# Bose - Einstein condensates 

B B Kadomtsev, M B Kadomtsev

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#### Abstract

The production of Bose-Einstein condensates in atomic gases of alkaline elements contained in magnetic traps has opened a new direction of research in the physics of superlow temperatures. This paper explains in plain words the theory of Bose - Einstein condensation in rarefied gases, and then offers a brief overview of the experimental results.


## 1. Introduction

The first three papers [1-3] concerned with the production and study of Bose-Einstein condensates in gases of alkaline elements at superlow temperatures were published in 1995. Condensates of diamagnetic atoms were contained in miniature magnetic traps, sometimes equipped with a laser stopper for controlling the shape of the potential well. In this way, new members with unusual and intriguing properties were added to the small family of Bose condensates which formerly included only superfluid helium II and exciton drops. A new page was opened in the physics of superlow temperatures [4, 10].

The production of condensates of alkaline atoms has been made possible by bringing together the new ideas and the techniques developed in other branches of physics. First of all, a magnetic trap is required for containing the condensate.

[^0]Such traps were first invented for the confinement of hightemperature plasma: from theoretical considerations regarding the stability of diamagnetic plasma in a magnetic field, Ioffe and colleagues designed a magnetic trap with a quadrupole stabilizing field [13]. It was in this trap that the possibility of stable magnetic containment of a diamagnetic plasma was demonstrated for the first time. Later Pritchard [14] proposed using traps of this type for the containment of neutral atoms in a diamagnetic state.

The magnetic traps are filled with gas with the aid of laser cooling. Let us recall the principle of the latter. Assume that we have lasers tuned to a frequency slightly below the absorption frequency of the gas atoms. An atom at rest will be not able to absorb the quanta of laser radiation. However, an atom which moves up the laser beam may absorb a photon owing to the Doppler effect. The atom then acquires the momentum of the quantum whose sign is opposite to the initial momentum of the atom. Some time later the excited atom will emit a quantum of light, on average isotropically in all directions. In this way, illuminating the atoms simultaneously from all sides, one can gradually slow them down in the magnetic trap.

Laser cooling permits the trap to be filled with gas at a very low temperature. The next step is evaporative cooling. The shape of the potential well can be adjusted in such a way with an added high-frequency field that the warmer atoms can easily escape. A combination of these techniques permits us to go down to sub-micro-kelvin temperatures. It is at such temperatures that the Bose-Einstein condensation of gases of alkaline elements takes place.

In this paper we first give a relatively simple treatment of certain theoretical aspects of the physics of Bose-Einstein condensates, and then briefly describe the relevant experimental results.

## 2. Ground state

Let us first of all describe the ground state of a rarefied Bose gas at zero temperature. It will be convenient to rely on Feynman's qualitative analysis [15].

Let the gas be contained in a vessel of large volume $V$. One particle of such a gas has a normalized wave function $\bar{\psi}=1 / \sqrt{V}$. Assume now that an entirely similar but fixed particle occurs at the point $r=0$. The existence of a second particle will slightly modify the wave function $\psi(\mathbf{r})$ of the first. Let us assume that the potential of interaction between the two particles $U(r)$ is the simplest potential of repulsion of two elastic spheres:

$$
U(r)= \begin{cases}\infty & \text { when } \quad r<a,  \tag{1}\\ 0 & \text { when } \quad r \geqslant a .\end{cases}
$$

Parameter $a$ corresponds to the closest distance between the particles, and is thus equal to the diameter of the elastic sphere.

The wave function $\psi(\mathbf{r})$ must be zero at $r=a$, and at $r>a$ it must satisfy the Schrödinger equation for free motion: $\Delta \psi=0$. Hence we find that

$$
\begin{equation*}
\psi(r)=\bar{\psi}\left(1-\frac{a}{r}\right), \quad \text { where } \quad \bar{\psi}=\frac{1}{\sqrt{V}} \tag{2}
\end{equation*}
$$

One may say that function (2) takes into account the $S$-wave scattered by the fixed site.

There is a certain infelicity in our choice of solution of the Schrödinger equation in the form (2). Strictly speaking, the wave function $\psi(\mathbf{r})$ must become zero on the boundary of the volume $V$. Because of this, in the expression for the mean kinetic energy $\varepsilon_{0}$ we may carry out integration by parts, so that

$$
\varepsilon_{0}=-\frac{\hbar^{2}}{2 m} \int \psi \Delta \psi \mathrm{~d} \mathbf{r}=\frac{\hbar^{2}}{2 m} \int \bar{\psi}^{2}\left(\nabla \frac{a}{r}\right)^{2} \mathrm{~d} \mathbf{r}
$$

where the integrals are taken over the volume $V$. Straightforward integration proves that $\varepsilon_{0}=2 \pi \hbar^{2} a / V$, where we have taken into account that $\bar{\psi}^{2}=1 / V$. The quantity $\varepsilon_{0}$ is obviously very small if the volume $V$ is macroscopic. Therefore, approximation (2) is fairly reasonable.

Assume now that our vessel of volume $V$ contains $N \gg 1$ identical particles. Assume also that the gas of these particles is rarefied, so that $N \ll V a^{-3}$. The wave function of any two particles located closely enough to each other is given by Eqn (2), where $r$ is the distance between the particles. The kinetic energy $2 \varepsilon_{0}$ of such particles may be regarded as the energy of their interaction. Multiplying $2 \varepsilon_{0}$ by the number of all possible pairs of particles $N(N-1) / 2 \cong N^{2} / 2$, we get the total energy of the particles $E=N 2 \pi \hbar^{2} n a / m$, where $n=N / V$ is the density of particles. The energy of each particle is $\varepsilon$, which differs from $\varepsilon_{0}$ in that the volume $V$ is replaced by the volume $n^{-1}$ pertaining to one particle.

Let us now look at the situation from a somewhat different standpoint. Obviously, the wave function of two close enough particles should not be too different from that given by Eqn (2). Accordingly, fixing one particle at the origin $\mathbf{r}=0$, we seek a solution of the Schrödinger equation for the other particle in the form

$$
\begin{equation*}
\psi(\mathbf{r})=\bar{\psi}[1-\varphi(\mathbf{r})] . \tag{3}
\end{equation*}
$$

Here $\mathbf{r}$ is the radius vector of the second (test) particle. Assume that all the remaining particles are uniformly distributed in the volume with a mean density $n=(N-2) / V \cong N / V$. For the sake of argument we temporarily fix the test particle at the point $\mathbf{r}=\{x, y, z\}$. By $\mathbf{r}_{j}$ we denote the radius vector of particle number $j$. Interaction of the test particle with particle number $j$ gives rise to a wave function of the form (2), with $\bar{\psi}$ replaced by $\bar{\psi}\left(\mathbf{r}_{j}\right)$ according to Eqn (3), and $r$ replaced by $\left|\mathbf{r}-\mathbf{r}_{j}\right|$. In this way we take into account the re-scattering of the wave of the test particle by particle number $j$. This brings us to

$$
\begin{equation*}
\psi(\mathbf{r})=\bar{\psi}\left[1-\varphi\left(\mathbf{r}_{j}\right)\right]\left[1-\frac{a}{\left|\mathbf{r}-\mathbf{r}_{j}\right|}\right] . \tag{4}
\end{equation*}
$$

Here the factor $\left[1-\varphi\left(\mathbf{r}_{j}\right)\right]$ is defined at point $\mathbf{r}_{j}$.
It is easy to see that the the expression $a / r$ looks like the electric potential of a 'charge' $a$ at a distance $r$ from the charge. If we want to find the effect of all particles on the wave function of a test particle, we must add together all the contributions from the many particles. Since the 'potential' $a / r$ falls off very slowly with the distance $r$, the remaining particles will give long-range contributions which must be diligently taken into account. The function $\varphi(r)$ will only coincide with $a / r$ when the particles are close enough to each other; when the distance is large, the function $\varphi(r)$ will be 'shielded' by the particles located in between. In accordance with Eqn (4) we get

$$
\begin{equation*}
\psi(r) \equiv \bar{\psi}\left[1-\frac{a}{r}-A+\sum_{j} \frac{a}{\left|\mathbf{r}-\mathbf{r}_{j}\right|} \varphi\left(\mathbf{r}_{j}\right)\right] . \tag{5}
\end{equation*}
$$

Here the second term in brackets takes care of the interaction between the test particle and the particle fixed at $r=0$, in accordance with Eqn (2). The sum over all the remaining particles describes the contribution from the latter arising from the term in Eqn (4) which is linear in $\varphi(\mathbf{r})$. Equation (5) also includes the constant $A$,

$$
\begin{equation*}
A=\sum_{j} \varphi\left(\left|\mathbf{r}-\mathbf{r}_{j}\right|\right) \tag{6}
\end{equation*}
$$

which will be calculated in due course.
If $n a^{3} \ll 1$, the summation in Eqn (5) must be carried out over a very large number of particles. Because of this, the sum $\sum_{j}$ may be approximated by the integral $\int n \mathrm{dr}^{\prime}$ with respect to $\mathbf{r}_{j} \equiv \mathbf{r}^{\prime}$. Given that the wave function $\psi(r)$ on the left-hand side of Eqn (5) is defined by Eqn (3), we get the integral equation for $\varphi(r)$ :

$$
\begin{equation*}
\varphi(r)=\frac{a}{r}-\int \frac{a n}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|} \varphi\left(\mathbf{r}^{\prime}\right) \mathrm{d} \mathbf{r}^{\prime}+A . \tag{7}
\end{equation*}
$$

Now we take advantage of the fact that the function $1 / r$ satisfies the equation $\Delta(1 / r)=-4 \pi \delta(\mathbf{r})$. Applying the $\Delta$ operator to Eqn (7), we rewrite it in the form

$$
\begin{equation*}
-\Delta \varphi+x_{0}^{2} \varphi=4 \pi a \delta(\mathbf{r}) \tag{8}
\end{equation*}
$$

where $\chi_{0}^{2}=4 \pi a n$.
The solution of Eqn (8) is well known:

$$
\begin{equation*}
\varphi(r)=\frac{a}{r} \exp \left(-\chi_{0} r\right) \tag{9}
\end{equation*}
$$

We see that the inclusion of contributions from other particles leads to the Debye - Huckel potential. For a rarefied gas the quantity $a^{2} \chi_{0}^{2}=4 \pi a^{3} N / V$ will be much less than one, and so the shielding of the charge occurs at a fairly large distance from each of the atoms. Since, by virtue of Eqn (2), (3), the function $\varphi(r)$ must be considered equal to unity when $r<a$, solution (4) can only be used for $r>a$. Accordingly, the 'charge' in Eqn (8) ought to be seen as localized on the surface of a sphere of radius $a$ rather than at its center. Of course, at $r>a$ the solution does not depend on the profile of 'charge' distribution.

In the course of the above discussion we occasionally had to 'fix' the charges. This was done solely to simplify the mathematical expressions. In reality, any two interacting particles, say, particles $i$ and $j$, are not fixed. The sum of the Laplacians $\Delta_{i}+\Delta_{j}$, however, can be represented as $(1 / 2) \Delta_{R}+2 \Delta^{\prime}$, where $\Delta_{R}$ is the Laplacian with respect to the coordinate of the center of mass $\mathbf{R}=\left(\mathbf{r}_{i}+\mathbf{r}_{j}\right) / 2$, and $\Delta^{\prime}$ is the Laplacian of the relative motion with the coordinate $\mathbf{r}_{i}-\mathbf{r}_{j}$. The factor two at $\Delta^{\prime}$ reflects the appearance of the reduced mass $m_{*}=m / 2$. In the case of the free motion of particles whose center of mass is at rest we have $\Delta_{R} \psi=0, \Delta^{\prime} \psi=0$, and so expressions (2), (3) and (9) still hold but now relate to the wave function of two particles at $r=\left|\mathbf{r}_{i}-\mathbf{r}_{j}\right|$ (duly accounting for the shielding caused by many particles).

In Eqn (6) for the constant $A$ we may again replace the sum by an integral. Then, in the gas approximation $\left(n a^{3} \ll 1\right)$, we get

$$
A=\frac{4 \pi a n}{x_{0}^{2}}=1
$$

Now let us find the energy of the test particle resulting from the interaction with the other $N-1 \cong N$ particles. This energy is

$$
\begin{equation*}
\varepsilon=N \int_{a}^{R_{0}}\left(\psi \frac{\hbar^{2}}{2 m} \Delta \psi\right) 4 \pi r^{2} \mathrm{~d} r . \tag{10}
\end{equation*}
$$

Here the radius $R_{0} \gg a$ is formally selected in such a way as to satisfy the normalization condition $4 \pi R_{0}^{2} \bar{\psi}^{2} / 3=1$. Since the wave function $\psi(r)=\bar{\psi}[1-\varphi(r)]$ goes to zero at $r=a$, and the function $\varphi(r)$ falls off fairly quickly as $r$ increases, one may carry out an integration by parts in Eqn (10), so that $\psi \Delta \psi$ will be replaced by $-\bar{\psi}^{2}(\nabla \varphi)^{2}$. In the gas approximation we get

$$
\begin{equation*}
\varepsilon=\hbar^{2} \varkappa_{0}^{2} / 2 m=\frac{2 \pi \hbar^{2} a n}{m} . \tag{11}
\end{equation*}
$$

This energy has been calculated with the second particle fixed. It would be more correct to calculate the energy of the two particles for a predefined function $\varphi\left(\left|\mathbf{r}_{i}-\mathbf{r}_{j}\right|\right)$ of their relative motion. Going over to the relative variable $\mathbf{r}_{i}-\mathbf{r}_{j}$, we find the energy with the reduced mass $m / 2$ and the interaction which is a function of $\mathbf{r}_{i}-\mathbf{r}_{j}$. As a result, the total energy (of the pair of particles at rest) will be doubled, but each of the particles will have the same energy $\varepsilon$ as before.

The total energy of the particles in the gas approximation is $E=N \varepsilon$, that is,

$$
\begin{equation*}
E=N \varepsilon=\frac{2 \pi \hbar^{2} a n^{2}}{m} V \tag{12}
\end{equation*}
$$

Sometimes it will be convenient to write it in the form

$$
\begin{equation*}
E=\frac{2 \pi \hbar^{2} a N^{2}}{m V} . \tag{13}
\end{equation*}
$$

We have written Eqn (3) for the case of interaction of the test particle with just one other particle (subsequently taking the shielding into account). In the general case the wave function of an individual particle $i$ must be written as

$$
\begin{equation*}
\psi\left(\mathbf{r}_{i}\right)=\bar{\psi}\left\{1+A-\sum_{j \neq i} \varphi\left(\left|\mathbf{r}_{i}-\mathbf{r}_{j}\right|\right)\right\} \tag{14}
\end{equation*}
$$

where $\bar{\psi}$ may be a function of $\mathbf{r}_{i}$, but in the homogeneous condensate $\bar{\psi}=$ const $=1 / \sqrt{V}$. We have included the constant $A$ to compensate for the mean value of the sum. In the gas approximation $A=1$.

We see that function (14) depends on the coordinates of all the other particles, and thus cannot be regarded as a truly independent one-particle function. Because the particles are indistinguishable, the overall wave function of $N$ particles could be represented as the product of individual wave functions of the form (14):

$$
\begin{equation*}
\Psi\left(\mathbf{r}_{1} \ldots \mathbf{r}_{N}\right)=\bar{\psi}^{N} \prod_{i}\left\{1+A-\sum_{j} \varphi\left(\mathbf{r}_{i}-\mathbf{r}_{j}\right)\right\} \tag{15}
\end{equation*}
$$

If, however, subscript $i$ runs through all possible values, each of the pairwise interactions will be included twice. We could have set $j>i$; then, however, it would be hard to see that function (15) is symmetrical. Therefore, we use a somewhat different approach. Given that in the gas approximation $A=1$, and $1-\sum_{j} \varphi\left(\mathbf{r}_{i}-\mathbf{r}_{j}\right)$ may be considered small, we represent the product (15) in the form

$$
\begin{equation*}
\Psi\left(\mathbf{r}_{1} \ldots \mathbf{r}_{N}\right)=\bar{\psi}^{N} \exp \left\{\frac{N}{2}-\frac{1}{2} \sum_{i \neq j} \varphi\left(\left|\mathbf{r}_{i}-\mathbf{r}_{j}\right|\right)\right\} \tag{16}
\end{equation*}
$$

where the first term in braces compensates for the mean value of the sum over $i \neq j$. Expression (16) is not quite exact when $\left|\mathbf{r}_{i}-\mathbf{r}_{j}\right| \sim a$, but it is useful for getting the hang of the structure of the many-particle wave function.

Expression (16) is completely symmetrical with respect to permutation of particles. Each of the functions $\bar{\psi}$ in the general case of a condensate in an external field must be considered as a function of only one of the coordinates - say, $\bar{\psi}\left(\mathbf{r}_{j}\right)$. In the case of a homogeneous condensate the exponent in Eqn (16) gives only a small correction to the normalization, and so, as before, we have

$$
\begin{equation*}
\bar{\psi}=\frac{1}{\sqrt{V}}=\frac{\sqrt{n}}{\sqrt{N}} \tag{17}
\end{equation*}
$$

for each of the identical particles.
It is easy to prove that function (16) satisfies the Schrödinger equation of free motion in the range $\left|\mathbf{r}_{i}-\mathbf{r}_{j}\right|>a$. Indeed, any operator of type $\Delta_{s}$ acting upon function (16) will select the terms $\varphi\left(\left|\mathbf{r}_{s}-\mathbf{r}_{j}\right|\right)+\varphi\left(\left|\mathbf{r}_{j}-\mathbf{r}_{s}\right|\right)$ of the sum, which will result in sums of the form

$$
\sum_{j} \Delta_{s} \varphi\left(\left|\mathbf{r}_{s}-\mathbf{r}_{j}\right|\right)-\sum_{j k} \nabla_{s} \varphi\left(\left|\mathbf{r}_{s}-\mathbf{r}_{j}\right|\right) \nabla_{s} \varphi\left(\left|\mathbf{r}_{s}-\mathbf{r}_{k}\right|\right) .
$$

In the first sum we may replace $\Delta_{s} \varphi$ by $\chi_{0}^{2} \varphi$ by virtue of Eqn (8) in the range $r>a$. In the second sum it is sufficient to
retain only the terms with $j=k$ (all the rest cancel out because of symmetry). If once again we replace the summation by integration, it immediately becomes clear that the total energy is zero. This is quite natural, since our solution has been constructed by rearranging the starting-point solutions of type (1) with zero energy. As regards the energy of interaction (12), it can be found by averaging the total Hamiltonian of the free motion of the particles, using function (16) and integrating by parts.

Since the individual functions (14) look exactly alike for all particles, it is possible to describe the condensate in terms of functions (14), dropping the subscript ' $i$ ' at $\mathbf{r}_{i}$. If we are dealing with an equilibrium or with the collective behavior of condensate, then function (14) is assumed to depend on coordinates and time.

## 3. Gas equilibrium

Having derived Eqn (13) for the total energy of the gas, we can find the gas pressure $p$ using the well-known thermodynamic relation

$$
\begin{equation*}
p=-\frac{\partial E}{\partial V}=\frac{2 \pi \hbar^{2} a N^{2}}{m V^{2}}=\frac{2 \pi \hbar^{2} a}{m} n^{2}=\varepsilon n . \tag{18}
\end{equation*}
$$

The adiabatic equation having the form $p V^{\gamma}=$ const, from Eqn (18) we see that the adiabatic exponent is $\gamma=2$.

We also find the squared speed of sound $c_{\mathrm{s}}^{2}$ :

$$
\begin{equation*}
c_{\mathrm{s}}^{2}=\frac{1}{m} \frac{\partial p}{\partial n}=\frac{4 \pi \hbar^{2} a}{m^{2}} n \tag{19}
\end{equation*}
$$

Hence it follows that the energy of the particle $\varepsilon$ can be represented in the form

$$
\begin{equation*}
\varepsilon=\frac{m c_{\mathrm{s}}^{2}}{2} . \tag{20}
\end{equation*}
$$

If we add one particle to the condensate at constant pressure, its energy will increase by the increment $\varepsilon+p / n$, since the volume per particle is $V / N=1 / n$. Hence, the chemical potential is $\mu=2 \varepsilon$. It is the chemical potential that must be considered equal to the energy per particle.

Assume that the gas in question is in an external potential $U(\mathbf{r})$. Then from the equilibrium condition

$$
\begin{equation*}
\nabla p+n \nabla U=0 \tag{21}
\end{equation*}
$$

we find that

$$
\begin{equation*}
n=\frac{m}{4 \pi \hbar^{2} a}\left(U_{0}-U\right) . \tag{22}
\end{equation*}
$$

The constant $U_{0}$ is found from the normalization condition $\int n \mathrm{~d} V=N$.

Let us first consider the simplest example of a gas in the field of gravity. Then $U=m g z$, where $z$ is the vertical coordinate. Then the density $n(z)$ falls off linearly with altitude:

$$
\begin{equation*}
n(z)=n_{0}-\frac{m^{2} g}{4 \pi \hbar^{2} a} z . \tag{23}
\end{equation*}
$$

where $n_{0}=n(z=0)$. At $z=H=4 \pi \hbar^{2} a n_{0} / m^{2} g$ the density of the gas vanishes. This means that the Bose gas can be poured into a vessel (Fig. 1), and its density will increase linearly with depth. We get something in between a liquid and a gas.


Figure 1. Rarefied Bose gas at zero temperature may be 'poured' into a vessel. Its density $n(z)$ varies linearly with $z$ from $n=n_{0}$ at the bottom to $n=0$ at the height $H$. The height $H$ is defined by the relation $H^{2}=8 \pi \hbar^{2} a N / m^{2} g S$, where $N$ is the total number of particles, and $S$ is the cross-sectional area of the vessel.

Using Eqn (23) it is easy to find the relation

$$
\begin{equation*}
H^{2}=\frac{8 \pi \hbar^{2} a N}{m^{2} g S} . \tag{24}
\end{equation*}
$$

which gives the thickness $H$ of the layer of gas poured into the vessel given the number of particles $N$ and the vessel area $S$. We see that the thickness of gas layer increases with $H / S$ as $H \sim(N / S)^{1 / 2}$.

A more realistic entity is the spherically symmetrical oscillator potential

$$
\begin{equation*}
U=\frac{\alpha r^{2}}{2} \tag{25}
\end{equation*}
$$

where $\alpha$ is the 'coefficient of elasticity'. Let $n_{0}$ be the density of gas at $r=0$, and $\varepsilon_{0}=2 \pi \hbar^{2} a n_{0} / m$ the particle energy at $r=0$. Then distribution (22) can be represented in the form

$$
\begin{equation*}
n=n_{0}-\frac{n_{0}}{2 \varepsilon_{0}} U=n_{0}\left(1-\frac{\alpha}{4 \varepsilon_{0}} r^{2}\right) . \tag{26}
\end{equation*}
$$

We see that the gas density falls off with $r$ according to a parabolic law, and goes to zero at $r=r_{0}=2\left(\varepsilon_{0} / \alpha\right)^{1 / 2}$. Using Eqn (26) and the normalization condition, we find

$$
\begin{align*}
& n_{0}=\frac{15^{2 / 5}}{8 \pi}\left(\frac{\alpha m}{\hbar^{2} a}\right)^{3 / 5} N^{2 / 5}  \tag{27}\\
& r_{0}^{2}=\frac{8 \pi \hbar^{2} a}{\alpha m} n_{0}=15^{2 / 5}\left(\frac{\hbar^{2} a}{\alpha m}\right)^{2 / 5} N^{2 / 5} . \tag{28}
\end{align*}
$$

We see that the density of gas in the center of the 'drop' increases with the number of particles as $N^{2 / 5}$, and the volume of the 'drop' of Bose gas increases as $N^{3 / 5}$.

Let us derive a few more relations. Using Eqn (11) and Eqn (22) we find

$$
\begin{equation*}
\varepsilon=\frac{1}{2}\left(U_{0}-U\right) . \tag{29}
\end{equation*}
$$

Hence the following quantity is preserved

$$
\begin{equation*}
\mu+U=2 \varepsilon+U=U_{0}=\mathrm{const} \tag{30}
\end{equation*}
$$

where $\mu=2 \varepsilon$ is the chemical potential.
Another interesting relation can be found from Eqn (20) and Eqn (29):

$$
\begin{equation*}
m c_{\mathrm{s}}^{2}+U=U_{0}=\text { const } . \tag{31}
\end{equation*}
$$

which implies that the speed of sound becomes zero at the boundary of the 'gas drop', at $U=U_{0}$. As we go deeper inside the 'drop', the speed of sound increases and attains its maximum at $U=0$ - that is, in the middle of the drop, $c_{\mathrm{s}}(0)=\sqrt{U_{0} / m}$.

## 4. Dense gas

In order to fathom the applicability of the simple relations above in the gas approximation, it will be worthwhile to look into what happens as the density increases. Without aspiring to achieve quantitative precision, we shall try instead to find approximations which will outline the nature of changes to be expected in our results. For the starting point we may take the idea of the 'shielding' of wave functions according to relations (3), (8), and (14). Also, it will be convenient to consider separately the various effects of increasing density, and then sum them up together.

The first effect we are going to consider consists in the following. Relations (4) and (5) were written under the assumption that the function $\varphi\left(\mathbf{r}_{j}\right)$ changes little over the length $a$, which allowed us to represent the response to the function $\varphi\left(\mathbf{r}_{j}\right)$ in the form $a \varphi\left(\mathbf{r}_{j}\right) /\left|\mathbf{r}-\mathbf{r}_{j}\right|$. If the parameter $\chi a$ is not too small, we must take into account the variation of $\varphi\left(\mathbf{r}_{j}\right)$ on the scale $\sim a$. If $\nabla \varphi\left(\mathbf{r}_{j}\right) \neq 0$, then the requirement that the wave function of the test particle must vanish on the boundary of sphere of radius $a$ at point $\mathbf{r}_{j}$ brings us to the expression

$$
\begin{align*}
\psi(r)= & \bar{\psi}\left[1-\frac{b}{r}+\sum_{j} \frac{a}{\left|\mathbf{r}-\mathbf{r}_{j}\right|} \varphi\left(\mathbf{r}_{j}\right)\right. \\
& \left.+\sum_{j} \frac{a^{2}\left(\mathbf{r}-\mathbf{r}_{j}\right)}{\left|\mathbf{r}-\mathbf{r}_{j}\right|^{3}} \nabla \varphi\left(\mathbf{r}_{j}\right)\right] \tag{32}
\end{align*}
$$

where $b$ is an unknown constant. In Eqn (32) we have taken into account the dipole contribution in addition to the centrally symmetrical contribution introduced earlier - in other words, we have added the $P$-wave to the scattered $S$ wave. If once again we replace the summation by integration, in lieu of Eqn (6) we get the integral equation

$$
\begin{equation*}
\varphi(r)=\frac{b}{r}-\int \frac{a n}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|} \varphi\left(\mathbf{r}^{\prime}\right) \mathrm{d} \mathbf{r}^{\prime}-\int \frac{n a^{3}\left(\mathbf{r}-\mathbf{r}^{\prime}\right)}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|^{3}} \nabla \varphi\left(\mathbf{r}^{\prime}\right) \mathrm{d} \mathbf{r}^{\prime}, \tag{33}
\end{equation*}
$$

where the parameter $b$ is found from the condition $\varphi(a)=1$.
To this equation we may again apply the $\Delta$ operation, accounting for the relation $\Delta\left(\mathbf{d r} / r^{3}\right)=4 \pi(\mathbf{d} \nabla) \delta(\mathbf{r})$, where $\mathbf{d}$ is an arbitrary constant vector (dipole moment). After integration with the $\delta\left(\mathbf{r}-\mathbf{r}^{\prime}\right)$ function we get:

$$
\begin{equation*}
-\left(1+\chi_{0}^{2} a^{2}\right) \Delta \varphi+\chi_{0}^{2} \varphi=4 \pi b \delta(\mathbf{r}) \tag{34}
\end{equation*}
$$

With a new notation

$$
\begin{equation*}
x^{2}=\frac{x_{0}^{2}}{1+x_{0}^{2} a^{2}} \tag{35}
\end{equation*}
$$

the solution of Eqn (34) may be written as

$$
\begin{equation*}
\varphi(r)=\frac{a}{r} \exp [-\chi(r-a)] . \tag{36}
\end{equation*}
$$

This formula appears as a refinement of Eqn (9). According to Eqn (35), the parameter $x$ increases linearly with density at
low densities, and at high densities it comes to saturation at $x=1 / a$.

Using function (36) it is easy to find the energy $\varepsilon$ of one particle:

$$
\begin{equation*}
\varepsilon=\frac{\hbar^{2} \chi_{0}^{2}}{2 m}\left(1+\frac{a \chi}{2}\right), \tag{37}
\end{equation*}
$$

where $\alpha$ is given by Eqn (35). For low densities the correction in parentheses is $a \not / 2 \cong a \chi_{0} / 2=\sqrt{\pi a^{3} n}$, for high densities the correction is $1 / 2$.

The above relations allow us to discuss some properties of superfluid helium on a qualitative level. The parameter $a^{2} \chi_{0}^{2}$ may be assumed to be large. and so the expression in parentheses in Eqn (37) is simply equal to 1.5. This expression describes that part of the kinetic energy of helium atoms which relates to the far end of the distribution of wave functions - that is, when the test atoms are located far from one another. In addition, however, we must take into account the potential energy of interatomic interaction and that part of the kinetic energy (the energy of zero oscillations) which is associated with the localization of atoms near the minima of the potential energy. This second part of the energy we shall call by convention the potential energy $U_{s}$; it may be considered as a function of the density of atoms. Near the minimum it may be represented as a parabolic function of the difference $n-n_{0}$, where $n_{0}$ corresponds to the density with the minimum potential energy. In this way, the total energy per atom may be written as

$$
\begin{equation*}
\varepsilon=\frac{3 \pi \hbar^{2} a n}{m}-U_{m}+\frac{3 \pi \hbar^{2} a n}{2 m n_{0}} B\left(n-n_{0}\right)^{2} . \tag{38}
\end{equation*}
$$

Here we have written the term of second order in $\left(n-n_{0}\right)$ in such a way as to make its dimensionality similar to that of the first term (constant $B$ is dimensionless).

From condition $\partial \varepsilon / \partial n=0$ it is easy to find expressions for the equilibrium density and the equilibrium potential energy $U_{0}$ :

$$
\begin{equation*}
\frac{n}{n_{0}}=1-\frac{1}{B}, \quad U_{0}=U_{m}-\frac{3 \pi \hbar^{2} a n_{0}}{3 m B} . \tag{39}
\end{equation*}
$$

The condition $\partial \varepsilon / \partial n=0$ is equivalent to the requirement that the external pressure be zero. Using Eqns (18), (19) and (38), we find a new expression for the squared speed of sound:

$$
\begin{equation*}
c_{\mathrm{s}}^{2}=\frac{n^{2}}{m} \frac{\partial^{2} \varepsilon}{\partial n^{2}}=\frac{3 \pi \hbar^{2} a n^{2} B}{m^{2} n_{0}}=\frac{3 \pi \hbar^{2} a n}{m^{2}}(B-1) . \tag{40}
\end{equation*}
$$

This expression contains one unknown parameter, $B$. Comparing Eqn (40) with the experimental value of $c_{\mathrm{s}}=2.4 \times 10^{4} \mathrm{~cm} \mathrm{~s}^{-1}$, and knowing the values $a=$ $2.7 \times 10^{-8} \mathrm{~cm}, n=2.2 \times 10^{22} \mathrm{~cm}^{-3}$ for superfluid helium, we find that $B \cong 5$, which seems reasonable. Now, using Eqns (22) and (40), we find:

$$
\begin{equation*}
U_{m}=\frac{4 \pi \hbar^{2} n a}{m}-\frac{3 \pi \hbar^{2} a n_{0}}{2 m B} \tag{41}
\end{equation*}
$$

Knowing the values of $n, a, B$, it is easy to check that the second term in Eqn (40) does not exceed $20 \%$ of the first term. The absolute value of $U_{m}$ seems quite reasonable, if we recall that each atom has six closest neighbors, and that the potential of pairwise attraction of the atoms is of the order of a few kelvins.

Note that the experimental value of the squared speed of sound in helium II is about three times the value given by Eqn (19), and so it is wrong to calculate the speed of sound in helium II without taking into account the short-range forces of attraction.

## 5. Collective motions

Let us now return to the case of rarefied gas. According to Eqn (16), the wave function of gas atoms may be represented in the form

$$
\begin{equation*}
\Psi=\Psi_{0} \Phi \equiv \bar{\psi}^{N} \exp \left[1+N-\sum_{i \neq j} \varphi\left(\left|\mathbf{r}_{i}-\mathbf{r}_{j}\right|\right)\right] . \tag{42}
\end{equation*}
$$

For a homogeneous gas at rest we have $\Psi_{0}=\bar{\psi}^{N}=V^{-N / 2}$. If, however, we set the gas in motion at a velocity $\mathbf{v}$, each of the wave functions $\bar{\psi}$ acquires a phase factor $\exp \left(\mathbf{i k r}_{i}\right)$, where $\mathbf{k}=m \mathbf{v} / \hbar$. The function $\Psi_{0}$ therefore becomes

$$
\begin{equation*}
\Psi_{0}=V^{-N / 2} \exp (\mathbf{i} \mathbf{K R}), \tag{43}
\end{equation*}
$$

where $\mathbf{R}=N^{-1} \sum \mathbf{r}_{j}$ is the coordinate of the center of mass, $\mathbf{K}=M \mathbf{v} / \hbar, M=m N$ is the mass of the entire fluid.

In case of the more complex flows the wave function $\Psi_{0}$ of the condensate may be presented in the form

$$
\begin{equation*}
\Psi_{0}=V^{N / 2} \exp \left[\frac{i}{\hbar} \sum_{i} \theta\left(\mathbf{r}_{i}\right)\right] \tag{44}
\end{equation*}
$$

where the function $\theta$ defines the velocity $\mathbf{v}$ as

$$
\begin{equation*}
\mathbf{v}=\frac{1}{m} \nabla \theta . \tag{45}
\end{equation*}
$$

In case of condensate at zero temperature the function $\Psi_{0}$ is equal to the product of $\bar{\psi}^{N}$, so the function $\Psi_{0}$ may be discussed in terms of the individual wave functions $\bar{\psi}$. In the general case $\bar{\psi}$ must be considered as a function of the time $t$ and the coordinate $\mathbf{r}$ of any of the identical particles. If $\bar{\psi}$ depends on $\mathbf{r}$, then it is $|\bar{\psi}(\mathbf{r})|^{2}$ that has to be normalized to unity. The quantity $N|\bar{\psi}(\mathbf{r})|^{2}$ must be assumed equal to the density of particles $n(\mathbf{r})$, so that $\int n \mathrm{~d} V=N$.

Let us now construct the Schrödinger equation for the wave function (42):

$$
\begin{equation*}
\mathrm{i} \hbar \frac{\partial \Psi}{\partial t}=H \Psi . \tag{46}
\end{equation*}
$$

Here the Hamiltonian $H$ is equal to the sum of the kinetic energies of $N$ particles with an added boundary condition: the wave function must vanish on the boundary corresponding to the closest approach of any two atoms. If the functions $\bar{\psi}$ vary little over a length of the order of $\varkappa^{-1}$, then for the function $\Phi$ we may use the stationary solution (42): $H \Phi=N \varepsilon \Phi$. In this case the flows of condensate are described by the function $\Psi_{0}$, which is equal to the product of the individual functions $\bar{\psi}$. In this way, the flows of condensate can be described in terms of the individual wave functions.

Let us now write the Schrödinger equation for the wave function of an individual particle $\bar{\psi}$. It might seem that we ought to include the collective contribution to its Hamiltonian only in the form of energy $\varepsilon$. However, this is not exactly so. As a matter of fact, the energy $N \varepsilon$ is found by adding together the 'pairwise interactions'. One may say that $N \varepsilon=N(N-1) 2 \varepsilon_{0} / 2$, where $2 \varepsilon_{0}$ is the energy of a paired
interaction - that is, the kinetic energy of two particles given that the wave function vanishes on the boundary between the most closely located particles. The energy of interaction of a particle with all the rest particles is $2 \varepsilon_{0}(N-1)=2 \varepsilon=\mu$ - that is, this energy is equal to twice the energy $\varepsilon$ or to the chemical potential. Accordingly, the equation for the wave function $\bar{\psi}$ of an individual particle must be written in the form

$$
\begin{equation*}
\mathrm{i} \hbar \frac{\partial \bar{\psi}}{\partial t}=-\frac{\hbar^{2}}{2 m} \Delta \bar{\psi}+\frac{4 \pi \hbar^{2} a}{m} N|\bar{\psi}|^{2} \bar{\psi}+U \bar{\psi} . \tag{47}
\end{equation*}
$$

This nonlinear Schrödinger equation is known as the Gross-Pitayevsky equation [16]. The quantity $U(\mathbf{r})$ in Eqn (47) is the potential energy of the particle, and $N|\bar{\psi}|^{2}$ is equal to the local density $n(\mathbf{r})$. The quantity $\sqrt{N} \bar{\psi}$ in Eqn (47) we represent as

$$
\begin{equation*}
\sqrt{N} \bar{\psi}=\sqrt{n} \exp \left[\frac{\mathrm{i}}{\hbar} \theta(\mathbf{r}, t)\right] . \tag{48}
\end{equation*}
$$

Now we substitute this into Eqn (47) and separate the real and imaginary parts, getting two equations:

$$
\begin{align*}
& \frac{\partial n}{\partial t}+\operatorname{div}(n \mathbf{v})=0  \tag{49}\\
& \frac{\partial \theta}{\partial t}+\frac{1}{2 m}(\nabla \theta)^{2}=-\frac{4 \pi \hbar^{2} a}{m} n-U+\frac{\hbar^{2}}{4 m} \frac{\Delta n}{n} \tag{50}
\end{align*}
$$

where $\mathbf{v}=\nabla \theta / m$.
Let us demonstrate that in the quasi-classical approximation we get the equations of ideal gas dynamics. For this purpose we drop the last term on the right-hand side of Eqn (50) and take the gradient of this equation. Then we get

$$
\begin{align*}
& \frac{\partial n}{\partial t}+\operatorname{div}(n \mathbf{v})=0  \tag{51}\\
& m\left[\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v} \nabla) \mathbf{v}\right]+\frac{4 \pi \hbar^{2} a}{m} \nabla n+\nabla U=0 \tag{52}
\end{align*}
$$

Hence it is clear that the condition of equilibrium of condensate is

$$
\begin{equation*}
\frac{4 \pi \hbar^{2} a n}{m}+U=2 \varepsilon+U=\text { const } \tag{53}
\end{equation*}
$$

which coincides with Eqn (30). This implies that the equilibrium of condensate discussed in Section 3 disregards the 'quantum pressure', described by the last term in Eqn (50). Note once again that equations (50) and (52) only apply to nonrotational flows: $m \mathbf{v}=\nabla \theta$. They may be written in the form of equations of gas dynamics with a pressure $p=2 \pi \hbar^{2} a n^{2} / \mathrm{m}$. Because of this, these equations have solutions of gas dynamic form.

For example, at $U \equiv 0$ one may seek solutions which only depend on $x$ and $t$. In particular, if we assume that the density $n$ is a function of the velocity $v$, we arrive at a solution of the Riemann wave or progressive wave. This wave is described by the equations

$$
\begin{align*}
& \frac{\partial v}{\partial t}+\left(v \pm c_{\mathrm{s}}\right) \frac{\partial v}{\partial x}=0,  \tag{54}\\
& \sqrt{n}=\sqrt{n_{0}} \pm v \sqrt{\frac{m^{2}}{16 \pi \hbar^{2} a}} . \tag{55}
\end{align*}
$$

In fact, here we have two sets of equations, one describing the wave propagating along the $x$-axis, and one describing the wave running in the opposite direction. If we align the $x$-axis with the direction of propagation of the wave, it is sufficient to retain only the plus sign in Eqns (54) and (55).

Equation (54) in the linear approximation describes a sonic wave; in the nonlinear case it leads to a Riemann wave. The leading edge in such a wave becomes steeper and almost 'topples over', so that the derivatives $\partial v / \partial x, \partial^{2} v / \partial x^{2}$ should exhibit a singularity. This implies that in the general case we have to retain the last term in Eqn (50), which takes care of the dispersion of the wave.

If the wave amplitude is not large (that is, the Mach number is $v / c_{\mathrm{s}} \ll 1$ ), then Eqn (49) may be used for finding the linkage between a small perturbation of the density $n^{\prime}$ and the velocity $v, n^{\prime}=n_{0} v k / \omega$. Substituting this expression into the linearized equation (50), for $k \ll \chi_{0}$ we find that $\omega=c_{\mathrm{s}} k+c_{\mathrm{s}} k^{3} / 8 \chi_{0}^{2}$ for a wave propagating to the right along the $x$-axis. The second term here arises because of the last term on the right-hand side of Eqn (50). Taking the nonlinear term into account, the equation for a weakly nonlinear wave with dispersion assumes the form

$$
\begin{equation*}
\frac{\partial v}{\partial t}+\left(v+c_{\mathrm{s}}\right) \frac{\partial v}{\partial x}=\frac{c_{\mathrm{s}}}{8 x_{0}^{2}} \frac{\partial^{3} v}{\partial x^{3}} . \tag{56}
\end{equation*}
$$

An equation of this kind is well known in hydrodynamics and plasma physics as the Korteweg - de Vries equation.

It describes, for example, waves in shallow water with positive dispersion [20]. Equation (56) is known to admit solutions in the form of solitary waves or solitons:

$$
\begin{equation*}
v=-v_{0} \cosh ^{-2}\left[\frac{x-\left(c_{\mathrm{s}}-c\right) t}{\Delta}\right] \tag{57}
\end{equation*}
$$

where $c=v_{0} / 3$, and $v_{0}$ is the amplitude of the soliton. The width of soliton $\Delta$ is linked with its amplitude by the following relation:

$$
\Delta^{2} v_{0}=\frac{3 c_{\mathrm{s}}}{2 \chi_{0}^{2}}=\text { const } .
$$

The velocity $v$ of the soliton (57) is negative, so the perturbation of density $n^{\prime}$ is also negative - in other words, the soliton looks like a small density trough running at a speed close to $c_{\mathrm{s}}$. If the amplitude of soliton $v_{0}$ is much less than the speed of sound, then $\Delta^{2} \chi_{0}^{2} \gg 1$, and the soliton is fairly broad.

Along with the one-dimensional solitons one may consider nonlinear solutions with a weak dependence on the transverse coordinate $y$. Such solutions are described by the Kadomtsev-Petviashvili equation. The Kadomtsev-Petviashvili equation for the dispersion described by Eqn (56) in the system of coordinates moving along the $x$-axis with the velocity $c_{\mathrm{s}}$ has the form

$$
\begin{equation*}
\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}+v \frac{\partial v}{\partial x}-\frac{c_{\mathrm{s}}}{8 x_{0}^{2}} \frac{\partial^{3} v}{\partial x^{3}}\right)=\frac{c_{\mathrm{s}}}{2} \frac{\partial^{2} v}{\partial y^{2}} . \tag{58}
\end{equation*}
$$

The solution of this equation has the form of a twodimensional soliton localized with respect to both $x$ and $y$ coordinates. Replacing the right-hand side of Eqn (58) by

$$
\frac{c_{\mathrm{s}}}{2}\left(\frac{\partial^{2} v}{\partial y^{2}}+\frac{\partial^{2} v}{\partial z^{2}}\right)
$$

one may also obtain solutions for three-dimensional solitons (see Appendix).

If $U \neq 0$, equations (49) and (50) can be used for describing the equilibrium and small oscillations of the condensate with respect to this equilibrium. It is especially easy to describe the dipole oscillations, when the condensate as a whole is displaced to a small distance $\xi$. Then $v=\dot{\xi}$ (the dot denotes the time derivative), and the potential is $\theta=\dot{\xi} x$. Assuming that $\hbar^{2}=0$, the density $n$ in Eqn (50) is perturbed by the increment $n^{\prime}=-\xi\left(\partial n_{0} / \partial x\right)$; the relevant perturbation of the first term is simply $-(\partial U / \partial x) \xi$ by virtue of the equilibrium condition (53). Hence, for the harmonic potential $U=(\alpha / 2) r^{2}$ at $\theta=\dot{\xi} x$ we get:

$$
m \ddot{\xi}=-\alpha \xi .
$$

We see that the 'drop of condensate' performs harmonic oscillations as a free particle in the potential $U$. It is not hard to obtain equations for other types of oscillations - for example, for spherical oscillations with $\theta=\dot{\xi} r^{2}$, or for quadrupole oscillations with $\theta=\dot{\xi}\left(x^{2}-y^{2}\right)$. One only has to construct the appropriate Lagrangian and write out the required equation of motion.

Let us also consider a solution in the form of a vortex filament. Assume that this filament is located along the $z$ axis of the cylindrical system of coordinates $r, \phi, z$. The function $\theta$ for this solution is $\theta=\hbar \phi$, and so function (48) remains the same when the angle $\phi$ is changed by $2 \pi$. For homogeneous condensate with $U=0$, given that

$$
\frac{\partial \theta}{\partial t}=-\frac{4 \pi \hbar^{2} a}{m} n_{0}
$$

the stationary equation ( 50 ) becomes

$$
\begin{equation*}
n\left[\frac{1}{r^{2}}+8 \pi \hbar^{2} a\left(n-n_{0}\right)\right]=\frac{\hbar^{2}}{2} \frac{1}{r} \frac{\mathrm{~d}}{\mathrm{~d} r}\left(r \frac{\mathrm{~d} n}{\mathrm{~d} r}\right) . \tag{59}
\end{equation*}
$$

Here $n_{0}$ is the density at $r \rightarrow \infty$. If we disregard the right-hand side of Eqn (59), we simply get $n-n_{0}=-1 / 8 \pi \hbar^{2} r^{2} a$. Hence it follows that the density of condensate vanishes at $r \leqslant\left(8 \pi \hbar^{2} a n_{0}\right)^{-1 / 2}$. The radius of the resulting cavity is the smaller, the larger the density $n_{0}$. Therefore, if $n_{0}$ depends on $z$, as in the case of condensate poured into a vessel (Fig. 1), the shape of the cavity (neglecting the term on the right-hand side of Eqn (59)) resembles the funnel in water going down a plughole (see Fig. 2).

The term on the right-hand side of Eqn (59) only slightly changes the shape of $n(r)$ (see Ref. [21]). More detailed calculations of the flows in rotating condensate in a magnetic trap were carried out in Ref. [22].


Figure 2. A vortex filament in the gas layer poured into a vessel in the field of gravity resembles a funnel in water going down a plug-hole.

## 6. Elementary excitations

Now let us consider small excitations in Bose condensate. We shall see that they look like a kind of quasi-particle, the phonon. At first sight it seems that small excitations can be described by the linearized equations (49) and (50), relating to collective flows. To better understand the linkage between such collective oscillations and quasi-particles, we shall indeed start with linearizing equations (49) and (50). The quantity $\theta$ must be considered small, and the density $n$ can be represented as $n=n_{0}+\tilde{n}$, where $n_{0}=$ const is the equilibrium density of homogeneous condensate, and $\tilde{n} \ll n_{0}$.

Consider the simplest solution in the form of a plane wave: $\tilde{n}=n^{\prime} \cos (\omega t-k x) ; \theta=\theta^{\prime} \sin (\omega t-k x)$. Then from Eqns (49) and (50) we get

$$
\begin{equation*}
m \omega n^{\prime}=n_{0} k^{2} \theta^{\prime} ; \quad m^{-1} n_{0} \omega \theta^{\prime}=c_{\mathrm{s}}^{2} n^{\prime}+\frac{\hbar^{2} k^{2} n^{\prime}}{4 m^{2}} \tag{60}
\end{equation*}
$$

where $c_{\mathrm{s}}^{2}=4 \pi \hbar^{2} n_{0} a / \mathrm{m}^{2}$.
From Eqn (60) we find the dispersion relation for the frequency:

$$
\begin{equation*}
\omega^{2}=k^{2} c_{\mathrm{s}}^{2}+\frac{\hbar^{2} k^{4}}{4 m^{2}}=\hbar^{2} k^{2}\left(\varkappa_{0}^{2}+\frac{k^{2}}{4}\right) m^{-2} . \tag{61}
\end{equation*}
$$

With the notation $\varepsilon_{\mathrm{B}}=\hbar \omega, p=\hbar k$, Eqn (61) becomes

$$
\begin{equation*}
\varepsilon_{\mathrm{B}}=\left[p^{2} c_{\mathrm{s}}^{2}+\frac{p^{4}}{4 m^{2}}\right]^{1 / 2} \tag{62}
\end{equation*}
$$

This is the well-known Bogolyubov's formula for the spectrum of elementary excitations in a Bose gas. The frequency $\omega$ in Eqn (61) we shall denote by $\omega_{\mathrm{B}}$, so $\varepsilon_{\mathrm{B}}=\hbar \omega_{\mathrm{B}}$. In Eqn (62) we have only retained the plus sign on the righthand side, although formally $\omega$ may have either sign.

As follows from Eqn (61), at $k^{2} \ll \chi_{0}^{2}$ the frequency is $\omega \cong k c_{\mathrm{s}}$ that is, we simply have a sonic wave. At $k^{2} \gg x_{0}^{2}$ we observe the transition to a free quantum particle: a particle with such a large momentum does not have a chance to 'notice' the correlation of its motion with perturbations of the condensate.

Expression (49) for $\bar{\psi}$ indicates that the wave function in the linear approximation is

$$
\begin{equation*}
\psi=\bar{\psi}\left(1+\frac{1}{2} \frac{\tilde{n}}{n_{0}}+\frac{\mathrm{i}}{\hbar} \theta\right) \equiv \bar{\psi}(1+\tilde{\psi}) \tag{63}
\end{equation*}
$$

where $\bar{\psi}=1 / \sqrt{V}$.
Since all the $\bar{\psi}\left(\mathbf{r}_{j}\right)$ are assumed to be identical, in the linear approximation we have

$$
\begin{equation*}
\Psi=\bar{\psi}^{N}\left[1+\sum_{j} \tilde{\psi}\left(\mathbf{r}_{j}\right)\right] . \tag{64}
\end{equation*}
$$

The functions $\tilde{\psi}\left(\mathbf{r}_{j}\right)$ in the linear approximation are assumed to be very small, so expression (64) can be rewritten in the equivalent form

$$
\begin{equation*}
\Psi=\bar{\psi}^{N} \exp \left[\sum_{j} \tilde{\psi}\left(\mathbf{r}_{j}\right)\right] . \tag{65}
\end{equation*}
$$

Hence it follows that the wave function of $N$ particles may be represented as the product of similar terms of the form
$\bar{\psi} \exp \left[\tilde{\psi}\left(\mathbf{r}_{j}\right)\right]$. Therefore, the linearized function (63) must satisfy the linearized equation for the collective mode. In other words, the wave function $\tilde{\psi}$ must satisfy the linearized equation (47).

In the zero approximation we simply get the equilibrium equation. When the number of particles $N$ is very large, this equation becomes $m c_{\mathrm{s}}^{2}+U=0$, which implies that the first term (the kinetic energy) on the right-hand side of Eqn (47) is small. In the next-order approximation we get

$$
\begin{equation*}
\mathrm{i} \hbar \frac{\partial \tilde{\psi}}{\partial t}=-\frac{\hbar^{2}}{2 m} \Delta \tilde{\psi}+m c_{\mathrm{s}}^{2}\left(\tilde{\psi}+\tilde{\psi}^{*}\right)+U \tilde{\psi} . \tag{66}
\end{equation*}
$$

If $U=0$, the solution of this equation may be sought in the form of a plane wave, so that the total wave function of individual particle becomes

$$
\begin{equation*}
\psi=\bar{\psi}\left[1+b_{1} \exp (-\mathrm{i} \omega t+\mathrm{i} k x)+b_{2} \exp (\mathrm{i} \omega t-k x)\right] . \tag{67}
\end{equation*}
$$

It would have been nice to regard this wave function as a superposition of three states: the condensate itself and two excited states with positive and negative frequencies. This straightforward interpretation, however, encounters a formal mathematical difficulty. The point is that Eqn (66) contains not only $\tilde{\psi}$, but also $\tilde{\psi}^{*}$, so $\tilde{\psi}$ will only satisfy the superposition principle with real-valued coefficients, combining different solutions of the form (67). The amplitudes $b_{1}, b_{2}$ in each of these solutions are linked by certain relations which follow from the homogeneous equations for these amplitudes. If, for example, $b_{1}$ and $b_{2}$ are real, then for $\omega>0$ it is easy to find that

$$
\begin{equation*}
\frac{b_{1}}{b_{2}}=-\left(1+\frac{k^{2}}{2 \varkappa_{0}^{2}}+\frac{k}{\varkappa_{0}} \sqrt{1+\frac{k^{2}}{4 \varkappa_{0}^{2}}}\right) . \tag{68}
\end{equation*}
$$

Hence it follows that at $k^{2} \gg x_{0}^{2}$ the amplitude $b_{2}$ is small, and so the wave function (67) only retains the first exponential term with positive frequency which describes a free particle. However, for $\varkappa^{2}<\chi_{0}^{2}$ we get $b_{2}=-b_{1}$, so that both amplitudes are equal in magnitude.

Nevertheless, the solutions with positive and negative frequencies may exist independently from each other. This can be proved by the following arguments.

It is easy to see that Eqn (47) is invariant with respect to the transformation $\bar{\psi} \rightarrow \bar{\psi} \exp (\mathrm{i} \alpha)$, where $\alpha$ is an arbitrary angle. Naturally, the solution (67) is also invariant with respect to this transformation. Now we take function (67) and use it for constructing a new solution $\psi_{\alpha}$ with the aid of the superposition

$$
\begin{equation*}
\psi_{\alpha}(x, t)=\exp (\mathrm{i} \alpha) \psi\left(x, t+\frac{\alpha}{\omega}\right)+\exp (-\mathrm{i} \alpha) \psi\left(x, t-\frac{\alpha}{\omega}\right) \tag{69}
\end{equation*}
$$

Obviously, this superposition is a solution of the initial linearized equation. Now it is easy to find that with $\alpha=\pi / 4$ solution (69) becomes the superposition of the wave functions of the condensate and the exponential term with positive frequency. In a similar way it is possible to construct a solution in the form of a superposition of the wave functions of the condensate and the exponential term with negative frequency. In this sense the wave function of the form (67) may indeed be regarded as a sort of superposition of the wave function of the condensate and two wave functions of the
form $\exp (-\mathrm{i} \omega t+\mathrm{i} k x)$ and $\exp (\mathrm{i} \omega t-\mathrm{i} k x)$. This superposition may actually arise when the condensate interacts with some external system at a frequency $\omega$.

Physically more realistic, however, is the situation when a perturbation with energy $\hbar \omega$ and momentum $\hbar \mathbf{k}$ is excited in the condensate. Then the only possible state will be represented by a condensate of $N-1$ particles and one phonon with a wave function $\bar{\psi} \exp (-\mathrm{i} \omega t+\mathrm{ikr})$. One may say that in the absence of rigid coherent coupling with an external system the Bose condensate may exist either in the ground state or in a state with one (or several) phonons.

If the phonons excited in the Bose gas have different wave numbers, they will behave independently of each other by virtue of the linearity of equations for small oscillations (if we disregard their nonlinear interactions).

In case of inhomogeneous condensate the description of phonons in the form of plane waves must be understood in the sense of a quasi-classical approximation in terms of wave packets. It is easy to envisage the following pattern of excitation. Assume that the initial state is represented by a condensate of $N_{s}$ particles. If we add to this condensate a particle with zero chemical potential, we get a condensate of $N_{s}+1$ particles. If, however, the added particle has the energy $\hbar \omega$, it will behave in the condensate as a phonon with the same energy $\hbar \omega$. Adding one particle after another, we may build a system of $N=N_{s}+N_{p}$ particles, in which $N_{s}$ particles are part of the condensate, and $N_{p}$ particles represent different excitations. If we bring the potential $U(\mathbf{r})$ at the periphery of the condensate to zero, then the $N_{p}$ excitations will escape as free particles, leaving behind only the $N_{s}$ particles of the condensate.

## 7. Phonons

Excitations with Bogolyubov spectra (62) are conventionally referred to as phonons. At $k^{2} \ll 4 \chi_{0}^{2}$ the energy of a phonon $\hbar \omega_{\mathrm{B}}$ is approximately equal to $\hbar c_{\mathrm{s}} k$. At $k^{2} \gg 4 \chi_{0}^{2}$ the energy $\hbar \omega_{\mathrm{B}}$ corresponds to the kinetic energy of the free particle: $\hbar \omega_{\mathrm{B}}=\hbar^{2} k^{2} / 2 m$. The transition from the sonic quantum to the particle may be loosely assumed to take place at $k=2 \chi_{0}$.

Assume that the gas is in a potential well $U(\mathbf{r})$, adjusted in such a way that $m c_{\mathrm{s}}^{2}+U(\mathbf{r})=0$. This implies that at the boundary of the condensate, where its density $n(\mathbf{r})$ vanishes, we have $U(\mathbf{r})=0$. In the middle of the condensate $U$ attains its minimum $-U_{0}$ - it is indeed a potential well. Hence it follows that the speed of sound is a function of the coordinates: it vanishes on the boundary of the condensate, and reaches its maximum in the middle of the well.

Consider now a phonon with energy $\hbar \omega$. The wave function of such phonon may be represented as a wave packet $a(\mathbf{r}) \exp (-\mathrm{i} \omega t+\mathbf{i k r})$, where the envelope of the packet $a(\mathbf{r})$ is much greater than $1 / k$. By virtue of the conservation of energy, the frequency is $\omega=$ const, and is thus a kind of adiabatic invariant. On the contrary, the momentum $\hbar \mathbf{k}$ of such packet is a function of the coordinates. For each value of frequency $\omega$ there exists a point for which

$$
\begin{equation*}
\omega=k c_{\mathrm{s}}=\frac{\hbar k^{2}}{2 m} . \tag{70}
\end{equation*}
$$

At this point we have $k=2 m c_{\mathrm{s}} / \hbar=\omega / c_{\mathrm{s}}$, that is,

$$
\begin{equation*}
c_{\mathrm{s}}^{2}=\frac{4 \pi \hbar^{2} n a}{m^{2}}=\frac{\hbar \omega}{2 m} . \tag{71}
\end{equation*}
$$

From this relation we find the density at the point of 'phonon - particle transformation':

$$
\begin{equation*}
n=\frac{m \omega}{8 \pi \hbar a} . \tag{72}
\end{equation*}
$$

As the energy $\hbar \omega$ increases, the point of phonon-particle transformation moves deeper and deeper into the condensate, until $n$ attains its maximum value of $n_{\max }=n(0)$. Thus, at frequencies

$$
\begin{equation*}
\omega>\frac{8 \pi \hbar n(0) a}{m} \tag{73}
\end{equation*}
$$

all elementary excitations are free particles.
In the region where the density is above the value given by Eqn (72) the excitations are phonons with the dispersion $\omega=c_{\mathrm{s}} k$. Hence it follows that the momentum of a phonon $p=\hbar k=\omega / c_{\mathrm{s}}$ increases as it moves from the middle of the well to its periphery. At the density defined by Eqn (72) the phonon becomes a free particle, and its momentum remains the same up to the boundary of the well, where $U=0$. Outside of the well the momentum starts to decrease in accordance with the law of conservation of energy:

$$
\begin{equation*}
\frac{p^{2}}{2 m}+U(\mathbf{r})=\hbar \omega=\text { const } . \tag{74}
\end{equation*}
$$

This relation can be used for finding the path of a given particle. Such particles form a gaseous halo around the condensate, if the temperature of the gas cloud is not zero. Thus, in terms of momentum the particle is in a certain effective potential well $U_{p}(\mathbf{r})$ shown in Fig. 3.

We see that beyond the points $A, A^{\prime}$ the excitation moves as a free particle, while inwards of the points $A, A^{\prime}$ the particle becomes a phonon whose momentum changes in such a way as though it were located on a potential hill (dashed line in Fig. 3).

When the gas is heated, the number of particles in the condensate will decrease, and the number of phonon excitations will increase. We can get a mental picture of such a state starting with a given number of particles in the condensate - say, $N_{s}$. The reference point for measuring


Figure 3. The effective potential well $U_{p}(x)$ for a phonon with energy $\hbar \omega=$ const. Beyond the points $A, A^{\prime}$ the elementary excitation becomes a free particle in the potential $U_{p}(x)$ shown by the dashed line. Between the points $A, A^{\prime}$ the momentum of particle is $p=\hbar k=\hbar \omega / c_{\mathrm{s}}$. Since the density of the gas in the condensate varies in such a way that the relation $m c_{\mathrm{s}}^{2}=-U(x)$ is satisfied, the squared momentum of the phonon is $p^{2}=\hbar^{2} \omega^{2} / c_{\mathrm{s}}^{2}=m \hbar^{2} \omega^{2} /|U(x)|$. Accordingly, in terms of momentum the effective 'well' $U_{p}(x)$ has the shape shown dashed line. When $\omega \rightarrow 0$, the points $A, A^{\prime}$ tend to the points $B, B^{\prime}$, and the dashed line approaches the segment $B B^{\prime}$ on the $x$-axis.
the potential energy $U(x)$ or $U(r)$ can be selected as shown in Fig. 3. Then the condensate will be located in the region $U<0$. All particles of the condensate have zero energy. In terms of particles this state appears as Bose condensate with zero energy. When any particle within the well is moved, the change in its potential energy is exactly compensated by the change in the energy $m c_{\mathrm{s}}^{2}$ resulting from paired interactions with other particles. Because of this, the shape of the potential well has no effect on the fact that the total energy of any particle of the condensate is zero.

We now add new 'warm' particles of the 'normal' gas phase. Their chemical potential must be zero. From this condition we may find the temperature dependence of the number of particles $N_{n}$ of the normal phase; the sum $N_{s}+N_{n}=N$ is equal to the total number of particles. The condensate itself appears as a set of $N_{s}$ particles with zero chemical potential.

## 8. Scattering lengths

All the arguments developed above were based on the simplest model of interaction between the atoms: the model of rigid spheres. In reality, this interaction is much more complicated. Thus, faraway atoms experience the Van der Waals force with the potential $U(r)=-b / r^{6}$, where $b$ is a certain constant. This potential is sufficiently deep and extensive, making it possible to use a quasi-classical approximation in a certain range of distances. The momentum of a particle with near-zero energy varies with the distance as $p^{2}=-m U$, and accordingly, the de Broglie wavelength $\lambda_{\mathrm{B}}$ depends on the radius as $\lambda_{\mathrm{B}} \sim r^{3}$. The derivative $\mathrm{d} \lambda_{\mathrm{B}} / \mathrm{d} r$ is proportional to $r^{2}$; therefore, as $r$ increases, a certain radius $r_{0}$ may be reached where this derivative becomes greater than unity. The quasi-classical approximation cannot be used in the neighborhood of this value.

At very large distances, where the kinetic energy is $\varepsilon \gg b / r^{6}$, the quasi-classical approximation holds again. In this way, the problem of scattering reduces to sewing together the two quasi- classical solutions in the region where $\varepsilon \simeq b / r^{6}$. Since the wave function has a large number of nodes in the region of interaction, the scattering length $a$ depends on the interplay of numerous particular features of $U(r)$.

The theoretical calculation of the scattering length is a formidable task. It is possible, however, to find the probability of the scattering length having a plus or minus sign from the condition of sewing of quasi-classical wave functions. It turns out (see Appendix) that the probability of the scattering length for an alkaline atom taken at random being positive is $3 / 4$, and being negative $1 / 4$. In particular, in three experiments carried out so far [1-6], the scattering length was positive for two substances, and negative for only one substance ( $\left.{ }^{7} \mathrm{Li}\right)$.

Now let us see how the magnitude of the scattering length affects the characteristics of the Bose condensate. We start with the equilibrium. Multiplying the right-hand side of Eqn (47) by $\bar{\psi}$ and integrating the result with respect to the volume, we find the magnitude of chemical potential $\mu$ at equilibrium. Given that at equilibrium, according to Eqn (48), we have $\bar{\psi}=\sqrt{n / N}$, we get

$$
\begin{equation*}
N \mu=\int\left(\frac{\hbar^{2}}{8 m n}|\nabla n|^{2}+g_{0} a n^{2}+n U\right) \mathrm{d} v \tag{75}
\end{equation*}
$$

where $g_{0}=4 \pi \hbar^{2} / m=$ const.

The equilibrium distribution of density $n(\mathbf{r})$ must be found from the condition that $\operatorname{Eqn}(75)$ is at minimum with the given number of particles $N=\int n \mathrm{~d} v$. From Eqn (75) we see that at $a=0$ the atoms of condensate fill the lower level in the potential well $U(\mathbf{r})$. The condensate then occupies a region of size $L_{0} \sim\left(\hbar^{2} / \alpha m\right)^{1 / 4}$ if $U=\alpha r^{2} / 2$. As $a>0$ increases, the atoms recede from one another, and if the first term in the integrand can be neglected, the condition of equilibrium assumes the simple form of Eqn (22): $a g_{0} n+U=$ const.

Now let us consider a condensate with two kinds of atoms. Let $n_{1}, n_{2}$ be their densities, $a_{1}, a_{2}$ the corresponding scattering lengths, and by $a_{12}$ we denote their mutual scattering length. The expression for the chemical potential becomes more complicated. Assume, for example, that $N_{1}=N_{2}=N$. Then we get:

$$
\begin{align*}
2 N \mu & =\int\left[\frac{\hbar^{2}}{8 m n_{1}}\left(\nabla n_{1}\right)^{2}+\frac{\hbar^{2}}{8 m n_{2}}\left(\nabla n_{2}\right)^{2}\right. \\
& \left.+g_{0}\left(a_{1} n_{1}^{2}+2 a_{12} n_{1} n_{2}+a_{2} n_{2}^{2}\right)+\left(n_{1}+n_{2}\right) U\right] \mathrm{d} v \tag{76}
\end{align*}
$$

Let us consider a few particular cases. If $a_{12}=0$, the two condensates are independent of each other and fill the trap in the same way. If $a_{12}^{2}=a_{1} a_{2}$, the trap is filled by gas with density $n=n_{1}+n_{2}$, while the concentrations $c_{1}=n_{1} / n$, $c_{2}=n_{2} / n$ may be entirely arbitrary functions of $\mathbf{r}$, the only condition being that $c_{1}+c_{2}=1$. Another interesting situation corresponds to $a_{12}^{2} \gg a_{1} a_{2}$. In terms of energy it is then advantageous to have $n_{1} n_{2}=0$. This implies that the condensates of atoms of kind 1 and kind 2 are segregated, and the middle of the trap is occupied by the phase whose scattering length is smaller (that is, whose 'elasticity' is lower).

Interesting effects are observed when the scattering length is negative. If the scattering length in Eqn (75) is negative, then the second term in the integrand describes attraction rather than repulsion of atoms. If we compare the second term with the first and assume that the characteristic size of the condensate is $L$, then (recall that $n \sim N / L^{3}$ ) we see that the first term is proportional to $L^{-5}$, and the second to $\sim L^{-6}$. Hence it follows that equilibrium is only possible when the number of atoms of condensate is not too large, $N<N_{\mathrm{c}} \sim L_{0} / a$, where $L_{0}$ is the characteristic size of the equilibrium condensate of not too high density. At $N>N_{c}$ the condensate must collapse into a point; this collapse was first studied by Zakharov [23] using the nonlinear Schrödinger equation of the form (47) as applied to a Langmuir soliton in a plasma (see also Refs [24-28]). Pitayevsky [29] demonstrated that an exactly similar collapse must take place in a Bose condensate. The outcome of such a collapse is not yet quite clear. Stoof [30] believes that the gas of atoms with negative scattering length must segregate into phases or build molecules before or instead of undergoing Bose condensation. Kagan, Shlyapnikov, Walraven [31] consider the feasibility of formation of dense clusters. Hopefully, this problem will be finally resolved by further experimental studies.

## 9. Magnetic confinement of Bose condensate

Gas condensates at superlow temperatures are created using atoms of alkaline elements in diamagnetic states, so they are most conveniently contained in magnetic traps.

Magnetic traps were first invented for confinement of high-temperature plasma. One of the simplest types of trap
was proposed by Budker [32] in this country and Post [33] in the USA. Budker called it a trap with magnetic stoppers. The simplest realization consists of a solenoid whose magnetic field is stronger at the ends than in the middle. Accordingly, in the longitudinal direction there is a 'magnetic well', a region where the field $B$ has a minimum. In the cross section, however, the magnetic field falls off towards the periphery. In a plasma trap this leads to instability of the plasma [11, 12]; for diamagnetic atoms such a device does not work as a trap. For the stable confinement of plasma or diamagnetic condensate there must be an absolute minimum of the magnetic field inside the trap. Such a trap was developed by Ioffe and colleagues [13], who were the first to prove the feasibility of stable confinement of plasma.

Later a trap of this type was used for the confinement of neutral diamagnetic atoms [14].

Let us explain the principle of this trap. It will suffice to consider the magnetic field near its minimum in the middle of the trap. Assume that the magnetic field has the form $\mathbf{B}=\nabla \phi$, where $\phi$ is the corresponding potential. In case of a homogeneous magnetic field we have $\phi=B_{0} z$, where $z$ is the coordinate in the cylindrical system of coordinates $(r, \varphi, z)$. Now we add the quantity $B_{0} z$ to $\alpha z^{3} / 3$. The magnetic field on the axis $r=0$ will then be $B_{0}+\alpha z^{2}$, and will thus increase to both sides from the point $z=0$. It turns out, however, that the magnetic field fall off in the transverse direction. The point is that because $\operatorname{div} \mathbf{B}=0$ the function $\phi$ must satisfy equation $\Delta \phi=0$. Therefore, in the neighborhood of $z=0, r=0$ the function $\phi$ must be represented as

$$
\begin{equation*}
\phi=B_{0} z+\frac{\alpha z^{3}}{3}-\frac{\alpha z r^{2}}{3} . \tag{77}
\end{equation*}
$$

Hence it follows that the field $B_{z}=B_{0}+\alpha z^{2}-\alpha r^{2} / 3$ falls off as $r$ increases, and it is not possible to confine a diamagnetic substance in the transverse direction. To overcome this impediment, the simple symmetrical trap was supplemented with a quadrupole magnetic field in Ref. [13]. Then the potential $\phi$ is given by

$$
\begin{equation*}
\phi=B_{0} z+\frac{\alpha z^{3}}{3}-\frac{\alpha z r^{2}}{3}+\frac{1}{2} \beta\left(x^{2}-y^{2}\right), \tag{78}
\end{equation*}
$$

where $x=r \cos \varphi, y=r \sin \varphi$.
According to Eqn (78), the squared magnetic field strength is

$$
\begin{equation*}
B^{2}=\left[B_{0}+\alpha z^{2}-\frac{\alpha r^{2}}{3}\right]^{2}+\beta^{2} r^{2} \tag{79}
\end{equation*}
$$

Hence it follows that at $\beta^{2}>2 \alpha B_{0} / 3$ the magnetic field grows in all directions away from $z=0, r=0$, and we have a trap with an absolute minimum of field $B$. Traps of this type are used in the experiments with Bose condensates of atoms of alkaline elements.

There are many more possible designs of magnetic traps. Cold plasma, for example, is sometimes confined with a 'magnetic grid' [34]. In its simplest form it consists of current-carrying rods arranged in a plane parallel to each other. The currents in adjacent rods run in opposite directions, so that the cross-sectional magnetic configuration has the shape shown in Fig. 4.

Magnetic grids may be combined to form 'magnetic vessels', which makes it possible to pour condensate into a


Figure 4. A 'Magnetic grid' is a set of conductors (current-carrying rods), with the current in adjacent rods going in the opposite directions. A region with a strong enough magnetic field is created in the neighborhood of the conductors.
vessel (see Fig. 1) in the laboratory. Here we shall mention just one such type of idealized vessel, the 'magnetic cup' shown in Fig. 5. It is a double helix with the current running in opposite directions in each spiral, forming the walls of the imaginary cup.


Figure 5. A 'Magnetic cup'

## 10. Experiments with Bose condensates

The first three condensates were prepared from atoms of rubidium-87 [1], sodium [2] and lithium-7 [3]. The scattering length is positive for rubidium and sodium, and negative for lithium. All condensates were prepared according to more or less similar routines which differed only in minor details.

The experiment starts by filling the magnetic trap with atoms in a diamagnetic state. The atoms of alkaline elements have integral spins; therefore, their quantum magnetic moments may be directed along the field, counter to the field, or with zero projection of the moment on the direction of the field. Only those atoms whose magnetic moments are opposite to the field are suitable for filling the trap. The energy $-\mu B$ of such atoms has a minimum in the middle of the trap.

Laser cooling is used during the first stage of filling the trap with the gas atoms. Then comes the second stage, the stage of induced cooling. A pulse of high-frequency magnetic field may rotate the spin of the atom in such a way that the atom will not be retained in the magnetic trap. The frequency of the high-frequency field can be adjusted so as to selectively eject the atoms from the periphery of the trap, where the 'warmest' particles are located. In this way it is possible to reduce the temperature and get into the sub-microkelvin range. It is then that Bose condensation takes place. With additional laser beams it is possible to control the shape of magnetic trap, thus changing the parameters of the Bose condensate.

In recent experiments with ${ }^{87} \mathrm{Rb}$ [35] the number of atoms in the condensate was brought up to $4 \times 10^{4}$ at a temperature
of 280 pK . It is interesting that the temperature of condensation $T_{\mathrm{c}}$ turned out to be close to the critical temperature of non-interacting atoms. The authors were able to excite and study the collective modes in the condensate [4]. These modes are well interpreted as collective coherent states. Two modes were observed: $m=0$, and $m=2$. The mode $m=0$ corresponds to axially symmetrical oscillations, when the radial contraction of the drop of condensate is accompanied by its elongation in the longitudinal direction. The mode $m=2$ corresponds to elliptic deformation of the drop in the transverse direction. The frequencies of oscillations were in good agreement with theoretical predictions. The amplitude of the mode $m=2$ was large enough to enable the detection of the nonlinear effect of the square-law frequency increase with amplitude. The experiment indicated that the time of damping of oscillations was fairly large, which calls for further theoretical analysis.

In experiments with condensates of sodium atoms $[2,5,6$, 36] the number of atoms in the condensate was made as large as $5 \times 10^{6}$. In accordance with the simplest theory (disregarding the effect of the expulsion of excitations from the condensate), the number of condensate atoms $N_{s}$ must vary with the temperature as $N_{s}=N-N\left(T / T_{\mathrm{c}}\right)^{3}$, where $N$ is the total number of atoms, and $T_{\mathrm{c}}$ is the critical temperature. Such a dependence was actually observed experimentally when the share of particles in the condensate was below $50 \%$.

The collective modes were also observed in experiments with sodium condensate [36]. These oscillations also died away rather slowly. The frequencies of two different modes observed in Ref. [36] were in good agreement with the theory of collective modes of condensate.

Keen theoretical interest in the condensates of ${ }^{7} \mathrm{Li}$ [3] is explained by the fact that the scattering length of lithium atoms is negative. This implies that the effective potential is negative - in other words, there is attraction between the atoms. From a theoretical standpoint the condensate under such conditions cannot be stable if the number of atoms in the condensate is large enough [29-31]. The initial reports [3] claimed that the experimentally achieved number of atoms in the condensate was above the critical value. Further experiments [37] revealed, however, that the highest number of particles $N_{0}$ in Bose condensate was only about 1300, which agrees very well with the theoretical limit of 1400 as found in Refs [22, 31, 38-40].

Coherent states in the form of Bose condensates open the possibility of studying many interesting quantum effects. For example, the cover of the March issue of Physics Today [41] featured a color picture of halfmoon-shaped falling drops of condensates, with the caption 'The Atom Laser'. This picture appeared courtesy of the authors of Ref. [42] concerned with the development of an atom laser using the ideas of Refs [43, 44].

Here we are actually dealing with a certain analogy between a Bose condensate and a laser. In a laser one may envisage a situation when laser radiation is stored in the resonator through atomic transitions from a higher excited level to a lower one. The release of a shutter eliminates one of the mirrors and lets out the pulse of light. The storage of cold atoms of Bose condensate in the trap is similar to the storage of photons in a laser cavity. If we switch off the trap or turn the magnetic moments of atoms with a pulse of highfrequency field, we may release either the entire condensate or part of it. It is exactly such a free drop of condensate that is portrayed on the front cover of Physics Today.

This analogy between laser and Bose condensate so far seems somewhat artificial. More sophisticated effects of the collective behavior of atoms, however, may take place in Bose condensate. For example, the formation of three-dimensional solitons may lead to the ejection of coherent clusters of matter, which fits in better with our intuitive ideas of coherent laser light pulses.

Another interesting effect was observed in Ref. [45] (see also Ref. [41]). It consists in the interference between two Bose condensates. The magnetic trap for sodium atoms in this experiment was designed in such a way that the Bose condensate had the shape of a long cigar. An additional laser beam of rectangular cross section was used to cut this condensate in two, with a certain distance $d$ between the resulting parts. Then the trap and the laser were turned off simultaneously, and the two halves of the condensate started to expand fast in the transverse direction, at the same time moving rather slowly towards each other. A clear interference pattern was formed in the overlap region, with the distance between the ripples equal to half the de Broglie's wavelength $\lambda$. The wavelength itself was well approximated by the formula $\lambda=h t / m d$, where $h=2 \pi \hbar, t$ is the observation time, $m$ is the mass of atom, and $d$ is the separation between the condensates prior to de-energizing the laser and the trap. The experimentally observed interference pattern of matter waves with the period of $15 \mu \mathrm{~m}$ corresponds to a kinetic energy of 0.5 nK , whereas the energy of the particles in the Bose condensate before was of the order of 100 nK . Owing to the radial expansion of the condensate its density decreased greatly, and so the $\psi$-waves which formed the interference pattern corresponded to free particles. This observation clearly demonstrates the indistinguishability of particles: the $\psi$-functions of the two condensates correspond to the common wave function of the same particles.

These experiments open new opportunities for studying coherent atomic beams. In particular, of special interest is the question of whether it is possible to observe and study the effects associated with a varying phase lag between two Bose condensates. The use of coherent phenomena may suggest new ideas for the development of atom lasers.

The prospects for the experimental studies of numerous physical effects in Bose condensates of alkaline elements are still quite vast.

## 11. Conclusions

The production of Bose condensates of atoms of alkaline elements in miniature magnetic traps has opened a new interesting domain of both experimental and theoretical studies. Progress in the experiments will to a large extent depend on the development of new diagnostic techniques and on those questions which will be addressed to the experimenters.

Theoretical studies may follow their own guidelines. First of all, the theory must be developed to describe and interpret the experimental results already available. Namely, it is necessary to describe in greater detail the kinetics of the normal phase - that is, the kinetics of phonons. The results of this treatment should enable one to calculate the damping of the collective modes. Dispersing gas clouds have been observed in some experiments [5]. The question is whether the expanding normal gas phase is associated with the compression of the condensate because of a kind of 'reactive recoil', or whether this effect is small.

More attention ought to be paid to the scenarios of the formation of condensates in gases with negative scattering length. Since this condensate is unstable when the number of particles is large, it would be desirable to carry out more detailed studies (compared with the theoretical works [2931]) concerned with the outcome of its evolution.

In the most recent experiments [46] microscopic explosions were observed in condensate where the number of atoms was $>1400$, accompanied by a rapid compression and scattering of the cloud of atoms [see UFN 167688 (1997)]. This micro-explosion points to a collapse - that is, a jump to a new quantum state with a higher density. The implosion of part of the condensate is accompanied by a release of heat and a bursting of the condensate. This effect resembles the explosion of supernova, where gravity 'overcomes' Pauli's principle. In the case of condensate, the attraction 'overcomes' Heisenberg's uncertainty principle.

The production of 'drops' of Bose condensates in magnetic traps also puts more general queries to the theoreticians. The point is that Bose gases are rather commonly described with the methods of quantum field theory using the operators of creation and destruction of particles. Bose condensation in this theory implies that the field operator has a mean value. Then the phase does not commute with the operator of the number of particles. This circumstance alone perplexes the experimenters [10], not to mention that this approach may imply spontaneous violation of symmetry. In the present paper this difficult problem is avoided by using only the Schrödinger equation at zero temperature. However, even when the formalism of the Schrödinger equation is used, the wave function of the condensate may be multiplied by an arbitrary factor of the form $\exp (\mathrm{i} \alpha)$, where $\alpha$ is the phase. Because of this, the phase difference between different condensates of identical particles may have a certain physical meaning. We believe that further experiments will enable one to detect and study the relevant effects.

The operator approach is found to be useful for describing a number of fine physical effects in Bose condensates. In particular, interference phenomena of the Anderson effect type may take place when two condensates come into contact with each other [46]. Hopefully, the improvements in experimental and observational techniques in the physics of superlow temperatures will permit the observation of these effects in direct experiments.

## 12. Appendices

### 12.1 Acoustic solitons

Solitons (wave formations in the shape of solitary nonlinear pulses) are well known in hydrodynamics and plasma physics [2]. Optical solitons are used for transmitting information through optical fibers. Acoustic solitons are less familiar, because there are not many media in which sonic waves exhibit dispersion with increasing wave number. However, equations (51), (52), and formula (61) indicate that acoustic waves in Bose condensate exhibit dispersion as the wave number approaches $\chi_{0}$. Accordingly, a weakly nonlinear plain wave described by Eqn (56) may give rise to solitons. A single one-dimensional soliton has the form of Eqn (57), where the amplitude $v_{0}$ is linked with the width of soliton $\Delta$ by Eqn (58). The dispersion of the form of Eqn (61), where the phase velocity $v_{\mathrm{p}}=\omega / k$ increases with $k$, is usually referred to
as positive. It is known [20] that, along with the onedimensional solitons, two and three-dimensional solitons may exist when the dispersion is positive. They are described by the Kadomtsev-Petviashvili equation. Let us briefly explain the point for those who have never encountered this equation before. In the linear approximation, Eqn (56) for a plane wave of the form $\exp (-\mathrm{i} \omega t+\mathrm{i} k x)$ leads to the dispersion relation

$$
\begin{equation*}
\omega=c_{\mathrm{s}} k+\frac{c_{\mathrm{s}} k^{3}}{8 \chi_{0}^{2}} . \tag{A.1}
\end{equation*}
$$

Assume that the wave resembles a one-dimensional wave, but is slightly different, so that it can be regarded as a superposition of plane waves of the form $\exp (-\mathrm{i} \omega t+\mathrm{ikr})$, with $k_{\perp} \ll k_{\|}$. Then the wave number $k$ in Eqn (A.1) may be assumed to be $k=\left(k_{x}^{2}+k_{\perp}^{2}\right)^{1 / 2} \cong k_{x}+k_{\perp}^{2} / 2 k_{x}$. The increment proportional to $k_{\perp}^{2}$ in the second term in Eqn (A.1) may be neglected, since $k^{2} \ll \chi_{0}^{2}$ by assumption. Therefore, Eqn (A.1) may be rewritten as

$$
\begin{equation*}
k_{x}\left(\omega-c_{\mathrm{s}} k_{x}-\frac{c_{\mathrm{s}} k_{x}^{3}}{8 x_{0}^{2}}\right)=\frac{c_{\mathrm{s}} k_{\perp}^{2}}{2} . \tag{A.2}
\end{equation*}
$$

This equation we multiply by $v$, and replace $\omega, k_{x}, k_{\perp}^{2}$ by the operators

$$
\omega=\mathrm{i} \frac{\partial}{\partial t}, \quad k_{x}=-\mathrm{i} \frac{\partial}{\partial x}, \quad k_{\perp}^{2}=-\Delta_{\perp}
$$

If we also take the nonlinear term into account, we get the Kadomtsev-Petviashvili equation (58). Introducing the appropriate scales along the $x, y, z$ axes, we may write Eqn (58) in the dimensionless form

$$
\begin{equation*}
\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}+v \frac{\partial v}{\partial x}+\frac{\partial^{3} v}{\partial x^{3}}\right)=\Delta_{\perp} v \tag{A.3}
\end{equation*}
$$

This equation admits a solution of the form $v(x-t)$, corresponding to a stationary nonlinear wave of the soliton type. In the case of a two-dimensional soliton, when $\Delta_{\perp} v=\partial^{2} v / \partial y^{2}$, the solution can be represented in a relatively simple analytical form

$$
\begin{equation*}
v=\frac{24\left(3-x^{2}+y^{2}\right)}{\left(3+x^{2}+y^{2}\right)^{2}} \tag{A.4}
\end{equation*}
$$

The analytical solution for the three-dimensional soliton is not known; it may well be, however, that it does not differ much from that given by Eqn (A.4). So we proceed as follows: we set $y=\alpha r$ and select the constant $\alpha$ so as to make the solution (83) close to a three-dimensional soliton. With this purpose we multiply

$$
\Delta_{\perp} v=\frac{\partial^{2} v}{\partial r^{2}}+\frac{1}{r} \frac{\partial v}{\partial r}
$$

by $v$, integrate the result with respect to $y$, and require that this expression be close to the integral of the function $v\left(\partial^{2} v / \partial y^{2}\right)$. This approach yields the following value of $\alpha^{2}$ : $\alpha^{2}=2 / 7$. This means that, as a function of the cylindrical radius $r$, the three-dimensional soliton is about twice as wide as the two-dimensional soliton (A.4).

The three-dimensional soliton looks like a 'pancake' with reduced density, moving at a velocity slightly below the speed of sound in the direction normal to the 'pancake'. The
thickness of the soliton is of the order of $\Delta \sim \chi_{0}^{-1}\left(c_{\mathrm{s}} / v_{0}\right)^{1 / 2}$, where $v_{0}$ is the amplitude of the soliton (in magnitude). The diameter of the three-dimensional soliton is of the order of $2 r_{0} \sim \Delta\left(c_{\mathrm{s}} / v_{0}\right)^{1 / 2}$. For small-amplitude solitons, $v_{0} \ll c_{\mathrm{s}}$, the transverse size $r_{0}$ of the three-dimensional soliton is much greater than its longitudinal dimensional $\Delta$.

### 12.2 Scattering length

Interaction of alkaline atoms aligned with the magnetic field, similarly to the interaction of hydrogen atoms, is determined by the triplet molecular term with ${ }^{3} \Sigma_{u}^{+}$symmetry. At close range the atoms are repelled because of antisymmetry of the coordinate wave function of electrons. At long range the atoms experience the Van der Waals attraction forces with the potential $U=-b / r^{6}$, where $r$ is the distance between the atoms. Because of higher polarizability, these forces for atoms of alkaline metals are much stronger than in the case of hydrogen. Accordingly, the Van der Waals well exhibits many levels of bound states. This circumstance affects both the magnitude and the sign of the scattering length.

In case of spherically symmetrical $S$-scattering, the Schrödinger equation for two alkaline atoms has the form

$$
\begin{equation*}
-\frac{\hbar^{2}}{2 m_{*}} \frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} \frac{\partial \psi}{\partial r}\right)+U(r) \psi=E \psi \tag{A.5}
\end{equation*}
$$

where $m_{*}=m / 2$ is the reduced mass, and all other notations are conventional.

Using the usual replacement $\chi=r \psi$, we rewrite Eqn (A.5) at $E=0$ and $U=-b / r^{6}$ in the form

$$
\begin{equation*}
r^{6} \frac{\mathrm{~d}^{2} \chi}{\mathrm{~d} r^{2}}+\gamma^{2} \chi=0 \tag{A.6}
\end{equation*}
$$

where $\gamma^{2}=m b / \hbar^{2}$.
If we now go over to the new variables $x=\gamma / 2 r^{2}$, $\chi=\varphi \sqrt{r}$, Eqn (A.6) becomes

$$
\begin{equation*}
\frac{\mathrm{d}^{2} \varphi}{\mathrm{~d} x^{2}}+\frac{1}{x} \frac{\mathrm{~d} \varphi}{\mathrm{~d} x}+\left[1-\frac{1}{(4 x)^{2}}\right] \varphi=0 \tag{A.7}
\end{equation*}
$$

which is the Bessel equation. Its general solution has the form

$$
\begin{equation*}
\varphi=C_{1} J_{-1 / 4}(x)-C_{2} J_{1 / 4}(x) . \tag{A.8}
\end{equation*}
$$

When $x$ is large (that is, when $r$ is small), the Bessel functions admit a quasi-classical asymptotic approximation in the cosines. If we now set the boundary condition requiring that Eqn (A.8) goes to zero at a certain value of $x_{0}$ (that is, at close range), we easily find that

$$
\begin{equation*}
\frac{C_{2}}{C_{1}}=\frac{\cos \left(x_{0}-\pi / 8\right)}{\cos \left(x_{0}-3 \pi / 8\right)} . \tag{A.9}
\end{equation*}
$$

Now with the aid of Eqn (A.8), (A.9) we find the wave function $\psi(r)$ at long range:

$$
\begin{equation*}
\psi(r)=\left(\frac{4}{\gamma}\right)^{1 / 4} \frac{1}{\Gamma(3 / 4)}\left(1-\frac{a}{r}\right) \tag{A.10}
\end{equation*}
$$

where the scattering length $a$ is given by

$$
\begin{equation*}
a=\frac{\sqrt{\gamma}}{2} \frac{\Gamma(3 / 4)}{\Gamma(5 / 4)} \frac{\cos \left(x_{0}-\pi / 8\right)}{\cos \left(x_{0}-3 \pi / 8\right)} . \tag{A.11}
\end{equation*}
$$

We see that the order of magnitude of scattering length is defined by the parameter, $\sqrt{\gamma}$. The scattering length for heavy atoms may be as large as hundreds of Angstrm. The sign of the scattering length is determined by the last term in Eqn (A.11). Since $x_{0} \gg 1$, the phase $x_{0}$ may be considered as a random quantity; then, according to Eqn (A.11), the 'probability' of the scattering length being positive is $3 / 4$.

The $S$-wave approximation and the concept of scattering length may be used as long as the energy of relative motion is small, or, more precisely, if

$$
\begin{equation*}
E \ll \frac{\hbar^{3}}{m^{3 / 2} b^{1 / 2}} . \tag{A.12}
\end{equation*}
$$

This implies that the temperature of the condensate must be considerably lower than $10^{-3} \mathrm{~K}$.

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[^0]:    B B Kadomtsev, M B Kadomtsev Russian Research Center
    "Kurchatov Institute"
    pl. Kurchatova 1, 123182 Moscow, Russia
    Tel. (7-095) 19698 14. Fax (7-095) 9430073
    E-mail: kadomtse@ufn.msk.su
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