In Figure 5a the thus calculated differential capacitance is depicted for temperatures $T = 4$ and 8 K. At the lower temperature the capacitance rises steeply and is clearly enhanced in the onset region. At higher temperature the enhancement is washed out. The origin of the enhancement can be identified with the help of Fig. 5b and 5c, where the Fermi energy and the energy separation between the first and second 1D subband are depicted, respectively. If the quantum wire just starts to become filled with electrons the Fermi energy drops at low temperature and the energy separation between the subbands increases. Neither feature is expected in a Hartree calculation [22]. However, a qualitatively similar behavior is predicted in analytical calculations that include exchange-correlation effects [23]. In addition, it is well known from experiments on 2D systems that many-particle effects can lead to so-called 'negative compressibility' giving rise to an enhanced capacitance [24, 25]. Such effects are observable in our 2D devices at high magnetic field and low temperatures [26]. The fact that they are less pronounced indicates that correlation effects are much more important in 1D electron systems.

In summary, we present capacitance measurements on electron channels of different widths in the range where we expect a transition from 2D to 1D behavior. In wide channels the typical 2D behavior is found. Pronounced and symmetric capacitance minima at even filling factors reflect the reduced DOS in the cyclotron gap that is dependent on impurity induced potential fluctuations in the bulk of the device. In contrast the capacitance of 300 nm wide channels is dominated by the influence of the confinement on the width of the compressible electron stripes at the channel edge. The onset of the capacitance is considerably smoother than in wide channels and the minimum at integer filling factors is asymmetric. At even smaller channel width the capacitance reflects the spatial quantization into one-dimensional subbands. The shape of the capacitance minima is strikingly different to that of wider channels, the onset becomes steep again and it exhibits additional structure. Simulation calculations indicate that the observed enhanced capacitance at the onset reflects the importance of the exchange energy in the ground state of quantum wires in the one-dimensional quantum limit.

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Pumping of energy into a mesoscopic ring. Exactly solvable model

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1. Introduction

The physical properties of mesoscopic systems are strongly influenced by the quantum interference of electronic states (see, e.g., Ref. [1] and references therein). Anderson localization of electrons, universal fluctuations of conductance as well as the periodic magnetic field dependence of thermodynamical and transport properties of multiply connected devices (e.g., metallic rings) are important examples. Previous extensive studies in mesoscopic physics were concentrated mainly on thermodynamics, as well as on the linear response of nanosrtuctures to dc or electrical and magnetic perturbations slowly varying in time. At the same time, relatively little is known about the *nonlinear* response of mesoscopic systems to a time-dependent bias. In general, an electron driven by an external time-dependent force does not conserve energy. In spite of the non-conservation of energy, interference processes remain crucially important if the phase breaking rate is much less than the rate characterizing *dynamical* redistribution of the electron wave function between different states in the energy space.

Below we consider an example of such a system, namely a single-channel mesoscopic ring subjected to a non-stationary perpendicular magnetic field, linearly dependent on time. We concentrate on the energy accumulation in such a system as a function of time. To investigate the role of interference, we take into account electron backscattering from a single potential barrier ('defect'), embedded in the ring. It is shown that by tuning either the time derivative of the external magnetic field variation, or the transmission amplitude through the barrier (by the gate potentials), one can influence the interference pattern, and in this way significantly change the dynamics. Impure conducting rings have been extensively discussed in connection with energy dissipation in mesoscopic *metallic* systems [2]. Gefen and Thouless [3, 4] have suggested that randomly distributed impurities lead to the so-called

dynamical localization of the electrons in *energy space*. This phenomenon, similar to the Anderson localization in the real space, should exist even in ballistic rings, i.e. when the elastic mean free path is much bigger than the ring's diameter. Specifically, the electron energy as a function of time *saturates* rather than infinitely increasing (as takes place in a perfect ballistic ring [5]). In the saturation regime the time-averaged electric current vanishes. Consequently, a slow-varying magnetic flux $\Phi(t)$ through the ring induces a circular slow-varying current *only in the presence of phase breaking processes*. The role of these processes was analyzed numerically in Refs [6, 7].

Dynamical localization in energy space, as well as Anderson localization in real space, occurs due to *destructive interference* of partial waves with random phases forming the electron state. However, in our case the nature of the randomness is dynamic (cf. with the case of so-called kicked rotator [8]). Consequently, the interference is crucially sensitive to the rate of magnetic flux variation, scattering amplitude against the barrier, etc. As was shown in Refs [9, 10], in the single-impurity case at a certain value of $\dot{\Phi}(t)$ the energy-space propagation of the electron can be mapped onto a real-space motion of a particle subjected to a *periodic* potential. Such a Bloch-like state results in the conductance behavior qualitatively similar to the one of pure rings. At the same time, according to numerical studies [7, 9], at other rates of magnetic flux variation the electron appears localized in energy space, the pumped energy being saturated as for the random systems. This result makes clear the crucial importance of the flux driven rate — by tuning the flux time derivative (i.e. the induced electro-motive force in the ring) one can cross over from one regime to the other, and in this way control the energy pumping. This issue, not addressed in a previous work, is the subject of the present study. We will show that the scenario of the cross-over is as follows. Consider the conductance of the ring, G, defined as the ratio between the circulating current and the electro-motive force $\mathcal{E} = -\dot{\Phi}/c$ induced in a ring of radius r_0 by a magnetic field linearly dependent on time. If the scattering is strong, $G \propto \mathcal{E}^{-2}$. As the scattering strength decreases, a set of peaks in the $G(\mathcal{E})$ dependence appears. The peaks correspond to rational values p/q of the dimensionless ratio $A \equiv \Delta/2e\mathcal{E}$, where $\Delta = \hbar^2 N_F/mr_0^2$. Here N_F is the number of filled electron states while m is the effective mass. The shape of the peaks as well as the distances between them are governed by the interplay between the height V of the potential barrier and the relaxation rate ν; the maximum value of q being determined by the condition $\tau^q/q \simeq \hbar\nu/e\mathcal{E}$. Here $\tau \equiv \exp(-\mathcal{E}_c/\mathcal{E})$ is the effective amplitude of Zener tunneling through the energy gaps in the electron spectrum, $\mathcal{E}_c = \pi^2 V^2/2\Delta e$. The peak structure near a maximum can be described by the interpolation formula

$$g = \tau^{2q}\frac{\hbar\nu\Delta}{(\hbar\nu)^2 + (e\mathcal{E}q)^2\varepsilon^2} + \eta q^2\frac{\hbar\nu\Delta}{(e\mathcal{E})^2}, \quad \varepsilon = A - \frac{p}{q}. \qquad (1)$$

Here $g \equiv G/G_0$, $G_0 = e^2/h$, while $\eta(\varepsilon)$ is a smooth function of ε. If $|\varepsilon| \lesssim \tau^q/q^2$ the function $\eta \sim 1$; beyond this region it decreases as $|\varepsilon|$ increases. As the barrier becomes more transparent, $\tau \to 1$, the inter-peak distance (determined by the maximum value of q) decreases. Finally, the peaks overlap forming the conductance $g = \Delta/\hbar\nu$ independent of the barrier's properties.

To understand the result conjectured above let us consider the electron energy levels in the vicinity of the Fermi level, E_F, where the energy dispersion can be considered linear. In a ballistic ring, one then has two sets of adiabatic energies $E_l(\Phi)$ corresponding to clockwise and counterclockwise motion. The scattering from the barrier opens gaps for the flux values $E_l = E_F + l\Delta/2$, $l = 0, \pm 1, \pm 2, \ldots$; the energy levels for clockwise and counterclockwise motion coincide. Consequently, the energy pumped into the system by a slowly varying magnetic flux can be mapped onto the one-dimensional motion of a quantum particle in the field of periodically placed scatterers (cf. Refs [3, 4, 9]). Landau–Zener tunneling (with the amplitude τ introduced above) through the gaps corresponds to forward scattering while reflection from the gaps is similar to backscattering. The important difference from the usual impurity problem is that there is no translational invariance at an arbitrary value of the driving force $\mathcal{E}$. This invariance is only present for rational values p/q of the dimensionless ratio A [9]. In this case we arrive at a superlattice containing q 'impurities' per unit cell. As a result, the motion along the E-axis is described by q allowed bands, the 'velocity' being $v_E \equiv \dot{E} \sim \Delta\tau^q/t_0$ (here $t_0 \equiv h/e\mathcal{E} = \Phi_0/\dot{\Phi}$ is twice the time interval between two sequential Landau–Zener scattering events). Since the upper bound of the Brillouin zone is $4\pi\hbar/q\Delta$, the corresponding bandwidth for motion along the E-axis is $W \simeq 4\pi\hbar\, v_E/q\Delta \simeq 4\pi\hbar\tau^q/t_0 q$. At rational values p/q of the quantity A the electron experiences $2p$ rotations around the ring while the enclosed magnetic flux changes by q quanta. As a result 'motion' of the system along the E-axis can be mapped onto the motion of a quantum particle in a one-dimensional periodic potential, the corresponding eigenstates being extended. If p/q is irrational the equivalent potential is *quasi-periodic*. It turns out that in such a case the corresponding states are then localized (see below) in spite of the fact that there is no real disorder in the system. The localization length *in energy space*, R_{loc}, can be estimated for $A = p/q + \varepsilon$, $|\varepsilon| \ll 1/q$ as follows. At finite ε the phase mismatch with respect to the case of rational $A = p/q$ can be ascribed to a quasiclassical potential $U(E) = \varepsilon\alpha E$ with $\alpha = 8\pi\hbar/\Delta t_0$. This potential gives rise to band bending which creates semiclassical turning points for the modes propagating along the E-axis. The localization length can be estimated as a half of the distance between the turning points produced by the upper and lower band edges:

$$R_{loc} \simeq \frac{W}{2\alpha|\varepsilon|} = \frac{\Delta\tau^q}{4q|\varepsilon|}.$$

Consequently, the localization time is $t_{loc} \sim 4R_{loc}/v_E \sim t_0/q|\varepsilon|$.

The manifestation of localization in the energy pumping depends on the product νt_{loc}. At $\nu t_{loc} \gg 1$ localization has no chance to develop and the band picture of energy pumping is relevant. The conductance is estimated as (cf. with Ref. [4]) $G = P/\mathcal{E}^2$, where P is the average energy accumulation rate. The quantity P, in turn, is determined as $\nu(\delta E)N_{eff}$. Here $\delta E \sim v_E/\nu$ is the energy accumulated by a single state, while $N_{eff}(\delta E) \sim \delta E/\Delta$ is the number of states involved. It follows that $g \sim \tau^{2q}(\Delta/\hbar\nu)$. If $\nu t_{loc} \ll 1$, on the other hand, G is determined by hops between intraband localized states. In this case, $\delta E \sim 2R_{loc}$, and we obtain $g \sim \hbar\nu\Delta\tau^{2q}/(e\mathcal{E}q\varepsilon)^2$. These estimates are consistent with the first term in Eqn (1).

2. Theory

2.1 The model and the average current

The following model is employed. The electron system is described by the Hamiltonian

$$\mathcal{H}(t,\varphi) = -\Delta\sigma_z\left(i\frac{\partial}{\partial\varphi}+\frac{t}{t_0}\right)+\mathcal{H}_{\rm imp}(\varphi)\,. \qquad (2)$$

Here σ_i are Pauli matrices. We are interested in the current, averaged over the time t_0,

$$I(t) = \frac{1}{t_0}\int_{t_-}^{t_+} \mathrm{d}t'\,\mathrm{Tr}\,\hat\rho\hat\jmath\,,$$

where $t_\pm = t \pm t_0/2$, $\hat\jmath \equiv (\mathrm{i}e/\hbar)[\mathcal{H},\hat\varphi]$ is the current operator.

The single-electron density matrix, $\hat\rho$, is calculated from the equation

$$\frac{\partial\hat\rho}{\partial t} = \frac{\mathrm{i}}{\hbar}[\hat\rho,\mathcal{H}] - \nu\big(\hat\rho - f_0\big[\mathcal{H}(t)\big]\big)\,,$$

where f_0 is the Fermi function.

The formal solution of the equation for $\hat\rho$ can be expressed in terms of the evolution operator $\hat u(t,t')$, which satisfies the equation $\mathrm{i}\hbar(\mathrm{d}/\mathrm{d}t)\,\hat u(t,t') = \mathcal{H}\hat u(t,t')$ with the initial condition $\hat u(t,t) = 1$. The average current can be expressed through the Heisenberg operators for the current, $\hat\jmath(t) = \hat u^\dagger(t,0)\,\hat\jmath\,\hat u(t,0)$, and the velocity in the energy space, $\hat v(t) = \hat u^\dagger(t,0)\hat f_0[\mathcal{H}(t)]\,\hat u(t,0)$. It is convenient to express the average current I as

$$\begin{aligned}
&I = \sum_{m=0}^{\infty}\exp(-2\tilde\nu m)\,\mathrm{Tr}(\hat T^{-m}\hat J\hat T^{m}\hat{\mathcal{F}}) - \mathrm{Tr}\,\hat J_1\,,\\
&\tilde\nu \equiv \frac{\nu t_0}{2}\,,\quad \hat T = \exp(-\mathrm{i}\hat\varphi)\,\hat u\,(t_0,0)\,,\\
&\hat J = \frac{1}{t_0}\int_{-t_0/2}^{t_0/2}\mathrm{d}t\exp(-\nu t)\,\hat\jmath\,(t)\,,\quad \hat{\mathcal{F}} = \int_{-t_0/2}^{t_0/2}\mathrm{d}t\,(\nu t)\,\hat v(t)\,,\\
&\hat J_1 = \frac{1}{t_0}\int_{-t_0/2}^{t_0/2}\mathrm{d}t\int_{t}^{t_0/2}\mathrm{d}t'\,\exp[-\nu(t-t')]\,\hat\jmath\,(t)\,\hat v(t')\,. \qquad (3)
\end{aligned}$$

Thus the average current is expressed in terms of the operators $\hat J$, $\hat{\mathcal{F}}$, $\hat J_1$ defined along the time interval $-t_0/2$, $t_0/2$ which are dependent on the dynamics between the successive Landau – Zener tunneling events, and by the unitary operator $\hat T$ which describes the long-time dynamics. Actually, the long-time behavior of the system is determined by the eigenstates $|\beta\rangle$ and eigenvalues $\exp(\mathrm{i}\beta)$ of the operator $\hat T$.

2.2 Analysis of the operator $\hat T$

We restrict ourselves to the case of weak scattering assuming that the relevant matrix element V (which corresponds to a momentum transfer $\sim 2p_{\rm F}$) is much less that the interlevel spacing, Δ. At $|V| \ll \Delta$ the impurity potential is important only near the crossings of 'clockwise' and 'counterclockwise' adiabatic terms, that takes place at the times $t_m = mt_0/2$. As a result of scattering, gaps are created in the adiabatic spectrum at $t = t_m$. Beyond the adiabatic approximation, these gaps can be penetrated by Landau – Zener tunneling.

Consequently, one can discriminate between the rather large time intervals of ballistic evolution (with duration $\sim t_0/2$) and the small intervals of Landau – Zener tunneling. The typical duration of the latter is $\lesssim\sqrt{t_0\hbar/\Delta}$ (cf. with Ref. [11]). Thus at $\Delta t_0/\hbar \gg 1$ the Landau – Zener tunneling is indeed essentially confined within narrow intervals and therefore can be described in terms of the scattering matrix

$$\hat S = \exp(\mathrm{i}\theta_0)\big[\sqrt{1-\tau^2}\exp(\mathrm{i}\theta_1\sigma_z) + \mathrm{i}\tau\sigma_x\big]\,. \qquad (4)$$

The physical meaning of τ is the probability amplitude for Landau – Zener 'forward scattering', i.e. to the transition into the state with the same angular moment while traversing the interval of non-adiabatic motion. It turns out that the quantities of interest here are independent of the phases θ_0 and θ_1. For simplicity we put $\theta_0 = \theta_1 = 0$.

Having in mind the periodicity of φ we introduce the vector basis

$$|n,\pm\rangle \equiv \frac{1}{\sqrt{2\pi}}\exp\big[\pm\mathrm{i}(N_{\rm F}+n)\varphi\big]\mathbf{s}_\pm\,,$$

where

$$\mathbf{s}_+ = \begin{pmatrix}1\\0\end{pmatrix},\quad \mathbf{s}_- = \begin{pmatrix}0\\1\end{pmatrix}.$$

In this representation, the operator $\hat T$ can be treated as an operator acting on the direct product $\hat N$ and pseudo-spin ($s \equiv \pm$) spaces:

$$\hat T = \hat S\Big[\sqrt{1-\tau^2}\exp(\mathrm{i}\pi a\sigma_z)\otimes\hat N + \mathrm{i}\tau\sum_{\pm}\sigma_\pm\otimes\hat N\hat R_\pm\Big]\hat N\,. \qquad (5)$$

The operators $\hat R_\pm$ and $\hat N$ are defined as $\hat R_\pm|n,s\rangle = |n\mp1,s\rangle$, $\hat N = \exp(2\mathrm{i}\pi a\hat n)$, $\hat n|n,s\rangle = n|n,s\rangle$, a is the fractional part of the quantity $A = \Delta/2e\mathcal{E}$ introduced above, while $\sigma_\pm = (\sigma_x \pm \mathrm{i}\sigma_y)/2$.

The most interesting situation is the case of weak relaxation, $\nu t_0 \ll 1$. This inequality means that the relaxation time is much longer than the interval between successive Landau – Zener transitions. It can be shown by a direct calculation that if this inequality is met then the operators $\hat J$ and $\hat{\mathcal{F}}$ acquire the form $\hat J = I_0\hat V\otimes\delta_{n,n'}$, $\hat{\mathcal{F}} = \hat V\otimes\delta_{0,n}\delta_{0,n'}$, where $I_0 = |e|\Delta/\hbar$ is the amplitude of the persistent current while

$$\hat V = \frac{1}{2}\big(\sigma_z - \hat S\sigma_z\hat S^\dagger\big) = \tau^2\sigma_z - \tau\sqrt{1-\tau^2}\sigma_y\,.$$

Under the same conditions $\mathrm{Tr}\,\hat J_1 = I_0\tau^2$.

The unitary operator $\hat T$ (as can be shown by direct calculation) possesses the properties $\hat R_-\hat T\hat R_+ = \exp(-4\mathrm{i}\pi a)\hat T$, $\sigma_y\hat T^*(-\hat n)\sigma_y = \hat T(\hat n)$. These properties result in the following relations between the eigenstates of the operator $\hat T$:

$$\boldsymbol{\varphi}_\beta(n+m) = \boldsymbol{\varphi}_{\beta-4\pi am}(n),\quad \sigma_y\,\boldsymbol{\varphi}^*_\beta(-n) = \boldsymbol{\varphi}_{-\beta}(n)\,. \qquad (6)$$

Here $m = 0,\pm1,\ldots$ At *irrational* a these relations allow one to generate *a complete set* of eigenstates provided ψ_{β_0} is known. Hence, in this case the spectrum of $\hat T$ can be expressed in the form $\exp(\mathrm{i}\beta_r^\pm)$, where $\beta_r^\pm = \pm\beta_0(a) - 4\pi ar$, $r = 0,\pm1,\pm2,\ldots$ At the same time it is evident that at *rational* $a = p/q$ the operator $\hat T$, according to Eqn (5) possesses the translational symmetry, $\hat R_\pm^q\hat T\hat R_\pm^{-q} = \hat T$. Consequently, the eigenstates of the operator $\hat T$ have the Bloch form while the spectrum of $\hat T$ can be represented by a band structure. In this case the relations (6) also generate *a complete set* of eigenstates provided structure of *one band* is known.

Using equations (3) one can express the average current in terms of eigenvalues and eigenfunctions of operator $\hat{T}$. In the limit $\tilde{v} \ll 1$ we get the following expression for the average current:

$$\frac{J}{J_0} = \sum_{\beta} |\Omega_\beta|^2 \frac{\tilde{v}}{\tilde{v}^2 + \sin^2 \Phi_\beta}, \quad \Omega_\beta = \sum_n \mathbf{n}_{\beta_0 \beta}. \tag{7}$$

Here we denote $\mathbf{n}_{\beta\beta'} \equiv \left(\boldsymbol{\varphi}_\beta(n), \tilde{V}\boldsymbol{\varphi}_{\beta'}(n)\right)$, $(\mathbf{a}, \mathbf{b})$ is the scalar product in pseudo spin space, $\Phi_\beta = (\beta - \beta_0)/2$ and $\boldsymbol{\varphi}_{\beta_0}(n)$ is any solution of the equation for eigenstates of the operator $\hat{T}$. To find the eigenstates and eigenvalues of the operator $\hat{T}$ explicitly it is convenient to introduce a new operator

$$\hat{L} = \sigma_-\sigma_+ \otimes \exp(\mathrm{i}\pi a n) + \hat{S}\sigma_+\sigma_- \otimes \exp\left(-\mathrm{i}\pi a n + \frac{\mathrm{i}\beta}{2}\right).$$

By direct calculations [12] one can obtain a set of equations for the auxiliary function $\mathbf{d}_\beta(n) = \hat{L}^{-1}\boldsymbol{\varphi}_\beta(n)$, which is equivalent to the set of equations for the components $d_{\pm,\beta}(n) \equiv \left(\mathbf{s}_\pm, \mathbf{d}_\beta(n)\right)$. Introducing the function $B(m)$ as $d_{+,\beta}(m/2)$ for even m and $d_{-,\beta}[(m+1)/2]$ for odd m one can reduce this set of difference equations to a single equation for $B(m)$:

$$B(m+1) + B(m-1) + K_m B(m) = 0,$$

where

$$K_m = \frac{2}{\tau}\exp\left(\frac{\mathrm{i}\pi a}{2}\right)\sin\left(\pi a m + \frac{\beta - \pi a}{2}\right).$$

Solution of this equation allows one to determine both the eigenstates and eigenvalues of $\hat{T}$. The results are different for the cases of rational and irrational a.

Irrational *a*. According to the analysis $\beta = \beta_r^\pm = \pm\pi a - 4\pi a r$, $r = 0, \pm1, \pm2, \ldots$ At $\beta = \pi a$ the eigenstate has the form

$$\boldsymbol{\varphi}_{\pi a} = \frac{\exp(\mathrm{i}\xi_\pm)}{2\pi}\int_0^{2\pi} \mathrm{d}k \, \exp\left[-\mathrm{i}k(2n-1)\right] \times \begin{pmatrix} \exp\left[-\mathrm{i}k + \mathrm{i}\chi(k)\right] \\ \exp\left[-\mathrm{i}\chi(k - \pi a)\right] \end{pmatrix}, \tag{8}$$

where $\xi_\pm = \mp\pi a n(2n \pm 1)$,

$$\chi(k) = -\sum_{l=1}^{\infty} \frac{\tau^l}{l\sin(\pi a l)} \cos\left[l\frac{k-\pi}{2}\right].$$

The infinite series (8) converges for almost all irrational values of a, and $\chi(k)$ is an analytic function. Consequently, all eigenfunctions are exponentially localized, the localization length $R_{\mathrm{loc}} \equiv \Delta(\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2)^{1/2}$ in *energy space* being determined by the expression

$$\left(\frac{2R_{\mathrm{loc}}}{\Delta}\right)^2 \approx \sum_{l=1}^{\infty} \frac{\tau^{2l}}{\sin^2(\pi a l)}.$$

One can see that in the vicinity of rational values p/q of a the localization length R_{loc} diverges as $\Delta\tau^q/(2\pi q|a - p/q|)$ in agreement with the qualitative estimates given above.

Rational *a* (= *p*/*q*). Since the problem is translation invariant in n-space the eigenstates can be labeled by a quasi-momentum $\mathcal{K}$ $(0 \leqslant \mathcal{K} < 2\pi/q)$. The spectrum is now given by the equation

$$\beta_r^\pm(\mathcal{K}) = 2\pi a\left(r + \frac{1}{2}\right) \pm \frac{2}{q} \times \arcsin\left\{\tau^q \sin\left[\frac{q}{2}(\mathcal{K} - 2\pi a r + \pi)\right]\right\},$$

where $r = 0, 1, \ldots, q-1$. The expression for the Bloch function $\boldsymbol{\varphi}_\beta(n)$ in the rational case is given elsewhere [12].

2.3 Final results

In the *strong localization limit* $(\tilde{v} \to 0)$ from Eqn (7) one obtains the following expression for the current and dimensionless conductance

$$I = I_0\tilde{v}\left(\frac{2R_{\mathrm{loc}}}{\Delta}\right)^2, \quad g = \pi h v \Delta\left(\frac{2R_{\mathrm{loc}}}{e\mathcal{E}\Delta}\right)^2. \tag{9}$$

Consequently, in the irrational case the average current tends to zero, when $\tilde{v} \to 0$. In the rational case with the help of Eqn (6) and the equation for $\beta(\mathcal{K})$ the current can be expressed as

$$\frac{I}{I_0} = \frac{1 - \sqrt{1 - \tau^{2q}}}{\tilde{v}} + \sum_{\pm, r=0}^{q-1} \int \mathrm{d}\mathcal{K} \, \frac{|\Omega_r^\pm|^2\tilde{v}}{\tilde{v}^2 + \sin^2\Phi_r^\pm}, \tag{10}$$

where

$$\Omega_{r\mathcal{K}}^\pm = \sum_{n=0}^{q-1} (\boldsymbol{\phi}_{\beta_0}, \hat{V}\boldsymbol{\phi}_{\beta_r^\pm}),$$

and $\boldsymbol{\phi}_{\beta_r^\pm}(n)$ is the Bloch amplitude corresponding to the eigenstate $\psi_{r\mathcal{K}}^\pm(n)$, $\Phi_r^\pm = [\beta_0^+(\mathcal{K}) - \beta_r^\pm(\mathcal{K})]/2$. The first term in Eqn (10) determines the contribution from the intraband transitions on the average current. Formally, it tends to infinity when $\tilde{v} \to 0$. Consequently, *the intraband transitions determine the singular behavior of the current* in the rational case. These conclusions are in agreement with the results of numerical calculations in Ref. [9]. An exact expression for the eigenfunction $\boldsymbol{\phi}_{\beta_r^\pm}(n)$ shows the limiting transition to expression (8) as q, $p \to \infty$, $p/q = \mathrm{const}$. The current calculated according to Eqn (10) also remains continuous. Thus, Eqn (10) with large enough p and q can be used as a good approximation for irrational a-values. The results of such a calculation is shown in Fig. 1.

3. Discussion and conclusion

The following two assumptions have been implicitly made in our consideration: (i) the electron dynamics are governed by a linear dispersion law; (ii) the energy gaps as well as the scattering matrices $\hat{S}$ are the same for all the energy levels involved. Assumption (i) is valid if the number of involved states (limited by the relaxation rate) is much less than N_{F}. This is the case if

$$\left(\frac{\hbar v}{\Delta}\right)^2 \frac{\Delta}{e\mathcal{E}} N_{\mathrm{F}} \gg 1.$$

The first factor in this product is small while the other two are large. However, it can be concluded that the criterion can be met under realistic experimental conditions.

Assumption (ii) is valid if the Fourier component of the impurity (barrier's) potential,

$$\int V(\varphi)\exp(2\mathrm{i}n\varphi)\,\mathrm{d}\varphi,$$

is essentially n-independent for relevant n. This is the case if the scattering potential is confined to a region of width

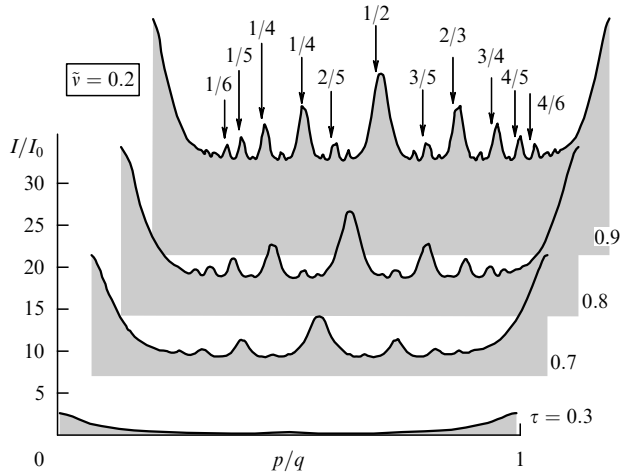

Figure 1. The normalized current I/I_0 as a function of $a = p/q$ at $\tilde{v} = 0.2$ for different Landau – Zener tunneling amplitudes, τ. $I_0 = |e|\Delta/\hbar$. Arrows indicate the positions of maxima.

$\delta\varphi \ll 2\pi\nu t_0$. Note that the inequality $\nu t_0 \ll 1$ is essential for maintaining a significant energy pumping.

Another approximation is that we have allowed for relaxation in the simplest possible way by using a single relaxation time in the equation for the density matrix. This assumption is adequate if the relaxation is caused by a transfer of the electrons in *real space* between the ring and a surrounding reservoir. If the electron energy spectrum in the reservoir is continuous, then the lifetime of an electron state in the ring with respect to this mechanism is almost independent of its quantum numbers. The mechanism discussed above allows us to describe electron states in the ring as pure quantum states, the relaxation rate being the decay through escape to the reservoir. The exact results obtained above are relevant for the case when such an 'escape' mechanism dominates. Internal inelastic relaxation processes in the ring can in principle lead to a significant difference between phase- and energy relaxation rates. Such a situation requires a separate treatment. However, in the most interesting case of efficient Landau – Zener tunneling, the intrinsic inelastic processes must involve a large momentum transfer and therefore they are strongly suppressed [5].

In conclusion, the quantum electron dynamics problem in a single-channel ballistic ring with a barrier, subjected to a linearly time-dependent magnetic flux has been solved exactly. Exponential localization in energy space has been proven. Finally, we have shown that the dc-current exhibits a set of peaks with fractional structure when plotted as a function of the induced electro-motive force. This structure is strongly sensitive to the barrier height, as well as to the relaxation rate.

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Interference effects in mesoscopic disordered rings and wires

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1. Introduction

The observation of persistent currents in mesoscopic metallic rings has revived interest in the comprehension of interference effects in coherent diffusive systems [1 – 5] (for a review and additional references see Ref. [6]). While transport quantities like the weak-localization correction were calculated in the '80s, the persistent current was studied more recently with the same techniques [7 – 17]. All these quantities can be related to Diffuson or Cooperon diagrams which describe the diffusive nature of the electronic motion, when the mean free path l_e is shorter than the typical size L of the system.

These diagrammatic calculations were then rewritten in a more transparent way which explicitly relates the quantities of interest to the return probability for a diffusive particle. In this paper, we summarize the derivation of these quantities, using a formalism which very simply relates all the quantities and which allows their calculation from the knowledge of a single function. We have recently used this formalism to calculate the mesoscopic magnetization in various geometries like connected rings. This may give a better understanding of the experimental situation as well as of the interplay between interaction and disorder in mesoscopic structures [15]. As examples, we present here the results for rings and wires which can be derived straightforwardly from this formalism.

2. Transport, thermodynamics and return probability

We characterize the diffusive motion by the quantity $p_\gamma(\mathbf{r}, \mathbf{r}', t)$, solution of the diffusion equation:

$$\left[\frac{\partial}{\partial t} + \gamma - D\left(\nabla + \frac{2ie\mathbf{A}}{\hbar c}\right)^2\right] p_\gamma(\mathbf{r}, \mathbf{r}', t) = \delta(\mathbf{r} - \mathbf{r}')\delta(t)\,, \quad (1)$$

where D is the diffusion coefficient. The scattering rate $\gamma = D/L_\phi^2$ describes the breaking of phase coherence. L_ϕ is the phase coherence length. γ will be compared to $1/\tau_D = D/L^2$ where τ_D is the diffusion time, i.e. the typical time to diffuse through a system of size L. This time is the inverse of the Thouless energy $E_c = \hbar D/L^2$.