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Single-cycle waveforms and non-periodic waves in dispersive media (exactly solvable models)

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Contents

Abstract. Exactly solvable models for impulse time domain electromagnetics of dispersive media are developed to describe the interaction of ultrashort (single-cycle) transients with certain classes of insulators and conductors. Transient-excited fields are described analytically by means on new, exact, non-periodic and non-stationary solutions to Maxwell's equations, obtained directly in the time domain without using Fourier-expansion or time-space separation methods. Such non-separable solutions form the mathematical basis of non-periodic waves optics. Extensions to spherical and MHD single-cycle transients, shock-excited distributed transmission lines, and some inhomogeneous and nonlinear media are presented. A flexible technique for modeling real transients by Laguerre functions is developed which enables the shape and duration dependence of the refraction and reflection features of single-cycle waveforms to be presented explicitly.

1. Introduction. The non-stationary electrodynamics of stationary media

Short duration transients may produce non-stationary electromagnetic fields in dispersive and conducting media, and there are a number of reasons why the shock excitation and propagation mechanisms of such fields are currently receiving attention in several areas of radiophysics and optics.

(1) Firstly, advances in single-cycle transient generation using wide band radars (with transient durations typically of 1 to 10 ns) [1] or picosecond optical systems ($t_0 \sim 0.1-1$ ps) [2, 3] have prompted much recent interest in potential application in transmitting information and power through continuous media. Such transients have a rather different structure from conventional modulated quasi-monochromatic signals with a rectangular or Gaussian envelope, namely:

(a) the envelope of a single-cycle transient contains few or only one field cycle whose shape is usually far from sinusoidal;

(b) the rising and falling edges of the transient are asymmetrical;

(c) the zeros of the envelope are unequally spaced.

Today, the short transient trend is also seen in femtosecond optics, with as few as 3 to 5 field cycles in compressor transients [4]; and indeed in the newly-emerging field of attosecond optics [5].

(2) The scattering and diffraction of finite duration transients on finite size targets exhibit a variety of new effects. In contrast to the habitual picture of stationary scattering indicatrices and diffraction patterns characteristic of long trains of sinusoidal waves, the field of a scattered single-cycle transient varies rapidly in time. Classical formulae for monochromatic waves scattered by a cylinder or diffracted by a circular aperture turn out to be just special cases of the expressions for the non-stationary interaction of single-cycle transients with such objects [6 – 9].

(3) The traditional approach to the solution of Maxwell's equations for continuous media is to represent a solution as a product of a coordinate-dependent by a time-dependent function (separable solutions), with the time dependence usually treated by means of Fourier transform. While this approach has been dominant over the decades in the study of quasi-monochromatic waves in optics, acoustics, and radiophysics, its application to the dynamics of short single-cycle transients interacting with dispersive media, in particular with

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plasmas, waveguides, and conductors, has run into unexpected difficulties of both conceptual and computational order.

(a) On application of the Fourier transform the envelope of a finite duration signal is averaged over an infinite time interval from $-\infty$ to ∞, with the envelope's fastest-changing portions omitted as a result; but it is precisely these portions which are crucial for signal detection in information systems. On the other hand, for the time envelope of a localized signal to be reconstructed by Fourier transform inversion, one needs to eliminate harmonic fields from outside the localization region; but the more accurately this region is determined the more harmonics must be taken into account.

(b) The distortion of a transient in a dispersive medium is known to be described, in the frequency domain, by expanding the phase in a power series in the ratio of the spectral transient width $\Delta\omega$ to the carrying frequency ω [9]. However, for short wideband transients of one or a few field cycles, this ratio does not represent a small parameter, so that the number of spectral components needed for obtaining the transient field in the bulk of the medium becomes prohibitively large and accordingly a number of computational difficulties arise [10].

(c) All terms in the $\Delta\omega/\omega$ expansion involve the refractive index $n(\omega)$ of the medium in their denominators. If the spectrum of the transient contains the medium's cut-off frequency ω_0 [for which $n(\omega_0) = 0$], the expansion of the phase produces a divergent series.

It should be stressed here that the above difficulties have nothing to do with Maxwell's equations themselves but rather derive from the traditional use of separation-of-variables and Fourier-transform techniques in their treatment. This, however, is only one of the possible approaches, well suited for describing quasi-monochromatic waves with slowly varying amplitude and phase but hardly applicable to the analysis of unsteady and non-harmonic fields.

It turns out, however, that a solution procedure exists for extracting information about such fields from Maxwell's equations which is carried out directly in the time domain and does not employ either the standard separation technique or Fourier expansions. Such non-separable, exact analytic solutions, free of the usual assumptions of small, slowly varying fields, provide a mathematical basis for dealing with rapidly varying non-periodic fields and short transients in a dispersive medium. The medium itself is assumed to be at rest and stationary, and the non-stationary nature of the time-space structure of the propagating field is due to its envelope changing significantly on the microscopic field relaxation scale represented, e.g., by the inverse cut-off frequency in the dielectric or the volume charge relaxation time in a conductor. It is the purpose of this paper to analyse the non-steady electrodynamics of stationary media.

Non-stationary electromagnetic fields are described here in the time domain using new exact solutions of Maxwell's equations

$$\operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}\,, \tag{1}$$

$$\operatorname{rot}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}\,, \tag{2}$$

$$\operatorname{div}\mathbf{H} = 0\,, \tag{3}$$

$$\operatorname{div}\mathbf{D} = 0\,. \tag{4}$$

The displacement $\mathbf{D}$ is related to the electric field $\mathbf{E}$ and the current $\mathbf{j}$ it induces by the well-known formula

$$\mathbf{D} = \varepsilon_\infty \mathbf{E} + 4\pi \int_{-\infty}^{t} \mathbf{j}\,\mathrm{d}t\,. \tag{5}$$

Here ε_∞ is the dielectric permittivity of the medium in the high frequency limit; it is assumed below that the medium is non-magnetic and isotropic.

For our further analysis it is expedient to express the components of the fields $\mathbf{E}$ and $\mathbf{H}$ in terms of the vector potential $\mathbf{A}$ by means of the familiar formula

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\,, \qquad \mathbf{H} = \operatorname{rot}\mathbf{A}\,. \tag{6}$$

The substitution of Eqn (6) into (1) turns this latter into an identity. For transversely polarized fields, to be discussed below, the space-time evolution of the vector potential $\mathbf{A}$ is described by Eqn (2) and the constitutional equation $\mathbf{j} = j(\mathbf{E})$.

While widely accepted in the optics of sinusoidal waves in the frequency domain, the above set of equations is not by any means the only possible way of developing wave optics from Maxwell's equations. Some of the information contained in these latter can be provided by solutions obtained directly in the time domain. It is this alternative approach, suitable for pulsed electromagnetic fields in some dispersive, conducting, and inhomogeneous media, which is developed here based on the non-separable solution of the Klein – Gordon and telegraph equations. In particular, the possibility is noted of using Laguerre and Hermite functions for flexibly modeling single-cycle transients that excite such fields. The non-separable solutions of Maxwell's equations for fields inside dispersive media are combined with the Laguerre representation of transient fields outside to form the exactly solvable single-cycle transient model which is considered in the present paper.

2. Non-separable solutions of Klein – Gordon equation in the optics of dispersive media

The physical basis and mathematical apparatus of the theory of non-stationary wave processes in a dispersive medium are graphically illustrated by examining the propagation of an electromagnetic field in an isotropic plasma. For simplicity, consider the one-dimensional problem of description of a linearly polarized plane wave traveling in a cold, collisionless, homogeneous, fully ionized gas plasma. In the linear approximation, the density $\mathbf{j}$ of the current induced by a field $\mathbf{E}$ in such a plasma is given by

$$\frac{\partial \mathbf{j}}{\partial t} = \frac{\Omega^2}{4\pi}\,\mathbf{E}\,. \tag{7}$$

Here Ω is the plasma frequency related by $\Omega^2 = 4\pi e^2 N m^{-1}$ to the density N, charge e, and mass m of the electrons. Expressing the wave field components E_x and H_y in terms of the vector potential component A_x defined by Eqn (6) and combining Eqns (2), (5), and (7) yields an equation for A_x $(A_y = A_z = 0)$ of the form

$$\frac{\partial^2 A_x}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 A_x}{\partial t^2} = \frac{\Omega^2}{c^2}\,A_x\,. \tag{8}$$

Equation (8) is known as the Klein – Gordon equation, which is traditionally solved to give harmonic waves of frequency ω and wave vector k,

$$A_x = A_0 \exp\left[\mathrm{i}(kz - \omega t)\right], \qquad kc = \sqrt{\omega^2 - \Omega^2}\,. \tag{9}$$

Apart from solution (9), the Klein–Gordon equation has large classes of exact, analytical, non-harmonic time-domain solutions. These are conveniently obtained by introducing a normalized vector potential f and the dimensionless variables η and τ defined by the equations

$$A_x = A_0 f(\eta, \tau)\,, \qquad \eta = z\Omega c^{-1}\,, \qquad \tau = \Omega t\,. \tag{10}$$

Equation (8) then takes the dimensionless form

$$\frac{\partial^2 f}{\partial \eta^2} - \frac{\partial^2 f}{\partial \tau^2} = f\,. \tag{11}$$

This form of the Klein–Gordon equation is of key importance in our further analysis. An exact non-periodic solution of Eqn (11) that describes non-stationary fields propagating from the boundary $\eta = 0$ into the bulk of the plasma ($\eta > 0$) may be written, for $\tau^2 \geqslant \eta^2$, in the form [11]

$$f = \sum_q a_q f_q(\eta, \tau)\,, \tag{12}$$

$$f_q(\eta, \tau) = \frac{1}{2}\left[\psi_{q-1}(\eta, \tau) - \psi_{q+1}(\eta, \tau)\right], \tag{13}$$

$$\psi_q(\eta, \tau) = \left(\frac{\tau - \eta}{\tau + \eta}\right)^{q/2} J_q\left(\sqrt{\tau^2 - \eta^2}\right). \tag{14}$$

Here J_q is the Bessel function of order q, and the constant coefficients a_q as well as the values of q will be determined from the boundary conditions at the plasma surface $\eta = 0$ (see Section 4).

The time and space derivatives of the above functions are calculated from the formulae

$$\frac{\partial \psi_q}{\partial \tau} = \frac{1}{2}(\psi_{q-1} - \psi_{q+1})\,, \tag{15}$$

$$\frac{\partial \psi_\eta}{\partial \tau} = -\frac{1}{2}(\psi_{q-1} + \psi_{q+1})\,. \tag{16}$$

Substituting Eqn (12) into (6) gives the electric and magnetic field components:

$$E_x = -\frac{A_0 \Omega}{c} \sum_q a_q e_q\,, \qquad H_y = -\frac{A_0 \Omega}{c} \sum_q a_q h_q\,, \tag{17}$$

$$e_q = \frac{1}{4}(\psi_{q-2} - 2\psi_q + \psi_{q+2})\,, \tag{18}$$

$$h_q = \frac{1}{4}(\psi_{q-2} - \psi_{q+2})\,. \tag{19}$$

The current density j is also expressed in terms of the functions ψ_q to give

$$j = \sum_q a_q j_q\,, \qquad j_q = \frac{1}{2}(\psi_{q-1} - \psi_{q+1})\,. \tag{20}$$

The solution to the Klein–Gordon equation thus represents the field and current in the plasma as sums of non-periodic harmonics expressed in terms of non-separable functions ψ_q. In contrast to Eqn (9), these harmonics cannot be written as a product of functions of time by functions of coordinates. Because of dispersion, the envelops of the field harmonics e_q, h_q, given by Eqns (18) and (19), rapidly distort when propagating in a lossless transparent medium (Fig. 1), and the space-time structure of these fields differ considerably from that for monochromatic waves in the same medium, namely:

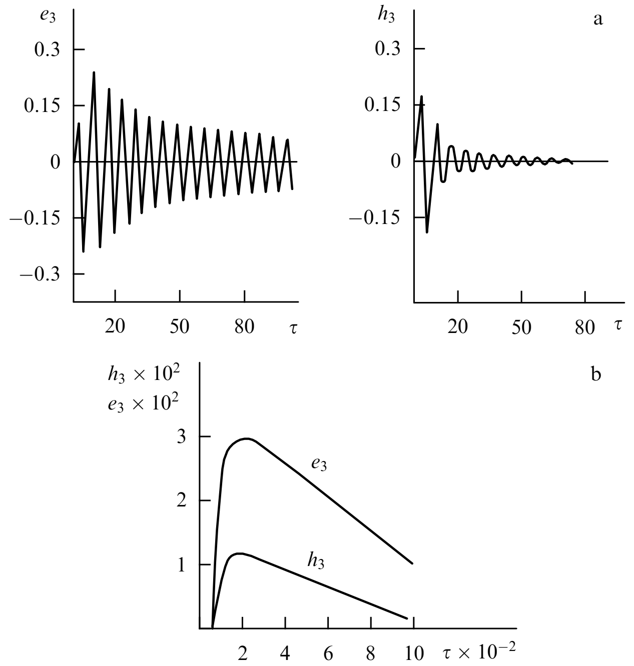

Figure 1. Envelops of the non-separable harmonics of the electric (e_3) and magnetic (h_3) field components in the (a) $\eta = 0$ and (b) $\eta = 3$ cross-sectional planes; $\tau = \Omega t$.

(a) the time spacing between the harmonic envelope zeros varies both for the electric and magnetic field, that is, the alternating field components E_x and H_y, Eqn (17), are non-periodic;

(b) the envelope extrema of the harmonics e_q and h_q vary in time, and the ratio $|h_{\max}|/|e_{\max}|$ is not constant;

(c) the harmonics e_q and h_q have greatly different dispersive distortion rates as they propagate into the plasma.

Some mathematical aspects of the field representation (17) are also noteworthy here:

(1) For all points $\tau = \eta$, the values of the function ψ_q, Eq. (14), are, for $z = ct$, as follows:

$$\psi_q\Big|_{\tau=\eta} = 0 \ (q > 0)\,, \qquad \psi_0\Big|_{\tau=\eta} = 1\,. \tag{21}$$

Using Eqn (21), the envelops of the e_q and h_q harmonics are found to be (for $q > 2$)

$$e_q\Big|_{\tau=\eta} = h_q\Big|_{\tau=\eta} = 0 \ (q > 2)\, e_2\Big|_{\tau=\eta} = h_2\Big|_{\tau=\eta} = \frac{1}{4}\,. \tag{22}$$

Thus, the edges of the e_q and h_q harmonics at $q > 2$ all move with velocity c.

(2) The envelops of the oscillation ψ_q, Eqn (14), at any point η decrease without bound as $\tau \to \infty$,

$$\lim_{\tau \to \infty} \psi_q\Big|_{\eta=\text{const}} = 0\,. \tag{23}$$

From the values of ψ_q at $\tau = \eta$, Eqn (21), and $\tau \to \infty$, Eqn (23), we obtain an important integral property of the quantities E_x, H_y, and j represented by the non-stationary harmonics (17) and (20). Using the recurrence formulae (15) and (16), the field and current harmonics may be written in the derivative forms

$$j_q = \frac{\partial \psi_q}{\partial \tau}, \quad e_q = \frac{\partial^2 \psi_q}{\partial \tau^2}, \quad h_q = \frac{\partial^2 \psi_q}{\partial \tau \, \partial \eta}. \tag{24}$$

Substituting Eqn (24) into expressions (17) and (20) and using the limits (21) and (23), it is found that for an arbitrary point η

$$\int_\eta^\infty E_x \, dt = \int_\eta^\infty H_y \, dt = \int_\eta^\infty j_x \, dt = 0. \tag{25}$$

(3) Asymptotic expressions for e_q and h_q for the 'periphery' of the envelops at $\tau \gg \eta$ ($\tau \gg 1$) are obtained from Eqns (18), (19) using the well known asymptotic expansion for the Bessel functions,

$$J_q(\tau)\Big|_{\tau \gg 1} = \sqrt{\frac{2}{\pi\tau}} \left[\cos\alpha_q - \frac{4q^2 - 1}{8\tau} \sin\alpha_q + O(\tau^{-2}) \right], \tag{26}$$

$$\alpha_q = \tau - \frac{\pi}{4} - \frac{\pi q}{2}. \tag{27}$$

Using Eqn (26), the 'tails' of the e_q and h_q harmonics may be represented at all points in the form

$$e_q\Big|_{\tau \gg 1} = -\sqrt{\frac{2}{\pi\tau}} \cos\left(\Omega t - \frac{\pi}{4} - \frac{\pi q}{2} \right), \tag{28}$$

$$h_q\Big|_{\tau \gg 1} = -\frac{1}{\tau} \sqrt{\frac{2}{\pi\tau}} \left[\eta \cos\alpha_q + q \sin\alpha_q \right]. \tag{29}$$

Thus, the evolution of the field harmonics leads to the formation of sinusoidal modes in the bulk of the plasma, their amplitudes decreasing in time and frequencies being equal to the harmonic wave cut-off frequencies. In all cross-sections, the electrical harmonics decrease as $\tau^{-1/2}$ and the magnetic ones as $\tau^{-3/2}$, i.e., somewhat faster. Note that the modes (28) and (29) are difficult to excite by sinusoidal waves of frequency $\omega = \Omega$ incident on the plasma boundary from outside because of the reflection these wave undergo at the boundary.

Interestingly, Eqns (28) and (29) involve the cut-off frequency as a natural time scale of a dispersive medium (see Section 4).

Time domain solutions reveal major differences in the dynamics of the electric and magnetic components of the non-periodic waves (17) in the transparent dispersive medium (see Fig. 1) modeling homogeneous plasmas, ionic crystals, and free carrier semiconductors. In contrast to the slow dispersion of a quasi-monochromatic narrow-band transient which may be described perturbatively [19] by expanding in powers of the small parameter $\Delta\omega/\omega$, the exact solutions (18) and (19) have nothing to do with the wave frequency and phase concepts and do not depend on assuming a weakly dispersive envelope. These solutions can be classified as

(a) non-sinusoidal,

(b) non-stationary, and

(c) non-separable.

Later on, similar fields in more complex media involving absorption, inhomogeneity, and nonlinearity will be discussed.

3. Exact solutions of the telegraph equation for non-periodic fields

The telegraph equation describes electromagnetic fields in conducting media in situations where the current induced by an alternating field is small compared to the conductivity current. In this case the constitutive equation of the medium, in contrast to Eqn (7), is given by Ohm's law

$$\mathbf{j} = \sigma \mathbf{E}, \tag{30}$$

where σ is the electrical conductivity. For transverse linearly polarized fields with components E_x and H_y, introducing the vector potential $\mathbf{A}$ ($A_x \neq 0$, $A_y = A_z = 0$) one obtains, using Eqn (30),

$$\frac{\partial^2 A_x}{\partial z^2} - \frac{\varepsilon_\infty}{c^2} \frac{\partial^2 A_x}{\partial t^2} = \frac{4\pi\sigma}{c^2} \frac{\partial A_x}{\partial t}. \tag{31}$$

This is the simplest example of the telegraph equation, widely used for the analysis of wave processes in dispersive media.

Introducing the field relaxation time

$$T = \frac{\varepsilon_\infty}{2\pi\sigma}, \tag{32}$$

the conventional solution to Eqn (31) describes a damped sinusoidal wave with wave vector

$$K = \frac{\omega}{v} \sqrt{1 + 2i(\omega T)^{-1}}; \quad v = c\varepsilon_\infty^{-1/2}. \tag{33}$$

Equation (31), along with (33) and the path integral solution (12), also describes non-sinusoidal fields in the time domain. Introducing the normalised variables

$$A = A_0 f, \quad \tau = tT^{-1}, \quad \eta = z(vT)^{-1}, \tag{34}$$

Eqn (31) can be rewritten in the dimensionless form

$$\frac{\partial^2 f}{\partial \eta^2} - \frac{\partial^2 f}{\partial \tau^2} = 2 \frac{\partial f}{\partial \tau}. \tag{35}$$

Exact analytical solutions of the dimensionless telegraph equation (35) for the vector potential of an alternating non-periodic field can be represented in the form similar to (12) – (14) as

$$f = \sum_q a_q \tilde{f}_q, \tag{36}$$

$$\tilde{f}_q = \frac{1}{2}(\Theta_{q-1} + \Theta_{q+1} - 2\Theta_q) = \frac{\partial \Theta_q}{\partial \tau}, \tag{37}$$

$$\Theta_q = \exp(-\tau) \left(\frac{\tau - \eta}{\tau + \eta} \right)^{q/2} I_q\left(\sqrt{\tau^2 - \eta^2} \right), \quad \tau \gg \eta. \tag{38}$$

Here I_q is the modified Bessel function, and q are found from the boundary conditions in the same way as in solving Eqns (12) – (14).

The electric and magnetic field of the conduction current are found by substituting Eqns (36) – (38) into (6) giving

$$E_x = -\frac{A_0}{vT} \sum_q a_q e_q, \quad H_y = -\frac{A_0}{vT} \sum_q a_q h_q. \tag{39}$$

The time and space derivatives of the vector potential A_x are calculated from Eqn (37), and

$$\frac{\partial \Theta_q}{\partial \eta} = -\frac{1}{2}(\Theta_{q-1} - \Theta_{q+1}) \,. \tag{40}$$

The e_q and h_q harmonics assume the forms

$$e_q = \frac{1}{4}\left(\Theta_{q-2} - 4\Theta_{q-1} + 6\Theta_q - 4\Theta_{q+1} + \Theta_{q+2}\right), \tag{41}$$

$$h_q = \frac{1}{4}\left(\Theta_{q-2} - 2\Theta_{q-1} + 2\Theta_{q+1} - \Theta_{q+2}\right). \tag{42}$$

Examples of the space-time evolution of q_q and h_q are shown in Fig. 2. There are some special features of the field formed by the harmonics (41) and (42) which should be noted.

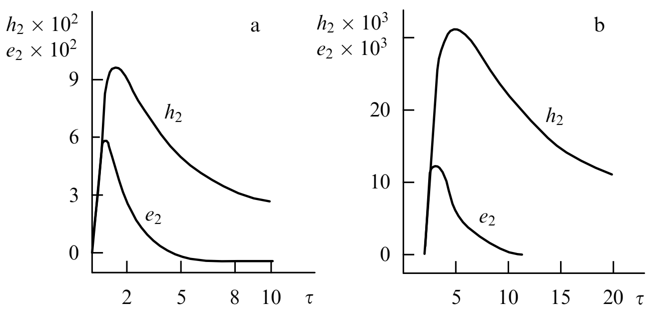

Figure 2. Non-sinusoidal harmonics of the electric (e_2) and magnetic (h_2) non-stationary field components at points (a) $\eta = 0$ and (b) $\eta = 2$; $\tau = tT^{-1}$.

(a) In contrast to harmonic waves in a conductor, with their electric and magnetic components equally damped, the maxima of the e_q and h_q harmonics decrease at different rates. As seen in Fig. 2, the $\eta = 2$ to $\eta = 0$ envelope peak ratio of e_2 is $\gamma_e = 0.2$, and that for h_2 is $\gamma_m = 0.3$. Note that the envelope peak ratio at $\eta = 2$ and $\eta = 0$ for sinusoidal waves of frequency ω for $(\omega T)^2 \geqslant 4$ [see Eqn (33)] is $\gamma_s = 0.136$, so that $\gamma_m > \gamma_e > \gamma_s$ showing that non-periodic harmonics may fall off more slowly than sinusoidal ones in a conductor.

(b) The field harmonic envelops rapidly disperse as they propagate.

(c) The electric and magnetic components of a non-harmonic field disperse at different rates as they propagate in a conductor.

The properties of the non-separable functions (39) describing the field in a conductor may be viewed as an extension of the corresponding field representations for a transparent medium (see Section 2):

(1) We have $\Theta_q(\tau, \eta)\Big|_{\tau=\eta} = 0 \quad (q > 0)$.

(2) Using the familiar asymptotic expansion of the functions $I_q(u)$,

$$I_q(u)\Big|_{u \gg 1} = \frac{\exp(-u)}{\sqrt{2\pi u}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2\pi)^n} \frac{\Gamma(q + 1/2 + n)}{\Gamma(q + 1/2 - n)} \,,$$

where Γ is the gamma function, the decrease of the field f_q, Eqn (37), in any section is, for $\tau \gg \eta$,

$$f_q\Big|_{\tau \gg \eta} = -\frac{1}{2\sqrt{2\pi}} \tau^{-3} \,.$$

(3) On substituting Eqns (41) and (42) into (39) it is readily seen that the integral property (25) for the non-separable representations of E_x and H_y, which was written earlier for a non-absorbing dielectric, also applies for any section of a conductor.

The important feature of non-stationary fields in a conductor is their natural time scale T, Eqn (32). A similar time quantity $t_c \sim \Omega^{-1}$, Eqn (7), is characteristic of a non-stationary field in a plasma. We shall discuss later the determining role time parameters play in the transient excitations of such fields in continuous media.

4. Laguerre optics of single-cycle transients

The penetration of a short single-cycle transient into a plasma or conductor causes the excitation of non-stationary electromagnetic fields in these media. Representing such fields by means of non-separable functions provides a simple analytical model for such transient processes. To see how the reflection and refraction properties of a pulsed signal depend on its duration and envelope shape, we must select a signal representation form capable of accounting in a flexible way for both the finite duration and a complex, possibly non-symmetric and non-periodic, shape. The widely used delta function or the Heaviside step function imply a zero signal duration and zero relaxation time and hence are not suitable for the purpose. On the other hand, more realistic models, for example the modulated Gaussian or rectangular transients

$$f_1 = \exp\left(-\frac{t^2}{2t_0^2}\right)\sin\omega t \,, \qquad f_2 = \begin{cases} \sin\omega t \,, & |T| \leqslant \dfrac{T}{2} \,, \\ 0 \,, & |T| > \dfrac{T}{2} \end{cases} \tag{43}$$

assume equally spaced zeros. As to the 'wavelet' signals with time-dependent frequency [13], these have little to do with the real, asymmetric, no-carrier-frequency transients generated by a mono-pulse radar or a picosecond mono-pulse source.

The proposed approach, in contrast, allows flexible modeling of real waveforms containing only one or a few field cycles and is capable of accounting for

(i) arbitrary transient steepness,
(ii) varying zero spacing, and
(iii) arbitrary envelope asymmetry.

To illustrate the reflection of such single-cycle transients from transparent and absorbing media, we take the medium to be a plasma or a conductor and examine two types of signal:

(1) Signals $E(t)$ with a distinct rising edge, i. e., with a fixed point $t = 0$ [$E(0) = 0$], defined in the region $0 \leqslant t < \infty$.

A set of functions orthogonal in this region is known to be given by the Laguerre functions

$$L_m(x) = \frac{\exp(x/2)}{m!} \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[\exp(-x)x^m\right], \qquad x = \frac{t - zc^{-1}}{t_0} \,, \tag{44}$$

with t_0 being the time scale of the signal. The properties of such signals can be analysed by considering an envelope formed by a linear combination of functions L_m,

$$E(x) = \sum_{m=0}^{\infty} b_m L_m(x)\,, \tag{45}$$

where b_m are real constants. The functions L_m are orthonormalised by

$$\int_0^{\infty} L_m(x) L_n(x)\,\mathrm{d}x = \delta_{mn} \tag{46}$$

and for small values of m are found from Eqn (44) to be

$$L_0 = \exp\left(-\frac{x}{2}\right), \qquad L_1 = (1-x)L_0\,,$$
$$L_2 = \left(1 - 2x + \frac{x^2}{2}\right)L_0\,, \quad L_3 = \left(1 - 3x + \frac{3x^2}{2} - \frac{x^3}{6}\right)L_0\,. \tag{47}$$

The behavior of the functions L_m near the leading edge of the transient ($x = 0$),

$$L_m(0) = 1\,, \quad \left.\frac{\partial L_m}{\partial x}\right|_{x=0} = -\left(m + \frac{1}{2}\right) \tag{48}$$

shows that while the steepness of the envelope L_m at point $x = 0$ is finite, the envelope values at this point are nonzero. Consequently, none of the functions L_m can represent a signal with a zero starting point and a finite steepness at this point. Such a signal can, however, be represented by a linear combination of functions L_m with a free parameter B,

$$E(t) = E_{\mathrm{in}} F_m(x)\,, \quad F_m(x) = B\big[L_m(x) - L_{m+2}(x)\big]\,, \tag{49}$$

where E_{in} is the amplitude of the incident transient.

The envelope functions F_m, Eqn (49), whose behavior at the boundary $z = 0$ $(x = tt_0^{-1})$ is shown in Fig. 3, have a number of properties suitable for modeling single-cycle transients. These are

(a) The function $F_m(x)$, Eqn (49), allows a series representation

$$F_m(x) = 2Bx\left(1 + \sum_{k=1}^{\infty} C_k x_k\right), \tag{50}$$

$$F_m(0) = 0\,, \tag{51}$$

(b) The steepness of the envelope's rising edge at $x = 0$ is specified by the free parameter

$$\left.\frac{\partial F_m(x)}{\partial x}\right|_{x=0} = 2B\,, \tag{52}$$

(c) The envelope $F_m(x)$ has $m + 2$ zeros and an exponentially decreasing 'tail,'

(d) The envelope defined by Eqn (49) possesses the integral property

$$\int_0^{\infty} F_m(x)\,\mathrm{d}x = 0\,, \tag{53}$$

(e) The envelope of a signal formed by a linear combination of functions F_m may have zero steepness at the initial point $x = 0$ (see Fig. 3):

$$K_m(x) = F_m(x) - F_{m+1}(x) = Mx^2\left(1 + \sum_{k=1}^{\infty} P_k x^k\right). \tag{54}$$

As can be seen from Fig. 3, combinations of Laguerre functions enable broad classes of single-cycle transients be flexibly modeled in the time interval $0 \leqslant t < \infty$.

(2) Turning to the normalized envelops of second type signals, which are defined over the interval $-\infty < t < \infty$ and correspond to the values

$$\left.E(t)\right|_{t=0} = E_{\mathrm{in}}\,, \quad \left.\frac{\partial E(t)}{\partial t}\right|_{t=0} = 0\,, \quad \left.E(t)\right|_{t\to\pm\infty} = 0\,, \tag{55}$$

these can also be represented by means of the Laguerre functions $L_m(x)$. An example of such an envelope shown in Fig. 4 is

$$E(x) = E_{\mathrm{in}}\begin{cases} L_0(x) + \dfrac{1}{2}\,L_1(x) - \dfrac{1}{2}\,L_2(x)\,, & x \geqslant 0\,, \\ L_0(-x) + \dfrac{1}{2}\,L_1(-x) - \dfrac{1}{2}\,L_2(-x)\,, & x \leqslant 0\,. \end{cases} \tag{56}$$

Note the similarity between transient (56) and the conventional symmetric 'Mexican hat' transient (see Fig. 4) that

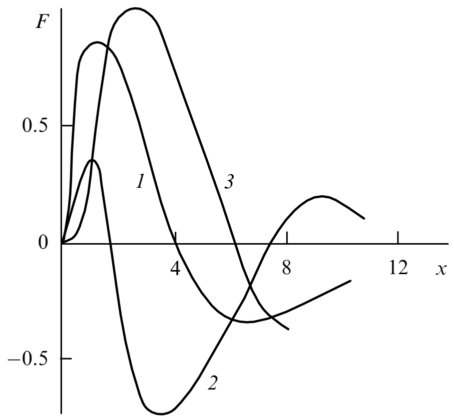

Figure 3. Envelops of the nonsymmetric single-cycle transients F_0 and F_1, Eqn (49), and K_0, Eqn (54), are shown as lines *1*, *2*, and *3*, respectively.

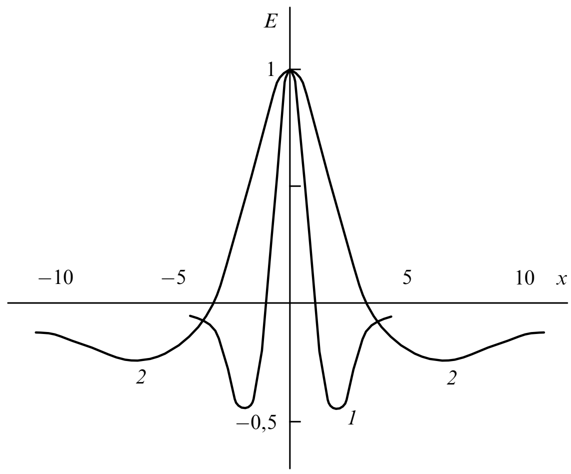

Figure 4. 'Mexican hat' envelope (curve *1*) and the symmetric envelope of the single-cycle transient (56), i.e., a linear combination of Laguerre functions (curve *2*).

dates back to Gabor [15]; in particular, both of them possess the integral property (53).

Now suppose a single-cycle transient described by Laguerre functions is reflected by the boundary of a dispersive medium. By representing an alternating field in the form of non-separable functions, simple analytical expressions for the reflection coefficients can be derived. Restricting the analysis to normally incident signals, it is expedient here to discuss separately the reflection from a non-absorbing medium (plasma) and that from an absorbing medium (conductor).

(1) Transient reflection from the plasma boundary.

The reflection coefficient is found as usual by imposing the continuity condition on the electrical and magnetic field components at the boundary of the medium ($z = 0$). The e_q and h_q harmonics of Eqns (18) and (19) at $z = 0$ reduce to combinations of Bessel functions

$$e_q\Big|_{z=0} = \frac{1}{4}\left[J_{q-2}(\tau) - 2J_q(\tau) + J_{q+2}(\tau)\right] h_q\Big|_{z=0} = \frac{1}{4}\left[J_{q-2}(\tau) - J_{q+2}(\tau)\right]. \tag{57}$$

Representing the Bessel functions in series form [16]

$$J_q(\tau) = \left(\frac{\tau}{2}\right)^q \left[\frac{1}{\Gamma(v+1)} - \left(\frac{\tau}{2}\right)^{q+2} \frac{1}{\Gamma(v+2)} + \left(\frac{\tau}{2}\right)^{q+4} \frac{1}{2!\Gamma(v+3)} - \ldots\right], \tag{58}$$

substituting into Eqns (57) and (17) and comparing with Eqn (50), the continuity conditions show that the lowest harmonic in sum (17) is given by $q - 2 = 1$; thus, the summation in Eqn (17) starts from $q = 3$ and runs over all positive numbers $q \geqslant 3$. A similar argument shows that for the incident transient (54) the excited field (17) contains harmonics with $q \geqslant 4$, and for transient (56), with $q \geqslant 2$.

We turn next to the reflection of transient (49) from a plasma. In this case the reflected wave components at $z = 0$ are represented in the form

$$E_x = -\frac{A_0\Omega}{c}\sum_{q=3}^{\infty} a_q \frac{1}{4}\left[J_{q-2}(\tau) - 2J_q(\tau) + J_{q+2}(\tau)\right], \tag{59}$$

$$H_y = -\frac{A_0\Omega}{c}\sum_{q=3}^{\infty} a_q \frac{1}{4}\left[J_{q-2}(\tau) - J_{q+2}(\tau)\right]. \tag{60}$$

Since the Bessel functions are not orthogonal in the interval $0 \leqslant \tau < \infty$, it is convenient to expand Eqns (59) and (60) in terms of Laguerre functions $L_m(tt_0^{-1})$ which are orthogonal over this interval, to give

$$E_x = -\frac{A_0\Omega}{c}\sum_{m=0}^{\infty} T_{1m} L_m(x), \qquad x = tt_0^{-1}, \tag{61}$$

$$H_y = -\frac{A_0\Omega}{c}\sum_{m=0}^{\infty} T_{2m} L_m(x), \tag{62}$$

$$T_{1m} = \sum_{q=3}^{\infty} a_q P_{mq}(\alpha), \qquad T_{2m} = \sum_{q=3}^{\infty} a_q Q_{mq}(\alpha), \qquad \alpha = t_0\Omega. \tag{63}$$

The matrix elements P_{mq} and Q_{mq} and the excitation by the mth Laguerre transient of the qth harmonic of the non-sinusoidal field are related by

$$P_{mq}(\alpha) = \int_0^{\infty} L_m(x) e_q(x\alpha)\,\mathrm{d}x, \tag{64}$$

$$Q_{mq}(\alpha) = \int_0^{\infty} L_m(x) h_q(x\alpha)\,\mathrm{d}x. \tag{65}$$

Now consider the incident transient

$$F_0(x) = B\left[L_0(x) - L_2(x)\right]. \tag{66}$$

Let the reflected single-cycle transient also be represented as a sum of products of Laguerre functions L_m with the appropriate reflection coefficients R_m. From the boundary conditions for the incident, Eqn (66), refracted, Eqn (61), and reflected fields, and using the orthonormality (46) of the Laguerre functions L_m, one obtains for each function a pair of equations of the form

$$E_{\mathrm{in}} B(1 + R_m) = -\frac{A_0\Omega}{c}\,T_{1m}, \tag{67}$$

$$E_{\mathrm{in}} B(1 - R_m) = -\frac{A_0\Omega}{c}\,T_{2m} \tag{68}$$

to represent the appropriate boundary conditions. The quantities T_{1m} and T_{2m} are defined by Eqn (63). Such pair equations for all values of m form an infinite set connecting the reflection coefficients R_m with the unknown coefficients a_q which determine the contribution of the qth harmonic to the electric and magnetic components of the refracted field (59) and (60). Solving the system (67) and (68) for R_m we find

$$R_m(\alpha) = \frac{1 - T_{2m}/T_{1m}}{1 + T_{2m}/T_{1m}}. \tag{69}$$

Comparing this with the well-known reflection coefficient expression in the frequency domain, $R = (1 - n)(1 + n)^{-1}$, where n is the refraction index of the medium, reveals an analogy between the T_{2m}/T_{1m} ratio and the quantity n. If the incident transient representation (49) does not contain the mth harmonic L_m, the corresponding pair of equations (67) and (68) reduces to

$$T_{1m} = 0, \qquad T_{2m} = 0. \tag{70}$$

We turn now to the problem of finding the reflection coefficient for the single-cycle transient (66). First we calculate the value of R_0, Eqn (69), for the reflection of the signal L_0 entering Eqn (66). The ratio T_{20}/T_{10} in Eqn (69) is calculated using the formula [17]

$$\int_0^{\infty} L_0(x) J_{q+2}(\alpha x)\,\mathrm{d}x = \left[\alpha D(\alpha)\right]^2 \int_0^{\infty} L_0(x) J_q(\alpha x)\,\mathrm{d}x, \tag{71}$$

$$D(\alpha) = 2\left(1 + \sqrt{1 + 4\alpha^2}\right)^{-1},$$

which when substituted into Eqns (64) and (65) yields

$$P_{0q}(\alpha) = \left[1 - (\alpha D)^2\right]^2 N_q, \tag{72}$$

$$Q_{0q}(\alpha) = \left[1 - (\alpha D)^4\right] N_q, \tag{73}$$

$$N_q = \frac{1}{4}\int_0^{\infty} L_0(x) J_q(\alpha x)\,\mathrm{d}x. \tag{74}$$

Finally, substituting Eqns (72)–(74) into (63) and (69), the ratio T_{20}/T_{10} and reflection coefficient R_0 are found to be

$$\frac{T_{20}}{T_{10}} = \frac{1+(\alpha D)^2}{1-(\alpha D)^2}\,, \qquad R_0 = -(\alpha D)^2\,. \tag{75}$$

An important point about the representation (66) is that the reflection coefficients of the signals must be equal because otherwise, in view of the equality $L_0(0) = L_2(0) = 1$, Eqn (48), in the case of a continuous signal incident from vacuum there would be an envelope discontinuity at the rising edge of the reflected signal. The extension of this property to the case of envelops (49) shows all these latter to have the same reflection coefficient (75) dependent on the ratio $\alpha = \Omega t_0$:

$$R(\alpha) = -(\alpha D)^2 = -\frac{4\alpha^2}{\left(1+\sqrt{1+4\alpha^2}\,\right)^2}\,. \tag{76}$$

As can be seen from Eqn (76), $0 > R > -1$. The behavior of $R(\alpha)$ is shown in Fig. 5. For a rarefied plasma ($\alpha \to 0$, $R \approx -\alpha^2$), the reflection coefficient is found to be proportional to the electron density, as would be expected. At the opposite extreme of a dense plasma ($\alpha \gg 1$) the reflection increases ($R \to -1$).

There is an interesting point to note about the field excited in a medium reflecting a transient with a central maximum of the form (56). The representations of the electric and magnetic components of such fields, Eqn (17), involve the e_2 and h_2 harmonics

$$e_2 = \frac{1}{4}\left[J_0 - 2\psi_2 + \psi_4\right] h_2 = \frac{1}{4}\left[J_0 - \psi_4\right]. \tag{77}$$

In propagating across the plasma the central maxima of these harmonics form narrow gaps of width $\Delta t \sim (1-2)\Omega^{-1}$ which conserve their peak amplitude in the no-absorption case (see Fig. 4). Since the peripheral portions of these harmonics disperse in the bulk of the medium, the contrast between the central peak and the envelope peripherals increases as the signal propagates. This effects may be of relevance to the transmission of a signal of the form (56) through a plasma-like medium.

(2) Reflection of a single-cycle transient from a conductor surface.

The analysis proceeds along much the same lines as in the preceding section. The harmonics e_q and h_q at the boundary $\eta = 0$ are expressed in terms of the modified Bessel functions $I_q(\tau)$ as

$$e_q = \frac{\exp(-\tau)}{4}\left[I_{q-2} - 4I_{q-1} + 6I_q - 4I_{q+1} + I_{q+2}\right], \tag{78}$$

$$h_q = \frac{\exp(-\tau)}{4}\left[I_{q-2} - 2I_{q-1} + 2I_{q+1} - I_{q+2}\right]. \tag{79}$$

Using instead of Eqn (71) the result [17]

$$\int_0^\infty \exp(-px) I_{q+1}(\beta x)\,\mathrm{d}x = \beta D \int_0^\infty \exp(-px) I_q(\beta x)\,\mathrm{d}x\,, \tag{80}$$

$$D = \left(p + \sqrt{p^2-\beta^2}\,\right)^{-1}, \qquad p = \beta + \frac{1}{2}\,, \qquad \beta = \frac{t_0}{T}\,,$$

the sums T_{10} and T_{20}, Eqn (63), become

$$T_{10} = (1-\beta D)^4 \sum_{q=3}^{\infty} a_q M_q\,, \tag{81}$$

$$T_{20} = (1-\beta D)^3 (1+\beta D) \sum_{q=3}^{\infty} a_q M_q\,, \tag{82}$$

$$M_q = \frac{1}{4}\int_0^\infty \exp(-px) I_{q-2}(\beta x)\,\mathrm{d}x\,. \tag{83}$$

Turning again to the reflection coefficient of the single-cycle transient (66), Eqn (69) yields

$$R = -\frac{2\beta}{1 + 2\beta + \sqrt{1+4\beta}}\,. \tag{84}$$

The dependence of the reflection coefficient on the ratio of the characteristic times is shown in Fig. 6, where it is seen that the transient reflection increases with the ratio $t_0 T^{-1}$ ($-1 < R < 0$).

The representation of fields in dispersive media by Bessel functions of integer order q, Eqns (59) and (60), is related to the optics of smooth Laguerre envelops with a zero starting point and a finite front steepness. The same approach allows one to address the reflection of shock-wave envelops, having an infinite front steepness near the zero starting point. Such signals can be represented by the generalized Laguerre

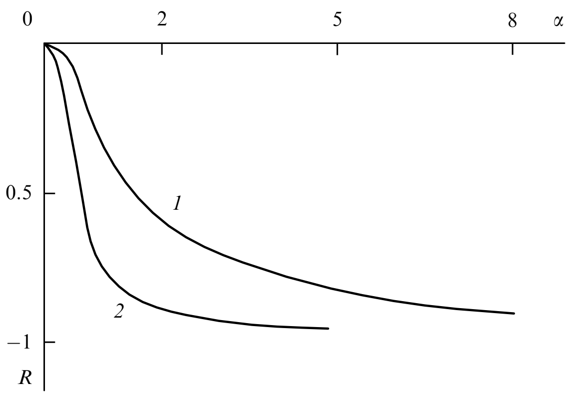

Figure 5. Coefficient of reflection R of the single-cycle transient F_0 (66) from a plasma; curves *1* and *2* are for the normal ($\gamma = 0$) and inclined ($\gamma = 75°$) incidence of an S-polarised transient; $\tau = \Omega t_0$.

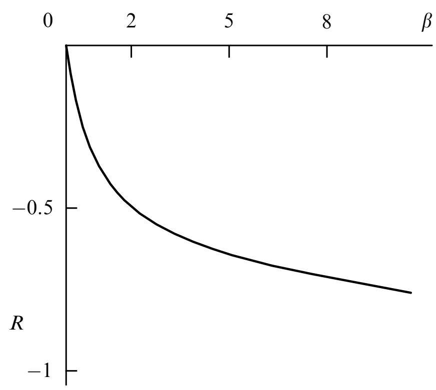

Figure 6. Coefficient of reflection R of a normally incident single-cycle transient F_0 (66) from a conductor as a function of the ratio of the characteristic transient and medium times $\beta = t_0 T^{-1}$.

functions

$$L_m^{(n)}(x) = \frac{\exp(x/2)}{m!} x^{-n/2} \frac{\partial^m}{\partial x^m} \left[\exp(-x) x^{m+n}\right], \quad 0 \leqslant n < 2 \tag{85}$$

of which the earlier used Laguerre functions (44) present the special case with $n = 0$. The behavior of the functions $L_m^{(n)}$ normalized over the interval $0 \leqslant x < \infty$, near the edge $x = 0$, is given by

$$L_m^{(n)}(0) = 0 , \qquad \left.\frac{\partial L_m^{(n)}(x)}{\partial x}\right|_{x=0} \sim x^{n/2-1} \to \infty , \tag{86}$$

showing that the leading edge steepness depends on the parameter n. Introducing a free parameter B, the linear combinations

$$F_0^{(n)}(x) = B\left[L_0^{(n)}(x) + L_1^{(n)}(x)\right] \tag{87}$$

of $L_m^{(n)}$ shown in Fig. 7 possess the same integral property (53) as the envelops of the usual Laguerre functions (49).

The reflection of the transients $L_m^{(n)}(x)$ can be treated as usual based on the field continuity condition at the boundary, with refracted fields described, in contrast to Eqns (59) and (60), in terms of Bessel functions of fractional index, and with the summation in Eqn (17) starting from $q = 2 + n/2$. Such 'shock' envelops differ from physical transient models with a finite rising time given by Eqn (49) and are not considered here.

In concluding this section, some features of the reflection of single-cycle transients from dispersive and conducting media should be noted:

(a) The reflection coefficients of Laguerre single-cycle transients are always real.

(b) The fact that the reflection coefficients of various envelops (49) are the same for given values of the parameters α, Eqn (76), or β, Eqn (84), is a characteristic property of the Laguerre signals not found in other types of signals (e.g., Hermite envelops) [18].

(c) For transients of duration t_0 reflected from media with field relaxation time T, it is found that one and the same transient may behave either as 'short' or 'long' depending on the medium it interacts with, i.e., on the value of the ratio $t_0 T^{-1}$.

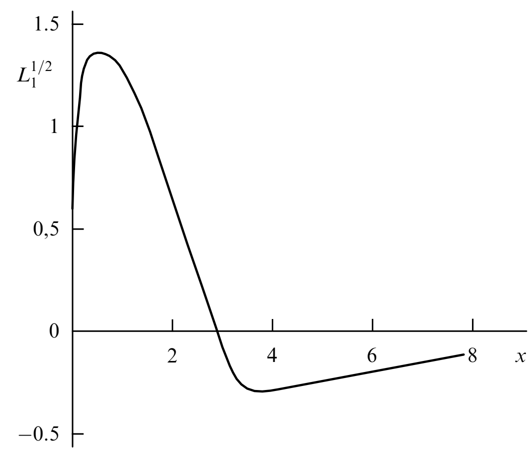

Figure 7. Envelope of a shock transient described by the generalised Laguerre function $L_m^{(n)}(x)$ $(m = 1, n = 0.5)$.

5. Spherical wave transients

The properties of non-sinusoidal electromagnetic plane waves discussed in Sections 2 through 4 can easily be extended to the spherical case. Thus, for a spherical wave having vector potential components $A_\varphi = A_0 F$ $(A_\theta = A_R = 0)$ and traveling along a radius R, referring to the spherical coordinate system (R, θ, φ) we have

$$\frac{\partial^2 F}{\partial R^2} + \frac{2}{R}\frac{\partial F}{\partial R} - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = \frac{\Omega^2}{c^2} F . \tag{88}$$

By making the substitution $F = fR^{-1}$, this is reduced to the one-dimensional Klein–Gordon equation for f of the form (11). The components E_φ and H_θ of the wave then follow from the equations

$$E_\varphi = -\frac{1}{cR}\frac{\partial A_\varphi}{\partial t} , \qquad H_\theta = -\frac{1}{R}\frac{\partial A_\varphi}{\partial R} . \tag{89}$$

Now suppose a single-cycle transient of spherical waves F_0, Eqn (49), emitted by a dipole source is reflected from the surface of a plasma. The geometry of the problem is shown in Fig. 8 for the case of S polarization. Representing the function f in Eqn (89) in the form of (12) and expressing the field components in the medium in terms of non-separable harmonics (17)–(19), the continuity condition for the E_φ, H_z, and H_ϱ components at the plasma boundary $z = 0$ can be written as

$$E_{\text{in}}(1 + R) = -\frac{A_0 \Omega}{c\sqrt{z_0^2 + \varrho^2}} T_{10} , \tag{90}$$

$$E_{\text{in}}(1 - R)\cos\gamma = -\frac{A_0 \Omega}{c\sqrt{z_0^2 + \varrho^2}} T_{20} \cos\delta , \tag{91}$$

$$E_{\text{in}}(1 + R)\sin\gamma = -\frac{A_0 \Omega}{c\sqrt{z_0^2 + \varrho^2}} T_{20} \sin\delta . \tag{92}$$

Here E_{in} is the amplitude of the incident single-cycle transient (49), R is the reflection coefficient, γ is the incident angle, δ is the refraction angle, and the sums T_{10} and T_{20} are defined by Eqns (63)–(65). Dividing Eqn (2) by (90) yields

$$\sin\gamma(\varrho) = \frac{T_{20}}{T_{10}} \sin\delta(\varrho) . \tag{93}$$

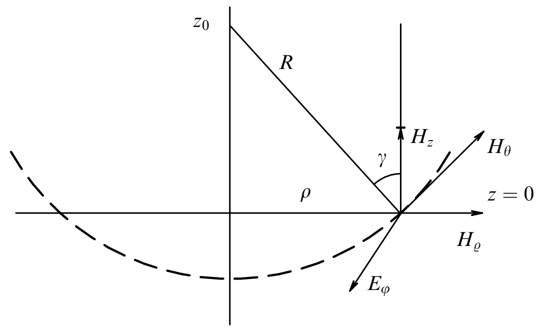

Figure 8. Reflection geometry of an S-polarized spherical wave transient at the dielectric surface $z = 0$. The source is at height z_0 above the surface; E_φ, H_ϱ, and H_z are the transient field components.

The ratio T_{20}/T_{10} is given by Eqn (75). The equality (93) expresses the refraction law for single-cycle transients, in which

$$\sin\gamma(\varrho) = \frac{\varrho}{\sqrt{z_0^2 + \varrho^2}} . \tag{94}$$

Combining Eqns (91) and (90) we obtain the reflection coefficient

$$R = \frac{\cos\gamma - \sqrt{(T_{20}/T_{10})^2 - \sin^2\gamma}}{\cos\gamma + \sqrt{(T_{20}/T_{10})^2 - \sin^2\gamma}} . \tag{95}$$

Thus, for all surface points specified by a given value of radius ϱ, the reflection coefficient depends only on the parameter $\alpha = \Omega t_0$ (see Fig. 5). In the case of normal incidence ($\gamma = 0$) the system (90)–(92) reduces to (67) and (68), and the reflection coefficient expression (95), to (76).

Expressions (93) and (95) may be viewed as extensions of Snellius' and Fresnel's formulae to the case of Laguerre's optics, with the ratio T_{20}/T_{10} as the refractive index. It should be noted that, from Eqn (75), this ratio always exceeds unity, so that such single-cycle transients do not suffer total internal reflection when coming from vacuum to the plasma boundary. Interestingly, total internal reflection is possible for a train of harmonic waves of frequency $\omega > \Omega$ incident on a plasma at an angle $\gamma > \gamma_0$,

$$\gamma_0 = \arccos(\Omega\omega^{-1}) , \tag{96}$$

thus indicating the possibility of real-time filtering of harmonic waves and single-cycle transients incident in the same direction on the boundary of a plasma.

6. Electric displacement in non-stationary fields

The electric displacement **D** defined by Eqn (5) is known to be related to the field relaxation dynamics in the medium. The contribution of such dynamic processes is accounted for by the integral term in Eqn (5), which for some types of field, in particular for sinusoidal waves, is usually evaluated by assuming the field and medium parameters to vary slowly with time. The non-separable function representation of alternating currents, in contrast, enables the integral term to be calculated explicitly without using this approximation. Substituting the plasma current expression (20) into Eqn (5), the electric displacement in a plasma may be written in a form analogous to Eqn (17),

$$D = -\frac{A_0\Omega}{c}\sum_{q=3}^{\infty} a_q d_q , \tag{97}$$

which, without any additional conditions, includes the time and space dependences of the displacement that follow from Eqn (5).

Comparing the d_q and e_q harmonics (Figs 1a, 9a) shows them to have markedly different envelope shapes. While the displacement (D_ω) and field (E_ω) Fourier components in the frequency domain are related through the dielectric permittivity $\varepsilon(\omega)$ by $D_\omega = \varepsilon(\omega)E_\omega$, the d_q and e_q harmonics are not proportional to each other. Thus, in order to describe the non-stationary electric displacement in the time domain there is no need to introduce the dielectric permittivity ε. Further-

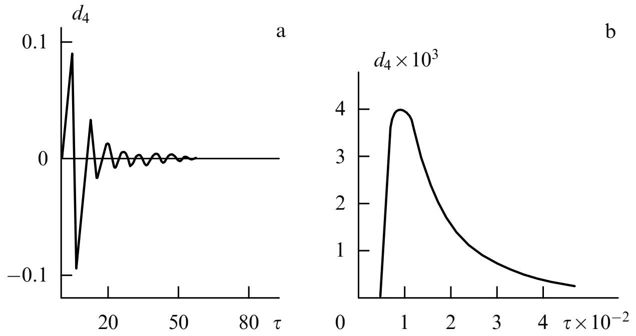

Figure 9. (a) Non-sinusoidal harmonic of the electric displacement d_4 in the cross-sectional plane $\eta = 0$, (b) envelope of the d_4 harmonic in the cross-sectional plane $\eta = 50$.

more, as seen from Sections 3 to 5, equally irrelevant to the optics of single-cycle transients are the refraction index n, related to ε by the well-known relation $n^2 = \varepsilon$, and the phase velocity $v_\varphi = cn^{-1}$. Thus, these fundamental concepts of conventional sinusoidal wave optics are not employed in the present treatment of non-harmonic fields in dispersive media.

Using the electric displacement expression (97) one can find the power flow velocity in the non-stationary field (17) in a transparent medium. The velocity may be defined by

$$\mathbf{v} = \frac{\mathbf{P}}{W} , \tag{98}$$

where **P**, the power flow density (Poynting vector), and W, the power density in a non-absorbing non-magnetic medium [9], are defined by the respective equations

$$\mathbf{P} = \frac{c}{4\pi}[\mathbf{EH}] , \tag{99}$$

$$W = \frac{1}{4\pi}\int E\,\frac{\partial D}{\partial t}\,\mathrm{d}t + \frac{|H|^2}{8\pi} . \tag{100}$$

Writing the derivative $\partial\mathbf{D}/\partial t$ in the form

$$\frac{\partial D}{\partial t} = -\frac{A_0\Omega^2}{c}\left(f + \frac{\partial^2 f}{\partial t^2}\right) \tag{101}$$

and using Eqn (6), the power flow velocity (98) can be written in the normalised form

$$\frac{v}{c} = \frac{2(\partial f/\partial\tau)(\partial f/\partial\eta)}{|f|^2 + |\partial f/\partial\tau|^2 + |\partial f/\partial\eta|^2} . \tag{102}$$

Using the vector potential of a running stationary harmonic wave in the form $f = \exp\left[\mathrm{i}(kz - \omega t)\right]$, $k = \omega n c^{-1}$, Eqn (102) yields the familiar expression for the group velocity v_{g} of a sinusoidal wave in a plasma,

$$v_{\mathrm{g}} = cn , \qquad n = \sqrt{1 - \Omega^2\omega^{-2}} .$$

From Eqn (102), using the vector potential expression for a non-stationary field (12) represented by the non-separable functions ψ_q, it is found that the power flow velocity depends in a complex way on the time and coordinates. The presence of the coefficients a_q from Eqn (12) in Eqn (102) indicates that, for a transient excitation of the medium, the power flow depends on the shape and duration of the transient.

Thus, the phase velocity v_φ, which assumes superluminal values for the stationary propagation of sinusoidal waves in a plasma, has no meaning for the non-stationary fields discussed here. In contrast, the power flow velocity defined by the general formula (98) is meaningful for both harmonic and non-harmonic fields and, as can be seen from Eqn (102), satisfies the relativistic restriction $|v| \leqslant c$.

7. Non-harmonic waves in transmission line

The distributed-parameter transmission line has been widely used over the years as a model for wave processes in electrical engineering and radiophysics. The dynamics of the current J and voltage U in such lines are described by the set of equations

$$\frac{\partial J}{\partial z} + C\,\frac{\partial U}{\partial t} + GU = 0\,, \tag{103}$$

$$\frac{\partial U}{\partial z} + L\,\frac{\partial J}{\partial t} + RJ = 0\,, \tag{104}$$

where C, L, R, and G are the capacitance, inductance, resistance, and leakage of the line (all per unit length). Introducing the characteristic time scales

$$t_1 = \frac{L}{R}\,, \qquad t_2 = \frac{C}{G}\,, \qquad T_0 = \frac{2t_1t_2}{|t_1 - t_2|}\,, \tag{105}$$

and defining the dimensionless variables τ and η and normalised values of the current i and voltage u,

$$\tau = \frac{t}{T_0}\,, \qquad \eta = z(vT_0)^{-1}\,, \qquad v = (LC)^{-1/2}\,, \tag{106}$$

$$i = AI\,, \qquad u = \frac{AU}{RT_0 v}\,, \tag{107}$$

the set (103) and (104) can conveniently be rewritten as

$$\frac{t_1t_2}{T_0^2}\,\frac{\partial i}{\partial \eta} + \frac{t_2}{T_0}\,\frac{\partial u}{\partial \tau} + u = 0\,, \tag{108}$$

$$\frac{\partial u}{\partial \eta} + \frac{t_2}{T_0}\,\frac{\partial i}{\partial \tau} + i = 0\,. \tag{109}$$

The solution of the system (103) and (104) is traditionally written in the form of a damped harmonic wave $\exp\left[\mathrm{i}(kz - \omega t) - \delta t\right]$, where the frequency ω is connected to the wave number k by the dispersion relation

$$\omega = \frac{i(t_1 + t_2)}{2t_1t_2} \pm \sqrt{(kv)^2 - T_0^{-2}}\,. \tag{110}$$

We wish now, instead, to find non-sinusoidal wave solutions for the transmission line problem. To this end, let us introduce an unknown function ψ defined by the equations

$$i = \frac{\partial \psi}{\partial \eta}\,, \qquad u = -\frac{t_1}{T_0}\,\frac{\partial \psi}{\partial \tau} - \psi\,. \tag{111}$$

On substituting this into the set (108) and (109), Eqn (109) becomes an identity and the function ψ is now found from Eqn (108). We have

$$\frac{\partial^2 \psi}{\partial \eta^2} - \frac{\partial^2 \psi}{\partial \tau^2} = \frac{T_0^2}{t_1t_2}\,\psi + \frac{T_0(t_1 + t_2)}{t_1t_2}\,\frac{\partial \psi}{\partial \tau}\,. \tag{112}$$

New solutions of Eqn (102) may be written in terms of the non-separable function $\tilde{\Theta}_q$ in the form

$$\psi = \sum_q a_q f_q\,, \qquad f_q = \frac{\partial \tilde{\Theta}_q}{\partial \tau}\,, \qquad \gamma = \frac{t_1 + t_2}{|t_1 - t_2|}\,, \tag{113}$$

$$\tilde{\Theta}_q = \exp(-\gamma\tau)\left(\frac{\tau - \eta}{\tau + \eta}\right)^{q/2} I_q\left(\sqrt{\tau^2 - \eta^2}\right)\,, \qquad \tau \geqslant \eta\,,$$

which when substituted into Eqn (111) yields the non-sinusoidal current and voltage envelops:

$$i = -\sum_q a_q i_q\,, \qquad u = -\sum_q a_q u_q\,, \tag{114}$$

$$i_q = \frac{1}{4}\left[\tilde{\Theta}_{q-2} - \tilde{\Theta}_{q+2} - 2\gamma(\Theta_{q-1} - \tilde{\Theta}_{q+1})\right]\,, \tag{115}$$

$$i_q = \frac{1}{4}\left(1 - \frac{t_2}{t_1}\right)\left[\tilde{\Theta}_{q-2} + \tilde{\Theta}_{q+2} - 2(1+\gamma)(\tilde{\Theta}_{q-1} - \tilde{\Theta}_{q+1}) + 2(1 + 2\gamma)\tilde{\Theta}_q\right]\,. \tag{116}$$

Eqns (115) and (116) describe the time and space evolution of non-sinusoidal current and voltage harmonics (Fig. 10), the rate of the evolution varying widely depending on the two time scales of the system, t_1 and t_2, Eqn (105). In the special case $t_1 \gg t_2$ ($\gamma \to 1$, $T_0 \to 2t_2$), the transmission line equation (112) reduces to the telegraph equation for a dissipative medium with a single time scale $T = 2t_2$, and the voltage (u_q) and current (i_q) harmonics go over to the electric and magnetic field harmonics, Eqns (41), (42),

$$u_q\Big|_{\gamma=1} = e_q\,, \qquad i_q\Big|_{\gamma=1} = h_q\,. \tag{117}$$

Pursuing the analogy further, we can require the current and voltage to be continuous at the end of the line $\eta = 0$ and to determine the reflection coefficient for a Laguerre transient there [14]. Thus, the standard transmission line equations

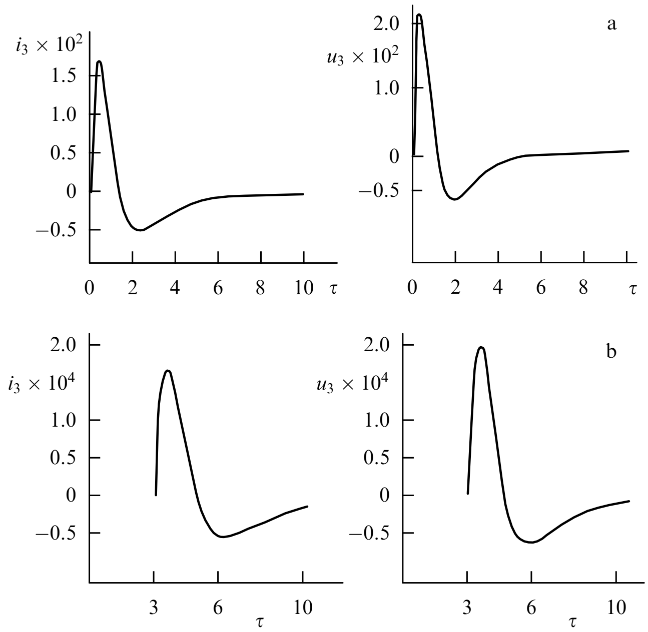

Figure 10. Non-separable harmonics of the current (i_3) and voltage (u_3) in a transmission line ($t_2 = 4t_1$), (a) at the end of the line $\eta = 0$; and (b) at the point $\eta = 3$.

(103) and (104) describe, along with damped sinusoidal waves, a wide variety of non-stationary regimes that arise from the transient excitation of the line, thus suggesting the possibility of using transmission lines for modeling the interaction of single-cycle transients with continuous media.

8. MHD transients in an inhomogeneous plasma

One of the most active areas of magnetohydrodynamics is the study of Alfven waves that couple magnetic field and plasma density vibrations. This concept is widely used both in cosmic and laboratory plasma problems in which magnetic field perturbations are often of a pulsed nature and the plasma density is strongly non-uniform along the direction in which these perturbations propagate. By modifying the method of non-separable solutions of the Klein – Gordon equation a number of exact analytical solutions can be obtained for such non-stationary non-uniform problems.

This approach is conveniently illustrated by considering an MHD transient propagating along a z-directed static magnetic field $\mathbf{H}_0$ in a non-dissipative plasma with a density profile $\varrho(z)$ defined by

$$\varrho(z) = \varrho_0 \begin{cases} 1 & \text{for} \quad z < 0\,, \\ U^2(z) & \text{for} \quad z \geqslant 0\,. \end{cases} \tag{118}$$

Here $U^2(z)$ is an unknown function that assumes positive values for $z \geqslant 0$. The propagating transient field is characterized by the magnetic field perturbation $\mathbf{H}_1$ and the plasma flow velocity $\mathbf{v}$, both the vectors lying in the plane (x, y) orthogonal to the direction of $\mathbf{H}_0$ if a longitudinal propagation is assumed. For simplicity, consider a linearly polarized wave whose H_{1x} and v_x components are related by the magnetohydrodynamics equations; for the assumed longitudinal propagation case these are

$$\frac{\partial H_{1x}}{\partial t} = H_0 \frac{\partial v_x}{\partial z}\,, \tag{119}$$

$$\frac{\partial v_x}{\partial t} = \frac{H_0}{4\pi \varrho_0 U^2(z)} \frac{\partial H_{1x}}{\partial z}\,. \tag{120}$$

We next introduce the normalised functions h and v and the Alfven velocity v_{A},

$$h = \frac{H_{1x}}{H_0}\,, \qquad v = \frac{v_x}{v_{\mathrm{A}}}\,, \qquad v_{\mathrm{A}} = \frac{H_0}{\sqrt{4\pi\varrho_0}}\,, \tag{121}$$

and define an unknown function ψ by

$$h = A_0 v_{\mathrm{A}} \frac{\partial \psi}{\partial z}\,, \qquad v = A_0 \frac{\partial \psi}{\partial t}\,. \tag{122}$$

By making the substitutions (121) and (122), the system (119) and (120) is reduced to just one equation

$$\frac{\partial^2 \psi}{\partial z^2} - \frac{U^2(z)}{v_{\mathrm{A}}^2} \frac{\partial^2 \psi}{\partial t^2} = 0\,. \tag{123}$$

The inhomogeneous wave equation (123) is widely used in describing stationary monochromatic waves in inhomogeneous media, when it is assumed that $\psi = \psi_1(z)\exp(-\mathrm{i}\omega t)$ and the function $\psi_1(\tau)$ solves the equation

$$\frac{\partial^2 \psi_1}{\partial z^2} + \frac{\omega^2 U^2(z)}{v_{\mathrm{A}}^2} \psi_1 = 0\,. \tag{124}$$

Exact solutions to Eqn (124) are available for some $U^2(z)$ profiles [20].

As opposed to the conventional variable separation approach (124), in what follows new classes of solutions to Eqn (123), unrelated to this method, are explored. Introducing new functions

$$y = \int_0^z U(z_1)\,\mathrm{d}z_1 \tag{125}$$

and

$$F = \psi \sqrt{U(z)}\,, \tag{126}$$

we transform Eqn (123) into

$$\frac{\partial^2 F}{\partial y^2} - \frac{1}{v_{\mathrm{A}}^2} \frac{\partial^2 F}{\partial t^2} = F\left[\frac{1}{2U^3} \frac{\partial^2 U}{\partial z^2} - \frac{3}{4U^4}\left(\frac{\partial U}{\partial z}\right)^2\right]. \tag{127}$$

The solution of this equation exhibits a great richness due to the unknown function on its right-hand side. Consider, for example, the case

$$\frac{1}{2U^3} \frac{\partial^2 U}{\partial z^2} - \frac{3}{4U^4}\left(\frac{\partial U}{\partial z}\right)^2 = p^2\,, \tag{128}$$

where p^2 is a real constant. The solution satisfying this nonlinear equation and the condition $U\big|_{z=0} = 1$, see Eqn (118), has the form

$$U(z) = \left(1 + s_1 \frac{z}{L_1} + s_2 \frac{z^2}{L_2^2}\right)^{-1}, \tag{129}$$

where the free parameters L_1 and L_2 are the characteristic lengths of the medium. Using these, the spatial dependence of the density is described by the two-parameter model (Fig. 11a)

$$\varrho = \varrho_0 \left(1 + s_1 \frac{z}{L_1} + s_2 \frac{z^2}{L_2^2}\right)^{-2}, \qquad s_1 = 0, \pm 1\,, \qquad s_2 = 0, \pm 1\,. \tag{130}$$

Thus, the wave equation (123) for an inhomogeneous non-dispersive medium reduces to the Klein – Gordon equation (127) for a homogeneous dispersive medium in (y, t) space,

$$\frac{\partial^2 F}{\partial y^2} - \frac{1}{v_{\mathrm{A}}^2} \frac{\partial^2 F}{\partial t^2} = p^2 F. \tag{131}$$

The dispersion of this medium is determined by the inhomogeneity parameter p^2 which, upon substituting Eqn (129) into (128), is expressed in terms of the characteristic lengths as

$$p^2 = \frac{1}{4L_1^2} - \frac{s_2}{L_2^2}\,. \tag{132}$$

The function F can be represented in terms of various non-separable solutions of the Klein – Gordon equations depending on the sign of the parameter p^2, Eqn (132). Thus, for $p^2 > 0$, solution (12) with $\eta = py$, $\tau = p v_{\mathrm{A}} t$ can be employed. Expressing the vector potential ψ from Eqn (126) we find the alternating magnetic field h and the plasma flow velocity v,

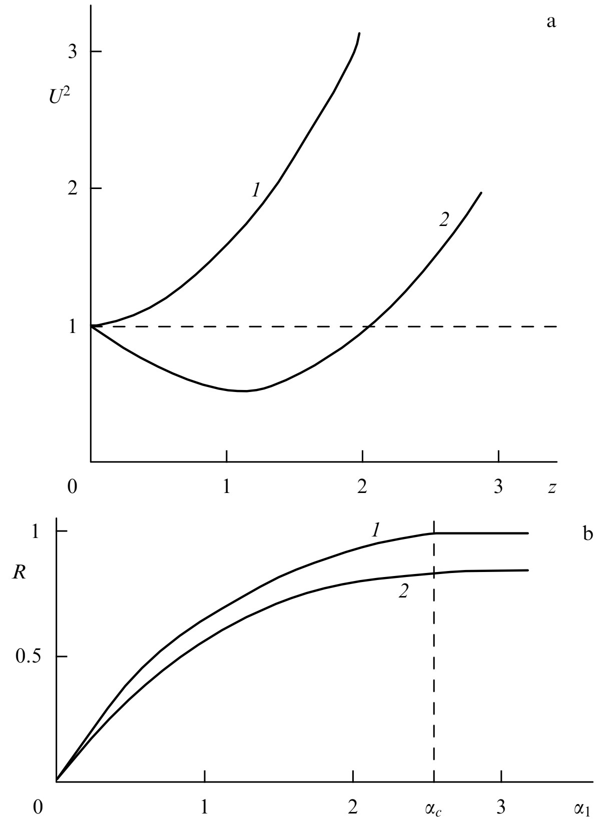

Figure 11. (a) Increasing density profiles U^2, Eqn (130), admitting an exact solution for a longitudinally traveling Alfven wave in an inhomogeneous plasma; curves: (*1*), $s_1 = s_2 = -1$; (*2*), $s_1 = 1$, $s_2 = -1$; $p^2 < 0$. (b) Coefficient of reflection of an Alfven transient F_0 [Eqn (66)] from an inhomogeneous plasma for $p^2 > 0$. Curves: (*1*), $K = 1.0015$; (*2*), $K = 1$. The parameter K is defined by Eqn (138), $\alpha_1 = pv_A t_0$.

Eqn (122). The electric field of the wave is expressed in terms of the velocity v_x as

$$E_y = \frac{H_0 v_x}{c} \,. \tag{133}$$

Finally, the electric and magnetic components of the Alfven wave in an inhomogeneous plasma are written in the form

$$E_y = -\frac{A_0 v_A}{c\sqrt{U(z)}} \sum_{q=3}^{\infty} a_q e_q \,, \; H_x = -\frac{A_0 v_A}{c\sqrt{U(z)}} \sum_{q=3}^{\infty} a_q h_q \,, \tag{134}$$

$$e_q = \frac{p}{4}(\psi_{q-2} - 2\psi_q + \psi_{q+2}) \,, \tag{135}$$

$$h_q = \frac{p}{4}\left[(\psi_{q-2} - \psi_{q+2}) - \frac{1}{pU}\frac{\partial U}{\partial z}(\psi_{q-1} - \psi_{q+1})\right] . \tag{136}$$

The variable η involved in the function ψ_q in Eqns (135) and (136) is easily calculated by substituting the function $U(z)$, Eqn (129), into Eqn (125).

Pursuing the analogy with the Klein–Gordon equation and using the calculational scheme of Section 4 we find the reflection coefficient of an MHD transient with the Laguerre envelope (49) reflected by the boundary of an inhomogeneous plasma, Eqn (129). For $s_1 = s_2 = 1$ we obtain ($p^2 > 0$)

$$R = -\frac{(\alpha_1 D_1)[\alpha_1 D_1 - K]}{1 - K\alpha_1 D_1} \,, \tag{137}$$

$$\alpha_1 = pv_A t_0 \,, \qquad K = \frac{s_1}{2pL_1} \,. \tag{138}$$

The function $D_1 = D(\alpha_1)$ is defined by Eqn (71). The reflection coefficient R is plotted in Fig. 11b. The reflection becomes total for $t_0 \geqslant t_c$, where t_c is obtained from the condition $R = 1$ giving

$$t_c = (pv_A)^{-1}\left(K - \sqrt{K^2 - 1}\right) . \tag{139}$$

To conclude this section we note that the plasma density profile (129) is an exact analytical solution for a single-cycle Alfven transient reflected from the boundary of an inhomogeneous plasma, a solution which does not require that the inhomogeneities be small in magnitude or vary slowly in time. Equations of the type of (123), governing wave propagation with a coordinate-dependent velocity, are used extensively in problems in radiophysics, acoustics, and optics, and the above analysis may therefore be of interest in the dynamics of various non-stationary and inhomogeneous fields occurring in many areas of physics research.

9. Nonlinear single-cycle transient dynamics

The interaction of ultrashort wave transients with dispersive media is currently a topical problem in nonlinear optics because it is in such transients where large peak radiation power is achieved. Analyses of nonlinear pulse processes usually consider a multicycle waveform whose phase and amplitude envelope evolve slowly in time and in which vibrations are assumed to be harmonic and the nonlinear perturbation of their refractive index Δn is small compared to the unperturbed 'linear' value n_0. This model, while underlying the physics of solitons and the theory of interacting and self-interacting waves, proves inadequate when faced with the self-action of single-cycle transients, with their non-sinusoidal vibrations and with the refractive index non-separable into linear and nonlinear parts. The construction of the optics of such essentially nonlinear transient fields therefore requires a first principle treatment. To illustrate this approach, the screening of a transient field by a nonlinear insulating medium is treated in what follows. Let us first write down Maxwell's equation for a linearly polarized wave in an isotropic conservative medium. Assuming the field relaxation time to be shorter than the transient duration, we represent the electric displacement D as a continuous function of the electric field E,

$$\frac{\partial E_x}{\partial z} = -\frac{1}{c}\frac{\partial H_y}{\partial t} \,, \qquad \frac{\partial H_y}{\partial z} = \frac{1}{c}\frac{\partial D}{\partial t} \,. \tag{140}$$

We next transform these equations as follows:

(1) Rewrite Eqns (140) treating the functions E_x and H_y as new independent variables and the variables z and t as new unknowns, $z = z(E_x, H_y)$, $t = t(E_x, H_y)$ (the hodograph transformation). The system (140) then becomes

$$\frac{\partial t}{\partial H} = -\frac{1}{c}\frac{\partial z}{\partial E} \,, \qquad \frac{\partial t}{\partial E} = -\frac{1}{c}\frac{\partial D}{\partial E}\frac{\partial z}{\partial H} \,, \tag{141}$$

$$E = E_x \,, \qquad H = H_y \,.$$

This approach provides a mathematical framework for understanding the nonlinear geometric optics of quasi-harmonic waves in a cubically nonlinear medium [21]. Here, in contrast, the hodograph transformation is employed to construct exact analytical solutions of Maxwell's equations (141) for non-harmonic fields in far more general models of nonlinear media.

(2) The hodograph transformation converts the nonlinear system (140) defined in (z, t) space into a linear but inhomogeneous system (141) in (E, H) space. The system (141) is formally analogous to the system (123) for wave propagation with a coordinate dependent velocity, the function $\partial D/\partial E$ in Eqn (141) corresponding to $U^2(z)$ in (123). This analogy suggests the following solution procedure for the system (141).

Define a new function ψ such that

$$t = -\frac{1}{c}\frac{\partial\psi}{\partial E}\,, \qquad z = \frac{\partial\psi}{\partial H}\,. \tag{142}$$

Substituting Eqn (142) into (141) we obtain an inhomogeneous wave equation in (E, H) space for this function,

$$\frac{\partial^2\psi}{\partial E^2} - \frac{\partial D}{\partial E}\frac{\partial^2\psi}{\partial H} = 0\,. \tag{143}$$

Specifically, let $\partial D/\partial E > 0$ so that

$$\frac{\partial D}{\partial E} = n_0^2 U^2(E)\,, \qquad U\Big|_{E=0} = 1\,, \tag{144}$$

where n_0 is the refractive coefficient in the weak field and short time scale limit,

$$n_0^2 = \varepsilon_\infty\,. \tag{145}$$

Introducing the function

$$F = \psi\sqrt{U(E)}\,, \qquad U \geqslant 0 \tag{146}$$

and the variable

$$\varphi = p\int_0^E U(y)\,\mathrm{d}y\,, \tag{147}$$

Eqn (143) yields the Klein – Gordon equation for F,

$$\frac{\partial^2 F}{\partial\varphi^2} - \frac{\partial^2 F}{\partial h^2} = F\,, \qquad h = \frac{pH}{n_0}\,, \tag{148}$$

where the constant p^2 is given by

$$p^2 = \frac{1}{2U^3}\frac{\partial^2 U}{\partial E^2} - \frac{3}{4U^4}\left(\frac{\partial U}{\partial E}\right)^2, \tag{149}$$

and depends on the nonlinear properties of the medium. The function U in Eqn (149) has the form

$$U = \left(1 + s_1\frac{E}{E_1} + s_2\frac{E^2}{E_2^2}\right)^{-1}, \qquad s_1 = 0, \pm 1\,, \qquad s_2 = 0, \pm 1\,, \tag{150}$$

$$p^2 = \frac{1}{4E_1^2} - \frac{s_2}{E_2^2}\,, \tag{151}$$

where E_1 and E_2 are certain field values characteristic of the material.

Using Eqns (149) and (150) we find the nonlinear electric displacement

$$D = n_0^2\int_0^E U^2(y)\,\mathrm{d}y \tag{152}$$

and the variable φ, Eqn (147), for various combinations of characteristic fields and signs.

(3) For simplicity, consider the one-parameter function $U(E)$

$$U = \left(1 - \frac{E^2}{E_2^2}\right)^{-1}, \qquad p^2 = \frac{1}{E_2^2}\,. \tag{153}$$

The corresponding electric displacement D and the variable φ are

$$D = \frac{E_2 n_0^2}{2}\left(\operatorname{artanh} x + \frac{x}{1 - x^2}\right), \qquad x = \frac{E}{E_2}\,, \tag{154}$$

$$\varphi = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right). \tag{155}$$

The values of the dimensionless parameter x lie in the interval $0 \leqslant x < 1$. For weak fields $E \ll E_2$ the dependence (153) corresponds to the cubic focusing nonlinearity model

$$D = n_0^2 E\left(1 + \frac{2\sqrt{n_0}}{3}\frac{E^2}{E_2^2}\right). \tag{156}$$

We next find the electrical and magnetic field components for a medium with nonlinear response (153). Let us represent the function ψ, Eqn (142), in the form

$$\psi = A\sqrt{1 - x^2}\,F\,, \tag{157}$$

where F satisfies the Klein – Gordon equation (148) and the boundary condition (142)

$$\frac{\partial\psi}{\partial h}\bigg|_{h=h_0,\,\varphi=\varphi_0} = 0\,. \tag{158}$$

Here $\varphi_0 = \varphi_0(t)$ and $h_0 = h_0(t)$ are the values of the functions φ, Eqn (147), and h, Eqn (148), at the boundary $z = 0$ at any time t; and A is the normalization constant.

If the solution to Eqn (148) is taken to be of the form

$$F = A\cosh\left(M\varphi_0 - \sqrt{M^2 - 1}\,h_0\right), \tag{159}$$

with M a free parameter ($M > 1$), then from Eqns (157) and (158) the functions φ_0 and h_0 are related by

$$\varphi_0 = h_0\sqrt{1 - M^{-2}}\,. \tag{160}$$

From Eqn (159), substituting the values of the nondimensional variables φ_0, Eqn (155), and h_0, Eqn (148), the relation between the electrical and magnetic field components at $z = 0$ can be written explicitly as

$$\frac{E_0}{E_2} = \tanh\left(\frac{H_0}{n_0 E_2}\sqrt{1 - M^{-2}}\right), \qquad E_0 = E(t)\Big|_{z=0}\,, \qquad H_0 = H(t)\Big|_{z=0}\,. \tag{161}$$

We next introduce the reflection coefficient of the incident signal R using the continuity condition at the boundary,

$$\frac{1+R}{1-R}=\frac{E_0(t)}{H_0(t)}\,, \tag{162}$$

and obtain from Eqn (160)

$$R=\frac{n_0\sqrt{1-M^{-2}}-G(x_0)}{n_0\sqrt{1-M^{-2}}+G(x_0)}\,,\qquad G(x_0)=\frac{\operatorname{artanh} x_0}{x_0}\,. \tag{163}$$

The envelope of the incident transient E_{in} is now found using the reflection coefficient R and the reflected wave field E_0 at the boundary, to give

$$\frac{E_{\text{in}}}{E_2}=\frac{n_0\sqrt{1-M^{-2}}+\operatorname{artanh} x_0}{2n_0\sqrt{1-M^{-2}}}\,. \tag{164}$$

Thus, the envelope of the incident transient is expressed in terms of the normalised envelope of the refracted electric field at the boundary $x_0=x_0(t)$. To obtain the variation of x_0 with time we substitute the solutions (157) and (159) into (142) at $z=0$, giving

$$x_0=\frac{ctL^{-1}}{\sqrt{1+(ctL^{-1})^2}}\,,\qquad L=-AE_2^{-1}\,, \tag{165}$$

where L is the spatial scale of the envelope.

In the bulk of the medium $z \geqslant 0$, the equations for the normalised envelops of the electric and magnetic field components $x=x(z,t)$ and $h=h(z,t)$ are found from Eqns (142), (157), and (159) to be

$$\frac{(1-M^{-2})x^2(1-x^2)}{n_0^2}=\left[\frac{z-vt(1-x^2)}{L}\right]^2-\frac{z^2}{L^2}\frac{x^2}{M^2}\,, \tag{166}$$

$$h=\frac{\varphi}{\sqrt{1-M^{-2}}}+\frac{1}{\sqrt{1-M^{-2}}}\operatorname{arsinh}\left(\frac{z}{L}\frac{n_0}{\sqrt{M^2-1}}\frac{1}{\sqrt{1-x^2}}\right)\,, \tag{167}$$

$$v=\frac{c}{n_0}\sqrt{1-M^{-2}}\,. \tag{168}$$

From the reflection coefficient expression R, Eqn (163), an important trend of the nonlinear field evolution described by Eqns (166)–(168) can be seen. For the limiting values of the reflection coefficient at the signal edge ($ct \ll L$) and away from the edge ($ct \gg L$)

$$\lim_{x_0\to 0} R=\frac{n_0\sqrt{1-M^{-2}}-1}{n_0\sqrt{1-M^{-2}}+1}\,,\qquad \lim_{x_0\to 1} R=1\,, \tag{169}$$

it is readily seen that the refection of the transient increases with the amplitude, the bulk electric field being much weaker than that on the surface (Fig. 12). Some features of this self-screening should be pointed out here.

(1) The electric field leading edge ($x=0$) moves with velocity v, Eqn (168), dependent on the material characteristic and the transient parameter M, the latter in turn depending on the steepness of the edge of the envelope

$$\frac{\partial}{\partial t}\left(\frac{E_{\text{in}}}{E_2}\right)\bigg|_{z=0}=\frac{c}{2L}\left(1+\frac{n_0}{\sqrt{1-M^{-2}}}\right)\,. \tag{170}$$

(2) The electric and magnetic envelops of a signal in a nonlinear medium have different distortion rates.

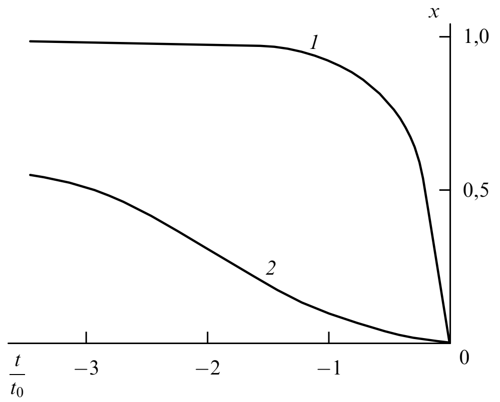

Figure 12. Self-screening of a field transient in a strongly nonlinear medium (154). Curves *1* and *2* represent the envelope of the transient (166) at the boundary and at the section $z=L$; $t_0=L/v$; $x=E/E_2$, Eqn (154); the parameters involved in Eqn (166) are: $M=\sqrt{2}$, $n_0=3.5$.

(3) The reflection coefficient R, Eqn (163), in the case $n_0\sqrt{1-M^{-2}}>1$ changes sign at the point x_0 defined by

$$\operatorname{artanh} x_0 = x_0 n_0\sqrt{1-M^{-2}}\,. \tag{171}$$

In this case, nonlinear reflection makes the reflected signal amplitude-modulated as well a changing the polarization of the peripheral portion of its envelope.

The present analysis of the nonlinear transient dynamics relies on the solution (157) which extends the non-separable treatment of the Klein–Gordon equation to the case $p^2>0$ [see Eqn (149)]; further extension enables the same equation to cover media with $p^2<0$ and those with nonlinear magnetization, when the dependence $B(H)$ is described by the function U^2, Eqn (150).

10. Conclusion. Non-separable field representations in transient optics

The study of rapidly varying non-periodic fields in a number of dispersive model media in Sections 2 to 9 has shown that such fields can be described in an 'alternative' way without relying on conventional harmonic analysis. It turned out, in particular, that certain concepts and material characteristics used in the harmonic analysis are not needed here. These are:

(a) dielectric permittivity and refractive index;

(b) vibration phase and phase velocity;

(c) cut-off frequency and damping factor.

At the heart of the present approach to the excitation and propagation of non-periodic fields in continua are the continuity of fields at the boundaries and the non-separable time domain solutions of the Klein–Gordon and telegraph equations. Since the time description techniques of transient optics are still in their infancy, a few example seem to be appropriate here to give further insight into non-separable solutions of Maxwell's equations.

(1) The existence of non-separable solutions is not unique to the Klein–Gordon equation. Solutions of the Helmholtz equation widely used in waveguide theory are of similar nature. In the simplest case of a hollow waveguide of rectangular cross section this equation takes the form

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=k_\perp^2 f\,. \tag{172}$$

Here x and y are coordinates in the waveguide section plane, and the parameter $k_{\perp}^2$ is related to the wave frequency and the propagation constant β by

$$k_{\perp}^2 = \frac{\omega^2}{c^2} - \beta^2 \,. \tag{173}$$

A one-mode separation-of-variables solution of Eqn (172) may be written, for example, in the form

$$f_0 = \cos(k_1 x)\cos(k_2 y)\,; \tag{174}$$

where the constants k_1 and k_2 for perfectly conducting walls of dimension a, b are given by

$$k_1 = \frac{\pi}{a}\,, \qquad k_2 = \frac{\pi}{b}\,, \qquad k_1^2 + k_2^2 = k_{\perp}^2\,. \tag{175}$$

Along with the separable solution f_0, Eqn (174), there are a infinite number of non-separable solutions f_n $(n \geqslant 1)$ to the Helmholtz equation. These are constructed from the recurrence relation [22]

$$f_{n+1} = \hat{L} f_n\,, \tag{176}$$

with the operator $\hat{L}$ of the form

$$\hat{L} = k_2 \frac{\partial}{\partial k_1} - k_1 \frac{\partial}{\partial k_2}\,. \tag{177}$$

For example, the simplest possible non-separable solution f_1 is

$$f_1 = k_2 \sin(k_1 x)\cos(k_2 y) - k_1 \cos(k_1 x)\sin(k_2 y)\,. \tag{178}$$

Linear combinations of the functions f_n are of interest for obtaining the eigenfunctions of waveguides of a complex curvilinear cross section [23].

(2) The general solution of the Klein – Gordon equation (11) can be written in the integral form [24]

$$f(\tau,\eta) = \frac{1}{2}\left[\varphi(\tau-\eta) + \varphi(\tau+\eta)\right] + \frac{1}{2}\int_{\tau-\eta}^{\tau+\eta} \mathrm{d}\theta\, \Phi(\theta,\tau,\eta), \tag{179}$$

$$\Phi(\theta,\tau,\eta) = \psi(\theta) I_0\left(\sqrt{(\tau-\theta)^2 - \eta^2}\right) + \varphi(\theta)\eta\, \frac{I_1\left(\sqrt{(\tau-\theta)^2-\eta^2}\right)}{\sqrt{(\tau-\theta)^2-\eta^2}}\,. \tag{180}$$

Here I_0 and I_1 are the modified Bessel functions, and the functions $\varphi(\theta)$ and $\psi(\theta)$ are related to the boundary equations

$$F\Big|_{\eta=0} = \varphi(\tau)\,, \qquad \frac{\partial F}{\partial \eta}\Big|_{\eta=0} = \psi(\tau)\,. \tag{181}$$

Although Eqns (179) – (181) provide the general solution of the problem, they do not generally represent explicitly the information about the fields excited by the external pulsed source in the bulk of a medium. There are two major reasons for this.

(a) The functions φ and ψ determining the field at the boundary are found from reflection coefficients whose calculation, in turn, requires a knowledge the field within the medium.

(b) Even if the functions (181), φ and ψ, are known, the integral conversion (179) does not generally admit of an analytical evaluation.

Considerable difficulties are encountered, for example, in applying this approach to the problem of Laguerre single-cycle transients reflected from a plasma; at the same time, as shown in Section 4, representing Klein – Gordon solutions in terms of non-separable functions yields explicit expressions for both the field structure and reflection coefficient.

(3) The same problem of the reflection of a Laguerre single-cycle transient F_0, Eqn (66), from a plasma can now be considered in the frequency domain. We will restrict ourselves to the normal incidence case and employ the reflection coefficient expression for a wave of frequency ω,

$$R(\omega) = \frac{1 - \sqrt{1 - \Omega^2\omega^{-2}}}{1 + \sqrt{1 - \Omega^2\omega^{-2}}}\,. \tag{182}$$

Taking the Fourier transform of the envelope F_0,

$$(F_0)_{\omega} = \frac{16 t_0 (1 + \mathrm{i}\omega t_0)}{(1 + 2\mathrm{i}\omega t_0)^3}\,, \tag{183}$$

the reflected signal can now be written in the frequency integral form

$$E_{\mathrm{ref}} = 16 E_{\mathrm{in}} t_0 \int_{-\infty}^{\infty} \frac{1 - \sqrt{1 - \Omega^2\omega^{-2}}}{1 + \sqrt{1 - \Omega^2\omega^{-2}}} \times \frac{1 + \mathrm{i}\omega t_0}{(1 + 2\mathrm{i}\omega t_0)^3} \exp(\mathrm{i}\omega t_0)\,\mathrm{d}\omega\,. \tag{184}$$

The fact that this integral cannot be performed analytically is a further indication of the advantage of the non-separable solutions which yield a simple algebraic expression for the reflection coefficient of the single-cycle transient (76).

(4) To proceed further with the analysis of the Fourier representation approach of transient electrodynamics, it is interesting to note here several properties of the spectral amplitudes of the non-separable representations (17) we use for the electric and magnetic components of a non-stationary field in a plasma. For simplicity we confine ourselves to symmetric field envelops at the boundary $z = 0$ $[E(t) = E(-t),\ H(t) = H(-t)]$ with only even harmonics $q = 2m$ $(m = 1, 2, 3 \ldots)$ present (see, e.g., Fig. 4). Let us calculate the sine and cosine Fourier transforms of these fields. Using the result [17]

$$\int_0^{\infty} \cos(\omega t) J_{2m}(\Omega t)\,\mathrm{d}t = 0 \tag{185}$$

in the transparency region $\omega > \Omega$, the cosine components of the e_{2m} and h_{2m} harmonics are, from Eqn (57),

$$\int_0^{\infty} \cos(\omega t) e_{2m}\,\mathrm{d}t = \int_0^{\infty} \cos(\omega t) h_{2m}\,\mathrm{d}t = 0\,. \tag{186}$$

The sine components of the Fourier transform of the even functions e_{2m} and h_{2m} vanish. Substituting these results into Eqn (17), we get the spectral amplitudes

$$E_{\omega}\Big|_{z=0} = H_{\omega}\Big|_{z=0} = 0 \tag{187}$$

of the non-separable representations of fields in the transparency region at the plasma boundary, which are indicating that the non-separable fields E and H, Eqn (17), cannot be excited by a monochromatic $\cos(kz - \omega t)$ wave from the transparency region ($\omega > \Omega$) incident on the plasma boundary.

(5) The methods of non-periodic analysis are also of relevance to a number of traditional problems in radiophysics and optics. Within the framework of the approach of Section 8 we can construct new exact solutions for the problem of harmonic waves in an inhomogeneous propagation media. To this end, consider, for example, how an Alfven wave of frequency ω traveling along the magnetic field $\mathbf{H}_0$ through a plasma is reflected by an inhomogeneity of the type (118) in the plasma density $\varrho(z)$. To find the wave field components in terms of the function ψ, Eqn (122), we use here instead of Eqn (134) the harmonic solutions of Eqn (131),

$$F = \exp\left[\mathrm{i}(k\eta - \omega t)\right] , \qquad k = \omega v_{\mathrm{A}}^{-1} n(\omega) , \tag{188}$$

$$n(\omega) = \sqrt{1 - \Omega_{\mathrm{p}}^2 \omega^{-2}} , \qquad \Omega_{\mathrm{p}} = p v_{\mathrm{A}} , \tag{189}$$

where v_{A} is the Alfven velocity (121), and the parameter p ($p^2 > 0$) is related to the structure of the inhomogeneous medium and is given by Eqn (132).

Representing the function ψ, Eqn (122), in the form

$$\psi = \frac{\exp\left[\mathrm{i}(k\eta - \omega t)\right]}{\sqrt{U(z)}} , \tag{190}$$

the wave field components H_x and E_y, Eqn (133), in an inhomogeneous medium are found to be

$$H_x = \frac{\mathrm{i} A_0 k v_{\mathrm{A}} H_0}{U\sqrt{U}} \left(U^2 + \frac{\mathrm{i}}{2k} \frac{\partial U}{\partial z} \right) \exp\left[\mathrm{i}(k\eta - \omega t)\right] ,$$

$$E_y = -\frac{\mathrm{i} A_0 \omega v_{\mathrm{A}} H_0}{c\sqrt{U}} \exp\left[\mathrm{i}(k\eta - \omega t)\right] . \tag{191}$$

The complex reflection coefficient of the wave is calculated by applying the standard boundary equations, giving

$$R = \frac{1 - M}{1 + M} , \qquad M = n(\omega)\left(1 + \frac{\mathrm{i} s_1}{2kL}\right) , \tag{192}$$

where the parameter $n(\omega)$ is defined by Eqn (189). Note that in the frequency range

$$\omega \leqslant \Omega_{\mathrm{p}} = p v_{\mathrm{A}} \tag{193}$$

the Alfven wave undergoes total internal reflection.

The exact solution we have obtained for the wave equation in an inhomogeneous medium does not involve the conventional slowness assumption on the field and material parameters. Note that Eqns (189) indicate a useful analogy between such inhomogeneous Alfven waves and waves in a homogeneous plasma, with the characteristic frequency Ω_{p}, Eqn (193), playing the role of the plasma cut-off frequency.

In conclusion, this work is only a brief sketch of the first steps on the way of the 'alternative' optics of non-stationary and non-periodic fields, in which no usual spectral concepts are needed. This approach provides exact analytical solutions capable of describing the interaction of single-cycle transients with a number of dispersive media directly in the time domain. In mathematical terms, this approach relies on the non-separable solutions of Maxwell's equations, and the treatment of these latter involves non-stationary extensions of the electric displacement and power flow velocity concepts and also of Snellius' and Fresnel's formulae. Since the present analysis is restricted to the Debye model of the insulator, the extension of the time domain approach to include non-stationary fields in Lorentz's free-oscillator model [25] and in mixed models [26] would be of particular importance. Since, further, the Klein – Gordon and telegraph equations also have non-optical applications and are widely used in describing of waves in radio engineering, acoustics, and geophysics, exact non-stationary solutions of these equations may be useful in treating impulse processes involving other types of physical fields and other dispersive media.

In the initial phase of this work, it was my happy opportunity and honor to discuss some of its aspects with the theoretical radiophysics' patriarch S M Rytov, and today, when Sergei Mikhailovich is sadly no longer among us, I am increasingly grateful to him for his advices and encouragement.

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Note added in proof

The transformation of the inhomogeneous wave equations to the homogeneous Klein – Gordon equation in new variables (see Section 8) is easily extended to the conventional electrodynamics equations with time-dependent parameters, giving rise to new exact analytical solutions for waves in non-stationary systems. As an example, one can consider the propagation of an electromagnetic field in a conductor whose conductivity varies with time as $\sigma = \sigma_0 U^2(t)$. Such a process is described by a non-stationary telegraph equation which, using the notation of Section 3, can be written

$$\frac{\partial^2 \Psi}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} = \frac{2}{T} U^2 \frac{\partial \Psi}{\partial t} , \qquad U^2(t)\Big|_{t=0} = 1 . \tag{194}$$

Difficulties encountered in solving Eqn (194) are discussed in Ref. [27], where a relevant perturbation theory is also constructed.

Alternatively, the non-stationary telegraph equation can yield a number of exact solutions if, in analogy with (126), the unknown function Ψ is represented in the form

$$\Psi = F \exp\left(-\frac{1}{T}\int_0^t U^2(x)\,\mathrm{d}x \right) . \tag{195}$$

Substitution of Eqn (195) into (194) gives the equation for F,

$$\frac{\partial^2 F}{\partial \eta^2} - \frac{\partial^2 F}{\partial \tau^2} = -F\left(\frac{\partial U^2}{\partial \tau} + U^4 \right) , \qquad \tau = tT^{-1} . \tag{196}$$

Equating the bracketed quantity to, say, a constant p^2, gives an equation from which the time dependence of the conductivity $U^2(t)$ can be found; in particular, for $p^2 > 0$ the 'saturation' of conductivity to the limiting value $U^2 = p$ with a characteristic time $t_0 = Tp^{-1}$ is described,

$$U^2(\tau) = \frac{p(1 + p\tanh p\tau)}{p + \tanh p\tau} . \tag{197}$$

The solution of Eqn (194) for the model function (197) can be written in the form

$$\Psi = \frac{F}{\sinh\left[p(1+\tau)\right]} \,. \tag{198}$$

The function F in Eqns (196) and (198) is represented by solutions to the Klein – Gordon equation (11) with constant coefficients. Equation (194) for $p^2 \leqslant 0$ is solved in an analogous fashion.

The exact solutions presented above are relevant to the electrodynamics of dissipative systems [9], Markoff processes [27], and impulse metal optics [28]. For example, the reflection coefficient of an electromagnetic wave of frequency ω incident on a non-stationary conductor (197) may be found for arbitrary values of the parameter ωT. If the solution of Eqn (196) for the conductivity model (197) is assumed in the form

$$F = \exp\left[\mathrm{i}\omega T(\eta N - \tau)\right] \,, \qquad N = \sqrt{1 + p^2(\omega T)^{-2}} \tag{199}$$

then, from continuity at $z = 0$, the complex reflection coefficient at normal incidence is found to be

$$R(t) = \frac{M\exp(-\mathrm{i}\theta) - 1}{M\exp(-\mathrm{i}\theta) + 1} \,, \qquad \theta = \varphi - \varphi_0 \,, \tag{200}$$

$$M(\tau) = Q(\tau)N^{-1}, \quad \cos\varphi(\tau) = Q^{-1}, \quad \cos\varphi_0 = Q^{-1}(0) \,,$$

$$Q(\tau) = \sqrt{1 + p^2(\omega T)^{-2}\tanh^{-2}\left[p(1+\tau)\right]} \,.$$

The time dependence of the reflection coefficient causes the non-stationary amplitude-phase modulation of the reflected wave as well as producing broadening of its spectrum, the amount of the broadening depending on the parameter ωT. For $\omega T \sim 1$, the broadening is quite strong (the localization region is only a few field cycles in size), and the peak in the reflection produces a short wide-band pulse against the background of the reflected wave. To separate the pulse from the background, a dispersive or diffracting system can be employed.

It is worth emphasizing that obtaining a non-stationary solution outside the framework of the WKB-approximation is not restricted by assumptions about the smallness or slowness of changes in the parameters of the field or medium.

References

1. Bakhrakh L D, Bliskavitskiĭ A A *Usp. Fiz. Nauk* **162** (12) 160 (1992) [*Sov. Phys. Usp.* **35** 1086 (1992)]
2. Pastol Y et al. *Appl. Phys. Lett.* **54** (4) 307 (1989)
3. Fattinger C, Grischkowsky D *Appl. Phys. Lett.* **54** (6) 490 (1989)
4. Stingl A et al. *Opt. Lett.* **19** (4) 204 (1994)
5. Akimoto K *J. Phys. Soc. Jpn.* **65** (7) 2020 (1996)
6. Ma J, Ciric I *Radio Science* **27** 561 (1992)
7. Bertolotti M, Ferrari A, Sereda L *J. Opt. Soc. Am. B* **12** (8) 1519 (1995)
8. Avakyan R A et al. *Opt. Spektrosk.* **77** 668 (1994) [*Opt. Spektrosc.* (Russia) (1994)]
9. Landau L D, Lifshitz E M *Electrodynamics of Continuous Media* (Oxford: Pergamon Press, 1960)
10. Sun E, Rusch W *IEEE Trans. Ant. Propag.* **42** (1) 9 (1994)
11. Shvartsburg A B *Dokl. Akad. Nauk* (Russia) **333** 324 (1993) [*Phys. Dokl.* (1993)]
12. Mugnai D et al. *Phys. Rev. Lett.* **68** (3) 259 (1992)
13. Kaizer G A *Friendly Guide to Wavelets* (New York: Birkhauser, 1994)
14. Shvartsburg A *Time-Domain Optics of Ultrashort Waveforms* (Oxford: Oxford University Press, 1996)
15. Gabor D "Theory of communications" *J. Inst. Electr. Eng.* **93** 429 (1946)
16. Janke E, Emde F, Lösch F *Tafeln Hönerer Funktionen* (Tables of Special Functions) (Stuttgart: Teubner, 1960)
17. Ditkin V A, Prudnikov A P *Integral'nye Preobrazovaniya i Operatsionnoe Ischislenie* (Integral Transformations and Operational Calculus) (Moscow: Fizmatgiz, 1961)
18. Shvartsburg A B *Usp. Fiz. Nauk* **164** 333 (1994) [*Phys. Usp.* **37** 311 (1994)]
19. Ginzburg V L, Rukhadze A A *Élektromagnitnye Volny v Plazme* (Electromagnetic Waves in a Plasma) (Moscow: Nauka, 1974)
20. Jackson J *Classical Electrodynamics* 2nd edition (New York: Wiley, 1975)
21. Shvartsburg A B *Non-Linear Pulses in Integrated and Waveguide Optics* (Oxford: Oxford University Press, 1993)
22. Moseley D S *Quart. Appl. Math.* **22** (2) 354 (1965)
23. Schoonaert D H, Luypaert P J *Electron. Lett.* **9** (11) 617 (1973)
24. Sobolev S L *Uravneniya Matematicheskoi Fiziki* (Equations of Mathematical Physics) (Moscow: Nauka, 1966)
25. Shvartsburg A B *Physica Scripta* **49** (4) 351 (1994)
26. Harmuth H F, Hussain G M *Propagation of Electromagnetic Signals* (Singapore: World Scientific, 1994)
27. Masoliver J, Weiss G H *Phys. Rev. E* **49** (5) 3852 (1994)
28. Kadin A M, Johnson M W *Appl. Phys. Lett.* **69** (25) 3938 (1996)