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Axisymmetric stationary flows in compact astrophysical objects

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Contents

Abstract. **A review is presented of the analytical results available for a large class of axisymmetric stationary flows in the vicinity of compact astrophysical objects. The determination of the two-dimensional structure of the poloidal magnetic field (hydrodynamic flow field) faces severe difficulties, due to the complexity of the trans-field equation for stationary axisymmetric flows. However, an approach exists which enables direct problems to be solved even within the balance law framework. This possibility arises when an exact solution to the equation is available and flows close to it are investigated. As a result, with the use of simple model problems, the basic features of supersonic flows past real compact objects are determined.**

1. Introduction

Stationary axisymmetric flows considered in terms of ideal magnetic hydrodynamics have been discussed for a long time in connection with many astrophysical sources. This class of flows involves accretion onto ordinary stars and black holes [1 – 3], axisymmetric stellar (solar) winds [4 – 7], jets from young stellar objects [8], and particle ejection from the magnetosphere of a rotating neutron star [9, 10]. The intense development of MHD models is also due to the theory of the structure of magnetospheres of supermassive black holes which are thought of as the 'central engine' in active galactic nuclei and quasars [11 – 13].

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We shall immediately note that many processes that may be of crucial importance in real astrophysical sources are beyond the scope of the approximation of ideal magnetic hydrodynamics. Such processes include the interaction of matter with self-radiation during accretion [14 – 17] and during stellar (solar) wind formation [6, 7], viscous forces [2, 3] and radiation transport during disc accretion [18, 19], as well as kinetic effects [20]. At the same time, the approximation of ideal hydrodynamics is sufficiently good in some cases. So, the radiation due to adiabatic heating of accreting matter turns out to be small compared to the Eddington luminosity, which allows the entropy of the matter to be regarded as constant [3]. Therefore, it is only the approximation of ideal (magnetic) hydrodynamics that is treated throughout the paper.

The MHD models are attractive first of all because they are comparatively simple. The point is that owing to axial symmetry and stationarity (as well as the freezing-in condition), in the general case there exist five 'integrals of motion' that are preserved on axisymmetric surfaces. In the first place, this is the energy flow (Bernoulli integral) and the z-component of angular momentum, as well as the electric potential, the entropy and the particle-to-magnetic field flux ratio. This remarkable fact allows a distinction to be made between the problem of poloidal field structure (the structure of a poloidal flow in hydrodynamics) and the problem of particle acceleration and the structure of electric currents. The solution of the latter problem in a given poloidal field is represented by fairly simple algebraic relations. It is of importance that such an approach is readily extendible to flows near a rotating black hole because the Kerr metric is also axisymmetric and stationary. As a result, it became possible to investigate quantitatively an extremely wide class of flows — from a magnetised stellar (solar) wind [21 – 24], jets from young stars [25, 26], and quasars and active galactic nuclei to the processes proceeding in a pulsar magnetosphere [28 – 31] and in supermassive black holes [32 – 35]. In particular, it has been shown that the energy release from a rotating black hole is possible in principle [36, 37]. Thus, undoubted progress has been made in this direction.

On the other hand, the problem of finding the structure of a poloidal magnetic field (the structure of a hydrodynamic flow) encounters much greater difficulties. This is connected in the first place with the involved structure of the equation describing stationary axisymmetric flows. In the general case, it appears to be a nonlinear mixed type equation that changes from elliptic to hyperbolic on singular surfaces and, in addition, contains integrals of motion in the form of free functions. Generally speaking, analogous equations taking root in the classical Trikomi equation have been discussed since the beginning of the century in connection with the problem of transonic hydrodynamic flows [38, 39]. In particular, for planar flows, the hodograph transformation method (leading to the linear Chaplygin equation) which provided better insight into the indicated processes is extremely fruitful [39, 40]. In the astrophysical literature, stationary axisymmetric trans-field equations bear the name of Grad – Shafranov who, in the late 1950s, derived an equation of this type related to the problem of controlled thermonuclear fusion [41, 42]. That equation referred, however, only to equilibrium statistical configurations and generally required a substantial change when extended to transonic flows (see, e.g., Refs [38, 43]).

Applied to astrophysics, Grad – Shafranov type equations (in the force-free approximation in the absence of gravitation) were first discussed widely in the 1970s in relation to the structure of radio pulsar magnetospheres [44 – 48]. In 1979, they were extended to the case of MHD flows [49] and were then investigated in numerous papers, see, for example, Refs [23 – 26, 50 – 54]. Finally, the case of the Schwarzschild metric was considered in Ref. [55] and the trans-field equation was written in the most general Kerr metric in recent papers [56, 57]. However, despite a large number of works devoted to this range of problems, no significant results have been achieved.

In our opinion, the difficulty is that the very statement of the problem within the method of the Grad – Shafranov equation appears to be non-trivial. For example, in the hydrodynamic limit, where only three integrals of motion exist, the problem requires four boundary conditions for the transonic flow. This means that on a certain surface two hydrodynamic functions should be defined, as well as two velocity components. However, to define the Bernoulli integral, without which the trans-field equation cannot of course be solved, all the three components must necessarily be known, which is impossible because the third velocity component must itself be found from the solution. This inconsistency is, in fact, one of the main difficulties of the method of the Grad – Shafranov equation. For this reason, in the majority of cases investigations were carried out using various self-similar substitutions [25, 27, 50, 58, 59] or numerical calculations [60 – 66].

Nevertheless, there exists an approach admitting the solution of direct problems within the method of the Grad – Shafranov equation. This is possible if the exact solution of the equation is known and the flows to be investigated differ only slightly from the known one. Such an exact solution, as will be seen below, is the spherically-symmetric accretion (ejection) of matter. As a result, knowing the structure of the flow in a zeroth approximation, one can determine to a required accuracy both the position of singular surfaces and all the integrals of motion directly from boundary conditions, which allows the solution of the trans-field equation in the direct statement.

It should be noted that the method is not new and has long been applied in hydrodynamic calculations [39]. Nonetheless, only two such solutions have been constructed to date in astrophysics, namely, the force-free model of a magnetosphere of a slowly rotating black hole [36] and the model of a relativistic magnetised wind from a slowly rotating star with a monopole magnetic field [67]. At the same time, as shown in the following section, the approach described above does allow a notable broadening of the class of exact solutions. Our review is devoted to a discussion of such exact solutions.

2. The method of the Grad – Shafranov equation

2.1 3 + 1-split

We shall consider an axisymmetric stationary plasma flow in the vicinity of a rotating black hole, i.e., in the most general axisymmetric stationary metric. This metric (the Kerr metric) in Boyer – Lindquist co-ordinates r, θ is known to have the form [13, 68]

$$\mathrm{d}s^2 = -\alpha^2\,\mathrm{d}t^2 + g_{ik}(\mathrm{d}x^i + \beta^i\,\mathrm{d}t)(\mathrm{d}x^k + \beta^k\,\mathrm{d}t)\,, \qquad (1)$$

where

$$\alpha = \frac{\rho}{\Sigma}\sqrt{\Delta}\,, \qquad \beta^r = \beta^\theta = 0\,, \qquad \beta^\varphi = -\omega = -\frac{2a\mathcal{M}r}{\Sigma^2}\,,$$

$$g_{rr} = \frac{\rho^2}{\Delta}\,, \qquad g_{\theta\theta} = \rho^2\,, \qquad g_{\varphi\varphi} = \varpi^2\,. \qquad (2)$$

Here α is a lapse function which is equal to zero at the horizon $r_{\mathrm{H}} = \mathcal{M} + (\mathcal{M}^2 - a^2)^{1/2}$, ω is the angular velocity of locally non-rotating observers (the so-called Lense-Thirring angular velocity) and

$$\Delta = r^2 + a^2 - 2\mathcal{M}r\,, \qquad \rho^2 = r^2 + a^2\cos^2\theta\,,$$

$$\Sigma^2 = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta\,, \qquad \varpi = \frac{\Sigma}{\rho}\sin\theta\,. \qquad (3)$$

As usual, $\mathcal{M}$ and a are respectively the mass and the angular momentum of a black hole per unit mass, i.e., $a = J/\mathcal{M}$. The Greek indices correspond to four and the Latin indices to three dimensions, uncapped ones standing for vector components with respect to the co-ordinate basis $\partial/\partial r, \partial/\partial\theta, \partial/\partial\varphi$ in an 'absolute' 3-space and capped ones for their physical components. The symbol ∇_k denotes covariant differentiation in an 'absolute' 3-space with the metric g_{ik} (2). Finally, except for specially specified cases, the system of units $c = G = 1$ is employed.

The technique of a 3 + 1-split [13] is used throughout the paper. In this approach, physical quantities are expressed in terms of three-dimensional vectors which in local experiments would be measured by observers moving around a rotating black hole at angular velocity ω. The convenience of the 3 + 1-language lies in the fact that in this technique many expressions are of the same form as they are in the flat case. So, the first two of Maxwell's equations (in which we immediately set the derivatives with respect to time $\partial/\partial t$ to be equal to zero)

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e\,, \tag{4}$$

$$\nabla \cdot \mathbf{B} = 0\,, \tag{5}$$

$$\nabla \times (\alpha\mathbf{E}) = \mathcal{L}_\beta \mathbf{B}\,, \tag{6}$$

$$\nabla \times (\alpha\mathbf{B}) = -\mathcal{L}_\beta \mathbf{E} + 4\pi\alpha\mathbf{j} \tag{7}$$

remain unaltered. In the second pair we only add the Lie derivative that acts according to the rule $\mathcal{L}_\beta \mathbf{A} \equiv (\beta\nabla)\mathbf{A} - (\mathbf{A}\nabla)\beta$. The appearance of this derivative is due to the rotation of locally non-rotating observers. Finally, the energy density ε, the energy flux $\mathbf{S}$ and the stress tensor T_{ik} for an electromagnetic field and particles are determined in exactly the same manner as in a flat space [13]. On the other hand, the temporal and spatial components of the energy–momentum conservation law, $\nabla_\nu T^{\mu\nu} = 0$, are written in a more complicated form

$$-\frac{1}{\alpha}(\beta\nabla)\varepsilon = -\frac{1}{\alpha^2}\nabla \cdot (\alpha^2\mathbf{S}) + H_{ik}T^{ik}\,, \tag{8}$$

$$\nabla_k T_i^k + \frac{1}{\alpha} S_\varphi \frac{\partial\omega}{\partial x^i} + (\varepsilon\delta_i^k + T_i^k)\frac{1}{\alpha}\frac{\partial\alpha}{\partial x^k} = 0\,. \tag{9}$$

Here $\mathbf{g} = -(1/\alpha)\nabla\alpha$ is the gravitational acceleration, $H_{ik} = (1/\alpha)\nabla_i\beta_k$ is the so-called gravimagnetic tensor field, and the quantities α and β_k are defined by relations (2). Finally, the continuity equation in the stationary case will be written as

$$\nabla \cdot (\alpha n\mathbf{u}) = 0\,, \tag{10}$$

where n is the concentration of particles in their own rest system and $\mathbf{u} = \mathbf{v}(1 - v^2)^{-1/2}$ is the spatial component of the 4-velocity of matter.

2.2 Motion in a given poloidal field

We shall show how five 'integrals of motion', constant on magnetic surfaces, appear in the general case for axisymmetric stationary flows. For this purpose, it is convenient to introduce a scalar function $\Psi(r, \theta)$ which represents magnetic flux. As a result, for a magnetic field we obtain

$$\mathbf{B} = \frac{\nabla\Psi \times \mathbf{e}_{\hat\varphi}}{2\pi\varpi} - \frac{2I}{\alpha\varpi}\mathbf{e}_{\hat\varphi} \tag{11}$$

[$I(r, \theta)$ is the total electric current inside the region $\Psi < \Psi(r, \theta)$], and Maxwell's equation $\nabla \cdot \mathbf{B} = 0$ holds automatically. It is readily seen that $\mathbf{B} \cdot \nabla\Psi = 0$, and thus the condition $\Psi(r, \theta) = \text{const}$ actually defines magnetic surfaces. The coefficient of proportionality in Eqn (11) is chosen so that the quantity Ψ coincides with the magnetic flux inside the tube $\Psi = \text{const}$, that is, $\mathrm{d}\Psi = \mathbf{B} \cdot \mathrm{d}\mathbf{S}$.

We shall assume, as usual, that in the magnetosphere there is enough plasma to satisfy the freezing-in condition, which in the 3 + 1-split will be written as in flat space in the form

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0\,. \tag{12}$$

On the other hand, in view of stationarity [as well as the condition of zero longitudinal electric field $\mathbf{E} \cdot \mathbf{B} = 0$ following from (12)], the field $\mathbf{E}$ can be written as

$$\mathbf{E} = -\frac{\Omega_F - \omega}{2\pi\alpha}\nabla\Psi\,. \tag{13}$$

Now substituting relation (13) into Maxwell's equation (6), one can readily make sure that $\mathbf{B} \cdot \nabla\Omega_F = 0$, i.e., the quantity Ω_F must be constant on magnetic surfaces (the Ferraro isorotation law [69]): $\Omega_F = \Omega_F(\Psi)$.

Owing to Maxwell's equation $\nabla \cdot \mathbf{B} = 0$, the continuity equation (10) and the freezing-in condition (12), we can write the 4-velocity of matter, $\mathbf{u}$, in the form

$$\mathbf{u} = \frac{\eta}{\alpha n}\mathbf{B} + \gamma(\Omega_F - \omega)\frac{\varpi}{\alpha}\mathbf{e}_{\hat\varphi}\,, \tag{14}$$

where $\gamma = (1 - v^2)^{-1/2}$ is the Lorentz-factor of the matter and the quantity η represents a particle flow along the magnetic flux. In view of the relation $\nabla \cdot (\eta\mathbf{B}) = 0$ implied by (5), (10) η must be constant on the magnetic surfaces $\Psi = \text{const}$, that is, $\eta = \eta(\Psi)$.

The following two integrals of motion are due to axial symmetry and stationarity of the flows, which lead to conservation of the energy flux E and the z-component of the angular momentum L_z:

$$E = E(\Psi) = \frac{\Omega_F I}{2\pi} + \mu\eta(\alpha\gamma + \omega u_\varphi)\,, \tag{15}$$

$$L = L(\Psi) = \frac{I}{2\pi} + \mu\eta u_\varphi\,, \tag{16}$$

where $\mu = (\rho_m + P)/n$ (ρ_m is the internal energy density and P is the pressure) is the relativistic enthalpy. The total loss of energy W_{tot} and angular momentum K_{tot} are determined by the simple relations

$$W_{tot} = \int_0^{\Psi_{max}} E(\Psi)\,\mathrm{d}\Psi\,, \tag{17}$$

$$K_{tot} = \int_0^{\Psi_{max}} L(\Psi)\,\mathrm{d}\Psi\,. \tag{18}$$

Finally, in the axisymmetric case the isoentropy condition gives $s = s(\Psi)$, so the entropy per particle $s(\Psi)$ is, in fact, the fifth integral of motion.

We shall now show how the toroidal magnetic field $B_{\hat\varphi}$ and all the other plasma parameters can be restored if the five integrals of motion $\Omega_F(\Psi)$, $\eta(\Psi)$, $s(\Psi)$, $E(\Psi)$, and $L(\Psi)$, as well as the poloidal magnetic field $\mathbf{B}_p$, are known. To do so, we shall use the conservation laws (15), (16) which, together with the φ-component of Eqn (14), give

$$\frac{I}{2\pi} = \frac{\alpha^2 L - (\Omega_F - \omega)\varpi^2(E - \omega L)}{\alpha^2 - (\Omega_F - \omega)^2\varpi^2 - M^2}\,, \tag{19}$$

$$\gamma = \frac{1}{\alpha\mu\eta}\,\frac{\alpha^2(E - \Omega_F L) - M^2(E - \omega L)}{\alpha^2 - (\Omega_F - \omega)^2\varpi^2 - M^2}\,, \tag{20}$$

$$u_{\hat\varphi} = \frac{1}{\varpi\mu\eta}\,\frac{(E - \Omega_F L)(\Omega_F - \omega)\varpi^2 - LM^2}{\alpha^2 - (\Omega_F - \omega)^2\varpi^2 - M^2}\,, \tag{21}$$

where

$$M^2 = \frac{4\pi\eta^2\mu}{n}\,. \tag{22}$$

It can readily be seen that the quantity M^2 is, to an accuracy of the coefficient α^2, the square of the Mach number of the poloidal velocity u_p with respect to the Alfven velocity $u_A = B_p(4\pi n\mu)^{-1/2}$, i.e., $M^2 = \alpha^2 u_p^2/u_A^2$. Further on, it will be convenient for us to employ only the quantity M^2 because it remains finite at the horizon of a black hole.

Since $\mu = \mu(n, s)$, definition (22) allows the expression of the concentration n (and, therefore, the specific enthalpy μ) as a function of η, s, and M^2. This means that along with the five integrals of motion, the expressions for I, γ and $u_{\hat{\varphi}}$ depend on a single additional quantity, i.e., the Mach number M. To determine the Mach number, it is convenient to use an obvious relation $\gamma^2 - \mathbf{u^2} = \mathbf{1}$ which, owing to expressions (20) and (21), can be rewritten as follows:

$$\frac{K}{\varpi^2 A^2} = \frac{1}{64\pi^4} \frac{M^4 (\nabla\Psi)^2}{\varpi^2} + \alpha^2\eta^2\mu^2 \,, \tag{23}$$

where

$$A = \alpha^2 - (\Omega_{\mathrm{F}} - \omega)^2\varpi^2 - M^2 \,, \tag{24}$$

$$K = \alpha^2\varpi^2(E - \Omega_{\mathrm{F}}L)^2\left[\alpha^2 - (\Omega_{\mathrm{F}} - \omega)^2\varpi^2 - 2M^2\right] + M^4\left[\varpi^2(E - \omega L)^2 - \alpha^2 L^2\right]. \tag{25}$$

Relations (19)–(21) and the Bernoulli equation (23) are precisely the algebraic constraints that allow the determination, although implicitly, of all the characteristics of a flow for a known poloidal field $\mathbf{B}_{\mathrm{p}}$ (i.e., by a known potential Ψ) and the known five integrals of motion. These equalities (19)–(21) and (23) have been analysed in an uncountable number of papers, beginning from those devoted to stellar (solar) wind [21–26], where their nonrelativistic limit was of course used, to the studies of relativistic pulsar wind [28–31], hydrodynamic and MHD accretion of matter onto black holes [32–35].

2.3 Singular surfaces

The algebraic relations (19)–(21), (23) allow the determination of the singular surfaces of MHD flows on which the poloidal velocity of the medium v_{p} is compared with the proper velocities of axisymmetric excitations capable of propagating in a plasma. Such singular surfaces are:

1. The Alfven surface A, determined from the condition of equality to zero of the denominator A (24) in the algebraic relations (19)–(21):

$$A = 0 \,. \tag{26}$$

From definitions (22) and (26) we obtain that on the Alfven surface the condition

$$u_{\mathrm{p}}^2 = u_{\mathrm{A}}^2\left[1 - \frac{(\Omega_{\mathrm{F}} - \omega)^2\varpi^2}{\alpha^2}\right] \tag{27}$$

must hold, which in the nonrelativistic limit naturally coincides with the Alfven velocity. As shown in Ref. [21], on a plane with co-ordinates u^r, r it is a higher order point than, for example, a saddle or a focus. On the other hand, it turns out that all the trajectories with a positive square of the energy E come through this point. This means that the algebraic relations (19)–(21) themselves contain no singularity, and the regularity conditions (zero numerators for nonzero denominators) are only defined by the position of the Alfven surface. At the same time, as shown below, the Grad–Shafranov equation itself has a singularity on the Alfven surface.

2. The fast and the slow magnetosonic surfaces F and S. The easiest way to define these surfaces is to treat them as singularities in the expression for the gradient of the Mach number M. Indeed, using relations (23)–(25), which can be rewritten in the form $(\nabla\Psi)^2 = F(M^2, E, L, \eta, \Omega_{\mathrm{F}}, \mu)$, where

$$F = \frac{64\pi^4}{M^4}\frac{K}{A^2} - \frac{64\pi^4}{M^4}\alpha^2\varpi^2\eta^2\mu^2 \,, \tag{28}$$

we obtain

$$\nabla_a M^2 = \frac{N_a}{D} \,, \tag{29}$$

where

$$N_a = -\frac{A}{(\nabla\Psi)^2}\nabla^b\Psi\cdot\nabla_a\nabla_b\Psi + \frac{A}{2}\frac{\nabla_a' F}{(\nabla\Psi)^2} \,. \tag{30}$$

Here and below, the indices a, b run through the values r, θ only and the operator ∇_a' acts on all the quantities except for M^2. The denominator D can be rewritten in the form

$$D = \frac{A}{M^2} + \frac{\alpha^2}{M^2}\frac{B_{\hat{\varphi}}^2}{B_{\mathrm{p}}^2} - \frac{1}{u_{\mathrm{p}}^2}\frac{A}{M^2}\frac{c_{\mathrm{s}}^2}{1 - c_{\mathrm{s}}^2} \,, \tag{31}$$

where $c_{\mathrm{s}}^2 = (1/\mu)(\partial P/\partial n)_{\mathrm{s}}$ is the velocity of sound and $\nabla\mu$ is represented as

$$\nabla\mu = \frac{c_{\mathrm{s}}^2}{1 - c_{\mathrm{s}}^2}\,\mu\left[2\frac{\nabla\eta}{\eta} - \frac{\nabla M^2}{M^2}\right] + \frac{1}{1 - c_{\mathrm{s}}^2}\left[\frac{1}{nm_{\mathrm{p}}}\left(\frac{\partial P}{\partial s}\right)_n + T\right]\nabla s \,. \tag{32}$$

We shall stress that relation (32) should be used in the differentiation of $\nabla_a' F$ in formula (30) because expression (28) for F contains the quantity μ. The condition of equality to zero of the denominator in the expression (29),

$$D = 0 \,, \tag{33}$$

determines the fast and slow singular surfaces. Indeed, making use of definition (22), we obtain that $D = 0$ for

$$(u_{\mathrm{p}})_{1,2}^2 = \frac{1}{2}Wu_{\mathrm{A}}^2 + \frac{1}{2}\frac{c_{\mathrm{s}}^2}{1 - c_{\mathrm{s}}^2} \pm \frac{1}{2}\left[\left(Wu_{\mathrm{A}}^2 + \frac{c_{\mathrm{s}}^2}{1 - c_{\mathrm{s}}^2}\right)^2 - 4\left(W - \frac{B_{\hat{\varphi}}^2}{B_{\mathrm{p}}^2}\right)\frac{c_{\mathrm{s}}^2}{1 - c_{\mathrm{s}}^2}u_{\mathrm{A}}^2\right]^{1/2}, \tag{34}$$

where

$$W = 1 - \frac{(\Omega_{\mathrm{F}} - \omega)^2\varpi^2}{\alpha^2} + \frac{B_{\hat{\varphi}}^2}{B_{\mathrm{p}}^2} \tag{35}$$

and, as before, $u_{\mathrm{A}} = B_{\mathrm{p}}/\sqrt{4\pi n\mu}$, that is, it is determined in terms of the poloidal component of the magnetic field. In the non-relativistic approximation we have, respectively, the well- known expressions [70]

$$(u_{\mathrm{p}})_{1,2}^2 = \frac{1}{2}(u_{\mathrm{a}}^2 + c_{\mathrm{s}}^2) \pm \frac{1}{2}\sqrt{(u_{\mathrm{a}}^2 + c_{\mathrm{s}}^2)^2 - 4c_{\mathrm{s}}^2u_{\mathrm{a}}^2\cos^2\theta} \,, \tag{36}$$

where now the equality $u_{\mathrm{a}} = B/\sqrt{4\pi\rho}$ holds (ρ is the density of matter) and θ is the angle between the wave vector and the magnetic field. For a cold plasma $c_{\mathrm{s}} = 0$, the slow magnetosonic velocity is zero [71], and so in this case the slow magnetosonic surface is absent altogether.

The fast and the slow surfaces, unlike the Alfven surface, are saddle points, i.e. transonic solutions only exist for a certain specific relation among the integrals of motion. These relations are obtained from the regularity conditions

$$N_r = 0\,, \qquad N_\theta = 0 \tag{37}$$

for $D = 0$. The regularity conditions (33) and (37), as we shall see below, are a crucial stage in the construction of analytical solutions of the trans-field equation.

3. It will be shown later that the trans-field equation has one more singular surface, namely, the **cusp surface** C which is determined from the condition $D = -1$. This singular surface is associated with the so-called cusp velocity appearing in the Friedrichs group diagram as a cusp for a slow magnetosonic wave [71]. As a result, we obtain for the corresponding velocity

$$u_{\mathrm{c}}^2 = u_{\mathrm{A}}^2\,\frac{\left[\alpha^2 - (\Omega_{\mathrm{F}} - \omega)^2\varpi^2\right]c_{\mathrm{s}}^2/(1 - c_{\mathrm{s}}^2)}{\left[\alpha^2 - (\Omega_{\mathrm{F}} - \omega)^2\varpi^2 + \alpha^2 B_\varphi^2/B_{\mathrm{p}}^2\right]u_{\mathrm{A}}^2 + \alpha^2 c_{\mathrm{s}}^2/(1 - c_{\mathrm{s}}^2)}\,. \tag{38}$$

In the non-relativistic limit for $B_\varphi = 0$, this expression is also transformed into the well-known expression $u_{\mathrm{c}} = u_{\mathrm{a}} c_{\mathrm{s}} (u_{\mathrm{a}}^2 + c_{\mathrm{s}}^2)^{-1/2}$ [71].

4. The light cylinder R_{L}, i.e., the surface on which the electric field $\mathbf{E}$ is equal in magnitude to the poloidal component of the magnetic field $\mathbf{B}_{\mathrm{p}}$. According to (11) and (13), far from gravitating bodies we have $\varpi = R_{\mathrm{L}} = 1/\Omega_{\mathrm{F}}$. In the case of a black hole magnetosphere, there is another 'light cylinder' located on the surface $\alpha = |\Omega_{\mathrm{F}} - \Omega_{\mathrm{H}}|\varpi_{\mathrm{H}}$. On the light cylinder there are no additional regularity conditions.

5. Finally, the characteristic surface is the **light surface** S_{L} on which the electric field $\mathbf{E}$ is equal in magnitude to the magnetic field $\mathbf{B}$. The light surface, like the limit line in pure hydrodynamics, determines the natural boundary of a continuous flow.

It can be readily seen that a moving plasma first intersects the Alfven surface, then the light cylinder, and finally the fast magnetosonic surface. The light surface (if existent) is located at even larger distances. On the other hand, the effects of the general theory of relativity lead to the appearance of a second family of singular surfaces in the vicinity of a black hole horizon. Exceedingly important is the fact that the external Alfven surface (traversed by all the trajectories, as mentioned above) corresponds to values $u^r > 0$, i.e., to an ejected plasma, whereas the internal Alfven point corresponds to a value $u^r < 0$, i.e., to accretion [34]. But this contradicts the assumption of constancy of the function η on a given field line $\Psi = \mathrm{const}$. Consequently, the plasma flow in the magnetosphere of a black hole (more precisely, on field lines through the horizon) cannot be continuous, and we have to introduce into consideration the regions of plasma generation in which the Grad–Shafranov equation does not hold. What has been said above does not, of course, refer to the hydrodynamic regime of accretion in which the Alfven surface is absent.

2.4 The trans-field equation

We shall now proceed to the discussion of the Grad–Shafranov equation — the trans-field equation of magnetic field lines. Writing the poloidal component of the energy-momentum conservation law (9), we make sure that this vector trans-field equation reduces to a scalar second-order equation multiplied by $\nabla_a\Psi$. It can be written in the form [57]

$$\frac{1}{\alpha}\,\nabla_k\left\{\frac{1}{\alpha\varpi^2}\left[\alpha^2 - (\Omega_{\mathrm{F}} - \omega)^2\varpi^2 - M^2\right]\nabla^k\Psi\right\} + \frac{\Omega_{\mathrm{F}} - \omega}{\alpha^2}\,(\nabla\Psi)^2\,\frac{\mathrm{d}\Omega_{\mathrm{F}}}{\mathrm{d}\Psi} + \frac{64\pi^4}{\alpha^2\varpi^2}\,\frac{1}{2M^2}\,\frac{\partial}{\partial\Psi}\left(\frac{G}{A}\right) - 16\pi^3\mu n\,\frac{1}{\eta}\,\frac{\mathrm{d}\eta}{\mathrm{d}\Psi} - 16\pi^3 nT\,\frac{\mathrm{d}s}{\mathrm{d}\Psi} = 0\,, \tag{39}$$

where

$$G = \alpha^2\varpi^2(E - \Omega_{\mathrm{F}}L)^2 + \alpha^2 M^2 L^2 - M^2\varpi^2(E - \omega L)^2\,. \tag{40}$$

Decomposing (39) with respect to $\nabla_a M^2$ according to definitions (29), (30), we finally obtain

$$A\left[\frac{1}{\alpha}\nabla_k\left(\frac{1}{\alpha\varpi^2}\,\nabla^k\Psi\right) + \frac{1}{\alpha^2\varpi^2(\nabla\Psi)^2}\,\frac{\nabla^a\Psi\cdot\nabla^b\Psi\cdot\nabla_a\nabla_b\Psi}{D}\right] + \frac{1}{\alpha^2\varpi^2}\,\nabla'_k A\cdot\nabla^k\Psi - \frac{A}{\alpha^2\varpi^2(\nabla\Psi)^2}\,\frac{1}{2D}\,\nabla'_k F\cdot\nabla^k\Psi + \frac{\Omega_{\mathrm{F}} - \omega}{\alpha^2}\,\frac{\mathrm{d}\Omega_{\mathrm{F}}}{\mathrm{d}\Psi}(\nabla\Psi)^2 + \frac{64\pi^4}{\alpha^2\varpi^2}\,\frac{1}{2M^2}\,\frac{\partial}{\partial\Psi}\left(\frac{G}{A}\right) - 16\pi^3\mu n\,\frac{1}{\eta}\,\frac{\mathrm{d}\eta}{\mathrm{d}\Psi} - 16\pi^3 nT\,\frac{\mathrm{d}s}{\mathrm{d}\Psi} = 0\,, \tag{41}$$

where the gradient ∇'_a again acts on all the quantities except for M^2, and the derivative $\partial/\partial\Psi$ acts on the integrals of motion only. So, formula (41) determines the trans-field equation of magnetic surfaces in the most general form.

It is noteworthy that equation (41) only contains the flow function Ψ and the five integrals of motion. Indeed, the thermodynamic quantities can be expressed, using the equations of state and definitions (22) and (32), as functions of entropy $s(\Psi)$, of the quantity $\eta(\Psi)$ and of the square of the Mach number M^2. By virtue of the Bernoulli equation (23), the quantity M^2 itself is expressed, although implicitly, in terms of the gradient $(\nabla\Psi)^2$ and the five integrals of motion. But, of course, the physical root of equation (23) should be chosen. As concerns the Grad–Shafranov equation itself,

$$r^2\sin^2\theta\nabla_k\left(\frac{1}{r^2\sin^2\theta}\,\nabla^k\Psi\right) + 16\pi^2 I\,\frac{\mathrm{d}I}{\mathrm{d}\Psi} + 16\pi^3 r^2\sin^2\theta\,\frac{\mathrm{d}P}{\mathrm{d}\Psi} = 0\,, \tag{42}$$

i.e., the equation describing, in the nonrelativistic case ($\alpha = 1$, $\omega = 0$), stable stationary ($\mathbf{v} = 0$, i.e., $\gamma = 1$) axisymmetric configurations, it is derived from (41) through the limiting transition $\Omega_{\mathrm{F}} \to 0$ (which corresponds to an infinitely distant light cylinder $R_{\mathrm{L}} \to \infty$), $L \to 0$ and $\eta \to 0$. In this case, as is seen from definitions (23) and (31), we also have $M^2 \to 0$, $D^{-1} \to 0$ and $E \to \mu\eta$, the current I and the enthalpy μ (and, therefore, any other thermodynamic function) becoming integrals of motion.

The trans-field equation (41) is a second-order equation, linear with respect to higher derivatives. In other words, it can be written in the canonical form

$$\mathcal{A}\,\frac{\partial^2\Psi}{\partial r^2} + 2\mathcal{B}\,\frac{\partial^2\Psi}{\partial r\,\partial\theta} + \mathcal{C}\,\frac{\partial^2\Psi}{\partial\theta^2} + \mathcal{F} = 0\,, \tag{43}$$

and $\mathcal{AC} - \mathcal{B}^2 = A^2 D(D+1)$. Hence, as should be expected, it changes from elliptic to hyperbolic on singular surfaces, on which the poloidal velocity of matter is equal either to the fast or slow magnetosonic velocity when we have $D = 0$, or to the cusp velocity when we have $D = -1$. On the Alfven surface $A = 0$, the form of the equation remains unaltered. Nevertheless, the Alfven surface is also a singular surface of the trans-field equation because the regularity condition on it implied directly from (41)

$$\frac{1}{\alpha^2 \varpi^2} \nabla_k' A \cdot \nabla^k \Psi + \frac{\Omega_{\rm F} - \omega}{\alpha^2} \frac{{\rm d}\Omega_{\rm F}}{{\rm d}\Psi} (\nabla \Psi)^2 + \frac{64\pi^4}{\alpha^2 \varpi^2} \frac{1}{2M^2} \frac{\partial}{\partial \Psi} \left(\frac{G}{A} \right) - 16\pi^3 \mu n \frac{1}{\eta} \frac{{\rm d}\eta}{{\rm d}\Psi} - 16\pi^3 n T \frac{{\rm d}s}{{\rm d}\Psi} = 0 \,, \tag{44}$$

must hold.

The following group of problems is connected with boundary conditions. On the one hand, the trans-field equation contains three integrals of motion in the hydrodynamic limit and five integrals of motion in the general case, which must generally be determined from boundary conditions. On the other hand, on singular surfaces the regularity conditions (37) and (44) must hold. Thus, at least for particularly simple topologies, when all the field lines (flow lines) intersect all s singular surfaces, the number of boundary conditions b can be written in the form

$$b = 2 + i - s \,, \tag{45}$$

where i is the number of integrals of motion. Note that since the trans-field equation can be rewritten as $D + K_1 N_r + K_2 N_\theta = 0$, relations (37) define only one regularity condition. The second condition will be fulfilled automatically on the solution of equation (41).

We shall now consider in detail the behaviour of the solution of the trans-field equation near the horizon of a rotating black hole. For physically reasonable solutions, when the velocity on the horizon is not equal to zero and accordingly $M^2(r_{\rm H}) \neq 0$, the quantity $D(r_{\rm H})$ can be rewritten in the form

$$D(r_{\rm H}) = -1 + \frac{\alpha^2}{M^2 B_{\rm p}^2} (B_{\hat\varphi}^2 - E_{\hat\theta}^2) \,. \tag{46}$$

On the other hand, as follows from the algebraic Bernoulli equation (23), for $M^2(r_{\rm H}) \neq 0$ on the horizon the condition

$$\frac{(E - \Omega_{\rm H} L)^2}{\left[(\Omega_{\rm F} - \Omega_{\rm H})^2 \varpi^2 + M^2 \right]^2} = \frac{1}{64\pi^4} \left(\frac{{\rm d}\Psi}{{\rm d}\theta} \right)^2 \frac{1}{\rho^2 \varpi^2} \,, \tag{47}$$

must hold. Here $\Omega_{\rm H} = \omega(r_{\rm H})$ is by definition the angular velocity of a rotating black hole. Owing to relations (11) and (13), condition (47) can be rewritten as $|B_{\hat\varphi}(r_{\rm H})| = |E_{\hat\theta}(r_{\rm H})|$. This result is in perfect agreement with the key point of the 'membrane paradigm', according to which a locally non-rotating observer must register only the φ-component of the magnetic field and the θ-component of the electric field that diverge as $1/\alpha$ [13]. As a result, on the horizon the condition

$$D(r_{\rm H}) = -1 \tag{48}$$

must hold, so that in the case $M^2(r_{\rm H}) \neq 0$ Eqn (41) near the horizon of a black hole must be hyperbolic. Thus, we arrive at an important conclusion that the trans-field equation (41) requires no boundary condition at the horizon, exactly as expected because by definition no signal can propagate from the horizon to external regions of the magnetosphere [33].

Indeed, we shall show in conclusion that condition (47) can be obtained immediately from the Grad – Shafranov equation (41). To this end, it is convenient to use Eqn (39). In the limit $\alpha \to 0$ we have

$$\frac{1}{\alpha} \nabla_k \left(\frac{1}{\varpi^2 \alpha} A \nabla^k \Psi \right) + \frac{\Omega_{\rm F} - \Omega_{\rm H}}{\alpha^2} (\nabla \Psi)^2 \frac{{\rm d}\Omega_{\rm F}}{{\rm d}\Psi} - \frac{32\pi^4}{\alpha^2} \frac{\partial}{\partial \Psi} \left[\frac{(E - \Omega_{\rm H} L)^2}{A} \right] = 0 \,. \tag{49}$$

Since $D = -1$ at the horizon, Eqn (39) here is parabolic. As a result, multiplying (49) by $(2A/\sin^2\theta)({\rm d}\Psi/{\rm d}\theta)$, we have

$$2A\varpi^2 (\Omega_{\rm F} - \Omega_{\rm H}) \frac{{\rm d}\Omega_{\rm F}}{{\rm d}\Psi} \left[\frac{1}{\varpi^2 \rho^2} \left(\frac{{\rm d}\Psi}{{\rm d}\theta} \right)^2 - \frac{64\pi^4 (E - \Omega_{\rm H} L)^2}{A^2} \right] + \frac{\rm d}{{\rm d}\theta} \left[\frac{A^2}{\varpi^2 \rho^2} \left(\frac{{\rm d}\Psi}{{\rm d}\theta} \right)^2 - 64\pi^4 (E - \Omega_{\rm H} L)^2 \right] = 0 \,, \tag{50}$$

which implies Eqn (47).

3. The hydrodynamic limit — classical accretion and ejection problems

3.1 Hydrodynamic limit of the trans-field equation

Accretion of matter onto black holes is one of the classical problems of present-day astrophysics. It is associated with the problem of activity of galactic nuclei and quasars [2, 3], the jet formation mechanism [12, 13] and the nature of some other galactic sources [72, 73]. First considered was the example of isentropy flow onto a moving gravitating centre (Bondi – Hoyle accretion [1]). The numerical calculations carried out subsequently by many authors [61 – 65, 74] allowed a sufficiently comprehensive analysis of all the basic properties of such flows. At the same time, even for adiabatic flows with zero viscosity, exact solutions up to now have been obtained only in some particular cases. What has been said above also refers to the problem of gas ejection from stars, for which exact two-dimensional solutions have not been found either, although numerically this question has been studied in much detail [75, 76].

First of all, therefore, we shall consider the hydrodynamic limit of the Grad – Shafranov equation in which we may ignore the electromagnetic field contribution. In this case it is convenient to introduce a new potential $\Phi(\Psi)$ satisfying the condition $\eta(\Psi) = {\rm d}\Phi/{\rm d}\Psi$. As can be readily verified, in (41) such a change corresponds to the conditions $\Psi \to \Phi$, $\eta \to 1$. According to (14), we obtain

$$\alpha n \mathbf{u}_{\rm p} = \frac{1}{2\pi\varpi} (\nabla \Phi \times \mathbf{e}_{\hat\varphi}) \,. \tag{51}$$

The lines $\Phi(r, \theta) = {\rm const}$ determine the flow lines of matter.

In the hydrodynamic limit there exist only three integrals of motion. They are the energy flux and the z-component of angular momentum

$$E(\Phi) = \mu(\alpha\gamma + \varpi\omega u_{\hat\varphi}) \,, \tag{52}$$

$$L(\Phi) = \mu\varpi u_{\hat\varphi} \,, \tag{53}$$

as well as the entropy $s = s(\Phi)$. The algebraic constraint (Bernoulli) equation (23) now has the form

$$(E - \omega L)^2 = \alpha^2\mu^2 + \frac{\alpha^2}{\varpi^2} L^2 + \frac{\hat{M}^4}{64\pi^4\varpi^2}(\nabla\Phi)^2 , \tag{54}$$

where the square of the 'Mach number' $\hat{M}^2$ is specified as $\hat{M}^2 = 4\pi\mu/n$. The Grad–Shafranov equation (41) will be rewritten as in Ref. [57]

$$- \hat{M}^2\left[\frac{1}{\alpha}\nabla_k\left(\frac{1}{\alpha\varpi^2}\nabla^k\Phi\right) + \frac{1}{\alpha^2\varpi^2(\nabla\Phi)^2}\frac{\nabla^a\Phi\cdot\nabla^b\Phi\cdot\nabla_a\nabla_b\Phi}{D}\right]$$
$$+ \frac{\hat{M}^2\nabla'_k\hat{F}\cdot\nabla^k\Phi}{2\alpha^2\varpi^2(\nabla\Phi)^2 D}$$
$$+ \frac{64\pi^4}{\alpha^2\varpi^2\hat{M}^2}\left[\varpi^2(E-\omega L)\left(\frac{dE}{d\Phi} - \omega\frac{dL}{d\Phi}\right) - \alpha^2 L\frac{dL}{d\Phi}\right]$$
$$- 16\pi^3 nT\frac{ds}{d\Phi} = 0 , \tag{55}$$

where we now have

$$D = -1 + \frac{1}{u_p^2}\frac{c_s^2}{1 - c_s^2} , \tag{56}$$

$$\hat{F} = \frac{64\pi^4}{\hat{M}^4}\left[\varpi^2(E-\omega L)^2 - \alpha^2 L^2 - \varpi^2\alpha^2\mu^2\right] . \tag{57}$$

Here, as in the general case, the derivative ∇'_k acts on all variables except for the Mach number $\hat{M}$. As we shall see, Eqn (55) contains only one singular surface determined from the equation $D = 0$. Finally, in the derivative $\nabla_k\hat{M}^2 = -\hat{M}^2 N_k/D$ the denominator N_k now (in the spherically symmetric case) looks like

$$N_k = -\frac{\nabla^i\Phi\cdot\nabla_i\nabla_k\Phi}{(\nabla\Phi)^2} + \frac{1}{2}\frac{\nabla_k\varpi^2}{\varpi^2} - \frac{1}{2}\frac{\mu^2\nabla_k\alpha^2}{E^2 - \alpha^2\mu^2} \tag{58}$$

and D is given by Eqn (56).

In the non-relativistic case, Eqn (51) has the form

$$n\mathbf{v}_p = \frac{1}{2\pi\varpi}(\nabla\Phi\times\mathbf{e}_{\hat\varphi}) , \tag{59}$$

and the first two integrals of motion (52) and (53) are

$$E(\Phi) = \frac{v^2}{2} + w(n,s) + \varphi_G(r,\theta) \tag{60}$$

(which is none other than the Bernoulli integral) and

$$L(\Phi) = rv_\varphi\sin\theta . \tag{61}$$

Here w is the non-relativistic enthalpy per particle. The Grad–Shafranov equation will now be written as

$$- \varpi^2\nabla_k\left(\frac{1}{\varpi^2}\nabla^k\Phi\right) - \frac{\nabla^i\Phi\cdot\nabla^k\Phi\cdot\nabla_i\nabla_k\Phi}{D(\nabla\Phi)^2} + \frac{\nabla\varpi^2\cdot\nabla\Phi}{2D\varpi^2}$$
$$- 4\pi^2\varpi^2 n^2\frac{\nabla\varphi_G\cdot\nabla\Phi}{D(\nabla\Phi)^2} - 4\pi^2 n^2\frac{D+1}{D}L\frac{dL}{d\Phi}$$
$$+ 2\pi^2\varpi^2 n^2\frac{\nabla\varpi^2\cdot\nabla\Phi}{D\varpi^4(\nabla\Phi)^2}L^2 + 4\pi^2\varpi^2 n^2\frac{D+1}{D}\frac{dE}{d\Phi}$$
$$- 4\pi^2\varpi^2 n^2\left[\frac{D+1}{D}\frac{T}{m_p} + \frac{1}{Dm_p n}\left(\frac{\partial P}{\partial s}\right)_n\right]\frac{ds}{d\Phi} = 0 . \tag{62}$$

Here m_p is the particle mass,

$$D = -1 + \frac{c_s^2}{v_p^2} , \tag{63}$$

$\varphi_G = \varphi_G(r,\theta)$ is Newton's gravitational potential and all the operators ∇_k act in a flat space: $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{\varphi\varphi} = \varpi^2 = r^2\sin^2\theta$. Finally, the algebraic constraint (Bernoulli) equation has the form

$$2E - 2\varphi_G(r,\theta) - 2w(n,s) - \frac{L^2}{r^2\sin^2\theta} = \frac{(\nabla\Phi)^2}{4\pi^2 r^2 n^2\sin^2\theta} . \tag{64}$$

Since the enthalpy w can be expressed as a function of n and s, Eqn (64) specifies, although implicitly, the concentration n and all the other thermodynamic quantities as functions of the potential Φ and of the three integrals of motion. The condition (29) will then become $\nabla_k n = nN_k/D$, where

$$N_k = -\frac{\nabla^i\Phi\cdot\nabla_i\nabla_k\Phi}{(\nabla\Phi)^2} + \frac{1}{2}\frac{\nabla_k\varpi^2}{\varpi^2} - 4\pi^2\varpi^2 n^2\frac{\nabla_k\varphi_G}{(\nabla\Phi)^2}$$
$$- 4\pi^2 n^2 L\frac{dL}{d\Phi}\frac{\nabla_k\Phi}{(\nabla\Phi)^2} + 2\pi^2 n^2\frac{\nabla_k\varpi^2}{\varpi^2(\nabla\Phi)^2}L^2$$
$$+ 4\pi^2\varpi^2 n^2\frac{\nabla_k\Phi}{(\nabla\Phi)^2}\frac{dE}{d\Phi}$$
$$- 4\pi^2\varpi^2 n^2\left[\frac{T}{m_p} + \frac{1}{m_p n}\left(\frac{\partial P}{\partial s}\right)_n\right]\frac{\nabla_k\Phi}{(\nabla\Phi)^2}\frac{ds}{d\Phi} . \tag{65}$$

As we can see, in the non-relativistic case the role of 'Mach numbers' is simply played by the concentration n.

Let us now present the main relations for the case of spherically symmetric accretion of matter onto a non-rotating (Schwarzschild) black hole. The velocity of the medium at infinity is naturally assumed to be zero, so that $\gamma_\infty = 1$. The flow in this case is completely described by two constants, for example, two thermodynamic functions at infinity, s_∞ and μ_∞, which, according to (60), also fix the Bernoulli integral E. The flow function Φ is then given by the trivial formula

$$\Phi = \Phi_0(1 - \cos\theta) . \tag{66}$$

The quantity Φ_0 itself (together with the accretion rate $2m_p\Phi_0$) must be found from the condition of passage through the sound point $D(r_0, c_0) = 0$ and $N_r(r_0, c_0) = 0$. Eventually using the Bernoulli equation (54), we obtain the well-known expressions for the radius of the sonic surface [3]:

$$r_0 = \frac{\mathcal{M}}{2}\left(\frac{1}{c_0^2} + 3\right) \tag{67}$$

[so for $c_0^2 \ll 1$ we simply have $r_0 = \mathcal{M}/(2c_0^2)$], for the velocity of sound on the sonic surface

$$c_0^2 = \frac{2}{5 - 3\Gamma}c_\infty^2 , \qquad \Gamma \neq \frac{5}{3} \tag{68}$$

and for the accretion rate

$$2m_p\Phi_0 = -\pi\left(\frac{2}{5 - 3\Gamma}\right)^{(5-3\Gamma)/(2\Gamma-2)}\frac{\mathcal{M}^2}{c_\infty^3}\rho_\infty . \tag{69}$$

We shall mention two other cases in which exact solutions to the trans-field equation (55) can be obtained. First of all, such a solution can be obtained for dust-like matter with $P = 0$ for which $c_s^2 = 0$ and, accordingly, $D = -1$ [63]. In this case, the motion of matter must coincide with the motion of test particles resting at infinity and possessing angular momentum with respect to the black hole. It is a known fact (see, e.g., Ref. [68]) that in the case of such a motion the flow lines must be straight lines ($\theta = \mathrm{const}$) whose density is not necessarily uniform because of the absence of pressure. It can be readily verified that the arbitrary function $\Phi = \Phi(\theta)$ for $D = -1$ is indeed a solution to the nonlinear trans-field equation of an arbitrarily fast black hole rotation.

Finally, a solution can also be obtained for a medium with $c_s^2 = 1$ where, according to Eqn (56), we have $D^{-1} = 0$. In this case, the trans-field equation (55) becomes linear:

$$\Delta \frac{\partial^2 \Phi}{\partial r^2} + \sin\theta \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\theta}\right) = 0\,. \tag{70}$$

The solution of the linear equation (70) can be expanded in eigenfunctions $Q_n(\theta)$ of the angular operator:

$$\mathcal{L}_\theta = \sin\theta \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right). \tag{71}$$

These eigenfunctions have the form

$$Q_0 = 1 - \cos\theta\,, \tag{72}$$

$$Q_m = \frac{2^m m!(m-1)!}{(2m)!}\sin^2\theta\, P'_m(\cos\theta)\,, \qquad m = 1, 2, \ldots \tag{73}$$

(P_m are Legendre polynomials). In particular, $Q_1 = \sin^2\theta$ and $Q_2 = \sin^2\theta\cos\theta$. The summand proportional to $g_0(1 - \cos\theta)$ corresponds to the case of spherically symmetric accretion. For the summand proportional to $g_1(r)\sin^2\theta$, it can easily be verified from Eqn (7) that at large distances ($r \gg \mathcal{M}$) from the black hole it has the form

$$\Phi = \pi n_\infty v_\infty r^2 \sin^2\theta\,, \tag{74}$$

which corresponds to a homogeneous flow of matter. So, for a Schwarzschild black hole we have

$$\Phi = \Phi_0(1 - \cos\theta) + \Phi_1(r^2 - 2\mathcal{M}r)\sin^2\theta\,. \tag{75}$$

In the case $c_s^2 = 1$ we can see that the accretion rate $2\Phi_0$ is arbitrary because the flow remains subsonic up to the black hole horizon. A solution equivalent to (75) was first obtained in Ref. [63] for arbitrary velocity of black hole rotation.

3.2 Accretion onto a black hole

We begin by considering accretion onto a slowly rotating black hole [77]. According to (45), such a problem generally requires four boundary conditions. We shall assume, however, that the accreting plasma does not possess intrinsic angular momentum. Since the quantity L is an integral of motion, everywhere we have $L(\Phi) = 0$, which, according to (53), means that $u_\varphi = 0$. The equality $u_\varphi = 0$ shows that the toroidal velocity of matter with respect to remote observers corresponds to rotation with Lense–Thirring angular velocity ω (2). Furthermore, the flow at large distances must clearly be spherically symmetric $\Phi \to \Phi_0(1 - \cos\theta)$. As a result, the flow, as in the case of spherically symmetric accretion, must be uniquely determined by two quantities, for example, two thermodynamic functions at infinity.

Now introducing the parameter $\varepsilon = a/\mathcal{M}$, we therefore have $\varepsilon \ll 1$ for the condition of slow rotation. As is seen from definitions (2) and (3), in the case $\varepsilon \ll 1$, corrections to the metric coefficients g_{ik} have the order of smallness ε^2. It is thus natural to seek the solution of the trans-field equation (55) in the form of a small correction to the spherically symmetric solution (66):

$$\Phi = \Phi_0\left[1 - \cos\theta + \varepsilon^2 f(r,\theta)\right], \tag{76}$$

where the quantity Φ_0 is specified, as before, by relation (69). Substituting now expression (76) into the trans-field equation (55), we obtain

$$-\varepsilon^2\alpha^2 D\frac{\partial^2 f}{\partial r^2} - \frac{\varepsilon^2}{\rho^2}(D+1)\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\theta}\right) + \varepsilon^2\alpha^2 N_r\frac{\partial f}{\partial r}$$
$$= \frac{a^2}{r^4}\left(1 - \frac{2\mathcal{M}}{r}\right)\left(1 - 2\frac{\mu^2}{E^2 - \alpha^2\mu^2}\frac{\mathcal{M}}{r}\right)\sin^2\theta\cos\theta\,, \tag{77}$$

where

$$N_r = \frac{2}{r} - \frac{\mu^2}{E^2 - \alpha^2\mu^2}\frac{\mathcal{M}}{r^2}\,. \tag{78}$$

We note first of all that according to condition (56) and likewise to the relation $u_p^2 = (E^2 - \alpha^2\mu^2)/\alpha^2\mu^2$, we have $D + 1 \propto \alpha^2$. Hence, the linearised trans-field equation (77) contains no singularities at the horizon of a black hole. On the other hand, the angular operator in Eqn (77) coincides with the angular operator of Eqn (70). Eventually, as shown in Ref. [77], the complete solution of Eqn (24) is written in the form

$$\Phi(r,\theta) = \Phi_0\big[1 - \cos\theta + \varepsilon^2 g_0(1 - \cos\theta)$$
$$+ \varepsilon^2 g_2(r)\sin^2\theta\cos\theta\big]\,, \tag{79}$$

where

$$g_0 = -2\frac{\mathcal{M}^3}{r_0^3} \tag{80}$$

(which corresponds to a decrease in the accretion rate) and $g_2(r)$ must be found as a solution of the ordinary differential equation

$$-D\frac{\mathrm{d}^2 g_2}{\mathrm{d}r^2} + N_r\frac{\mathrm{d}g_2}{\mathrm{d}r} + 6\frac{\mu^2}{E^2 - \alpha^2\mu^2}\frac{c_s^2}{1 - c_s^2}\frac{g_2}{r^2}$$
$$= \frac{\mathcal{M}^2}{r^4}\left(1 - 2\frac{\mu^2}{E^2 - \alpha^2\mu^2}\frac{\mathcal{M}}{r}\right) \tag{81}$$

with the boundary conditions $g_2(\infty) = 0$ and $g_2(r_0) = -\mathcal{M}^2/2r_0^2$. Moreover, in the asymptotic region $r \ll r_0$ we have

$$g_2(r) = -G(\Gamma)\frac{\mathcal{M}^2}{r_0^2}\left(\frac{r}{r_0}\right)^{(1-3\Gamma)/2} \tag{82}$$

$\left[|G(\Gamma)| \sim 1\right]$, the asymptotics (82) being completely determined by the inhomogeneous solution of Eqn (81), which means that it does not depend on the boundary conditions. As can be seen from Eqn (82), rotation leads to the formation of a disc in the equatorial plane.

Note that under the physically reasonable condition $c_\infty^2 \ll 1$ the sonic surface radius r_0 (67) substantially exceeds the black hole radius $2\mathcal{M}$. That is why the effects of rotation in this region prove to be extremely small. Consequently, the shape of the sound surface will be distorted only slightly, and so, under real conditions the above-mentioned effects may hardly be of practical interest. Thus we think that the main result lies in the mathematical plane. It has been shown, in fact, that the well-known spherically symmetric conditions of accretion are stable under small perturbations due to black hole rotation.

At the same time, as was recently shown by Pariev [78], in the problem of accretion onto a rotating black hole, one can overstep the limits of small perturbations. The point is that the condition $c_\infty^2 \ll 1$ normally holds to a good accuracy. On the other hand, as has already been said, the spherically symmetric solution for a cold plasma remains valid for any velocity of black hole rotation. As a result, for a small quantity $c_\infty \ll 1$ the structure of the flow was also established for accretion onto a rapidly rotating black hole. As demonstrated in Fig. 1, in this case, too, the flow differs slightly from being spherically symmetric.

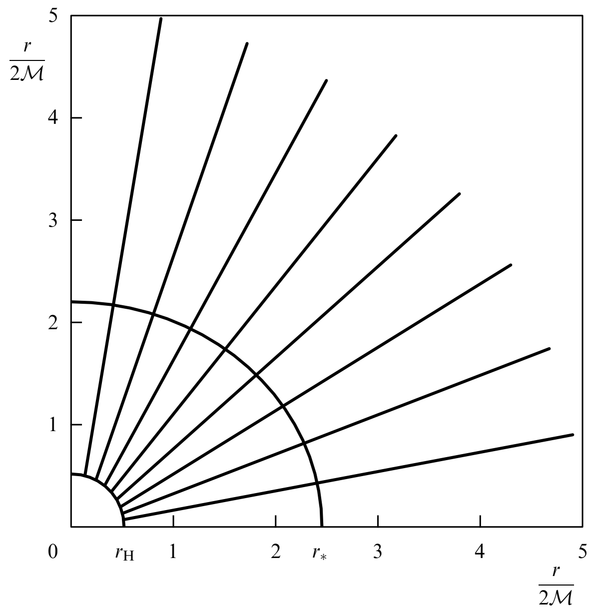

Figure 1. Structure of flow onto a rapidly rotating black hole for $c_\infty^2 = 0.1$, $a = \mathcal{M}$ and $\Gamma = 4/3$ [78].

We shall now turn to the problem of accretion onto a Schwarzschild black hole moving at a velocity $v_\infty \ll c_\infty$ relative to the medium (Bondi – Hoyle accretion) [77]. The small parameter here is the quantity

$$\varepsilon_1 = \frac{v_\infty}{c_\infty} . \tag{83}$$

We shall immediately go over to the frame of reference in which the black hole is at rest and the medium moves relative to it at a velocity v_∞. As a result, as in the preceding section, the solution of the trans-field equation (55) can be written in the form

$$\Phi(r, \theta) = \Phi_0 \left[1 - \cos\theta + \varepsilon_1 g_1(r) \sin^2\theta\right] , \tag{84}$$

the asymptotics (84) as $r \to \infty$ corresponding to a homogeneous oncoming flow (74). As we can see, to the first order in ε_1 the accretion rate remains unchanged. The shape of the sonic surface will be given by the relation

$$r_*(\theta) = r_0 \left[1 + 2\varepsilon_1 k \cos\theta\right] , \tag{85}$$

where the constant $k \sim 1$ depends on the polytropic index Γ [77]. On the other hand, for $r \ll r_0$ the function $g_1(r)$ appears to have the asymptotic behaviour

$$g_1(r) = K_{\text{in}}(\Gamma) \left(\frac{r}{r_0}\right)^{-1/2} , \tag{86}$$

where the constant $K_{\text{in}} \sim 1$ depends on the polytropic index Γ. We can see that in the case

$$\varepsilon_1 > \frac{1}{|K_{\text{in}}|} \left(\frac{\mathcal{M}}{r_0}\right)^{1/2}$$

at distances

$$2\mathcal{M} < r < r_0 \varepsilon_1^2 K_{\text{in}}^2 \tag{87}$$

the perturbed term $\Phi_0 \varepsilon_1 g_1(r) \sin^2\theta$ becomes larger than the summand $\Phi_0(1 - \cos\theta)$ that corresponds to spherically symmetric accretion. This means that in this region the linear approximation is no longer applicable, so here one should solve the complete nonlinear equation (55). However, a violation of the linear approximation takes place in the hyperbolic region which does not affect the behaviour of the solution for $r > r_0$.

We note immediately that expansion (84) is formally valid only at small distances $r \ll r_0 \varepsilon_1^{-1/2}$ from the black hole because at large distances the perturbation exceeds the summand $\Phi_0(1 - \cos\theta)$ which describes the spherically symmetric accretion. Nevertheless, the expansion (84) correctly describes the behaviour of the flow function over the entire space. The complete proof can be found in Ref. [77]. This conclusion can however be easily understood from the following simple arguments. The point is that for spherically symmetric accretion the density of matter at distances $r \gg r_0$ may be thought of as constant [3]. The density in a homogeneous oncoming plasma flow will also be constant. On the other hand, the constant density corresponds to an infinitely high velocity of sound for which the Grad – Shafranov equation (55) becomes linear. It is therefore not surprising that the sum of the two solutions (66) and (74) in the asymptotics $r \gg r_0$ appears to be a solution of equation (55) too.

Figure 2 shows the structure of the flow (84) and the form of the sonic surface (85) for the case $\Gamma = 4/3$, $\varepsilon_1 = 0.6$. The dashed lines indicate the flow lines and the sonic surface obtained numerically in Ref. [60]. As one can see, the solution (84) nearly coincides with the result of numerical calculation in spite of the fact that the parameter ε_1 is close to unity in the case under consideration.

At the same time, the conclusion that in the case of Bondi – Hoyle accretion the flow may also differ appreciably from spherically symmetric at small distances (87) was not known earlier. Since the sign of the coefficient $K_{\text{in}}(\Gamma)$ depends on the polytropic index Γ, the region of thickening of the flow lines will either appear on the front side for $\Gamma > 1.27$ or on the rear side for $\Gamma < 1.27$. This effect would be of interest for both galactic black holes of solar mass and a binary system of black holes existent supposedly in active galactic nuclei. However, the discussion of this effect requires a more thorough analysis that goes beyond the scope of the perturbation method.

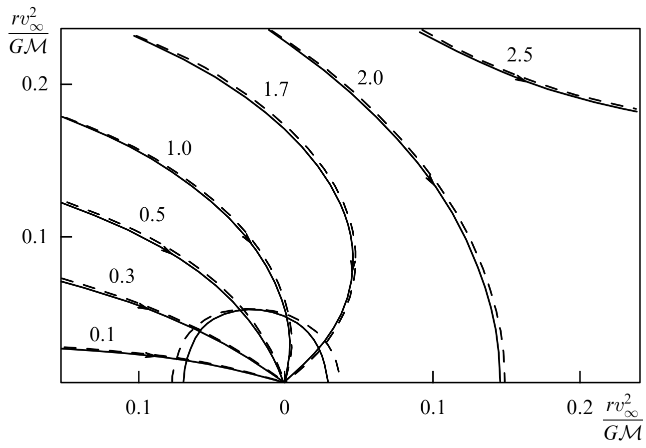

Figure 2. Structure of flow and form of sonic surface onto a moving black hole for the case $\Gamma = 4/3$, $\varepsilon = 0.6$ [77]. Numbers are Φ/Φ_0 values, the dashed lines are the lines of flow and the sonic surface obtained numerically in Ref. [60].

Let us finally consider the problem of the accretion of matter with angular momentum onto a non-rotating black hole [79]. According to the computational method developed earlier, we shall assume the angular momentum of the accreting matter L to be sufficiently small, so that the radial velocity of the medium $v_{\hat{r}}$ be always greater than the azimuthal velocity $v_{\hat{\varphi}}$. In this case it is natural to assume that the structure of the flow will differ slightly from spherically symmetric accretion. Moreover, we suppose that the entropy s is the same for all the flow lines. Then the equation describing axisymmetric stationary accretion has the form

$$-\alpha\varpi^2\nabla_k\left(\frac{1}{\alpha\varpi^2}\nabla^k\Phi\right) - \frac{1}{(\nabla\Phi)^2}\frac{\nabla^i\Phi\cdot\nabla^k\Phi\cdot\nabla_i\nabla_k\Phi}{D}$$
$$+\frac{1}{2D}\frac{(E^2\nabla_k\varpi^2 - L^2\nabla_k\alpha^2 - \mu^2\nabla_k\varpi^2\alpha^2)\nabla^k\Phi}{\varpi^2E^2 - \alpha^2L^2 - \varpi^2\alpha^2\mu^2}$$
$$+\frac{1}{D}\left(\varpi^2E\frac{dE}{d\Phi} - \alpha^2L\frac{dL}{d\Phi}\right)\frac{(\nabla\Phi)^2}{\varpi^2E^2 - \alpha^2L^2 - \varpi^2\alpha^2\mu^2}$$
$$+\frac{64\pi^4}{M^4}\left(\varpi^2E\frac{dE}{d\Phi} - \alpha^2L\frac{dL}{d\Phi}\right) = 0\,, \qquad (88)$$

and D is defined, as before, by formula (56).

We shall assume that for $r = R$ the gas rotates as a single whole, i.e., $v_{\hat{\varphi}} \propto \sin\theta$, $v_{\hat{\theta}} = 0$. In this case $L = L_0\sin^2\theta$,

$$E = E_0 + \frac{L_0^2}{2R^2E_0}\sin^2\theta\,,$$

and so the quantity $\varepsilon_L = L_0/E_0r_H$ appears to be a small parameter. As a result, as shown in Ref. [79], the flow function can be written in the form

$$\Phi(r,\theta) = \Phi_0\left[(1+\varepsilon_L^2g_0)(1-\cos\theta) + \varepsilon_L^2g_2(r)\cos\theta\sin^2\theta\right]\,, \qquad (89)$$

where $g_0 = -(16/3)c_0^2$ and the radial function $g_2(r)$ is determined from an ordinary differential equation equivalent to Eqn (81). It turns out that for small r the solution of this equation is universal and the flow function near a black hole can be written as

$$\Phi(r,\theta) \simeq \Phi_0\left[(1+\varepsilon_L^2g_0)(1-\cos\theta) - 2\varepsilon_L^2\frac{r_H}{r}\cos\theta\sin^2\theta\right]\,, \qquad (90)$$

that is, it depends neither on the conditions on the external boundary, nor on the polytropic index Γ. Since $g_0 < 0$, the rotation diminishes the accretion rate. Finally, the gas concentration near the black hole horizon has the form

$$n = \frac{|\Phi_0|}{2\pi\sqrt{r_Hr^3}}\left\{1 - \frac{\varepsilon_L^2}{2}\frac{r_H}{r}\left[13\cos^2\theta - 5 + \frac{r_H}{r}(1-\cos^2\theta)\right]\right\}\,. \qquad (91)$$

In Figure 3, the flow lines $\Phi(r,\theta) = \text{const}$ are constructed for the ε_L^2 values 0.1 and 0.3. As can be seen, because the matter has angular momentum, the gas density near the equator becomes higher than the density near the rotation axis. This confirms the well-known result obtained earlier in numerical calculations. As to the region of parameters $\varepsilon \gtrsim 1$, at distances $r \simeq r_g\varepsilon_L^2$ in the framework of ideal hydrodynamics, there must inevitably appear a gas stop point $v_{\hat{r}} = 0$, and so the approximation $v_{\hat{r}} \gg v_{\hat{\varphi}}$ is violated in this case.

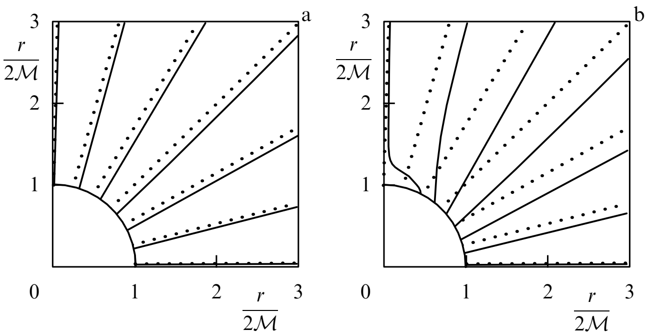

Figure 3. Line of matter flow for the accretion of matter with angular momentum. The parameter ε_L^2 is equal to 0.1 (a) and 0.3 (b). For comparison, the dots show the flow lines for spherically symmetric accretion [79].

3.3 Ejection from slowly rotating stars

The structure and characteristics of stellar wind, observed first of all in stars of early spectral classes, also represent one of the fundamental questions in modern astrophysics [6, 7]. In particular, there exist a lot of confirmations of the fact that gas outflow must be transonic, i.e., subsonic near a star surface and supersonic at large distances from it. Gas outflow from the surface of rapidly rotating B_e-stars leads, as is also seen from direct observations, to the formation of a rather dense disc in the equatorial plane. Numerical calculations are now available to testify in favour of the existence of such a disc [75, 76]. On the other hand, the question of the construction of a consistent analytical theory for a two-dimensional transonic flow remains open. There exists a single analytical solution — the Parker solution [4] for a spherically symmetric outflow from the surface of a non-rotating star (see also Ref. [59]). In the case of outflow from a rotating star, the structure of flow lines is unknown and should be defined as a solution of the trans-field equation. As a result, the problem is again reduced to a Grad – Shafranov type equation.

So, we shall now consider a non-relativistic gas outflow from a slowly rotating star [80]; in this section, dimensionality is involved for the sake of definiteness. The small parameter of our problem will be the quantity

$$\varepsilon_*^2 = \frac{\Omega^2 R^3}{G\mathcal{M}}\,, \tag{92}$$

where Ω is the characteristic angular frequency of star rotation. In particular, as is well known, the perturbation of the star surface $r_R(\theta)$ can be written in the form [6]

$$r_R(\theta) = R\left[1 + \varepsilon_*^2 \rho(\theta)\right]\,, \tag{93}$$

where $\rho(\theta) \simeq 1$, and the thermodynamic quantities, for example, the temperature per particle $t_R(\theta)$ and concentration $n_R(\theta)$ will be represented as

$$t_R(\theta) = t_R\left[1 + \varepsilon_*^2 \tau(\theta)\right]\,, \tag{94}$$

$$n_R(\theta) = n_R\left[1 + \varepsilon_*^2 \eta(\theta)\right]\,. \tag{95}$$

In view of the thermodynamic relation

$$\mathrm{d}s = \frac{1}{\Gamma - 1}\,\frac{\mathrm{d}t}{t} - \frac{\mathrm{d}n}{n}\,,$$

we have

$$\delta s(\theta) = \varepsilon_*^2 \left[\frac{1}{\Gamma - 1}\,\tau(\theta) - \eta(\theta)\right]\,. \tag{96}$$

Next, we shall find the angular velocity of the star surface rotation as

$$\Omega(R, \theta) = \Omega\omega_1(\theta)\,, \tag{97}$$

where $\omega_1(\theta)$ is the dimensionless angular velocity describing differential rotation. This means that the azimuthal component of velocity on the surface will be written as

$$v_\varphi(R, \theta) = \varepsilon_* \left(\frac{G\mathcal{M}}{R}\right)^{1/2} \omega_1(\theta) \sin\theta\,. \tag{98}$$

Finally, the radial velocity on the star surface should be written in the form

$$v_r(R, \theta) = v_R\left[1 + \varepsilon_*^2 h(\theta)\right]\,. \tag{99}$$

Clearly, in the case of a slow star rotation, the solution of the complete trans-field equation (62) can also be sought as a small correction to the spherically symmetric flow (66):

$$\Phi(r, \theta) = \Phi_0\left[1 - \cos\theta + \varepsilon_*^2 f(r, \theta)\right]\,. \tag{100}$$

Substituting relation (100) into the complete trans-field equation for flow lines (62), we obtain in the first order with respect to the small quantity ε_*^2

$$\begin{aligned} &- \varepsilon_*^2 \Phi_0 D\,\frac{\partial^2 f}{\partial r^2} - \frac{\varepsilon_*^2}{r^2}\,\Phi_0 (D+1) \sin\theta\,\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\,\frac{\partial f}{\partial\theta}\right) \\ &\quad + \varepsilon_*^2 \Phi_0 N_r \frac{\partial f}{\partial r} = 4\pi^2 n^2 (D+1) L\,\frac{\mathrm{d}L}{\mathrm{d}\Phi} \\ &\quad - 4\pi^2 n^2\,\frac{\cos\theta}{\Phi_0 \sin^2\theta}\,L^2 - 4\pi^2 n^2 r^2 \sin^2\theta (D+1)\,\frac{\mathrm{d}E}{\mathrm{d}\Phi} \\ &\quad + 4\pi^2 n^2 r^2 \sin^2\theta \left[(D+1)t + \frac{\Gamma - 1}{\Gamma}\,c_\mathrm{s}^2\right] \frac{\mathrm{d}s}{\mathrm{d}\Phi}\,; \end{aligned} \tag{101}$$

the integrals of motion E and s now depend, generally speaking, on the flow lines Φ, and the angular momentum L is non-zero.

Because we are dealing with flows that differ little from being spherically symmetric, the perturbations of the integrals of motion $\delta E(\theta)$ and $L^2(\theta)$ as functions of the angle θ coincide to within terms of the order of ε_*^2 with their values on the star surface. We have

$$\begin{aligned} \delta E(\theta) &= \varepsilon_*^2 v_R^2 h(\theta) + \frac{1}{2}\,\varepsilon_*^2\,\frac{G\mathcal{M}}{R}\,\omega_1^2(\theta) \sin^2\theta \\ &\quad + \varepsilon_*^2\,\frac{\Gamma}{\Gamma - 1}\,t\tau(\theta) + \delta\varphi_\mathrm{G}(R, \theta)\,, \end{aligned} \tag{102}$$

$$L^2(\theta) = \varepsilon_*^2 R^2\,\frac{G\mathcal{M}}{R}\,\omega_1^2 \sin^2\theta\,, \tag{103}$$

and $\delta s(\theta)$ is defined by relation (96). In particular, for a star whose whole mass is concentrated in the centre, the gravitational potential perturbation $\delta\varphi_\mathrm{G}(R, \theta)$ on the surface will be written as

$$\delta\varphi_\mathrm{G}(R, \theta) = \varepsilon_*^2\,\frac{G\mathcal{M}}{R}\,\rho(\theta)\,. \tag{104}$$

The solution of Eqn (101) can ultimately be again decomposed with respect to the eigenfunctions of operator (71), and the radial functions in dimensionless variables

$$x = \frac{r}{r_0}\,, \qquad u = \frac{n}{n_0}\,, \qquad a = \frac{c_\mathrm{s}^2}{c_0^2}\,, \tag{105}$$

must satisfy the equation

$$\begin{aligned} &(1 - x^4 a u^2)\,\frac{\mathrm{d}^2 g_m}{\mathrm{d}x^2} + 2\left(\frac{1}{x} - x^2 u^2\right) \frac{\mathrm{d}g_m}{\mathrm{d}x} \\ &\quad + m(m+1) x^2 a u^2 g_m = \kappa_m\,\frac{R^2}{r_0^2}\,x^4 a u^4 - \lambda_m\,\frac{R^2}{r_0^2}\,u^2 \\ &\quad - \sigma_m x^6 a u^4 + \frac{1}{\Gamma}\,v_m x^6 a^2 u^4 + \frac{\Gamma - 1}{\Gamma}\,v_m x^2 a u^2\,. \end{aligned} \tag{106}$$

Here $-m(m+1)$ are eigenvalues of operator (71) and the numerical coefficients κ_m, λ_m, σ_m, and v_m are coefficients of the decomposition in the angular functions $Q_m(\theta)$:

$$\sin\theta\,\frac{\mathrm{d}E}{\mathrm{d}\theta} = \varepsilon_*^2 c_0^2 \sum_{m=0}^{\infty} \sigma_m Q_m(\theta)\,, \tag{107}$$

$$\frac{\cos\theta}{\sin^2\theta}\,L^2 = \varepsilon_*^2 c_0^2 r_0^2 \sum_{m=0}^{\infty} \lambda_m Q_m(\theta)\,, \tag{108}$$

$$\frac{L}{\sin\theta}\,\frac{\mathrm{d}L}{\mathrm{d}\theta} = \varepsilon_*^2 c_0^2 r_0^2 \sum_{m=0}^{\infty} \kappa_m Q_m(\theta)\,, \tag{109}$$

$$\sin\theta\,\frac{\mathrm{d}s}{\mathrm{d}\theta} = \varepsilon_*^2 \sum_{m=0}^{\infty} v_m Q_m(\theta)\,. \tag{110}$$

They must be determined from boundary conditions on the star surface.

We shall now turn to the boundary conditions on Eqn (106). According to Eqn (45), to determine the two-dimensional structure of transonic flow in the problem under consideration it is necessary to specify four boundary conditions on the body surface $\rho(\theta)$, i.e., not only two thermodynamic functions, for example, $t_R(\theta)$ and $n_R(\theta)$ and an azimuthal velocity $v_\varphi(R,\theta)$, but also one more velocity component, e.g., $v_r(R,\theta)$ or $v_\theta(R,\theta)$. So, having specified the radial velocity, we have by definition (59)

$$g_m\left(\frac{R}{r_0}\right) = \frac{(2m)!}{2^m(m+1)!m!}(\eta_m + h_m + 2\rho_m)\,, \tag{111}$$

where g_m, η_m, h_m, and ρ_m are coefficients of the expansion in Legendre polynomials. If we specify the meridian velocity component $v_\theta(R,\theta)$ on the star surface then, using definition (59), we obtain

$$nv_\theta(R,\theta) = -\frac{\partial\Phi}{\partial r}\frac{1}{2\pi R\sin\theta} = \varepsilon_*^2\frac{\Phi_0}{2\pi R\sin\theta}\sum_m\left(\frac{\mathrm{d}g_m}{\mathrm{d}r}\right)_{r=R}Q_m(\theta)\,, \tag{112}$$

owing to which the quantity $v_\theta(R,\theta)$ defines, in fact, the derivative g_m' on the star surface. In particular, in the absence of meridional convection, $v_\theta(R,\theta)=0$, we simply have

$$g_m'\bigg|_{x=R/r_0} = 0\,. \tag{113}$$

The second boundary condition on the radial functions $g_m(r)$, associated with the condition of smooth passage of the solution through the sonic surface, can be obtained from the regularity condition $N_\theta(r_*)=0$. As a result, we have

$$\frac{2^m(m+1)!m!}{(2m)!}g_m(1) = \frac{1}{\varepsilon_*^2c_0^2}\,\delta E_m - \frac{\delta s_m}{\varepsilon_*^2} - \frac{1}{2\varepsilon_*^2c_0^2r_0^2}\left(\frac{L^2}{\sin^2\theta}\right)_m, \tag{114}$$

where the indices m correspond to the harmonics of expansion of the functions $\delta E(\theta)$, $L^2(\theta)/\sin^2\theta$, and $\delta s(\theta)$ in Legendre polynomials $P_m(\cos\theta)$. We can see that the radial functions $g_m(1)$ for $r=r_0$ are completely defined by the behaviour of three integrals of motion whose values are, in turn, determined by the variation of parameters on the star surface.

Thus, relations (111) and (114) [or (112) and (114)] define two boundary conditions on the differential equation (106) at the points $x=R/r_0$ and $x=1$, respectively. As we can see, they actually depend on the four functions that must be defined on the star surface $r_R(\theta)$. Along with two thermodynamic functions, for example, the concentration $n_R(\theta)$ (95) and the velocity of sound $c_R(\theta)$, as well as the azimuthal velocity of rotation $v_\varphi(R,\theta)$, it is necessary to define in addition either the radial component of outflow velocity $v_r(R,\theta)$ (99) in case the boundary condition (111) is used, or the meridional component of velocity $v_\theta(R,\theta)$ if the boundary condition (112) is used [80, 81]. The latter circumstance seems surprising at first glance because in the spherically symmetric case the radial component of velocity v_R and the flow rate Φ_0 are completely defined by the two thermodynamic functions n_R and c_R^2 on the star surface. But as a matter of fact, there is no contradiction. The point is that in the case of a rotating star the zeroth harmonic h_0 entering in the definition of the velocity $v_r(R,\theta)$ (99) and, therefore, the total accretion rate

$$\Phi(\pi) = 2\Phi_0\left[1+\varepsilon_*^2(2\rho_0+h_0+\eta_0)\right] \tag{115}$$

cannot, as before, be defined arbitrarily. Formally, this is connected with the fact that as distinct from higher harmonics g_m the zeroth harmonic g_0 must be defined by only one of the two proper solutions of Eqn (196), namely, the solution $g_0=\mathrm{const}$, as can be readily verified. The second proper solution of the equation for $m=0$, which depends on the radius $f=g_0(r)(1-\cos\theta)$, would lead to a singularity of the complete solution on the axis $\theta=\pi$. As a result, the value of $g_0(R/r_0)$ must coincide with the value $g_0(1)$:

$$g_0\left(\frac{R}{r_0}\right) = g_0(1)\,. \tag{116}$$

The equality (116) together with relations (11) and (114) defines the quantity h_0 as a function of thermodynamic quantities given on the star surface.

As an example, we shall consider a transonic gas outflow from the surface of a uniformly rotating star $\left[\omega_1(\theta)=1\right]$ in the absence of meridional convection. In this case, a slow rotation leads to the following distortion of the form of the surface:

$$r_R(\theta) = R\left(1+\frac{1}{2}\frac{\Omega^2R^3}{G\mathcal{M}}\sin^2\theta\right). \tag{117}$$

Furthermore, we shall make use of von Zeipel's condition [5], according to which the temperature on the star surface satisfies the condition

$$T(R,\theta) = T_R\left(\frac{|\nabla\varphi_{\mathrm{eff}}|}{|\nabla\varphi_{\mathrm{G}}|}\right)^{1/4}, \tag{118}$$

where

$$\varphi_{\mathrm{eff}} = \varphi_{\mathrm{G}} - \frac{1}{2}\,\Omega^2r^2\sin^2\theta\,. \tag{119}$$

As a result, we have

$$\sigma_0 = 0\,,$$

$$\sigma_m = -\frac{(2m)!}{2^m m!(m-1)!}\frac{v_R^2}{c_0^2}h_m \quad \text{when} \quad m\neq 0,2\,,$$

$$\lambda_m = \kappa_m = \nu_m = 0 \quad \text{for} \quad m\neq 2\,,$$

$$\sigma_2 = 2\frac{r_0}{R} - \frac{5-3\Gamma}{2(\Gamma-1)} + \frac{1}{2}\frac{v_R^2}{c_0^2} - 3\frac{v_R^2}{c_0^2}h_2\,, \tag{120}$$

$$\lambda_2 = 2\frac{R}{r_0}\,,\qquad \kappa_2 = 4\frac{R}{r_0}\,,\qquad \nu_2 = -\frac{\Gamma}{\Gamma-1}\,, \tag{121}$$

and so the perturbation of the flow function will again be defined only by the zeroth and second harmonics of the operator (71).

In order to obtain the complete solution of the problem, it is now necessary to solve the ordinary differential equation (106) for $m=2$ with the boundary conditions

$$g_2(1) = \frac{1}{2}\frac{v_R^2}{c_0^2}h_2 + \frac{1}{3}\left(\frac{R}{r_0}-\frac{r_0}{R}\right) - \frac{1}{12}\frac{v_R^2}{c_0^2} - \frac{5}{12}\,,$$

$$g_2\left(\frac{R}{r_0}\right) = \frac{1}{2}(h_2-1)\,,$$

$$g_2'\left(\frac{R}{r_0}\right) = 0\,. \tag{122}$$

It is precisely this solution that determines the constant h_2. On the other hand, from (111), (114), and (116) we get

$$h_0 = -\frac{1}{6} + \frac{2}{3}\left(\frac{r_0}{R} - \frac{R}{r_0}\right)\left(1 - \frac{v_R^2}{c_0^2}\right)^{-1} . \tag{123}$$

According to (115), the ejection rate can be given by

$$\Phi = 2\Phi_0\left[1 + \frac{\Omega^2 R^3}{G\mathcal{M}}(1 + h_0)\right] . \tag{124}$$

Since $1 + h_0 > 0$, we make sure that rotation increases the ejection rate.

The complete solution of the trans-field equation was derived in Ref. [80]. The asymptotics of the solution at large distances can be obtained analytically. Indeed, it can be readily verified that for $r \gg r_0$ the quantity $g_2(r)$ does not depend on the radius r. As a result we arrive at

$$\Phi = \Phi_0\left[(1 - \cos\theta) + \frac{\Omega^2 R^3}{G\mathcal{M}}(1 + h_0)(1 - \cos\theta) + \frac{\Omega^2 R^3}{G\mathcal{M}}\, q_2 \sin^2\theta \cos\theta\right] , \tag{125}$$

$$n(r, \theta) = n_0 \frac{c_0}{v_\infty}\frac{r_0^2}{r^2}\left[1 + \frac{\Omega^2 R^3}{G\mathcal{M}}\, b_0 + \frac{1}{2}\frac{\Omega^2 R^3}{G\mathcal{M}}\, b_2(3\cos^2\theta - 1)\right], \tag{126}$$

where the constants b_0, b_2 and q_2 are determined in Ref. [80]. It is obvious that the asymptotic (125) does not depend on the radius r. This means that the flow lines at infinity are the straight lines $\theta = \text{const}$. As shown in Ref. [80], in all cases we have $b_2 < 0$ and $g_2 < 0$. Hence, star rotation is actually responsible for the formation of a disc in the equatorial plane. On the other hand, under certain conditions the gas flow velocity in the equatorial plane may even become lower than in the spherically symmetric case.

Thus, we have shown how the presence of an exact analytical solution — either spherically symmetric accretion or ejection — also allows the construction of the solution of the direct problem for two-dimensional transonic flows that possess axial symmetry. In particular, it has been shown that rotation causes the formation of a disc in the equatorial plane. As to a more detailed comparison of the theoretical predictions with observational data, it seems irrelevant. As has already been mentioned, an important role must be played by dissipative processes (viscosity, heat conduction, and radiation pressure) that cannot consistently be taken into account in the ideal hydrodynamics approximation.

4. The force-free approximation — the magnetosphere of radio pulsars

4.1 The force-free limit of the trans-field equation in a flat space

Radio pulsars (rotating neutron stars of radius $R \sim 10^6$ cm, mass $\mathcal{M} \sim \mathcal{M}_\odot$, and rotation period $P \sim 0.0016-1$ s with a surface magnetic field $B_0 \sim 10^{12}$ G) are unique astrophysical sources. They are the only cosmic objects in which the rotation deceleration mechanism (and, therefore, evolution) is determined by electrodynamic forces [82–84]. On the other hand, although gravitational forces near neutron stars are extremely strong, they still appear to be by many orders of magnitude smaller than the electromagnetic interaction. This allows us to restrict our consideration with high accuracy to the trans-field equation in a flat space. At the same time, in some cases the effects of the general theory of relativity may become substantial [85–87].

As we shall see, the energy loss of radio pulsars is completely determined by the ponderomotive action of electric current circulating in the magnetosphere, which can, in turn, be determined only together with the entire magnetic field structure. That is why a consistent solution of the question of radio pulsar evolution can only be obtained in the framework of the general problem which, for the axisymmetric magnetosphere, leads us again to the trans-field equation.

So, we shall consider the force-free limit of the Grad–Shafranov equation in which we should now ignore the contribution of particles; here, for the sake of definiteness we again apply to dimensionality. If, according to Michel, the magnetisation parameter [28] is $\sigma = E/\mu\eta$, the conditions of smallness of the contribution of particles $T_{ik}^{\text{matter}} \ll T_{ik}^{\text{em}}$ can be written in the form

$$\sigma \gg \gamma_{\text{in}} . \tag{127}$$

Here γ_{in} is the characteristic Lorentz factor of plasma near the star surface. As can be readily verified, when condition (127) holds, the condition $M^2 \ll 1$ will hold up to the light cylinder $R_{\text{L}} = c/\Omega$. The force-free limit, however, does not of course imply a complete absence of particles because in that case the longitudinal electric field would not vanish. It is a known fact that for screening a longitudinal electric field the charge density must be close to the so-called Goldreich density [82]

$$\rho_{\text{GJ}} = -\frac{\mathbf{\Omega}\cdot\mathbf{B}}{2\pi c} . \tag{128}$$

Thus, for the particle concentration we have

$$n \gg n_{\text{GJ}} = \frac{\Omega B}{2\pi c|e|} . \tag{129}$$

For radio pulsars the conditions (127), (129) hold to a good accuracy [88–90]. Indeed, the plasma filling the magnetosphere is secondary with respect to the magnetic field. Particle (electron and positron) birth is due to one-photon conversion of hard gamma-quanta in a strong magnetic field in polar regions of a neutron star, where the longitudinal electric field is not equal to zero [84, 88].

Passing now in the general equation (41) to the limit $M^2 \to 0$, $s \to 0$, we obtain [45–48, 91–93]

$$-\left(1 - \frac{\Omega^2\varpi^2}{c^2}\right)\nabla^2\Psi + \frac{2}{\varpi}\frac{\partial\Psi}{\partial\varpi} - \frac{16\pi^2}{c^2} I \frac{\mathrm{d}I}{\mathrm{d}\Psi} + \frac{\varpi^2}{c^2}\,\Omega_{\text{F}}(\nabla\Psi)^2\frac{\mathrm{d}\Omega_{\text{F}}}{\mathrm{d}\Psi} = 0 , \tag{130}$$

where ∇^2 is the Laplacian, and we now have $\varpi = r\sin\theta$, $E(\Psi) = \Omega_{\text{F}}(\Psi)L(\Psi)$ and the longitudinal current I according to (15) and (16) also becomes an integral of motion.

As we can see, the force-free equation (130) has only two integrals of motion. These are the angular velocity $\Omega_{\text{F}}(\Psi)$ and the longitudinal current $I(\Psi)$. As distinguished from the complete trans-field equation (41), Eqn (130) is elliptic and is sure to have only one singular surface — the Alfven surface which in the force-free limit is coincident with the light cylinder. Consequently, according to Eqn (45), Eqn (130) requires three boundary conditions. As such boundary conditions one typically takes the values of two integrals of motion, $\Omega_{\rm F} = \Omega_{\rm F}(\Psi)$ and $I = I(\Psi)$, and also the normal component of the magnetic field on the surface of a neutron star $r = R$ or, which is the same, the value of the potential $\Psi = \Psi(R, \theta)$. For example, for a dipole magnetic field we have

$$\Psi(R, \theta) \simeq M_{\rm m}\,\frac{\sin^2\theta}{R}\,. \tag{131}$$

Here $M_{\rm m}$ is the magnetic moment of a neutron star. As concerns the light surface (which in the force-free limit coincides with a fast magnetosonic surface), its existence, as shown below, is closely related to the value of the longitudinal current.

4.2 Energy losses of pulsars

Before proceeding to a discussion of the energy losses of a neutron star, we recall some relations that refer to the quasi-stationary generalisation of the above equations already describing the magnetosphere of an incline rotator. All the quantities are assumed to be dependent on the time t and the angular coordinate φ only in the combination $\varphi - \Omega t$. In this case [44]

$$\mathbf{E} + \left[\boldsymbol{\beta}_{\rm R} \times \mathbf{B}\right] = -\nabla\psi\,, \tag{132}$$

where $\boldsymbol{\beta}_{\rm R} = \left[\boldsymbol{\Omega} \times \mathbf{r}\right]/c$ and ψ represents the electric potential in a rotating co-ordinate system. In particular, for an ideally conducting star, where $\mathbf{E}_{\rm in} + \left[\boldsymbol{\beta}_{\rm R} \times \mathbf{B}_{\rm in}\right] = 0$, we have $\psi = 0$. However, for the case of a zero longitudinal electric field $(\mathbf{E}\cdot\mathbf{B}) = 0$ we have $(\mathbf{B}\cdot\nabla\psi) = 0$. Thus, the potential ψ must be constant on magnetic surfaces

$$\psi = \psi(\Psi)\,, \tag{133}$$

and according to (13) and (132), in the axisymmetric case the angular velocity will be written simply as

$$\Omega_{\rm F} = \Omega - \frac{{\rm d}\psi}{{\rm d}\Psi}\,. \tag{134}$$

Therefore, in the region of closed field lines (i.e., field lines not overstepping the light cylinder) we simply have $\psi = 0$, whereas in the region of open field lines, which are separated from the neutron star by the region of the longitudinal electric field, the potential ψ will be non-zero. The potential ψ just leads to particle acceleration in the region of the longitudinal electric field. The magnitude of the potential $\psi(P, B_0)$ will be determined by a particular mechanism of particle production. For convenience, we further introduce a dimensionless accelerating potential $\beta_0 = \psi(P, B_0)/\psi_{\rm max}$, where

$$\psi_{\rm max} = \left(\frac{\Omega R}{c}\right)^2 R B_0 \tag{135}$$

is the maximum potential drop in the region of acceleration [82, 88]. As a result, the angular velocity $\Omega_{\rm F}$ over the acceleration region, where a secondary plasma screens the longitudinal electric field (and where the method of Grad–Shafranov equation can thus be used), is simply specified as $\Omega_{\rm F} = (1 - \beta_0)\Omega$. As to the longitudinal currents, they are conveniently normalised to the Goldreich current density $j_{\rm GJ} = c\rho_{\rm GJ}$. We can therefore write $I(\Psi) = i_0 I_{\rm GJ}$, where

$$I_{\rm GJ} = \frac{B_0 \Omega^2 R^3}{2c} \tag{136}$$

is the characteristic total current through the polar cap surface.

The appearance of longitudinal currents in pulsar magnetosphere is crucial because it is precisely the longitudinal currents that determine the deceleration of a neutron star. Indeed, the total current running out of the pulsar surface must be equal to zero. As a result, currents $\mathbf{J}_{\rm s}$ must flow along the pulsar surface that close the longitudinal currents flowing in the magnetosphere. The ponderomotive action of these currents leads to the observed slowing down of radio pulsars [10, 82]. To show this, we shall write the energy loss rate in the form

$$W_{\rm tot} = -\boldsymbol{\Omega}\cdot\mathbf{K}\,, \tag{137}$$

where

$$\mathbf{K} = \frac{1}{c}\int\left[\mathbf{r} \times [\mathbf{J}_{\rm s} \times \mathbf{B}]\right]{\rm d}S \tag{138}$$

is the torque associated with the Ampere force of currents flowing along the surface. It is directed antiparallel to the angular momentum of the neutron star. Employing now definitions (137) and (138), we come to

$$W_{\rm tot} \simeq \frac{B_0^2 \Omega^4 R^6}{c^3}\, i_0 \cos\chi\,, \tag{139}$$

where χ is the angle between the magnetic axis and the axis of rotation. As we shall see, in the analytical expression the loss (139) coincides with the magnetic loss [94]

$$W_{\rm md} = \frac{1}{6}\,\frac{B_0^2 \Omega^4 R^6}{c^3}\,\sin^2\chi\,. \tag{140}$$

However, the magnetodipole loss (140) is absent in the axisymmetric case $\chi = 0$. At the same time, the loss (139) is proportional to the electric current i_0 circulating in the magnetosphere. Nevertheless, for the majority of radio pulsars the dimensionless current is $i_0 \sim 1$, and so the simplest magnetodipole formula (140) on the whole gives a correct estimate of the total rotation energy loss by a neutron star [10].

It should be stressed that the current loss $W_{\rm tot}$ (139) does not coincide with the energy flux (17) which in the force-free case is associated with the Poynting vector flux only:

$$W_{\rm em} = \frac{1}{2\pi}\int I(\Psi)\Omega_{\rm F}(\Psi)\,{\rm d}\Psi\,. \tag{141}$$

Indeed, formulae (137), (138) can be transformed to become [10]

$$W_{\rm tot} = W_{\rm em} + W_{\rm matter}\,, \tag{142}$$

where the second summand

$$W_{\rm matter} = \int \psi\mathbf{j}_{\rm e}\cdot{\rm d}\mathbf{S} = \frac{1}{2\pi}\int I(\Psi)\left[\Omega - \Omega_{\rm F}(\Psi)\right]{\rm d}\Psi\,, \tag{143}$$

corresponds, according to Eqn (132), to the energy gained by primary particles in the acceleration region.

On the other hand, the loss of the angular momentum (18) is totally due to the electrodynamic loss (138):

$$K_{\mathrm{tot}} = \frac{1}{2\pi}\int I(\Psi)\,\mathrm{d}\Psi \tag{144}$$

exactly as it should be because the angular momentum of photons $\mathcal{L}_{\mathrm{ph}}$ radiated near the star surface is much less than $\Omega^{-1}\mathcal{E}_{\mathrm{ph}}$. As a result, in view of Eqns (141) and (143), the condition $W_{\mathrm{tot}} = \Omega K_{\mathrm{tot}}$ (which holds for a rotating neutron star by definition) appears to hold identically for outgoing radiation. At the same time, as we have seen, this relation cannot be derived without the additional summand (143). The attempt to solve the problem of losses in the framework of the force-free approximation inevitably leads to confusion [95, 96]. Note that the total loss W_{tot} can be written as $W_{\mathrm{tot}} = \psi_{\max} I$, where I is the total current circulating in the magnetosphere. As shown in Ref. [10], relations (141)–(144) remain valid for the case of an incline rotator.

4.3 Exact solutions

The trans-field equation (130) is nonlinear, but now the nonlinearity is totally associated with the integrals of motion. In particular, in the absence of longitudinal current and with a constant angular velocity $\Omega_{\mathrm{F}}(\Psi) = \Omega$ it becomes linear. Since Eqn (130) does not explicitly contain a cylindrical variable z, its solution can be sought by the method of separation of variables [48, 92]:

$$\Psi(\varpi, z) = M_{\mathrm{m}}\int_0^{\infty} \varphi(\lambda) Q(\varpi, \lambda)\cos(\lambda z)\,\mathrm{d}\lambda\,. \tag{145}$$

This property allowed a rather quick construction of the solution of Eqn (130) for a dipole (and a monopole) magnetic field of a star in the absence of longitudinal current, i.e., when the only currents in the magnetosphere are the corotation currents $\Omega\varpi\rho_{\mathrm{GJ}}\mathbf{e}_{\varphi}$ [46, 48] (Fig. 4, $\chi = 0°$). For these solutions, the range of applicability of the trans-field equation extends only to within the light cylinder which in this case coincides with both the Alfven and the light surfaces. It turned out that the corotation currents lead to a concentration of magnetic surfaces near the equatorial plane, and the magnetic field along the axis of rotation decreases exponentially. An interesting property of the magnetosphere is also the fact that on some open field lines, where $\mathbf{\Omega}\cdot\mathbf{B} = 0$, the charge density reverses sign according to Eqn (128). Clearly, the charge- separated plasma flowing out of a star could not have provided fulfilment of the condition $\rho_{\mathrm{e}} = \rho_{\mathrm{GJ}}$. So, it has been hypothesised that there exists an 'external gap' near the line $\rho_{\mathrm{GJ}} = 0$, in which the appearing longitudinal electric field also causes production of a secondary electron-positron plasma; here, because of the weak magnetic field, the principal mechanism of particle production is the two-photon conversion $\gamma + \gamma \rightarrow \mathrm{e}^+ + \mathrm{e}^-$ [97]. In real conditions, the plasma flowing out of the magnetosphere certainly contains particles of both signs, and thus the condition $\rho_{\mathrm{e}} = \rho_{\mathrm{GJ}}$ might, in principle, be fulfilled at the expense of a slight variation of the longitudinal velocities of particles. However, such a problem generally requiring a kinetic consideration remains unsolved [98].

Solutions for the case of an incline rotator (see Fig. 4) were considered in the same fashion in Ref. [92]. This became possible because for $i_0 = 0$, $\beta_0 = 0$ the quasi-stationary trans-field equation also remains linear. We thus confirmed the result of Ref. [99] according to which the magnetic field on the light-cylinder surface has solely the ϖ-component and the electric field has solely the φ-component. Consequently, in the absence of a longitudinal current and an accelerating potential, the Poynting vector flux through the light-cylinder surface must be zero. In other words, the corotation currents in the magnetosphere completely screen the magnetodipole radiation of the neutron star. This means that in the case of an incline rotator all energy losses will be associated with the longitudinal current circulating in the magnetosphere.

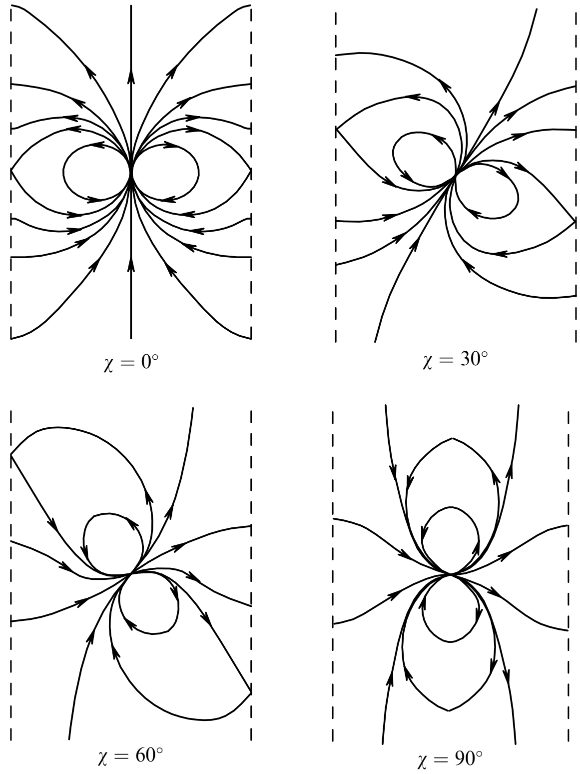

Figure 4. Structure of the magnetosphere of an incline rotator for zero longitudinal currents [10].

Finally, Michel found another remarkable analytical solution for the case of a monopole magnetic field of a star [91]. It turns out that with a special choice of longitudinal current

$$I(\Psi) = I_{\mathrm{M}} = \frac{\Omega_{\mathrm{F}}}{4\pi}\left(2\Psi - \frac{\Psi^2}{\Psi_0}\right) \tag{146}$$

and for $\Omega_{\mathrm{F}} = \mathrm{const}$, a monopole magnetic field

$$\Psi(r,\theta) = \Psi_0(1-\cos\theta) \tag{147}$$

is an exact solution of the nonlinear trans-field equation (130), including the region outside the light cylinder. To put it differently, for $I(\theta) = I_{\mathrm{M,A}}\sin^2\theta$ (which, in fact, coincides with the Goldreich current $j_{\mathrm{GJ}} = c\rho_{\mathrm{GJ}}$) the effects of longitudinal and corotation currents compensate for each other completely. As follows from relations (146), (147), in Michel's solution the electric field is equal in magnitude to the toroidal component of the magnetic field

$$B_{\hat{\varphi}} = E_{\hat{\theta}} = B_0\,\frac{\Omega R}{c}\,\frac{R}{r}\sin\theta\,, \tag{148}$$

which at large distances becomes much larger than the poloidal magnetic field $B_{\rm p} = B_0(R/r)^2$. On the other hand, in this solution the total magnetic field remains larger than the electric field up to infinity, which also extends the light surface to infinity. Although artificial, Michel's solution plays an important role in the theory of the magnetosphere of black holes.

As an illustration, we shall consider a model problem of a small perturbation of Michel's monopole solution. As has already been said, Eqn (130) requires three boundary conditions. We shall assume the angular rotation rate $\Omega_{\rm F}$ to remain the same as in Michel's solution, and the potential $\Psi(R, \theta)$ on the star surface to be unaltered too. As to the longitudinal current $I(R, \theta)$, we shall assume it to differ only slightly from the equilibrium current (146):

$$I = I_{\rm M}(\theta) + l(\theta) = I_{\rm M,A}\sin^2\theta + l(\theta)\,, \tag{149}$$

and so $l(\theta)/I_{\rm M,A} \ll 1$. Since the perturbations are assumed to be small, relation (149) also determines the magnitude of the current as a function of the potential Ψ.

Writing again the solution of Eqn (130) in the form $\Psi(r, \theta) = \Psi_0\left[1 - \cos\theta + \varepsilon f(r, \theta)\right]$, in the first order of the small parameter $\varepsilon = l/I_{\rm M}$ we obtain

$$\begin{aligned}&\varepsilon(1 - x^2\sin^2\theta)\frac{\partial^2 f}{\partial x^2} + \varepsilon(1 - x^2\sin^2\theta)\frac{\sin\theta}{x^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\theta}\right)\\ &\quad - 2\varepsilon x\sin^2\theta\frac{\partial f}{\partial x} - 2\varepsilon\sin\theta\cos\theta\frac{\partial f}{\partial\theta} + 2\varepsilon(3\cos^2\theta - 1)f\\ &\quad + \frac{1}{\sin\theta\, I_{\rm M,A}}\frac{\rm d}{{\rm d}\theta}(l\sin^2\theta) = 0\,.\end{aligned} \tag{150}$$

Here $x = \Omega_{\rm F} r/c$. Equation (150), just as expected, has a singularity $x\sin\theta = 1$ on the light cylinder. As a result, we obtain for $l(\theta) = I_{\rm M,A}h\sin^2\theta$, $h = {\rm const} \ll 1$ (and $R \ll R_{\rm L}$) [100]

$$\Psi(r, \theta) = \Psi_0\left[1 - \cos\theta + hr^2\Omega_{\rm F}^2\sin^2\theta\cos\theta\right]. \tag{151}$$

Unfortunately, in the general case the solution of even the linearised equation (150) is extremely cumbersome.

Solution (151) just shows that for $I < I_{\rm M}$ ($h < 0$) the magnetic field lines constrict towards the equator ($\delta\Psi < 0$ for $\theta < \pi/2$). The light surface is then at a finite distance

$$\varpi_{\rm C} = |h|^{-1/4}R_{\rm L} \tag{152}$$

at which the perturbation of the monopole field may as yet be thought of as small. Consequently, for $I > I_{\rm M}$ ($h > 0$) magnetic field lines develop towards the axis of rotation ($\delta\Psi > 0$ for $\theta < \pi/2$) and the light surface is reached, as before, at infinity.

We shall now proceed to the crucial point of this section, i.e., the discussion of the structure of a neutron-star magnetosphere in the presence of a longitudinal current I and an accelerating potential ψ. As mentioned above, the trans-field equation (130) becomes nonlinear in this case, which substantially hampers its analysis. However, by a special choice of the longitudinal current and the potential, Eqn (130) can be reduced to being linear. This becomes possible if the values of $\Omega_{\rm F}(\Psi)$ and $I(\Psi)$ are taken in the form

$$\Omega_{\rm F} = \Omega(1 - \beta_0)\,, \tag{153}$$

$$I(\Psi) = \frac{c}{4\pi}\, i_0\Psi\,, \tag{154}$$

where i_0 and β_0 are constant. In the region of open field lines (in dimensionless variables $x = \Omega\varpi/c$), the trans-field equation will assume the form

$$-\nabla^2\Psi\left[1 - x^2(1 - \beta_0)^2\right] + \frac{2}{x}\frac{\partial\Psi}{\partial x} - i_0^2\Psi = 0\,, \tag{155}$$

whereas in the region of closed field lines we simply have

$$-\nabla^2\Psi(1 - x^2) + \frac{2}{x}\frac{\partial\Psi}{\partial x} = 0\,. \tag{156}$$

As a result, all the nonlinearity will be contained in the thin transition layer near the separatrix whose position should be found from the solution. We note that as distinct from the previous papers, we do not require the zero point of the magnetic field to lie on the light-cylinder surface $x = 1$.

We shall now specify the boundary conditions on the system of equations (155) and (156). Clearly, in the presence of a longitudinal current (i.e., for $B_\varphi \neq 0$) the light surface does not already coincide with the light cylinder. On the other hand, the Alfven surface, on which the regularity condition (44) must be satisfied, coincides as before with the light cylinder $x_{\rm L} = 1/(1 - \beta_0)$. In the region of open field lines, Eqn (155) requires, according to Eqn (45), three boundary conditions. First of all, such conditions are the quantities i_0 and β_0 determined on the star surface. Moreover (and this is exceedingly important), it is required that the magnetic field be decreasing at infinity along the axis of rotation as $z \to \infty$. The necessity to introduce an 'additional' boundary condition is merely due to the fact that a magnetic field line going to infinity along the axis of rotation does not intersect the light cylinder, and therefore for this field line no additional regularity condition appears. A violation of this condition leads to non-physical solutions not decreasing at infinity [101]. The third boundary condition is not only the value of the potential $\Psi(R, \theta)$ on the star surface (131), but also the value of the potential Ψ on the surface of the separatrix between the regions of open and closed magnetosphere [47]. The regularity condition (44) on the light cylinder $x = x_{\rm L}$ will now be written like

$$\frac{2}{x}\frac{\partial\Psi}{\partial x} - i_0^2\Psi = 0\,. \tag{157}$$

As regards the region of closed field lines, which in the general case does not reach the light cylinder at all, the role of additional boundary conditions must be played by the conditions of 'matching' of closed and open field line regions. Such conditions should first of all be the coincidence of the position of the separatrix field line $z = z_*(x)$ for both the regions

$$\Psi^{(1)}\left[x, z_*(x)\right] = \Psi^{(2)}\left[x, z_*(x)\right] \tag{158}$$

and also continuity of the quantity $L_{\rm b} = \mathbf{B}^2 - \mathbf{E}^2$

$$L_{\rm b}^{(1)}\left[x, z_*(x)\right] = L_{\rm b}^{(2)}\left[x, z_*(x)\right]. \tag{159}$$

The latter conditions can readily be obtained by integrating the force-free trans-field equation, written in the form $\mathbf{E}\nabla\mathbf{E} + \left[\nabla\times\mathbf{B}\right]\times\mathbf{B} = 0$, across the transition layer [47, 102].

Figure 5 presents the structure of magnetic surfaces for a non-zero longitudinal current i_0 and an accelerating potential β_0, which was obtained in the solution of Eqns (155) and (156) [103]. We note, however, that the solution of the problem cannot be constructed for any i_0 and β_0 values. The point is that for some parameters i_0 and β_0 the solution of Eqn (155) for the region of open field lines shows that the zero magnetic field line is located outside the light cylinder $x_L = 1$. Clearly, in this case the solution cannot be matched with the region of closed magnetosphere because it cannot be extended to $x > 1$. As is shown in Fig. 6, on the plane of parameters i_0, β_0, the forbidden region corresponds to sufficiently low i_0 values.

Thus, the existence in the magnetosphere of a neutron star of closed magnetic field lines not intersecting the Alfven surface imposes certain restrictions on the longitudinal currents circulating in the magnetosphere. Furthermore, detailed calculations show that the total energy of an electromagnetic field proves to be minimum precisely near the boundary line $\beta_0 = \beta_0(i_0)$ when, as it happens, the zero point of the magnetic field lies in the vicinity of the light cylinder $x = 1$ [103]. One may suppose, accordingly, that equilibrium of a radio pulsar magnetosphere is realised only

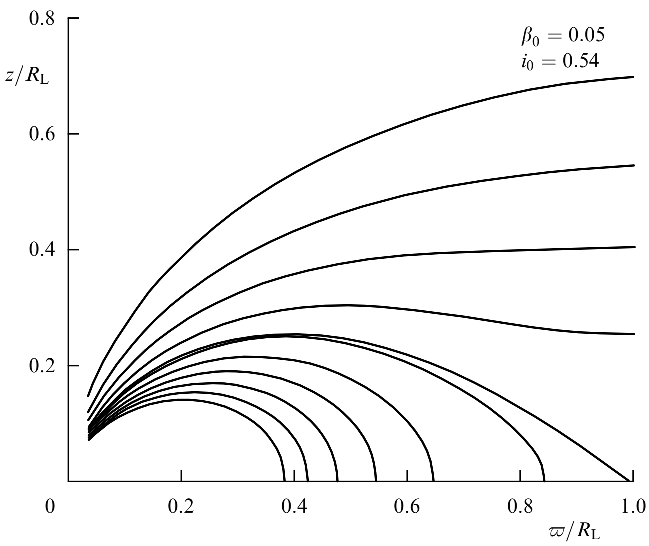

Figure 5. Structure of magnetosphere of a axisymmetric rotator when longitudinal currents are non-zero. The values of i_0 and β_0 correspond to the consistency relation (160) [103].

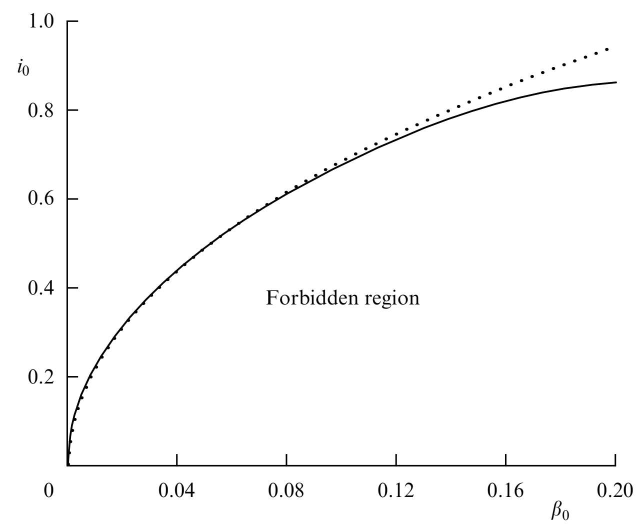

Figure 6. Range of parameters i_0, β_0 for which a solution can be constructed. The dots indicate the line (160) [103].

for a certain relation between the accelerating potential $\psi(P, B_0)$ and the longitudinal current.

The existence of such an 'Ohm's law' is undoubtedly a very important conclusion. Indeed, as shown above, it is the longitudinal currents which determine the rotation energy loss of a neutron star. Hence, if there exists a connection between the longitudinal current and the accelerating potential, the energy loss of a neutron star must by fully determined by a concrete particle production mechanism near a pulsar surface.

Note that the consistency relation describing the 'Ohm's law' can be obtained immediately from the trans-field equation. Indeed, assuming the field line $\Psi = \Psi_*$ corresponding to the solution of Eqn (155) in the region of open magnetosphere to pass near the zero point (where $B_z \sim \partial\Psi/\partial x = 0$) located on the light cylinder (where $x = 1$), we have

$$\beta_0(i_0) = 1 - \left(1 - \frac{i_0^2}{i_{\max}^2}\right)^{1/2}, \tag{160}$$

where $i_{\max} = \left(|\nabla^2\Psi|/\Psi_*\right)^{1/2} \approx 1.58$ [92]. As can be seen from Fig. 6, the analytical estimate (160) is in good agreement with numerical calculations. It is shown in Ref. [10], that relation (160) remains valid on the whole for the incline rotator.

Analysis of astrophysical data lies beyond the scope of this paper. Nevertheless we shall briefly recall the main observational results that might be explained on the basis of the theory presented above.

(1) If we understand correctly the nature of pulsars, their radio emission must be associated with the secondary plasma produced in the region of longitudinal electric field near a star surface [9, 10]. Consequently, the condition $\psi(P, B) = \psi_{\max}$ may be regarded as that for the maximum period up to which a neutron star is registered as a radio pulsar. For example, in the Ruderman–Sutherland model (a non-free particle outflow from the surface, a dipole magnetic field) [88, 89] we have $P_{\max} \sim 1$ s, which exactly corresponds to the observations [104, 105]. The relation $P < P_{\max}$ can be rewritten in the form $Q < 1$, where [106]

$$Q = 2\left(\frac{P}{1\text{c}}\right)^{11/10}\left(\frac{\dot{P}}{10^{-15}}\right)^{-4/10}. \tag{161}$$

The parameter $Q \simeq i_0$ determined, as we can see, directly from observations, appears to describe very conveniently the basic characteristics of radio pulsars [106–108]. So, pulsars with $Q \sim 1$, for which the directivity pattern is a sufficiently thin cone, must have predominantly a two-hump mean profile of radio emission; it is in just such pulsars that radio emission exhibits various irregularities, e.g., nulling of radio emission, mode switching, etc. Pulsars with $Q \ll 1$ must on the contrary be characterised by stable radio emission, their mean profiles must for the most part have a single hump. It is well known that this is the situation actually observed [10].

(2) A very important parameter of radio pulsars that bears information on the character of rotation deceleration is the so-called braking index n_{br} [105] which can be determined directly from observations: $n_{\text{br}} = \ddot{\Omega}\Omega/\dot{\Omega}^2$. Unfortunately, at the present time the braking index has been determined only for a few pulsars [105], among them $n_{\text{br}} = 2.24$ for pulsar PSR 0540–693 and $n_{\text{br}} = 2.84$ for pulsar PSR 1509–58. For the magnetodipole loss (140) we have $n_{\text{br}} = 3 + (\tan\chi)^{-2}$, which does not agree with observations. On the other hand, for the

current loss [10] we have $n_{\rm br} = 1.93 + 1.5\tan^2\chi$, which does not contradict observations.

(3) In the standard model with a non-free particle outflow [88], the longitudinal current is equal to the Goldreich current. Accordingly, the total energy of particles bombarding the neutron star surface turns out to be rather large, which leads to a contradiction with the observed X-ray radiation [90]. However, according to Eqn (160), in fast radio pulsars (which alone exhibit X-ray radiation) the longitudinal current must be substantially smaller than the Goldreich current. As a result, the polar cap heating due to secondary particles must be much smaller than that given by standard estimates [88]. Recall that in the Arons model [90] there was no such problem.

Thus, the key predictions of the theory (based on the model with a non-free particle outflow from a neutron star surface and on the above hypothesis on the relation between longitudinal current and acceleration potential) do not contradict observational data. Moreover, more indirect conclusions have lately been confirmed, such as the statement of the absence of evolution of the magnetic field of a neutron star within the period of its activity as a radio pulsar, as well as the statistical conclusion concerning the fact that the initial period of radio pulsars is most often not 1 – 10 ms, as in the case of well-known young Crab and Vela pulsars, but only 0.1 – 1 s [106]. Of crucial importance would of course be a direct measurement of the evolution of the inclination angle χ of the axes, which must tend to $\pi/2$ in the model of current loss. But at the moment such an experiment is hardly possible.

5. The force-free approximation — the magnetosphere of black holes

5.1 The force-free limit of the trans-field equation in the Kerr metric

As is well known, the most popular 'central engine' model in active galactic nuclei and quasars is a supermassive black hole of mass $\mathcal{M} \sim (10^8 - 10^9)\mathcal{M}_\odot$ [12, 109 – 114]. Indeed, it is accretion onto such compact objects that provides understanding of both the nature of their extremely intense energy release $L \sim 10^{44} - 10^{46}$ erg s^{-1} and a sufficient stability of observed jets [115].

It should be noted that there exist two radical distinctions between the magnetosphere of a black hole and the magnetosphere of a neutron star. First of all, the intrinsic magnetic field of a black hole must vanish within a dynamic time $\sim r_{\rm H}/c$ [13], which for galactic nuclei makes up only $\sim 10^3$ s. For this reason the magnetic field must be constantly maintained by external sources, namely, currents flowing in the accretion disc. According to both the early estimates [13, 37] and the most recent result [116, 117], the magnetic field near the surface of a black hole may reach the so-called Eddington values

$$B_{\rm Edd} \sim 10^4 \left(\frac{\mathcal{M}}{10^9 \mathcal{M}_\odot}\right)^{-1/2} {\rm G}\,, \tag{162}$$

when the magnetic field energy density coincides with the density of accreting plasma responsible for Eddington luminosity [115]. Furthermore, it is clear that in the accreting disc itself, where viscous forces are predominant, the method of the Grad – Shafranov equation becomes inapplicable. At the same time, in polar regions of the magnetosphere the energy density of a magnetic field may appreciably exceed the energy density of particles. In the present section, we therefore lay special emphasis specifically on the region of the magnetosphere which is connected by field lines with the horizon of a black hole, the more so as alternative models, in which the leading part in jet formation is played by field lines traversing the accretion disc, are rather thoroughly investigated both within the method of self-similar solutions [27, 55, 118] and numerically [119 – 121].

Let us now proceed to a discussion of the magnetosphere of a rotating black hole. To begin, we shall write the basic equations of the force-free approximation. The trans-field equation will now be written as [37, 68]

$$\frac{1}{\alpha}\nabla_k\left\{\frac{\alpha}{\varpi^2}\left[1 - \frac{(\Omega_{\rm F}-\omega)^2\varpi^2}{\alpha^2}\right]\nabla^k\Psi\right\} + \frac{\Omega_{\rm F}-\omega}{\alpha^2}(\nabla\Psi)^2\frac{{\rm d}\Omega_{\rm F}}{{\rm d}\Psi} + \frac{16\pi^2}{\alpha^2\varpi^2}I\frac{{\rm d}I}{{\rm d}\Psi} = 0\,. \tag{163}$$

This is a second-order elliptic equation containing only two integrals of motion. As has already been said, it contains two singular surfaces, namely, Alfven surfaces which are again coincident with light cylinders. One of these surfaces, which corresponds to the ejected plasma, is fully equivalent to the light cylinder in a neutron star magnetosphere, while the second, internal surface on which $\alpha \approx (\Omega_{\rm H} - \Omega_{\rm F})\varpi_{\rm H}$ is due to the effects of the general theory of relativity. The condition on the horizon (47), which in the force-free approximation can be rewritten as [37]

$$4\pi I(\Psi) = (\Omega_{\rm H} - \Omega_{\rm F})\sin\theta\,\frac{r_{\rm H}^2 + a^2}{r_{\rm H}^2 + a^2\cos^2\theta}\,\frac{{\rm d}\Psi}{{\rm d}\theta}\,, \tag{164}$$

as shown above in the general case, is simply obtained by integration of the trans-field equation (163) and cannot therefore be regarded as a boundary condition.

As has already been mentioned, in the general magnetohydrodynamic approach, on field lines traversing the horizon there inevitably appears a region of generation of matter that separates the accretion and ejection regions where the Grad – Shafranov equation itself is already inapplicable. However, if in the generation region the potential drop is much smaller than the characteristic potential difference and the surface currents are much smaller than the longitudinal currents, one may assume that $\Omega_{\rm F}^+ = \Omega_{\rm F}^-$ and $I^+ = I^-$. As a result, the values of the two integrals of motion will be equal along the entire magnetic field line connected with the black hole.

On the other hand, the magnetosphere of a black hole turns out to be similar in many respects to the magnetosphere of a radio pulsar. First of all, the conditions (127) – (129), necessary for the validity of the force-free approximation prove to hold for the polar regions of a black hole magnetosphere. The basic result here is the Blandford – Znajek effect [36] owing to which a rotating black hole can now lose energy and angular momentum at the expense of longitudinal currents circulating in the magnetosphere. This conclusion is essentially based on the fact that far from the star, where the space is flat, one can use as before relations (141) and (144). Moreover, within the so-called membrane paradigm, the notions of 'surface charge' and 'surface current' of a black hole may be introduced in order that the energy loss might be written in the form fully equivalent to relation (138) for the ponderomotive action of surface currents flowing along the neutron star surface [13].

There exists however one more important distinction between the magnetospheres of a black hole and a neutron star. The point is that the black hole energy is contained not only in the rotation energy, but also in the so-called irreducible mass proportional to the surface area of the black hole [13]. Hence, the energy balance of a black hole is other than that described by the relation $W_{\rm tot} = \Omega K_{\rm tot}$ holding for neutron stars. When written in the thermodynamic form, it looks like

$$\frac{{\rm d}E}{{\rm d}t} = T\frac{{\rm d}S}{{\rm d}t} - \Omega_{\rm H}\frac{{\rm d}J}{{\rm d}t}\,. \tag{165}$$

Here ${\rm d}E/{\rm d}t = -W_{\rm tot}$, ${\rm d}J/{\rm d}t = -K_{\rm tot}$, and the 'temperature' T and the 'entropy' S are connected with the internal parameters of the black hole (for more details see Ref. [13]).

Now using relations (141), (144) and (164), we obtain for the total energy loss

$$W_{\rm tot} = k_1\Omega_{\rm F}(\Omega_{\rm H} - \Omega_{\rm F})\frac{B_0^2 r_{\rm H}^4}{c}\,, \tag{166}$$

where $k_1 \simeq 1$. Consequently, the energy loss will be positive provided that the conditions

$$0 < \Omega_{\rm F} < \Omega_{\rm H} \tag{167}$$

hold, the highest loss being attained for $\Omega_{\rm F} = \Omega_{\rm H}/2$. We ultimately have $W_{\rm tot} \sim W_{\rm BZ}$, where

$$W_{\rm BZ} = \frac{1}{4}\frac{\Omega_{\rm H}^2 B_0^2 r_{\rm H}^4}{c} = 10^{45}\left(\frac{B_0}{10^4\,{\rm G}}\right)^2\left(\frac{\Omega_{\rm H} r_{\rm H}}{c}\right)^2\left[\frac{\rm erg}{\rm s}\right]. \tag{168}$$

Thus, for a rapidly rotating black hole the theory definitely suggests an explanation of the observed jet energies [13, 37].

We see, however, that the determination of the energy loss of a rotating black hole is related not so much to the magnitude of the longitudinal current, as was the case with a radio pulsar magnetosphere, as to the value of the angular rotation velocity of the magnetic field lines $\Omega_{\rm F}(\Psi)$. Indeed, in the case of a black hole magnetosphere the angular velocity $\Omega_{\rm F}(\Psi)$ is generally in no way connected with the angular velocity of the black hole $\Omega_{\rm H}$ and must itself be determined from the solution of the complete problem. The magnitude of the longitudinal current I can then be found from relation (164).

Note that for the simplest case of an infinitely thin disc in the problem of the structure of a black hole magnetosphere a great role is played by the simplest monopole solution (146), (147) obtained by Michel for a pulsar magnetosphere [46]. Indeed, in the simplest vacuum model in which magnetic field lines do not penetrate into the thin accretion disc of internal radius b, the magnetic field structure (in the absence of a black hole) is described by the potential

$$\Psi(r,\theta) = \Psi_0\left[1 - \sqrt{\frac{1}{2}\left(1-\frac{r^2}{b^2}\right) + \sqrt{\frac{1}{4}\left(1-\frac{r^2}{b^2}\right)^2 + \frac{r^2\cos^2\theta}{b^2}}}\,\right]. \tag{169}$$

In the limit $b \to 0$ (with conservation of the total flux Ψ_0), formula (169) passes over to the monopole field (147) in each hemisphere. The jump of the magnetic field at the equator is due to the currents flowing along the disc surface.

5.2 Behaviour of the solution in the vicinity of the horizon

Let us now discuss the basic properties of a magnetosphere near the horizon of a black hole, where the effects of the general theory of relativity play a dominating role. As was shown above, in the general magnetohydrodynamic case over the horizon of a black hole the general equation (41) has a hyperbolic region, in view of which it requires no additional boundary conditions. According to Eqn (48), even in the limit $M^2 \to 0$ on the horizon the condition $D = -1$ must hold. On the other hand, Eqn (163) remains elliptic up to the very horizon. That is why it is necessary to consider the limiting transition to the case of a force-free magnetosphere in more detail.

To do this, we shall begin with analysing the regularity conditions (33) and (37) as $M^2 \to 0$ on a fast magnetosonic surface. As can be seen from Eqn (31), the very position of the surface as $M^2 \to 0$ tends to the horizon of a black hole. On the other hand, the whole singularity in the quantity D is contained in the factor $1/M^2$. The quantities $N'_a = \nabla_a M^2 D/M^2 = N_a/A$ remain finite on the horizon as $M^2 \to 0$. Therefore, as the limit on the regularity conditions (37) we may consider the values N'_a for $M^2 = 0$ and $r = r_{\rm H}$.

To begin with, we shall write the condition $N'_\theta(r_{\rm H}) = 0$ which can alternatively be written as

$$\frac{\rm d}{{\rm d}\theta}\left[64\pi^4\,\frac{(E-\Omega_{\rm H}L)^2}{(\Omega_{\rm F}-\Omega_{\rm H})^4\varpi^2} - \frac{1}{\rho^2(r_{\rm H})}\left(\frac{{\rm d}\Psi}{{\rm d}\theta}\right)^2\right] = 0\,. \tag{170}$$

It is easy to see that (170) coincides with condition (164) and, consequently, does not impose any additional restrictions. Similarly, the equality $M^2 D(r_{\rm H}) = 0$ also leads to condition (164). This can be readily verified. Finally, after elementary, although cumbersome transformations the condition $N'_r(r_{\rm H}) = 0$ can be reduced to the form

$$r_{\rm H}\frac{\partial}{\partial r}\left[(\nabla\Psi)^2 - \frac{16\pi^2 I^2}{(\Omega_{\rm F}-\omega)^2\varpi^2}\right]_{r_{\rm H}} + \left(\frac{{\rm d}\Psi}{{\rm d}\theta}\right)^2\frac{2\xi + (\Omega_{\rm H}-\Omega_{\rm F})\varpi\left[\xi^2 + 1/(\alpha\gamma)^2\right]}{(\Omega_{\rm H}-\Omega_{\rm F})\varpi\rho^2} = 0\,, \tag{171}$$

where

$$\xi = \left(\frac{u_{\hat\varphi}}{\alpha\gamma}\right)_{r=r_{\rm H}}\,, \tag{172}$$

and all metric quantities in Eqn (171) are taken for the black hole horizon.

As we can see, along with the 'force-free' quantities $I(\Psi)$ and $\Omega_{\rm F}(\Psi)$, condition (171) also contains the plasma parameters γ and $u_{\hat\varphi}$. In the case of a relativistic plasma, $\alpha\gamma \gg 1$, we only take the ratio of these quantities ξ (172). Consequently, the force-free limit of the general magnetohydrodynamic regularity condition (37) on an internal fast magnetosonic surface does not impose any additional restrictions upon the force-free equation (163) but only fixes the ratio ξ on the black hole horizon. Since

$$\frac{\partial}{\partial r}(\nabla\Psi)^2 \sim \frac{\Psi_0^2}{r_{\rm H}^3}\,,$$

we have

$$u_{\hat\varphi}(r_{\rm H}) \sim \alpha\gamma\Omega_{\rm H}\varpi_{\rm H}\,, \tag{173}$$

which fully agrees with the general equations (20) and (21).

We shall now turn to the regularity condition (44) on the Alfven surface as the angular rotation velocity of the black hole tends to zero. As can be readily seen from definition (24), the Alfven surface will also tend to the horizon surface (on the axis of rotation the Alfven surface in the force-free approximation always touches the black hole surface). As a result, having written the force-free limit of the regularity condition (44), we have as $\alpha \to 0$

$$\frac{\partial^2 \Psi}{\partial \theta^2} - \frac{\cos\theta}{\sin\theta}\frac{\partial \Psi}{\partial \theta} + r\frac{\partial \Psi}{\partial r} = 0\,. \tag{174}$$

But the latter condition is coincident with the leading term of Eqn (163) (for $\Omega_{\rm F} = 0$ and $I = 0$) describing the vacuum magnetic field near the black hole surface. Thus, condition (164) may be regarded as a boundary condition on the horizon only providing the potential Ψ corresponds to the vacuum magnetic field.

5.3 Exact solutions

The first example of an exact solution of Eqn (163) for the case of an infinitely thin accretion disc was constructed by Blandford and Znajek [36] who took the magnetosphere of a non-rotating black hole with a monopole magnetic moment as the zeroth approximation. It was shown that for a slowly rotating black hole ($\Omega_{\rm H} r_{\rm H} \ll 1$) under the condition

$$\Omega_{\rm F} = \frac{\Omega_{\rm H}}{2}\,, \qquad I(\Psi) = I_{\rm M} = \frac{\Omega_{\rm F}}{4\pi}\left(2\Psi - \frac{\Psi^2}{\Psi_0}\right) \tag{175}$$

the monopole field $\Psi = \Psi_0(1-\cos\theta)$ is also an exact solution of the nonlinear equation (163). It is an extension of Michel's solution [46] to the case of a slowly rotating black hole. The two boundary conditions determining this solution are the zero normal component of the magnetic field on the disc surface and the monopole behaviour of the field at infinity. In the same paper [36] they constructed another solution for the case where in the absence of rotation the potential Ψ has the form $\Psi(r,\theta) = r(1-\cos\theta)$.

Note that in the paper by Blandford and Znajek the inverse problem was in fact analysed, where the values of the longitudinal current $I(\Psi)$ and angular velocity $\Omega_{\rm F}(\Psi)$ were determined for a particular structure of poloidal magnetic field. More precisely, they found values of $I(\Psi)$ and $\Omega_{\rm F}(\Psi)$ such that the monopole field remains a solution to the trans-field equation over the entire space. Indeed, the conditions (175) can be obtained immediately from the algebraic relations characterising the behaviour of the solution near the horizon of a black hole and at infinity. As the first relation one should choose the condition on the horizon (164). The second condition will be the relation

$$4\pi I(\Psi) = \Omega_{\rm F}(\Psi)\left(2\Psi - \frac{\Psi^2}{\Psi_0}\right). \tag{176}$$

As can be well checked, if condition (176) is fulfilled, the magnetic field is a solution of the trans-field equation (163) for any function $\Omega_{\rm F}(\Psi)$. Finally, for low angular velocities of rotation one can set $\Psi = \Psi_0(1-\cos\theta)$ so that the system of equations (164) and (176) lead to the expressions (175).

We ultimately note that relation (176) is a reflection of a more general statement according to which the Grad–Shafranov equation can be integrated when for some reason or other it becomes one-dimensional. It is easy to check that in an asymptotically distant region $r \to \infty$ for canonical magnetic surfaces $\Psi = \Psi(\theta)$ the trans-field equation (163) leads to the relation [122, 123]

$$4\pi I(\theta) = \Omega_{\rm F}(\theta)\sin\theta\,\frac{{\rm d}\Psi}{{\rm d}\theta} \tag{177}$$

which generalises the equality (176). For the cylindrical solution in a flat space in which all the quantities depend on the co-ordinate ϖ only, we have [124]

$$\Omega_{\rm F}^2(\Psi)\varpi^4 B_z^2 = \varpi^2 B_\varphi^2 + \int_0^{\varpi} x^2\,\frac{\rm d}{{\rm d}x}\,B_z^2\,{\rm d}x\,. \tag{178}$$

In particular, the homogeneous magnetic field $B_z = {\rm const}$ will be a solution of the nonlinear equation (163) for any values of the integrals of motion $\Omega_{\rm F}(\Psi)$ and $I(\Psi)$ that satisfy the equation

$$4\pi I(\Psi) = 2\Omega_{\rm F}(\Psi)\Psi\,. \tag{179}$$

Thus, in the one-dimensional case the definition of any two quantities completely defines the solution of the trans-field equation. One should bear in mind, however, that a solution of the direct problem [i.e., the determination of the poloidal magnetic field from the given quantities $B_\varphi(\varpi)$ and $\Omega_{\rm F}(\varpi)$] is not always existent. Indeed, for $B_\varphi = 0$ and $\Omega_{\rm F} = {\rm const}$ we obtain

$$B_z(\varpi) = \frac{B_z(0)}{1 - \Omega_{\rm F}^2\varpi^2}\,, \tag{180}$$

in which case the solution cannot be extended outside the light cylinder. But with a given magnetic field $B_z(\varpi)$ one can always find such values of $B_\varphi(\varpi)$ and $\Omega_{\rm F}(\varpi)$ that the field is the solution of the trans-field equation. In this case, however, the magnitude of the longitudinal current must be close to the Goldreich current $I_{\rm M}$.

5.4 The structure of the magnetosphere

To begin with, we shall consider the possibility of plasma generation in polar regions of a magnetosphere in connection with relativistic effects in the magnetosphere of a black hole. According to Eqn (13), the exact relativistic formula for the Goldreich charge density $\rho_{\rm GJ}$ looks like this

$$\rho_{\rm GJ} = \frac{1}{8\pi^2}\,\nabla\left(\frac{\Omega_{\rm F} - \omega}{\alpha c}\,\nabla\Psi\right). \tag{181}$$

In particular, near the axis of rotation we simply have

$$\rho_{\rm GJ} \approx -\frac{(\Omega_{\rm F} - \omega)B}{2\pi c\alpha}\,. \tag{182}$$

Thus, the effects of general relativity require that the Goldreich density becomes zero for $\Omega_{\rm F} \simeq \omega$. This becomes possible if the condition $0 < \Omega_{\rm F} < \Omega_{\rm H}$ (167) holds which is responsible for the release of rotational energy of the black hole. For example, for a monopole magnetic field the surface $\rho_{\rm GJ} = 0$ is a sphere of radius $r_0 = 2^{1/3} r_{\rm H} \approx 1.26 r_{\rm H}$. As can be seen, this surface is located not so close to the black hole horizon. To find the surface $\rho_{\rm e} = 0$ in the general case, it is necessary to know the magnetic field structure $\Psi(r,\theta)$ and the dependence of the angular velocity $\Omega_{\rm F}$ on the potential Ψ.

As a result, in the black hole magnetosphere a region appears absolutely similar to the external gap in the magnetosphere of radio pulsars [97]. In this region, as mentioned above, longitudinal electric fields may appear because the flow of a charge-separated plasma cannot fulfil the condition $\rho_e = \rho_{GJ}$. The longitudinal electric field causes particle acceleration which, as it turns out, can just initiate the generation of a secondary electron-positron plasma at the expense of the two-photon process $\gamma + \gamma \to e^+ + e^-$. Hard γ-quanta are due to the interaction between accelerated particles and the background radiation (for more details see Refs [125, 126], where it is also shown that under real conditions the dimension of the acceleration region is much smaller than the size of the system, and so the acceleration region does not affect the global structure of the magnetosphere). One-photon particle production, as in the case of an external gap in the magnetosphere of radio pulsars, is impossible because the magnetic field (162) is too weak.

Let us now examine the structure of the black hole magnetosphere; together with this, we shall find the values of the angular velocity $\Omega_F(\Psi)$ and the current $I(\Psi)$, and therefore, the form of the surface $\rho_{GJ} = 0$. As distinct from the case of a monopole magnetic field, we shall deal with what seems to us a more realistic case where a black hole is in the centre of a well-conducting disc of internal radius b. As in Ref. [36], we shall assume the magnetic field lines not to penetrate into the accretion disc.

In the absence of a black hole, the potential Ψ, as mentioned above, is described by formula (169). Expanding now expression (169) around $r = 0$ in a series of multipoles and 'sewing' each harmonic with its asymptotics near the black hole horizon, one readily obtains an expression for the potential Ψ in the vacuum approximation (and, therefore, for the magnetosphere of a non-rotating black hole) [125]. Thus, for $r_H \ll b$ we have

$$\Psi_v = \Psi_0\left[\frac{1}{2}\frac{r^2\sin^2\theta}{b^2} - \right.$$
$$\left. - \frac{1}{120}\frac{r^4}{b^4}F(r)(4\sin^2\theta\cos^2\theta - \sin^4\theta) + \ldots\right], \quad (183)$$

where $F(-4, -2, 1, 1 - r_H/r)$ is a hypergeometric function. Consequently, near a black hole the magnetic field is homogeneous:

$$\Psi_v \approx \frac{1}{2}\Psi_0\frac{r^2\sin^2\theta}{b^2}\,. \quad (184)$$

We shall now consider the case of a slowly rotating black hole. It is easy to see that in the wide region $r_H(\Omega_H r_H)^2 \ll r - r_H \ll 1/\Omega_H$ the trans-field equation (163) coincides to within the small quantity $\sim (\Omega_H r_H)^2$ with the vacuum equation. Hence, as in the paper by Blandford and Znajek [36], the problem of determination of corrections to the vacuum magnetic field, in fact, stands aside from the problem of determination of the longitudinal current I and the angular velocity Ω_F. The two boundary conditions for their definition are the conditions on the external and internal Alfven surfaces. It can readily be verified that the potential Ψ (169) gives a monopole magnetic field $\Psi = \Psi_0(1 - \cos\theta)$ at large distances from the black hole $r \gg b$. As a result, this field will remain the solution of the nonlinear equation (163) (and, in particular, will not have a singularity on the Alfven surface) when the condition

$$4\pi I(\Psi) = \Omega_F(\Psi)\left(2\Psi - \frac{\Psi^2}{\Psi_0}\right) \quad (185)$$

is met. But we already know that near the horizon the regularity condition on the internal Alfven surface (provided that the magnetic field (184) itself is a solution of the vacuum equation) coincides with the condition on the horizon (164), which gives

$$4\pi I(\Psi) = 2\Psi\sqrt{1 - \frac{\Psi}{\Psi_*}}\left[\Omega_H - \Omega_F(\Psi)\right]. \quad (186)$$

Here $\Psi_* = 0.5\Psi_0 r_H^2/b^2$ is the magnetic flux through the black hole surface. As a result, combining relations (185) and (186), we obtain

$$\Omega_F(\Psi) = \frac{\sqrt{1 - \Psi/\Psi_*}}{1 - \Psi/2\Psi_0 + \sqrt{1 - \Psi/\Psi_*}}\,\Omega_H\,, \quad (187)$$

$$4\pi I(\Psi) = 2\Psi\left(1 - \frac{\Psi}{2\Psi_0}\right)\frac{\sqrt{1 - \Psi/\Psi_*}}{1 - \Psi/2\Psi_0 + \sqrt{1 - \Psi/\Psi_*}}\,\Omega_H\,. \quad (188)$$

Relations (187) and (188) determine the structure of the electric fields and longitudinal currents in the magnetosphere of a black hole.

We immediately see that $I(\Psi_*) = 0$. Consequently, in the model constructed above, in the field lines traversing the horizon of a black hole, a reverse current inevitably appears, and so the total current in the region $\Psi < \Psi_*$ is automatically zero. Accordingly, the condition $\Omega_F(\Psi_*) = 0$ shows that the total electric charge is zero, too. Now, Fig. 7 demonstrates that the region of longitudinal current forms a 'jet' at an angle

$$\theta_j = \arccos\left(1 - \frac{\Psi_*}{\Psi_0}\right) \approx \frac{r_H}{b}\,,$$

and the energy transferred by the electromagnetic field

$$E = \frac{1}{2\pi}\int \Omega_F(\Psi)I(\Psi)\,d\Psi$$

inside the 'jet' is close to the maximum possible energy (168) $W_{tot} = 0.489\,W_{BZ}$ because the angular velocity (187) is close to $\Omega_H/2$.

Figure 7 also shows the form of the surface $\rho_{GJ} = 0$. We can see that each magnetic field line traversing the horizon of a black hole also traverses the surface $\rho_{GJ} = 0$ where the secondary plasma filling up the magnetosphere is produced. Different branches of this surface correspond to different directions of the longitudinal current $j \propto dI/d\Psi$. This result (which does not depend on the angular velocity of a black hole) can be well explained. The point is that near the horizon where, according to (164), the electric field tends to the magnetic field $|\mathbf{E}| \to |\mathbf{B}|$ and all the particles move in the direction of the hole at a velocity close to the velocity of light, the drift component begins to dominate

$$\mathbf{j} = c\rho_e\frac{[\mathbf{E}\times\mathbf{B}]}{B^2} \approx -c\rho_e\frac{\mathbf{r}}{r}$$

which is proportional to the charge density ρ_{GJ}.

We have thus constructed the model of a force-free magnetosphere for the case of a quasi-homogeneous magnetic field near the horizon of a black hole. We have shown that as distinct from the quasi-monopole Blandford – Znajek model, both the total charge and the total electric current in the region of force lines through the horizon of a black hole are identically zero.

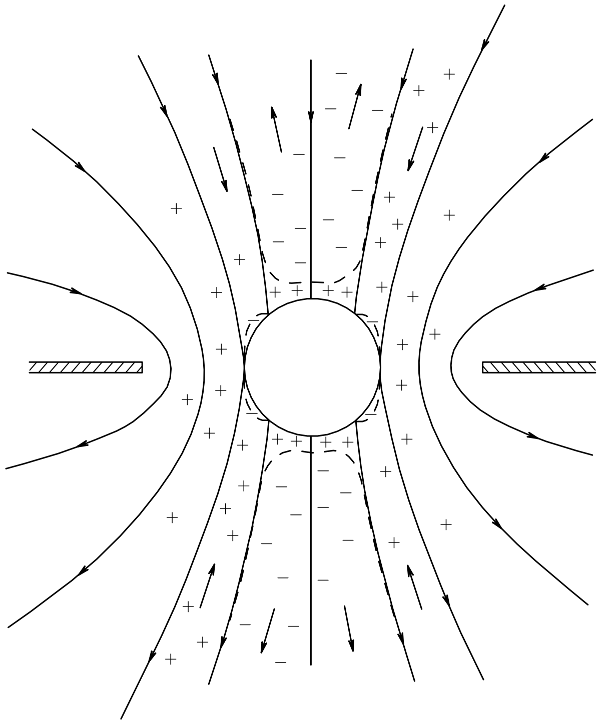

Figure 7. Structure of the magnetosphere of a black hole and the form of the surface $\rho_{GJ} = 0$.

6. The general MHD case — jet formation

6.1 The cold plasma approximation

Concluding, we shall consider the properties of ejected plasma whose radiation is registered in jets from active galactic nuclei and quasars [12, 115], as well as from young stars [8, 127] and supernova remnants (plerions) containing radio pulsars [128 – 131]. As mentioned above, the most preferable model to account for the nature of such jets is currently the magnetohydrodynamic model [115] providing a rather clear insight into the energy source of their activity (a rotating compact object, e.g., an accreting black hole), and into the mechanism of energy and angular momentum release. Furthermore, in the framework of such models, the mechanism of energy transfer from the field to particles may become ineffective, which would, in principle, explain the energy transfer from the 'central engine' to radiation regions in radio galaxies [12]. It is therefore not surprising that quite a number of papers, beginning with the investigation of the possibility of collimation of magnetic surfaces [25, 27, 51, 67, 132, 133] and their real structure [134 – 137] to the study of their stability [138, 139] and the origin of the observed radiation [12, 115] have been devoted to the magnetohydrodynamic jet model. A great many numerical calculations were carried out [140 – 142] that advanced the understanding of the jet structure and, in particular, the mechanism of energy release in active regions. However, as we have already said, no consistent analytical results have as yet been obtained. On the other hand, as to young radio pulsars, it has long been known that at distances small as compared with the dimensions of supernova remnants, the bulk energy is transferred not by an electromagnetic wave but by particles [129, 143], although inside the light cylinder the situation is opposite. Hence, in radio pulsars there must exist an effective energy transfer from the electromagnetic field to particles.

In the theoretical context, the question of the magnitude of longitudinal electric current circulating in the magnetosphere is crucial. As we have seen, when the currents exceed the Goldreich current (146), a collimation of magnetic field lines along the axis of rotation becomes possible. Recall that in the force-free approximation the longitudinal current is a free quantity. However, in the general magnetohydrodynamic case, when the number of singular surfaces increases, a smooth passage through the singular surfaces becomes possible if the choice of the longitudinal current is fixed.

For the sake of simplicity, we shall consider the case of a cold plasma because, as we know, the thermal processes in the magnetosphere of radio pulsars do not play a dominant role. As to the jets from active galactic nuclei, this approximation is applicable in those magnetosphere regions where the accreting gas density is not high. This, apparently, holds true for field lines passing through the surface of a black hole as considered in the preceding section.

So, we shall examine an axisymmetric stationary flow of a cold relativistic plasma. Clearly, such an approximation determines, in fact, one of the integrals of motion because now we have $s(\Psi) = 0$. Consequently, in this approximation there are only four integrals of motion. These are the fluxes of energy and angular momentum, as well as the angular velocity of rotation $\Omega_F(\Psi)$ and the particle-to-magnetic flux ratio $\eta(\Psi)$. The enthalpy μ is now coincident with the particle mass and is therefore constant. As a result, although the trans-field equation, where we now have

$$D = \frac{A}{M^2} + \frac{\alpha^2}{M^2}\frac{B_\varphi^2}{B_p^2}\,, \tag{189}$$

and the Bernoulli equation seem outwardly to almost coincide with Eqns (23) and (41), they are, in fact, much simpler. This is primarily due to the fact that the derivative ∇_k' does not any longer affect the μ values. Moreover, the Bernoulli equation (23) for $\mu = \text{const}$ becomes a fourth-order algebraic equation for the quantity M^2. It will be shown below that in some cases this will make it possible to obtain analytical asymptotics of Eqn (23), i.e., to define explicitly M^2 as a function of the four integrals of motion and of the potential Ψ. In addition, in the cold plasma approximation, the slow and the cusp singular surfaces are absent [71] which also makes the analysis of the trans-field equation much simpler. Hence, according to Eqn (45), the trans-field equation (41) requires four boundary conditions.

It is of interest that as in the force-free approximation, the trans-field equation in the one-dimensional case can be integrated [26]. In fact, for conical magnetic surfaces $\Psi = \Psi(\theta)$ in an asymptotically distant region $r \to \infty$ the trans-field equation (41) can alternatively be written in the form

$$-\frac{g}{g+1}\frac{d}{d\theta}\left[\frac{A^2}{64\pi^4 r^4 \sin^2\theta}\left(\frac{d\Psi}{d\theta}\right)^2\right] - \frac{\mu^2\eta^2}{\Omega_F^2 g}\frac{d\Omega_F^2}{d\theta} + \frac{g}{g+1}\frac{dE^2}{d\theta} - \frac{d}{d\theta}(\mu^2\eta^2) = 0\,, \tag{190}$$

where

$$g = \frac{M^2}{\Omega_F^2 \varpi^2}\,. \tag{191}$$

Making use of the asymptotics of the Bernoulli equation (23), we ultimately have

$$\frac{d}{d\theta}\left(\frac{\mu^2\eta^2}{g^2\Omega_F^2}\right) = 0\,. \tag{192}$$

In particular, for $\eta = \text{const}$, $\Omega_F = \text{const}$ we simply obtain

$$g = \text{const}\,. \tag{193}$$

We stress that relations (190)–(193) are invalid as $\theta \to 0$ because they were obtained for the assumption that $\varpi \gg R_L$.

Finally, the trans-field equation (41) (for the case of slow rotation and in the absence of gravitation) was integrated by Bogovalov in the problem of relativistic electron-positron plasma outflow from a sphere with a monopole magnetic field [67]. He applied exactly the same method as is used widely in the present paper. Indeed, in the absence of rotation (i.e., for $\Omega_F = 0$, $L = 0$) the monopole magnetic field $\Psi = \Psi_0(1 - \cos\theta)$ is, clearly, again a solution of the trans-field equation. The four boundary conditions on the sphere of radius R will be the quantities

$$\begin{aligned} &\Omega_F(R,\theta) = 0\,,\\ &\gamma(R,\theta) = \gamma_{in} = \text{const}\,,\\ &n(R,\theta) = n_{in} = \text{const} \end{aligned} \tag{194}$$

and also the monopole magnetic field on the surface $\Psi(R,\theta) = \Psi_0(1 - \cos\theta)$. At the distance

$$r_a = R\,\frac{B_p}{\sqrt{4\pi n_{in} m_e c^2}} \tag{195}$$

the plasma flow smoothly intersects the Alfven surface and the magnetosonic surface coincident with the former for $I = 0$.

We now suppose that the sphere starts rotating at a small angular velocity Ω, and so the parameter

$$\varepsilon_B = \frac{\Omega r_a}{c} \tag{196}$$

remains much less than unity. So, the equilibrium solution can again be sought in the form

$$\Psi(r,\theta) = \Psi_0\left[1 - \cos\theta + \varepsilon_B g_2(x)\sin^2\theta\cos\theta\right],$$

where $x = r/r_a$ and the radial function $g_2(x)$ should be determined from the equation

$$(x^2 - 1)\frac{d^2 g_2}{dx^2} + 2x\frac{dg_2}{dx} + \frac{6}{x^2}\,g_2 - \frac{2}{\sigma^2} = 0\,. \tag{197}$$

An exact solution of Eqn (197) can be found in the paper by Bogovalov [67]. The asymptotic behaviour of the solution as $r \to \infty$, as in the majority of problems considered in Section 3, is completely determined by the inhomogeneous solution of Eqn (197), i.e., is independent of the boundary conditions. As a result, we have at a distance $r \gg r_a$

$$\Psi(r,\theta) = \Psi_0\left[1 - \cos\theta + 2\varepsilon_B^2\,\frac{1}{\sigma^2}\,\ln\left(\frac{r}{r_a}\right)\sin^2\theta\cos\theta\right]. \tag{198}$$

We can see that although magnetic force lines have a tendency to collimation along the axis of rotation (the correction $\delta\Psi > 0$ for $\theta < \pi/2$), this process proceeds extremely slowly, and so even in the region of the light cylinder, which in this problem is much farther than the singular surfaces (195), $R_L \approx \varepsilon_B^{-1} r_a$, the perturbation of the magnetic field $\sim \varepsilon_B^2 \ln(1/\varepsilon_B)$ turns out to be much less than unity. As regards the particle energy, in the problem under consideration it remains practically unchanged as compared to the initial energy $m_e c^2 \gamma_{in}$.

We shall note another very important circumstance concerning the determination of the magnitude of the longitudinal current I (or the integral L). As has already been said, the magnitude of the longitudinal current here is not a free function and is to be determined from the condition of smooth passage through singular surfaces. It can be readily verified that for small quantities ε_B a smooth transition from a subsonic to a supersonic branch is possible only provided the position of the Alfven surface, which is determined as before by the relation

$$r_a^2\sin^2\theta = \frac{L}{\Omega_F E} \tag{199}$$

(here $E = \gamma_{in}\mu\eta = \text{const}$), in the zeroth order in the small quantity ε_B coincides with the position of the fast magnetosonic surface (195). As a result we have

$$L(\theta) = \frac{\Omega_F}{8\pi^2}\,\Psi_0\sin^2\theta\,, \tag{200}$$

and so the longitudinal current (19) is here almost coincident with the Michel's current I_M (146). It is not therefore surprising that for such a current there is practically no collimation of magnetic surfaces along the axis of rotation.

It is of interest that Bogovalov's solution can be readily extended to the case of accretion-ejection onto a slowly rotating black hole. Indeed, the monopole solution (147) remains exact for a non-rotating black hole as well. The only point which makes the problem more complicated is the necessity of introducing a region of generation of matter that provides both the outflow and accretion, as well as the appearance of the second family of singular surfaces for accreting matter due to effects of general relativity. Employing now Eqn (19) to find the quantity α^2 on the Alfven surface as $\Omega_H \to 0$,

$$\alpha_A^2 = \frac{(\Omega_H - \Omega_F^-)\varpi_H^2 E^-}{L^-}\,, \tag{201}$$

and also the condition $\alpha_F^2 = M^2(r_H)$ for Alfven and fast magnetosonic surfaces for $\Omega_H = 0$, we have

$$L^-(\theta) = \frac{\Omega_H - \Omega_F^-}{8\pi^2}\,\Psi_0\sin^2\theta\,. \tag{202}$$

As for the ejected matter, we can use as before Eqns (199) and (200). If we believe that in the plasma generation region the potential drop does not change the angular velocity of rotation $\Omega_F \approx \Omega_F^-$ and, in addition, that $L \approx L^-$ (which means the absence of surface currents along the plasma generation region), we eventually obtain

$$\Omega_F \approx \frac{\Omega_H}{2}\,. \tag{203}$$

As we can see, here, too, the angular velocity appeared to equal half of that of the black hole rotation when the release of rotation energy is most effective.

6.2 Particle acceleration

We shall begin our consideration with the case of a small longitudinal current $i_0 < 1$ when the light surface is near the light cylinder. As emphasised above, the light surface is a natural limit of the Grad – Shafranov equation because the electric field is by definition compared here in magnitude with the magnetic field, and so the freezing-in condition (12) can no longer be fulfilled. As a result, in this region particles can be accelerated perpendicularly to the magnetic surfaces, which leads to a sharp increase in their energy. Consequently, in investigating the particle motion in this region it is necessary to formulate a complete system of equations, including equations of motion of individual particles. As shown in Refs [10, 92], for sufficiently small longitudinal currents in a thin transition layer located near the light surface, almost all the energy transferred in the internal regions by the electromagnetic field is given to plasma particles. On the other hand, in the transition layer closure of the longitudinal current circulating in the magnetosphere also occurs.

We shall now analyse more closely another limiting case where the longitudinal current flowing in the magnetosphere is sufficiently large, so that the light surface goes to infinity. For simplicity we shall consider the outflow of a relativistic cold plasma from a rotating sphere with a monopole magnetic field. As mentioned above, such a model may well serve as a first approximation for the external region of the magnetosphere of a black hole. After that, as a zeroth approximation we shall consider the force-free Michel's solution [46] which gives a monopole field for a special choice of longitudinal current $I_{\mathrm{M}}(\theta)$ (146) and the angular velocity of rotation Ω_{F}. In other words, we shall assume the magnetisation parameter to be $\sigma \gg 1$.

According to (45), this problem requires four boundary conditions, and as such we can choose the condition $\Psi = \Psi(R, \theta)$ and the values of another three physical quantities on the body surface. For simplicity we assume

$$\Omega_{\mathrm{F}}(R,\theta) = \Omega_{\mathrm{F}} = \mathrm{const}\,, \tag{204}$$

$$\gamma(R,\theta) = \gamma_{\mathrm{in}} = \mathrm{const}\,, \tag{205}$$

$$n(R,\theta) = n_{\mathrm{in}} = \mathrm{const}\,. \tag{206}$$

Now introducing perturbations to the force-free values of the integrals of motion L_0 (146) and $E_0 = \Omega_{\mathrm{F}} L_0$ in the form

$$E(\Psi) = E_0(\Psi) + b(\Psi)\,, \tag{207}$$

$$L(\Psi) = L_0(\Psi) + l(\Psi)\,, \tag{208}$$

we come to

$$\eta = \frac{n_{\mathrm{in}}}{B_{\mathrm{p}}}\,, \tag{209}$$

$$e = E - \Omega_{\mathrm{F}} L = b - \Omega_{\mathrm{F}} l = \frac{B_{\mathrm{p}}}{4\pi}\, M^2(R) = \gamma_{\mathrm{in}} \mu\eta\,, \tag{210}$$

the integrals of motion η, Ω_{F}, and e being constant over the entire space. In relation (210), $M^2(R)$ is the Mach number (22) on the star surface, $n(r,\theta) = n_{\mathrm{in}}/\gamma_{\mathrm{in}}$, and $B_{\mathrm{p}} = \Psi_0/2\pi R^2$ is the radial magnetic field on the surface. Recall that according to Eqns (146) and (147), the quantity E_0 can be represented in the form $E_0 = E_{\mathrm{A}} \sin^2\theta$ and we just have $\sigma = E_{\mathrm{A}}/\mu\eta$. In view of the condition $\sigma \gg 1$ we also have $e/E \ll 1$. Finally, according to Eqn (210) we can write

$$b(\Psi) = e + \Omega_{\mathrm{F}} l(\Psi)\,. \tag{211}$$

As to the quantity $l(\Psi)$ [which for small perturbations of the monopole field can be interpreted as a function of the angle θ, i.e., $l = l(\theta)$], it is to be defined from the regularity condition (37) on a fast magnetosonic surface.

As before, we shall seek the solution of Eqn (41) in the form $\Psi(r,\theta) = \Psi_0\left[1 - \cos\theta + \varepsilon f(r,\theta)\right]$, where $\varepsilon \sim \gamma_{\mathrm{in}}\sigma^{-1} \ll 1$. Substituting this decomposition into Eqn (41), we are led to

$$\begin{aligned}
&\varepsilon A\,\frac{\partial^2 f}{\partial r^2} + \varepsilon A\,\frac{D+1}{Dr^2}\,\sin\theta\,\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\,\frac{\partial f}{\partial\theta}\right) - 2\varepsilon\Omega_{\mathrm{F}}^2 r\sin^2\theta\,\frac{\partial f}{\partial r}\\
&- 2\varepsilon\Omega_{\mathrm{F}}^2 \sin\theta\cos\theta\,\frac{\partial f}{\partial\theta} + 2\varepsilon\Omega_{\mathrm{F}}^2 (3\cos^2\theta - 1) f\\
&+ \frac{8\pi^2\Omega_{\mathrm{F}}}{\Psi_0}\,\frac{1}{\sin\theta}\,\frac{\mathrm{d}}{\mathrm{d}\theta}(l\sin^2\theta) - 2\varepsilon\,\frac{A}{Dr^2}\,\frac{\cos\theta}{\sin\theta}\,\frac{\partial f}{\partial\theta}\\
&+ 2\varepsilon\,\frac{A}{Dr^2}\,\frac{1+\cos^2\theta}{\sin^2\theta}\,f - \frac{64\pi^4 A}{\Omega_{\mathrm{F}}^2 D\Psi_0^2}\,\frac{\cos\theta}{\sin^2\theta}\,\frac{e^2}{M^4}\\
&+ 2A\cos\theta\,\frac{1-M^2}{D\Omega_{\mathrm{F}}^2 r^4\sin^2\theta} + \frac{16\pi^2 A\cos\theta}{\Omega_{\mathrm{F}}^2 r^2\sin^2\theta}\,\frac{e}{D\Psi_0}\\
&- \frac{8\pi^2 A\sin\theta}{D\Omega_{\mathrm{F}} r^2\Psi_0}\,\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{l}{\sin^2\theta}\right) - \frac{1}{Dr^2}\,\frac{A\cos\theta}{\Omega_{\mathrm{F}}^2 r^2\sin^2\theta}\\
&+ 2\Omega_{\mathrm{F}}^2\sin^2\theta\cos\theta\,\frac{M^2}{A} - 16\pi^2\Omega_{\mathrm{F}}^2 r^2\sin^2\theta\cos\theta\,\frac{e}{A\Psi_0} = 0\,.
\end{aligned} \tag{212}$$

We shall begin with analysing the position of the fast magnetosonic surface which, as all the other parameters of the flow, is to be determined from the solution of Eqn (212) with the boundary conditions (204) – (206). To determine the position of the fast magnetosonic surface $r = r_{\mathrm{F}}$ (as well as the physical root M^2), it is necessary to solve the fourth-order equation (23). To do this, it is convenient to employ the quantity g (191). According to Eqns (191) and (211), we have $g \ll 1$ for $\sigma \gg 1$, and therefore far from the Alfven surface the equality

$$\gamma = \frac{E}{\mu\eta}\, g \tag{213}$$

holds. Consequently, g is in fact the ratio of the particle energy flux to the total energy flux and for $g \ll 1$ the main contribution to the energy transfer will be made by the electromagnetic field. Thus, the algebraic equation (23) can now be written as

$$g^3 - \frac{1}{2}\left(\xi + \frac{1}{\Omega_{\mathrm{F}}^2 r^2\sin^2\theta}\right) g^2 + \frac{\mu^2\eta^2}{2E^2} + \frac{e^2}{2\Omega_{\mathrm{F}}^2 r^2 E^2\sin^2\theta} = 0\,, \tag{214}$$

where we omitted the summand g^4 (which gives a nonphysical root $g < 0$) and introduced the quantity

$$\begin{aligned}
\xi &= 1 - \frac{\Omega_{\mathrm{F}}^4 r^2\sin^2\theta(\nabla\Psi)^2}{64\pi^4 E^2}\\
&= 2\,\frac{e + \Omega_{\mathrm{F}} l}{E} - \frac{2\varepsilon}{\sin\theta}\,\frac{\partial f}{\partial\theta} + 4\,\varepsilon\,\frac{\cos\theta}{\sin^2\theta}\,f\,.
\end{aligned} \tag{215}$$

We can easily check that $\xi = 0$ for the force-free Michel's solution (146) and $\xi \ll 1$ for $\sigma^{-1} \ll 1$. We stress that it is just the dependence of ξ on εf that allows a self-consistent analysis of our problem.

Clearly, a fast magnetosonic surface corresponds to the intersection of two roots of Eqn (214) at one point. On the other hand, Eqn (214) has three real roots only for $Q \leqslant 0$, where the discriminant of the cubic equation (for $r \approx r_{\rm F}$) is

$$Q = \frac{1}{16}\frac{\mu^4\eta^4}{E^4} - \frac{1}{16\cdot 27}\frac{\mu^2\eta^2}{E^2}\left(\xi + \frac{1}{\Omega_{\rm F}^2 r^2 \sin^2\theta}\right)^3 . \quad (216)$$

Given this, the regularity conditions of the solution near the fast magnetosonic surface ($r = r_{\rm F}$) now have the form

$$Q = 0 , \quad \frac{\partial Q}{\partial r} = 0 , \quad \frac{\partial Q}{\partial \theta} = 0 . \quad (217)$$

As can be readily checked, the conditions (217) coincide exactly with the regularity conditions $D = 0$ (33), $N_r = 0$ and $N_\theta = 0$ (37). This leads us to

$$r_{\rm F}(\theta) \approx R_{\rm L}\sigma^{1/3}\sin^{-1/3}\theta \quad (218)$$

for $\theta > \sigma^{-1/2}$ and (cf. [144])

$$r_{\rm F} \approx R_{\rm L}\left(\frac{\sigma}{\gamma_{\rm in}}\right)^{1/2} \quad (219)$$

near the axis of rotation. Moreover, since the values of the roots of Eqn (214) at the moment of their coincidence do not depend on the quantity ξ, we have exactly

$$g(r_{\rm F}, \theta) = \left(\frac{\mu\eta}{E}\right)^{2/3} . \quad (220)$$

Hence,

$$\gamma(r_{\rm F}, \theta) = \left(\frac{E}{\mu\eta}\right)^{1/3} = \sigma^{1/3}\sin^{2/3}\theta , \quad (221)$$

which coincides with Michel's result [28] (see also Ref. [145]). The only difference between them is that the energy (221) is already attained at a finite distance $r_{\rm F}$ (218). Positions of the Alfven and the fast magnetosonic surfaces are shown in Fig. 8. In the case $\gamma_{\rm in} > \sigma^{1/3}$, the fast magnetosonic surface is a sphere of radius $r_{\rm F} = (\sigma/\gamma_{\rm in})^{1/2} R_{\rm L}$.

Let us now consider the internal region $r\sin\theta < \gamma_{\rm in} R_{\rm L}$, $r < r_{\rm F}$. In this region, the physical root of Eqn (214) (i.e., the root corresponding to a subsonic flow $D > 0$) is

$$g = \frac{e}{E} . \quad (222)$$

Consequently, according to Eqn (213), there is no particle acceleration in this region:

$$\gamma(r, \theta) = \gamma_{\rm in} . \quad (223)$$

In particular, for $\sigma < \gamma_{\rm in}^3$ the particle energy remains constant up to the fast magnetosonic surface.

Next, in the intermediate region $\gamma_{\rm in} R_{\rm L} < r\sin\theta$, $r < r_{\rm F}$, which exists for $\sigma > \gamma_{\rm in}^3$, the physical root $D > 0$ of the algebraic equation (214) has the form

$$g = \frac{\mu\eta}{E}\,\Omega_{\rm F} r\sin\theta . \quad (224)$$

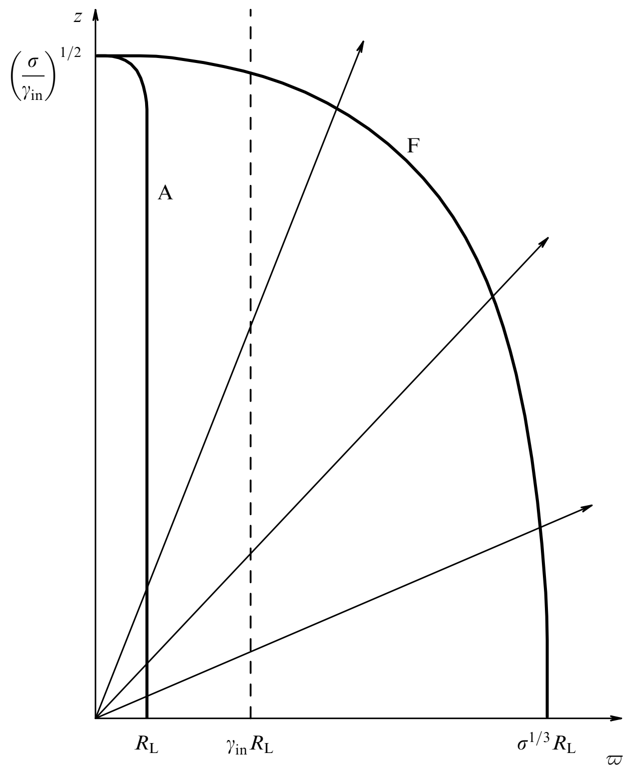

Figure 8. Positions of Alfven and fast magnetosonic surfaces for $\sigma > \gamma_{\rm in}^3$ [100].

We thus arrive at

$$\gamma(r, \theta) = \Omega_{\rm F} r\sin\theta , \quad (225)$$

$$M^2(r, \theta) = \sigma^{-1}\Omega_{\rm F}^3 r^3 \sin\theta \quad (226)$$

and $D = M^{-2}$. As we can see, in this region all the physical characteristics of the flow are independent of ξ and, therefore, of the perturbation of the field $\varepsilon f(r, \theta)$. That is why, for their determination, we need not know an exact solution of the trans-field equation (212). However, comparing the solution (151) with the regularity conditions on a fast magnetosonic surface (217), we obtain

$$\frac{l}{L_0} \sim \sigma^{-4/3} . \quad (227)$$

Thus, for a smooth passage of a fast magnetosonic surface the longitudinal current must nearly coincide with the force-free current $I_{\rm M}$ (146). This confirms our conclusion of $I = 2\pi L_0(\Psi)$ as the zeroth approximation. Finally, according to Eqns (215) and (227) we have

$$\varepsilon f(r_{\rm F}) \sim \sigma^{-2/3} . \quad (228)$$

Hence, the magnetic field perturbation also remains small up to the fast magnetosonic surface $r = r_{\rm F}$.

As far as the asymptotic region $r \gg r_{\rm F}$ is concerned, the physical root of Eqn (214) that corresponds to the supersonic flow $D < 0$ is

$$g = \frac{\xi}{2}\left(1 - 4\,\frac{\mu^2\eta^2}{\xi^3 E^2}\right) . \quad (229)$$

Given this, it can easily be checked that the corresponding value of the particle energy $\gamma = g(E/\mu\eta)$ coincides with the

Lorentz-factor $\gamma = (1 - U_{\rm dr}^2)^{-1/2}$ determined by the drift velocity $U_{\rm dr} = |\mathbf{E}|/|\mathbf{B}| \approx |\mathbf{E}|/B_\varphi$. On the other hand, the trans-field equation (212) can be written here in a simple form

$$\varepsilon r^2 \frac{\partial^2 f}{\partial r^2} + 2\varepsilon r \frac{\partial f}{\partial r} - \sin\theta \frac{D+1}{D} \frac{\partial g}{\partial \theta} = 0\,, \qquad (230)$$

where g is defined by formula (229), and

$$D + 1 = 8 \frac{\mu^2 \eta^2}{E^2 \xi^3} \ll 1\,. \qquad (231)$$

So, for the radial flow $\partial f/\partial r = 0$ we have $\mathrm{d}g/\mathrm{d}\theta = 0$ which corresponds to the integral (193) [26]. However, as can be seen, in the complete equation (230) the derivative $\partial g/\partial\theta$ has a small coefficient $D + 1 \ll 1$. Consequently, this summand does not actually play any role in the complete equation (230). As a result, according to Eqn (230), in the asymptotically distant region $r \gg r_{\rm F}$ the perturbation of the magnetic field εf becomes constant

$$\varepsilon f(r, \theta) = \sigma^{-2/3} a(\theta), \qquad a(\theta) \sim 1\,. \qquad (232)$$

Accordingly, by virtue of Eqns (213) and (229), the particle energy stops increasing:

$$\gamma(\infty) \approx \sigma^{1/3} \sin^{2/3}\theta\,. \qquad (233)$$

Thus, we have, in fact, constructed an example of a solution where the regularity conditions on singular surfaces restrict the value of the longitudinal current. As a result, outside the fast magnetosonic surface both collimation and particle acceleration become ineffective. On the other hand, we showed that the expression, obtained by Michel [28], for the particle energy at infinity $\gamma \sim \sigma^{1/3}$ (within the model in question) remains valid.

6.3 The jet structure

Finally, we shall consider the collimation of magnetic surfaces and the internal structure of jets. As has already been said, the collimation is the key question in the entire problem of the construction of the magnetosphere of a compact object, and a considerable number of papers [25–27], including numerical calculations [146, 147] have been devoted to it. At the same time, in most cases attention was primarily paid to proper collimation in the sense that the influence of the external medium was assumed to be insignificant. Such a situation is however possible only for a nonzero longitudinal current I flowing inside a jet [148], and so, the question arises of its closure in the external parts of the magnetosphere. On the other hand, the magnitude of the longitudinal current is limited by regularity conditions on singular surfaces, which, as shown in the preceding section, does not lead to the sufficiently large longitudinal currents necessary for collimation.

It is quite obvious that the question of collimation cannot be solved without involving external conditions (see, e.g., Ref. [138]). Moreover, as is well known on an example of moving cosmic bodies (Jupiter's satellites or artificial Earth's satellites [149]), an external magnetic field may serve as an efficient transmitter that sometimes determines the total energy losses of a system. Therefore, the question of construction of a self-consistent model of the magnetosphere of compact objects embedded in an external magnetic field in our opinion is of undeniable interest.

The question of the existence of an external regular magnetic field in the vicinity of compact objects is undoubtedly disputable. As is known, the regular magnetic field in our Galaxy (i.e., the field constant on scales comparable with the Galaxy itself), $B_{\rm ext} \sim 10^{-6}$ G, is practically coincident with the chaotic component of the magnetic field which changes over scales of several parsecs [150]. However, if the collimation were assumed to be merely due to the presence of an external field, it would be possible to estimate very easily the transverse jet dimension $r_{\rm j}$. Indeed, assuming the magnetic field in a jet to be close to the external magnetic field, we obtain from the condition of conservation of magnetic flux

$$r_{\rm j} \sim R \left(\frac{B_{\rm in}}{B_{\rm ext}}\right)^{1/2}, \qquad (234)$$

where R and $B_{\rm in}$ are respectively the radius and the magnetic field of a compact object. For instance, for active galactic nuclei ($B_{\rm in} \sim 10^4$ G, $R \sim 10^{13}$ cm) we have $r_{\rm j} \sim 1$ pc, which corresponds to the observed transverse dimensions of jets [12]. Accordingly, for young stellar objects ($B_{\rm in} \sim 10^2$ G, $R \sim 10^{10}$ cm) [8] we have $r_{\rm j} \sim 10^{16}$ cm, which also agrees with observations. That is why it seems interesting to consider as an example a one-dimensional jet embedded in an external homogeneous magnetic field [151].

So, let us consider the structure of a one-dimensional jet. As everywhere in this section, the temperature of the matter is assumed to be zero. Then, far from gravitating bodies the trans-field equation (41) is written in the form

$$\frac{1}{\varpi} \frac{\mathrm{d}}{\mathrm{d}\varpi} \left(\frac{A}{\varpi} \frac{\mathrm{d}\Psi}{\mathrm{d}\varpi}\right) + \Omega_{\rm F} (\nabla\Psi)^2 \frac{\mathrm{d}\Omega_{\rm F}}{\mathrm{d}\Psi} + \frac{64\pi^4}{\varpi^2} \frac{1}{2M^2} \frac{\mathrm{d}}{\mathrm{d}\Psi} \left(\frac{G}{A}\right) - \frac{64\pi^4}{M^2} \mu^2 \eta \frac{\mathrm{d}\eta}{\mathrm{d}\Psi} = 0\,, \qquad (235)$$

where $\varpi = r\sin\theta$ and G is given as before by relation (40).

Equation (235) contains four integrals of motion. Clearly, on the jet boundary $\Psi = \Psi_*$, where the longitudinal motion of matter is absent, all four integrals of motion must vanish. As we have seen, the integrals of motion $\Omega_{\rm F}(\Psi)$ (187), $L(\Psi) = I(\Psi)/2\pi$ (188) and $E(\Psi) = \Omega_{\rm F}(\Psi) L(\Psi)$ obtained for a force-free magnetosphere of a black hole meet these conditions and can, therefore, be used directly in a study of the jet structure. The only, but as shown below, exceedingly important specification is here that for a finite σ value near the axis of rotation, i.e., as $\Psi \to 0$, it is necessary that the contribution of particles should be added to the force-free value of the energy integral $E(\Psi)$ because on the axis of rotation the energy flux of an electromagnetic field must inevitably vanish. As a result, according to Eqns (15) and (188), we have for $\Psi \ll \Psi_*$

$$E(\Psi) \approx \mu\eta \left(\gamma_{\rm in} + 2\sigma \frac{\Psi}{\Psi_*}\right). \qquad (236)$$

Thus, the contribution of the electromagnetic field becomes dominant only for $\Psi > \Psi_{\rm in}$, where

$$\Psi_{\rm in} = \frac{\gamma_{\rm in}}{\sigma} \Psi_*\,. \qquad (237)$$

For low Ψ values, the bulk energy will be transferred by relativistic particles, and their energy, as follows directly from relation (236), remains constant and equal to their initial energy $\gamma_{\rm in}$.

It might be well to point out that the very possibility of using the integrals of motion obtained in the analysis of internal magnetosphere regions is not trivial. Indeed, the formation of a one-dimensional flux may only be due to the interaction with the external medium. For this reason, in the regions of interaction, where the conditions of applicability of ideal hydrodynamics will definitely be violated, a considerable redistribution of the energy E and angular momentum L is possible. Nevertheless, the integrals of motion $E(\Psi)$ and $L(\Psi)$ as functions of the flux Ψ are assumed here for simplicity to remain exactly the same as they are in the internal regions of the magnetosphere.

We now pass over to the solution of the trans-field equation. As for canonical magnetic surfaces, it is convenient to reduce at once the second-order equation (235) to a system of first-order equations for the quantities $\Psi(\varpi)$ and $M^2(\varpi)$. This becomes possible owing to the indicated property of the Grad–Shafranov equation, according to which it can be integrated in the one-dimensional case. As a result, again multiplying Eqn (235) by $(2M^2/64\pi^4)(\mathrm{d}\Psi/\mathrm{d}\varpi)$ and using the Bernoulli equation (23), we come to

$$\frac{2}{M^4}\left[(e^2-\mu^2\eta^2)+\mu^2\eta^2\Omega_{\mathrm{F}}^2\varpi^2\right]\frac{\mathrm{d}M^2}{\mathrm{d}\varpi} = \frac{4}{M^2}\,\mu^2\eta^2\Omega_{\mathrm{F}}^2\varpi - \frac{2}{AM^2}\,e^2\Omega_{\mathrm{F}}^2\varpi + 2\,\frac{M^2}{A}\,\frac{L^2}{\varpi^3} - \frac{1-\Omega_{\mathrm{F}}^2\varpi^2}{M^2}\,\frac{\mathrm{d}}{\mathrm{d}\varpi}(\mu^2\eta^2) + \frac{1}{M^2}\,\frac{\mathrm{d}}{\mathrm{d}\varpi}\,e^2 + \frac{\mu^2\eta^2\varpi^2}{M^2}\,\frac{\mathrm{d}}{\mathrm{d}\varpi}\,\Omega_{\mathrm{F}}^2\,, \tag{238}$$

where we again have $e = E - \Omega_{\mathrm{F}}L$. The second equation is the Bernoulli equation (23) itself, which should now be regarded as the equation for the derivative $\mathrm{d}\Psi/\mathrm{d}\varpi$:

$$\left(\frac{\mathrm{d}\Psi}{\mathrm{d}\varpi}\right)^2 = \frac{64\pi^4}{M^4}\left(\frac{K}{A^2} - \varpi^2\eta^2\mu^2\right). \tag{239}$$

Let us discuss the general properties of the system (238), (239). We note first of all that in the relativistic case under consideration one can put $\gamma = u$ to a good accuracy. After this, at large distances from the axis of rotation $\varpi \gg \gamma_{\mathrm{in}}R_{\mathrm{L}}$ Eqn (239) can be rewritten in the form

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\varpi} = \frac{8\pi^2E(\Psi)}{\varpi\Omega_{\mathrm{F}}^2(\Psi)}\,. \tag{240}$$

As we can see, Eqn (240) does not contain the quantity M^2 and can, therefore, be integrated independently. Assuming in (240) $B_{\mathrm{p}}(\Psi_*) = B_{\mathrm{ext}}$, we obtain, in particular, for the dimension of a jet

$$r_{\mathrm{j}}^2 = \lim_{\Psi\to\Psi_*}\frac{4\pi E(\Psi)}{\Omega_{\mathrm{F}}^2(\Psi)B_{\mathrm{ext}}} \sim \frac{\Psi_*}{\pi B_{\mathrm{ext}}}\,, \tag{241}$$

which is, in fact, coincident with the estimate (234).

It can be readily seen that for the considered values of the integrals of motion (187) and (188) the solution of Eqn (240) actually gives a constant magnetic field

$$B_{\mathrm{p}}(\varpi) = B_{\mathrm{ext}}\,, \tag{242}$$

but only under the condition $M_0^2 < \gamma_{\mathrm{in}}^2$ where the value of the Mach number on the axis $M_0^2 = M^2(0)$ is related to the magnetic field as

$$B_{\mathrm{p}} = \frac{4\pi\mu\eta\gamma_{\mathrm{in}}}{M_0^2} \sim \frac{\gamma_{\mathrm{in}}}{\sigma M_0^2}\,B(R_{\mathrm{L}})\,. \tag{243}$$

Here $B(R_{\mathrm{L}}) \sim \Psi_*/R_{\mathrm{L}}^2$ is the characteristic magnetic field on the light cylinder. In this case, the magnetic field flux exceeds Ψ_{in} (237) for $\varpi > \gamma_{\mathrm{in}}R_{\mathrm{L}}$, and so the predominant contribution to the quantity E is made here by the electromagnetic field. If $M_0^2 > \gamma_{\mathrm{in}}^2$ [i.e., $\Psi(\gamma_{\mathrm{in}}R_{\mathrm{L}}) < \Psi_{\mathrm{in}}$], then at distances $\varpi > \gamma_{\mathrm{in}}R_{\mathrm{L}}$ the Bernoulli integral E, according to (236), is on the contrary completely determined by the contribution of particles. In this case, the quantity $E(\Psi)$ should be thought of as constant and equal to $\mu\eta\gamma_{\mathrm{in}}$. As a result, for $x = \varpi/R_{\mathrm{L}} > \gamma_{\mathrm{in}}$ the solution of Eqn (240) yields a power-law decrease of the magnetic field [152, 153]:

$$B_{\mathrm{p}}(x) \approx B_{\mathrm{p}}(0)\,\frac{\gamma_{\mathrm{in}}^2}{x^2}\,. \tag{244}$$

Consequently, the magnetic flux will increase very slowly (logarithmically) with increasing distance from the axis of rotation. But this, in turn, contradicts the estimate (241) of the jet dimension. This suggests that the magnetic field on the axis of rotation cannot be smaller than

$$B_{\min} = \frac{4\pi\mu\eta}{\gamma_{\mathrm{in}}} \sim \frac{1}{\sigma\gamma_{\mathrm{in}}}\,B(R_{\mathrm{L}})\,. \tag{245}$$

In the general case, as has already been mentioned, the structure of a poloidal magnetic field is determined by a concrete choice of the integrals $E(\Psi)$ and $\Omega_{\mathrm{F}}(\Psi)$.

The solution of Eqn (238) determining the value of the Mach number $M^2(\varpi)$ can be estimated in a similar simple manner. In particular, for $x = \varpi/R_{\mathrm{L}} > \gamma_{\mathrm{in}}$ we have

$$M^2(x) = \frac{M_0^2}{\gamma_{\mathrm{in}}}\,x\,. \tag{246}$$

Accordingly,

$$g(x) = \frac{M_0^2}{\gamma_{\mathrm{in}}}\,\frac{1}{x} \tag{247}$$

and, therefore,

$$\gamma(x) = x\,, \qquad x > \gamma_{\mathrm{in}}\,. \tag{248}$$

Thus, we can see that for $\varpi > \gamma_{\mathrm{in}}R_{\mathrm{L}}$ the particle energy has universal asymptotic behaviour (248). Such simple asymptotics (analogous to the law (225) for particle energy in a monopole magnetic field in the intermediate region $\gamma_{\mathrm{in}}R_{\mathrm{L}} < r_{\mathrm{F}}$) can naturally be obtained from simpler considerations [115]. Indeed, using the freezing-in equation (12), we obtain for the drift velocity

$$U_{\mathrm{dr}}^2 = \frac{|\mathbf{E}|^2}{|\mathbf{B}|^2} = \left(\frac{B_{\hat\varphi}^2}{|\mathbf{E}|^2} + \frac{B_{\mathrm{p}}^2}{|\mathbf{E}|^2}\right)^{-1}. \tag{249}$$

But in our case, in view of Eqns (11) and (13) we have $B_{\hat\varphi}^2 \approx |\mathbf{E}|^2$ and $|\mathbf{E}|^2 \approx x^2B_{\mathrm{p}}^2$, which immediately leads to the asymptotics (248).

Figure 9 illustrates the particle dependence $\gamma(x)\mu\eta$ obtained by the integration of Eqns (238) and (239) for the integrals of motion (187) and (188). The dashed line shows the

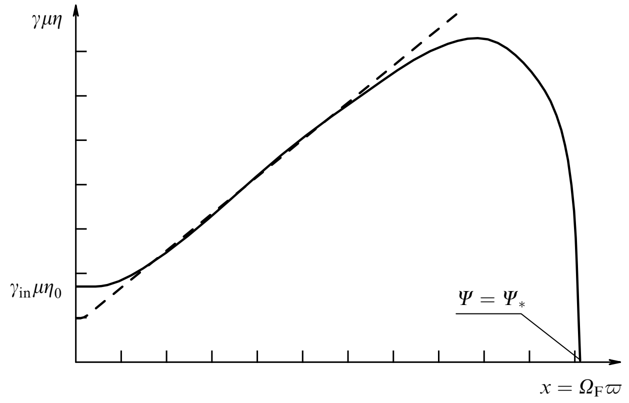

Figure 9. Dependence of particle energy $\gamma(x)\mu\eta$ on the coordinate x. The dashed line demonstrates the behaviour of this quantity following from the analytical estimate (248) [151].

behaviour of this quantity implied by the analytical estimate (248). As we can see, for sufficiently small values of x the solution virtually coincides with the analytical asymptotics (248). In particular, for $M_0^2 \sim \gamma_{\rm in}^2$ we obtain for the maximum energy

$$\gamma_{\rm max} \sim (\gamma_{\rm in}\sigma)^{1/2}, \tag{250}$$

which yields the estimate $\gamma_{\rm max} \sim 10^4 - 10^5$ for jets from active galactic nuclei. At the same time, comparing the estimate $M_0^2 < \gamma_{\rm in}^2$ with expression (247) for the quantity g we obtain that far from the light cylinder the parameter g must be much less than unity. So, for $B_{\rm ext} \sim B_{\rm min}$ we have

$$g \sim \left(\frac{\gamma_{\rm in}}{\sigma}\right)^{1/2}. \tag{251}$$

Consequently, the contribution of particles to the resultant energy flux balance turns out to be insignificant. Finally, the solution shows that for $B_{\rm ext} > \gamma_{\rm in} B_{\rm min}$ in internal jet regions $\varpi < \varpi_{\rm F}$, where

$$\varpi_{\rm F} \sim \frac{B_{\rm ext}}{B_{\rm min}} R_{\rm L}, \tag{252}$$

a region with a subsonic flow inevitably occurs (for details see Ref. [151]).

All this suggests the conclusion that the Grad – Shafranov equation does actually allow the construction of a self-consistent model of jets embedded in a homogeneous external magnetic field. As in the force-free case, the resultant electric current along a jet, the same as its total electric charge, is automatically equal to zero. The magnetic field homogeneity inside a jet here is a consequence of the choice of the integrals of motion (187) and (188). Finally, the energy transferred by particles is shown to be only a small fraction ($\sim \sigma^{-1/2}$) of the total energy flux transferred by the electromagnetic field.

Concluding we recall briefly, as in the case of a radio pulsar magnetosphere, the main predictions of the theory of magnetised wind from compact sources (see also Ref. [115]).

(1) If the longitudinal current through a radio pulsar magnetosphere is actually not large, the ejected plasma will inevitably intersect the light surface at distances comparable with the light cylinder. During this process, particles are further accelerated, as a result of which outside the light surface almost all the energy will already be transferred by a relativistic plasma. This completely solves the key question of the mechanism of conversion of electromagnetic field energy to particle energy [128, 143]. It simultaneously becomes clear how the closure of the electric current through the magnetosphere proceeds.

(2) As shown in Section 5, the effects of the general theory of relativity inevitably lead to the appearance of a surface near a black hole with properties similar to the 'external gap' in a radio pulsar magnetosphere. Near this surface, secondary particles may actually be generated. This provides a solution to the problem of the plasma filling of polar regions of a black hole magnetosphere and, simultaneously, the source of electron – positron plasma flowing out of active galactic nuclei also becomes clear. The existence of such a plasma has recently been weightily confirmed [111].

(3) An important result in our opinion is the statement that, accounting for an external regular magnetic field the equations of magnetic hydrodynamics allow the construction of a self-consistent jet model in which the total longitudinal electric current $I(\Psi_*)$ (the same as the resultant electric charge) is automatically zero. Given this, a homogeneous magnetic field coincident with the magnetic field of the external medium may also be a solution for internal jet regions. As has already been emphasised, this accounts for the magnitude of the transverse jet dimension. Energy channelling from a compact object in a region of energy release can be explained by the small energy conversion from the electromagnetic field to particles. At the same time, as shown above, an extension of the hydrodynamic solution to the region of jet needs extremely high particle energies, $\sim 10^4$ MeV, which have not yet been registered. However, a consistent analysis of the question of ejected plasma energy requires a correct account of the interaction between particles and the surrounding medium (e.g., background radiation) which may cause an appreciable change in the particle energy [12].

We stress that since for active galactic nuclei the transverse jet dimension is always several orders of magnitude greater than the radius of the light cylinder $r_{\rm j} \sim (10^4 - 10^5) R_{\rm L}$, the toroidal magnetic field inside a jet must exceed the poloidal magnetic field to the same extent [155]. That is why the discovery of such a strong toroidal component would be a very important result to confirm the picture discussed here. Unfortunately, the determination of real physical conditions in jets is presently encountering great difficulties, although some indications of the existence of such magnetic field structure in BL Lac type objects have appeared of late [154].

7. Conclusions

It has been shown above how direct problems of magnetic hydrodynamics referring to a very wide class of astrophysical compact objects can be formulated and solved in a fairly simple way. Except for slow magnetosonic and cusp surfaces, we considered all singular surfaces of the general trans-field equation (41) and investigated the role of all the five integrals of motion. It was shown that in many cases fairly simple model solutions were obtained that suggested judgements of the basic physical characteristics of the indicated flows and thus demonstrated an exceptional efficiency of the method of Grad – Shafranov equation in the investigation of many astrophysical objects.

At the same time, it must be borne in mind that the method of the Grad – Shafranov equation definitely leads to a deadlock in many respects. First of all there is no possibility of extending this equation to the case where dissipative processes (viscosity, heat conductivity, emission of plasma and its interaction with radiation, kinetic effects, etc.) may play a dominant role. This happens because the approach is based on the existence of integrals of motion. Breakdown of this assumption immediately leads to the impossibility of reducing the complete system of equations to a single second-order equation. Similarly, it is practically impossible to extend this method to the case of non-axisymmetric nonstationary flows, and therefore the solutions obtained cannot be investigated for stability.

As has already been mentioned, the possibility of solving direct problems is intimately related to the existence of an exact analytical solution to the equilibrium equation. In this sense, astrophysical problems appear to be simpler than other applied problems because the simplest spherically symmetric flow in many cases describes a real situation well enough. In the general case, when an exact solution is unknown, the direct problem cannot be formulated and solved. For this reason, in numerical calculations one typically has to solve a time-dependent problem which differs in its very statement from the method of the Grad – Shafranov equation.

Nonetheless, it may be hoped that the results obtained above are of quite definite interest. First of all, all the principal features of transonic flows near real compact objects have been described inside the framework of sufficiently simple model problems. This allowed us to obtain rather simple analytical expressions for practically all the key quantities characterising such flows. Furthermore, it may be hoped that many of the results obtained in the framework of model solutions (the absence of self-collimation upon outflow of a magnetised relativistic wind, the connection between current and potential in radio pulsar magnetosphere, and the generation of secondary plasma in the magnetosphere of a black hole) give a qualitatively correct description of real processes, the more so as many other solutions, for example, the formation of a disc upon transonic accretion and ejection, confirmed previous results. Finally, the exact analytical solutions may provide a good verification in various numerical calculations.

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