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An introduction to matrix superstring models

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Contents

Abstract. The recently developed matrix approach to the superstring concept and M-theory is introduced. The Banks – Fishler – Shenker – Susskind matrix theory, built as a supersymmetric matrix quantum mechanics, is considered. Two supersymmetric matrix models providing a non-perturbative formulation of the IIB superstring problem are discussed. Applications to non-perturbative string theory are reviewed.

1. Introduction

String theory (or superstring theory [4], to be precise) is currently the sole consistent attempt to integrate all fundamental interactions including gravity†. In this theory, various particles correspond to internal oscillation quanta of a one-dimensional extended object, a string. It appears that the string state spectrum almost inevitably comprises gravitons and massless gauge fields. Moreover, the string theory predicts supersymmetry which may be considered to explain the existence of fermions.

String interaction leads to their decoupling (or coupling). Such an essentially geometric description allows for the development of a divergence-free perturbation theory whose zeroth order at low energies reproduces Einstein's equations in the case of closed strings and the Yang – Mills equations in the case of open ones. An important breakthrough in superstring theory has been the solution of the ultraviolet divergence problem and the construction of finite and anomaly-free quantum gravity in the framework of perturbation theory. However, string interaction at Planck scales is no longer weak and non-perturbative effects can not be neglected. The development of methods unrelated to perturbation theory constitutes one of the main objectives in modern superstring theory.

Superstring theory can be consistently formulated in ten-dimensional space – time. There are several self-consistent string theories, not to mention different variants of compactification into a space of fewer dimensions. In the weak coupling domain where perturbation theory is applicable, different superstring theories appear to be virtually unrelated. Considering the string theory as the theory of everything, such a large number of alternative variants looks quite unnatural, at least from the aesthetic point of view. However, it has been argued that all superstring models beyond the limits of the perturbation theory are, in a sense, equivalent. Available observations appear to support this hypothesis. There is strong evidence in favor of a single universal theory encompassing all superstring models as different limiting cases.

That all superstring models are limiting cases of a certain unified theory is confirmed by discrete symmetries with respect to duality transformations which, generally speaking, relate different superstring theories‡. These symmetries can translate the weak coupling domain of one theory into the strong coupling domain of another. In duality transformations, normal string excitations, their Kaluza – Klein modes, and soliton-like states undergo interconversion. All this gives reason to believe that string excitations are no more

† See Ref. [5] for a brief review of physical aspects of the modern superstring theory.

‡ See Refs [6, 7].

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fundamental degrees of freedom than non-perturbative states and that the theory can be formulated in such a way as to ensure the description of all degrees of freedom (and different types of superstrings) on a footing of equality. Extensive superstring duality studies have created a somewhat paradoxical situation. On the one hand, string behavior in the strong coupling domain is known fairly well. Moreover, considerable progress has been made in the research of non-perturbative degrees of freedom. On the other hand, a consistent approach to the superstring concept unrelated to perturbation theory is still lacking.

It is quite possible that an interpretation of string theory as the quantum mechanics of one-dimensional extended objects is adequate only in the weak coupling domain whereas fundamental degrees of freedom should be viewed from a different standpoint. Moreover, non-perturbative string theory should not necessarily be ten-dimensional. In fact, it has been suggested that a unique dynamical description in eleven dimensions is equally applicable to all current string theories [8 – 10]. Different types of superstrings are realized in different kinematic regimes. Such a hypothetical construction is referred to as M-theory† (see Ref. [6] and references therein). The low-energy limit of M-theory is consistent with eleven-dimensional supergravity. The considerations underlying this hypothesis are beyond the scope of the present review. However, it is worthwhile to note that eleven-dimensional supergravity is maximal in terms of supersymmetry because no other consistent supersymmetric theory exists in space – time with a larger number of dimensions [11]. This fact accounts in part for the singularity of eleven-dimensional space.

A variety of indirect considerations allows for the identification of certain variants of M-theory compactification with (compactified) superstring theories. The simplest example is compactification of M-theory on a circle along one of the spatial directions. The resultant ten-dimensional theory is actually a IIA type superstring‡, and the perturbative string is in correspondence with the small compactification radius. Conversely, in the strong coupling domain, the radius tends to infinity, and the theory is neither compact nor eleven-dimensional.

The only thing known about the state spectrum in the M-theory is that it contains the graviton and its superpartners as the sole massless states along with extended objects, such as membranes and the five-branes dual to them, carrying electric and magnetic charges relative to the third rank tensor field included in the eleven-dimensional gravity supermultiplet. M-theory offers no general considerations to say anything about fundamental degrees of freedom. Generally speaking, M-theory dynamics at Planck scales must be very complicated because the theory contains no free parameters. The coupling constant in M-theory, the 11D-gravity constant, has a dimension and can be eliminated by an appropriate choice of units of measurement. Therefore, a small parameter can arise in the theory only kinematically. For this reason, it is not easy to offer even the quantum-mechanical formulation of M-theory and the problem awaits solution. One hypothesis suggests that supermembranes play the role of fundamental degrees of freedom in M-theory [12, 13, 9] in the same sense as superstrings are fundamental objects in ten-dimensional quantum supergravity. However, a consistent quantum supermembrane theory remains to be developed.

† The term 'M-theory' is believed to have been coined from 'membrane theory' although opinions differ.

‡ See Glossary in the end of this paper for superstring classification.

An alternative approach to the dynamical formulation of M-theory referred to as matrix theory§ has been suggested by Banks, Fishler, Shenker, and Susskind (BFSS) [1]. This approach is based on quantization in the light cone frame with one of the cone coordinates compactified on a circle of radius $R \to \infty$. The momentum along the compactification direction p_- is quantized in units of R^{-1}: $p_- = N/R$. The association of M-theory compactified on a circle with type IIA superstring allows the singling out of degrees of freedom important in the given kinematic regime [1]. These degrees of freedom turn out to be a sort of string soliton. The dynamics of these solitons is described with the help of the non-commutative coordinates X^i in the form of $N \times N$ Hermitian matrices and their superpartners (see Section 2 for details). An allegedly less fundamental but more definitive explanation of matrix origin ensues from the close relationship between matrix theory and supermembranes. X^i_{IJ} matrices appear to arise simply as Fourier modes of transverse membrane coordinates $X^i(\sigma_1, \sigma_2)$ in the light cone gauge.

The transition to non-compact 11D space – time, $R \to \infty$, must be characterized by the $N \to \infty$ limit which is necessary for the correct normalization of the longitudinal momentum. Therefore, the number of degrees of freedom in the matrix theory is really infinite. It has been shown in Ref. [1] that quantized excitations of matrix coordinates X^i describe an eleven-dimensional graviton and its superpartners as well as their scattering states. This, taken together with the intrinsic non-linearity of matrix theory, allows for the interactions to be taken into consideration without secondary quantization.

Similar ideas have been developed with respect to type IIB superstring theory [2]. The authors proposed to regard string instantons as fundamental objects and interpret matrices as Fourier modes of the string world sheet coordinates.

The objective of the present paper is to review applications of matrix models in M-theory and non-perturbative superstring theory. However, it appears appropriate to preface the discussion of supersymmetry with an outline of bosonic strings to which the matrix approach was successfully applied and became a standard tool to exceed the limits of perturbation theory. This method is based on discretization of the functional integral over the string world sheet.

1.1 Dynamical triangulation

It is well-known that in the first quantized formulation, summation over string trajectories actually reduces to a two-dimensional quantum gravity on the world sheet [4]. In this case, the string interaction corresponds to topologic fluctuations:

$$Z_{\mathrm{s}} = \sum_{n=0}^{\infty} g_{\mathrm{s}}^{2n-2} \int [\mathrm{d}h_{ab}][\mathrm{d}X^{\mu}] \times \exp\left[- \int_{\mathcal{M}_n} \mathrm{d}^2\sigma \sqrt{h} \left(\frac{1}{2} h^{ab} \partial_a X^{\nu} \partial_b X_{\nu} + \mu \right) \right], \quad (1.1)$$

where $h_{ab}(\sigma_1, \sigma_2)$ is the metric on the string world sheet and coordinates $X^{\nu}(\sigma_1, \sigma_2)$ ensure the embedding of the string surface into a D-dimensional Euclidean space. Parameter g_{s} is the string coupling constant and μ is the cosmological constant of two-dimensional gravity having a square-of-

§ The term M(atrix) theory can also be encountered in the literature.

mass dimension. The world sheet $\mathcal{M}_n$ in the nth order of the string perturbation theory is topologically a sphere with n handles. It should be recalled that the Euler characteristic χ of the surface and its genus n are related by the formula $\chi = 2 - 2n$.

A major property of string action is reparametrization invariance. The right choice of the integration measure in Eqn (1.1) is crucial. Collectively, gauge fixing and the Faddeev–Popov method allow for the description of string coordinate fluctuations and metrics to be reduced to the Liouville theory [14, 15] and integration over the moduli-space of the Riemann surface [16–18]. Topological fluctuations are taken into account in terms of excitation theory, order by order. Another approach altogether equivalent to the previous one is more or less analogous to lattice regularization in the field theory. Similar to the field theory, it does not require gauge fixing. Discretization of the continuum integral (1.1) consists in the replacement of integration over internal metrics by summation over string world sheet triangulations [19–21].

Let us consider a simplest example of two-dimensional quantum gravity with $D = 0$ in Eqn (1.1)†:

$$Z_{2\mathrm{D}} = \sum_{n=0}^{\infty} g_{\mathrm{s}}^{2n-2} \int [\mathrm{d}h_{ab}] \exp\left(- \int_{\mathcal{M}_n} \mathrm{d}^2\sigma \sqrt{h}\, \mu \right) . \tag{1.2}$$

Such a model describes fluctuations of the internal metric and topology on the 'zero-dimensional string' world sheet.

The essence of dynamical surface triangulation is the approximation of the surface of genus n by a combination of equilateral triangles. Each vertex need not necessarily host six triangles, as on the plane, because the surface in question may have internal curvature. The statistical sum (1.2) is approximated by

$$Z_{\mathrm{DT}} = \sum_{n=0}^{\infty} g_{\mathrm{s}}^{2n-2} \sum_{T_n} \exp\left(-\Lambda n_t \right) , \tag{1.3}$$

where T_n denotes a certain triangulation, i.e. a system of n_t triangles which form the surface of genus n. It is very important that the number of triangles n_t is not predetermined but represents a dynamical constant. Also, summation over triangulations T_n implies summation over n_t.

The exponential dependence on n_t is responsible for the convergence of the sum over triangulations, at least for sufficiently large values of the parameter Λ. However, with decreasing Λ, the sum can undergo divergence at a certain value of $\Lambda = \Lambda_{\mathrm{c}}$, due to the entropy factor, i.e. the number of different graphs with given n_t. Crucial in the approach based on dynamical triangulations is the fact that the total number of graphs of genus n formed by p triangles increases only with increasing p as [22]:

$$\sum_{T_n} \delta(n_t - p) = \exp(\Lambda_{\mathrm{c}}\, p) p^{-b_n} \left[1 + O(p^{-1}) \right] , \tag{1.4}$$

where Λ_{c} is independent of genus n; n-dependence is present only in the superscript b_n. It is worthwhile noting that the factorial p-dependence of the total number of graphs arises due to the summation over genera.

The continuum limit of sum (1.3) over triangulations is reached when

$$\Lambda \to \Lambda_{\mathrm{c}} + 0 . \tag{1.5}$$

In this case, contributions from all genera to, say, the string susceptibility

$$f = \frac{\partial^2}{\partial \Lambda^2} Z_{\mathrm{DT}} \sim \sum_n g_{\mathrm{s}}^{2n-2} (\Lambda - \Lambda_{\mathrm{c}})^{-\gamma_n} , \qquad \gamma_n = -b_n + 3 , \tag{1.6}$$

simultaneously become singular at $\Lambda \to \Lambda_{\mathrm{c}} + 0$. At this point, the discretized statistical sum Z_{DT} reproduces continuous $Z_{2\mathrm{D}}$. The critical index γ_n in Eqn (1.6) is referred to as the string susceptibility index; it is an important characteristic of the string. It is worthy of note that any complex surface can be taken into consideration by such a transition to the continuum limit in which n_t is a dynamical constant. This distinguishes the method in question from discretization with fixed n_t.

1.2 Matrix models

The sum (1.3) over random triangulations has a convenient analytical representation in the form of a matrix model. Let us consider a one-matrix model

$$Z = \int \mathrm{d}\Phi \, \exp\left[-N \operatorname{tr} V(\Phi) \right] , \tag{1.7}$$

where

$$\mathrm{d}\Phi = \prod_{i>j}^{N} \mathrm{d}\,\mathrm{Re}\,\Phi_{ij} \, \mathrm{d}\,\mathrm{Im}\,\Phi_{ij} \prod_{i=1}^{N} \mathrm{d}\Phi_{ii} \tag{1.8}$$

is the measure of integration over $N \times N$ Hermitian matrices and

$$V(\Phi) = \frac{1}{2}\Phi^2 - \frac{1}{3}\alpha\Phi^3 \tag{1.9}$$

is the cubic potential. Propagators and vertices in Feynman diagrams for the zero-dimensional field theory (1.7) may be depicted by double lines, one for each matrix index arising in the simplest Gaussian matrix integral

$$(2\pi)^{-N^2/2} \int \mathrm{d}\Phi \, \exp\left(-\frac{N}{2} \operatorname{tr} \Phi^2 \right) \Phi_{ij}\Phi_{kl} = \frac{1}{N} \delta_{il}\delta_{kj} . \tag{1.10}$$

A fragment of a typical diagram is shown in Fig 1. Each diagram may be considered to be composed of two-dimensional objects, e.g. polygons bound by index loops. These polygons are glued together thus giving rise to an oriented two-dimensional surface. This makes the difference from conventional Feynman diagrams. Lines drawn through the centers of polygons produce a diagram known as 'dual' to the original. It is made up of triangles by virtue of cubic interaction in the matrix model potential (1.9). The shape of the dual diagram is exactly that needed for dynamical triangulation of Riemann surfaces.

There is close relationship between the two approaches due to the fact that topological classification of diagrams in matrix theories is naturally related to an expansion in powers of $1/N^2$ as was first noticed by t'Hooft [23] for quantum

† Using the Gauss–Bonnet theorem, this formula may be rewritten in the standard-for-gravity form:

$$Z_{2\mathrm{D}} = \int \mathrm{d}h \, \exp\left\{ -\int \mathrm{d}^2\sigma\sqrt{h}\left[\mu - (4\pi G)^{-1} R \right] \right\} ,$$

where R is the scalar curvature and $G^{-1} = -\ln g_{\mathrm{s}}$.

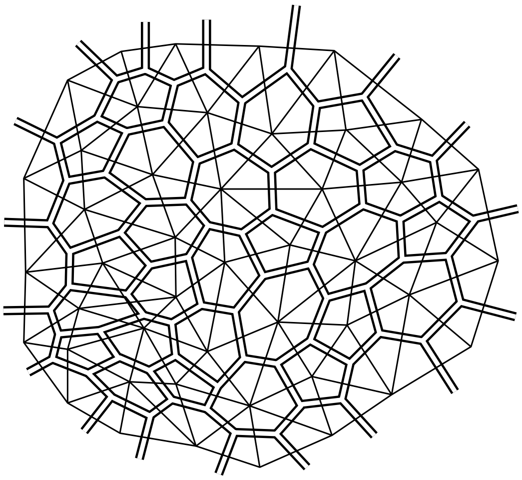

Figure 1. Fragment of planar diagram in the matrix model (double lines) and the corresponding dual diagram (single lines) which is a certain surface triangulation.

chromodynamics. Simple counting of powers of N indicates that the diagram with the topology of a Riemann surface of genus n has the order N^{2-2n}. Therefore, each term in $1/N$ expansion of free energy

$$F \equiv \ln Z = \sum_{n=0}^{\infty} F_n N^{2-2n} \tag{1.11}$$

is given by the sum of connected diagrams each topologically representing a sphere with n handles:

$$F_n = \sum_G \frac{\alpha^{v(G)}}{\#\mathrm{Aut}\,(G)}\,, \tag{1.12}$$

where $v(G)$ is the number of vertices in diagram G and $\#\mathrm{Aut}\,(G)$ is the order of its symmetry group. At $N \to \infty$, planar diagrams with spherical topology predominate.

To summarize, the transition from Feynman diagrams to dual diagrams establishes the relationship between $1/N$ expansion in the matrix model (1.7) and the expansion in genera (1.3) for dynamical triangulation of random surfaces since Eqn (1.12) can be regarded as the sum over all triangulations of the surface of genus n. It readily appears from the comparison of Eqns (1.3) and (1.7) that

$$F = Z_{\mathrm{DT}} \tag{1.13}$$

provided the size of matrices is $N = 1/g_s$ and the cubic coupling constant α of the matrix model is related to Λ by the formula $\alpha = \exp(-\Lambda)$. Equality (1.13) holds so far as the number of diagram G vertices, $v(G) = n_t$, and the number of triangles in the dual diagram coincide. The aforementioned divergence of the sum over genera in Eqn (1.3) is apparent in the matrix model as the divergence of the integral over $\mathrm{d}\Phi$ at a finite N.

Also, there is every reason to consider the matrix model (1.7) with the potential of a more general form

$$V(\Phi) = \sum_{j>1} \alpha_j \Phi^j\,, \tag{1.14}$$

which is a polynomial of arbitrary power. It is understandable from the analysis of the corresponding Feynman diagrams and dual diagrams that such a matrix model conforms to discretization of random surfaces by regular polygons with $j \geqslant 3$ vertices, the area of the j-gon being $(j-2)$ times that of the equilateral triangle in case of dynamical triangulation. This is the most general case of random surface discretization.

1.3 Continuum limit

In a previous section, we discussed how the sum (1.3) over triangulations describes a continuous string in the limit (1.5). In terms of the matrix model (1.7), this must correspond to a phase transition at a certain value of $\alpha_c = \exp(-\Lambda_c)$, as is common in lattice theories. At a finite N, such a phase transition is infeasible because the system has a finite number of degrees of freedom. However, the phase transition† in this matrix model is possible when the number of degrees of freedom becomes infinite. This was first demonstrated in Ref. [24]. Therefore, the continuum limit is reached at $N \to \infty$ and $\alpha \to \alpha_c$.

In the transition to the continuum limit, it must be assumed that

$$\Lambda = \mu a^2\,, \tag{1.15}$$

where a^2 is interpreted as the area assigned to the constituent triangles. When the area is small, summation over triangulations may be interpreted as a discretized variant of integration over internal metrics. Because the coupling constant of the matrix model is related to the priming cosmological constant μ by the expression $\alpha = (-\mu a^2)$, the Boltzmann weight of each triangulation is $\alpha^v = \exp(-\mu \times \text{area})$. The perturbation theory series undergoes divergence near the phase transition point, and the main contribution is made by diagrams with $v \sim (\alpha_c - \alpha)^{-1}$. If the α-dependence of a is such that $a^2(\alpha_c - \alpha)^{-1}$ remains finite at $a \to 0$, the sum over triangulations is dominated by surfaces with the finite area $\mathcal{S} = va^2$. Renormalization of the cosmological constant upon transition to the continuum limit consists in computation of the quadratic divergence $\mu = \Lambda_c/a^2 + \mu_R$.

It follows that the matrix model provides the analytical description of lattice regularization of the zero-dimensional string theory. The matrix integral contains information about all orders of the perturbative string theory. Evidently, it can be regarded as a non-perturbative definition of two-dimensional quantum gravity.

Well-developed methods of $1/N$ expansion for matrix models make it possible to give the non-perturbative description of topologic fluctuations in two-dimensional quantum gravity. Topological expansion appears to be defined by a single parameter, the renormalized string coupling constant λ, expressed through priming parameters by the relation

$$\lambda \equiv \left[(\alpha_c - \alpha)^{5/4} N\right]^{-1}\,. \tag{1.16}$$

Transition from discretized random surfaces to continuous ones occurs in the double scaling limit [25–27], with $\alpha \to \alpha_c$ and concurrently $N \to \infty$, while the quantity λ given by formula (1.16) remains finite. This approach was used to construct an expansion in genera in two-dimensional quantum gravity and demonstrate the important role of non-perturbative effects. Specifically, the renormalized theory is

† This phase transition for the one-matrix model (1.7) satisfies $\gamma_0 = -1/2$ and is characterized by singularity of (1.6)-derivative rather than that of string susceptibility itself. It is a third order phase transition.

generally speaking characterized by two constants; that is, in addition to λ, another parameter shows up which is inapparent in any finite order of the perturbation theory. Applications of matrix models to two-dimensional quantum gravity have been reviewed in several papers [28–30].

1.4 $D = 1$ barrier

One-matrix model (1.7) admits natural generalization to the case of several matrices; accordingly, the strings they describe are embedded into a space of greater dimensionality. The simplest example is a two-matrix model

$$Z = \int d\Phi_1\, d\Phi_2\, \exp\left[-N \operatorname{tr} V(\Phi_1) + N \operatorname{tr} \Phi_1\Phi_2 - N \operatorname{tr} V(\Phi_2)\right] \tag{1.17}$$

and its generalization to an (open) chain of q matrices

$$Z = \int \prod_{i=1}^{q} d\Phi_i\, \exp\left[-\sum_{i=1}^{q} N \operatorname{tr} V(\Phi_i) + N \sum_{i=1}^{q-1} \operatorname{tr} \Phi_i\Phi_{i+1}\right]. \tag{1.18}$$

Such matrix models describe the discretization of bosonic strings with $0 < D \leqslant 1$ or two-dimensional quantum gravity with matter†. For instance, the multi-matrix model (1.18) corresponds to $D = 1$ at $q \to \infty$.

At $q \to \infty$, index i may be substituted by a continuous variable x and the matrix chain with $q = \infty$ regarded as 'lattice regularization' of the statistical sum of one-dimensional matrix model

$$Z = \int \prod_x d\Phi(x) \times \exp\left\{-\int dx\, N \operatorname{tr}\left[\frac{1}{2}\dot{\Phi}^2 + \frac{m^2}{2}\Phi^2 + V_{\mathrm{int}}(\Phi)\right]\right\}, \tag{1.19}$$

where $\dot{\Phi} = d\Phi/dx$. Each perturbation theory diagram in this model has the corresponding Feynman integral which, in the coordinate representation, has the form

$$F_G = \int \prod_i \left(\frac{dX_i}{2m}\right) \exp\left(-m \sum_{\langle ij\rangle} |X_i - X_j|\right). \tag{1.20}$$

Therefore, for the cubic interaction potential, the contribution of diagrams of genus n to the free energy is

$$F_n = \sum_G \frac{\alpha^{v(G)}}{\#\mathrm{Aut}\,(G)} F_G\,. \tag{1.21}$$

Integration variables X_i can be identified with the values of the string coordinate X in diagram vertices. This accounts for the correspondence (in the continuum limit) between summation of diagrams and integration over X_i, on the one hand, and functional averaging over metrics and field X in (1.1), on the other. In this case, the one-dimensional propagator reproduces the kinetic term for the bosonic field‡.

† In the context of two-dimensional quantum gravity, the space dimension D is equal to the central charge of matter fields in the Virasoro algebra. In such a broad interpretation, D is not necessarily an integer.

‡ Strictly speaking, the regularization of Eqn (1.1) at $D = 1$ requires that the propagator be proportional to $\exp\left[-m(X - X')^2\right]$, in conformity with a matrix model non-local in time. However, analysis of such a model would be an unnecessary complication because in the continuum limit it is included in the same universality class as the local theory [31].

A similar approach may be employed to build up a matrix model equivalent to the string theory at any D [32]. At $D > 1$, however, one should consider the matrix field theory rather than zero-dimensional integrals and quantum mechanics. In this case, neither the exact solution of the problem at $N \to \infty$ nor the construction of $1/N$ expansion is possible. Thus, $D = 1$ is a sort of barrier in the bosonic string theory.

At $D > 1$, the qualitative behavior changes dramatically due not so much to technical complication of the theory as to instability of the perturbative string vacuum. The point is that the squared mass of the lowest string excitation is proportional to $1 - D$, and the ground state at $D > 1$ becomes a tachyon. The presence of infrared divergences leads to instability of the perturbative vacuum and is manifested, for example, in that the critical index γ_0 acquires an imaginary part. Numerous analytical and numerical studies of discretized random surfaces have demonstrated (see Ref. [33]) the absence of the string phase at $D > 1$. Due to infrared instability, the string world sheet undergoes degeneration to a quasi-one-dimensional object, a branching polymer; as a result, the bosonic string fails to describe a system with an infinite number of degrees of freedom. Such is the tachyon problem solution at $D > 1$: there is neither a tachyon over the new (stable) vacuum nor a string.

1.5 Generalization to superstrings

The aspect of the tachyon problem in the superstring context is a little bit different. In this case, the tachyon may be excluded from the system of physical states (at least order by order of the perturbation theory) with the aid of GSO-*projection*. This confirms that the superstring remains in the string phase even at $D > 1$, in compliance with the fact, well-known from statistical mechanics, that fermions are able to smooth the dynamical behavior of the system.

Further development of the random triangulation method for superstrings encounters some difficulties related in the first place to supersymmetry discretization. Attempts at supersymmetric generalization of Riemann surfaces and matrix models date from Ref. [34], but a certain degree of progress was reported only in Ref. [35] for the simplest case of two-dimensional supergravity although its formulation in the form of a supersymmetric matrix model has never been proposed.

In autumn 1996, another approach to the non-perturbative description of M-theory and ten-dimensional superstrings was proposed based on different ideas. Nevertheless, the fundamental objects used in this approach are also specific forms of matrix models, and these matrices have a space–time interpretation.

Section 2 of the present review comprises an introduction to the BFSS matrix theory [1].

Section 3 is concerned with the zero-dimensional matrix model suggested by Ishibashi, Kawai, Kitazawa, and Tsuchiya (IKKT) [2] to directly describe IIB superstrings.

A modified IKKT model [3] which makes it possible to reproduce a matrix analog of the Nambu–Goto action at the quantum level is discussed in Section 4.

The Glossary at the end includes some terms frequently encountered in the present paper but unexplained in the text (see also Ref. [36]).

2. Banks – Fishler – Shenker – Susskind matrix theory [1]

The BFSS matrix theory [1] for the dynamical description of M-theory looks very much like the first quantized formulation of the theory of superparticles and superstrings. The most important difference lies in the possibility of making the equation of motion for superparticles and superstrings linear by choosing the gauge. Due to this, the action in the first quantized formalism describes the distribution of free particles or free string states. The interaction can be taken into account by considering Feynman diagrams for particles or world sheets with non-trivial topology for strings. Equations of motion in the matrix theory are essentially non-linear; therefore, even the first quantized theory turns out to be interacting. According to the BFSS hypothesis, these interactions settle the problem, and secondary quantization is superfluous.

It was suggested in Ref. [1] to consider eleven-dimensional M-theory in a special reference frame sometimes called the infinite momentum frame or light cone frame. Of course, a part of the eleven-dimensional Lorentz covariance in this reference frame is no longer explicit and needs verification. However, all degrees of freedom in the matrix theory can be explicitly described only in the light cone gauge. In the first quantized theory of particles and strings, reparametrization invariance of action allows one of the light cone coordinates to be identified with time. The second coordinate is fixed by the links resulting from the gauge choice. This makes the action dependent only on transverse coordinates. A covariant formulation of the matrix theory is yet unavailable, and the BFSS model is interpreted as a theory with a fixed light cone gauge; therefore, the action has no longitudinal coordinates, by definition.

Transverse coordinates of eleven-dimensional M-theory and their superpartners are described in the matrix theory by supersymmetric quantum mechanics of $N \times N$ Hermitian matrices. The matrix size N is identified with a positive integer which defines the total longitudinal momentum of the system. If eleven-dimensional M-theory is to be described, the matrix size must tend to infinity in order to get a correct normalization of the longitudinal momentum. Such is the mechanism of origin of the infinite number of degrees of freedom in the matrix theory. Space – time coordinates arise in the matrix theory as parameters characterizing degenerate potential energy minima.

2.1 Infinite momentum frame

The eleven coordinates $x^{\mu} = (x^0, x^i, x^{10})$ of M-theory are actually nine *transverse* coordinates x^i, $i = 1, \ldots, 9$ (or $x^{\perp}$) and two light cone coordinates $t \equiv x^+ = (x^0 + x^{10})/\sqrt{2}$ and $x^- = (x^0 - x^{10})/\sqrt{2}$, one standing for time and the other being regarded as spatial. Conjugate variables are identical with the energy $E \equiv p_+$ and *longitudinal* momentum p_- respectively. An advantage of the light cone frame is in that both the energy and longitudinal momentum on the mass surface are positive. The metric in the light cone coordinates has the form $p^2 = 2p_+p_- - p_{\perp}^2$ and the dependence of the massless particle energy on the transverse momentum is given by the non-relativistic dispersion law

$$p_+ = \frac{p_{\perp}^2}{2p_-} . \qquad (2.1)$$

The longitudinal coordinate x^- is supposed to be *compact*:

$$x^- \equiv x^- + 2\pi R .$$

The compactification radius R serves as an infrared regularization parameter which should be pushed to infinity at the end of calculations. By virtue of compactness, the longitudinal momentum is quantized in units of R^{-1}:

$$p_- = \frac{N}{R} \qquad (N = 0, 1, 2, \ldots) . \qquad (2.2)$$

For a system with finite longitudinal momentum in the non-compact eleven-dimensional space, N must tend to infinity to ensure that relation (2.2) remains fixed. It should be emphasized that there is no need to take into account states with negative p_- values which greatly facilitates the construction of the fundamental M-theory Lagrangian in the infinite momentum frame.

2.2 11D and 10D

Compactification of M-theory on a circle leads to the theory of type IIA superstrings. Although the compactification radius is used solely as an infrared regularization parameter, all of its quantities are essentially ten-dimensional by origin. It therefore seems appropriate to briefly discuss how the parameters and degrees of freedom are related in ten and eleven dimensions.

M-theory is characterized by the Planck length l_{p} and the string theory by the dimensionless coupling constant g_{s} and tension

$$T = \frac{1}{2\pi\alpha'} . \qquad (2.3)$$

It is sometimes convenient to use the length dimension parameter $l_{\mathrm{s}} \equiv \sqrt{2\alpha'}$ instead of tension. The compactification radius and string length are related to 11D-quantities by the formulae

$$R = g_{\mathrm{s}}^{2/3} l_{\mathrm{p}} , \qquad l_{\mathrm{s}} = g_{\mathrm{s}}^{-1/3} l_{\mathrm{p}} . \qquad (2.4)$$

These relations suggest that M-theory may be regarded as the strong coupling limit of the type IIA string theory. At $g_{\mathrm{s}} \to \infty$, the compactification radius tends to infinity and the theory actually has eleven dimensions.

Massless states in M-theory form a fundamental supermultiplet of 256 states which correspond to an eleven-dimensional graviton and its superpartners, the gravitino and antisymmetric tensor field of rank 3. The number 256 is easy to find since the 11D supercharge has 32 components one half of which trivially act on the massless states [37]. The irreducible representation of the algebra of the 16 remaining supercharge components has the dimension $2^{16/2} = 256$.

Following M-theory compactification, each state from the gravity supermultiplet gives rise to an infinite set of Kaluza – Klein modes in ten dimensions. There is no problem as far as zero-modes are concerned for they have corresponding massless degrees of freedom in the type IIA superstring theory, and it is well-known that the low-energy effective theory describing them, 10D non-chiral $\mathcal{N} = 2$ supergravity, can be obtained by dimensional reduction from eleven dimensions.

The interpretation of massive modes is much less straightforward. The respective states in the superstring theory are essentially non-perturbative because their masses in string units are inversely proportional to the coupling constant $N/R \longrightarrow N/g_s$. In the ten-dimensional context, the integer N looks like a persistent charge whose gauge field arises from the R – R superstring sector. None of the perturbative string states carries RR-charges.

However, the superstring theory contains soliton-like states with necessary properties, D0-branes. Elementary D0-branes carry unit RR-charge and satisfy the BPS (Bogomol'ny – Prasad – Sommerfield) condition. This accounts for the equality of masses and charges of D0-brane bound states up to a coefficient. For the same reason, they form a supermultiplet of 256 rather than 256^2 states as is the case with ordinary massive states in $\mathcal{N} = 2$ $D = 10$ supergravity (e.g. superstring massive modes).

To sum up, an eleven-dimensional graviton with longitudinal momentum $p_- = 1/R$ looks like a D0-brane in the ten-dimensional theory. The bound state of N D0-branes corresponds to a graviton with momentum $p_- = N/R$.

2.3 How matrices arise

There are various soliton-like states in the superstring theory. In the general case, they have the form of extended objects, p-branes. A p-brane propagating in space – time covers a $(p+1)$-dimensional world volume. Ten-dimensional gravities offer solutions for this type of classical equations of motion (see Ref. [38]). They have finite tension (mass per unit spatial volume) and, unlike perturbative string states, carry RR-charges. Reference [39] showed the possibility of describing solitonic states with RR-charges in the framework of the string theory as D(irichlet)-branes. Specifically, such states are exemplified by D0-branes (D-particles) corresponding to the Kaluza – Klein modes of eleven-dimensional graviton.

The dynamical description of D-branes is organized in the following way. In the presence of a RR-source, fundamental closed strings may undergo disjunction and conversion to open strings rigidly bound to a D-brane. The ends of the strings can move freely along the D-brane but can not detach from it. Mathematically, this corresponds to the superposition of Dirichlet boundary conditions on the string coordinates which are orthogonal to the D-brane world volume and Neumann conditions on the tangential coordinates. Quantum fluctuations of the string are described in the usual way. It turns out that the Dirichlet boundary conditions are compatible with supersymmetry and the GSO-projection for even-dimensional D-branes in the type IIA superstring theory and odd-dimensional ones in the IIB theory.

As usual, when the characteristic scale of a problem is much larger than the Planck length, the exact string description has no sense because the infinite system of massive string states makes a negligibly small contribution, and it is possible to be confined to massless or very light degrees of freedom. Low-energy interactions of massless degrees of freedom in such an approximation can be described by an effective field theory with a local Lagrangian. Low-energy excitations of Dirichlet strings spread only along a D-brane because the string end points are rigidly attached to it. Therefore, in the low-energy approximation, a D-brane is described by the effective field theory on its world volume.

Dirichlet strings differ from 10D open superstrings only by boundary conditions whereas quantum fluctuations in both theories are identical. For this reason, the effective Lagrangian for a D-brane is the same as for open strings. Gauge fields and their superpartners are massless degrees of freedom of an open string. In other words, the low-energy effective theory on the D-brane world volume is derived from the 10D U(1)-gauge theory by dimensional reduction, i.e. by disregarding dependences of all fields on coordinates orthogonal to the world volume.

Let us assume for certainty that D-brane lives on a hyperplane $x^{p+1} = 0, \ldots, x^9 = 0$ and its world volume is parametrized by coordinates $\xi^0, \ldots, \xi^p$ as shown in Fig. 2. The components of the vector potential $A^a(\xi)$ $(a = 0, \ldots, p)$ tangential to the D-brane describe internal gauge fields while the remaining $(9-p)$ components look like scalars from the standpoint of the field theory on the world volume. It can be shown that they play the role of coordinates describing D-brane fluctuations [40]:

$$X^i(\xi) = 2\pi\alpha' A^i(\xi) \qquad (i = p+1, \ldots, 9)\,. \tag{2.5}$$

The state carrying an RR-charge equal to N can be represented as the superposition of N D-branes. Let us consider, for example, N parallel static D-branes in hyperplanes $x^{p+1} = X_I^{p+1}, \ldots, x^9 = X_I^9$, $I = 1, \ldots, N$. Each D-brane has a corresponding U(1) field, and the gauge group of the low-energy effective theory is $[\mathrm{U}(1)]^N$. The D-brane coordinates X_I^i can be identified with average scalar field values (2.5).

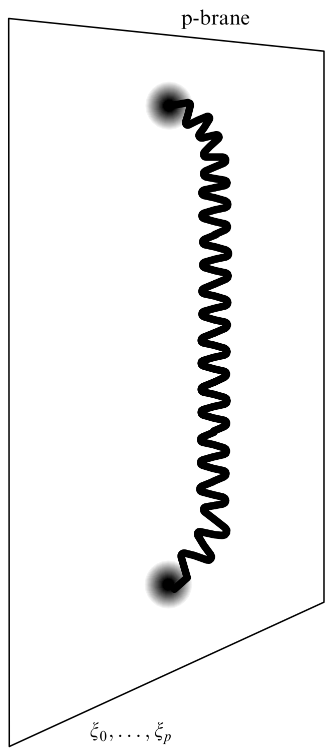

Figure 2. Dp-brane (depicted as a hyperplane parametrized by coordinates $\xi_0, \ldots, \xi_p$) and fundamental (open) string with end points attached to the D-brane.

In fact, the number of string degrees of freedom in the presence of N D-branes increases N^2 rather than N times just because fundamental strings may begin on one D-brane and end on another as shown in Fig. 3 for the case of $N = 2$. However, the strings possess tension, and the energy

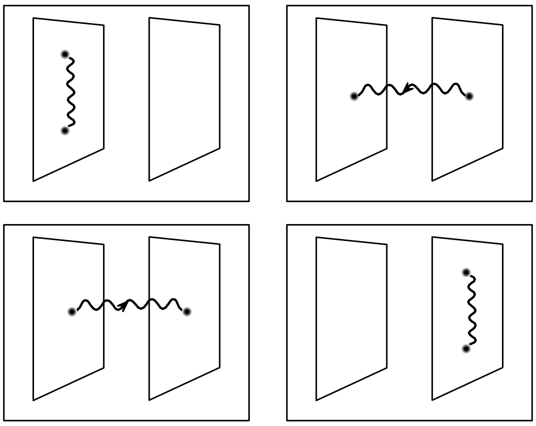

Figure 3. Emergence of matrices for the case of a bound state of two parallel D-branes ($N = 2$). A fundamental string can start and end either on the same or different branes. Since the string is oriented, there are four massless vector states for the case when D-branes practically lie on top of each other. They form a U(2)-group representation.

necessary for the creation of a string connecting different branes must be at least proportional to the distance between them. Hence, the corresponding field is massive:

$$M_{IJ} \sim T|X_I - X_J|\,, \tag{2.6}$$

if the branes are separated in space.

However, the masses of strings connecting D-branes become small if the branes are close enough to each other. This necessitates taking into account the relevant degrees of freedom in the low-energy theory, when examining bound states. In an extreme case of coincident D-branes, all N^2 states corresponding to different strings are massless, and the gauge symmetry increases to the U(N) group [41].

Such a picture of D-brane interactions is mathematically described in the following way. In the presence of N D-branes, each string is a match for integers I and J numbering the D-branes on which the string arises and ends. The situation in which discrete indices are ascribed to string ends is familiar; it is in this way that non-Abelian gauge symmetry is introduced into the open string theory with the help of Chan – Paton factors. In the low-energy approximation, superstrings with Chan – Paton factors are described by the Yang – Mills supersymmetric theory. When strings connecting all possible pairs of D-branes are taken into account, fields in the low-energy effective theory become Hermitian matrices with I and J indices running from one to N. The Lagrangian of the effective theory is obtained by simple reduction of the 10D supersymmetric U(N)-gauge theory to $(p+1)$-dimensional space:

$$S_{\mathrm{eff}} = \frac{1}{g_{\mathrm{s}}^2}\int \mathrm{d}^{p+1}\xi\,\mathrm{tr}\left\{-\frac{1}{4}F_{ab}^2 + \frac{1}{2}(\mathrm{D}_a X^i)^2 + \frac{1}{4}[X^i, X^j]^2 + \frac{\mathrm{i}}{2}\bar{\psi}\Gamma^a \mathrm{D}_a\psi + \frac{1}{2}\bar{\psi}\Gamma^i[X_i, \psi]\right\}. \tag{2.7}$$

When the charge RR $= 1$, the scalar fields X^i play the part of D-brane transverse coordinates. For a D-brane with charge N, the scalar fields are $N \times N$ Hermitian matrices. The space – time interpretation of such 'non-commutative coordinates' is based on the fact that the potential for scalar fields (arising from the term proportional to F_{ij}^2 in the action of the 10D Yang – Mills theory) has flat directions. Indeed, the potential vanishes if the X^i matrices commute and at the same time are amenable to diagonalization:

$$X^i = \begin{pmatrix} X_1^i & & \\ & \ddots & \\ & & X_N^i \end{pmatrix}. \tag{2.8}$$

When the scalar fields acquire vacuum averages of the form (2.8)†, the gauge symmetry is spontaneously broken to $[\mathrm{U}(1)]^N$. The diagonal components X_I^i of X^i matrices should be identified with the coordinates of N D-branes. In this case, off-diagonal components of scalar and gauge fields acquire masses of order (2.6), as expected.

2.4 Matrix theory Lagrangian

According to the BFSS hypothesis (see also Ref. [42]), the sole M-theory fundamental degrees of freedom in the light cone gauge are D0-branes. This means that *all* physical states in M-theory are built of D0-branes. The number of D0-branes is N or, in other words, their RR-charge must tend to infinity because in terms of M-theory it must be equal to the longitudinal momentum in units of R^{-1}. Similarly, R should be taken to infinity leaving the longitudinal momentum $p_- = N/R$ fixed. The matrix theory action is extracted from the low-energy effective action (2.7) for D0-branes by field restretching and transition to 11D Planck units.

The fundamental Lagrangian of the matrix theory has the form

$$L = \mathrm{tr}\left\{\frac{1}{2R}(\mathrm{D}_t X^i)^2 + \frac{R}{4}[X^i, X^j]^2 + \theta \mathrm{D}_t\theta + \mathrm{i}R\,\theta\gamma_i[X^i, \theta]\right\}, \tag{2.9}$$

where $X^i(t)$ stands for the nine $N \times N$ Hermitian bosonic matrices while the 16-component nine-dimensional spinor $\theta^\alpha(t)$ ($\alpha = 1, \ldots, 16$) is composed of $N \times N$ Hermitian fermionic matrices. The nine-dimensional Dirac γ_i matrices satisfy the standard anticommutation relations

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}\,, \tag{2.10}$$

and Planck units, $l_{\mathrm{p}} \equiv 1$, are used.

The covariant derivative D_t in Eqn (2.9) is defined as

$$\mathrm{D}_t = \frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}[A, \cdot\,]\,. \tag{2.11}$$

The gauge field A is not a dynamical degree of freedom. In principle, it can be eliminated by the gauge choice, but in this case the action is no longer explicitly supersymmetric since supersymmetry transformations also affect A:

$$\delta X^i = \sqrt{R}\,\epsilon\gamma^i\theta\,, \tag{2.12}$$

$$\delta\theta = \left(\frac{\mathrm{i}\sqrt{R}}{4}[X^i, X^j]\gamma_{ij} - \frac{1}{2\sqrt{R}}\mathrm{D}_t X^i\gamma_i\right)\epsilon + \epsilon'\,, \tag{2.13}$$

$$\delta A = R\sqrt{R}\,\epsilon\theta\,, \tag{2.14}$$

ϵ and ϵ' spinors give rise to two sets of 16 independent transformation parameters unrelated to the time t.

† Strictly speaking, degenerate minima of the potential correspond to different vacua only in the field theory, i.e. at $p \geqslant 1$. At $p = 1$, it is not quite correct either because of infrared divergences; see Ref. [41] for details.

It is important to keep in mind the critical difference between the effective action (2.7) for D0-branes and the matrix theory Lagrangian (2.9). While the former is an effective action and has sense only at the tree level, the latter is the *quantum* theory Lagrangian for which quantum loop effects are essential.

At first sight, matrix quantum mechanics with Lagrangian (2.9) seems too simple to describe eleven-dimensional quantum supergravity. However, in the $N \to \infty$ limit when the infinite number of degrees of freedom becomes a reality, matrix theory dynamics turns out to be sufficiently complicated. Specifically, phase space of the matrix theory at $N = \infty$ describes a very large collection of states. This can be illustrated by the following example. At $N = \infty$, the difference between matrices and operators disappear. Let us assume X^9 to be an operator (infinite matrix) of the special form $X^9 = -\mathrm{i}R_9(\partial/\partial\sigma) - \mathcal{A}$, where σ is the variable which changes in the range from 0 to 2π. Matrix $\mathcal{A}$ as well as X^i with $i < 9$ depends only on σ. Substitution of such an ansatz into formula (2.9) converts it into the Lagrangian of the two-dimensional Yang–Mills theory with $\mathcal{N} = 8$ supersymmetry. This construction is consistent with the compactification of the 9th dimension on a circle of radius R_9 which is used to describe the string limit of the matrix theory [43].

X^i_{IJ} and θ^α_{IJ} $(I, J = 1, \ldots, N)$ matrices in the matrix theory play roughly the same role as world sheet coordinates $X^i(\sigma, t)$, $\theta^\alpha(\sigma, t)$ in the string theory excepting one important difference, i.e. that the string Lagrangian in the light cone gauge describes the distribution of free string states whereas the matrix theory Lagrangian contains *all* dynamical information, in compliance with the BFSS hypothesis; specifically, it describes scattering states and their interactions.

True, the physical sense of non-commuting X^i quantities, unlike that of string coordinates, is not immediately apparent. There are two possibilities of interpreting X^i matrices in terms of space–time, one ensuing from ten-dimensional interpretation of the matrix theory Hamiltonian and the other borrowed from the theory of supermembranes.

2.5 Matrix quantum mechanics

The Hamiltonian corresponding to Lagrangian (2.9) has the form

$$H = R\,\mathrm{tr}\left\{\frac{1}{2}\,\Pi_i\Pi_i - \frac{1}{4}[X^i, X^j]^2 - \mathrm{i}\theta\gamma_i[X^i, \theta]\right\}, \qquad (2.15)$$

where Π_i and X^i give a pair of canonically conjugate variables. As usual, θ^α_{IJ} satisfy anticommutation relations, one half of them playing the role of momenta and the other that of coordinates.

Group $\mathrm{U}(N)$ can be represented in the form of the direct product $\mathrm{U}(1) \otimes \mathrm{SU}(N)$. The Abelian part of X^i should be identified with the center-of-mass (cm) coordinate

$$x^i(\mathrm{cm}) = \frac{1}{N}\,\mathrm{tr}\,X^i\,. \qquad (2.16)$$

Such an identification is quite natural both from the viewpoint of D0-branes [see Eqn (2.8)] and in the membrane interpretation of the matrix theory action discussed in Section 2.6.

In compliance with equality $p_- = N/R$, the transverse center-of-mass momentum

$$p_i(\mathrm{cm}) = \mathrm{tr}\,\Pi_i = \frac{N}{R}\,\dot{x}_i(\mathrm{cm})\,, \qquad (2.17)$$

is related to velocity by the conventional formula

$$\frac{1}{p_-}\,p_i(\mathrm{cm}) = \dot{x}_i(\mathrm{cm})\,. \qquad (2.18)$$

Momentum dependence of the center-of-mass energy ensues from Eqn (2.15):

$$p_+ \equiv E = \frac{R}{N}\,\frac{p_\perp^2(\mathrm{cm})}{2} = \frac{p_\perp^2(\mathrm{cm})}{2p_-}\,. \qquad (2.19)$$

The last expression fairly well coincides with Eqn (2.1) which holds for a massless particle in eleven dimensions.

It is worthwhile noting that the Hamiltonian is totally independent of fermionic superpartners of the center-of-mass coordinates. Therefore, a state with a definite momentum is degenerate and forms representation of the algebra of 16 $\mathrm{tr}\,\theta^\alpha$ components. The dimension of this representation is

$$2^{16/2} = 2^8 = 256\,.$$

This exactly corresponds to the 256 states of an eleven-dimensional graviton. For excellent agreement with supergravity, the $\mathrm{SU}(N)$-part of the Hamiltonian must have a zero-energy normalizable vacuum state. The delicate question of the reality of such a state has been discussed by several authors [44, 45].

It can be inferred that the vacuum state of the matrix theory Hamiltonian is consistent with a supergravity of longitudinal momentum N/R. Excited states have a continuous spectrum originating from zero†. It arises as a sequel of flat directions of the potential for the X^i discussed in Section 2.3‡. The existence of the continuous spectrum in matrix theory appears to be quite natural because it is necessary for the description of graviton scattering states. It should be noted that only a small part of the continuous spectrum survives in the $R \to \infty$, $N \to \infty$ limit. The compactification radius R enters the Hamiltonian only as the common multiplier; therefore, the energy eigenvalues have the form $R\mathcal{E}$, where $\mathcal{E}$ is independent of R. Only states with $\mathcal{E} \sim 1/N$ have finite energy of order $R/N = 1/p_-$ in the non-compact eleven-dimensional limit.

Supergraviton scattering states can be described in the matrix theory as block-diagonal matrices. For example, it is natural to compare superpositions of two matrix gravitons

$$X^i = \begin{pmatrix} X^i_{(1)} & 0 \\ 0 & X^i_{(2)} \end{pmatrix}, \qquad (2.20)$$

where the $X^i_{(I)}$ blocks are of size $N_I \times N_I$. The longitudinal and transverse graviton momenta are N_I/R and $\mathrm{tr}\,\dot{X}^i_{(I)}/R$ respectively. The standard background field method is suitable for the investigation of intergraviton interactions in the matrix theory. Results of such calculations are in excellent agreement with eleven-dimensional supergravity [1, 47, 48].

2.6 Relation to membranes

The infinite matrix size limit allows for the relationship between matrix theory and supermembranes to be established. Such a correspondence is underlain by somewhat

† Positive-definite Hamiltonian (2.15) ensues from its equality to the square of the supercharge.

‡ The existence of the continuous spectrum is a non-trivial fact despite the presence of flat directions; it also follows from supersymmetry [46].

formal but sufficiently general considerations playing an important role in superstring matrix models discussed below. The present section concerns such considerations for membranes with toroidal topology.

The idea is to expand dynamical variables of the BFSS model, i.e. X^i and θ^α matrices, around the special basis in gl(N). It is introduced in the following way [49]. At first, two unitary matrices g and h which satisfy the relations

$$hg = \exp\left(\frac{2\pi i}{N}\right) gh\,, \tag{2.21}$$

$$h^N = 1 = g^N \tag{2.22}$$

are considered. In the representation in which matrix g is diagonal, h acts as the shift operator:

$$g|n\rangle = \exp\left(\frac{2\pi i n}{N}\right)|n\rangle\,, \tag{2.23}$$

$$h|n\rangle = |n-1\rangle\,. \tag{2.24}$$

In this case, $|0\rangle \equiv |N\rangle$. Any matrix of size $N \times N$ can be represented as a function of g and h:

$$Z = \sum_{n,m=0}^{N-1} Z_{n,m}\, g^m h^n\,. \tag{2.25}$$

Upon transition to the $N \to \infty$ limit, the space in which g and h act becomes infinitely dimensional and equality (2.22) is no longer essential while the permutation relation (2.21) can be realized by expressing g and h through coordinate and momentum operators:

$$g = \exp iq\,, \qquad h = \exp ip\,, \tag{2.26}$$

$$[q, p] = \frac{2\pi i}{N}\,. \tag{2.27}$$

Relation (2.27) suggests that parameter N plays a dual role. It defines the matrix size while $2\pi/N$ has the sense of the Planck constant. Hence, $N \to \infty$ is a quasi-classical limit. In the quasi-classical limit, the q and p operators turn into c-numbers which makes it natural to apply correspondence to the function of two variables in the matrix of the general form (2.25) at $N \to \infty$:

$$Z \Rightarrow Z(q,p) = \sum_{n,m=0}^{N/2} Z_{n,m} \exp(imq + inp) + \sum_{\substack{n,m=0\\(n,m)\neq(0,0)}}^{N/2} Z_{N-n,N-m} \exp(-imq - inp)\,. \tag{2.28}$$

The function $Z(q,p)$ of both variables is periodic and is thus defined on the torus.

In the quasi-classical approximation, the commutators are replaced by Poisson brackets

$$[X, Y] \Rightarrow \frac{2\pi i}{N}(\partial_q X \partial_p Y - \partial_p X \partial_q Y)\,, \tag{2.29}$$

and the matrix trace changes to an integral over phase space:

$$\operatorname{tr} Z = N \int_0^{2\pi} \frac{dp\, dq}{(2\pi)^2}\, Z(p,q)\,. \tag{2.30}$$

True, the latter equality contains no approximation.

The above calculations with the substitution of infinite matrices by the functions of two variables are equally applicable to the matrix theory Lagrangian (2.9). In the gauge $A = 0$, it turns into expression

$$L = \int \frac{dp\, dq}{(2\pi)^2}\left[\frac{p_-}{2}\dot{X}^2 - \frac{\pi^2}{p_-}(\partial_q X^i \partial_p X^j - \partial_p X^i \partial_q X^j)^2 + p_-\theta\dot{\theta} - 2\pi\theta\gamma_i(\partial_q X^i \partial_p \theta - \partial_p X^i \partial_q \theta)\right], \tag{2.31}$$

which coincides with the supermembrane Lagrangian in the light cone gauge [50].

The relationship between matrix theory and supermembranes offers an elegant solution of the problem of the supermembrane continuous spectrum [46]. The continuous spectrum of supermembrane theory looks a serious problem if it is to be interpreted in the spirit of string theory. By contrast, the presence of the continuous spectrum in the matrix theory is quite natural because it corresponds to the graviton scattering states discussed in the previous section.

The description of membranes in the matrix theory may be approached from the other side by simply identifying them with classical solutions without regard for the origin of Lagrangian (2.31). Membranes in the matrix theory may be interpreted as statistical solutions of classical equations of motion

$$[X^i, [X^j, X^i]] = 0\,. \tag{2.32}$$

A membrane in a (x^8, x^9) plane has the form [1]

$$X^8 = R_8\sqrt{N}p\,, \qquad X^9 = R_9\sqrt{N}q\,, \qquad \text{all the remaining} \quad X^i = 0\,. \tag{2.33}$$

Here, p and q are infinite matrices (operators) satisfying canonical commutation relations and R_8 and R_9 play the role of compactification radii; they must be large enough to allow for the periodicity in x^8, x^9 to be neglected. The commutator $[X^8, X^9]$ is equal to the c-number; therefore, (2.33) is actually the solution of Eqns (2.32).

The long-range interaction between these membrane configurations (and more general classical solutions corresponding to even-dimensional Dp-branes†) has been investigated in the framework of matrix theory [51 – 55] and compared with the results for IIA superstrings. Details of these calculations are not presented here since they are similar to those for type IIB strings below.

3. Ishibashi – Kawai – Kitazawa – Tsuchiya matrix model [2]

The matrix theory has been suggested as a model providing a full quantum-mechanical description of M-theory in eleven dimensions. Strings arise in the matrix theory only after compactification.

In another approach, the IKKT hypothesis [2], the matrix model is directly connected with superstrings. The action of this model is derived from the ten-dimensional supersymmetric gauge theory by reduction to a point, i.e. by considering fields altogether independent of coordinates. According to the IKKT hypothesis, this zero-dimensional matrix model can be considered in the infinite matrix size limit

† Strictly speaking, with a presumably magnetic field living on them.

as a non-perturbative definition of the type IIB superstring theory.

The IKKT matrix model has much in common with BFSS matrix theory. Specifically, it is closely related to the D-brane concept. Unlike type IIA theory, the string IIB theory deals with odd-dimensional D-branes. The minimal dimension, $p = -1$, is intrinsic in D-instantons for which the low-energy effective action is extracted by reducing the Yang–Mills theory to a point. This explains why D-instantons play the role of elementary degrees of freedom in the IKKT model by analogy with the role of D0-branes in the matrix theory. Moreover, fundamental extended objects in the IKKT model are strings instead of membranes, due to diminishing the dimensionality by one.

The IKKT approach is based on the same line of reasoning which connects matrix theory with supermembranes. As a result of the calculations described in Section 2.6 for membranes, the action of the IKKT model is reduced to that of the Green–Schwarz superstring as formulated by Schild or, vice versa, the Lagrangian of the IKKT model may be regarded as a matrix analog of the Schild action for a superstring. The Schild formulation is equivalent, at least at the level of classical equations of motion, to more familiar approaches to string theory based on the Nambu–Goto action or two-dimensional gravity. Selected properties of the Schild action are discussed in the next section.

3.1 Schild action

Usually, the starting point for a string theory is a Nambu–Goto-type action. Geometrically, it is the area of its world sheet. By introducing auxiliary fields, it is possible to formulate the variation principle of the string theory in several different ways. All of them are equivalent to the Nambu–Goto formulation, at least at the classical level. The standard approach implies introducing an auxiliary metric on the world sheet [14, 56–58] although alternative options are available, e.g. formulation in terms of a Schild action.

The Schild action contains an auxiliary field, i.e. a positive-definite function $\sqrt{g}(\sigma_0, \sigma_1)$, which enters the action without derivatives:

$$S_{\text{Schild}} = \int \mathrm{d}^2\sigma \left(-\frac{\alpha}{4\sqrt{g}} \{X^\mu, X^\nu\}^2 - \frac{\mathrm{i}}{2}\, \bar{\Psi}\Gamma^\mu \{X_\mu, \Psi\} + \beta\sqrt{g} \right). \tag{3.1}$$

Poisson brackets { , } are defined as usual:

$$\{X, Y\} \equiv \varepsilon^{ab} \partial_a X \partial_b Y. \tag{3.2}$$

Indices μ and ν running from 0 to 9 are raised or lowered using the Minkowski metric $\eta^{\mu\nu} = (+\,-\ldots-)$. The Γ^μ matrices satisfy anticommutation relations

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}. \tag{3.3}$$

The fermionic field $\Psi_\alpha(\sigma_1, \sigma_2)$ is a scalar on the world sheet and a Majorana–Weyl spinor in space–time, i.e. $\bar{\Psi} = \Psi\Gamma^0$ and $\Gamma_{11}\Psi = \Psi$. It should be noted that the metric needs to be pseudo-Euclidean if Majorana–Weyl spinors are to exist in ten dimensions. In principle, α and β constants may be included in field normalization, but it is convenient to retain them further.

The auxiliary field $\sqrt{g}$ can be disposed of by classical equations of motion. This will result only in a change of the bosonic part of the action because its fermionic term does not depend on $\sqrt{g}$. Variation of Eqn (3.1) over $\sqrt{g}$ yields

$$\sqrt{g} = \frac{1}{2}\sqrt{\frac{\alpha}{\beta}}\sqrt{-\{X^\mu, X^\nu\}^2}. \tag{3.4}$$

In this equality, the expression under the radical is, up to two, the determinant of a metric induced on the string world sheet:

$$\{X^\mu, X^\nu\}^2 = (\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu)^2 = 2 \det_{ab} G_{ab}, \tag{3.5}$$

$$G_{ab} = \partial_a X^\mu \partial_b X_\mu. \tag{3.6}$$

It should be emphasized that the radicand in Eqn (3.4) cannot be negative. Otherwise, the induced metric would be positive-definite which is possible only for a string propagating faster than light.

Substituting Eqn (3.4) into (3.1) and changing the fermionic field normalization leads to the Nambu–Goto-type action for a Green–Schwarz string with fixed κ-symmetry [2]:

$$S_{\text{NG}} = \int \mathrm{d}^2\sigma \left(\sqrt{2\alpha\beta}\sqrt{-\det_{ab} G_{ab}} - 2\mathrm{i}\varepsilon^{ab}\partial_a X^\mu \bar{\Psi}\Gamma_\mu \partial_b \Psi \right). \tag{3.7}$$

Quantum theory requires integration over the $\sqrt{g}$ field. It has been argued (see Ref. [59]) that the functional integral for a string with the Schild action is equivalent to that in the conventional Polyakov formulation provided the choice of the integration measure is correct and the conformal anomaly is canceled, i.e. in the critical dimension $D = 10$.

Action (3.1) shows space–time $\mathcal{N} = 2$ supersymmetry:

$$\delta\Psi_\alpha = -\frac{1}{2}\sqrt{g}\{X_\mu, X_\nu\}(\gamma^{\mu\nu}\epsilon)_\alpha + \xi_\alpha,$$
$$\delta X^\mu = \mathrm{i}\bar{\epsilon}\gamma^\mu\Psi. \tag{3.8}$$

The parameters of this transformation, ξ and ϵ, are real spinors of the same chirality as Ψ, independent of the world sheet coordinates.

Of course, the Schild action possesses reparametrization invariance. During general coordinate transformations, the auxiliary field $\sqrt{g}$ behaves as scalar density as follows, for instance, from Eqn (3.5). A unique class of world sheet transformations, symplectic diffeomorphisms, deserves special comment. Symplectic transformations are characterized by a unit Jacobian, i.e. they leave the element of the world sheet area invariant. In classical mechanics, phase space area-preserving transformations are usually referred to as canonical. It is well-known that they may be given by a generating function:

$$\delta\sqrt{g} = \{\sqrt{g}, \Omega\},$$
$$\delta X^\mu = \{X^\mu, \Omega\},$$
$$\delta\Psi_\alpha = \{\Psi_\alpha, \Omega\}. \tag{3.9}$$

Formulae (3.9) resemble gauge transformations in the Yang–Mills theory and therefore play an important role in the matrix model.

3.2 Matrix formulation of the IIB superstring problem

The matrix model action is derived from the superstring action by the substitutions

$$X_\mu(\sigma_0, \sigma_1) \Longrightarrow A_\mu^{IJ}\,, \tag{3.10}$$

$$\Psi_\alpha(\sigma_0, \sigma_1) \Longrightarrow \psi_\alpha^{IJ}\,, \tag{3.11}$$

where A_μ^{IJ} and ψ_α^{IJ} are $N \times N$ Hermitian bosonic and fermionic matrices respectively. Transition from functions to infinite matrices and back may be formalized by expansion around the bases J^{IJ}_{m1,m_2} in $\mathrm{gl}(\infty)$ and $j_{m_1,m_2}(\sigma_0,\sigma_1)$ in the function space.

Transition from one representation to another is realized by convolution with the matrix function

$$L(\sigma_0, \sigma_1)^{IJ} = \sum_{m_1, m_2} j_{m_1,m_2}(\sigma_0, \sigma_1) J^{IJ}_{m_1,m_2}\,, \tag{3.12}$$

namely,

$$A_\mu = \int \mathrm{d}^2\sigma\, N X_\mu L\,, \tag{3.13}$$

$$X_\mu = \mathrm{tr}\, A_\mu\, L\,. \tag{3.14}$$

Matrix traces change to integrals while commutators at $N \to \infty$ are defined by Poisson brackets

$$\mathrm{tr} \Longrightarrow \int \mathrm{d}^2\sigma\, N\,, \tag{3.15}$$

$$[\,\cdot,\cdot\,] \Longrightarrow \frac{\mathrm{i}}{N}\{\,\cdot\,,\,\cdot\,\}\,. \tag{3.16}$$

The latter formula holds only for smooth configurations. The term 'smooth' refers to fields or matrices for which the amplitudes of high-frequency modes with $m_i \sim N$ expanded in the basis $j_{m_1,m_2}(\sigma_0,\sigma_1)$ or $J^{IJ}_{m_1,m_2}$ are small.

Formulae describing transformation from matrices to functions on a torus from Section 2.6 present a specific case of the above construction. Generally speaking, the expression for basic functions are known in an explicit form for both the sphere and the torus; the general case of the Riemann surface of genus g has also been discussed in a series of papers (see review [60]).

As a result of substitution (3.10) and (3.11), the Schild action (3.1) changes to

$$S = \mathrm{tr}\left(\frac{\alpha}{4}[A_\mu, A_\nu]^2 + \frac{1}{2}\,\bar\psi\Gamma^\mu[A_\mu,\psi] + \beta\right). \tag{3.17}$$

the IKKT model contains no matrix analog of the auxiliary field $\sqrt{g}$. Its role is assumed by the matrix size N. Indeed, after substitution of the integral for the trace and Poisson brackets for commutators, N enters the action exactly as the field $\sqrt{g}$ does. Therefore, the IKKT matrix model is defined by the integral over matrices of variable size which is thus also a dynamic variable.

Since the transition to Euclidean metric poses some problems raised by Majorana – Weyl spinors, it is natural to define the model by a vacuum amplitude

$$Z = \sum_{N=1}^{\infty} \int \mathrm{d}A_\mu\, \mathrm{d}\psi_\alpha \exp(\mathrm{i}S) \tag{3.18}$$

instead of a statistical sum. If this amplitude is dominated by large-N and smooth A_μ and ψ_α matrices, the matrix model may serve as the non-perturbative definition of the type IIB superstring.

Action (3.17), excepting its last term dependent only on N, has the form of the Lagrangian of the ten-dimensional supersymmetric Yang – Mills theory in which derivatives of all fields are omitted. A similar effective action arises in the description of bound states of N D-instantons in the type IIB superstring theory [cf. with Eqn (2.7) at $p = -1$]. This makes the IKKT model distinct from the D-instanton matrix model only in the summation over N.

The relationship between the IKKT model and the ten-dimensional Yang – Mills theory can also be interpreted in terms of the Eguchi – Kawai reduction [61] (see also review [62]). In the $N \to \infty$ limit, any matrix field theory is equivalent to the reduced zero-dimensional matrix model. True, in a pure non-supersymmetric gauge theory the straightforward variant of reduction, in which derivatives are simply dropped from the action, is not applicable because of a spontaneous breakdown of invariance with respect to shifts to constant matrices

$$A_\mu^{IJ} \to A_\mu^{IJ} + c_\mu \delta^{IJ}\,. \tag{3.19}$$

Special care is needed to restore this symmetry using external fields of a definite form. However, supersymmetry precludes violation of $\mathbf{R}^D$-invariance (3.19) as was first noticed in the four-dimensional $\mathcal{N} = 1$ supersymmetric Yang – Mills theory [63]. This fact is crucial for the IKKT model because the gauge potentials A_μ play the role of space and time-like coordinates and symmetry (3.19) corresponds to translational invariance.

Action (3.17) is invariant with respect to $\mathcal{N} = 2$ supertransformations

$$\delta\psi_\alpha^{IJ} = \frac{\mathrm{i}}{2}\,[A_\mu, A_\nu]^{IJ}(\Gamma^{\mu\nu}\epsilon)_\alpha + \delta^{IJ}\xi_\alpha\,, \tag{3.20}$$

$$\delta A_\mu^{IJ} = \mathrm{i}\bar\epsilon\gamma_\mu\psi^{IJ}\,, \tag{3.21}$$

analogous to Eqns (3.8). One of the supersymmetries, a U(1)-shift of U(1)-components of fermionic matrices, has a kinematic origin since action ξ is totally independent of $\mathrm{tr}\,\psi$. The other supersymmetry can be viewed as ten-dimensional supertransformations for coordinate-independent fields.

Also, the action of the IKKT model does not change in the case of gauge $\mathrm{U}(N)$-transformations

$$\delta A_\mu = \mathrm{i}[A_\mu, \omega]\,,$$
$$\delta\psi_\alpha = \mathrm{i}[\psi_\alpha, \omega]\,, \tag{3.22}$$

which turn into symplectic transformations (3.9) on the string world sheet upon replacement of the commutators by Poisson brackets. This explains why the matrix model automatically ensures invariance with respect to area-preserving diffeomorphisms due to gauge symmetry. Complete reparametrization invariance in the matrix model remains implicit and must arise dynamically in Eqn (3.18) during summation over N.

3.3 D-strings as classical solutions

Matrix models (both BFSS and IKKT) are maintained to provide the full non-perturbative description of string dynamics. Non-perturbative effects in the string theory are most conspicuously manifest in that the spectrum of physical degrees of freedom contains solitonic p-branes having finite tension and carrying electric and magnetic charges relative to tensor gauge fields present in the gravity supermultiplet in ten and eleven dimensions. A general formalism for the description of p-branes in matrix models was developed in Ref. [64] and is based on two observations:

(1) supersymmetry algebra at $N=\infty$ includes central charges of a non-trivial tensor structure;

(2) classical equations of motion for matrix models have operator-like solutions which may be interpreted as p-branes of various dimensions.

The presence of central charges in the matrix theory were reported in Ref. [64]. For the IKKT model, they were calculated in Ref. [65]. The discussion of central charges is beyond the scope of the present review which is concerned only with solutions of classical equations.

The classical equations of motion ensuing from Eqn (3.17) have the form

$$\left[A^{\mu},[A_{\mu},A_{\nu}]\right]=0\,, \tag{3.23}$$

$$[A^{\mu},(\Gamma_{\mu}\psi)_{\alpha}]=0\,. \tag{3.24}$$

The former equation is actually the Yang – Mills equation for coordinate-independent fields. Equations of this type have been considered in early studies in connection with the master-field in multicolor quantum chromodynamics [66] and also at finite N [67]. Solutions describing D-branes in matrix models exist only at $N=\infty$. Equation (3.24) defines fermionic zero modes against the classical configuration of bosonic fields. The problem of fermionic zero modes in matrix models remains to be investigated. Here, we shall also discuss only equations of motion for bosonic degrees of freedom (3.23).

The simplest solution has the form of a diagonal matrix

$$A_{\mu}^{\mathrm{cl}}=\begin{pmatrix} x_{\mu}^{1} & & \\ & \ddots & \\ & & x_{\mu}^{N}\end{pmatrix}. \tag{3.25}$$

Classical vacua of this type can play an important role in the IKKT matrix model. Interpretation of the IKKT model in terms of an effective theory for D-instantons makes it natural to regard $x^1,\ldots,x^N$ as space and time-like coordinates of N D-instantons (see Section 2.3). In the context of the Eguchi – Kawai reduction, the space – time dynamics is also described by diagonal components of gauge fields.

Solution (3.25) for D-instantons exists at any N whereas for the description of extended objects the infinite matrix size limit is essential. At $N=\infty$, the number of solutions for the equations of motion (3.23) markedly increases because the infinite matrix size limit allows the solutions of classical equations to play the role of any operator. The operator-like solution interpreted as a D-string was suggested in Ref. [2]:

$$A_{\mu}^{\mathrm{cl}}=\left(\frac{T}{2\pi}\,q,\,\frac{L}{2\pi}\,p,\,0,\ldots,0\right). \tag{3.26}$$

Here, p and q are infinite matrices satisfying canonical commutation relations (2.27), T is the periodicity interval in time, and $L/2\pi$ is the compactification radius. These two quantities must be large enough to make compactification effects imperceptible. Solution (3.26) has the same form as the classical membrane [1] in the matrix theory (2.23). Its match in a ten-dimensional space is a static string extending along the x_1 axis.

In the quasi-classical limit,

$$A_0^{\mathrm{cl}}\Longrightarrow X_0^{\mathrm{cl}}=\frac{T}{2\pi}\,\sigma_0\,, \tag{3.27}$$

$$A_1^{\mathrm{cl}}\Longrightarrow X_1^{\mathrm{cl}}=\frac{L}{2\pi}\,\sigma_1\,. \tag{3.28}$$

Solution (3.26) actually has a number of properties characteristic of a D-string. It keeps half the supersymmetry and is thus a BPS state. Moreover, it is possible to calculate interaction potential in the matrix model the between two strings in the form (3.26). At large distances, the answer is in excellent agreement with D-string interaction in supergravity [2]. In addition, solution (3.26) is easy to generalize to D-branes with $p>1$.

3.4 Dp-branes

BPS states, i.e. solutions preserving half of the supersymmetry, are singled out from all solutions of the classical equation of motion (3.23). This corresponds to cancellation of the first and second terms in the formula for fermion variation (3.20) provided the parameter ξ is specially selected. The matrix structure of the second term is trivial which makes the cancellation feasible only when the first term is proportional to a unit matrix, i.e. at

$$\left[A_{\mu}^{\mathrm{cl}},A_{\nu}^{\mathrm{cl}}\right]=\mathrm{i}c_{\mu\nu}\mathbf{1}\,, \tag{3.29}$$

where $c_{\mu\nu}$ are arbitrary numbers. Equation (3.29) plays the role of the BPS condition in the matrix model [2,64]. As usual, any classical configuration meeting the BPS condition solves the equations of motion (3.23).

Using the Lorentz transformation, the $c_{\mu\nu}$ matrix is readily reduced to the canonical Jordan form

$$c_{\mu\nu}=\begin{pmatrix} 0 & \omega_1 & & & \\ -\omega_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \omega_5 \\ & & & -\omega_5 & 0\end{pmatrix}. \tag{3.30}$$

The solution for which the $(p+1)/2$ coefficient of ω_k from 5 (where $p=1,3,5,7,9$) is other than zero may be expressed through a $(p+1)/2$ pair of canonically conjugate operators:

$$A_{\mu}^{\mathrm{cl}}=\left(Q_1,P_1,\ldots,Q_{(p+1)/2},P_{(p+1)/2},0,\ldots,0\right), \tag{3.31}$$

which follows directly from the commutation relations (3.29):

$$[Q_k,P_k]=\mathrm{i}\omega_k\,. \tag{3.32}$$

In the specific case $p=1$, the D-string is obtained from a previous section on the assumption that $Q_1=Tq/2\pi$, $P_1=Lp/2\pi$ and

$$\omega_1=\frac{TL}{2\pi N}\,. \tag{3.33}$$

In the case of arbitrary p, the solution (3.31) describes an extended p-dimensional object. Interpretation of this solution as a Dp-brane in the type IIB theory inevitably comes to mind. Arguments in favor of this solution can be found in Refs [65, 68]. It is worthwhile noting that the dimension p is automatically odd. The authors of Refs [69, 70] adhere to a different opinion and choose to interpret (3.31) as a Dp-brane with a constant magnetic field on its world volume. Such an object can be regarded as the bound state of a Dp-brane and N D-instantons.

If the solution (3.31) is to exist, the ten-dimensional space must be assumed to be compactified along axes $x_0, \ldots, x_{\mathrm{p}}$ on a circle with radii $L_0/2\pi, \ldots, L_{\mathrm{p}}/2\pi$. Hence, eigenvalues of Q_k, P_k matrices are uniformly distributed over intervals $[-L_{2k-2}, L_{2k-2}]$ and $[-L_{2k-1}, L_{2k-1}]$ respectively. The space in which A_μ matrices act is naturally broken into a tensor product of $(p+1)/2$ spaces on which Q_k, P_k operators are defined. This means that for the p-brane solution, Q_k and P_k act in a $N^{2/(p+1)}$-dimensional space [64]; hence, they have $N^{2/(p+1)}$ different eigenvalues. The compactification radii being proportional to $N^{1/(p+1)}$ at $N \to \infty$ [1, 2, 64], the distance $L_a N^{-2/(p+1)}$ between neighbor eigenvalues tends to zero as $N^{-1/(p+1)}$.

The quantities ω_k in commutation relations (3.23) play the role of Planck's constant. They are not independent parameters and can be expressed through the compactification radii L_a and matrix size N. According to the Bohr–Sommerfeld quantization rule, the number of quantum states is proportional to the phase space volume with coefficient $(2\pi\hbar)^{-1}$. In our case, $\hbar = \omega_k$ and eigenvalues of Q_k and P_k operators are in the ranges from $-L_{2k-2}/2$ to $L_{2k-2}/2$ and from $-L_{2k-1}/2$ to $L_{2k-1}/2$, respectively. Thus, the phase space volume is $L_{2k-2}L_{2k-1}$. The number of states, $N^{2/(p+1)}$, depends on the size of the Q_k and P_k matrices. The Bohr–Sommerfeld rule leads to

$$\omega_k = \frac{L_{2k-2}L_{2k-1}}{2\pi N^{2/(p+1)}} \,. \tag{3.34}$$

Formula (3.34) is a generalization of relation (3.33) and is easy to derive from commutation relations for Q_k, P_k by the Fourier transform, without regard for the quasi-classical Bohr–Sommerfeld rule. This formula indicates that ω_k remain finite in the $N \to \infty$ limit. By multiplying relations (3.34) for all k, the matrix size can be expressed through the p-brane world volume,

$$V_{p+1} \equiv L_0 L_1 \ldots L_p \,, \tag{3.35}$$

and the parameters ω_k:

$$N = V_{p+1} \prod_{i=1}^{(p+1)/2} (2\pi\omega_i)^{-1} \,. \tag{3.36}$$

The above solutions play the role of elementary generators which can be used to construct more complicated classical configurations. The general method for building up superpositions of arbitrary states in matrix models has been described in Ref. [1]. Let each state be characterized by matrices of a definite form. Then, the superposition is described by block-diagonal matrices into which elementary constituents are embedded as blocks. Symmetries with respect to block permutations reflect the statistics of the states forming.

Now, let us consider two similar p-branes coincident in a space. Evidently, such a configuration corresponds to the matrix of two similar blocks. The relative orientation of p-branes can be made arbitrary using different Lorentz transformations to act on either of them. Let us further suppose that translation along the x_{p+1} axis results in a configuration composed of two parallel p-branes:

$$\begin{gathered}
A_a^{\mathrm{cl}} = \begin{pmatrix} B_a & 0 \\ 0 & B_a \end{pmatrix}, \qquad a = 0, \ldots, p\,, \\
A_{p+1}^{\mathrm{cl}} = \begin{pmatrix} \dfrac{b}{2} & 0 \\ 0 & -\dfrac{b}{2} \end{pmatrix}, \\
A_i^{\mathrm{cl}} = 0\,, \qquad i = p+2, \ldots, 9\,.
\end{gathered} \tag{3.37}$$

Here, B_μ denotes solution (3.31) for one p-brane:

$$B_0 \equiv Q_1\,, \quad B_1 \equiv P_1\,, \quad \ldots\,, \quad B_{p-1} \equiv Q_{(p+1)/2}\,, \quad B_p \equiv P_{(p+1)/2}\,. \tag{3.38}$$

Parameter b has the sense of the distance between p-branes. It is quite understandable that both (3.37) and its constituent p-branes satisfy the BPS condition.

The configuration with rotated p-branes can be obtained from parallel ones, for example by rotating in the (x_p, x_{p+2}) plane through an angle θ which leads to

$$\begin{gathered}
A_a^{\mathrm{cl}} = \begin{pmatrix} B_a & 0 \\ 0 & B_a \end{pmatrix}, \qquad a = 0, \ldots, p-1, \\
A_p^{\mathrm{cl}} = \begin{pmatrix} B_p \cos\dfrac{\theta}{2} & 0 \\ 0 & B_p \cos\dfrac{\theta}{2} \end{pmatrix}, \\
A_{p+1}^{\mathrm{cl}} = \begin{pmatrix} \dfrac{b}{2} & 0 \\ 0 & -\dfrac{b}{2} \end{pmatrix}, \\
A_{p+2}^{\mathrm{cl}} = \begin{pmatrix} B_p \sin\dfrac{\theta}{2} & 0 \\ 0 & -B_p \sin\dfrac{\theta}{2} \end{pmatrix}, \\
A_i^{\mathrm{cl}} = 0\,, \qquad i = p+3, \ldots, 9\,.
\end{gathered} \tag{3.39}$$

Two rotated p-branes no longer form the BPS state but satisfy the classical equations of motion, as before.

Parallel-moving p-branes may be obtained by a hyperbolic rotation in (x_0, x_{p+2}) the plane. In a center-of-mass system, the following solution is consistent with such a configuration:

$$\begin{gathered}
A_0^{\mathrm{cl}} = \begin{pmatrix} B_0 \cosh\epsilon & 0 \\ 0 & B_0 \cosh\epsilon \end{pmatrix}, \\
A_a^{\mathrm{cl}} = \begin{pmatrix} B_a & 0 \\ 0 & B_a \end{pmatrix}, \qquad a = 1, \ldots, p\,, \\
A_{p+1}^{\mathrm{cl}} = \begin{pmatrix} \dfrac{b}{2} & 0 \\ 0 & -\dfrac{b}{2} \end{pmatrix}, \\
A_{p+2}^{\mathrm{cl}} = \begin{pmatrix} B_0 \sinh\epsilon & 0 \\ 0 & -B_0 \sinh\epsilon \end{pmatrix}, \\
A_i^{\mathrm{cl}} = 0\,, \qquad i = p+3, \ldots, 9\,.
\end{gathered} \tag{3.40}$$

The velocity of each p-brain is related to parameter ϵ by means of the conventional equation:

$$v = \tanh\epsilon\,. \tag{3.41}$$

3.5 One-loop effective action and p-brane interactions

A comparison of D-brane interaction potentials computed in the matrix model and superstring theory turns out to be the simplest way to verify the IKKT hypothesis. The classical solutions for p-brane superposition in the matrix model having the form of block-diagonal matrices, the classical theory is devoid of interactions. However, the interaction potential may be generated by quantum corrections unless considerations of supersymmetry interfere. Specifically, the BPS state should not undergo renormalization.

Association of the IKKT model with the ten-dimensional Yang–Mills theory allows the standard background field method to be employed to calculate quantum corrections for classical solutions. Bosonic variables are expanded in the vicinity of a classical solution into the background constituent and fluctuations:

$$A_\mu = A_\mu^{\mathrm{cl}} + \alpha^{-1} a_\mu \,. \tag{3.42}$$

The contribution of quantum fluctuations is computed from perturbation theory. It should be borne in mind that in calculations to follow, the matrix size is assumed to be fixed (and infinite). The procedure of calculating quantum corrections in the IKKT model regardless of the summation over N tacitly suggests that such a summation is saturated by a certain saddle point at infinite N.

As usual, it is convenient to select the background gauge for computing quantum corrections in the external field:

$$P_\mu a^\mu = 0 \,, \tag{3.43}$$

where P_μ is the zero-dimensional covariant derivative in the adjoint representation

$$P_\mu = \left[A_\mu^{\mathrm{cl}}, \cdot\,\right] . \tag{3.44}$$

The matrix model action in this gauge up to terms of order α^{-1} has the form

$$S = S^{\mathrm{cl}} - \mathrm{tr}\left[\frac{1}{2}\, a^\mu (P^2 \eta_{\mu\nu} - 2\mathrm{i}F_{\mu\nu}) a^\nu - \frac{1}{2}\,\bar\psi \Gamma^\mu P_\mu \psi + \bar c P^2 c\right] + O(\alpha^{-1}) \,. \tag{3.45}$$

The last term in square brackets contains Faddeev–Popov ghosts. The gauge field strength in the adjoint representation is denoted through $F_{\mu\nu}$:

$$F_{\mu\nu} = \mathrm{i}\left[P_\mu, P_\nu\right] = \mathrm{i}\left[[A_\mu^{\mathrm{cl}}, A_\nu^{\mathrm{cl}}], \cdot\,\right] . \tag{3.46}$$

Integration over a_μ, ψ and c gives the effective action for the background fields. In the lowest order of perturbation theory in α^{-1}, i.e in the one-loop approximation, the effective action has the form [2]:

$$W = \frac{1}{2}\,\mathrm{Tr}\ln(P^2\delta_{\mu\nu} - 2\mathrm{i}F_{\mu\nu}) - \frac{1}{4}\,\mathrm{Tr}\ln\left[\left(P^2 + \frac{\mathrm{i}}{2}\,F_{\mu\nu}\Gamma^{\mu\nu}\right)\frac{1+\Gamma_{11}}{2}\right] - \mathrm{Tr}\ln P^2 \,. \tag{3.47}$$

This result is obtained following quadration of the Dirac operator and a Wick rotation to Euclidean space. Superfluous $1/2$ multipliers in front of the first and second terms arise from the Hermiticity of A_μ and ψ matrices. The contribution of fermions includes a projector taking into account chirality. The 'minus' signs before the second and third terms are explicit.

Let us first consider a case when the background field meets the BPS condition. This means that $F_{\mu\nu} = 0$. The contributions of bosons, fermions, and ghosts are mutually canceled [2]:

$$W = \left(\frac{1}{2}\times 10 - \frac{1}{4}\times 16 - 1\right)\mathrm{Tr}\ln P^2 = 0 \,. \tag{3.48}$$

As expected, the BPS states undergo no renormalization and fail to interact.

An important sequel of the absence of BPS renormalization is the zero effective potential for diagonal matrices of the form (3.25) which accounts for the persisting uniform distribution of x_μ^I eigenvalues [63,2]. This is not true of non-supersymmetric theories where the one-loop attraction potential for eigenvalues appear and $\mathbf{R}^D$ symmetry (3.19) turns out to be spontaneously broken by quantum corrections.

Let us now consider a one-loop effective action computed on classical configurations from the previous section. This action has the sense of an interaction energy for static p-branes and that of a phase shift for p-branes propagating past one another.

For parallel p-branes, the effective potential becomes zero, as expected, because such a configuration meets the BPS condition. Therefore, the interaction energy for parallel branes becomes identically zero. In supergravity, such a cancellation occurs due to the compensation of gravitational attraction by the electric or magnetic repulsion of p-branes.

Other classical solutions from the previous section fail to satisfy the BPS condition, and the effective action for them does not vanish. It can be calculated in a closed form which allows for the comparison between interaction energies of different p-brane configurations in the matrix model with string results. The calculation of the interaction potential between p-branes is technically feasible because classical fields for one p-brane (3.38) in the 'coordinate representation' have the form of multiplication and derivative operators:

$$B_0 = q_1 , \quad B_1 = -\mathrm{i}\omega_1\partial_1 , \quad \ldots , \quad B_{p-1} = q_{(p+1)/2} , \quad B_p = -\mathrm{i}\omega_{(p+1)/2}\partial_{(p+1)/2} \,. \tag{3.49}$$

To sum up, the one-loop effective action (3.47) after the substitution of classical solutions is expressed through the determinants of certain second-order differential operators. We report here the final results of computation of the effective action [2, 65, 3] without giving the derivation.

The interaction energy of two antiparallel p-branes is

$$W = -2N\int_0^\infty \frac{\mathrm{d}s}{s}\exp(-b^2 s)\prod_{i=1}^{(p+1)/2}\frac{1}{2\sinh 2\omega_i s} \times\left[\sum_{i=1}^{(p+1)/2}(\cosh 4\omega_i s - 1) - 4\left(\prod_{i=1}^{(p+1)/2}\cosh 2\omega_i s - 1\right)\right] . \tag{3.50}$$

At large distances, the potential decreases as $1/b^{7-p}$:

$$W = -\frac{1}{16} N\Gamma\left(\frac{7-p}{2}\right)\left[2\sum_{i=1}^{(p+1)/2}\omega_i^4 - \left(\sum_{i=1}^{(p+1)/2}\omega_i^2\right)^2\right] \times \prod_{i=1}^{(p+1)/2}\omega_i^{-1}\left(\frac{2}{b}\right)^{7-p} + O\left(\frac{1}{b^{9-p}}\right). \quad (3.51)$$

This result agrees with supergravity, $1/b^{7-p}$ being none other than the Coulomb potential between p-dimensional planes in a nine-dimensional space. Naturally, the string result for the potential [39, 40, 71] has the same asymptotics at large distances. However, the full answer taking into account the exchange of all modes of a closed string including massive ones is at variance with formula (3.50) obtained in the matrix model†.

The effective action for rotated p-branes has the form

$$W = -4N^{2p/(p+1)}\frac{1}{\cos(\theta/2)}\prod_{a\neq p-1}L_a^{-1} \times \int_0^\infty \frac{\mathrm{d}s}{s}\left(\frac{\pi}{s}\right)^{p/2}\exp(-b^2 s)\tanh\left(\omega_{(p+1)/2}s\sin\frac{\theta}{2}\right) \times \sinh^2\left(\omega_{(p+1)/2}s\sin\frac{\theta}{2}\right). \quad (3.52)$$

Its asymptotics at $b\to\infty$,

$$W = -\Gamma\left(\frac{6-p}{2}\right)\frac{4V_{\mathrm{p}}}{(4\pi)^{p/2}}\,\omega_{(p+1)/2}^4 \times \prod_{i=1}^{(p+1)/2}\frac{1}{\omega_i^2}\frac{\sin^3(\theta/2)}{\cos(\theta/2)}\frac{1}{b^{6-p}} + O\left(\frac{1}{b^{8-p}}\right), \quad (3.53)$$

correctly reproduce angular and distance dependences for the energy of interaction between statistically rotated p-branes in ten-dimensional gravity.

The phase shift undergone by propagating p-branes when scattered at one another is found from the interaction potential of crossed branes by means of the analytical extension $\theta/2 \to \mathrm{i}\epsilon$:

$$\delta = -\frac{V_p}{(2\pi)^p}\,\omega_1\prod_{i=1}^{(p+1)/2}\frac{1}{\omega_i^2}\int_0^\infty\frac{\mathrm{d}s}{s}\left(\frac{\pi}{s}\right)^{p/2}\exp(-b^2 s) \times \frac{\left[\cos(4\omega_1 s\sinh\epsilon) - 4\cos(2\omega_1 s\sinh\epsilon) + 3\right]}{\cosh\epsilon\sin(2\omega_1 s\sinh\epsilon)}. \quad (3.54)$$

The integrand function in this formula has poles on the real axis so that the integration contour needs to be shifted to the complex plane. Residues in the poles define the imaginary part of the phase shift. The magnitude of these residues coincides with the string result [73] at small velocities for the assumption that all $\omega_i = 2\pi\alpha'$. The real part of the phase shift is in concord with string calculations at large distances when

$$\delta = -4V_p(4\pi)^{-p/2}\frac{\sinh^3\epsilon}{\cosh\epsilon}\,\Gamma\left(\frac{6-p}{2}\right)\omega_1^4 \times \prod_{i=1}^{(p+1)/2}\frac{1}{\omega_i^2}\frac{1}{b^{6-p}} + O\left(\frac{1}{b^{8-p}}\right). \quad (3.55)$$

† It was suggested in Ref. [72] that the effects of senior loops in the matrix model can restore the agreement with the superstring for intermediate distances.

The above examples demonstrate that the interaction between p-branes in the IKKT model agrees with superstring results in a certain kinematic region when the contribution of massless states becomes essential and massive modes can be neglected. This suggests that the truncated variant of the IKKT model including no summation over N which was used in the calculations reproduces superstrings in the low-energy approximation, i.e. at low velocities or large distances.

4. Matrix model with non-Abelian Born – Infeld action [3]

It has been mentioned in a previous paragraph that the gauge symmetry of the IKKT matrix model is responsible for string action invariance with respect to area-preserving diffeomorphisms. They represent only a part of general coordinate transformations, and complete reparametrization invariance of the string theory in the Schild formulation has to be dynamically restored by integration over the auxiliary field $\sqrt{g}$. In the IKKT model, integration over $\sqrt{g}$ is substituted by summation over the matrix size N and the limiting $N\to\infty$ transition is assumed to occur dynamically.

On the other hand, it is possible to modify the matrix model by introducing, from the very beginning, an additional degree of freedom, a $Y^{(IJ)}$ matrix, which is a direct field $\sqrt{g}$ analog in the Schild formulation. Such a modification was proposed in Ref. [3] and is discussed below. The matrix size N becomes a parameter, rather than a dynamical variable as in the IKKT model, which allows for a direct transition to the $N\to\infty$ limit. If integration over Y reproduces a matrix analog of the Nambu – Goto action, the matrix model may be expected to keep the reparametrization invariance of the string theory at the quantum level.

The matrix analog of the Nambu – Goto action has the form

$$S = \sqrt{\alpha\beta}\,\mathrm{tr}\sqrt{[A_\mu, A_\nu]^2}\,. \quad (4.1)$$

Indeed, the transition to string variables and substitution of Poisson brackets for commutators at $N\to\infty$ leads to

$$[A_\mu, A_\nu]^2 \Longrightarrow -\frac{1}{N^2}\{X_\mu, X_\nu\}^2 = -\frac{2}{N^2}\det_{ab}\partial_a X^\mu\,\partial_b X_\mu\,, \quad (4.2)$$

hence,

$$\mathrm{tr}\sqrt{[A_\mu, A_\nu]^2} \Longrightarrow \int \mathrm{d}^2\sigma\sqrt{-\{X_\mu, X_\nu\}^2}\,. \quad (4.3)$$

Expression (4.1) looks like the non-Abelian generalization of the Born – Infeld action (NBI) in the strong field limit‡. It would be possible to handle action (4.1) directly were it not for its non-analyticity which causes much inconvenience. Specifically, in the case of classical equations of motion for Eqn (4.1), the introduction of an auxiliary field

$$Y \propto \sqrt{[A_\mu, A_\nu]^2}. \quad (4.4)$$

seems opportune. In what follows, we consider a matrix model for which Eqn (4.4) ensues from the equations of

‡ It is worthy of note that in the string theory the effective Born – Infeld action for open strings and D-branes of a different structure is more familiar.

motion and which reproduces the non-Abelian Born – Infeld action (4.1) at both classical and quantum levels.

4.1 Matrix description of the Schild string

The action of the NBI matrix model is defined by formula

$$S = \mathrm{tr}\left(\frac{\alpha}{4}\, Y^{-1}[A_\mu, A_\nu]^2 + \frac{1}{2}\,\bar{\psi}\Gamma^\mu[A_\mu, \psi] + \beta Y\right), \qquad (4.5)$$

where Y^{IJ} is the positive-definite $N \times N$ Hermitian matrix considered to be an independent dynamical variable and a field $\sqrt{g}$ analog in the Schild action (3.1).

The vacuum amplitude of the NBI matrix model is given by the integral

$$Z_{\mathrm{NBI}} = \int \mathrm{d}A_\mu\, \mathrm{d}\psi_\alpha\, \mathrm{d}Y\, (\det Y)^{-\gamma} \exp(\mathrm{i}S) \qquad (4.6)$$

over all fields with action (4.5), where parameter γ which defines the measure of integration over the Y matrix is related to matrix size by the formula

$$\gamma = N - \frac{1}{2}\,. \qquad (4.7)$$

The NBI model is supersymmetric in the infinite matrix size limit. The supertransitions

$$\begin{aligned}
\delta\psi &= \frac{\mathrm{i}}{4}\left[Y^{-1}, [A_\mu, A_\nu]\right]_+ \Gamma^{\mu\nu}\epsilon + \xi\,,\\
\delta A_\mu &= \mathrm{i}\bar{\epsilon}\Gamma_\mu\psi\,,\\
\delta Y &= 0
\end{aligned} \qquad (4.8)$$

leave (see Ref. [3]) action (4.5) invariant up to terms which vanish upon replacement of the commutators by Poisson brackets. In the formula for fermion variations, $[\cdot,\cdot]_+$ denotes the matrix anticommutator.

The action of the NBI model (4.5) changes to the Schild action for a superstring (3.1) following the replacement of functions by matrices according to Eqns (3.10) and (3.11) and substitution of

$$Y^{IJ} \Longrightarrow \sqrt{g}(\sigma_0, \sigma_1)\,. \qquad (4.9)$$

Similarly, the matrix analog of the Nambu – Goto action may be obtained by removing the auxiliary field Y using the classical equations of motion

$$-\frac{\alpha}{4}\left(Y^{-1}[A_\mu, A_\nu]^2 Y^{-1}\right)^{IJ} + \beta\delta^{IJ} = 0\,, \qquad (4.10)$$

the solution of which is given by the formula

$$Y = \frac{1}{2}\sqrt{\frac{\alpha}{\beta}}\sqrt{[A_\mu, A_\nu]^2}\,. \qquad (4.11)$$

This solution is the only one provided the Y matrix is positive-definite as in the case being considered.

Substitution of the classical solution (4.11) into the action (4.5) leads to

$$S_{\mathrm{NBI}} = \sqrt{\alpha\beta}\ \mathrm{tr}\sqrt{[A_\mu, A_\nu]^2} + \frac{1}{2}\,\mathrm{tr}\left(\bar{\psi}\Gamma^\mu[A_\mu, \psi]\right). \qquad (4.12)$$

This expression is the supersymmetric generalization of the non-Abelian Born – Infeld action (4.1).

Thus, the NBI matrix model has an important advantage over the IKKT model because it has a simple matrix analog of the connection between the Schild formulation and the Nambu – Goto action at the classical level.

Evidently, both the action and the integration measure in the NBI model are gauge invariant and, similar to (3.22), Y is transformed over the adjoint $\mathrm{U}(N)$-group representation:

$$\delta Y = -\mathrm{i}\left[Y, \omega\right]. \qquad (4.13)$$

However, a transition to the string limit results in enhanced symmetry of the theory because in the Schild formulation the string is invariant with respect not only to area-preserving diffeomorphisms which correspond to gauge transformations but also to the complete group of reparametrization transformations. Strictly speaking, the matrix model is lacking in reparametrization analogs which do not preserve area. For example, in gauge transformations $\sigma \longrightarrow \lambda\sigma$, $\sqrt{g} \longrightarrow \lambda^{-2}\sqrt{g}$, both the integration measure on the string world sheet $\mathrm{d}^2\sigma$ and Poisson brackets $\{\cdot,\cdot\} \to \lambda^{-2}\{\cdot,\cdot\}$ are changed. In the matrix model, this would have correspondence not only in auxiliary field stretching $Y \longrightarrow \lambda^{-2} Y$ but also in the transformation of traces $\mathrm{tr} \to \lambda^2\,\mathrm{tr}$ and commutators $[\cdot,\cdot] \to \lambda^{-2}[\cdot,\cdot]$. Certainly, such transformations do not represent symmetry in the full sense of the word. Nevertheless, it appears natural to require that the measure of integration over Y be invariant with respect to gauge transformations. This unambiguously fixes the parameter γ in Eqn (4.6): $\gamma = N$. Indeed, suffice it to assume that $\gamma = N + \eta$, where η is any number of unity order which is of no significance whatever because, after all, N must be kept tending to infinity.

This line of reasoning facilitates an explanation of formula (4.7) corresponding to $\eta = -1/2$. It will be shown in the next section that at this value of parameter γ the integral over Y in definition (4.6) is taken *exactly* which eventually leads to a type (4.2) action. In case of arbitrary γ, the effective action can be found in the leading order of $1/N$ expansion [74]. It turns out to be non-local and non-reparametrizationally invariant at $\gamma \neq N + O(1)$.

4.2 Effective action and measure

The Y matrix, as an analog of the auxiliary field $\sqrt{g}$, plays the part of a Lagrange multiplier whose integration induces the effective action for A_μ fields and their superpartners. The action does not change for fermions because they fail to interact with Y. In the quasi-classical approximation, when α and β simultaneously tend to infinity faster than γ, the effective action has the form (4.12). It becomes local and reparametrizationally invariant after the transition to string variables. These properties are not explicitly conserved at the quantum level. This question is discussed at greater length below.

Since fermions may be neglected, an analytical extension to Euclidean space poses no special problem. Transition to the Euclidean metric makes it possible to deal with a statistical sum instead of vacuum amplitude as is the case with matrix models. Matrix Y is transformed in the same manner as the $\sqrt{g}$ field and is substituted by $\mathrm{i}Y$ upon transition to Euclidean space.

It is convenient to introduce the notation

$$G = -[A_\mu, A_\nu]^2\,. \qquad (4.14)$$

Then, following the Wick rotation, the integral over Y in Eqn (4.6) assumes the form

$$\exp\left[-S_{\text{eff}}(G)\right] = \int \mathrm{d}Y \exp\left(-\frac{\alpha}{4}\operatorname{tr} Y^{-1}G - \beta \operatorname{tr} Y - \gamma \operatorname{tr}\log Y\right). \quad (4.15)$$

This integral may be viewed as the Hermitian one-matrix model in the external field.

The standard method for calculating matrix integrals with the external field is based on the reduction of the Y matrix to diagonal form by the unitary transformation

$$Y = \Omega^{\dagger} \operatorname{diag}(y_1, \ldots, y_N)\Omega\,; \quad (4.16)$$

y_I has positive eigenvalues. The integration measure in new variables takes the form

$$\mathrm{d}Y = \mathrm{d}\Omega \prod_{I=1}^{N} \mathrm{d}y_I\, \Delta^2(y)\,, \quad (4.17)$$

where $\Delta(y)$ is the Vandermond determinant.

$$\Delta(y) = \prod_{I>J}\left(y_I - y_J\right). \quad (4.18)$$

The integral over the unitary matrix Ω is calculated as described in Ref. [75]. The resulting integral over eigenvalues is

$$\exp\left[-S_{\text{eff}}(G)\right] \propto \frac{1}{\Delta(g)}\int_0^{\infty}\prod_{I=1}^{N}\frac{\mathrm{d}y_I}{y_I^{1/2}}\,\Delta(y)\exp\left[-\sum_I\left(\frac{\alpha g_I}{4y_I}+\beta y_I\right)\right], \quad (4.19)$$

where g_I are eigenvalues of the G matrix. The calculation of integral (4.19) gives [3]

$$\exp\left[-S_{\text{eff}}(G)\right] \propto \frac{\Delta(\sqrt{g})}{\Delta(g)}\exp\left[-\sqrt{\alpha\beta}\sum_I\sqrt{g_I}\right]. \quad (4.20)$$

The exponent in the right-hand side of Eqn (4.20) coincides with Euclidean variant of the non-Abelian Born–Infeld action:

$$\sqrt{\alpha\beta}\sum_I\sqrt{g_I} = \sqrt{\alpha\beta}\operatorname{tr}\sqrt{G} = \sqrt{\alpha\beta}\operatorname{tr}\sqrt{-[A_\mu, A_\nu]^2}\,. \quad (4.21)$$

An additional pre-exponential factor is

$$J(G) = \frac{\Delta(\sqrt{g})}{\Delta(g)} = \prod_{I<J}\frac{1}{\sqrt{g_I}+\sqrt{g_J}}\,. \quad (4.22)$$

It has already been noted that action (4.21) is a matrix analog of the Nambu–Goto string action (4.1) (here, in Euclidean space). Therefore, the Nambu–Goto action is reproduced in the NBI model at the quantum level too. In the quasi-classical limit, the pre-exponential factor (4.22) is unessential as follows from the results in the previous section. In the quantum case, this additional factor has the sense of the measure of integration over A_μ field induced by averaging over Y field fluctuations.

It would be interesting to find a correspondence for this induced measure in the $N \to \infty$ limit where it is possible to pass from matrices to string coordinates and

$$G \to \frac{1}{N^2}\left\{X_\mu, X_\nu\right\}^2 = \frac{2}{N^2}\det_{ab}\partial_a X^\mu \partial_b X_\mu\,. \quad (4.23)$$

Expression (4.22) proves to turn into

$$J[G] = \text{const} \times \left(\operatorname{Det}\sqrt{G}\right)^{-1/2}. \quad (4.24)$$

Two alternative derivations of this formula have been described in Refs [74] and [76].

Another interesting aspect is the possibility of obtaining the Nambu–Goto action by integration over Y. This action has reparametrization invariance which remains in part implicit in the matrix formulation as shown before. The reparametrization invariance is not broken by the induced measure of integration over string coordinates which is defined by the expression

$$[\mathrm{d}X_\mu] = \prod_\sigma \mathrm{d}X_\mu(\sigma)G^{-1/4}(\sigma) \quad (4.25)$$

according to Eqn (4.24). Substitution of world sheet coordinates, $\sigma \to \sigma'$, implies multiplication of the integration measure and a constant:

$$[\mathrm{d}X_\mu] \to \text{const} \times [\mathrm{d}X_\mu]\,, \quad (4.26)$$

which does not depend on the fields and is thus canceled in all correlation functions. It should be emphasized that both the effective action and the induced measure are local which does not follow from general considerations.

4.3 A note on classical solutions

It seems natural to suggest that non-perturbative string states may be described by the solutions of classical equations of motion in the matrix model. Equations of motion for the NBI model have the form

$$Y^2 = \frac{\alpha}{4\beta}[A_\mu, A_\nu]^2\,, \quad (4.27)$$

$$\left[A^\mu, \left[Y^{-1}, [A_\mu, A_\nu]\right]_+\right] = 0\,, \quad (4.28)$$

$$\left[A_\mu, (\gamma^\mu\psi)_\alpha\right] = 0\,. \quad (4.29)$$

The classical configurations (3.31) identified with p-branes in the IKKT model may just as well be solutions in the BNI model. This is due to their meeting the BPS condition (3.29) which makes their Y matrix defined by the classical equation (4.27) proportional to the unit matrix. For this reason, it may be merely taken out of the brackets after which Eqn (4.28) assumes the same form as in the IKKT model.

A more general statement appears to be relevant in the $N \to \infty$ limit. It reads that any solution of classical equations for the IKKT model is at the same time true for classical solutions in the NBI model. It is easy to demonstrate by rewriting equation (4.28) in the form

$$\left[Y^{-1}, \left[A^\mu, [A_\mu, A_\nu]\right]\right]_+ + \left[\left[A^\mu, Y^{-1}\right], [A_\mu, A_\nu]\right]_+ = 0\,. \quad (4.28)$$

The first term in the left-hand side vanishes for any solution of the IKKT model. When the commutators are replaced by Poisson brackets, the second term is also zero. This follows from Ref. [77] where the IKKT equations of motion have been shown to turn at infinite N into classical equations for the Schild string which results in a stationary Y: $\partial_a Y^2 = 0$. Therefore, the Poisson bracket $\{A_\mu, Y^{-1}\}$ becomes identically zero.

The structure of the classical equations for the NBI model is in a sense wealthier. Specifically, it provides for solutions with a non-trivial distribution of matrix Y eigenvalues [78] which is generally speaking typical of matrix models in the $N \to \infty$ limit.

5. Conclusions

We have considered three matrix models maintained to provide a non-perturbative description of superstrings. It should be emphasized that they are closely related despite outward differences and may in the end prove to be absolutely identical. Suffice it to say that application of the Eguchi–Kawai reduction to the matrix theory leads to the IKKT model (without summation over N). On the other hand, matrix theory may be derived from IKKT by the choice of special vacuum configurations in the infinite matrix size limit [2]. The NBI matrix model contains additional degrees of freedom as compared with IKKT, but it has been argued [72] that the two models lie in the same universality class at $N \to \infty$.

The BFSS matrix model appears to be the most promising of the three due to its direct relation to M-theory. This model claims to correctly describe dynamical degrees of freedom at small distances in M-theory which are in fact non-perturbative excitations if considered in the context of ten-dimensional superstrings. Verification studies conducted till now appear to confirm the hypothesis that BFSS matrix theory actually describes M-theory. Specifically, interactions between classical solutions are in agreement with eleven-dimensional supergravity, and the connection with the supermembrane theory is established in the natural way. Moreover, matrix theory compactification on a circle constitutes the mechanism of perturbative string creation from the BFSS matrix model in the light cone gauge.

We have given only the formulation of matrix theory here, and the discussion in Section 2 is presented largely by way of illustration. The matrix model has been formulated only in the light cone frame while the covariant approach remains to be developed.

It appears premature to make a conclusion as regards the validity of the approach to superstring theory based on matrix models. A definitive check of this approach might be provided by a comparison with the conventional string perturbation theory. From this standpoint, the NBI model looks very promising since it allows the supersymmetric version of the Nambu–Goto action to be reproduced. Being non-perturbative superstring formulations, supersymmetric matrix models may be expected to answer the question put in the Introduction: does a superstring live in the string phase? In any case, matrix models of superstrings need further studies which should throw new light on the string theory at large.

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6. Glossary

Green–Schwarz action — an explicitly covariant and supersymmetric action for a superstring:

$$S_{\mathrm{GS}} = \int \mathrm{d}^2\sigma \left[-\frac{1}{2}\sqrt{-h}\, h^{ab}\eta_{\mu\nu}\Pi^\mu_a \Pi^\nu_b \right.$$
$$- \mathrm{i}\varepsilon^{ab}\,\partial_a X^\mu \left(\bar\theta^1 \Gamma_\mu \partial_b \theta^1 - \bar\theta^2 \Gamma_\mu \partial_b \theta^2 + \right.$$
$$\left.\left. + \varepsilon^{ab}\,\bar\theta^1 \Gamma^\mu \partial_a \theta^1\, \bar\theta^2 \Gamma_\mu \partial_b \theta^2 \right)\right], \tag{6.1}$$

where h^{ab} is the internal metric on the string world sheet, θ^1 and θ^2 are ten-dimensional Majorana–Weyl spinors for IIB superstrings of similar chirality, and

$$\Pi^\mu_a = \partial_a X^\mu - \mathrm{i}\bar\theta^1 \Gamma^\mu \partial_a \theta^1 - \mathrm{i}\bar\theta^2 \Gamma^\mu \partial_a \theta^2\,. \tag{6.2}$$

Action (6.1) is characterized by parametrization invariance and shows ten-dimensional $\mathcal{N} = 2$ supersymmetry. It is also invariant with respect to local fermionic κ-symmetry which allows half the fermionic degrees of freedom to be excluded. The large number of symmetries in the Green–Schwarz action is responsible for a complicated system of constrains and hampers its covariant quantization.

Nambu–Goto action — the standard action functional in the bosonic string theory. The Nambu–Goto action gives a clear idea of a string as a strand with energy proportional to its length. Therefore, the action equals the area covered by the string world sheet up to a coefficient. If the string trajectory coordinates are labeled as $X^\mu(\sigma_1, \sigma_2)$, the area is expressed through the metric induced on the world sheet in the standard formula from differential geometry:

$$S_{\mathrm{NG}} = \frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma \sqrt{\left|\det_{ab} G_{ab}\right|}\,, \tag{6.3}$$

where

$$G_{ab} = \partial_a X^\mu \partial_b X_\mu\,. \tag{6.4}$$

The Nambu–Goto action is by definition independent of the choice of coordinate system on the world sheet. In other words, it is invariant with respect to arbitrary reparametrizations $\sigma \to f(\sigma)$.

Kaluza–Klein modes — an infinite set of fields arising from one field as a result of compactification of higher dimensions. Kaluza–Klein modes have masses of the order of the inverse compactification radius.

Compactification — contraction of space–time dimensionality by wrapping a part of the dimensions around a compact manifold of small radius. In theories of the Kaluza–Klein type, the product $\mathbf{R}^d \times K^{D-d}$ is considered to be the multidimensional space–time manifold. Each $\Psi(x, y)$ field on $\mathbf{R}^d \times K^{D-d}$ has corresponding infinite set of fields in the compactified theory:

$$\Psi(x, y) = \sum_n \psi_n(x) \phi_n(y) , \tag{6.5}$$

where x and y denote coordinates on $\mathbf{R}^d$ and the compact manifold K respectively. Basis $\phi_n(y)$ provides a complete set of wave operator eigenfunctions for the field Ψ on the K^{D-d} manifold; the corresponding eigenvalues of λ_n define masses of $\psi_n(x)$ fields: $m_n^2 = m^2 + \lambda_n$, where m is the mass of the Ψ field in the uncompactified theory.

A simple example is the compactification of a massless scalar field on a circle of radius R. In this case, $\phi_n(y) = \exp(\mathrm{i}ny/R)$, $\lambda_n = n^2/R^2$ and masses of Kaluza–Klein modes are $m_n = |n|/R$.

Reference [4] comprises an introduction to the compactification of higher dimensions with special reference to string theory.

M-theory — a quantum theory of eleven-dimensional supergravity. As a field theory, supergravity in eleven dimensions is strongly non-renormalizable. Therefore, a consistent theory changing to supergravity at large distances must contain, at Planck scales, additional degrees of freedom which contract divergences in the same manner as in the superstring theory.

From a different viewpoint, M-theory is a universal theory which contains all known string theories as different limiting cases. The simplest connection is between M-theory and type IIA superstrings which are created by compactifying one of the ten spatial dimensions on a circle [8–10]. The compactification radius R is expressed through the Planck length l_{p} and string coupling constant g_{s} in the relation $R = g_{\mathrm{s}}^{2/3} l_{\mathrm{p}}$ while the string length $l_{\mathrm{s}} \equiv \sqrt{2\alpha'}$ is related to the eleven-dimensional scale by the formula $l_{\mathrm{s}} = g_{\mathrm{s}}^{-1/3} l_{\mathrm{p}}$. This allows M-theory to be viewed as the strong coupling limit for type IIA superstrings since the compactification radius tends to infinity at $g_{\mathrm{s}} \to \infty$ and the theory actually becomes an eleven-dimensional one.

Massless degrees of freedom in M-theory give rise to an eleven-dimensional gravity supermultiplet which contains the metric $g_{\mu\nu}$, gravitino ψ_μ^α, and antisymmetric tensor field $A_{\mu\nu\lambda}$. The presence of the gauge field suggests that M-theory must have membranes carrying an electric charge relative to the $A_{\mu\nu\lambda}$ field and magnetically charged five-branes dual to membranes. There are two closely related approaches to the consistent dynamical construction of M-theory. According to one hypothesis, M-theory is a quantum theory of eleven-dimensional supermembranes [12, 13, 9]. The alternative approach is based on the BFSS matrix model [1].

Matrix theory — quantum mechanics of $N \times N$ Hermitian matrices with action

$$S = \int \mathrm{d}t \, \mathrm{tr} \left\{ \frac{1}{2R} (\dot{X}^i)^2 + \frac{R}{4} [X^i, X^j]^2 + \theta\dot{\theta} + \mathrm{i}R\theta\gamma_i[X^i, \theta] \right\}, \tag{6.6}$$

which is supposed to provide a complete dynamical description of M-theory in the limit of $N \to \infty$, $R \to \infty$ [1]. X^i variables play the role of nine transverse coordinates in the light cone gauge and fermionic matrices θ are their superpartners. The time t is identified with one of the light cone coordinates while the other coordinate is considered to be compactified on a circle of radius R so that $p_- = N/R$ has the sense of longitudinal momentum.

Matrix models — in the broad sense of the term — are statistical systems of random matrices. They were first introduced by E Wigner in 1951 for the description of excited nuclear levels and are extensively used in statistical physics. In the theory of elementary particles, the term 'matrix models' is frequently used in connection with the application of random matrices to fluctuations of geometry. This approach proved to be of special value in two-dimensional quantum gravity (see Refs [28–30]) and Section 1 of this review) where averaging over metrics and summation over topologies can be substituted by integration over matrices in the limit in which their dimension becomes infinite. Sometimes, the term 'matrix models' is attributed to random matrix systems in two-dimensional gravity.

Membranes — extended two-dimensional objects having three-dimensional world volume. According to one hypothesis, supermembranes are fundamental degrees of freedom in the quantum theory of eleven-dimensional supergravity [12, 13, 9], by analogy with the ten-dimensional case where the part of fundamental objects is played by strings. One of the principal propositions of this hypothesis is that κ-symmetry of the classical action of an eleven-dimensional supermembrane in a curved space imposes restrictions on background fields. These constraints have the form of differential equations which exactly coincide with the classical equations of motion for supergravity [12, 13]. The quantum supermembrane theory encounters certain difficulties, presumably purely technical ones, related to the non-linearity of the field theory on the world volume. Specifically, this theory contains normal ultraviolet divergences. Moreover, supermembranes possess a continuous spectrum which also constituted a problem until matrix theory was suggested.

p-brane tension — mass per unit spatial volume.

Planck scale — length, time, energy or mass scale defined by gravity constant (in combination with other world constants). Gravitational interaction at Planck scales is no longer weak.

Eguchi–Kawai reduction — an operation to reduce any matrix field theory to a zero-dimensional matrix model in the $N \to \infty$ limit. In the general form, the operation is as follows. Fields in the theory action are transformed according to the rule

$$\Phi(x) \to \exp(-\mathrm{i}Px) \Phi \exp(\mathrm{i}Px) , \tag{6.7}$$

where Φ is already independent of x. P_μ matrices are diagonal, $P_\mu = \mathrm{diag}\,(p_\mu^1, \ldots, p_\mu^N)$, and regarded as fixed when correlation functions are calculated. Averaging over p_μ^i phases with uniform distribution results in their transformation to intermediate momenta in Feynman diagrams. It can be shown that such a procedure restores conventional perturbation theory at $N = \infty$, i.e. the sum of planar diagrams. Moreover, Eguchi–Kawai reduction may use Dyson–Schwinger equations which are similar at $N \to \infty$ in the original and reduced theories.

The situation is somewhat different in gauge symmetry theories where the dependence on momenta in Eqn (6.7) can be taken up by the gauge transformation. This turns the reduction into a mere consideration of coordinate-independent fields while the part of p_μ^i momenta is played by diagonal components of the gauge fields themselves. However, the uniform distribution of p^i_μ is usually distorted by quantum corrections which requires special measures to restore it, e.g. quenching the diagonal p^i_μ components during averaging over gauge fields. The aforesaid does not refer to supersymmetric gauge theories in which violation of $\mathbf{R}^D$ invariance $p^i_\mu \to p^i_\mu + c_\mu$ is precluded by supersymmetry and there is no need to quench the p^i_μ momenta [63].

Superstrings. A supersymmetric string theory can be consistently formulated in ten-dimensional space–time where the conformal anomaly on the world shift is canceled. There are a few non-contradictory ten-dimensional superstring theories. Firstly, there are two types of closed strings with $\mathcal{N} = 2$ supersymmetry, IIA and IIB, which differ in terms of chirality (IIA theory is chiral whereas IIB one is not). Their low-energy limit is chiral and non-chiral $\mathcal{N} = 2$ supergravity respectively. Type I theory may contain both open and closed strings. Massless states of the open string are described by the Yang–Mills supersymmetric theory. Unitarity and the absence of anomalies keep the gauge group in the type I superstring theory unambiguously fixed: it can not be other than SO(32). Along with type I, IIA, and IIB superstrings, there are heterotic strings which are actually a hybrid of supersymmetric and 26-dimensional bosonic strings. They possess intrinsic gauge symmetry with group SO(32) or $E_8 \times E_8$.

Types of superstrings. There are five self-consistent ten-dimensional superstring theories including those of type I, IIA, IIB, and heterotic ones with gauge groups SO(32) or $E_8 \times E_8$. Type II theories are characterized by expanded ($\mathcal{N} = 2$) supersymmetry, with supercharges of opposite chirality in IIA theory and of similar chirality in IIB theory. Type II theories contain only closed strings whereas type I theory also includes the open string sector. The left and right modes of string coordinates in a heterotic string are described in a totally different way.

Central charges in the algebra of supersymmetry — generators which arise in supercharge anticommutation relations but commute with all algebra elements. In each irreducible representation, central charges assume certain numerical values. In field theory, they frequently have the form of topologic quantities of the type of integrals from full derivatives. In this case, non-zero central charges are carried by solitonic states. Central charges may have a tensor structure and are thus connected with extended objects.

Another characteristic example of the emergence of central charges in the algebra of supersymmetry is related to compactification of additional dimensions. The appearance of central charges in this case can be schematically illustrated in the following way. Let the $(D+1)$-dimensional theory be compactified on a circle of radius R. Then, the momentum component along the compact dimension is quantized in units of R^{-1} and turns into a conserved charge in the D-dimensional theory. The algebra of supersymmetry for Kaluza–Klein modes turns out to be centrally expanded:

$$\{\bar{Q}_\alpha, Q_\beta\} = P_M \Gamma^M_{\alpha\beta} = P_\mu \Gamma^\mu_{\alpha\beta} + \frac{N}{R}\,\Gamma^D_{\alpha\beta}\,, \tag{6.8}$$

because supercharges anticommute on the momentum operator. Further analysis depends on the parity of the number of dimensions D. An important moment is that the central charge, i.e. coefficient at the last term in the right-hand side of Eqn (6.8), is equal to the mass of a Kaluza–Klein particle arising from the massless state of the uncompactified theory.

Electromagnetic duality — a symmetry transformation switching the places of electric and magnetic fields and, accordingly, charges. The meaning of the terms 'electric' and 'magnetic charges' needs elucidation. The wave function of a charged particle propagating in an external electric field acquires an additional phase $\exp(\mathrm{i}eS)$; in other words, the action on the particle's world line has a term

$$S = \int \mathrm{d}\sigma^\mu A_\mu\,. \tag{6.9}$$

The coefficient in front of this term may be regarded as the definition of the particle's electric field. On the other hand, the magnetic charge is related to the magnetic field flux across the sphere around the point where the charge is present:

$$\Phi = \oint \mathrm{d}\sigma^{ij} F_{ij}\,. \tag{6.10}$$

In the four-dimensional case, both electric and magnetic charges relative to the vector field are point objects. However, it is possible to consider a more general situation when gauge potentials form an antisymmetric tensor of rank r in D-dimensional space–time. Then, electric charges are carried by p-branes with $p = r - 1$ because their world surface has the dimensionality necessary to integrate the gauge potential:

$$S = \int \mathrm{d}\sigma^{\mu_1\ldots\mu_r} A_{\mu_1\ldots\mu_r}\,. \tag{6.11}$$

At the same time, the gauge field strength

$$H_{\mu_0\ldots\mu_r} = \partial_{[\mu_0} A_{\mu_1\ldots\mu_r]} \tag{6.12}$$

is responsible for the antisymmetric tensor of rank $r+1$ while the magnetic flux is defined by the $(r+1)$-multiple integral

$$\Phi = \oint \mathrm{d}\sigma^{i_1\ldots i_{r+1}} H_{i_1\ldots i_{r+1}}\,. \tag{6.13}$$

The surface surrounding a rectilinear infinite p-brane in the $(D-1)$-dimensional space is a cylinder $\mathbf{R}^p \times S^{D-p-2}$. Therefore, the p-brane magnetic charge is defined by the integral (6.13) over a $(D-p-2)$-dimensional sphere. Hence, magnetic branes have the dimensionality $p = D - r - 3$. This means, that in the D-dimensional theory branes with dimensionalities p and $D-p-4$ are reciprocally dual. It is worthwhile mentioning important specific cases of particles (0-branes) in four dimensions and 2- and 5-branes in the eleven-dimensional theory.

BPS (Bogomol'ny–Prasad–Sommerfield) state in supersymmetric theories — a state preserving a part of the supersymmetries. At the classical level, this state is most often realized topologically by stable solutions of the equations of motion. The supermultiplet formed by BPS states has a smaller dimensionality than the usual massive supermultiplet. BPS states saturate constraints imposed by the algebra of supersymmetry on masses, i.e. for them inequality-type constraints turn into equalities. This accounts for the rigid connection of BPS state masses with central charges in the algebra of supersymmetry; in the simplest case, the mass of a BPS state is simply equal to the charge. If supersymmetry is not spontaneously broken, this statement remains exact when all quantum corrections are taken into consideration, which allows it to be viewed as the non-renormalization theorem. Stable BPS states do not interact because the BPS state of charge 2, for example, has the same energy as the superposition of two charge 1 states. Examples of BPS states are BPS monopoles in four-dimensional theories with expanded supersymmetry, extreme black holes, domain walls in supersymmetric theories [79, 80], and D-branes. The Kaluza – Klein modes of massless fields are also BPS states.

D(irichlet)-branes — soliton-like extended objects in string theory. Their dynamical description is organized in the following way. Open string end-points can move freely on the D-brane world volume. This imposes the Neumann boundary conditions on string coordinates tangential to the D-brane and Dirichlet conditions on the orthogonal coordinates. An open string connected with the D-brane covers a certain surface in space – time. In the dual description, it may be regarded as the world sheet of a closed string emitted by the D-brane. Therefore, D-branes interact with gravitons since gravitons correspond to the massless state of a closed string. Hence, D-branes possess tension. Moreover, D-branes carry RR-charges and are BPS states. In the IIA theory, D-branes have even dimensionality: $p = 0, 2, \ldots$, whereas in the IIB theory their dimensionality is odd: $p = -1, 1, 3, \ldots$. (-1)-branes are point objects, D-instantons.

D-instanton — D-brane of dimensionality $p = -1$.

D-string — D-brane of dimensionality $p = 1$.

D-particle — D-brane of dimensionality $p = 0$.

GSO (Gliozzi – Scherk – Olive) projection — superstring generalization of chiral projection which consists in discarding string states whose so-called G-parity is negative. For massless modes of an open string, G-parity merely coincides with chirality. In the massless sector of a closed string, the signs of GSO-projection for left and right modes may or may not coincide. This determines whether the theory is chiral (IIB) or non-chiral (IIA). The GSO-projection allows one to get rid of a tachyon and also ensures the space – time supersymmetry of an NSR-string.

NSR (Neveu – Schwarz – Ramond) string — the formulation of a superstring theory with local supersymmetry on the world sheet. The NSR-string action is given elsewhere (see, for instance, Ref. [4]). Suffice it to say that the physical degrees of freedom in the NSR-formalism are the string coordinates X^μ and their fermionic superpartners (relative to supersymmetry on the world sheet) ψ^μ_A, i.e. Majorana spinors, which also carry a certain vector space – time index. The critical dimension of a NSR-string is 10. At this dimension, it shows space – time supersymmetry following the GSO-projection and is equivalent to the Green – Schwarz superstring.

NS (Neveu – Schwarz)-sector in the space of NSR-string states corresponds to the expansion of fermionic fields on the world sheet in semi-integer harmonics; for a closed string, this meets antiperiodic boundary conditions.

R (Ramond)-sector occurs in the space of NSR-string states if fermions on the world sheet are expanded in a Fourier series with integer harmonics. In other words, the R-sector for a closed string corresponds to periodic boundary conditions for fermions.

RR (Ramond – Ramond)-charges. In the perturbative state space of a closed superstring, there are four sectors: R – R, R – NS, NS – R, and NS – NS, corresponding to different boundary conditions, both periodic and antiperiodic, for left and right fermionic modes on the string world surface in the NSR-formalism. Bosonic degrees of freedom emerge from NS – NS and R – R sectors. Specifically, the R – R sector contains massless vector and tensor fields. These fields enter the low-energy effective Lagrangian only through their intensities. Generally speaking, none of the string perturbative states is charged relative to R – R sector fields. Nevertheless, solitonic states may carry both electric and magnetic RR-charges. In the general case, these states correspond to extended objects, depending on the dimension of the gauge field relative to which they are charged.

In the IIA theory, the massless R – R sector consists of a vector field and an antisymmetric tensor field of rank 3. Therefore, electric p-branes have the dimensionality $p = 0$ and $p = 2$ and magnetic ones that of $p = 6$ and $p = 4$. Massless states from the R – R sector of the IIB theory correspond to pseudoscalar, rank two tensor, and rank four tensor with self-dual strength. Accordingly, RR-charges in this theory carry 1 and 5-branes dual to each other and self-dual 3-branes.

κ-symmetry — local fermionic symmetry of the covariant action of a superparticle, superstring, and supersymmetric p-branes at large which allows an extra part of the fermionic degrees of freedom to be gauged. κ-symmetry plays an important role in the formulation of Green – Schwarz superstrings with explicit space – time supersymmetry.

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