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Attractors and frozen-in invariants in turbulent plasma

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Abstract. Work on turbulent equipartitions in a plasma (i.e. attractors characterized by Lagrangian invariants) is reviewed. Although such attractors also exist in the convective zone of the Sun and in atmospheres, the primary emphasis is on turbulent transport in tokamaks. By extending the hydrodynamic concept of freezing-in to Vlasov's equation, it is explained why the magnetic field topology in a collisionless plasma is conserved even though the conventional hydrodynamic description breaks down. Arguments are presented to support the conjecture that the canonical profiles of tokamak plasma are due to an attractor with a plasma frozen into the poloidal magnetic field. In fact, the exclusion from the conventional set of frozen-in integrals of the one for the toroidal magnetic field is all what is needed. The reason for the breakdown of this invariant is the poloidal noninvariancy of the magnetic field, an effect to which trapped particles are particularly sensitive. The predictions of the attractor and of two attraction basin boundaries (H-mode and transport suppression by the reversed shear) are confirmed experimentally to a reasonable accuracy.

1. Introduction

It is well known that even ordinary nonlinear differential equations are in general globally nonintegrable. The more so it is true for partial differential equations of turbulence. There are many definitions of nonintegrability but the exponential instability of at least a fraction of the solutions can be taken as its major indication, because the global solution cannot be presented in this case as a smooth function involving a required number of arbitrary constants (arbitrary functions for partial differential equations). Exact specific solutions, like fully integrable equations, exhibit atypical behaviour and describe idealized models. It is quite understandable that integrable examples fill up scientific journals but we must not forget about the more general nonintegrable case. It may seem that nonintegrability leads to incognisability of the object at hand. However the general solution is not required because nonintegrable systems spend almost all the time near attractors.

The idea behind attractors is not new and is expressed in the old law of mechanics stating that the energy of a system tends to a minimum. Aristotle made an absolute of this law and wrote that a motion proceeds until there is a driving force. The equations of a nonlinear two-dimensional oscillator with friction are nonintegrable but an attraction to the bottom of the potential well is clear and quite sufficient for a qualitative understanding. A potential minimum is the simplest example of an attractor; in the presence of driving forces more complex strange attractors with fractal structures have been revealed in recent decades [1 – 3]. In the current review we confine ourselves to another type of attractors known for a long while. In modern terminology the Maxwell – Boltzmann – Gibbs thermodynamic equilibrium is none other than a statistical attractor for which the law of conservation of energy and Lagrangian invariant (Liouville's theorem) hold. The adjective *statistical* emphasizes that the solution can deviate, though slightly, from such an attractor as a result of fluctuation. We are reminded that Liouville's theorem does not prohibit statistical attractors in the case of many degrees of freedom. In physics, dissipation appears as a result of approximating the Hamiltonian systems with many degrees

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of freedom (it is required for common and strange attractors to exist). With turbulence, the conservation of energy may be replaced by other invariants and the statistical attractor will be referred to as a turbulent equipartition (TEP).

Although the importance of invariants has long been known, I cannot refrain from dithyrambs. Invariants, as well as the results derived from them, are more fundamental and more reliable than equations of motions and the ensuing findings. For example, energy is conserved in many situations when the equations of motion are unknown or they are known only approximately. Practically all the valuable equations in physics were derived from laws of conservation of previously known invariants. 'Derivation' of invariants from equations of motion offers usually just an ordinary check. Scarce results are the fee for the reliability of the method. The more complex the system, the more important the role of invariants. Relations between people are purely described by equations of motion but they are governed and described by such an invariant as money.

In this review, turbulent plasma in tokamaks is the main object to which I apply the method of invariants. Canonical plasma profiles will be interpreted as turbulent equipartitions and this is the reason why we confine ourselves to one type of attractor only.

The tokamak is not nearly the first thermonuclear reactor. TEPs have long been known in natural fusion reactors with turbulent energy transport to the surface: stars. Energy released in thermonuclear reactions in the vicinity of the star centre is transferred by radiative heat conduction and, with a temperature decrease towards the surface, radiative heat transfer grades into convection. Convection conserves the Lagrangian invariant, specific entropy, and mixing results in turbulent equipartition, i.e. constant distribution of specific entropy in space, not only along trajectories:

$$\frac{T}{n^{2/3}} = \text{const}\,.$$

Here we assume full ionization. In combination with the hydrostatic equilibrium condition, TEP immediately yields the universal canonical density profile. Near the edge of a star the force of gravity may be considered to be constant and this leads to the hydrostatic equilibrium equation $\mathrm{d}(n^{5/3})/\mathrm{d}x \sim n$ and to the density distribution $n \propto x^{3/2}$, where the x distance is measured from the upper boundary of the atmosphere. In contrast to an exponential, isothermal atmosphere, an isentropic atmosphere has a distinct upper boundary and it may be seen without a theory, merely by looking at the Sun.

Recently the Sun profile was measured with high precision by helioseismology methods [4]. The frequencies of natural oscillations of the Sun can be found from measurements of spectral line shifts and the radial distribution of the speed of sound can be inferred from the frequency spectrum. With due regard for the equilibrium condition we obtain temperature and density profiles. It turned out that the adiabaticity condition is satisfied in the convective zone with an accuracy better than one percent. The accuracy is unbelievable for the theory of turbulence and it is reached because the Lagrangian invariant — specific entropy — is conserved with high accuracy under convection. In an isentropic dry earth's atmosphere the temperature would drop by 10 degrees per kilometre but meteorologists count only 6 degrees per kilometre for the standard atmosphere. Zaslavskiĭ evidences that artillery-men when calculating the distance to their object also assume the gradient of 6 degrees per kilometre in the absence of direct measurements. This deviation from the simplest model is due to vapor condensation and radiative heat transfer, as a result of which the simplified Lagrangian invariant is not conserved with the accuracy we expect. Getting somewhat ahead I say that the accuracy of the results in tokamak is closer to that for the Earth than for the Sun.

Turbulent transport has been declared the main and most complicated problem to be solved on the way to the controlled thermonuclear synthesis and tokamaks have been declared the leading type of installations. There are plenty of reviews on turbulent transport in tokamaks (see, for example, Refs [5–8]) but the topic is not settled. Wagner and Stroth [7] emphasized that neither the type of turbulence, nor the nature of transport, nor the stabilizing elements were known as far as several years ago. Recent experiments, especially transport suppression by reversed magnetic shear [9–11] make it possible to select a better theoretical model. However, this choice is not unique and the opinions of specialists are controversial. In our opinion, advents in theory, and especially in experiment, have verified that the convection of trapped particles is the chief cause of transport as it had been suggested in the very first review on this subject [5]. Large-scale magnetohydrodynamic (MHD) instabilities are usually not coupled with the problem of turbulent transport and are considered separately [12–14].

The traditional scheme of studying the turbulent transport in tokamaks usually starts with the linear analysis of instabilities on the base of equations of motion; then transport coefficients are evaluated by general nonlinear speculations; and finally, the transport equations are solved. This is a good direct program and it could be the best one if it was realized with simple means. Though there have been thousands of works and all the power of computers has been harnessed, a satisfactory accuracy has been attained only in individual cases and new enhanced confinement modes were found basically in experiments (one exception is transport suppression by reversed shear predicted from various ideas [5, 15, 16]).

The aim of this review is to start from invariants (instead of equations of motion) and then to find an attractor and its attraction basin.

All the method of magnetic confinement is based on the invariants of 'freezing-in', pasting of plasma particles into magnetic field lines. If freezing-in is ideal and field lines lie on embedded tori then there is no turbulent transport. Analysis of turbulent transport essentially reduces to an analysis of the breakdown of the frozen-in invariants. Here at the outset, a researcher meets an unpleasant surprise: there are no correct three-dimensional MHD equations for weak-collisional and collisionless plasmas, and hence, the question arises of how we can find invariants? The magnetized plasma provides a very complex multidimensional system. In a tokamak or a slellarator the distribution function depends on at least six distinct variables: small radius, toroidal angle, poloidal angle, time, longitudinal velocity, and transverse velocity. In a six-dimensional strongly anisotropic space many different objects may be placed. Therefore, it is quite natural that there are no simple MHD equations that adequately describe turbulent transport. If we try to write down MHD equations, then it turns out that the pressure is a poor-defined tensor rather than a scalar function and the notion of freezing-in cannot be inserted theoretically although frozen-in effects are found in experiments.

Frozen-in invariants can, however, be conserved even in the absence of MHD equations of motion. The extension of the notion of freezing-in to the Vlasov equation using the Poincare invariant yields a new most important tool of analysis. Along with the assumption of the integrable behaviour of a bunch of plasma particles, this leads to a new important general conclusion that the topology of the magnetic field is conserved in collisionless plasma.

Using the toroidal symmetry of the tokamak we shall show that over transport times the freezing-in of plasma is preserved in the poloidal field and breaks down in the toroidal field as has long been known for trapped particles [5], and thus that convection becomes possible. If freezing-in is conserved in the poloidal field, then the quantity nr/B_p is a Lagrangian invariant and leads to the attractor $n \propto 1/q$. This canonical density profile explains the turbulent pinching paradox and is close to that observed in supershots and with the L-mode. If we adopt an unfounded, but natural polytrope distribution of temperature, then the attractor profiles (density, temperature, and magnetic field) in the ohmic regime are uniquely defined and very close to those of experiment.

Turbulent equipartitions are fully described by invariants; and without any exaggeration, invariants show up as rapid rails through the swamp of variables. The knowledge of the invariants of turbulent attractors makes it possible to predict under which conditions turbulent transport is suppressed. If the atmosphere is heated from above or if the force of gravity is reversed, then there are no reasons for convection. In tokamaks, the negative sign (i.e. the inversion) of magnetic shear means the contraction of plasma in its motion towards the outside, but this contraction is unfavourable in terms of energy and suppresses convection. This 30-year old conclusion has been recently confirmed by experiments [9–11].

The toroidal component of the law of freezing-in is broken down by trapped particles. If plasma rotates in the poloidal direction and trapped ions are absent, then there is no cause for freezing-in to break down and, as a consequence, turbulent transport is suppressed. This prediction of the H-mode has been also made 30 years ago [17, 18] but for some reasons it was not discussed after the experimental discovery of the pattern [19].

Thus, the analysis of the breakdown of one invariant makes it possible to explain three totally different regimes of confinement in tokamaks corresponding to an attractor and two boundaries of its attraction basin. This economy of means is usually an indication of a correct choice although it cannot be considered a proof.

2. Turbulent equipartition

The evolution of nonlinear systems with a large number of degrees of freedom depends on the shape of the hypersurface in the phase space. At the same time the latter shape is defined by the laws of conservation. If the hypersurface is open and its area is concentrated at infinity, then the evolution of the system is normally described in terms of energy fluxes and other invariants after the Kolmogorov model of turbulence [20]. This case is known as mostly widespread but it fails to be the only one possible. If the hypersurface is closed then the fluxes cannot flow anywhere and there appears the natural attraction to the statistical attractor, or an equipartition of integrals of motion on the hypersurface. Quantities that are constant along trajectories (Lagrangian invariants) give a principal tool for the study of equipartitions. In this section I shall deduce the diffusion equation describing relaxation to spatially inhomogeneous TEPs. The difference between this diffusion equation and the one conventionally used for tokamaks is that it includes some additional fluxes.

2.1 Salt in dough

Humanity had its first experience with turbulent equipartition thousands of years ago. It is sufficient to knead a salted dough long enough to get a uniformly salted dough, although the stirring intensity may be nonuniform in space. Ancient people knew this although they did not know the causes of equipartition. Today we can formulate these causes explicitly.

1. The total amount of salt is conserved.

2. The motion is incompressible, namely $\operatorname{div} \mathbf{v} = 0$. It follows from these two suppositions that the salt concentration c is conserved along trajectories, i.e. the Lagrangian invariant

$$\frac{\mathrm{d}c}{\mathrm{d}t} = 0 \tag{1}$$

is involved. It should be emphasized that the Lagrangian invariant carries more information than the integral one because it is conserved along an infinite number of trajectories. The integral invariant (the total amount of salt) follows from the Lagrangian invariant but not vice versa. When the dough is kneaded, molecular diffusion smooths out sharp salt density oscillations. If the stirring and diffusion combined have once made $c = \text{const}$, then the density cannot be changed by any incompressing stirring, i.e. the solution $c = \text{const}$ is an attractor. These considerations make it possible to write out the diffusion equation governing relaxation to the equipartition in the local approximation. (The equipartition itself does not require the local approximation to be true.)

But the matter does not disappear, and hence

$$\frac{\partial c}{\partial t} + \operatorname{div} \mathbf{q} = 0 .$$

The $\mathbf{q}$ flux must vanish on equipartition, so that assuming also isotropy we arrive at

$$\mathbf{q} = -D\nabla c .$$

Finally one obtains

$$\frac{\partial c}{\partial t} = \operatorname{div} D \cdot \nabla c . \tag{2}$$

The diffusion coefficient D is determined by the turbulence features but the equation structure is dictated by the Lagrangian invariant.

In nature, though, not only incompressible motions occur. Atmospheric turbulence yields the next simple, but important example provided that the turbulent pulsations are slow in comparison to the speed of sound. Air expands on elevating and it contracts on being sunk. Therefore, air density is no longer a Lagrangian invariant. Temperature also experiences adiabatic changes, so that mixing does not produce an isothermic atmosphere. The specific entropy s represents, however, a Lagrangian invariant and it satisfies both the exact transport equation $\mathrm{d}s/\mathrm{d}t = 0$ and the averaged

transport equation

$$\frac{\partial s}{\partial t} = \operatorname{div} D \cdot \nabla s \,.$$

The relaxed equilibrium state corresponds to the isentropic atmosphere neutrally stable to convection. The value of this example is that when compressibility comes into play the turbulent heat flux vanishes by no means on the background of the isothermic distribution and the system does not relax to it. Consideration of nondiagonal fluxes is insufficient. This fact is usually ignored in discussions on heat and particle fluxes in tokamaks but it has been long known in atmospheric physics (see, for example, Ref. [21]).

For an isentropic profile to establish it is essential that the atmosphere is heated from below, though instability develops only when the temperature gradient exceeds the critical value. When the atmosphere is heated from above, there is no cause for instability and there will be no TEPs unless an additional source of turbulence such as wind appears.

2.2 Structure of the turbulent transport matrix and the Onsager symmetry

If a system is multidimensional and anisotropic, then a tensor replaces the scalar diffusion coefficient, its structure depending on Lagrangian invariants. In this section we shall show that turbulence not only breaks the Onsager symmetry but also brings about fluxes in the absence of the gradients of thermodynamic variables.

If turbulent transport is ignored, then the major contribution to transport processes is made by Coulomb collisions, and neoclassical coefficients [22, 23] feature Onsager symmetry [24], i.e. the fluxes of particles, heat, charge, toroidal momentum and other quantities are proportional to the gradients of thermodynamic variables:

$$q_i = a_{ik} \nabla \varphi_k \,, \tag{3}$$

where the factors a_{ik} are either symmetric or antisymmetric [24]. The turbulent contribution is usually allowed for by adding a term to the transport matrix:

$$q_i = (a_{ik} + T_{ik}) \nabla \varphi_k \,. \tag{4}$$

There exist a dozen papers in which turbulent transport in tokamaks features a symmetry, and there are a dozen papers in which the authors show that the symmetry is absent (see discussions in Ref. [25]). We shall show that symmetry can present for simplified models of turbulence but that in general the symmetry is broken, and moreover, the matrix equation (4) itself does not hold because turbulence brings about fluxes in the absence of gradients of thermodynamic variables [16]. This conclusion is the main result of the present section.

The availability or absence of the symmetry is not specific for turbulence in tokamaks: this pertains to a general problem of turbulent motion. Therefore, we shall try to consider it using a minimal number of assumptions. The major limitation results from the phase flow incompressibility as it follows from the Hamiltonian behaviour of the system (Liouville's theorem). First we shall see how the one-dimensional transport equation appears. Let the incompressible rearrangements be accomplished in the two-dimensional plane, i.e.

$$\frac{\mathrm{d} f(x, y)}{\mathrm{d} t} = 0 \,,$$

and the mean distribution function f_0 depends solely on x. Then for the flux q one obtains

$$q_x = \langle \delta f \, \delta x \rangle = \frac{\partial f_0}{\partial x} \langle \delta x^2 \rangle \,,$$

whence

$$\frac{\partial f_0}{\partial t} = \frac{\partial q}{\partial x} = \frac{\partial}{\partial x} D_{xx} \frac{\partial f_0}{\partial x} \,.$$

This is true, for example, for quasi-linear particle diffusion conditioned by waves [26]. Notice that the quasi-linear diffusion coefficient is uniquely determined from the condition of energy conservation

$$\gamma W = D_{xx} \frac{m v^2}{2} \frac{\partial f}{\partial v} \,,$$

where γ is the Landau damping factor, and W is the wave energy density.

If diffusion is essentially two-dimensional and turbulence is anisotropic but not gyrotropic, then there occur fundamental rearrangements like those in Fig. 1a, from which a general incompressible rearrangement can be constructed. Here one has

$$q_x = \langle \delta f \, \delta x \rangle = \frac{\partial f_0}{\partial x} \langle \delta x^2 \rangle + \frac{\partial f_0}{\partial y} \langle \delta x \delta y \rangle \,,$$

$$q_y = \langle \delta f \, \delta y \rangle = \frac{\partial f_0}{\partial x} \langle \delta x \delta y \rangle + \frac{\partial f_0}{\partial y} \langle \delta y^2 \rangle \,.$$

We see that in this special case the transport matrix is symmetric and can be brought to a diagonal form by rotation.

In the special case of two-dimensional gyrotropic and yet isotropic turbulence the fundamental rearrangement looks like a local rotation by a small angle and it is presented in Fig. 1b. Similar evaluations show that in this case the transport matrix is antisymmetric. In a more general case the sum of the symmetric and antisymmetric matrices has no symmetry. In the case of the original Onsager symmetry, the

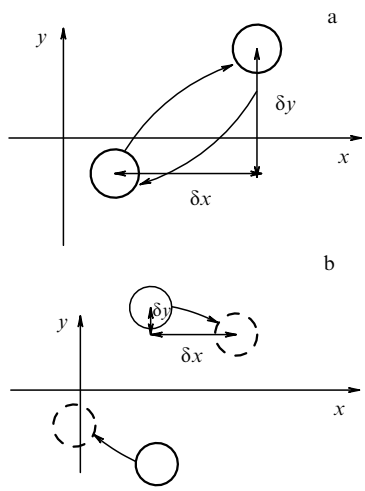

Figure 1. (a) An incompressible rearrangement typical for anysotropic and yet nongyrotropic turbulence; (b) an incompressible rearrangement typical for isotropic and yet gyrotropic turbulence.

change of the magnetic field sign is provided for; analogous suggestions can be made for turbulence. The resultant relationships appear not very useful for experiment. The real situation, however, is even worse. The reason is that in the multidimensional case (the Vlasov equation is governed in a six-dimensional phase space), the two-dimensional rearrangements can be compressible because of a compensating compression in other directions. As a result, fluxes become possible at zero gradients of the thermodynamic variables and can even be directed against the gradients. This does not seem contrary to the basic principles of thermodynamics (see Ref. [27]). The reader should remember that the matrix elements can depend on gradients of the thermodynamic variables quadratically or otherwise.

Thus, turbulent fluxes in a tokamak for a half dozen variables (density, electron and ion heat, toroidal rotation velocity, poloidal magnetic field, radial electric field, etc.) generally depend on the full matrix plus terms in the absence of gradients, i.e. on several tens of independent factors, which are impossible to be calculated. Fortunately, an attractor can be examined without knowledge of these coefficients.

2.3 Marginal stability and proximity to an attractor

Potentially, marginal stability yields more information than TEP but the latter approach is simpler.

The marginal stability principle supposes that the fluxes responsible for the deviation from equilibrium are so small that even a weak instability would suffice to reach a new equilibrium. Instability appears when the gradients exceed some critical values (critical gradients can be found from the condition that instability increments are zero). In the simplest case for the Sun, the marginal instability causes the specific entropy to be constant as well as TEP does. Proximity to the attractor is evident as the energy flux is small in comparison with the gas-dynamic flux (the product of the thermal energy density into the speed of sound).

For a tokamak the marginal stability principle was prescribed in Ref. [28]. It has still not been realized because of the complexity of the linear stability problem, though fast-evolving numerical methods are much promising [29–32]. Kadomtsev and Pogutse derived a confinement time close to the Bohm estimate due to trapped particles [5] when gradients exceed their critical values by a factor of the order of unity. The heat fluxes due to heating in experiments are much less than the Bohm fluxes and this is sometimes erroneously considered as an argument against the Kadomtsev–Pogutse theory. In fact this only means that gradients are close to the critical ones. That the observed static transport coefficients are small in comparison to the dynamic ones is a more evident indication of proximity to the attractor. The static coefficients can even coincide with the dynamic ones but in any case the gradients should be measured from the critical gradients.

The attractor that coincides with marginal stability was considered by Pastukhov for the transient layer in a magnetoelectrostatic trap [33].

The well-known plateau appearing in the one-dimensional quasi-linear relaxation of particles on waves yields another example of TEP. Thus, turbulent equipartition is closely related to marginal stability and increment vanishes when equilibrium relaxation cannot proceed any further. In the simplest cases TEP and marginal stability yield the same results but they are different when we allow for decay and excitation of waves by collisions and by nonlinear beats. Analysis of TEP, however, is simpler because it suffices to know the invariants of motion. For this reason throughout the paper we shall apply the TEP method and turn to marginal stability only if necessary.

2.4 A simple two-dimensional TEP model in planar geometry and in a z-pinch

Even the simplest two-dimensional model, in which the magnetic field has one component, reveals important features of TEP: particle and heat pinching, and its relationship with frozen-in invariants. A similar model is feasible also in z-pinches.

We shall consider below four examples of TEP of increasing complexity.

The simplest example is a home refrigerator. It produces heat fluxes in the absence of an initial temperature gradient. As with turbulence, the refrigerator is active and consequently such fluxes are not prohibited.

The refrigerator example may seem artificial. We remember therefore the adiabatic agitation of an initially isothermic atmosphere in thermodynamically equilibrium state from the Introduction. The raised volumes of air expand and cool; the lowered volumes contract and heat up. Hence, a downward heat flux arises. Certainly, an isothermal atmosphere is convectively stable and requires an external turbulent source, for example, a horizontal wind. The equation for the heat flux can easily be derived from the condition that the flux vanishes in an isentropic atmosphere $s = \text{const}$:

$$q = D\,\frac{\partial s}{\partial z}\,.$$

The coefficient D depends on a turbulent behaviour. Of course, we do not pretend that this process is significant for the atmosphere but it is rather clear and physical. In tokamaks a similar process would be referred to as a thermal pinch.

The two-dimensional idealized model of magnetic confinement is of fundamental interest. We assume the magnetic field to be given and that it possesses only a z-component and depends on two other coordinates, i.e. $B = B_z(x, y)$. We also assume that the plasma is described by an ideal single-fluid MHD. This implies that the Lagrangian invariant n/B is frozen-in. If plasma is mixed by drift flows, then the spatially inhomogeneous TEP naturally appears for the equipartition of the invariant:

$$n(x, y) \propto B(x, y)\,.$$

The simplicity, with which we obtained this answer, should not be deluding. Pinching in this mechanism is ignored in almost all papers on turbulent transport. The numerical verification of the attraction to this attractor was performed in Ref. [34] in the context of MHD approach and the result follows the analytical theory with a high accuracy. A similar result was obtained through simulating the Vlasov equation [35]. Figure 2 shows the steady-state density profile and the model magnetic field profile (turbulence was simulated by a random set of harmonics). Obviously, the profiles coincide though the hydrodynamic frozen-in invariant, from which the analytical result $n/B = \text{const}$ was obtained, is not applicable. The answer is in the extension of frozen-in effect to the Vlasov equation (see the next section).

Equipartitions were also considered in z-pinches. First Kadomtsev [36] obtained the marginal stability criterion in

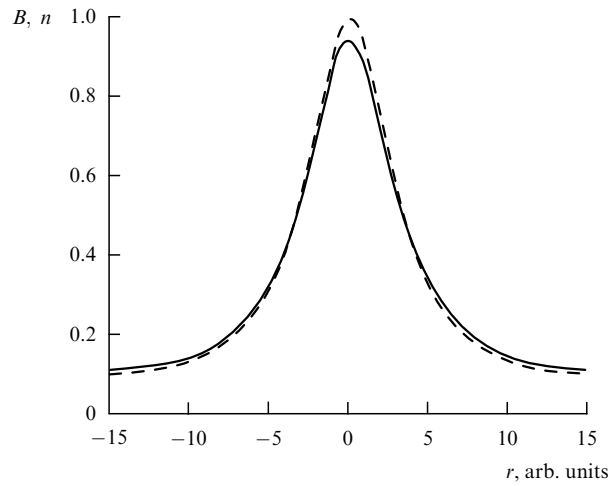

Figure 2. Magnetic field (dashed line) and density (solid line) profiles obtained from the numerical simulation of kinetics in Ref. [35].

the form

$$\frac{rp^{3/5}}{B} = \text{const} \tag{5}$$

for the interchange instability of an axially symmetric z-pinch in the context of an ideal MHD model.

This criterion has a clear physical meaning. The thermal pressure per unit magnetic flux produces an instability if it is larger near the pinch axis. This is in full analogy with convection in the atmosphere.

Then Sasorov [37] supposed that a region in which invariant (5) has approximately the same value is formed near the axis when a sausage-type instability develops as the pinch wrenches inside out. Experimental data on plasma foci agree rather well with the hypothesis. In fact, this is the TEP method and it raises the following question. In contrast to the atmospheric case, the pinch has two independent Lagrangian invariants: the frozen-in function (5) and the specific entropy. Why do we ignore the entropy? The answer is clear: marginal stability and MHD are sensitive only to a single invariant (5); therefore, the trend to equipartition will manifest itself more strongly. In other words, there are passive invariants besides the active invariants as, for example, salt in dough. The absence of TEP for specific entropy does not entail MHD instabilities but if the turbulence is intensive, then entropy TEP can also arise. In fact, MHD equations have a third Lagrangian invariant which is the particle density per unit magnetic flux nr/B depending on the two other invariants. Therefore the turbulent diffusive flux, which for simplicity we write out in the pinch corona where the magnetic field drops down inversely proportional to the radius, has the form

$$q_n = D\,\frac{\partial(nr^2)}{\partial r}\,.$$

Here one unit in the exponent of r^2 appears because of dependence of the magnetic field on radius, and the other does because of cylindrical symmetry. The equipartition is reached on a strongly inhomogeneous profile; in tokamaks this effect is referred to as a particle pinching. However, there is no marginal instability for specific entropy and density but in the context of the model adopted they play a role similar to salt in dough. These invariants are passive scalars.

Equipartitions based on hydrodynamic Lagrangian invariants are relatively simple. They are close to the familiar MHD theories of relaxed states by Taylor [38] and Kadomtsev [39] with a lower dimensionality of invariant surface and to the less known and more complex equipartitions for nonintegrable nonlinear wave equations with a higher dimensionality. In the last case the statistical attractors are solitary waves (solitons) [40, 41].

The examples we have considered in this section were studied earlier in MHD in connection with plasma foci [37] and ion diodes [42], and also with the problem of the Onsager symmetry and pinching in tokamaks [43]. These simple examples show that Onsager symmetry is broken by turbulence [25, 44, 45], and moreover, the equilibrium itself should be reconsidered. Equilibria, or in other words attractors are described by Lagrangian invariants.

3. Frozen-in integrals in collisionless plasma

The ordinary law of freezing-in is not applicable to collisionless plasma because thermal pressure forces have a tensor nature. The extension of the frozen-in notion to the Vlasov equation shows that the magnetic field topology can be conserved even if thermal forces are present, but in general plasma is no more frozen in this field. In a toroidally symmetric tokamak the plasma is frozen in the poloidal field: $nr/B_{\rm p} = \text{const}$.

3.1 The frozen-in notion and Poincare invariants

The fundamental cause, for which the law of freezing-in appears in plasma, is that forces are potential. This law is related to the conservation of vorticity in an ideal fluid, to the canonical form of Hamilton's equations for a fluid particle, and to Poincare's invariant. Poincare's invariant helps to generalize the law of freezing-in.

In nature, forces are usually potential, and as a result, angular momentum is conserved. In condensed media the potential nature of forces manifests itself in the conservation of vorticity in an ideal fluid, in freezing of the magnetic field in a plasma, and in generalized vorticity. The stability of a whipping top and magnetic confinement have the same nature. Unfortunately, this question is barely touched in most textbooks and magnetic field freezing-in is presented as a special property of the equations of motion. In the differential formulation the freezing-in means that the evolution of a magnetic field $\mathbf{B}$ is governed by the equation

$$\mathbf{B}_{\rm t} = \boldsymbol{\nabla} \times \left[\mathbf{v} \times \mathbf{B}\right], \tag{6}$$

where $\mathbf{v}$ is the plasma velocity. In the equivalent integral formulation the magnetic field flux $\mathbf{B}$ through any closed contour is conserved if the contour travels with the plasma velocity $\mathbf{v}$.

Equation (6) can be deduced by taking the rotor of the hydrodynamic equation of the electron motion, and by setting zero current velocity and zero electron mass. In the mean time, even if the mass is finite, the equation of motion of each plasma component can be represented in the form of conservation of the generalized vorticity $\boldsymbol{\Omega}$:

$$\boldsymbol{\Omega}_{\rm t} = \boldsymbol{\nabla} \times \left[\mathbf{v} \times \boldsymbol{\Omega}\right], \tag{7}$$

where $\boldsymbol{\Omega} = \boldsymbol{\nabla} \times \mathbf{p}$, and $\mathbf{p} = m\mathbf{v} + e\mathbf{A}/c$ is the generalized momentum of a particle. If the mechanical component

dominates in the generalized momentum, then we arrive at the Kelvin theorem on circulation in an ideal fluid, known since last century; if the electromagnetic component dominates, then particles are frozen into the magnetic field. On introducing the finite electron mass, the integral of motion $\mathbf{\Omega}$ does not disappear, but its definition is slightly changed. Consequently, there is neither plasma leakage, nor collisionless reconnection. Studies in this field frequently overlook the conservation of generalized vorticity.

The conservation of generalized vorticity was introduced by Dirac, and in plasma, by Sudan [46, 47]. In mathematical terms the relationship between vorticity and the Hamiltonian is set forth in monographs [48, 49].

This new notation helps to clarify the fundamental reason for which the notion of freezing-in is introduced. It is not mere chance that the generalized vorticity is conserved: the reason should be sought in the canonical form of the Hamiltonian equation for a fluid particle. In fact, the hydrodynamic equations of motion for pressure depending solely on density can be derived from the Hamiltonian

$$H = P(\mathbf{q}) + e\phi(\mathbf{q}) + \frac{(\mathbf{p} - e\mathbf{A}/c)^2}{2m}\,, \tag{8}$$

where $P(\mathbf{q})$ is the normalized pressure, $\phi(\mathbf{q})$ and $\mathbf{A}(\mathbf{q})$ are the electrostatic and vector potentials. The equations have a canonical form

$$\dot{\mathbf{p}} = -\frac{\delta H}{\delta \mathbf{q}}\,, \qquad \dot{\mathbf{q}} = \frac{\delta H}{\delta \mathbf{p}}\,. \tag{9}$$

It follows then from (9) that the relative integral Poincare invariant

$$I = \oint \mathbf{p}\,\mathrm{d}\mathbf{q} \tag{10}$$

is conserved if the integration contour is carried by the phase flow. (This exact invariant should not be confused with the approximate adiabatic invariant, which is derived from the Poincare invariant having the same form. They differ in integration contour: in the case of the adiabatic invariant the integral is taken over the periodic trajectory of a particle. The periodic trajectory coincides only approximately with the contour that the phase flow carries.) Since in hydrodynamics the generalized momentum is a function of coordinates and time, the six-dimensional phase space can be projected onto ordinary three-dimensional space. Then the contour integral can be transformed into the flux $\mathbf{\nabla} \times \mathbf{p}$ through the surface stretched over the contour. Thus, we arrive at the integral formulation of the freezing-in for the quantity

$$\mathbf{\Omega} = \mathbf{\nabla} \times \mathbf{p}\,.$$

The electric field is not potential because the magnetic field is variable. Therefore, it may seem that the momentum circulation should not be conserved. The answer is that the generalized momentum change due to this mechanism is hidden in its magnetic part and the canonical form of the Hamilton equation is not disturbed.

Analogously, the law of freezing-in can also be introduced into the general theory of relativity when the magnetic field acquires a weight and space is curved. The law of freezing-in can also be introduced for Yang–Mills plasma.

3.2 Lagrangian invariants and the frozen-in property for the Vlasov drift equation

A Lagrangian invariant is introduced for the Vlasov drift equation, based on the relative integral Poincare invariant and also on the transverse and longitudinal adiabatic invariants. This section includes no calculations but some efforts are required to generalize the frozen-in notion introduced in hydrodynamics to kinetics. These efforts are necessary to gain a complete understanding of the frozen-in peculiarities in collisionless plasma.

The hydrodynamic frozen-in condition is justified purely for the motion of plasma with velocities being of the order of a thermal drift velocity of particles but it can be extended to the Vlasov equation. The Poincare invariant defined in six-dimensional phase space can also be considered in the space of the Vlasov equation, the difference being that the fields in the Vlasov equation are known. It is right to say that the Vlasov equation for electrons governs the sum of an infinite number of electron hydrodynamics with a zero thermal pressure, in each of which its own invariant (10) is conserved. The initial integration contour is arbitrary and the result depends on what choice we make. The value of an integral is its ability to confine the motion. If the contour remains on a two-dimensional surface, then the number of particles inside the contour would be preserved because (a) the line divides a surface, and (b) the line sticks to particles.

A line does not divide the six-dimensional space but the dimensionality can be lowered to two by introducing adiabatic invariants natural for magnetized plasma. The transverse adiabatic invariant, or the magnetic moment

$$\mu = \frac{v_\perp^2}{B} \tag{11}$$

is especially simple and its conservation lowers the dimensionality by two dimensions at once. If we now consider the quasi-periodical motion of the particle between magnetic mirrors and introduce conservation of the longitudinal adiabatic invariant

$$J = \oint v_\parallel\,\mathrm{d}l\,, \tag{12}$$

then the dimensionality of space lowers by yet another two units and the centres of banana orbits move onto a two-dimensional hypersurface and this fact alone imposes severe restrictions. Inside the contour not only the flux Bs but also the number of particles $n_{\mu,J}\,s$ is conserved, and thus, the first quantity can be divided by the second and the integration contour can be 'eliminated'. Since we have already passed to the drift approximation, only the magnetic momentum component should be retained in the Poincare invariant and the contour integral takes the remarkably simple form of a Lagrangian invariant. The quantity

$$L = \frac{B_\perp}{n_{\mu,J}} \tag{13}$$

is conserved along trajectories. Here the magnetic field component $B_\perp$ is perpendicular to the surface of the centres of banana drift orbits. This Lagrangian invariant extends the frozen-in notion to the Vlasov equation and is very convenient to analyze the structure of the turbulent transport equations. This is the principal tool we use in this paper. We

recall that invariant (13) and both adiabatic invariants are obtained from a single Poincare invariant: the difference is in integration contours.

We shall consider now a simple example where the magnetic field has only a z-component. In this case we can take advantage of the conservation of the transverse adiabatic invariant

$$\mu = \frac{v_\perp^2}{B} ,$$

and now the Vlasov drift equation preserves the Poincare invariant (10) and the Lagrangian invariant n_μ/B, which is an analogue of (13). The relevant TEP

$$n_\mu(x, y) \propto B(x, y)$$

differs from the gas-dynamic one only in that the same TEP is valid on each hypersurface $\mu = \text{const}$. Consequently, averaging over different μ is equivalent to omitting the index.

This result may be derived less rigorously, but more clearly. A cold particle experiences only an electrostatic drift along equipotential lines. Since the drift velocity is proportional to B^{-1}, the particle spends more time in regions where the magnetic field strength is higher. A hot particle experiences an additional drift along the lines of constant magnetic field but the density of the above TEP is constant on these lines and therefore is not perturbed.

Thus, in this special case the hydrodynamic and Vlasov TEPs fully coincide. The reason is that the Vlasov TEP is independent of μ. This example explains why in numerical simulation of kinetics in Ref. [35] the authors obtained the same turbulent equipartition as in numerical simulation of hydrodynamics in Ref. [34]. Note one important and attractive property: the density distribution follows the collisionless TEP even if the transverse adiabatic invariant is broken by collisions. The reason is that collisions do not change the magnetic component of the momentum and only the fact that the large-scale Poincare invariant is conserved proves to be principal. Two-dimensional TEPs are considered in more detail in Ref. [50].

In the general three-dimensional case the hydrodynamic and Vlasov TEPs are not, broadly speaking, the same.

3.3 Semiideal MHD and Ing-Yan particles

The notion of freezing-in is used in collisionless plasma in general and in tokamaks in particular without sufficient reason since the nonpotential component of the thermal pressure forces can break the magnetic field topology over transport times. In this section we shall show that particles with integrable behaviour provide an excellent longitudinal plasma conductivity, and hence, conservation of the magnetic topology, while particles with nonintegrable behaviour make the main contribution to transverse leakage.

In the first tokamaks the temperature was low and classic transport was less than turbulent transport, though not negligible. As the plasma conductivity increased with the electron temperature, a paradox manifested itself increasingly more: the magnetic field diffusion became less than the particle diffusion, although in MHD only one process — electron-ion friction — is responsible for the two diffusions. Attempts to explain it by magnetic surface splitting predict a more extensive diffusion of fast electrons but it seems that this prediction contradicts the experiment.

This paradox has a fundamental significance and subtle physical causes but it was not discussed until recently. Boozer formulated it explicitly in Ref. [51]: "It is well known that in laboratory plasma the by-pass voltage depends on the Spitzer conductivity though the transverse transport can be greater by a factor of 10^3 over the classic resistance prediction." Boozer has also shown that if the longitudinal conductivity is infinite then the magnetic topology is preserved and the magnetic field obeys the equation

$$\mathbf{B}_t = \nabla \times (\mathbf{u} \times \mathbf{B}) , \tag{14}$$

where $\mathbf{u}$ is the velocity of an abstract frozen-in preserver, differing from the plasma velocity $\mathbf{v}$. The correlation between the magnetic field topology conservation and the longitudinal conductivity was also noted in Ref. [52] and Eqn (14) was derived in Ref. [53] from the Poincare invariant for the collisionless Vlasov equation.

Boozer set forth two possible reasons why longitudinal and transverse conductivities differ so much. The first reason is that perturbations in magnetized plasma are very elongated along the magnetic field, $k_\perp \gg k_\parallel$. If $k_\perp/k_\parallel \simeq 10^3$, then the main momentum transfer to particles occurs transversely and a larger transverse transport can be explained, although very small-scale perturbations should be considered.

The second reason is more subtle and fundamental: "The longitudinal conductivity is close to the classical conductivity if a fraction of passing particles has a low dissipation." This cause acts even for $k_\perp \simeq k_\parallel$. The same proposition was made in Refs [16, 53] from an analysis of the Poincare invariant and frozen-in presence in the Vlasov equation.

The coexistence of regions in phase space with integrable and nonintegrable behaviour (nonresonance and resonance particles in linear theory) is typical for Hamiltonian mechanics [48, 49]. For brevity we shall refer to particles with integrable trajectories as Ing-particles, and to particles with nonintegrable trajectories as Yan-particles. In ancient Chinese philosophy Ing-Yan symbolized opposing forces, female and male, passive and active. This is in accord with the meanings of the two types of behaviour of particles in plasma, moreover Ing sounds like integrable and as it can be easily remembered.

The methods by which Ing- and Yan-particles are described totally differ. A long time ago Ptolemaeus took advantage of the integrable motion of the planets in his theory of epicycles. In this case the consequence of integrability is that trajectories discretely expand into Fourier series. At the same time weak planet interactions and nearly circular orbits result in rapid convergence and the successful application of epicycles. This method can also be applied to describe Ing-particles in tokamaks.

To describe Yan-particles, diffusion and TEP are employed. If we want to distinguish Ing-particles from Yan-particles, we must consider not only the linear Landau resonance but also all nonlinearities, for which convective cells, magnetic field imperfection, etc. are responsible. The implications of the coexistence of Ing- and Yan-particles are very important for collisionless plasma in tokamaks. Like two-component superfluid helium and electrons in superconductors, electrons in plasma can be considered to be comprised of Ing-particles, or the frozen-in preservers [51, 53] and Yan-particles responsible for transverse transport.

Integrability is more easily broken for trapped particles and it seems that they comprise the larger fraction of Yan-

particles and make the major contribution to plasma leakage. Passing Ing-particles provide excellent longitudinal conductivity. Certainly, their fraction cannot be too small but they must be able to carry all the current almost without resistance. It follows from the proximity of the experimental longitudinal conductivity to that of Spitzer that the fraction of Ing-particles cannot be much less than half, and this result seems quite natural from the theoretical standpoint. Wave and vortex dissipations are closely related to nonintegrability because the continuous spectrum corresponds to Yan-particles and any perturbations exhibit their own resonances.

The Poincare integral contour for Ing-particles is simple as the small mechanical component of momentum can be neglected. As a result the vector potential circulation and magnetic field isofreezing-in are preserved. The Poincare invariant is preserved for Yan-particles as well but in this case the integration contour is exponentially elongated and becomes entangled, and therefore, it is impossible to neglect the mechanical component of momentum and the magnetic field cannot be recovered from the invariant (the contour and particle locations cannot be recovered from the magnetic field either).

MHD with an infinite longitudinal conductivity of the frozen-in preserver which satisfies Eqn (14), was called a 'semiideal MHD' [54]. A semiideal MHD preserves magnetic topology and the poloidal magnetic field is frozen into a toroidal one. The equation of plasma motion is not defined in this case and the evolutionary magnetic field equation also includes an indeterminate velocity. However, valuable information can be extracted even in such a situation. For example, in a tokamak the magnetic field is fairly 'rigid' relative to the thermal pressure. Hence, we can neglect the evolution of the magnetic field and consider the diffusion of particles in a given field.

This interpretation differs somewhat from the original interpretation of Boozer [51] in that the tensor nature of thermal pressure (the direction of the hydrodynamic force is current-independent), rather than finite transverse conductivity, is declared the cause of transverse transport.

Thus, the Ing-Yan dualism, and as a consequence, the semiideal nature of MHD can be considered as an experimental and theoretical fact. As for the nonideal part (plasma motion), little is known about this item. We shall only suppose that plasma is better frozen in a poloidal field than in a toroidal field (see the next section).

3.4 Plasma frozen in a poloidal magnetic field

Due to the action of the nonpotential component of the thermal pressure forces the plasma freezing in the magnetic field is disturbed over transport times. In kinetics terminology, trapped particles are only frozen in a poloidal magnetic field while passing particles do not take part in the transport and they are frozen in both components of the field. This prompts the hypothesis that plasma is more strongly frozen in poloidal than in toroidal fields.

In a weakly-collisional plasma in a tokamak, turbulent perturbations of thermal pressure forces necessarily have a tensor form and disturb the frozen-in property. There is no way to express these perturbations correctly but the frozen-in invariants of passing and trapped particles can be examined. Following Kadomtsev and Pogutse [5] we assume that all the passing particles are Ing-particles and do not take part in transport while a fraction of the trapped particles are Yan-particles. The frozen-in property for trapped particles in a poloidal field was introduced into the Vlasov equation with the Lagrangian invariant

$$L = \frac{n_{\mu,J}}{B_{\mathrm{p}}}\,.$$

Coulomb collisions change only the small mechanical component of the generalized momentum and slightly affect the total momentum. The role of collisions reduces to mixing of trapped and passing particles. The natural hypothesis is that only the poloidal component of the frozen-in law survives as it was preserved by both groups of particles in collisionless plasma. The relevant Lagrangian hydrodynamic invariant, in which the density is defined in ordinary space instead of the hypersurface of adiabatic invariants, takes the form

$$L = \frac{nr}{B_{\mathrm{p}}}\,. \tag{15}$$

The toroidal component is not conserved because physically this corresponds to the well-known nonpotential poloidal mirror forces (they appear when a plasma is rotated in the poloidal direction).

It is yet unclear whether only one direction of positive Ing-Yan duality and natural small parameters of the theory are sufficient for this derivation of invariance. We recall that there are no correct MHD equations by which turbulent transport is described adequately.

4. The attractor in a tokamak and its attraction basin

The turbulent attractor and two boundaries of its attraction basin correspond to the three different confinement modes in tokamaks. It appears that all three modes are observed in experiment. Plasma freezing in the poloidal magnetic field at ordinary positive magnetic shear causes instability, TEP, and canonical profiles to appear. At reversed shear instability and turbulent transport are suppressed and the profiles do not follow TEPs. Elimination of trapped ions by poloidal plasma rotation restores the full law of freezing-in and also suppresses turbulence.

4.1 TEP and the minimal model of canonical profiles for supershots and L-mode

The fact that plasma is frozen in a poloidal magnetic field and the assumption of turbulent equipartition explain the canonical profiles (for the ohmic regime) and the pinching paradox.

It is interesting to note that some interpretations of an experiment, almost equivalent to the observation of an attractor, were made before a theory based on first principles; they are known as canonical, universal, or resilient profiles [55–60]. The particle pinch paradox is closely related to canonical profiles. The pinching phenomenon is that a density profile is formed and sustained with a peak at the centre although the source of particles is at the periphery and the ordinary diffusion is directed outward. Particle pinching can conveniently be described phenomenologically by convective particle transport to the centre with velocity v [58, 61]:

$$Q = -D\,\frac{\partial n}{\partial r} + nv\,. \tag{16}$$

This convective flow does not contradict the basic principles, and moreover, it is natural to inhomogeneous TEPs. The idea

of inhomogeneous TEPs provides an explanation of convection and suggests that the flux should be rewritten in a form in which the inhomogeneous equilibrium $n_0(r)$ is emphasized:

$$Q = -D_1 \frac{\partial(n/n_0)}{\partial r} . \tag{17}$$

Here $n_0(r)$ is a dimensionless presentation of TEP, $D_1 = Dn_0$ and deviations from it result in fluxes. In Ref. [60] it was verified that the quantity D/av, where a is the small tokamak radius, equals approximately 0.5 in various conditions although diffusion and velocity vary over wide ranges. It seems that this situation is typical for all tokamaks otherwise density profiles would be flat or narrow.

The Vlasov equation with a collisional term has too many dimensions and in addition it is anisotropic in order that the causes of abnormal transport in tokamaks should be found for sure from pure theoretical considerations. Any instability enhances transport, therefore experimental data on the confinement times do not make the choice easier. Paradoxical particle pinch is a different matter. It exists in all tokamaks but no simple explanations could be found for a long time. (We do not consider Ware pinch [62], which is small in comparison with turbulent diffusion, as well as other types of neoclassical transport [22, 23].)

In what follows we shall be interested in the equilibrium conditions and $n_0(r)$ but pay hardly any attention to fluxes and D_1. Turbulent equipartition is specified by plasma freezing in poloidal field (15) and it leads to the density profile [63 – 65]

$$n_0 \propto \frac{B_\mathrm{p}}{r} \propto q^{-1} , \tag{18}$$

where q is the safety factor.

This is a principal formula in this review. Therefore we shall deduce it in two more ways. This result can be explained very simply by considering a pure toroidal electric field as in a Galeev – Ware pinch. Trapped particles drift radially with a velocity $v/c = E/B_\mathrm{p}$. Formula (18) follows directly from the fact that the radial flux is constant: $nvr = \mathrm{const}$. Of course, the constant toroidal electric field in the tokamak is weak because the conductivity along the magnetic field is too high. But variable electrostatic field is not prohibited along the magnetic field and it makes the major contribution in mixing. With such a rough estimate it however remains unclear whether collisions should be taken into account as well as the fact that the fraction of trapped particles varies with radius.

Collisionless distributions are frequently the same as collisional ones. The reason is that the Poincare invariant is not lost for one Coulomb collision when the integration contour is wide, because the electromagnetic momentum makes a major contribution. This is especially clear in the following derivation of TEP. As Pastukhov has noted in Ref. [33] TEPs are similar to a quasi-linear plateau [26]. In a tokamak the toroidal direction is invariant. Therefore, we consider a plateau on the generalized toroidal momentum distribution function:

$$f = \frac{\mathrm{d}N}{\mathrm{d}p} = \mathrm{const}, \qquad p = mv + \frac{eA}{c} . \tag{19}$$

By neglecting the small mechanical component of momentum, differentiating (19) in radius and substituting $\mathrm{d}A/\mathrm{d}r = B_\mathrm{p}$, $\mathrm{d}N/\mathrm{d}r = 2\pi rn$ we obtain

$$nr \propto B_\mathrm{p} ,$$

i.e. the familiar already distribution. Here it is quite natural to neglect Coulomb collisions because they vary only the small mechanical component of momentum. Notice however that assumptions based on which plateau was obtained have to be examined carefully. These assumptions are that noninvariant directions can be neglected and that the Liouville theorem is valid in the invariant direction.

In 1980 Coppi put forward a hypothesis that in tokamaks the electron temperature profiles are canonical [55]. TEP yields almost the same profiles as in experiment not only for the temperature, but also for the density and safety factor.

Let us consider plasma profiles in tokamak under the assumption that the density depends on the TEP we have found above:

$$n \propto \frac{B_\theta}{r} \propto q^{-1} , \tag{20}$$

and that the temperature is described by a polytropic curve

$$T \propto n^{\gamma-1} \propto q^{-\gamma+1} ,$$

where γ is still arbitrary. Then the conductivity is also a known function of q:

$$\sigma \propto T^{3/2} \propto q^{(1-\gamma)3/2} .$$

If we assume that all the current in plasma is inductive and that the electric field is constant, i.e. the profiles are steady, then it follows from Ohm's law that

$$j \propto \frac{1}{r}\frac{\mathrm{d}(rB_\theta)}{\mathrm{d}r} = \frac{1}{r}\frac{\mathrm{d}(r^2/q)}{\mathrm{d}r} \propto \sigma E \propto q^{(1-\gamma)3/2} .$$

This first-order ordinary differential equation is readily integrated:

$$q = q_0(1 + r^{2\beta})^{1/\beta} , \qquad \beta = \frac{3}{2}(\gamma - 1) - 1 .$$

The important fact is that this profile is close for any β to the profile $q = q_0(1 + r^2)$ that experimenters use. Hence the specific heat ratio can be selected rather arbitrarily without introducing a large error. If we recall that mainly the longitudinal energy of trapped particles is changed in mixing, then the one-dimensional adiabatic curve for temperature

$$T \propto n^2 \propto q^{-2}$$

can be used. As a result we arrive at the minimal model of canonical profiles [16]:

$$q = q_0(1 + r^4)^{1/2} , \qquad T \propto (1 + r^4)^{-1} , \qquad n \propto (1 + r^4)^{-1/2} , \tag{21}$$

where the radius is normalized in such a way as to obtain the right value of safety factor at the boundary.

The model is minimal in that the simplest assumptions were made whenever required, and thus different tokamaks are not distinguished. For example, in divertor tokamaks the

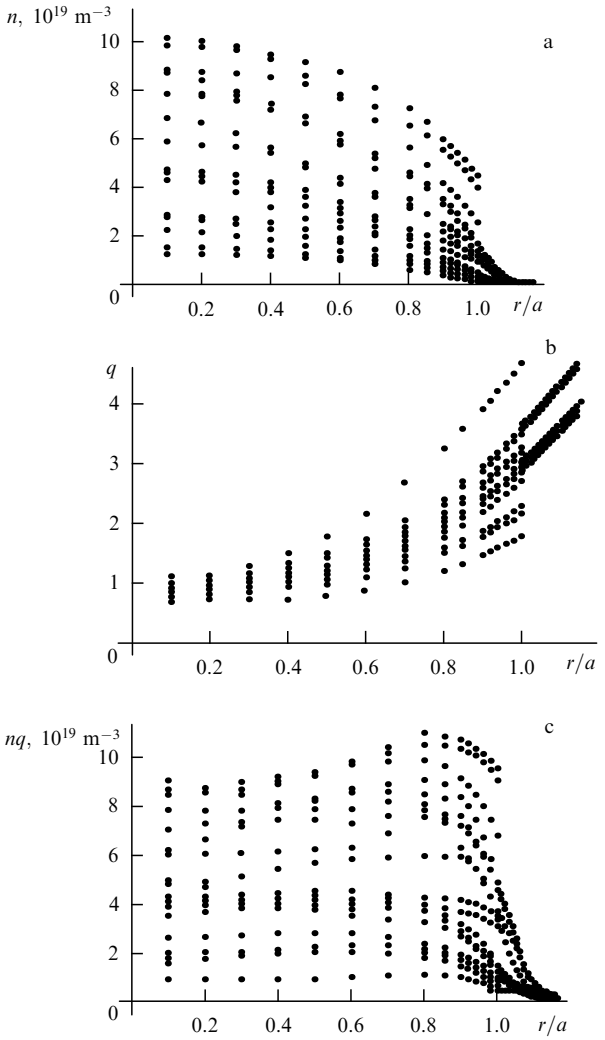

Figure 3. Profiles of the density n, the safety factor q, and the Lagrangian invariant nq in the TEXT tokamak (shots 88127 – 124778) [67].

circular cross-section approximation $q = q_0(1 + r^4)^{1/2}$ is known to be invalid near a separatrix where the magnetic shear is divergent. Formula in (19) can easily be extended to the case when a circular torus has a noncircular cross-section. To this end it suffices to replace the moment A in (19) with the angular moment $M = Ar$. Surfaces on which the angular moment is constant coincide with magnetic surfaces. Introducing the volume bounded by a magnetic surface $V(M)$ we arrive at the formula for the density

$$n \propto \frac{\mathrm{d}M(V)}{\mathrm{d}V} \,. \tag{22}$$

This formula, as well as the formula $n \propto q^{-1}$, is also applicable in divertor tokamaks and in experiment the density and temperature gradients do increase near the separatrix. Moreover, in the JET tokamak it was independently revealed by data adjustment [66] that the density profile depends mainly on the q profile and can be approximated by the law $n \propto q^{-1/2}$.

It had been discovered even earlier that in many tokamaks the pressure and temperature profiles look similar when they are normalized using the boundary value q_a [56, 57]. Moreover, in reviews [56, 57] the peak of $n(0)/\langle n\rangle$ increases with q_a in qualitative accord with (18).

An in-depth comparison with theoretical predictions requires simultaneous measurements of radial profiles of density and poloidal magnetic field. Figure 3 presents results for the TEXT tokamak (my special thanks to Isichenko who helped me to retrieve them from the magnetic confinement database described in Ref. [67]). Plots of density n, q, and nq are shown as functions of a small radius. Obviously, $nq = \text{const}$ with a good accuracy although we have done calculations only for the equilibrium state but thermal fluxes and saw-tooth oscillations must perturb it. I recall that comparison with the attractor is appropriate only for modes which are free of ELMs, MARFE, L – H transitions and MHD activity.

In the TFTR tokamak it was possible to obtain supershots in which there are no saw-tooth oscillations [68, 69]. In this mode nq is constant to a reasonable degree (Fig. 4a and 4b). Theoretical and experimental results coincide similarly in

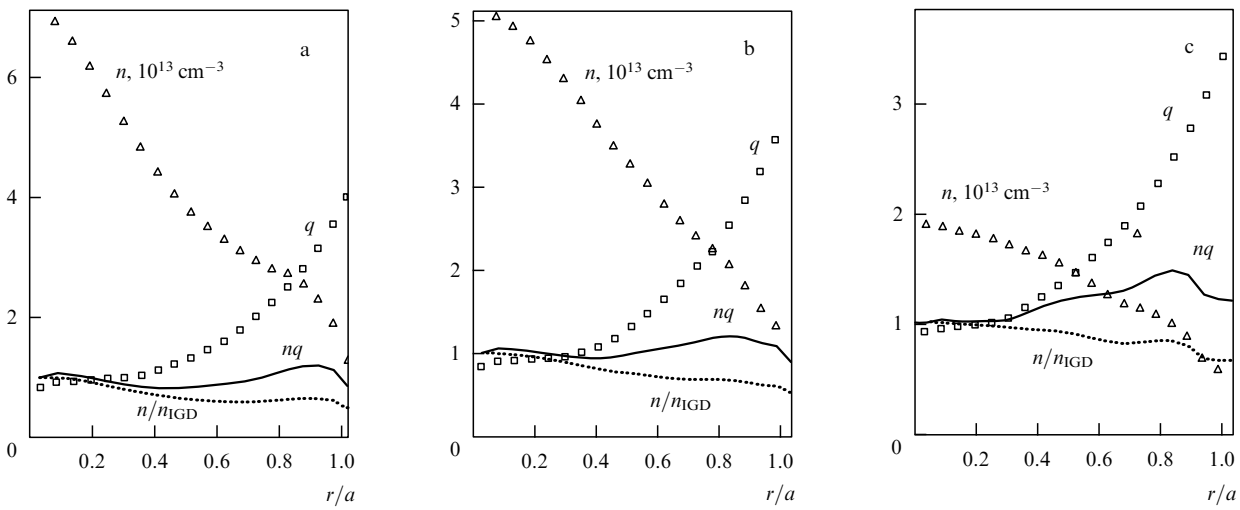

Figure 4. Profiles of the density n, the safety factor q, and the Lagrangian invariant nq in the TFTR tokamak [69]: (a) supershot 76770, temperature peak; (b) the same supershot after swithing-off the heating before crash; (c) L-mode. Here, n_{IGD} is the profile suggested in [77].

other available supershots. In L-mode in the TFTR, the density profiles are usually flattened because of saw-tooth oscillations. Thus it is natural that there appears a deviation from profile (18) (see comments in Ref. [70]). Moreover, even in L-mode the available density profiles are only slightly more flattened than $nq = \text{const}$ curves, i.e. they deviate in the same direction as the theory predicts (Fig. 4c).

Thus, density profile peaking in tokamaks clearly correlates with the peaking of $1/q$. When there exist detailed measurements, peaking follows a quasi-linear plateau $nq = \text{const}$ or is weaker (the last fact is quite natural). Particle pinch is difficult to explain while there occur many flattening factors.

To explain the deviations from canonical profiles the transport matrix should be considered and the analytical theory loses its simplicity. Numerical codes are more convenient for this goal [29 – 32].

Canonical temperature and current profiles (21) agree rather well with experiment though more comprehensive comparisons are needed.

If particle pinch is explained, then thermal pinch [71] presents no problem because heating is inevitable, at least under the adiabatic law. It is interesting to note that the thermal pinch was first discovered and interpreted in the Earth's radiation belts [72]. In heating both longitudinal and transverse invariants are conserved in a dipole magnetic field $B \propto r^{-3}$, and the Sun's energy provides mixing. Hasegawa used this idea in his proposal for a dipole reactor with low temperature and plasma density near the wall [73, 74]. Pressure in a dipole trap drops very sharply as $p \propto r^{-20/3}$. Recently Kadomtsev stated his belief that the physics of radiation belts and the physics of tokamaks are similar [27, 72]. In a tokamak particle pinch is dependent on q and is not so pronounced as in a dipole trap.

The pinch effect can be seen from the ordinary quasi-linear diffusion equation for trapped particles in a tokamak [75, 76]. This analysis was performed in Ref. [77] and showed that particle pinch depends on the q profile. This approach is far from straightforward because the diffusion coefficient is unknown.

The method of invariants does not require comprehensive information on the turbulence modes although large-scale convection preserves plasma freezing in the poloidal field in a more natural way. Short-wave electrostatic perturbations are discussed more frequently because it is easier to measure and study them theoretically. However, some measurements of large-scale convection are also known [78].

It is essential that the theory of canonical profiles is applicable only for a positive magnetic shear. For a negative shear the used law of freezing-in prohibits turbulence and this conclusion is corroborated by experiment as will be shown in the next section.

4.2 Transport suppression by reversed shear

Experiments using a changed heat flux vector or a partially inverted profile of q give new confirmation to the principle that 'plasma is frozen only in a poloidal field'.

If the atmosphere is heated not from below but at some altitude then there is no turbulence source below this altitude. A similar phenomenon was observed in tokamaks in experiments with off-axis electron-cyclotron heating, when transport decreased substantially in the central part without perceptible changes in the profiles [59]. Turbulence suppression is also obtainable not with displacing the heat source in the atmosphere but 'turning over' the gravitational force. In a tokamak this is achieved by a decrease of the safety factor with radius and in fact an impressive suppression of transport processes was observed in recent experiments [9 – 11].

The poloidal magnetic field profile can substantially diverge from the canonical profile over times smaller than the skin time when the fraction of bootstrap current is large or when current is maintained by a noninductive method. Thus, a region in which magnetic shear is negative can be formed. In the very first works on the instability of trapped particles Kadomtsev and Pogutse pointed out that negative shear suppresses this instability [5]. The energy of trapped particles is sensitive to the profile $q(r)$ since $q(r) = rB_\varphi/(RB_\theta)$ defines the distance between the points of reflection. This distance is about qR. Here r and R are small and large radii, respectively. If the shear is negative, $\mathrm{d}q/\mathrm{d}r < 0$, then trapped particles contract along the field when they drift outside and the transverse (μ) and longitudinal (J) adiabatic invariants are conserved. The longitudinal energy therewith increases and the instability is suppressed. This estimate is true only for a typical particle, and in addition, the transverse energy is also perturbed. Thus, the criterion is very approximate.

If we take the advantage of the Lagrangian invariant nq, then the plasma contracts in its motion outward in the negative shear region, it heats and the energy stability criterion is uncommonly simple: $\mathrm{d}q/\mathrm{d}r < 0$ [16], and experiments [9 – 11] follow this criterion. Recall that this criterion is derived from the principle that plasma is frozen in the poloidal field.

Linear stability analysis is more complex and Kadomtsev and Pogutse have pointed out that the condition

$$\frac{\mathrm{d}(\ln q)}{\mathrm{d}(\ln r)} < -\frac{3}{2}$$

leads to stability [5]. A more comprehensive analysis can be made by numerical codes, in which case Kessel et al. showed in Ref. [15] that negative shear stabilized trapped ion modes. Some numerical codes yield other results and there is discord in the analytical conclusions. All these indicate that the problem is rather complex.

Energy analysis can be performed for collisionless plasma in hydrodynamics as well as in kinetics using the Taylor energy principle [79]. The total kinetic energy of particles can be written as

$$E = \int Wf\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{v}\,, \tag{23}$$

where W is the energy per particle, and f is the distribution function. We shall consider the mixing event when two volumes exchange places in phase space. A sufficient condition of stability is that the energy E increases as a result of any such event. Let us assume that the adiabatic invariants μ and J are conserved. Two elements $\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{v}$ must have the same volume because the phase flow is incompressible, and f is a Lagrangian invariant by the Liouville theorem. Stability exists if

$$\left(\frac{\partial W}{\partial r}\right)_{\mu,J}\left(\frac{\partial f}{\partial r}\right)_{\mu,J} < 0\,. \tag{24}$$

The Taylor energy principle [79] says that plasma is stable if in the phase space the density is greater where the energy is at

minimum. In a tokamak it is desirable that the energy minimum was at the centre of the plasma, $(\partial W/\partial r)_{\mu,J} > 0$. According to (24) the instability is totally suppressed if $(\partial f/\partial r)_{\mu,J} < 0$.

The sign of $(\partial W/\partial r)_{\mu,J}$ depends essentially on the sign of the magnetic shear. The dependence of the kinetic energy W of a trapped particle on the small radius r and the first two adiabatic invariants μ and J provides the energy source for turbulence. This dependence is related to the velocity of 'bananas' toroidal drift through the Hamilton equation

$$v_\varphi = \frac{\partial H}{\partial p_\varphi} = -\frac{1}{B_\theta}\left(\frac{\partial(W+\Phi)}{\partial r}\right)_{\mu,J}. \tag{25}$$

Here H is a Hamiltonian, Φ is the electrostatic potential averaged over the banana orbit, and p_φ is the toroidal canonical moment. The poloidal field takes the form

$$B_\theta = -\frac{\partial A_\varphi}{\partial r} = -\frac{\partial p_\varphi}{\partial r},$$

where we have neglected the mechanical component of the moment. The toroidal drift has been calculated in Ref. [5] whence it follows that

$$\left(\frac{\partial W}{\partial r}\right)_{\mu,J} = \frac{\mu B_0}{R}\left\{2\left[1+\frac{2rq'(r)}{q}\right] \times \left[\cos^2\frac{\alpha}{2} - \frac{E(\sin(\alpha/2))}{K(\sin(\alpha/2))}\right] - \cos\alpha\right\}. \tag{26}$$

Here α is the poloidal angle of the point of reflection, and E and K are complete elliptic integrals. The sign of the energy change depends essentially on the sign of the shear $\mathrm{d}q/\mathrm{d}r$.

It is very instructive to consider the energy W as a function of radius at fixed μ and J. Figure 5 presents the same dependence for the shot 23100 in JET [80], for which the shear was negative inside the region of $r = 0.5$. It is seen that $(\partial W/\partial r)_{\mu,J} < 0$ in the region of the positive shear for practically all trapped particles, and $(\partial W/\partial r)_{\mu,J} > 0$ for almost all particles in the region of the negative shear. Not all questions are solved by majority but this energy disadvantage agrees well with the observed confinement improvement in the negative shear region. Collisions of particles introduce some averaging over the distribution function but it is difficult to calculate. In tokamaks collisions are not frequent enough for perturbations to be considered locally Maxwellian, and not rare enough to be entirely neglected (especially for electrons).

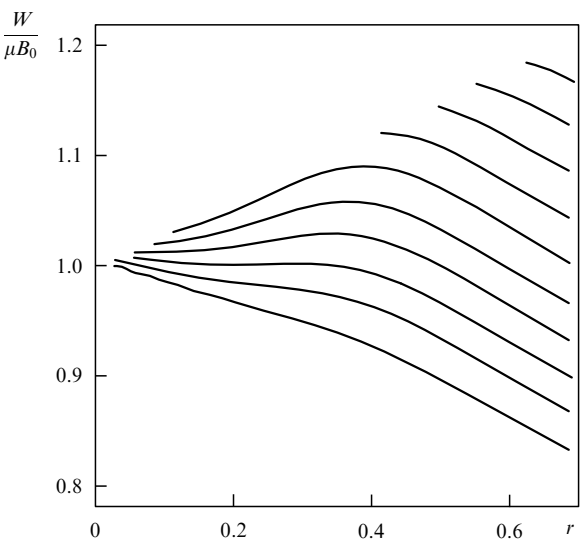

Figure 5. Radial dependence of the energy $W(r)$ of trapped particles at fixed adiabatic invariants.

Although there are no strict results, the simple hydrodynamic and collisionless kinetic models give a clear indication of stabilization by negative shear. Recall that the conservation of the invariant in the hydrodynamic model follows also from the accord between the TEP theory and the canonical profiles for a positive shear.

In modern experiments the electron temperature, and hence, the conductivity and skin time are large. Thus, the profiles of q depend on Ohm's law as well as on the shot history, noninductive current maintenance, etc. In this case profiles can deviate far from canonical ones and they can have a negative shear at the central part of the plasma column [80, 81]. The large values of β in these experiments are usually associated with the second zone of stability for ballooning modes [14] but the transport suppression is also important.

Impressive results in the reversed shear zone were recently obtained in several tokamaks simultaneously [9–11]. In TFTR heat and particle diffusion dropped by a factor of 40 to the neoclassical level and even lower [10]. Many believe that the transport cannot be less than the neoclassical [22, 23] but this is not quite so. Poloidal rotation can eliminate trapped ions and the neoclassical transport along with them (see Section 4.3). In the RTP tokamak [11], off-axis electron-cyclotron heating caused a shear inversion, hollow electron profiles, and abnormally low electron transport to occur. Notice that experiments in TFTR indicate the strong suppression of ion transport and give little data on electron transport.

Since in tokamaks the transport of toroidal moment is abnormal [82], turbulent transport suppression by a negative shear naturally results in plasma spinning by an unbalanced injection as was, perhaps, observed in Ref. [9]. Many believe that this rotation makes a serious contribution to stabilization although rotation is a natural consequence of stabilization and the dependence may be reversed. In addition, the strong transport suppression is observed some time after the negative shear has appeared: a barrier formed by some uncertain parameter is overcome. In TFTR the barrier is related to the critical power, while in DIII-D it is related to the size of the transition zone which is close to the thickness of the ion banana.

Although the suppression of instabilities with trapped particles by negative shear was predicted in various ways [5, 15, 16], transport suppression in experiment is so impressive that the question arises as to where are the rest kinetic instabilities that the theory predicted to be insensitive to magnetic shear? The answer is yet to be found but the following general hypothesis can be set forth. Particles with nonintegrable behaviour are difficult to describe; therefore theorists usually make assumptions to eliminate these particles from consideration or retain only the linear Landau resonance. Meanwhile, as was noted in the analysis of frozen-in invariants, nonintegrability entails dissipation, and possibly, additional attenuation of waves. This means that weak instabilities can disappear if the theory gives up a simplification of the planar geometry type. Experimental small-scale

turbulence can be created by large gradients, for which large-scale convection is responsible as is the case in the atmosphere. In this situation small-scale turbulence must disappear along with convection.

In general, the hypothesis that plasma is frozen in the poloidal field agrees rather well with experiments on positive shear as well as on reversed shear although its manifestations are totally different.

4.3 H-mode

Canonical profiles of L-mode and supershots can be explained by an attractor arising from breakdown of the toroidal component of the law of freezing-in through trapped particles. Transport suppression by negative shear relates to energy prohibition of instabilities with trapped particles and defines one of the boundaries of the attraction basin. After that it is natural to expect that elimination of trapped ions by fast poloidal rotation suppresses turbulence and defines another boundary of the attraction basin. This boundary corresponds with the phenomenology of the H-mode, and moreover, was predicted 30 years ago.

The enhanced confinement mode, or H-mode, was revealed in the ASDEX tokamak in 1982 [19]. Since then it was obtained and investigated in many tokamaks [82 – 84]. The confinement time in this mode is two or three times greater than in L-mode, and the enhancement is due first and foremost to a transport barrier on the plasma surface with a slightly greater thickness than that of the ion banana. In these layers noise is typically suppressed while the plasma rotates. Hence, the radial electric field is much stronger here than in the bulk of plasma. The dominant opinion is that transport is suppressed as a result of decorrelation of turbulent fluctuations attributable to the electric drift shear $\mathbf{E} \times \mathbf{B}$ [85, 86].

We shall concentrate our attention on another mechanism related directly to trapped ions and frozen-in invariants. An H-like mode was proposed in 1967 by Berk and Galeev [17] and by Galeev, Sagdeev, and Wong [18] long before experimental discovery but it had rarely been mentioned in literature till [87]. In Refs [17, 18] it was assumed that as a result of ion loss from the surface layer, with a thickness comparable with that of the ion banana, the plasma surface layer starts to rotate, trapped ions disappear and the trapped ion instabilities that Kadomtsev and Pogutse [5] had just predicted disappear with them. Following [87] we compare the Berk – Galeev – Sagdeev – Wong idea with experimental findings.

If a plasma rotates along the magnetic field with a velocity v larger than the ion thermal velocity, $v > v_{T\mathrm{i}}$, then only ions on the tail of the Maxwellian distribution remain trapped. In experiment the plasma rotates not only along the magnetic field but a criterion for the main part of ions not to be trapped can be formulated by noting that a radially uniform toroidal rotation is almost invariant and it does not change the fraction of trapped ions. Thus, only the poloidal component of velocity is important. By projecting the velocity onto the poloidal direction we transform the criterion $v > v_{T\mathrm{i}}$ into the principal formula of this section, or in other words into the criterion for the poloidal ion Mach number

$$M = \frac{v_\theta B}{v_{T\mathrm{i}} B_\theta} > 1\,. \tag{27}$$

This condition is in qualitative agreement with experiment. Figure 6 presents data for the JFT-2M tokamak [88], where

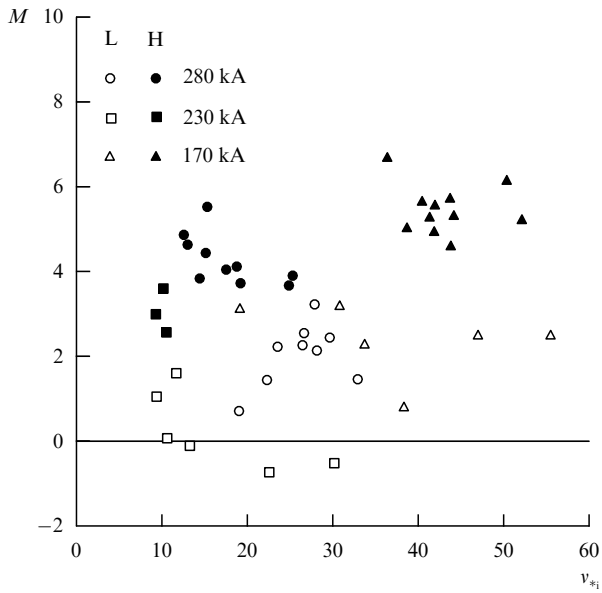

Figure 6. The poloidal Mach number as a function of ion collision rate. Data for the JFT-2M tokamak [88]. Open data points refer to L-mode, solid data points refer to H-mode.

the values $M > 3$ correspond to H-mode. Unfortunately, the velocity of hydrogen ions rotation was not measured in any tokamak. In JFT-2M the poloidal velocity of impurities was only measured by the spectral line shift. In the DIII-D tokamak, the rotational parameters were measured for the helium plasma, and the data in Fig. 4 from Ref. [89] indicate clearly that the confinement mode correlates with the poloidal Mach number: $M < 1$ in L-mode, and $M > 1$ in H-mode. It would be interesting to measure the poloidal rotation velocities in VH-mode [90].

There exist also some other indications that trapped particles play an important role in H-mode. For example, in L-mode turbulence and transport are much stronger on the outer side of the torus, i.e. where trapped particles are concentrated. Transport suppression in the L – H transition affects mainly the outer side of the torus [91].

We do not discuss here the reasons why the plasma rotates because they are covered in the review [82]. This may be the loss of ions on the plasma boundary [85, 92, 93] as it has been already assumed in Ref. [17]. Another mechanism was considered in Ref. [94]. Plasma can be spun by applying the force $\mathbf{j} \times \mathbf{B}$ using biasing electrodes [95]; the resistance measured yields additional information [82]. The reader should remember that the threshold $M \sim 1$ appears in many theories since the poloidal friction (very similar to the Landau damping of a magnetic wave as Galeev and Sagdeev pointed out when the first neoclassic work had just appeared) peaks near $M \sim 1$. For larger M the fraction of trapped particles and the friction decrease [82, 84].

The hydrodynamic criterion $M \sim 1$ loses its accuracy if the thickness of the transport barrier is comparable with that of the ion banana. The turbulent transport of the angular moment widens the transport barrier to some extent [96].

Recall that the traditional models of the H-mode concentrate on the rotation shear and on the change in radial electric field taking the form

$$E_\mathrm{r} = \frac{\mathrm{d}p}{\mathrm{d}r}\,\frac{1}{nZe} - v_\theta B_\varphi + v_\varphi B_\theta\,, \tag{28}$$

where θ is the poloidal angle, and φ is the toroidal angle. The reader should remember that two components of velocity cannot be uniquely determined from the electric field even for a zero pressure gradient, while the field is uniquely defined by the velocity.

The difference between the traditional mechanism and the mechanism Berk, Galeev, Sagdeev and Wong proposed is important for the development of fusion reactors since the latter does not require the velocity of rotation to increase when the barrier widens. The V-H mode and the barrier on the rational surface $q = 3$ [97] show the peculiarities of the widening.

4.4 Summary for tokamaks

We used a relatively new tool, namely, the frozen-in property for the Vlasov equation to study turbulent equipartitions in tokamaks. Here three results appear clear and strict, and the other three appear plausible.

(1) If turbulence preserves invariants and if these invariants depend on the magnetic field, then an attractor can appear and this attractor also depends on the magnetic field, and hence, is inhomogeneous in space.

(2) The traditional structure of transport matrix is simplified too much. If there is some level of turbulence, then fluxes appear even in the absence of gradients of thermodynamic quantities.

(3) The frozen-in property for the Vlasov drift equation is followed by the trapped particle pinching and this pinch is defined by the profile of q.

We set forth nonstrict arguments that collisional plasma in tokamaks ceases to be frozen in the toroidal magnetic field under the action of thermal pressure forces of a tensor nature and the single invariant $nr/B_\theta = \text{const}$ is conserved. Three confinement modes (attractor and two boundaries of the attraction basin) follow from this invariant and from the limits of its applicability, and all three are in proper agreement with experiment.

(4) The turbulent attractor (TEP) $nq = \text{const}$ is predicted for tokamaks with positive magnetic shear.

(5) Transport suppression follows for reversed shear.

(6) Poloidal rotation can eliminate trapped ions, restore the frozen-in property in full measure, and suppress the turbulent transport at velocities of rotation typical for the H-mode.

Taken together, these results return us to the position of the first review on turbulent transport in tokamaks [5]. Of course, at that time nothing was known about attractors, and more importantly there were not impressive experiments to shed light on physics of confinement.

Small-scale turbulence has a rich theory but the latter rather disagrees with experiment, especially in explaining the radial dependence of transport and now in explaining transport suppression by magnetic shear. In my opinion, it is quite possible that small-scale turbulence in experiment is caused by abrupt gradients attributable to the large-scale convection by trapped particles. In this case it is a secondary phenomenon like small-scale whirlwinds in the atmosphere, where they are caused by large-scale convection. The simplest example of this theory is the generation of ship waves [98].

Attractors and particle pinching can be studied also without invariants using equations [77, 99].

The agreement between experiment [9–11, 80, 81], modelling [15, 31], and analytical theory suggests an important role of negative shear in future tokamaks. Negative shear can provide a large share of the bootstrap current and it also stabilizes ballooning modes and favours large β [14]. Unfortunately, large currents near the boundary are inevitable in the case of reversed shear and they destabilize kink modes [100], though kink modes can be stabilized by plasma rotation [101, 102]. A review of the physics of future tokamaks is given in Ref. [103].

Unfortunately, shear in tokamaks is difficult to reverse globally because of the low conductivity of boundary plasma and because of the kink modes but one can try to reach a similar effect in a quasi-symmetric stellarator [104, 105]. In this case the radial derivative of $-B/r$ has to be considered instead of the shear. Here B is a magnetic field component perpendicular to the quasi-symmetry direction, and it is an analogue of the poloidal field in a tokamak.

There exists also a less revolutionary way. Turbulence may not be suppressed but the turbulent attractor peaking must be enhanced through a sharper peaking of q as for supershots in TFTR or when q increases near the separatrix as in JET, in DIII-D, and especially in spheromaks (nearly spherical tokamaks).

5. The history of turbulent equipartitions

In the textbook published in 1937 Fermi illustrated the adiabatic law by the relaxation of atmosphere to an isentropic attractor and thus he explained why temperature drops with altitude [21]. The adiabatic law and the experimental fact were known long before this publication, so the history of isentropic attractor is lost in remote ages. In the twentieth century the isentropic attractor became a common approximation for a convective zone as soon as scientists started to study combustion in stars.

In plasma the idea of turbulent equipartition appeared initially in the form of a one-dimensional plateau in quasi-linear relaxation [26]. The first spatially inhomogeneous TEP in near perfect form was considered by Pastukhov as applied to a magnetoelectrostatic trap [33]. Sasorov considered TEP in a z-pinch [37] and this TEP was very close to that in tokamaks.

The well-known Taylor theory of relaxation in pinches with a reversed field is based on a helicity invariant [38]. This invariant is not Lagrangian, and therefore the attractor differs from TEP in type: this is a simpler energy minimum. The Taylor theory is not discussed in this review because it is widely known.

It should be noted that the idea of marginal stability in z-pinches by Kadomtsev [36] and in tokamaks by Manheimer and Antonsen [28, 106] is close to TEP.

As applied to tokamaks the idea of canonical profiles was set forth by Coppi in 1980 [55], but the cause of this phenomenon was not quite clear. The Coppi hypothesis was then strongly supported by experiments of Esipchuk and Razumova [56] and Murakami et al. [57]: the plasma profile peaking enhances with the safety factor profile peaking, i.e. the plasma profiles are defined by the geometry of the magnetic field. Models of canonical profiles in tokamaks were posed by Kadomtsev [39] and Taylor [107]. The Kadomtsev model included a somewhat arbitrary assumption of the pressure profile $p \propto q^{-2}$. This assumption was based on experimental data but later a similar profile was derived from the theory of turbulent equipartitions. Heat and particle fluxes in the absence of gradients of thermodynamic quantities are an integral part of TEP and they were used

many times in phenomenological simulation by the team under the supervision of Dnestrovskiĭ (see, for example, Ref. [58]). Lallia et al. [108] proposed that heat fluxes would appear when the electron temperature gradient exceeded a critical value. In the analytical theory of tokamaks, Weiland and Norman [29] encountered fluxes which are not proportional to gradients. The discovery of the H-mode [19] and other confinement modes, whose profiles are different from canonical ones, shook at first our confidence in canonical profiles but the modern theory shows that H-mode and transport suppression by reversed shear appear exactly when the turbulent equipartition condition becomes no longer applicable. The same range of applicability in theory and in experiment additionally support the physical pattern of how canonical profiles appear.

The description of TEPs in tokamaks is based on the frozen-in notion. The most important observation implies that trapped particles are frozen in the poloidal magnetic field and that they are practically insensitive to the toroidal field; it was made by Kadomtsev and Pogutse in 1967. They proposed convection by trapped particles as the principal cause of turbulent transport. Almost at once Berk and Galeev [17], Galeev, Sagdeev, and Wong [18] suggested the elimination of trapped ions by plasma rotation, and thus they predicted the H-mode and that the plasma would recover its frozen-in property in the surface layer in full measure. Boozer put forward in essence a semiideal hydrodynamics to describe an experimental transport and he indicated the coexistence of resonant and nonresonant particles as the cause of its semiideal character [51].

6. Conclusions

The reader has surely paid attention to the fact that in the method of invariants and attractors the formulae are very short. The fee for simplicity of turbulent equipartitions is that we can describe only profiles, not fluxes. Attractors possess high noise immunity, especially if invariants describing attractors are obtained from general principles rather than from equations of motion. Therefore, it is quite natural that the theoretical predictions are very close to the experimental results.

The frozen-in property for the Vlasov equation, the existence of Ing-Yan particles, and the consequences for the law of freezing in collisionless plasma is interesting not only for tokamaks and magnetic confinement. The study of collisionless reconnection or of other plasma dynamics is not complete without the use of these invariants.

The reader should not think that there is a common opinion on above problems. If there is people's voice in science, then it is surely the voice of an anonymous reviewer, and therefore it deserves a special attention. Reviewer's comments are generally formulated in terms of linear modes and instabilities and this is a clear indication that they do not quite believe in invariants and attractors. One may conclude that the majority tends to discuss integrable phenomena because more general nonintegrable case is difficult to study, and therefore the position is as if it does not exist. The nonintegrable case cannot boast strict results but the accuracy of linear methods should not be overestimated either.

Most of my work on attractors in tokamaks, without which this review would be impossible, was performed together with J Nycander, to whom I am very obliged. I am pleased to thank K V Chukbar and Yu N Dnestrovskiĭ for reading the manuscript and for correcting many inaccurate points.

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