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Magnetodynamics of antiferromagnets

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Abstract. A procedure for calculating antiferromagnetic resonance (AFMR) frequencies is discussed in which a new form of the magnetohydrodynamic equations is employed. As an example, orthorhombic antiferromagnets with an orthoferrite-type magnetic exchange structure are considered. The application to antiferromagnets with other crystallographic and magnetic structures is discussed.

1. Introduction

During recent years extensive research has been in progress in the field of the dynamics of magnetic materials, especially antiferromagnets whose symmetry allows weak ferromagnetism of relativistic origin (against the background of a collinear exchange magnetic structure). Some reviews, discussions, and important old publications can be found in Refs [1–10]. Various types of equations are used to describe the motion of the magnetizations of the sublattices, starting from those based on the equal-modulus approximation, in which in the two-sublattice model the magnetizations meet the conditions $\mathbf{M}_1^2 = \mathbf{M}_2^2 = M_0^2$. Such equations are the well-known Landau – Lifshitz equations [11] as applied to two magnetic sublattices of an antiferromagnet. These equations have obvious limits when they are used to describe experiments (this is especially true of regions near orientational phase transitions [3]), so other phenomenological approaches are employed (in these approaches the equal-modulus property is not postulated). Examples are the Onsager method of examining the linear dynamics originating in nonequilibrium thermodynamics and the Lagrange formalism of describing small oscillations of the dynamic variables of the magnetic material [3, 5]. The advantage of the Lagrange formalism is that the method allows a nonlinear generalization. Another example is the Andreev – Marchenko formalism [2], whose popularity stems from the fact that it allows a relatively simple description of extremely complex magnetic systems, such as amorphous, multiple-sublattice, and exchange-noncollinear magnetic materials.

Discussions of the different approaches to magnetic dynamics, their features, advantages and drawbacks, and interrelationships — can all be found in the papers we mentioned earlier, although we believe that the problem requires a more thorough unified interpretation. We plan to do this in future papers. In this review, we primarily pursue methodological goals and formulate a limited problem, i.e., we discuss another 'variant' of spin-dynamics equations, which we call the Vlasov – Ishmukhametov equations†. We compare these equations with the Landau – Lifshitz equations, which are still the most widely used equations in magnetic (spin) dynamics. In view of what we have just said, we augment the Vlasov – Ishmukhametov equations written for a two-sublattice antiferromagnet in terms of the mechanical (spin) densities $\mathbf{S}_1(\mathbf{r})$ and $\mathbf{S}_2(\mathbf{r})$ by the following equal-modulus condition:

$$\mathbf{S}_1^2 = \mathbf{S}_2^2 = S_0^2 = \text{const}\,. \tag{1}$$

In the case of the Landau – Lifshitz equations, condition (1) is a corollary of these equations (as the integral of motion), while for the Vlasov – Ishmukhametov equations this is not so. On the other hand, the Vlasov – Ishmukhametov equations allow for the anisotropy of the magnetomechanical tensor $\hat{g} = \hbar\hat{\gamma}/\mu_B$. The relationship between the magnetic and spin densities is given by the equations $\mathbf{M}_1 = \hat{\gamma}_1\mathbf{S}_1$ and $\mathbf{M}_2 = \hat{\gamma}_2\mathbf{S}_2$, where the specific form of the tensors $\hat{\gamma}_1$ and $\hat{\gamma}_2$ and the relationships between their components are determined by the crystallochemical symmetry and the exchange magnetic structure (EMS) of the antiferromagnet (these relationships follow from the invariance of these equations with respect to the symmetry elements in the EMS code) [13, 20].

† We assume that Golovenchits et al. [12] used similar equations when they calculated the antiferromagnetic resonance (AFMR) frequencies for $NaNiF_3$ and $YCrO_3$.

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The result of comparing of the Vlasov – Ishmukhametov and Landau – Lifshitz equations is somewhat unexpected: the Landau – Lifshitz equations written in terms of the spin densities can be applied (after the scalar g factor has been canceled out) to a magnetic material with an anisotropic g factor if this factor is taken into account in the thermodynamic potential via the Zeeman energy. Thus, in this respect the Vlasov – Ishmukhametov equations have no advantages over the Landau – Lifshitz equations, but in some cases their use is preferable since the procedure of calculating the vibration spectrum for the Vlasov – Ishmukhametov equations proves to be simpler than that for the Landau – Lifshitz equations.

We also discuss some methodological aspects. In particular, we examine the problem of separating dynamic spin variables in independent spin-wave representations, which immediately makes it possible to separate the equations corresponding to each normal mode for the magnetic phase in question.

The approach based on the Vlasov–Ishmukhametov equations is examined using, as an example, the calculation of the AFMR frequencies for an EMS of the G type characteristic of orthoferrites and some other orthorhombic antiferromagnets ($NaNiF_3$, in particular). What is important is that in this approach all three possible mechanisms of weak antiferromagnetism are taken into account: the antisymmetric and symmetric Dzyaloshinskiĭ interactions and the anisotropy of the g factor. We also compare the results obtained through this approach with the experimental data on $NaNiF_3$ [12].

2. The Vlasov – Ishmukhametov equations

In the case of a single-sublattice ferromagnet with an anisotropic g factor, the equation of motion for $\mathbf{S}$ derived by Vlasov and Ishmukhametov [14] has the following form:

$$\dot{\mathbf{S}} \times \mathbf{S} = -\frac{\partial \Phi}{\partial \mathbf{S}} S^2 . \tag{2}$$

Here Φ is the density of the thermodynamic potential expressed in terms of the spin density. The anisotropy of the g factor manifests itself in the expression for the Zeeman energy.

Note that Eqn (2), in contrast to the Landau – Lifshitz equation, has no constant of motion of the type $\mathbf{S}^2 = S_0^2 = \text{const}$. If we require that such a constant of motion exist, the variational problem of finding the unconditional extremum of the action J in Ref. [14] is replaced by a conditional-extremum problem. The latter can be solved by the Lagrange method (e.g., see p. 227 in Ref. [15]), in which the density of the Lagrangian function $\mathcal{L}$ is replaced by $\tilde{\mathcal{L}} = \mathcal{L} + \lambda(\mathbf{S}^2 - S_0^2)$, where $\lambda = \lambda(t)$ is an unknown function. Accordingly, Φ is replaced by the function $\Psi = \Phi + \lambda(\mathbf{S}^2 - S_0^2)$ and, instead of Eqn (2), we have the following system of equations:

$$\dot{\mathbf{S}} \times \mathbf{S} = -\frac{\partial \Psi}{\partial \mathbf{S}} S^2 , \qquad \mathbf{S}^2 - S_0^2 = 0 . \tag{3}$$

On the other hand, we can immediately reduce the number of independent dynamic variables in the action J by resolving the condition $\mathbf{S}^2 = S_0^2$ with respect to one of these variables and excluding this variable from J [15]. In this case, the variational procedure in any (e.g., angular) *independent* variables ζ_α, $\alpha = 1, 2$, leads to equations of the type

$$[\dot{\mathbf{S}} \times \mathbf{S}] \cdot \frac{\partial \mathbf{S}}{\partial \zeta_\alpha} = -S_0^2 \frac{\partial \tilde{\Phi}}{\partial \zeta_\alpha} . \tag{4}$$

Here $S_i = S_i(\zeta_1, \zeta_2)$ and

$$\tilde{\Phi} = \tilde{\Phi}(\zeta_1, \zeta_2) \equiv \Phi\big[S_1(\zeta_1, \zeta_2), S_2(\zeta_1, \zeta_2), S_3(\zeta_1, \zeta_2)\big] .$$

In the case of a two-sublattice antiferromagnet, we write an equation of type (2) for each sublattice and, instead of using $\mathbf{S}_1$ and $\mathbf{S}_2$, we introduce the spin vectors of ferromagnetism ($\boldsymbol{\mathcal{M}}$) and antiferromagnetism ($\boldsymbol{\Lambda}$) with the formulas

$$\boldsymbol{\mathcal{M}} = \mathbf{S}_1 + \mathbf{S}_2 , \qquad \boldsymbol{\Lambda} = \mathbf{S}_1 - \mathbf{S}_2 .$$

We then have the following system of equations:

$$\begin{aligned} \dot{\boldsymbol{\mathcal{M}}} \times \boldsymbol{\mathcal{M}} + \dot{\boldsymbol{\Lambda}} \times \boldsymbol{\Lambda} &= -\frac{\partial \Phi}{\partial \boldsymbol{\mathcal{M}}}(\boldsymbol{\mathcal{M}}^2 + \boldsymbol{\Lambda}^2) - 2\frac{\partial \Phi}{\partial \boldsymbol{\Lambda}}(\boldsymbol{\Lambda} \cdot \boldsymbol{\mathcal{M}}) , \\ \dot{\boldsymbol{\mathcal{M}}} \times \boldsymbol{\Lambda} + \dot{\boldsymbol{\Lambda}} \times \boldsymbol{\mathcal{M}} &= -2\frac{\partial \Phi}{\partial \boldsymbol{\mathcal{M}}}(\boldsymbol{\Lambda} \cdot \boldsymbol{\mathcal{M}}) - \frac{\partial \Phi}{\partial \boldsymbol{\Lambda}}(\boldsymbol{\mathcal{M}}^2 + \boldsymbol{\Lambda}^2) . \end{aligned} \tag{5}$$

These equations ignore the equal-modulus condition (1), i.e., they are similar to Eqn (2) for a ferromagnet. Of course, we can attempt to find the AFMR frequencies by solving system (5). This, however, would require incorporating in our studies all the problems of magnetic dynamics mentioned earlier and the problem of the ground state with allowance for a finite magnetic susceptibility $\chi_\parallel \neq 0$ (along the vector $\boldsymbol{\Lambda}$). We would be forced to compare the calculations based on (5) with those obtained in the Onsager and Lagrange approaches and also in the model with $\boldsymbol{\Lambda}^2 = \text{const}$ (see Ref. [3]) to allow for longitudinal vibrations and relaxation, etc. All this, of course, is important, but we limit ourselves to the problems formulated in the introduction. Here, in deriving the equations of motion, we are forced to impose extraneous constraints (1), which after the introduction of the relative vectors $\mathbf{m} = \boldsymbol{\mathcal{M}}/2S_0$ and $\mathbf{l} = \boldsymbol{\Lambda}/2S_0$ assume the form

$$\mathbf{m}^2 + \mathbf{l}^2 - 1 = 0 , \qquad \mathbf{m} \cdot \mathbf{l} = 0 . \tag{6}$$

As a result, instead of (5) we have the following system of equations:

$$\begin{aligned} \dot{\mathbf{m}} \times \mathbf{m} + \dot{\mathbf{l}} \times \mathbf{l} &= -\frac{\partial \Psi}{\partial \mathbf{m}} , \\ \dot{\mathbf{m}} \times \mathbf{l} + \dot{\mathbf{l}} \times \mathbf{m} &= -\frac{\partial \Psi}{\partial \mathbf{l}} , \end{aligned} \tag{7}$$

where the function

$$\Psi \equiv \Phi + \lambda_1(\mathbf{m}^2 + \mathbf{l}^2 - 1) + \lambda_2(\mathbf{m} \cdot \mathbf{l}) \tag{8}$$

contains two additional unknown functions $\lambda_1(t)$ and $\lambda_2(t)$, and Eqns (7) together with (6) must be considered as a united system of equations. Strictly speaking, a factor equal to $(2S_0)^{-1}$ must appear on the right-hand sides of Eqns (7), but we have incorporated it into the potential Φ, so that the latter has the dimension of frequency (s^{-1}).

As in the case of a ferromagnet, we can immediately reduce the number of equations by introducing the independent variables ζ_α, $\alpha = 1, 2, 3, 4$:

$$-\frac{\partial \tilde{\Phi}}{\partial \zeta_\alpha} = \frac{\partial \mathbf{m}}{\partial \zeta_\alpha} \cdot (\dot{\mathbf{m}} \times \mathbf{m} + \dot{\mathbf{l}} \times \mathbf{l}) + \frac{\partial \mathbf{l}}{\partial \zeta_\alpha} \cdot (\dot{\mathbf{m}} \times \mathbf{l} + \dot{\mathbf{l}} \times \mathbf{m}) . \tag{9}$$

Here, $\tilde{\Phi} \equiv \Phi\big[\mathbf{m}(\zeta_1, \zeta_2, \zeta_3, \zeta_4), \mathbf{l}(\zeta_1, \zeta_2, \zeta_3, \zeta_4)\big]$.

Let us now compare Eqns (6) and (7) with the Landau–Lifshitz equations for an antiferromagnet, which can be written in terms of the vectors of total magnetization $\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$ and total antiferromagnetism $\mathbf{L} = \mathbf{M}_1 - \mathbf{M}_2$:

$$\dot{\mathbf{M}} = -\gamma\left(\mathbf{M} \times \frac{\partial \Phi}{\partial \mathbf{M}} + \mathbf{L} \times \frac{\partial \Phi}{\partial \mathbf{L}}\right),$$

$$\dot{\mathbf{L}} = -\gamma\left(\mathbf{M} \times \frac{\partial \Phi}{\partial \mathbf{L}} + \mathbf{L} \times \frac{\partial \Phi}{\partial \mathbf{M}}\right), \qquad (10)$$

where γ is the scalar magnetomechanical ratio. If we introduce the spin densities $\mathbf{S}_1 = \mathbf{M}_1/\gamma$ and $\mathbf{S}_1 = \mathbf{M}_1/\gamma$, Eqns (10) for the variables $\mathbf{m} = (\mathbf{S}_1 + \mathbf{S}_2)/2S_0$ and $\mathbf{l} = (\mathbf{S}_1 - \mathbf{S}_2)/2S_0$ assume the following form:

$$\dot{\mathbf{m}} = -\left(\mathbf{m} \times \frac{\partial \Phi}{\partial \mathbf{m}} + \mathbf{l} \times \frac{\partial \Phi}{\partial \mathbf{l}}\right),$$

$$\dot{\mathbf{l}} = -\left(\mathbf{m} \times \frac{\partial \Phi}{\partial \mathbf{l}} + \mathbf{l} \times \frac{\partial \Phi}{\partial \mathbf{m}}\right). \qquad (11)$$

Equations (11), as Eqns (7), do not incorporate the magnetomechanical ratio explicitly; it enters into Φ only through the Zeeman energy. What is interesting is that Eqns (11) can be obtained from Eqns (7) (provided that conditions (6) are met) by performing simple algebraic transformations after vector multiplication of both equations by $\mathbf{m}$ or $\mathbf{l}$.

But does the aforesaid imply that the Landau–Lifshitz equations in the form (11), obtained as a corollary of Eqns (7) and (6), allow for the anisotropy of the g factor to the same degree?

Apparently, the situation is similar to that encountered in the derivation of the equations of motion in the Onsager approach [3, 5] with allowance for the magnetocrystalline anisotropy. The latter is taken into account only in the thermodynamic potential, while the kinetic equations in these equations are written in the isotropic (exchange) approximation.

Finally, we note that Eqns (11) are derived directly from the variational principle if from the start we write $\mathcal{L}$ in terms of the spin densities $\mathbf{S}_i$ $(i = 1, 2)$, assuming all along that the only variations $\delta\mathbf{S}_i$ that are realized are those perpendicular to $\mathbf{S}_i$, i.e., $(\mathbf{S}_i \cdot \delta\mathbf{S}_i) = 0$.

The answer to the question posed above is yes. This is also corroborated by the fact that the results of calculations of AFMR frequencies via Eqns (11) with allowance for the anisotropy of the g factor (through the Zeeman energy) agree perfectly with the results obtained in Ref. [13] from the spin Hamiltonian by the second quantization method.

The important difference between the Vlasov–Ishmukhametov equations and the Landau–Lifshitz equations [even in the form (11)] is that the equal-modulus conditions (6) do not follow from Eqns (7) but are equations augmenting (7), so that, strictly speaking, only (7) together with (6) can be considered to represent the Vlasov–Ishmukhametov equations.

It is this system of equations [or the equivalent system (9)] that is used below. A specific feature of the system is that it simplifies calculations of the AFMR frequency spectrum. We use this example to demonstrate the effectiveness of the Vlasov–Ishmukhametov equations. In the linear theory (which is needed in our reasoning), the right-hand factors in the vector products on the left-hand sides of Eqns (7) can be immediately replaced by their steady-state (equilibrium) values $\mathbf{m}^{(0)}$ and $\mathbf{l}^{(0)}$. On the right-hand sides, it is enough to isolate in Φ the terms Φ_2 quadratic in the independent dynamic variables $\Delta\mathbf{m} \equiv \mathbf{m} - \mathbf{m}^{(0)}$ and $\Delta\mathbf{l} \equiv \mathbf{l} - \mathbf{l}^{(0)}$. Finally, as shown below, for each vibration mode (there are two such modes in the model if we ignore dissipation), we need only calculate two derivatives (with respect to two independent variables) of Φ_2. But first we must isolate the dynamic variables corresponding to each of these two modes.

3. The spin-wave representation

The calculation of the natural vibration frequencies is much simpler if for the ground state ('phase') considered here we know, *a priori*, how the dynamic ('vibrational') variables are distributed among the modes. Generally speaking, this problem can be solved by using the theory of representations of the magnetic group, which describes the symmetry of the magnetic structure involved [16, 17]. Unfortunately, it is impossible to explain this theory here, but for the particular case of an antiferromagnet of an orthorhombic symmetry the conclusions of the theory can be exposed with extreme clarity. There are reasons to believe that the results can be generalized to crystal systems of higher symmetries, such as tetragonal and trigonal antiferromagnets (see the summary at the end of the paper).

To solve this problem, we turn to the well-known table that lists the possible transformations of the components of $\mathbf{m}$ and $\mathbf{l}$ [7, 13, 18], bearing in mind that we are dealing with a two-sublattice model with an EMS of type G with an antiferromagnetism vector $\mathbf{G} \equiv \mathbf{l} = (\mathbf{S}_1 - \mathbf{S}_2 + \mathbf{S}_3 - \mathbf{S}_4)/2S_0$. This table corresponds to the space group $Pbnm \equiv D_{2h}^{16}$ for ions in the $4b$ position, which is characteristic of, say, iron in orthoferrites. A reduced variant of this table, which does not include rare-earth ions and in which the two other antiferromagnetic vectors for ions in the $4b$ position are zeros $\mathbf{A} = \mathbf{C} = 0$ (which leads to the two-sublattice approximation), has the form

Table 1.

Γ_ν	$\bar{1}$	2_{1x}	2_{1y}	2_{1z}	Basis functions	Magnetic group
Γ_1	+1	+1	+1	+1	l_y	$m_x m_y m_z$
Γ_2	+1	+1	−1	−1	$m_x\ l_z$	$m_x m'_y m'_z$
Γ_3	+1	−1	+1	−1	m_y	$m'_x m_y m'_z$
Γ_4	+1	−1	−1	+1	$l_x\ m_z$	$m'_x m'_y m_z$

Table 1 lists the results of the action of the elements of the crystallochemical symmetry group on the components of $\mathbf{m}$ and $\mathbf{l}$: +1 corresponds to conservation of the component's sign, and −1 to the reversal of the sign. The components belonging to one row (they are transformed in the same way when the symmetry elements act on them) form the irreducible representations $(\Gamma_1 - \Gamma_4)$ of the group $Pbnm$. To each representation there corresponds a particular orientational state ('phase' in Refs [7, 17]); it is specified by the components of $\mathbf{m}$ and $\mathbf{l}$ transformed according to this representation, while the other components of the given phase are zeros. For instance, for the phase Γ_1, this is the state with $\mathbf{l} \parallel Y$ and $\mathbf{m} = 0$; for the phase Γ_2, this is the state with $\mathbf{l} \parallel Z$ and $\mathbf{m} \parallel X$; etc. The rightmost column in Table 1 lists, for each phase, the magnetic symmetry group (which

does not change the components of the vectors in a row) in terms of symmetry planes. The symbol m' stands for the planes augmented by the time-reversal operation†. Bearing in mind that $m_i = \bar{1} \cdot 2_i$ $(i = x, y, z)$, in the case at hand (where $\bar{1}$ is also a symmetry element in the magnetic group) we define the magnetic groups in terms of the 2_i axes with the same arrangement concerning the primes (one must distinguish between the symmetry planes with the labels x, y, z and the corresponding components of **m**).

We now classify the six oscillation variables Δm_i, Δl_i $(i = x, y, z)$ for all four phases $\Gamma_{1,2,3,4}$ into such sets that each set corresponds to one of the normal vibration modes of the phase under consideration. In the literature (e.g. see Refs [16, 17]) such a set of variables is called the spin-wave representation of the magnetic group of this phase. An obvious indication that a variable from Δm_i, Δl_i belongs to the spin-wave representation is the invariance of the equations of motion (see below) written only for these variables (with all the other variables being identically zero) with respect to the magnetic group of the corresponding phase.

Of course, it is desirable to classify the variables Δm_i, Δl_i according to mode before dealing with the equations of motion. There exists a law that enables this to be done. Let us take the phase $\Gamma_2(m_x l_z)$ for example. The variables Δm_i, Δl_i can be divided into two triples, $\Gamma_{12}(\Delta l_y, \Delta m_x, \Delta l_z)$ and $\Gamma_{34}(\Delta m_y, \Delta m_z, \Delta l_x)$. The variables in each triplet are selected from the rows Γ_1 and Γ_2 (or Γ_3 and Γ_4) in such a way that the product of the results of the action of the elements of the *magnetic* group $mm'm'$ of the phase Γ_2 on the variables belonging to these rows (a) are the same and (b) coincide with the results of the action of the elements of the *crystallochemical* group on the basis functions represented in row Γ_2 of Table 1. Schematically this can be expressed by the following chain of equalities‡:

$$(+\,-\,-)\cdot(+\,+\,+) = (-\,-\,+)\cdot(-\,+\,-) = (+\,-\,-)\,,$$

where 'pluses' and 'minuses' correspond to a transformation of the basis functions that is induced by the elements m and m', respectively (the functions remain the same or reverse their sign).

The two sets of variables that obey this rule, Γ_{12} and Γ_{34}, form the spin-wave representation corresponding to the two normal modes for the phase $\Gamma_2(m_x l_z)$. The representation $\Gamma_{34}(\Delta m_y, \Delta m_z, \Delta l_x)$ corresponds to the quasiferromagnetic mode (mode 3), for which the vector **m** precesses about the X axis, so that this mode is excited by a variable field $\mathbf{h}_\omega \perp \mathbf{m}^0 \parallel X$. Another normal mode, $\Gamma_{12}(\Delta m_x, \Delta l_y, \Delta l_z)$, corresponds to vibrations $\Delta\mathbf{m}$ that are longitudinal with respect to $\mathbf{m}^{(0)} \parallel X$ and is excited by a field of the same direction. This is the quasiantiferromagnetic mode (mode 4).

Reasoning in a similar manner, we can use this rule to identify the spin-wave representation for the phase $\Gamma_4(l_x m_z)$ with the symmetry $m'm'm$, a case that is most often encountered (structure G, state $\mathbf{l} \parallel X$) in orthoferrites. The natural modes are the quasiferromagnetic mode $\Gamma_{23}(\Delta l_z, \Delta m_x, \Delta m_y)$ (mode 1) and the quasiantiferromagnetic mode $\Gamma_{14}(\Delta l_y, \Delta l_x, \Delta m_z)$ (mode 2).

The phase $\Gamma_3(m_y)$ belongs to the ferromagnetic structure with $\mathbf{m} \parallel Y$. In this case, the spin-wave representations that satisfy the above rule correspond to the modes $\Gamma_{13}(\Delta l_y, \Delta m_y)$ and $\Gamma_{24}(\Delta m_x, \Delta l_z, \Delta l_x, \Delta m_z)$. However, the first mode must be excluded in view of the adopted equal-modulus conditions (6).

The situation with the antiferromagnetic phase $\Gamma_1(l_y)$, which in contrast to Γ_2 and Γ_4 has no weak ferromagnetism, is somewhat different. The magnetic group mmm of this phase coincides with the crystallochemical group. Hence, the spin-wave representation of mmm can be generated by the basis functions of any one row in $\Gamma_1 - \Gamma_4$. This is quite obvious, since these are the rows that realize the representations of the crystallochemical group, with this group coinciding in this case with the magnetic group. The oscillations corresponding to the rows Γ_1 and Γ_3 are not realized, again because of the equal-modulus conditions. Hence, there are only two possibilities: $\Gamma_2(\Delta l_z, \Delta m_x)$ and $\Gamma_4(\Delta l_x, \Delta m_z)$ (we note again that here all the representations are spin-wave).

Our analysis of the spin-wave representations for orientational states (phases) of an EMS of type G refers to a situation in which there is no field **H**, since, generally speaking, for $\mathbf{H} \neq 0$ the symmetry of the system changes. Several cases are exceptions, however. In these cases the vector **H** is directed along specific symmetry axes and does not violate the initial symmetry of the phase. Here, one must bear in mind that **H** is transformed in the same way as the spin (or magnetic) density **m**. Among such cases are the phases $\Gamma_2(m_x l_z)$ and $\Gamma_4(l_x m_z)$ if **H** is applied along $X \parallel 2_{1x}$ and $Z \parallel 2_{1z}$, respectively.

Below we examine the case where the field **H** breaks the symmetry, and this leads to a mixture of the initial modes (with unbroken symmetry): the case of phase $\Gamma_1(l_y)$ with $\mathbf{H} \parallel Y$. The point is that, according to its transformation properties, H_y lands not in the row Γ_1 of Table 1 but in the row Γ_3 with m_y. It is with this case that we begin our calculation of AFMR spectra. The reader will be able to judge how much simpler the calculations based on Eqns (6), (7) or (9) are (provided that we have specified the spin-wave representations beforehand) compared to calculations based on the Landau – Lifshitz equations (even if one ignores the anisotropy of the g factor in the latter).

4. Thermodynamic potential and AFMR frequencies

For the initial density, we take the well-known thermodynamic potential density [13, 18]

$$\begin{aligned}\Phi = \frac{1}{2}\,\mathcal{E}m^2 + \frac{1}{2}\,K_a l_x^2 + \frac{1}{2}\,K_c l_z^2 - d_a(l_x m_z - l_z m_x)\\ - d_s(l_x m_z + l_z m_x) - h_x(m_x + \tau_1 l_z)\\ - h_y m_y - h_z(m_z + \tau_3 l_x)\,.\end{aligned} \tag{12}$$

† Note that there are several misprints at this point in Refs [7, 18] and other papers, which obviously migrate from paper to paper. More than that, the notation also differs: $m' \equiv \underline{m}$. To avoid misunderstanding, the symmetry planes in Table 1 are equipped with indices of the normals to the planes (x, y, or z); although the order (xyz) in which these labels follow is the common one, it is the same as for the crystallochemical elements (see Ref. [19]). Bearing all this in mind, below we discard the normal label on the symmetry plane m.

‡ Incidentally, since the presence of the prime in m' has the same effect on both factors in each product in this chain of equalities, instead of the elements of the magnetic group we can use in this situation the crystallochemical elements. This is even simpler, since the results of the action of the latter elements are listed directly in Table 1, while for the magnetic elements we must also allow for the change in sign related to the presence of the prime in m'. In the next section, we discuss the differences that emerge when the invariance of the equations of motion is inspected (when the crystallochemical or magnetic symmetry is used).

Here, the antisymmetric and symmetric parts of the Dzyaloshinskiĭ interaction, d_a and d_s are written explicitly and we have introduced the following notation: $h_i = \gamma_{ii} H_i$, $\tau_1 = \gamma_{xz}/\gamma_{xx}$, and $\tau_3 = \gamma_{zx}/\gamma_{zz}$. All constants in (12), like h_i, have the dimension of frequency (s^{-1}). The quantities τ_1 and τ_3 account for the off-diagonal components of the anisotropic g factor ($g_{ij}\mu_B = \gamma_{ij}\hbar$) for an antiferromagnet with a structure of type G in the two-sublattice approximation in an orthorhombic crystal with *Pbnm* symmetry. In (12) we have partially allowed for conditions (6).

Let us first assume that $\mathbf{H} = \mathbf{h} = 0$. Then for the phase $\Gamma_1(l_y)$ there are two normal modes corresponding to the spin-wave representations $\Gamma_2(\Delta m_x, \Delta l_z)$ and $\Gamma_4(\Delta m_z, \Delta l_x)$. Accordingly, for the first mode we should put $m_z = l_x = 0$ and for the second, $m_x = l_z = 0$. Moreover, in both cases $m_y = 0$ and $l_y \approx l_y^{(0)} = 1$. All this implies that for the mode Γ_2 there remain only two independent variables, for which we can take, say, $\zeta_1 = m_x$ and $\zeta_2 = l_z$. Similarly, for Γ_4 there remains only a pair of equations with $\zeta_1 = m_z$ and $\zeta_2 = l_x$. According to (9), for Γ_2 we have

$$\dot{m}_x = -\left[K_c l_z + (d_a - d_s) m_x\right],$$
$$\dot{l}_z = \mathcal{E} m_x + (d_a - d_s) l_z, \tag{13}$$

and for mode Γ_4,

$$\dot{m}_z = K_a l_x - (d_a + d_s) m_z,$$
$$\dot{l}_x = -\mathcal{E} m_z + (d_a + d_s) l_x. \tag{14}$$

The search for the solutions of these equations in the form $\propto \exp(-i\omega t)$ leads us to AFMR frequencies of the form

$$\omega_1^2 = \mathcal{E} K_c - (d_a - d_s)^2, \tag{15}$$

$$\omega_2^2 = \mathcal{E} K_a - (d_a + d_s)^2. \tag{16}$$

The motion of the vectors $\mathbf{l}$ and $\mathbf{m}$ amounts to their rolling in the XZ plane. More precisely, $\mathbf{l}$ is deflected in the direction Z (X) for the first (second) mode while $\mathbf{m}$ emerges (simultaneously) in the perpendicular direction X (Z). The first mode is excited by a variable field $\mathbf{h}_\omega \parallel X$ and the second, by a variable field $\mathbf{h}_\omega \parallel Z$. Respectively, the finite component of the magnetic susceptibility is $\chi_{xx}(\omega)$ or $\chi_{zz}(\omega)$.

If we apply a field $\mathbf{H} \parallel Y$, these two modes mix. Due to the occurrence in Φ of a term of type

$$-h_y m_y \approx (m_x l_x + m_z l_z) h_y, \tag{17}$$

all four variables l_z, l_x, m_z, and m_x are coupled. Here we have allowed for the second condition in (6) and assumed that $l_y \approx l_y^{(0)} = 1$. Under these circumstances, Eqns (9) lead to the system

$$\dot{m}_x = -\left[K_c l_z + (d_a - d_s) m_x + h_y m_z\right],$$
$$\dot{m}_z = K_a l_x - (d_a + d_s) m_z + h_y m_x,$$
$$\dot{l}_x = -\left[\mathcal{E} m_z - (d_a + d_s) l_x + h_y l_z\right],$$
$$\dot{l}_z = \mathcal{E} m_x + (d_a - d_s) l_z + h_y l_x. \tag{18}$$

The solution of this system determines a new pair of AFMR frequencies:

$$\omega_\pm^2 = \frac{1}{2}\left[\mathcal{E}(K_a + K_c) - 2(d_a^2 + d_s^2)\right] + h_y^2 \pm \left\{\frac{1}{4}\left[\mathcal{E}(K_a - K_c) - 4 d_a d_s\right]^2 + 2h_y^2\left[\mathcal{E}(K_a + K_c) - 2d_a^2\right]\right\}^{1/2}. \tag{19}$$

Naturally, at $h_y = 0$ Eqns (19) again become (15) and (16). The vectors $\mathbf{m}$ and $\mathbf{l}$ now precess about $\mathbf{h} \parallel Y$, and the precession cones are elliptic rather than circular. The bases of the cones corresponding to $\mathbf{m}$ and $\mathbf{l}$ point in the same direction for one mode and in opposite directions for the other. The finite components of the magnetic susceptibility are $\chi_{xx}(\omega)$, $\chi_{zz}(\omega)$, $\chi_{xz}(\omega)$, and $\chi_{zx}(\omega)$.

To establish some features of the adopted approximations and the role of the equal-modulus conditions (6), we follow (with the same detail) the calculation of AFMR frequencies for one more phase. Let this be phase $\Gamma_4(l_x m_z)$ in a field $\mathbf{H} \parallel Z$. The reader will recall that such a field does not alter the magnetic symmetry of the phase Γ_4 and its spin-wave representations $\Gamma_{23}(\Delta l_z, \Delta m_x, \Delta m_y)$ and $\Gamma_{14}(\Delta l_x, \Delta l_y, \Delta m_z)$.

We begin with the mode Γ_{23}. For this mode, we must put, in the linear approximation, $\Delta l_x = l_y = \Delta m_z = 0$ and isolate in the spin-wave representation the quadratic form Φ_2 in the variables l_z, m_x, and m_y from Φ given by (12). Note, however, that in view of the equal-modulus conditions (6), dynamic parts l_x and m_z (which are determined by terms quadratic in the above spin-wave variables) also contribute to Φ_2. At this stage we use, by way of an example, the Lagrange method of multipliers. Here, the sought correction to Φ_2 from Δl_x and Δm_z is

$$\Delta\Phi_2(\Delta l_x, \Delta m_z) = \left(\frac{\partial \Phi}{\partial l_x}\right)_0 \Delta l_x + \left(\frac{\partial \Phi}{\partial m_z}\right)_0 \Delta m_z = -2\lambda_1 (l_0 \Delta l_x + m_0 \Delta m_z). \tag{20}$$

The derivatives are taken at the point of equilibrium with $l_x^{(0)} = l_0$ and $m_z^{(0)} = m_0$ and are represented in (20) in accordance with the definition (8) for Ψ and the equilibrium conditions $(\partial\Psi/\partial l_x)_0 = (\partial\Psi/\partial m_z)_0 = 0$ as follows:

$$\left(\frac{\partial \Phi}{\partial l_x}\right)_0 = -2\lambda_1 l_0, \qquad \left(\frac{\partial \Phi}{\partial m_z}\right)_0 = -2\lambda_1 m_0. \tag{21}$$

These two equations together with (12) can be used to determine λ_1 and m_0:

$$2\lambda_1 = (d_a + d_s) m_0 l_0 + h_z \tau_3 l_0 - K_a,$$
$$m_0 = \frac{(d_a + d_s) l_0 + h_z}{\mathcal{E} - K_a}. \tag{22}$$

Next, using (6) and the condition $m_0^2 + l_0^2 = 1$, we find that

$$-2(l_0 \Delta l_x + m_0 \Delta m_z) = m_x^2 + m_y^2 + l_z^2, \qquad m_x = -\frac{m_0}{l_0} l_z.$$

Substituting these relationships and λ_1 into Eqn (20), we obtain at the following expression for the total quadratic form $\Phi_2 = \Phi_2(m_x, m_y, l_z) + \Delta\Phi_2(\Delta l_x, \Delta m_z)$ in terms of the independent dynamic variables $\zeta_1 = l_z$ and $\zeta_2 = m_y$:

$$\Phi_2 = \frac{1}{2}\,\mathcal{E}\left[m_y^2 + \left(\frac{m_0}{l_0}\right)^2 l_z^2\right] + \frac{1}{2}(K_c - K_a) l_z^2 + \frac{1}{2}\,\frac{m_0}{l_0}\left[(d_a + d_s) m_y^2 + (3d_s - d_a) l_z^2\right] + \frac{1}{2}\,\frac{h_z}{l_0}\,\tau_3 (m_y^2 + l_z^2)\,. \tag{23}$$

Now it is convenient to use the linearized variants of Eqns (9). There will be only two such equations, which correspond to two finite derivatives $\partial\Phi_2/\partial l_z$ and $\partial\Phi_2/\partial m_y$. The resulting equations are

$$\dot{m}_y l_0 = l_z\left[K_c - K_a + \mathcal{E} m_0^2 - \frac{m_0}{l_0}(d_a - 3d_s) + \frac{h_z}{l_0}\,\tau_3\right],$$
$$\dot{l}_z l_0 = -m_y \mathcal{E}\,. \tag{24}$$

In deriving Eqns (24), we consistently ignored, in the right- and left-hand sides, the terms of order $K_a/\mathcal{E}$, $(m_0/l_0)^2$, and $h_z\tau_3/\mathcal{E}$ (small in comparison to unity) in the coefficients of l_x and m_y. Here it was assumed that $l_0^2 = 1$, so that l_0 may have different signs ($l_0 = \pm 1$), depending on the sign and value of the constants d_a, d_s, and τ_3.

Clearly, Eqns (24) are indeed invariant under transformations of both the crystallochemical and magnetic symmetries. In the first case, this is due to the fact that, according to Table 1, the function $\Delta l_z \equiv l_z$ is transformed as the product of $\Delta m_y \equiv m_y$ by $l_x^{(0)} \equiv l_0$, and m_y is transformed as the product $l_z l_x^{(0)}$ (the time derivative plays no role here). As for the magnetic group $m'm'm$ of the $\Gamma_4(l_x m_z)$ phase, its elements act only on the variables of the spin-wave representation Γ_{23} remaining in Eqns (24), i.e., l_z and m_y. Here we must bear in mind that the presence of the prime in the symmetry plane m' leads to a change of sign in the time derivatives $\dot{m}$ and $\dot{l}$ (since the prime stands for the time-reversal operation $t \to -t$). As a result, the right- and left-hand sides of the equations are transformed in the same way.

The solution of Eqns (24) with allowance for m_0 in (22) yields the following expression for the frequency of the quasiferromagnetic mode 1:

$$\omega_1^2 = \mathcal{E}(K_c - K_a) + 4d_s(d_s + d_a) + \tilde{h}_z(d_a + 5d_s + \mathcal{E}\tau_3) + \tilde{h}_z^2\,, \tag{25}$$

where $\tilde{h}_z = h_z l_0$.

We see that (25) coincides almost perfectly with the formula obtained in earlier work (e.g., see Refs [12, 13]), the only difference being that (25) contains l_0 as a factor of h_z, which ensures the validity of the formula in both states: $l_0 = +1$ and $l_0 = -1$. If we now write the formula for the total magnetization [7, 13],

$$M_z = 2\gamma_{zz} S_0 (m_z^{(0)} + \tau_3 l_x^{(0)}) = \frac{2\gamma_{zz} S_0}{\mathcal{E}}\left[(d_a + d_s + \mathcal{E}\tau_3) l_0 + h_z\right], \tag{26}$$

we see that the latter case ($l_0 = -1$) occurs if $d_a + d_s + \mathcal{E}\tau_3 < 0$.

Here we would like to mention the specifics of calculating the AFMR frequency for the second mode $\Gamma_{14}(\Delta l_x, \Delta l_y, \Delta m_z)$ of the same phase $\Gamma_4(l_x m_z)$. We can immediately go over to independent oscillatory variables. For these we take l_y and Δm_z, while Δl_x is again excluded via the equality $\Delta l_x = -(l_y^2 + 2m_0\Delta m_z)/2l_0$, which follows from (6). However, the second term ($\sim \Delta m_z$) provides a contribution to Φ_2 proportional to $(d_a + d_s) m_0 (\Delta m_z)^2$ (via the Dzyaloshinskiĭ interaction), which enters together with the exchange term $\mathcal{E}(\Delta m_z)^2$ and can be discarded as a small term. The final result is

$$\omega_2^2 = \mathcal{E} K_a^* + \tilde{h}_z(d_a + d_s + \mathcal{E}\tau_3)\,, \tag{27}$$

where

$$K_a^* = -K_a + \frac{(d_a + d_s)^2}{\mathcal{E}}\,.$$

Reasoning along similar lines, we can calculate the AFMR frequencies for the modes $\Gamma_{34}(\Delta l_x, \Delta m_y, \Delta m_z)$ (mode 3) and $\Gamma_{12}(\Delta m_x, \Delta l_y, \Delta l_z)$ (mode 4), which correspond to the phase $\Gamma_2(m_x l_z)$, when $\mathbf{H} \parallel X$. However, there is no need to proceed with actual calculations in this case. The point is that the phases Γ_4 and Γ_2 are very similar: the second phase can be obtained from the first by rotating $\mathbf{m}^{(0)}$ and $\mathbf{l}^{(0)}$ through 90° about the Y axis (together with the field $\mathbf{H}$). If we carefully examine the potential Φ in (12), we see that the spectrum for the phase Γ_2 (the frequencies ω_3 and ω_4) can be obtained from the spectrum for the phase Γ_4 (the frequencies ω_1 and ω_2) if in (25) and (27) we do the following substitutions: $z \to x$, $K_c \rightleftarrows K_a$, $d_a \to -d_a$, and $\tau_3 \to \tau_1$. As a result, we have

$$\omega_3^2 = \mathcal{E}(K_a - K_c) + 4d_s(d_s - d_a) + \tilde{h}_x(5d_s - d_a + \mathcal{E}\tau_1) + \tilde{h}_x^2\,, \tag{28}$$

$$\omega_4^2 = \mathcal{E} K_c^* + \tilde{h}_x(d_s - d_a + \mathcal{E}\tau_1)\,, \tag{29}$$

where $K_c^* = -K_c + (d_s - d_a)^2/\mathcal{E}$ and $\tilde{h}_x = h_x l_0$ with $\mathbf{l}^{(0)} \parallel Z$, and again $l_0 \equiv l_z^{(0)}$ may be either $+1$ or -1.

Naturally, if at $\mathbf{H} = 0$ the phase $\Gamma_4(l_x m_z)$ is the stable state, the phase $\Gamma_2(m_x l_z)$ can be attained only as a result of an orientational phase transition of the spin flop type, a transition caused by a fairly high field $H \geqslant H_{sf}$ applied along the axis $X \parallel \mathbf{l}_s$ of the initial phase. Generally speaking, this may be either a second-order phase transition or a first-order phase transition (the latter case occurs when there is anisotropy of the fourth order or higher). If there is a second-order phase transition (more precisely, two second-order phase transition points: one at $H = 0$ and the other, after the spin flop has been completed at $H = H_{sf}$), in the latter case the field $H = H_{sf}$ can be found by putting to zero expression (28) for ω_3^2. This equation yields the following solution for $\tilde{h}_{sf} = \gamma_{xx} \tilde{H}_{sf}$:

$$\tilde{h}_{sf} = -\frac{1}{2}(5d_s - d_a + \mathcal{E}\tau_1) \pm \left\{\frac{1}{4}(5d_s - d_a + \mathcal{E}\tau_1)^2 + \left[\mathcal{E}(K_c - K_a) - 4d_s(d_s - d_a)\right]\right\}^{1/2}. \tag{30}$$

The sign in front of the root is chosen such that h_{sf} is positive, which, clearly, depends on the choice of the sign of l_0. Incidentally, the expression in square brackets is positive only if the initial phase Γ_4 is stable (at $H = 0$), so that on the whole the expression with the square roots is larger in absolute value than the remainder.

All the transformations that we have done so far in connection with the structure G can be performed in a similar manner for the structures A and C by using the full table of transformations for all the basis functions **G**, **A**, and **C** and again passing to the two-sublattice model [7, 13, 18]. For instance, if we are dealing with the structure A, we must put

$\mathbf{G} = \mathbf{C} = 0$. Note, finally, that the aforesaid can also be applied to rare-earth orthoferrites *R*FeO if the frequency of transitions of the rare-earth ion *R* proper is much higher than the AFMR frequency of the iron subsystem [7]. Then we may assume that the paramagnetic *R* subsystem instantaneously follows the Fe subsystem, so that by minimizing over the variables of the *R* subsystem and excluding it, we arrive at an effective thermodynamic potential of type (12) in which, however, the coupling constants $\mathcal{E}$, *K*, etc. are renormalized by the *R*–Fe interaction. What is interesting here is that even if the *g* factor of the Fe subsystem is isotropic, the effective *g* factor (the tensor $\hat{\gamma}$) is anisotropic, since terms similar to τ_1 and τ_3 in (12) emerge. Of course, the use of the equal-modulus model in this case becomes problematic.

5. An example that uses the Vlasov–Ishmukhametov equations

In the previous sections, we examined a variant of the equations of the magnetic dynamics of antiferromagnets [the Vlasov–Ishmukhametov equations (6), (7) or (9)] that presents certain advantages when one has to calculate the AFMR frequencies. First, Eqns (7) make it possible to automatically allow for the anisotropy of the *g* factor, including the effective *g* tensor related to the interaction of the ions in the 4*b* position and the rare-earth metal ions. Second, these equations are much more convenient for practical calculations, since they lead to results much faster, especially if one separates, in advance, the dynamic variables with respect to the spin-wave representations of the magnetic group of the ground state under consideration. We use the term 'variant' since the Landau–Lifshitz equations in the form (11) for the spin densities are obtained from (7) with allowance for (6) via simple algebraic transformations. These advantages of the Vlasov–Ishmukhametov equations will be demonstrated by employing the example of an orthorhombic antiferromagnet with a structure of type *G*. Some features of the calculation method will also be discussed.

Expression (12) for the thermodynamic potential enables us to account for all three possible mechanisms of weak ferromagnetism: the antisymmetric and symmetric Dzyaloshinskiĭ interactions and the anisotropy of the *g* factor. It would be interesting to apply the results to a specific antiferromagnet in which (we hope) all three mechanisms manifest themselves. For such an antiferromagnet, we take $NaNiF_3$ ($T_N = 150$ K). This antiferromagnet has the required symmetry of type *G* and the phase $\Gamma_4(l_x m_z)$ in the ground state in a zero magnetic field **H**. This example will also demonstrate the importance of the sign of l_0 (+1 or −1).

The AFMR in this antiferromagnet has been studied both theoretically and experimentally [12], and this includes the study of the orientational phase transition from state $l_x m_z$ to state $l_z m_x$, a transition induced by a field $\mathbf{H} \parallel X$. The magnitude of the field at which this transition terminates, H_{sf}, is given by formula (30)†.

As noted earlier, the formulas for the AFMR frequencies and the field H_{sf} are expressed in terms of energy constants whose dimension is that of frequency (s^{-1}). To compare them with the experimental data (and the formulas of other researchers), we must introduce effective fields. For the case of the frequencies ω_3 (28) and ω_4 (29) and the field H_{sf} (30), we introduce the following fields and constants:

$$2H_E = \frac{\mathcal{E}}{\gamma_{xx}} = 4200 \text{ kOe}, \qquad H_A = \frac{K_a - K_c}{\gamma_{xx}} = -1.1 \text{ kOe},$$
$$H_{Da} = \frac{d_a}{\gamma_{xx}} = 162 \text{ kOe}, \qquad H_{Ds} = \frac{d_s}{\gamma_{xx}} = 12 \text{ kOe},$$
$$\tau_1 \approx \tau_3 \approx -0.012\,,$$
$$2H_E\tau = -50 \text{ kOe}, \qquad g = \frac{\hbar\gamma_{xx}}{\mu_B} \approx \frac{\hbar\gamma_{zz}}{\mu_B} \approx 2.14\,. \tag{31}$$

Here, in addition to specifying the fields, we also list their values found from the experimental data on AFMR at $T = 77$ K [12]. Clearly, the most important constant for weak ferromagnetism, as in orthoferrites [7, 18], is d_a, the antisymmetric exchange constant. Nevertheless, the contribution of the anisotropy of the *g* factor to the appropriate terms in (28) and (30) amounts to about 30%. Note, however, that these estimates do not take into account the measurement errors specified in Ref. [12], which in some cases may exceed 10%.

The only verification of the fact that the theoretical results agree with the experimental data in this case can be carried out if we calculate H_{sf} by (30), which was not used in obtaining the data listed in (31). The form of Φ specified in Eqn (12) implies that in the state with $\mathbf{l} \parallel Z$ (in a field $H_x \geqslant H_{sf}$) the minimum in the energy (for $d_a > 0$ and $m_x^{(0)} > 0$) corresponds to $l_z^{(0)} \equiv l_0 = -1$, so that we must put a 'minus' in front of the root in (30) (bearing in mind that the first term on the right-hand side of Eqn (30) is positive and smaller than the second in absolute value). As a result, we find that $H_{sf} = 15.4$ kOe, which is in satisfactory agreement with the experimentally observed value of order 15–20 kOe [12] if we allow for the fact that the accuracy of measurements is only moderate (a large spread of the points near H_{sf}).

We believe that this example shows that there is great potential for AFMR studies via the Vlasov–Ishmukhametov equations in applications.

6. Summary

We would like to conclude this review with several remarks that make it possible to substantiate the usage of the results in the majority of exchange-collinear antiferromagnets, including other (uniaxial) crystal systems most often studied by researchers.

First, as noted earlier, a structure of type *A* or *C* can also be examined by this method. Clearly, if we write the codes [13, 20] of all three antiferromagnetic structures, we see that they differ only in one respect, i.e., the even symmetry axis 2(+) in these structures are the 2_x and 2_z axes, respectively, while for the structure *G* such an even symmetry axis is 2_y (to the purpose, for this reason it would be more logical to call it a structure of type *B*). In all the cases the remaining two symmetry axes are called odd symmetry axes‡. This means that the thermodynamic potential Φ, the spin-wave variables, and all subsequent results for the structures *A* and *C* can be obtained from similar quantities for the structure *G* by a cyclic permutation of the coordinates *x*, *y*, and *z*.

† Golovenchits et al. [12] give an entirely different formula for H_{sf} (the origin of the ideas on which their derivation was based is hazy, to say the least).

‡ The reader will recall that a symmetry element is said to be even if it couples spins belonging to lattices with the same direction (in the exchange approximation) of spin densities, and odd if it couples sublattices with opposite directions of spin densities.

Moreover, the above formulas for the AFMR frequencies (usually in the limits of the bilinear approximation for Φ) lead to results, which are simply partial cases of these formulas, for some antiferromagnets of uniaxial crystal structures (whose point group incorporates the subgroup with the symmetry *mmm* characteristic of orthorhombic antiferromagnets). Here, sending the Z axis along the principal axis, we must first rewrite the above results by introducing the substitutions $x \to y \to z$ in a structure in which the even symmetry axis is 2_z (i.e., a structure of type C). Then for structures with even principal axes [$4_z(+)$, $3_z(+)$, or $6_z(+)$] we must, in accordance with the formulas of this review, put $K_c = K_a$, $d_s = 0$, and $\tau_1 = -\tau_3$. At the same time, for the structure with the code $\bar{1}(+)4_z(-)2_x(-)$ we must put $K_a = K_c$, $d_a = 0$, and $\tau_1 = \tau_3$. For the structure $\bar{1}(+)4_z(-)2_x(+)$ [or $2_d(-)$], we must first rotate the system of coordinates through 45° about the Z axis, so that the weakly ferromagnetic invariant $m_x l_x - m_y l_y$ (characteristic of these structures) could become $m_x l_y + m_y l_x$ [13].

Here we must note that the separation of the dynamic variables $\Delta\mathbf{l}$ and $\Delta\mathbf{m}$ in the modes done in Section 2 remains valid for the cases of crystal structures discussed above only if we allow for an appropriate transformation of coordinates. The point is that in the simple geometrical situations discussed in this paper such a separation is possible only because the natural modes can be separated into quasiferromagnetic and quasiantiferromagnetic.

The specific calculations discussed in this paper were based on the potential Φ being linear in $\mathbf{m}$ and $\mathbf{l}$. However, neither the equations themselves [(6), (7) or (9)] nor the spin-wave representations change if Φ incorporates higher-order terms (of the fourth or higher order). Such terms must simply be included in the picture when we find the quadratic form Φ_2 in the independent oscillatory variables $\Delta\mathbf{m}$ and $\Delta\mathbf{l}$ corresponding to the mode under consideration.

Several remarks concerning relaxation in the equations of motion are in order. In our discussion we ignored relaxation. However, it may be the reason not only for the broadening of the AFMR line. It may also transform equal-modulus equations (of the Landau – Lifshitz type) into unequal-modulus equations [6] and lead to the existence of an additional (relaxation) mode [3]. But on the whole, as is known [4], the problem of relaxation terms in the equations of magnetic dynamics is even more complicated than the problem of the type of the dissipation-free equations themselves. And there seems to be no reason why for the Vlasov – Ishmukhametov equations this problem should be solved differently than for other equations of motion. Hence, in accordance with the goals formulated in the introduction, we ignored the problem of relaxation terms in this paper.

A final remark. In this review we centrally discussed symmetric EMS's [$\bar{1} \to \bar{1}(+)$], but of course Eqns (6) and (7) can be applied to other magnetic structures — centrally antisymmetric or without symmetry center at all — in which a magnetoelectric effect may occur.

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