefully acknowledged.

References

1. Lifshitz I M, Azbel' M Ya, Kaganov M I *Zh. Eksp. Teor. Fiz.* **31** 63 (1956) [*Sov. Phys. JETP* **4** 41 (1957)]
2. Lifshitz I M, Peschanskiĭ V G *Zh. Eksp. Teor. Fiz.* **35** 1251 (1958) [*Sov. Phys. JETP* **8** 875 (1959)]
3. Lifshitz I M, Peschanskiĭ V G *Zh. Eksp. Teor. Fiz.* **38** 188 (1960) [*Sov. Phys. JETP* **11** 137 (1960)]
4. Alekseevskiĭ N E, Gaidukov Yu P *Zh. Eksp. Teor. Fiz.* **35** 554 (1958) [*Sov. Phys. JETP* **8** 383 (1959)]
5. Alekseevskiĭ N E, Gaidukov Yu P *Zh. Eksp. Teor. Fiz.* **36** 447 (1959) [*Sov. Phys. JETP* **9** 311 (1959)]
6. Alekseevskiĭ N E, Gaidukov Yu P *Zh. Eksp. Teor. Fiz.* **37** 672 (1959) [*Sov. Phys. JETP* **10** 481 (1960)]
7. Gaidukov Yu P *Zh. Eksp. Teor. Fiz.* **37** 1281 (1959) [*Sov. Phys. JETP* **10** 913 (1960)]
8. Lifshitz I M, Kaganov M I *Usp. Fiz. Nauk* **69** 419 (1959) [*Sov. Phys. Usp.* **2** 831 (1960)]
9. Lifshitz I M, Kaganov M I *Usp. Fiz. Nauk* **78** 411 (1962) [*Sov. Phys. Usp.* **5** 411 (1962)]
10. Lifshitz I M, Azbel' M Ya, Kaganov M I *Elektronnaya Teoriya Metallov* (Electron Theory of Metals) (Moscow: Nauka, 1971) [Translated into English (New York: Consultants Bureau, 1973)]
11. Abrikosov A A*Osnovy Teorii Metallov* (Fundamentals of the Theory of Metals) (Moscow: Nauka, 1987) [Translated into English (Amsterdam: North-Holland, 1998)]
12. Novikov S P *Usp. Matem. Nauk* **37** 3 (1982) [*Russian Math. Surveys* **37** (1982)]
13. Novikov S P *Trudy MIAN* **166** 201 (1984) [*Proc. Steklov Inst. Math.* 1 (1986)]
14. Novikov S P *Topological Methods in Mathematics* (Proc. Conference dedicated to the 60th birthday of J. Milnor, June 15 – 22, 1991) (S.U.N.Y: Stony Brook, 1993)
15. Novikov S P *Proc. Conf. of Geometry* (December 15 – 26, 1993) (Tel Aviv: Tel Aviv University, 1995)
16. Zorich A V *Usp. Matem. Nauk* **39** 235 (1984) [*Russian Math. Surveys*]
17. Dynnikov I A *Usp. Matem. Nauk* **47** 161 (1992) [*Russian Math. Surveys* **51** (3) 172 – 173 (1992)]
18. Dynnikov I A *Usp. Matem. Nauk* **48** 179 (1993) [*Russian Math. Surveys* **48** (2) 173 – 174 (1993)]
19. Dynnikov I A *Mat. Zametki* **53** 57 (1993) [*Math. Notes* **53** (5) 495 – 501 (1993)]
20. Dynnikov I A "Semiclassical motion of the electron. A proof of the Novikov conjecture in general position and counterexamples", in *Solitons, Geometry, and Topology*: *On the Crossroad* (Advances in the Mathematical Sciences, 33; American Mathematical Society Translations, 0065–9290, Ser. 2, Vol. 179) (Eds V M Buchstaber, S P Novikov) (Providence, R.I.: American Mathematical Society, 1997)
21. Dynnikov I A *Surfaces in 3-Torus: Geometry of plane sections* (Proc. of ECM2, BuDA, 1996)
22. Dynnikov I A Dissertation for the degree of Candidate of Physical and Mathematical Sciences (Moscow: Izd. MGU, 1996)
23. Tsarev S P *Private communication* (1992 – 1993)
24. Novikov S P, Mal'tsev A Ya *Pis'ma Zh. Eksp. Teor. Fiz.* **63** 809 (1996) [*JETP Lett.* **63** 855 (1996)]
25. Pippard A B *Philos. Trans. R. Soc. London Ser. A* **250** 325 (1957)
26. Dynnikov I A, Mal'tsev A Ya *Zh. Eksp. Teor. Fiz.* **112** 371 (1997) [*JETP* **85** 205 (1997)]
27. Mal'tsev A Ya *Zh. Eksp. Teor. Fiz.* **112** 1710 (1997) [*JETP* **85** 934 (1997)
28. Cohen M H, Falicov L M *Phys. Rev. Lett.* **7** 231 (1961)