Physics – Uspekhi **40** (4) 359 – 386 (1997)

REVIEWS OF TOPICAL PROBLEMS PACS numbers: **03.50.–z**, 03.50.Kk, 52.35.Py, 52.35.Sb

Solitons and their interactions in classical field theory

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Contents

Abstract. Effects of nonlinearity in the classical field theory for non-integrable systems are considered, such as soliton scattering, soliton bound states, the fractal nature of resonant structures, kink scattering by inhomogeneities, and bubble collapse. The results are presented in both $(1+1)$ and higher dimensions. Both neutral and charged scalar fields are considered. Possible application areas for the nonlinearity effects are discussed.

To the memory of Igor' Kobzarev

1. Introduction

This review is a summary of our current knowledge of interaction of solitary waves in non-integrable field systems.

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Received 11 December 1996
Uspekhi Fizicheskikh Nauk **167** (4) 377 – 406 (1997)
Translated by A S Dobroslavskiĭ; edited by L V Semenova

Almost all the main material of the review has been stimulated by the ideas of spontaneous violation of symmetry, the Higgs mechanism [1], and the electroweak Weinberg – Salam theory [2, 3]. The reader will find detailed discussions of these problems and the relevant references in Ref. [4].

We do not touch upon the issues associated with the interactions of monopoles, skyrmions and instantons for various field models. These indubitably interesting questions have received comprehensive treatment in other reviews and original papers [4 – 8].

The great interest in the theory of nonlinear exactly integrable equations bearing a concrete physical content, such as the Korteweg – de Vries equation (KdV), the nonlinear Schrödinger equation (NSE), the sine-Gordon equation (sG) and the like, was associated with the development of the method of inverse scattering problem (MISP) (see, for example, Refs [9 – 11] in the late 1960s and early 70s. The presence of the relativistic-invariant sG-equation on this list enhanced the interest to this range of problems from the standpoint of the field theory. Using the MISP-based canonical transform, it was possible to express the Hamiltonian which generates the sG-equation as a function of action–angle variables [12]. The Hamiltonian in these new variables was represented as a sum of contributions from the initial particles, as well as solitons and their bound states known as breathers. In this way, the spectrum of the theory turned out to be broad enough without introducing any additional fields: along with the point particles it also included extended objects — solitons.

The scattering S-matrix of particles in the sG-equation, however, is rather peculiar: it does not take into account the birth of new particles [13], whereas collisions are always associated with the generation of new particles in the real physics of elementary particles. Because of this, starting from the mid-70s much attention has been paid to studying the properties of solutions of the classical field equations which are not exactly integrable [14, 15], under the assumption that the information obtained by solving the classical nonlinear equations of motion can be extended to the quantum case — for example, in the approximation of the weak coupling [16].

Unfortunately, the existence of three-dimensional time-independent stable solitons is prohibited by the Hobbard–Derrick (HD) theorem [17, 18] (see also Refs [19, 20]). Here it is worth recalling the main methods of bypassing the HD prohibition.

The first option is to seek stationary topologically nontrivial solutions of the field theory like a 'hedgehog' or a monopole [21]. Conservation of the topological quantum numbers then guarantees separation of the sector of solutions with a given topological number from solutions in the vacuum sector. The solutions found in the sector with a given nontrivial topology may, however, turn out to be unstable. Then the stability can be achieved by introducing terms with higher derivatives of Skyrmion type into the Lagrangian [22]. Under the scale transforms (dilatations) such terms behave differently from the kinetic and the potential terms, and so the HD prohibition is evaded.

Another method consists in considering the classical solutions for fields with constraints of the σ-field type. In this case, if the stationary solution of equations of motion is represented in terms of σ-fields, the straightforward application of the dilatation transform may be not possible without leaving the class of functions which satisfy the constraints. If the constraints are actually resolved, the Lagrangians in terms of the transformed fields assume a more complicated form, and do not fall directly under the HD prohibition [23].

The HD theorem for stationary solutions can also be evaded for fields defined on compacted manifolds or in a space with nontrivial metric [24].

The HD theorem in the case of stationary solutions imposes serious restrictions on the theories which permit the existence of solitons. There are no such restrictions, however, when the solutions depend on time. The latter may be divided into two main classes.

Firstly, it is the breather type solutions, which are stationary or quasi-stationary periodical solutions for real fields. The evolution of collapsing domain bubble was studied in the pioneering paper by Ya B Zel'dovich, I Yu Kobzarev, L B Okun' [25]. This paper (see also Refs [26, 27]) stimulated a whole series of studies which led to discovery of breather solutions in both one-dimensional and three-dimensional cases.

Secondly, it is the solutions for complex fields with time-dependent phase:

$$\psi(x,t) = \exp(\mathrm{i}\omega t)\psi(x,0)\,. \tag{1}$$

The solutions of this type were first discussed in Ref. [28], where it was demonstrated that the existence of additional conservation laws, like the charge conservation law, apart from those of energy and momentum, may lead to stable soliton-like solutions.

In this review we discuss in detail the time-dependent soliton-like solutions for various Lorentz-invariant classical field theories. Sections 2–4 deal with the search for and the properties of time-dependent solutions in $(1+1)$-dimensional field theories; in Sections 5 and 6 the same is done for $(2+1)$- and $(3+1)$-dimensional cases.

Section 2 is devoted to one-dimensional pulsating solitons of the Ginzburg–Landau–Higgs equation (GLH). We show that the routine search for new solutions by means of numerical integration of equations of motion has led to the discovery of an absolutely new phenomenon in the physics of solitons which became known as the mechanism of resonant redistribution of energy (RRE).

In Section 3 we consider other equations which admit long-lived breather-type solutions, and which also display the RRE effect.

The behavior of solitons near an inhomogeneity also bears a resonant nature. In particular, a soliton may be bounced back by an attracting impurity, which is not feasible in the potential models. These problems are discussed in Section 4.

In Section 5 we consider pulsating breather-type three-dimensional and two-dimensional solutions which arise in connection with the problem of collapsing bubbles.

Finally, Section 6 is devoted to the solutions of Q-ball type for charged fields, and to their ranges of stability.

Although the studies have been heavily biased from the start towards field theory and the physics of elementary particles, the results go far beyond the limits of these branches and can be employed in cosmology, astrophysics, solid state physics, the physics of low temperatures, etc.

We conclude by listing the main applications of the phenomena discussed in the review and giving a glossary of terms used in this review and in the relevant original papers.

This study differs from earlier topical reviews [6, 7, 15, 16, 29, 30] in that here for the first time we discuss the mechanism of RRE in the collisions of solitons, the fractal nature of capture and bouncing of colliding solitons, the resonant mechanism of reflection of solitons from impurities. For the first time we discuss in detail the properties of two-dimensional pulsating solutions, and the resonant mechanism of pulsating solutions in the three-dimensional case.

2. Resonance interaction of solitons for scalar Higgs field (one-dimensional case)

The ideas of spontaneous symmetry violation in field theory, and the requirement of the renormalizability of the theory, have drawn attention to the real scalar Higgs field [1]. This model of field theory (in the static limit) has been discussed in the paper of V L Ginzburg and L D Landau [31] dealing with the phenomenological theory of phase transitions of the second kind. In this section we shall be mainly concerned with the classical solutions of equations of motion derived from the Lagrangian of the $\lambda\phi_2^4$ theory.

2.1 Kink of the $\lambda\phi_2^4$ theory

It is convenient to start from the classical solutions of the $\lambda\phi^4$ theory in two-dimensional space–time; this case is denoted hereafter by $\lambda\phi_2^4$. The density of Lagrangian in this theory is

$$\mathcal{L}(x,t) = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_x\phi)^2 + \frac{1}{2}m^2\phi^2 - \frac{1}{4}\lambda\phi^4 \qquad (2)$$

[note the sign of the term with m^2 in Eqn (2)]. This model is often used as the paradigm of spontaneous violation of symmetry. Observe that the classical solutions discussed below can be reasonably extended to the quantum systems only in the limit of the weak constraint, $\lambda/m^2 \to 0$.

The equation of motion for the field $\phi(x,t)$ (hereafter referred to as GLH) is

$$\phi_{tt} - \phi_{xx} - m^2\phi + \lambda\phi^3 = 0\,, \qquad (3)$$

and the Hamiltonian $H[\phi]$ of the system is

$$H[\phi] = \frac{1}{2}\int \mathrm{d}x\,\left\{\pi^2 + \phi_x^2 - m^2\phi^2 + \frac{\lambda}{2}\phi^4\right\}, \qquad \pi \equiv \phi_t\,. \quad (4)$$

The constant solutions of Eqn (3)

$$\phi_\pm = \pm\frac{m}{\sqrt{\lambda}} \qquad (5)$$

correspond to the degenerate absolute minima of the Hamiltonian $H[\phi]$, and the solution $\phi_0 \equiv 0$ corresponds to the unstable state with non-violated symmetry. A more comprehensive inventory of known solutions of Eqn (3) can be found, for example, in Refs [32–34]. The most interesting from the standpoint of field theory are the classical solutions with finite energy. In the first place, such are the vacuum solutions. Small deviations from the vacuum solutions $\phi = \phi_\pm + \eta$, $|\eta| \ll |\phi_\pm|$ are described by the linearized equation

$$\eta_{tt} - \eta_{xx} + 2m^2\eta = 0\,. \qquad (6)$$

The solutions of Eqn (6) are plane waves with a dispersion law $\omega^2 = k^2 + 2m^2$; in other words, the excitations above the vacua correspond to particles of mass $\mu = m\sqrt{2}$.

A static solution of Eqn (3) with finite energy is a solution in the form of a solitary wave hereafter referred to as a kink (antikink) [$\mathrm{K}(\bar{\mathrm{K}})$]:

$$\phi_{\mathrm{K}(\bar{\mathrm{K}})} = \pm\frac{m}{\sqrt{\lambda}}\tanh\frac{m(x-x_0)}{\sqrt{2}}\,, \qquad (7)$$

where x_0 is the coordinate of the center of the kink. In three dimensions this solution looks like a wall which divides the space into two regions with different vacuum values $\phi_\pm$ [25]. The density of energy for solution (7) differs only locally (in the neighborhood of $x = x_0$) from the vacuum value, and the total energy exceeds the vacuum value by

$$M_{\mathrm{K}} = E[\phi_{\mathrm{K}}] - E[\phi_+] = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda}\,. \qquad (8)$$

It is convenient to call M_{K} the mass of the kink; it becomes large compared to the mass μ of particles excited above the vacuum in the limit of the weak constraint $\lambda/m^2 \to 0$.

The solution

$$\phi_{\mathrm{K}(\bar{\mathrm{K}})} = \pm\frac{m}{\sqrt{\lambda}}\tanh\left\{m(x-x_0-Vt)\left[2(1-V^2)\right]^{-1/2}\right\}, \quad (9)$$

obviously corresponds to a kink traveling at a speed V. Then the field momentum

$$P_{\mathrm{K}} = -\int \phi_t\phi_x\,\mathrm{d}x = M_{\mathrm{K}}V\gamma\,, \qquad \gamma \equiv (1-V^2)^{-1/2} \qquad (10)$$

and the field energy (minus the energy of vacuum)

$$E_{\mathrm{K}} = M_{\mathrm{K}}\gamma \qquad (11)$$

are linked by the common relativistic relation between energy and momentum

$$E_{\mathrm{K}}^2 = M_{\mathrm{K}}^2 + P_{\mathrm{K}}^2\,. \qquad (12)$$

This relation justifies using solutions of the kink type (7) as the basis for constructing a heavy composite particle in two-dimensional field theory [21, 35–38].

The kink (7) realizes a local extreme of the Hamiltonian $H[\phi]$. Indeed, let us see what happens with the energy of the system when we add small perturbations to the kink (7) [6]. We seek solutions for small deviations from ϕ_{K} of the form

$$\phi(x,t) = \phi_{\mathrm{K}}(x) + \eta(x,t), \qquad |\eta| \ll |\phi|\,. \qquad (13)$$

Then in the linear approximation with respect to η, from Eqn (13) we get the following equation for the function $\eta(x,t)$:

$$\eta_{tt} - \eta_{xx} - m^2\eta + 3m^2\tanh^2\left(\frac{mx}{\sqrt{2}}\right)\eta = 0\,. \qquad (14)$$

Substituting

$$\eta = \exp(-\mathrm{i}\omega t)\chi(x)\,,$$

for $\chi(x)$ we get an equation which coincides with the stationary Schrödinger equation:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 3m^2\cosh^{-2}\frac{mx}{\sqrt{2}}\right)\chi(x) = (\omega^2 - 2m^2)\chi(x)\,. \quad (15)$$

Among the solutions of Eqn (15) are two which correspond to the bound states:

$$\chi_0(x) = \left(\frac{3m}{4\sqrt{2}}\right)^{1/2}\mathrm{sech}^2\frac{mx}{\sqrt{2}}\,, \qquad (16)$$

which is the ground state of the system with $\omega_0 = 0$ (zero or translation mode), and

$$\chi_1(x) = \left(\frac{3m}{2\sqrt{2}}\right)^{1/2}\tanh\frac{mx}{\sqrt{2}}\cosh^{-1}\frac{mx}{\sqrt{2}}\,, \qquad (17)$$

which is the first excited state, corresponding to $\omega_1^2 = (3/2)m^2$. The solution of Eqn (14)

$$\eta_1(x,t) = \exp(-\mathrm{i}\omega_1 t)\chi_1(x)$$

is the discrete mode of excitation of the kink. This solution is localized on the kink and plays an important role in the dynamics of kinks discussed below. Apart from solutions (16) and (17), Eqn (15) admits solutions corresponding to the continuum with $\omega_k^2 \geqslant 2m^2$.

Observe that all $\omega_i^2 \geqslant 0$; this implies that solution (7) is stable: small deviations from this solution do not grow with time. When small perturbations are added to the kink, the energy of the system increases. The only exception is the zero mode $\chi_0(x)$. Observe that $\chi_0(x)$ [see Eqn (16)] is the derivative of the kink (17). Because of this, when a small perturbation proportional to $\chi_0(x)$ is added to the kink, it gives rise to a new kink which is displaced with respect to the old one. Accordingly, we have the following profile of the energy functional in the small neighborhood of the kink: the energy increases in all directions in the functional space except one — the energy of the system does not change in the direction of $\chi_0(x)$.

2.2 Non-topological solutions of small amplitude

The kink (7) is also stable because of the nontrivial topology by virtue of the boundary conditions

$$\phi_K \to \phi_{\pm}\,, \quad x \to \pm\infty\,. \tag{18}$$

The same boundary conditions do not allow the expansion of the solution for the kink into a Fourier series with finite coefficients with respect to excitations over the vacuum solutions [39]. In other words, one may say that the number of conventional particles in the kink is infinite. This gives rise to the problem of searching for solutions of Eqn (3) with the same boundary conditions at $\pm\infty$ with respect to x. A solution of this kind for the sG-equation

$$u_{tt} - u_{xx} + \sin u = 0 \tag{19}$$

was found in Ref. [40], and has been discussed at length [41–44]. Such a solution, known as a breather, has the form

$$u = 4\arctan\left\{(\tau^2-1)^{1/2}\cos\frac{t}{\tau}\,\mathrm{sech}\left[(\tau^2-1)^{1/2}\,\frac{x}{\tau}\right]\right\}, \tag{20}$$

and is a periodical (with period $T = 2\pi\tau$) space-localized solution. Depending on the parameter τ, the amplitude of solution (20) varies from 2π to 0.

A consistent method of finding localized small-amplitude breather-type solutions for one-dimensional nonlinear equations of the general form

$$u_{tt} - u_{xx} - f(u) = 0 \tag{21}$$

was developed in Ref. [45]. The Bogolyubov – Mitropol'skiĭ method [46] was generalized in this paper for systems with an infinite number of degrees of freedom. If, for example, $f(u) = -u + \beta u^3$, the solution of Eqn (21) is sought in the form

$$u(x) = A(x)\cos\omega t + B(x)\cos 3\omega t + \ldots\,. \tag{22}$$

Assuming that the series converges — that is, $|A| \gg |B|$ — in the leading approximation we get a sequence of equations

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} - (1-\omega^2)A + \frac{3}{4}\,\beta A^3 = 0\,, \tag{23}$$

$$\frac{\mathrm{d}^2B}{\mathrm{d}x^2} + (9\omega^2-1)B = -\frac{1}{4}\,\beta A^3\,. \tag{24}$$

Then there is a unique localized solution for the function $A(x)$:

$$A(x) = \left(\frac{8}{3\beta}\right)^{1/2}\varepsilon\cosh^{-1}\varepsilon x\,, \qquad \varepsilon \equiv (1-\omega^2)^{1/2}\,, \tag{25}$$

whereas all the other functions $B, C, \ldots$, and the corrections to $A(x)$, can be found by the method of successive approximations with respect to parameter $\varepsilon \ll 1$. A procedure somewhat improved upon that used in Ref. [45] was explicitly implemented for the GLH equation (3). Redefining the field as

$$\phi = \left(\frac{m^2}{\lambda}\right)^{1/2}(1+z)\,,$$

for the field $z(x,t)$ we get the following expansion:

$$z(x,t) = \varepsilon^2 g_1(\xi) + \sum_{n=0}^{\infty}\left[\varepsilon^{2n+1}f_{2n+1}(\xi)\sin(2n+1)\tau + \varepsilon^{2n+2}g_{2n+2}(\xi)\cos(2n+2)\tau\right], \tag{26}$$

where

$$\tau = \left(\frac{2}{1+\varepsilon^2}\right)^{1/2}, \qquad \xi = \varepsilon x\left(\frac{2}{1+\varepsilon^2}\right)^{1/2},$$

and $\varepsilon \ll 1$. Substituting Eqn (26) into the equation of motion, and equalizing the terms of similar time harmonics, we come to the following solution of the problem in the lowest approximation in ε for each term:

$$f_1(\xi) = \frac{2}{\sqrt{3}}\cosh^{-1}\xi\,, \qquad g_1 = -\frac{3}{4}\,f_1^2\,,$$
$$g_2 = -\frac{1}{4}\,f_1^2\,, \qquad f_3 = -\frac{1}{16}\,f_1^3\,, \ldots\,. \tag{27}$$

It is also possible to find analytically the subsequent terms in ε^2 for each of the functions f_i and g_i. In this way we have obtained a strong argument in favor of the existence of a time-periodical (bion) small-amplitude solution of the GLH equation. In the approximation of expansion (26) the solution is stable, and its energy is

$$E_{\mathrm{bion}} = \frac{2\sqrt{2}\,m^3}{3\lambda}\left(2\varepsilon + \frac{37}{27}\,\varepsilon^3\right) + O(\varepsilon^5)\,. \tag{28}$$

Observe that expansion (26) is asymptotical, and only holds for small bion amplitudes, $\varepsilon \ll 1$. Shortly afterwards the Kosevich method of Ref. [45] was extended to the three-dimensional case in Refs [47, 48]. The counterpart of Eqn (23) for the leading function of the expansion in ε in the three-dimensional case is

$$\Delta A - A + \frac{3}{2}\,A^3 = 0\,. \tag{29}$$

Equation (29) with boundary conditions

$$\underset{r\to\infty}{A(r)} \to 0\,, \qquad \left.\frac{\mathrm{d}A}{\mathrm{d}r}\right|_{r=0} = 0$$

leads, unlike the one-dimensional equation, to an infinite set of solutions [49–51]. Because of this, the number of possible oscillating solutions for the GLH equation in the three-dimensional case is much greater.

A somewhat more general assumption was made in Ref. [52] concerning the possible existence of localized bion-type solutions of many-dimensional equations of the type

$$\Box u = V'(u) \tag{30}$$

when the function $V(u)$ is expanded in even powers of the field u. Observe, however, that this general assumption was subjected to constructive criticism in Ref. [53], where evidence was given of the possible loss of self-localization (instability) of the solutions discussed in Ref. [52].

The issue of stability of small-amplitude bions was first raised in Ref. [54], where it was indicated that the flux of radiation may be proportional to $\exp(-c/\varepsilon)$, thus not belonging to the asymptotic series of Kosevich – Kovalev.

The instability of a small-amplitude bion for the GLH equation was proved in Ref. [55], where a term exponentially small with respect to the field was explicitly added to the solution (26),

$$\delta\phi_{\rm rad} = 4v_2 \exp\left(-\frac{\pi\sqrt{6}}{2\varepsilon}\right)\left[\sin(\sqrt{6}\,x + 2\omega t) + O(\varepsilon)\right], \quad (31)$$

where the coefficient $v_2 \cong -(4.5 \pm 1.0) \times 10^{-3}$ was obtained numerically. As a consequence, it was found that as $t \to \infty$ the total energy of the localized solution falls off slowly as

$$E \sim \frac{\rm const}{\ln t}\,. \quad (32)$$

This implies that, on the one hand, there is no stable solution (breather) in the mathematical sense. On the other hand, there exists a long-lived solution (bion) for the $\lambda\phi_2^4$ theory. Since the radiation correction is small, the approximate description of bion quantization in Ref. [43] on the basis of the classical solution (26) is probably justified.

2.3 Quantization of small-amplitude bions

The procedure of quasi-classical quantization of the breather of the sG-equation was carried out in Refs [41, 43, 56 – 59]. The exact equation for the quantum $\hat{\rm S}$-matrix of soliton – antisoliton scattering was derived in Ref. [60], where it was also shown that the formulas obtained in the quasi-classical approximation for the breather mass spectrum are correct.

In addition, the quasi-classical quantization of the bion of the $\lambda\phi_2^4$ theory was performed in Ref. [43]. The reported spectrum of small-amplitude quantum bions is

$$E_n = nm\sqrt{2} - 3\sqrt{2}\,\lambda^2\,\frac{n^3}{32m^3} + O(\lambda^4)\,. \quad (33)$$

Thus, in the limit of the weak coupling, and when n is not too large, the excited state of the quantum bion may be regarded as a bound state of n nonrelativistic particles. A detailed description of the Wentzel – Kramers – Brillouin method (WKB) in the field theory can be found in Ref. [6]. The quantization of bion was also discussed in Ref. [61].

2.4 Discovery of large-amplitude bions in the $\lambda\phi_2^4$ field theory

So, small-amplitude bions for the GLH equation in the classical approximation have been discovered and studied in the one-dimensional case. A localized oscillating solution of large amplitude was discovered in Ref. [62] by numerical integration of Eqn (3). The initial condition had the form of two walls (a $\mathrm{K\bar{K}}$ system). This statement of the problem is a one-dimensional analogue of the problem of the collapsing bubble, discussed for the case of three dimensions in Ref. [25]. The solution was found to be long-lived (in other words, the radiation from the region of localization of the large-amplitude solution was low). Similar time-dependent solutions in $(2+1)$- and $(3+1)$-dimensional spaces will be discussed in Section 5.

Actually, in Ref. [62] it was the Cauchy problem that was solved for the GLH equation (3) with the initial conditions

$$\phi(x,0) = \frac{m}{\sqrt{\lambda}}\Big\{\tanh\left[m(x+x_0)2^{-1/2}(1-V^2)^{-1/2}\right] + \tanh\left[m(-x+x_0)2^{-1/2}(1-V^2)^{-1/2}\right] - 1\Big\}\,, \quad (34)$$

and the time derivative $\phi_t(x,0)$ corresponding to kinks traveling towards each other at a speed V. The function $\phi(x,0)$ is plotted in Fig. 1. Observe that the initial condition for the function $\phi(x,0)$ coincides with the vacuum ϕ_- at $|x| \to \infty$. Because of this, one might expect that this state will annihilate quickly as soon as the kinks collide, and will take the form of small oscillations with respect to the vacuum ϕ_- (see, for example, Ref. [6]).

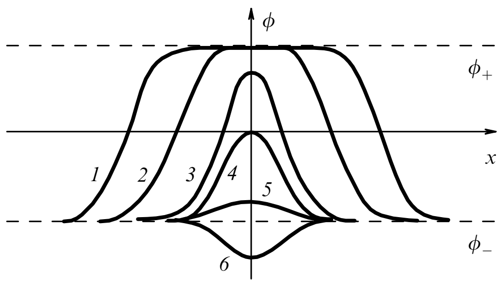

Figure 1. Field solution $\phi(x, t_i)$ vs. x for selected times t_i $(i = 1, \ldots, 6)$.

The actual situation, however, was unexpected. For $V = 0.1$ (the speed of light is $c = 1$) the $\mathrm{K\bar{K}}$ was found to result in a long-lived bound state. The solutions $\phi(x,t)$ for a few selected characteristic time values t_i are shown in Fig. 1. Figure 2 shows the time dependence of the field function at the origin $\phi(0,t)$. The period of the large amplitude oscillations was found to be of the order of m^{-1}. It was also found that fast-moving kinks bounce off after collision with little loss of energy, and a bound state is not formed. Observe that the phenomenon of reflection of kinks by itself, which was not observed in the case of sG-equation, does not yet imply that the system is not completely integrable. An example of such a solution is a completely elastic reflection in the soliton – antisoliton system for the integrable classical Gross – Nevue model [63].

The formation of a bion was also soon discovered in Ref. [32]. A more detailed numerical analysis of the patterns of $\mathrm{K\bar{K}}$

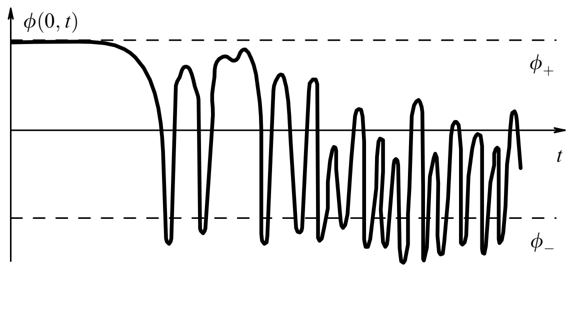

Figure 2. Time dependence of field $\phi(0, t)$ at $x = 0$.

collisions was carried out in Refs [64, 65]. It was found that the time dependence of the solution steadies after the initial rather irregular pulsations, although the main harmonic remains slightly modulated. At the same time the loss of energy by radiation was found to be quite small: assuming exponential dependence of the bion energy at large times, Getmanov [65] found that the energy of localized oscillations decreased to one-half of its initial value over the time $T \approx 1750m^{-1}$.

Subsequently the damping of bions over larger periods of time was studied in greater detail [66]. It was found that the damping decrement depends on the amplitude itself, and decreases as the amplitude falls off. Challenging the assumption of Ref. [65], the system was found to lose energy slower than by the exponential law.

Thus, the discovered localized oscillating weakly damped large-amplitude solution (bion) in the $\lambda\phi_2^4$ theory is an analogue of the breather in the sG-theory, and in the limit of small amplitudes seems to coincide with the solution (27). The weak damping of this solution, found numerically, is also in general agreement with the conclusion of Ref. [55].

2.5 Search for other solutions of the Ginzburg – Landau – Higgs equation

The discovery of the bion stimulated the search for other relatively stable solutions of the GLH equation. A relatively long-lived excited oscillating state of a kink was soon discovered [65]. Having been discovered in the $\mathrm{K\bar{K}}$ collision in the field of a kink at rest, it became known as a triton. Sometimes this solution is also referred to as a wobbling kink. The lifetime of this solution, however, is less than that of the bion. Its energy as function of time may be approximated by

$$E(t) = M_{\mathrm{K}}\left[1 + 2\exp(-\delta_1 t)\right], \qquad (35)$$

where $\delta_1 = (3.0 \pm 0.1) \times 10^{-2} m$.

The potential description of the triton solution in the symmetric configuration was discussed in Ref. [67].

The Kosevich – Kovalev expansion [45] as applied to the wobbling kink was discussed in Ref. [68]. Obviously, in the limit of small-amplitude oscillations this solution becomes a kink with an added first vibration mode (17).

It was not possible to discover the birth of the supplementary $\mathrm{K\bar{K}}$ pair in the kink – antikink collisions at near-threshold energies. Observe that in the quantum approximation the birth of the $\mathrm{K\bar{K}}$ pair of the GLH equation may be detected when the external field is strong enough [69].

2.6 Potential approximation

The first attempt to explain the formation of bions using numerical experiments has already been made in Ref. [62]. By assumption, at low energies it is possible to introduce the effective potential of $\mathrm{K\bar{K}}$ interaction $U(X)$. It is defined as the energy of the static configuration of kink and antikink separated by a distance X:

$$U(X) = \frac{1}{2}\int \mathrm{d}x \left(\phi_x^2 - m^2\phi^2 + \frac{\lambda}{2}\,\phi^4\right). \qquad (36)$$

The configuration function was function (34) with $X = 2x_0$. The potential of interaction then has the form shown in Fig. 3. When the distance is infinitely large, the potential energy is constant and equal to the mass of the two kinks. The potential is zero when $X = 0$. When X is negative, the $\mathrm{K\bar{K}}$ system displays strong repulsion which grows linearly with increasing $|X|$.

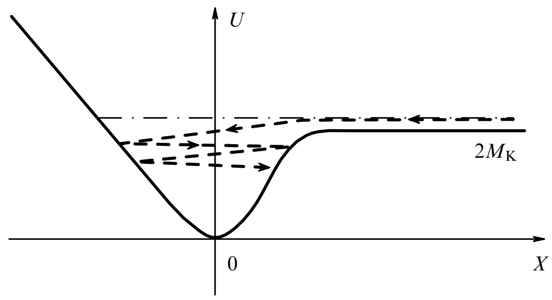

Figure 3. The potential of soliton – antisoliton interaction in the GLH theory.

It is interesting that a similar potential description has been employed in Ref. [40] for the two-soliton solutions of the sG-equation. The method of collective coordinates in the field theories with extensive objects — for example, with the kinks in the $\lambda\phi^4$ theory — was developed in Refs [70 – 73]. A somewhat different method of introducing the potential between two solitons on the basis of a coherent state was proposed in Ref. [74] for the sG-system. A more consistent definition of the nonrelativistic potential leading to the correct behavior of the S-matrix of soliton – soliton scattering for the sG-equation was discussed in Ref. [75].

Observe, however, that the definition of potential (36) is quite sufficient for the qualitative description of the nature of motion of the kinks. Indeed, if we assume that the behavior of the $\mathrm{K\bar{K}}$ system reduces to a nonrelativistic problem of two point particles with mass M_{K} and potential of interaction $U(X)$, then we must first of all note that there are two kinds of motion in this problem: finite motion in a well (an analogue of a bion) at $E < 2M_{\mathrm{K}}$, and infinite motion at $E > 2M_{\mathrm{K}}$ which corresponds to the scattering problem. As suggested in Ref. [62], the presence of friction in the system (that is, emission of small-amplitude waves) may lead to the capture of a kink by the potential, which was duly observed in the numerical experiment. Note that the $\mathrm{K\bar{K}}$ potential was subsequently calculated with greater precision in a number of studies [61, 76 – 78].

It ought to be emphasized once again that the potential approximation taking into account the possible loss of energy due to friction generally portrays the results of numerical experiments in a quite satisfactory way.

Note also that there have been other attempts at explaining the discovered long-lived states. In Ref. [79], for example, the general reason d'etre of long-lived pulsating objects in field theories was attributed to their sufficient extensiveness in space. The pulse distribution of particles in the object is then soft, and the properties of the system are close to nonrelativistic. The system exhibits an additional approximate integral of motion: the number of particles in the system. As follows, for example, from Fig. 2, the characteristic rates of change of field in the bion are quite relativistic. Because of this, the observation made in Ref. [79] cannot be applied literally to the bion of the GLH equation. A convenient approximate parametric representation of the bion of the GLH equation, suitable also for large-amplitude bions, was proposed in Ref. [80].

2.7 Discrepancy found in the potential description

At the same time, not all phenomena discovered in the early numerical simulations can be explained within the framework of the proposed potential approach. And indeed, it is hard to explain the irregular oscillations of the bion which are readily discernible in Fig. 2 if there is only friction in the system.

Apart from that, even in the early papers there was already some disagreement concerning the value of the critical capture velocity. In Ref. [62] it was assumed to be equal to 0.25; in Ref. [65] it was defined as $V_{\rm cr} = 0.2 \pm 0.01$. One could attribute such differences in the value of $V_{\rm cr}$ to certain inaccuracies in the calculations. The situation, however, turned out to be much more complicated.

As early as in Ref. [32] it was found that capture takes place not at any arbitrary initial velocity of colliding kinks below a certain $V_{\rm cr}$. In particular, capture was reported in Ref. [32] at $V = 0.25$, whereas scattering of kinks occurred at $V = 0.22$ and $V = 0.26$.

Another peculiar feature in the behavior of kinks near the capture threshold was discovered in Ref. [81]. In particular, it was found that when the initial velocity of the kinks is $V_{\rm in} = 0.3$, after collision they recoil with final velocity $V_{\rm f} = 0.135$. At $V_{\rm in} = 0.25$ the kinks merge to form a bion. At $V_{\rm in} = 0.2$ the kinks again recoil with $V_{\rm f} = 0.155$. We see that the elasticity of collision in the bounce window near $V_{\rm in} = 0.2$ is higher than at $V_{\rm in} = 0.3$! This resonance-like effect points to the existence of a nontrivial mechanism of capture, different from the initially assumed capture due to energy loss by radiation (friction).

2.8 Bounce windows and mechanism of resonant redistribution of energy

The 'bounce windows' of Refs [32, 81] turned out to be just the tip of the iceberg, and led to the discovery of the complex structure of the transition region between the continuous range of $V_{\rm in} > V_{\rm cr} \cong 0.2598$, where the kinks always rebound, and the range of $V_{\rm in} < 0.192575$, where the kinks always merge to form bions.

Studies in the wake of Ref. [81] resulted in both the discovery of new bounce windows, and in the more precise definition of their location [77, 82, 83] in the transition range of initial velocities.

The peculiar behavior of kinks in bounce windows was detected in Refs [77, 82]. While at $V > V_{\rm cr}$ the kinks collide just once, having once collided in the bounce windows the kinks retreat to a considerable distance, stop, and collide again, whereupon they rebound to infinity. The coordinates of the kink center as a function of time for $V > V_{\rm cr}$ and for a bounce window are plotted in Fig. 4a, b. The range of velocities where the kinks rebound after a double collision is referred to as a two-bounce window.

A careful study of bounce windows which threw light on the reasons of their existence in the transition region was carried out in Ref. [84] (see also Ref. [85]). Nine two-bounce windows were discovered in Ref. [84]. After the first collision in the bounce windows the kinks rebound to a considerable distance but cannot leave the sphere of interaction. After the second collision, however, the separation of the kinks becomes possible. Accordingly, it was necessary to propose a mechanism by which the kinetic energy of the kinks could be removed in the first collision and returned at the time of the second collision.

Such a mechanism of redistribution and conservation of energy was suggested in Ref. [84]. The following explanation was put forward: the discrete mode (17) may be excited in the kinks receding after the first collision. This mode is associated with the kink — that is, it is a localized solution which stores energy near the center of the kink. The frequency of oscillation of this solution is fixed, $\omega_1^2 = 3/2$. If at the time of the second collision the phase of the discrete mode and the relative motion of the kinks are tuned appropriately, some of the energy of the discrete mode may be reconverted into kinetic energy of the kinks to ensure that they rebound to infinity. If this explanation is correct, then the energy is mainly stored in the discrete mode of the kink. Figure 5 shows the function $\phi(0, t)$ in the region of the first eight two-bounce windows. The solutions for different initial velocities $V_{\rm in}$ in Fig. 5 differ in the number of periods of small oscillations occurring between the collisions of the kinks themselves, which correspond to large-amplitude oscillations. If this hypothesis is correct, the period of small-amplitude oscillations must be close to the period of the first discrete mode of the kink. The numerical value of the period of small oscillations was found to be $T_{\rm exp} \cong 5.2$, which is very close to $T_1 = 2\pi/\omega_1 = 5.13$! Thus, the proposed mechanism of the energy being stored in the discrete mode of kink excitation was confirmed by numerical experiment.

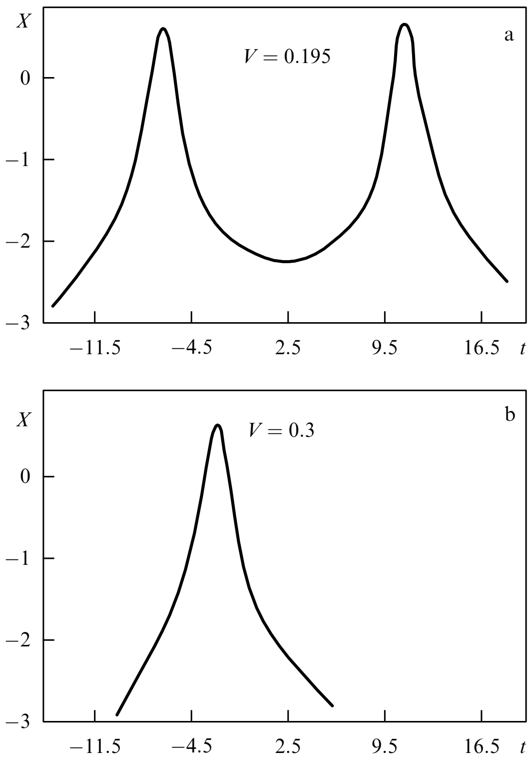

Figure 4. Time dependence of the coordinate of the center of the colliding kink for the initial velocity corresponding to (a) the bounce window; (b) $V > V_{\rm cr}$.

2.9 Verification of implications of mechanism of resonant energy transfer

First of all we ought to mention that the importance of the discrete mode of kink excitation in $K\bar{K}$ collisions had been noted earlier in Ref. [78]. In the same study the critical velocity of capture was calculated on the basis of the hypothesis of the predominance of the discrete mode, and

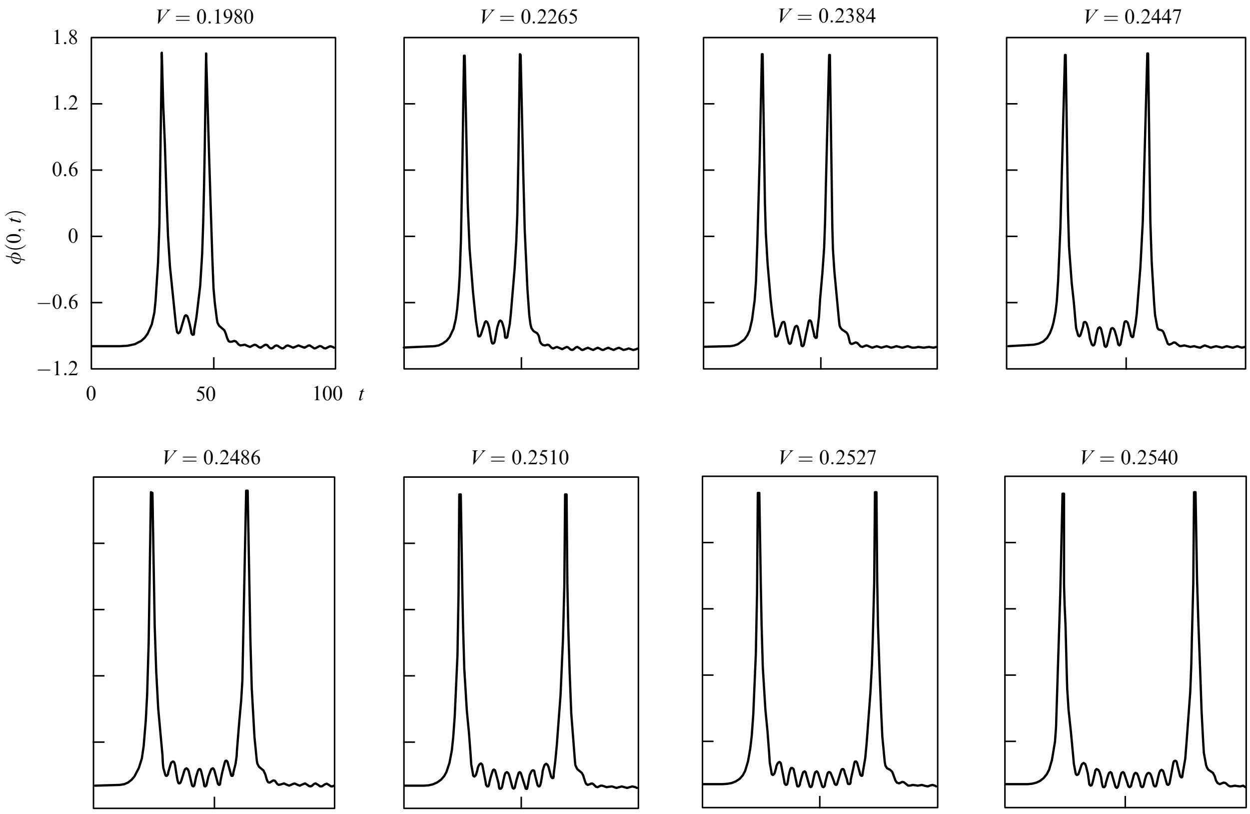

Figure 5. The function $\phi(0, t)$ in the region of two-bounce windows (first eight windows).

was found to be $V_{\rm cr} \cong 0.25$, which is close to the exact value of $V_{\rm cr} = 0.2598\ldots$.

As an implication of the proposed mechanism, the following phenomenological condition was written out in Ref. [84] for the feasibility of the reconversion of energy into kinetic energy of the $\mathrm{K\bar{K}}$ system:

$$\omega_1 T_{12}(V_n) = \delta + 2\pi n\,, \tag{37}$$

where n is a whole number, and $T_{12}(V_n)$ is the time between the two collisions of the kinks. This formula was checked by numerical simulation, which resulted in the experimental value of $T_{12}^{\rm exp} \cong 5.2$, quoted in Section 2.8.

Further, starting with the asymptotic expression for the potential of the $\mathrm{K\bar{K}}$ interaction [6, 78], phenomenological relations were obtained in Ref. [84] between the initial velocity of collisions in the nth window, and the critical velocity:

$$V_n^2 = V_{\rm cr}^2 - 1.37(2n+1)^{-2}\,. \tag{38}$$

Relation (38) was found to be in good agreement with the experiment for all bounce windows. It was also demonstrated that the width of the bounce windows is accurately described by the condition

$$\Delta V_n = \vartheta(2n+1)^{-3}\,, \tag{39}$$

which follows from the potential approximation (ϑ is a numerical constant).

Numerically it was confirmed that the energy loss by radiation is small. The energy contained in the discrete mode of kink excitation dominates up to very high initial velocities $V_{\rm in} \sim 0.7-0.8$. At $V_{\rm in} \leqslant 0.4$ it is greater than the radiation energy by an order of magnitude.

2.10 Discovery of multi-bounce windows. Quasi-fractals

The study of bounce windows was continued in Refs [86, 87]. For instance, 35 two-bounce windows were discovered in Ref. [86]. Their arrangement and width as a function of the number n of small oscillations between collisions of the kinks are shown in Fig. 6. From the diagram it follows that $V_{\rm cr}$ looks like a condensation point of resonances. Apart from that, multi-bounce windows of multiple $\mathrm{K\bar{K}}$ collisions were discovered in Ref. [86], whose feasibility had been discussed in Ref. [84]. The solutions corresponding to bounce windows

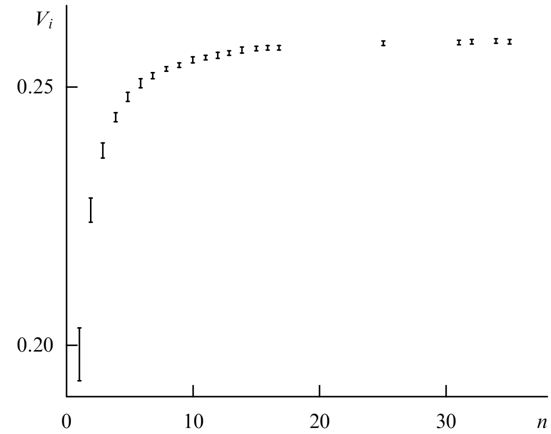

Figure 6. Locations and widths of regions of two-bounce windows vs. n.

were termed, according to Ref. [86], quasi-closed orbits. The field in these solutions behaved much differently from the chaotic pattern in a bion. It was noted that the three-bounce windows are located near the two-bounce windows — in other words, there is a kind of hierarchy in the arrangement of the bounce windows. A typical three-bounce window is shown in Fig. 7.

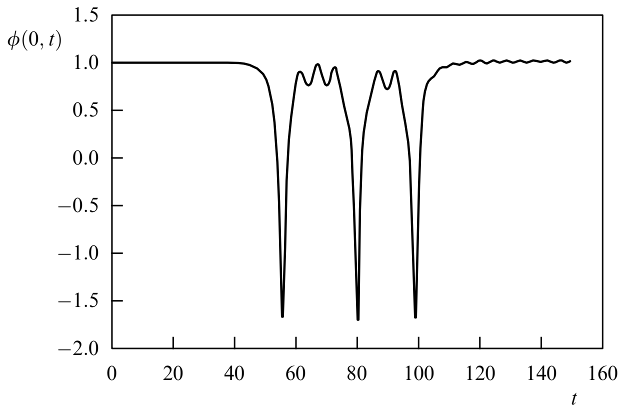

Figure 7. Time dependence of function $\phi(0,t)$ for a typical three-bounce window.

Further study of multi-bounce windows in Ref. [87] revealed that their arrangement is fairly regular: each two-bounce window is associated with a series of three-bounce windows; each three-bounce window is associated with a series of four-bounce windows, etc. In this way, the arrangement of multi-bounce windows displays an obvious fractal structure. The scale change in the transitions

$$n\text{-bounce} \to (n+1)\text{-bounce} \to (n+2)\text{-bounce} \to \ldots .$$

is shown schematically in Fig. 8. Figure 8a shows the black two-bounce windows with a continuous black band to the right, which corresponds to the continuum of single-bounce collisions at $V_{\text{in}} > V_{\text{cr}}$. The region in the box is shown to a larger scale in Fig. 8b. The black region on the left corresponds to the two-bounce window, and the black bands to the right mark the locations of the three-bounce windows. The region in the box is again shown to a larger scale in Fig. 8c, which shows the arrangement of the four-bounce windows. In connection with the discovered fractal structure of the multi-bounce windows, a certain universal formula was proposed in Ref. [87] for calculating the width of bounce windows depending on the number n of small

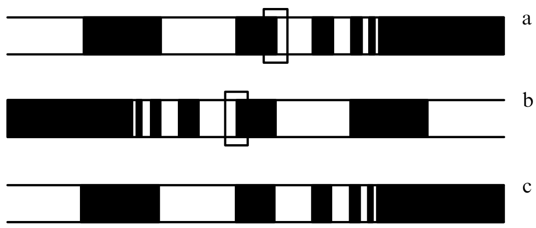

Figure 8. The fractal structure of locations and widths of bounce windows and capture regions.

oscillations between collisions of the kinks:

$$\Delta V_n \propto n^{-\beta} , \tag{40}$$

where β is a certain universal constant, which does not depend either on n or the order of the bounce window. Normalizing to the least n in each series, it was possible to obtain a good description of the width of two-bounce and three-bounce series using the same value of β.

Another point discussed in Ref. [87] was whether the motion in the bion region (that is, outside the bounce windows) is chaotic. The solution was studied for $V_{\text{in}} = 0.18$, where there are no bounce windows, and a bion is formed. Observations were based on the time dependence of the function $\phi(0,t)$. According to Refs [88–90], this dependence was used for constructing a series of d-dimensional vectors V_i,

$$V_i = \left\{ V_1(t_i), V_2(t_i), \ldots, V_d(t_i) \right\} ,$$

where d is the assumed dimensionality of the attractor, $V_k(t_i) = V\left[t_i + (k-1)\tau\right]$, and τ is the time lag. In fact, the time dependence of the distance between a pair of vectors V_0 and V_0' was studied such that initially

$$\| V_0 - V_0' \| < \delta_0 , \tag{41}$$

where $\delta_0 < 10^{-6}$. The following quantity was calculated at each time step:

$$\lambda_j = \log_2 \frac{\delta_{j+1}}{\delta_j} . \tag{42}$$

The largest Lyapunov index λ_{I} was calculated as the average of λ_j over the time interval Δt:

$$\lambda_{\text{I}} = \frac{\mathrm{d}t}{\Delta t} \sum_j \lambda_j . \tag{43}$$

For large times, λ_{I} is close to $\lambda_{\text{I}} \cong 0.31$. This is an indication that oscillations in a bion are chaotic, because a positive Lyapunov index λ_{I} implies that any two initially close points in the phase space of a dynamic system diverge exponentially with time.

Further studies of the quasi-fractal behavior of bounce windows for $\mathrm{K\bar{K}}$ scattering have led to the conclusion that exact fractals would only have been possible in the absence of radiation [91].

2.11 Effective Lagrangian for bion

The method of collective coordinates [70–73] applied to the $\mathrm{K\bar{K}}$ system in an approximation improved over the potential approximation of Refs [62, 76, 61, 77] was first discussed in Ref. [78]. Neglecting the excitation of the continuum, the field configuration at any time is assumed to have the form

$$\begin{aligned} \phi(x,t) = {} & \phi_{\text{K}}\left[x + X(t)\right] - \phi_{\text{K}}\left[x - X(t)\right] - 1 + A(t) \\ & \times \left\{ \chi_1\left[x + X(t)\right] - \chi_1\left[x - X(t)\right] \right\} , \end{aligned} \tag{44}$$

where $\chi_1(x)$ is the solution (17) corresponding to a discrete mode of kink excitation. The variable $X(t)$ is the collective coordinate corresponding to the translation mode — that is, one half of the distance between the kink and the antikink. The variable $A(t)$ characterizes the excitation of the discrete mode. The effective Lagrangian is defined as

$$L_{\text{eff}} = \int \mathrm{d}x\,\mathcal{L}(x, t, X, A)\,, \tag{45}$$

where $\mathcal{L}(x, t, X, A)$ is given by Eqn (2) with $\phi(x, t)$ as defined by Eqn (44) [78, 86]. Integration of Eqn (45) with respect to x leads to the following expression for L_{eff}:

$$L_{\text{eff}}(X, \dot{X}, A, \dot{A}) = \big[M_{\text{K}} + I(X)\big]\dot{X}^2 - U(X) + \dot{A}^2 - \omega_1^2 A^2 + 2F(X)A + 2C(X)\dot{A}\dot{X}\,, \tag{46}$$

where $\omega_1^2 = 3/2$ is the frequency of the discrete mode of kink excitation, and the functions $I(X)$, $U(X)$, $F(X)$, and $C(X)$ are written out explicitly (see, for example, Ref. [78]). This Lagrangian is a natural generalization of the potential approximation proposed in Ref. [62] and discussed in Section 2.6. Observe that the higher terms in A and X are dropped in Eqn (46).

The Lagrangian (46) was used in Ref. [78] for evaluating the critical velocity of capture. The motion with respect to X was assumed to be classical, whereas the excitation of oscillator $A(t)$ was treated as a quantum problem given the motion of the kinks (adiabatic approximation). Accordingly, the return of energy from the discrete mode $A(t)$ into the $X(t)$ mode in Ref. [78] was neglected, and the bounce windows were not discovered.

The importance of the exact solution of the classical equations of motion defined by the Lagrangian (46) was first pointed out in Ref. [92]. This idea was successfully implemented in Ref. [86]. It was found that even a system simplified with respect to Eqn (46) ($C = 0$) beautifully portrays all the features of the solution: the region of single-bounce collisions of the kinks above the critical velocity, the two and three-bounce windows below V_{cr}, and the formation of bions. The behavior of the function $\phi(0, t)$ in Eqn (44), reconstructed with the aid of equations of motion for $X(t)$ and $A(t)$, is shown for different initial velocities in the model problem in Fig. 9, and differs little from similar solutions of the exact problem.

In Ref. [87] it was proved that the hierarchy in the arrangement of bounce windows, discovered for the exact field problem, is faithfully reproduced for the reduced problem (46), and the retrieved parameter β (40) is very close to that for the exact problem.

In Ref. [87] it was pointed out that a bion is not formed in the reduced problem because of the absence of friction in the transition region: for any initial velocity there exists such a time after which the kinks will scatter apart. This constitutes an important distinction from the exact field problem, in which a bion may also be formed in the transition region. At the same time, when the initial velocity is low, the behavior of solutions below the region of bounce windows for the reduced system (46) is close to chaotic up to very large times. This is confirmed by the everywhere-dense Poincaré mapping in the (A, A) plane for the initial velocity $V_{\text{in}} = 0.02$. Simultaneously, the maximum Lyapunov index, $\sigma_{\text{I}} \cong 0.32$, was calculated for the reduced system, which turned out to be close to the value $\lambda_{\text{I}} \cong 0.31$ for the exact system.

In this way, the analysis of the $\text{K}\bar{\text{K}}$ interaction has led to the discovery of a number of new phenomena in the classical field theory: long-lived solutions (bions), bounce windows and their hierarchical fractal nature, chaotic motion in the

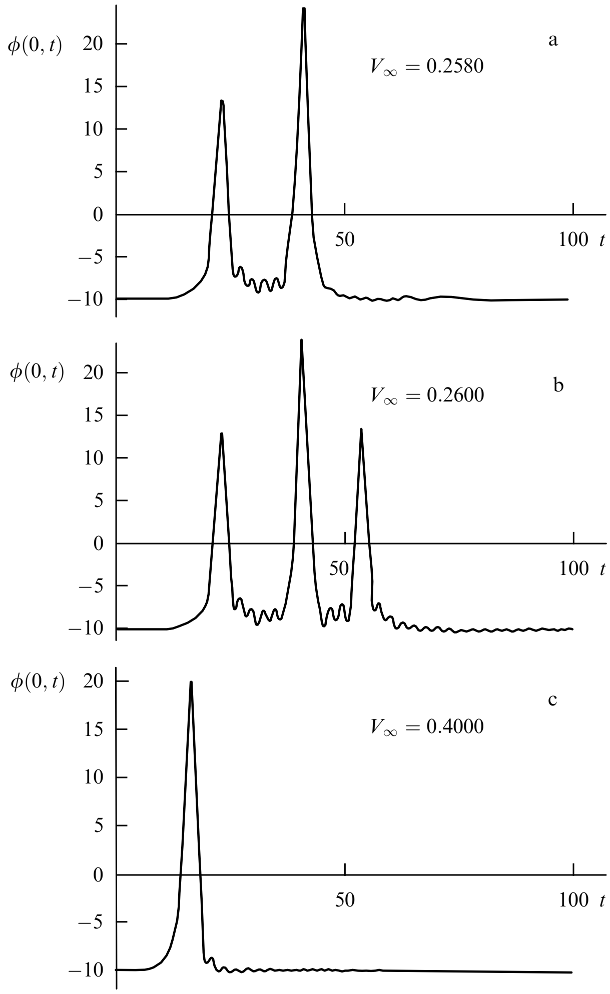

Figure 9. The behavior of function $\phi(0, t)$, reconstructed from the solution of equations for the effective Lagrangian: (a) two-bounce window; (b) three-bounce window; (c) scattering of kinks in continuum.

bion, and the feasibility of reducing the exact problem in field theory to the description based on the effective Lagrangian with a finite number of degrees of freedom. An important role in shaping the peculiar properties of the $\text{K}\bar{\text{K}}$ system in the $\lambda\phi_2^4$ theory is played by the discrete mode 2 of kink excitation. Further on we shall demonstrate that the presence of the discrete mode leads to a pattern, similar to that discovered in the $\lambda\phi_2^4$ GLH theory, in many other systems as well.

3. Some examples of problems with kink interactions

3.1 Kink – antikink interaction in the modified sine-Gordon (MsG) equation

Consider a system defined by the Lagrangian

$$\mathcal{L}(\phi, \phi_t) = \int \mathrm{d}x \left[\frac{1}{2}\,\phi_t^2 - \frac{1}{2}\,\phi_x^2 - U(\phi)\right] \tag{47}$$

with the potential $U(\phi)$:

$$U(\phi, r) = (1 - r^2)(1 - \cos\phi)\big[1 + r^2 + 2r\cos\phi\big]^{-1}\,,$$

where $r \in (-1, 1)$. This system was first discussed in Ref. [93]. Analytical solutions for kinks were found in Ref. [94]. For any r, $\phi_K(x, r) \to 0$ when $x \to -\infty$, and $\phi_K(x, r) \to 2\pi$ when $x \to +\infty$. Certain interactions in the $K\bar{K}$ system were studied in Ref. [95], and some evidence of a bion being formed at $r = 0.5$ was found. The system in question was analyzed in greater detail in Ref. [96].

First of all, the spectrum of small excitations near $\phi_K(x, r)$ was studied in Ref. [96]. (A similar task for the GLH equation was discussed in Section 2.) It was found that the spectrum of excitations in the system depends considerably on the parameter r. When r is positive, the spectrum exhibits one discrete mode, corresponding to the translation mode with $\omega_0 = 0$, and the continuum. When r goes negative, the number N of discrete excitation modes starts to increase, and $N \to \infty$ as $r \to -1$. The spectrum of excitations of the kink, obtained numerically, is shown in Fig. 10. In this way, studying the interactions in the $K\bar{K}$ system at different values of r, it is possible to derive information about the role of discrete excitation modes. Observe that $r = 0$ corresponds to a system described by the exactly integrable sG-equation, whose soliton–antisoliton solution has been obtained analytically.

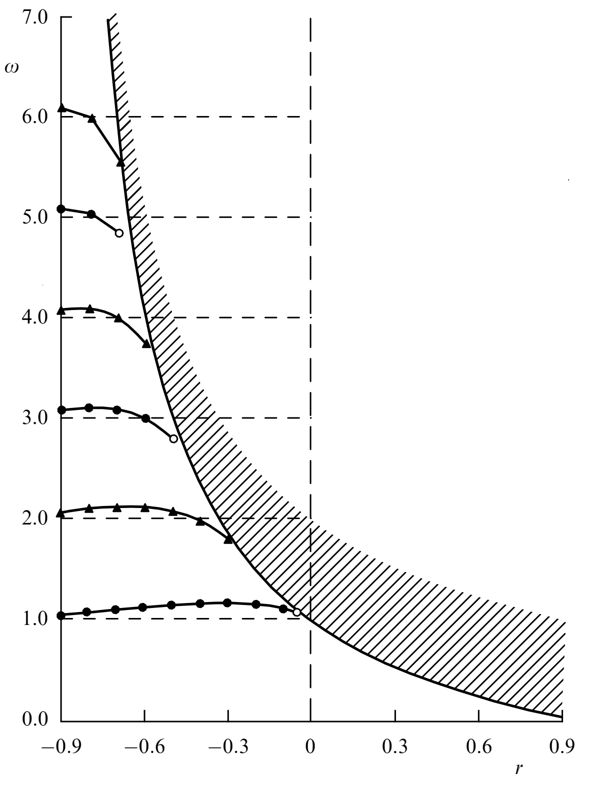

Figure 10. Spectrum of kink excitations in the theory of the MsG-equation (52) as function of parameter r.

3.1.1 The case of small negative r. The case of $r = -0.1$ has been considered. The system is then close to that described by the sG-equation. At the same time, the excitation spectrum of the kink already contains one discrete excited level with $\omega_1 = 1.1205$ (see Fig. 10). The continuous spectrum starts from $\omega_{\text{cont}} = 1.2222$. As in the $\lambda\phi_2^4$ theory, in the case of $K\bar{K}$ the interaction there is a certain critical velocity above which the kinks pass losing part of their energy. In addition, bounce windows and a passage window were discovered below $V < V_{\text{cr}}$. In the bounce windows the behavior of the kinks is similar to the two-bounce collisions in the $\lambda\phi_2^4$ theory. The kinks collide, pass through each other to a considerable distance, stop and collide again. Accordingly, these solutions correspond to back scattering of the kinks. In the passage window the kinks collide three times, and the result corresponds to the forward passage of the kink.

3.1.2. The case of small positive r. In the case of $r = 0.1$ the continuum starts at $\omega_{\text{cont}} = 0.8182$, and there is only one trivial discrete mode with $\omega_0 = 0$. As in the case of $r = -0.1$, capture occurs below, and scattering above the critical velocity $V_{\text{cr}} = 0.234$. This time, however, the passage and scattering windows are not observed. In the capture region, as opposed to the $\lambda\phi_2^4$ theory, the $K\bar{K}$ system does not form a long-lived state, and soon disintegrates into non-localized divergent waves.

In case of $r = 0.05$, owing to the smallness of r, the properties of the system ought to be similar to those of the integrable system. As a result, the critical velocity is low, $V_{\text{cr}} = 0.112$. The capture of kinks occurs below V_{cr}. There are no passage or scattering windows. It should be noted that, by contrast to the case of $r = 0.1$, the bion at $r = 0.05$ is rather stable. In addition, in Ref. [96] it was found that at certain velocities below V_{cr} the periods of the first few oscillations in the bion are notably greater than at other values of V. Since scattering of kinks does not take place, this pattern of bion formation at certain initial velocities was named a quasi-resonance. Initial velocities corresponding to quasi-resonances are given by the simple phenomenological formula

$$V_n = V_{\text{cr}} - A(2n+1)^{-2}, \tag{48}$$

where $A = 5.84$, and $n = 4, \ldots, 15$ is the index of the quasi-resonance. Since the discrete mode of kink excitation is absent, one may only conjecture that quasi-resonances result from the interaction of the translation mode of the kinks with a certain distinguished frequency of excitation of continuum $\tilde{\omega}_{\text{cont}}$. In our present case we have $r = 0.05$, and $\tilde{\omega}_{\text{cont}} = 1.159$, which is beyond the limit of the continuum $\tilde{\omega}_{\text{cont}}^{\min} = 0.9048$. For large positive r the pattern of $K\bar{K}$ interactions remains essentially the same: there are no passage or scattering windows below V_{cr}.

3.1.3. The case of large negative r. The case of $r = -0.5$ leads to a much more sophisticated pattern of $K\bar{K}$ interaction, which is shown in Fig. 11. Below $V_{\text{cr}} \cong 0.34$ there are two scattering windows at $V_i = 0.315$ and $V_i = 0.317$. In addition, however, there is a passage band in the range $0.176 < V_i < 0.278$, corresponding to the three-bounce interaction. We ought to note that at $r = -0.5$ the spectrum of kink excitations contains three discrete levels, and the pattern of interactions between these levels can be quite complicated.

We see that the existence of passage and scattering windows for the MsG-equation generally depends on the availability of discrete modes of kink excitation. The newly discovered phenomenon is the existence of quasi-resonances at small positive values of r, which correspond to the excitation of a distinguished frequency in the continuous spectrum. The nature of this phenomenon is not yet entirely clear.

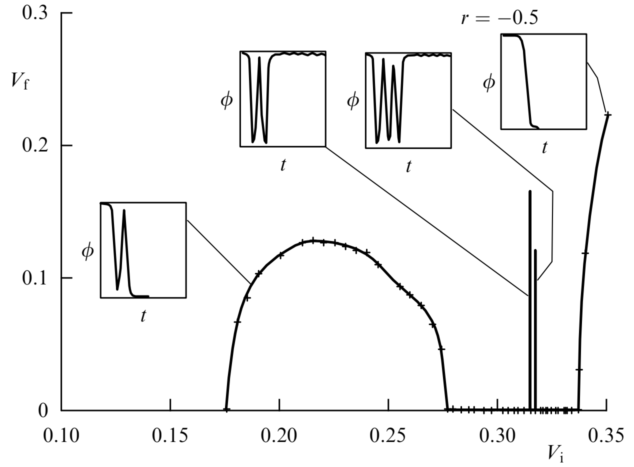

Figure 11. The complex pattern of zones and passage windows for large negative values of r.

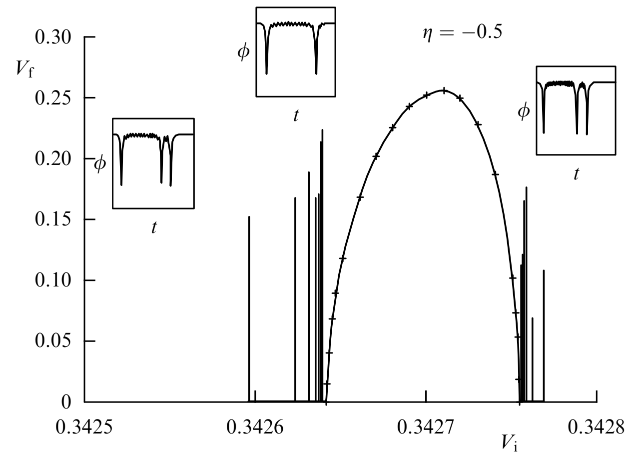

Figure 12. Structure of three-bounce windows near a two-bounce window at $\eta = -0.5$.

3.2 Kink – antikink interaction for the double sine-Gordon equation

Interactions in the kink – antikink system were studied in Ref. [97] for the so-called double sine-Gordon (DsG) equation

$$\phi_{tt} - \phi_{xx} + 2\big(1 + |4\eta|\big)^{-1}\left(-\sin\frac{\phi}{2} + 2\eta\sin\phi\right) = 0\,, \quad (49)$$

where the parameter η may be assigned any arbitrary real value ($-\infty < \eta < +\infty$). This equation corresponds to the potential of field self-action in the form

$$U[\phi] = -4\big(1 + |4\eta|\big)^{-1}\left(-\cos\frac{\phi}{2} + \eta\cos\phi\right). \quad (50)$$

As $\eta \to \pm\infty$ (as well as at $\eta = 0$), the potential $U[\phi]$ becomes that of the sG-equation. In the range $\eta < -1/4$ the potential exhibits a set of degenerate minima separated by non-equivalent barriers. Accordingly, there are two types of kinks: 'major' and 'minor'. The spectrum of small excitations of kinks contains translation modes with $\omega_0 = 0$, as well as one discrete mode for the 'minor' kink. Owing to the fact that the model potential $U[\phi]$ with $\eta < -1/4$ was used in the studies of superfluid ^{3}He in phase B, the $K\bar{K}$ interactions have been investigated extensively [98 – 103]. In particular, the critical velocity of capture of a 'minor' $K\bar{K}$ pair into a bion was calculated in Ref. [103] (for $-3.6 \leqslant \eta \leqslant -0.31$). In addition, the conversion of a minor pair into a major pair was observed at large velocities $V \geqslant 0.92$ ($\eta = -1$) [100], and the conversion of a major pair into a minor pair at any velocity [100, 103]. The structure of the solution for the collision of the minor $K\bar{K}$ pair at $\eta = -0.50$ is in perfect agreement with the model of resonant energy transfer into the discrete mode considered above for the $\lambda\phi_2^4$ theory. Above the critical velocity the kinks experience single-bounce collisions. Twenty two-bounce windows were discovered below $V_{\rm cr}$, and their arrangement is in good agreement with the phenomenological description. Three-bounce windows are grouped near the two-bounce windows (Fig. 12). The hierarchy of states can be detected at the next level: clusters of four-bounce windows are located near the three-bounce windows. The situation remains essentially the same when other negative values of η are used.

Consider a kink – antikink scattering for $-1/4 < \eta < 0$. In this range the potential of the DsG-model is topologically similar to the potential of the sG-model, and there is just one type of kink that links the vacuum solutions. The kink excitation spectrum in this range does not contain any nontrivial discrete levels. Calculations made for this range indicate, as ought to be expected, that there are no bounce windows below the critical capture velocity.

At the same time, as in the case of the MsG-equation, the lifetime of the bion increases as $|\eta|$ becomes smaller — that is, as we are approaching the exactly solvable sG-equation. It seems that as $\eta \to 0$ the behavior of the bion is strongly influenced by the existence of the breather of the sG-equation ($\eta = 0$).

In the case of $\eta > 0$ the kink has one discrete excitation level. While at $0 < \eta < 1/4$ the potential remains topologically similar to the potential of the sG-equation, at $\eta > 1/4$ the maxima at $\phi = 4\pi n$ display local recessions (more detailed information on the solutions of the DsG-equation can be found in Ref. [104]). For $\eta > 0$ it is convenient to replace η with another variable R,

$$\eta = \frac{1}{4}\sinh^2 R\,. \quad (51)$$

The soliton of the DsG-equation in terms of R becomes very simple:

$$\phi_{K(\bar{K})} = 2n(2\pi) \pm 4\arctan\frac{\sinh x}{\cosh R}\,. \quad (52)$$

At $R = 1.2$ the pattern is very similar to the standard case of kink scattering in the presence of the discrete mode of kink excitation. Above $V_{\rm cr} = 0.2305$ there is a continuous spectrum of pass-through kinks. Two-bounce windows are found below $V_{\rm cr}$.

In the case of $R = 0.5$, which formally is no different from the case of $R = 1.2$, scattering windows below the critical velocity $V_{\rm cr} = 0.117$ have not been found. It follows that the mechanism of energy transfer into the discrete mode does not work in this example even though a discrete mode is available! Observe, however, that the frequency of the discrete mode $\omega_1 = 0.96692$ in this case is very close to the continuous spectrum ($\omega_{\rm cont} \geqslant 1$). At the same time, one may argue that a certain mechanism of energy transfer still works, since the formation of the bion is not a monotonic function of V_{in}. This can be seen in Fig. 13, where the period between the second and third collisions of the kink in the bion is by no means a steady function of the initial velocity. This phenomenon resembles the quasi-resonances discussed above for the MsG-equation in Section 3.1.2.

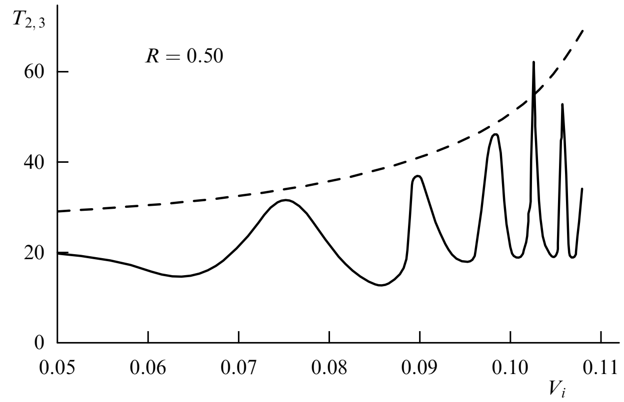

Figure 13. The time $T_{2,3}$ between the second and third collisions vs. the initial velocity for $R = 0.5$. The dashed line shows the time $T_{1,2}$.

For large values of $R > 2.4$, the critical velocity of capture becomes small, and the scattering windows are no longer observed. Here, however, the $\mathrm{K\bar{K}}$ collisions display the same quasi-resonances as in case of $R = 0.5$. The attempt to describe quasi-resonances by the standard formula

$$\tilde{\omega}_1 T = 2\pi n + \delta \qquad (53)$$

is only successful for the series of quasi-resonances with $\tilde{\omega}_1 = 1.0456$, whereas the actual frequency of the discrete mode of kink excitation is entirely different, $\omega_1 = 0.24822$. The resonances at large values of R are apparently associated with the excitation of a group of levels in the continuous spectrum. So far there is no reliable interpretation of this effect.

The study of the kink–antikink interaction in the DsG-equation generally confirms the established concept of the mechanism of energy transfer to and storage in the discrete excitation mode, and at the same time reveals a number of certain new properties of the problem: it is the existence of long-lived bions at small negative values of η, and the accumulation of convincing evidence concerning the existence of quasi-resonances which are observed in the MsG-equation and correspond to the excitation of levels in the continuous spectrum.

4. Resonant interaction of solitons and kinks with impurity

It would be interesting to note that the discrete mode of kink excitation may manifest itself not only in the case of kinks interacting with one another, but also in the one-kink configuration when the problem contains an inhomogeneity. Moreover, some types of inhomogeneities may give rise to certain specific discrete excitation modes in the system (impurity modes), which may influence the interaction between a kink or a soliton and the impurity.

The problem of interaction of solitons or kinks with impurities has a history of its own. Let us discuss it in brief.

4.1 Potential approximation

The dynamic behavior of the soliton of the sG-equation in the presence of a disturbance generated by a field interacting with an inhomogeneity was discussed in Ref. [105]. This problem has many applications.

As a matter of fact, in Ref. [105] the behavior of a soliton for inhomogeneous sG-equation of the form

$$u_{tt} - u_{xx} + m^2 \sin u = \lambda\left[\delta(x + x_0) - \delta(x - x_0)\right] . \qquad (54)$$

was considered. The constant λ was assumed to be small. Recall that the spectrum of small deviations from the soliton solution for the homogeneous sG-equation has only one discrete mode with $\omega_0 = 0$ (the zero mode), and the continuous spectrum with $\omega^2 \geqslant m^2$. The time evolution of the soliton solution $u_{\mathrm{S}}(x - vt)$ was studied in the presence of disturbance described by the right-hand side of Eqn (54). It was found, neglecting excitations of the continuum, that the action of impurity upon the soliton is similar to that of a potential. Indeed, in the context of problem (54) the energy of a static soliton depends on the distance to the impurity ξ. The potential of interaction between the soliton and the impurity, calculated in Ref. [105], has the form

$$U(\xi) = 4\lambda \arctan \frac{\sinh z_0}{\cosh \xi} , \qquad (55)$$

where $z_0 = mx_0$, and $\xi = mX$. The potential is attractive or repulsive depending on the sign of λ.

When the velocity of the soliton is high, the potential $U(\xi)$ of Eqn (55) may be regarded as a disturbance, and the independently calculated energy loss due to excitation of the continuum modes may be neglected. At low energies, however, the energy loss due to excitation of the continuum modes becomes considerable, and the theory predicts the possible trapping of the traveling soliton by the impurity. As predicted in Ref. [105], the soliton may be trapped by the potential of the impurity if the kinetic energy of the soliton at infinite distance from the impurity, $E_{\text{kin}}(\infty)$, satisfies the condition

$$E_{\text{kin}}(\infty) < \left|U(\xi)\right| , \qquad \text{at} \;\; U(\xi) = 0 . \qquad (56)$$

The critical velocity of capture of soliton by the impurity at early stages was also discussed in Refs [106–108].

The behavior of the soliton trapped by an impurity was also discussed in Ref. [105]. Then the center of the soliton oscillate harmonically when the amplitude is small. When the amplitude is large, these oscillations are described by elliptic functions.

4.2 Adiabatic perturbation theory

Further investigations of this problem were mainly concerned with more accurate calculation of the critical velocity of capture of a soliton by an impurity. The foundation was laid in Ref. [109], where the general formalism of the adiabatic perturbation theory was developed for finding solutions for equations of evolution of the form

$$u_t = S[u] + \varepsilon R[u] , \qquad \varepsilon \ll 1 , \qquad (57)$$

which for $\varepsilon = 0$ are exactly integrable by the MISP. Within the framework of perturbation theory for a small parameter

ε, equations were constructed for the transmission and reflection coefficients and residues in the poles of the S-matrix of the inverse problem, which furnished a consistent procedure for finding the corrections to solutions of the unperturbed equations. In particular, corrections were obtained for the one-soliton solutions of the KdV equation and the NSE. In addition, the evolution equations for polynomial integrals of motion $I_n[u, u^*]$ of the unperturbed equations in the presence of a disturbance were formulated in Ref. [109] in the form

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \varepsilon \int_{-\infty}^{\infty} \left\{ \frac{\delta I_n}{\delta u} R\big[u(x)\big] + \frac{\delta I_n}{\delta u^*} R^*\big[u(x)\big] \right\}, \qquad n = 1, 2, 3, \ldots, \tag{58}$$

which were used later in some form or other for practical purposes.

It ought to be mentioned that a simpler method for calculating the loss of energy in solitons is based on the analysis of the evolution equations for integrals of motion of the type of Eqn (58). For example, the integral of momentum of the field for the soliton solution of the sG-equation is

$$P = -\int_{-\infty}^{\infty} u_x u_t \,\mathrm{d}x = 8V(1 - V^2)^{-1/2}, \tag{59}$$

where V is the velocity of the unperturbed soliton. On the other hand, one could take into account the non-conservation of momentum of the soliton because of the disturbance:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\varepsilon \int_{-\infty}^{\infty} \mathrm{d}x\, u_x P[u]. \tag{60}$$

In terms of velocity, equation (60) reconstructs the law of variation of velocity of the center of soliton (see, for example, Refs [110, 111]). In this way, the developed formalism of the adiabatic perturbation theory, based on the assumption that the system is close to integrable, allowed more precise calculation, for example, of the critical velocities of capture — that is, the phenomenological friction in the soliton configuration. In Ref. [112], for example, a formula was derived for the critical velocity of capture of a soliton of the sG-equation by a δ-shaped impurity (see Section 4.6). Numerical simulation of the field problem revealed, however, that the applicability of adiabatic perturbation theory is very limited when the impurities have their own discrete modes, as will be discussed below.

4.3 Instability of perturbations around soliton localized on an inhomogeneity

The perturbation theory discussed above can be used because the Lagrangian system under investigation differs little from the exactly integrable one. The limits of applicability of this perturbation theory, however, have not been analyzed. In the meantime, a number of independent studies appeared, where the stability of solutions of classical nonlinear equations with sources was discussed from the standpoint of the theory of bifurcations and the theory of catastrophes (see, for example, Refs [113, 34, 114]). References [115–117] were concerned with the stability of a localized soliton for the equation

$$\phi_{tt} - \phi_{xx} - \left[1 - \sum_i \mu_i \delta(x - x_i)\right] \sin\phi = \alpha\phi_t, \tag{61}$$

which describes the behavior of fluxons for long Josephson junctions. The behavior of the spectrum of small oscillations was studied with respect to the stable points of such an equation in the space of solutions

$$\phi(x,t) = \phi_{\mathrm{stab}}(x) + \exp\left(-\frac{\alpha t}{2}\right) \times \sum_n \chi_n(x)\big[a_n \exp(-\mathrm{i}\bar{\omega}_n t) + a_n^* \exp(\mathrm{i}\bar{\omega}_n t)\big]. \tag{62}$$

The analysis was based on the expansion in eigenfunctions $\chi_n(x)$ in the neighborhood of the stable solution (fluxon), localized on an inhomogeneity. It was found that when the external parameters of the problem are varied (for example, the coefficients μ_i), some coefficients a_n in the expansion (62) may exhibit a sharp increase at certain values of μ_i. This may be associated with the possible vanishing of eigenfrequencies — for example, the frequency ω_0:

$$\omega_0^2 = \bar{\omega}_0^2 + \frac{\alpha^2}{2}. \tag{63}$$

Numerical deviations of the exact solutions from the potential oscillatory regime, discussed in Ref. [105], were discovered in Refs [118–120]. The question of the changing shape of a soliton passing over an impurity was discussed in Ref. [118]; it was found that the distortion of the soliton becomes stronger as the interaction constant increases.

The problem of trapping a slowly moving soliton by a stand-alone impurity was considered in Refs [119, 120] for Eqn (61) with the boundary condition $\phi_x(\pm l) = 0$, where $2l$ is the length of the transition layer. For small values of μ it was found that, as anticipated, the soliton is attracted by the microscopic impurity and starts to oscillate around the impurity with very little distortion. However, as μ increases ($\mu = 0.5-0.6$), the pulsations of the soliton shape are quite considerable. The nature of the pulsations allowed the singling out of several harmonics, multiples of the main frequency of oscillations of the center of soliton in the potential created by the impurity. Possible manifestations of the discovered instability of a soliton in a system with a large number of inhomogeneities (Josephson lattice) were further discussed in Ref. [121].

We see that the model of potential approximation with friction, discussed in Sections 4.1 and 4.2, is only good for the qualitative explanation of certain aspects of the interaction between solitons and impurities. At the same time, the deviations of the potential model from the numerical simulation [119, 120] have been observed for a problem with specially designed boundary conditions, which could have augmented the effect. At the same time, the frequency of pulsations did not quite correspond to the frequency

$$\omega_0^2 \cong \frac{\mu}{2}\left[\left(1 + \frac{\mu^2}{16}\right)^{1/2} - \frac{\mu}{2}\right] \tag{64}$$

suggested in Ref. [117]. Because of this, further efforts were necessary to elucidate the causes of deviations from the potential approximation, and to predict other possible phenomena resulting from the non-potentiality of the problem.

4.4 Behavior of kinks in the $\lambda\phi_2^4$ theory near an inhomogeneity

The problem of the interaction of a kink of the $\lambda\phi_2^4$ theory with an inhomogeneity was first discussed in Ref. [122] (see also Ref. [123]). While the cause of strong pulsations of the soliton shape in the sG-equation theories with inhomogeneities is not quite clear; the $\lambda\phi_2^4$ theory exhibits a discrete level of kink excitation. The availability of this degree of freedom allows for the RRE as in the case of the $\mathrm{K\bar{K}}$ interaction discussed in Section 2.

The problem of scattering an initially undistorted kink by an inhomogeneity was considered in Ref. [122], which amounted to solving the Cauchy problem for the equation

$$\phi_{tt} - \phi_{xx} + (\phi^3 - \phi)\left[1 - \mu\delta(x - x_0)\right] = 0 \qquad (65)$$

with the boundary conditions

$$\phi(x,0) = \tanh\left\{(x - x_0)\left[2(1 - V^2)\right]^{-1/2}\right\}, \qquad (66a)$$

$$\phi_t(x,0) = -V\left[2(1 - V^2)\right]^{-1/2} \times \cosh^{-2}\left\{(x - x_0)\left[2(1 - V^2)\right]^{-1/2}\right\}, \qquad (66b)$$

where V is the initial velocity of the undistorted kink, and x_0 is the initial distance between the kink and the impurity. The problem may be solved either by straightforward numerical integration of Eqn (65), or by using the approximate method of collective coordinates discussed in Section 2.10 and in Ref. [86] for $\mathrm{K\bar{K}}$ interactions. By assumption, the solution of the problem (65), (66) at any time has the form

$$\phi(x,t) = \tanh\left[\frac{(x - X)}{\sqrt{2}}\right] + A\chi_1\left[\frac{(x - X)}{\sqrt{2}}\right], \qquad (67)$$

where $X(t)$ and $A(t)$ are the collective coordinates, and $\chi_1(x)$ is the solution (17) of Eqn (14) for the discrete mode of the kink. A procedure similar to that performed in Section 2.10 yields the following nonrelativistic effective Lagrangian:

$$\mathcal{L}(X, \dot{X}, A, \dot{A}) = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_{\mathrm{int}}, \qquad (68)$$

where

$$\mathcal{L}_0 = M_{\mathrm{K}}\left(\frac{\dot{X}^2}{2} - 1\right) - V(X), \qquad V(X) = -\left(\frac{\mu}{4}\right)\cosh^{-4}(\alpha X),$$

$$\mathcal{L}_1 = \frac{\dot{A}^2}{2} - \frac{\omega_1^2 A^2}{2},$$

$$\mathcal{L}_{\mathrm{int}} = -\frac{\mu A\sqrt{3/2}}{\sqrt[4]{2}}\tanh^2(\alpha X)\cosh^{-3}(\alpha X) + \frac{\mu A^2(3/4)}{\sqrt{2}}\tanh^2(\alpha X)\cosh^{-2}(\alpha X)\left[3\tanh^2(\alpha X) - 1\right],$$

where M_{K} is the mass of the kink. The standard variation procedure for the Lagrangian (68) leads to the dynamic equations in $X(t)$ and $A(t)$, by analogy with the procedure used earlier for the $\mathrm{K\bar{K}}$ interactions.

Retaining only the term with $\mathcal{L}_0$ in Eqn (68), we get the potential model of interaction between the kink and the impurity. In this approximation the kink either passes over the impurity, or is engaged in a finite motion about the impurity. The term $\mathcal{L}_1$ represents the Lagrangian which describes the free oscillation of the first discrete mode of kink excitation with $\omega_1^2 = 3/2m^2$. The term $\mathcal{L}_{\mathrm{int}}$ describes the interaction between the zero and first modes due to the presence of the impurity. The time evolution of the functions $X(t)$ and $A(t)$ was studied complying with the initial conditions

$$X(0) = 1.5, \qquad \dot{X}(0) = V, \qquad A(0) = \dot{A}(0) = 0, \qquad (69)$$

where $\dot{X}(0)$ is the initial velocity of the kink. The solution of the equations of motion for $X(t)$ and $A(t)$ was analyzed for different values of the constant of interaction with the impurity μ ($\mu = 0.3$, 0.5,1.0, 2.0). Special attention was paid to the case $\mu = 0.3$, for which the range of initial velocities $0.025 \leqslant V \leqslant 0.075$ was calculated with a step of $\Delta V = 2.0 \times 10^{-4}$. The relative accuracy in terms of energy was 10^{-6}.

It was found that for the model problem (68), (69), in contrast to the potential approximation, the kink may be trapped or scattered by the impurity.

It follows, in spite of the attractive nature of the impurity, that the kink may be bounced off.

Depending on the initial kink velocity, the following situations may be realized in the model problem:

(a) passage of kink over the impurity at $V > V_{\mathrm{cr}}(\mu)$;

(b) windows of transmission and reflection, and a capture band at $V < V_{\mathrm{cr}}(\mu)$.

The behavior of the center of mass of the kink for the exact field problem (65), (66) depending on the initial velocity of the kink V is shown in Fig. 14. The distinction between capture and scattering zones becomes clearer if we look at the Poincaré mappings. Capture corresponds to a much more uniform occupation of the mapping plane, which indicates that the system is very close to chaotic.

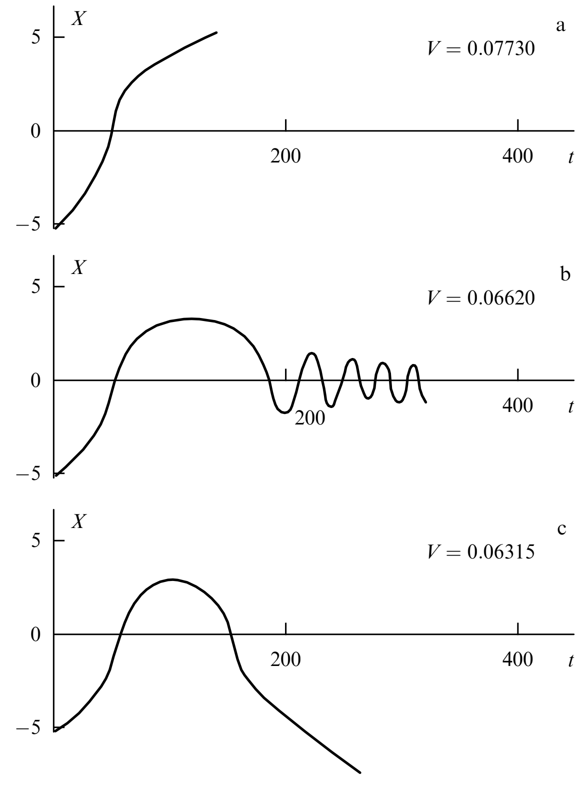

Figure 14. Paths of center of mass of the kink (a) passing over the impurity; (b) captured by the impurity; (c) bounced off the impurity. The exact field problem, $\mu = 0.3$.

As noted in Ref. [120], the numerical solution of the exact field problem (65), (66) is a quite complicated feat. As in Ref. [120], the δ-function for the exact field problem in Ref. [122] was replaced by a Gaussian. Calculations were performed for different values of the Gaussian parameter α, and the stability of the results was checked for each value of α. The model calculations (68), (69) for the exact field problem were generally confirmed. For some of the observables — for example, for $V_{\rm cr}$ — the model problem and the exact calculations were in good quantitative agreement. For example, for $\mu = 0.3$, the critical velocity is $V_{\rm cr} = 0.0686$ for the model, and $V_{\rm cr} = 0.0690$ for the exact problem with $\alpha = 5$. At the same time, only one scattering window was found in the field problem with $\mu = 0.3$ and $\mu = 0.5$, the step in the initial velocity being $\Delta V_{\rm in} = 2 \times 10^{-3}$. Passage windows were observed below $V_{\rm cr}$. It must be also noted that the period of oscillations of the discrete mode of kink excitation in the discovered scattering and passage windows, in accordance with the observations made with the field problem, was many times less than the period of oscillations of the kink about the impurity. Accordingly, some of the passage and scattering windows in the exact problem could have really disappeared because of friction (radiation), becoming zones of capture.

We see that the impurity, whose action upon the kink is generally attractive, may sometimes bounce the kink back by a mechanism of resonant exchange between the kinetic energy of the kink and its discrete excitation mode. This phenomenon, predicted for the model effective Lagrangian, was confirmed by calculations of the exact field problem. This result completely disagrees with the notion of the potential nature of interaction between the kink and the impurity taking into account the energy loss by friction.

At the same time, the observed vigorous pulsations of the soliton shape for the inhomogeneous sG-equation with the dissipative term (61) do not fit in with the above mechanism of energy transfer, since the soliton has no discrete excitation mode. It was found, however, that, as in the system described by Eqn (65), the spectrum of excitations of the system described by Eqn (61) exhibits an additional discrete level associated with the impurity.

4.5 Discrete impurity mode

It is well known that solitons were discussed not only for the continuous theories like the sG- and KdV equations, but also in the lattice approximation. An example is the Toda lattice model [124], which admits exact soliton solutions. Since lattices simulate the solid state, the question about the role of impurities comes up quite naturally. For example, a defect may be associated with an atom of different mass implanted in the lattice or chain; or there may be defects on the boundary of the lattice. More complex dislocations are also possible. The existence of extrinsic vibration modes in periodic lattices is well known. At the same time, the existence of such modes in nonlinear systems depends on a number of conditions (see, for example, Ref. [125]). In the course of time, however, discrete extrinsic modes were found in many discrete models. Moreover, it was discovered that the interaction between a soliton and an impurity in some theories is not trivial. For example, the interaction for the Toda lattice [124] was studied numerically in Ref. [126]. The soliton was found to lose some of its energy, and the vibration mode of the impurity becomes strongly excited after the passage of soliton. The soliton either travels past the impurity or is bounced off.

The problem of the influence of the impurity mode on the motion of kink in the lattice $\lambda\phi_2^4$ theory was formulated and studied in Ref. [127]. It was found that, depending on the mass of the impurity atom, the kink may either go past the impurity or be bounced back. The vibration mode of the impurity becomes strongly excited after the passage of kink. The amplitude of the excited discrete impurity mode is a non-monotonic function of the velocity of the incident kink. Thus, the discrete mode of the impurity is excited when the kink passes, taking away some of the energy of the kink.

Observe that the discrete impurity mode is not directly associated with a kink or soliton, and may exist in their absence. Let us consider the solution for the impurity mode for the GLH $\lambda\phi_2^4$ theory with an impurity [see Eqn (65)]. We seek a solution in the form of small deviations from the vacuum solution ϕ_+, that is

$$\phi(x,t) = \phi_+ + \delta\phi(x,t)\,, \qquad \text{where} \qquad |\delta\phi| \ll \phi_+\,. \tag{70}$$

Linearizing Eqn (65), we get

$$\delta\phi_{tt} - \delta\phi_{xx} + 2\delta\phi = 2\mu\delta\phi\delta(x) \tag{71}$$

(assuming that the impurity is located at $x = 0$). If the solution of Eqn (71) is sought in the form $\delta\phi(x,t) = \exp(-\mathrm{i}\tilde{\omega}t)\chi(x)$, then for the function $\chi(x)$ we get the stationary Schrödinger equation with the δ-function potential:

$$-\chi_{xx} - 2\mu\delta(x)\chi = (\tilde{\omega}^2 - 2)\chi\,. \tag{72}$$

This equation admits a unique discrete normalizable solution $\chi(x) = A\exp\bigl(-\mu|x|\bigr)$, and the frequency is given by $\tilde{\omega}^2 = 2 - \mu^2$. Observe that Eqn (72) also admits solutions corresponding to the continuous spectrum at $\tilde{\omega}^2 \geqslant 2$. These solutions, however, are not localized on the impurity and therefore cannot account for the conservation of energy in the kink – impurity system.

The discrete impurity mode may also exist outside the linearized approximation used above. This was first studied in Ref. [128], and was also discussed in Ref. [129] within the framework of the nonrelativistic approximation. As different from the linearized approach, the frequency $\tilde{\omega}$ in the nonlinear approximation depends on the amplitude of the oscillations.

We see that the spectrum of excitation of the GLH system with an impurity, like that of the sG-system with an impurity, displays a spatially localized characteristic excitation mode with a discrete frequency. This mode may play a role similar to that of the discrete mode of kink excitation in the $\lambda\phi_2^4$ theory.

4.6 Critical velocity of soliton capture by impurity

The problem of the role of the discrete extrinsic mode in the interaction of the soliton of the sG-equation with an impurity

$$u_{tt} - u_{xx} + \bigl[1 - \varepsilon\delta(x)\bigr]\sin u = 0 \tag{73}$$

was studied in Ref. [130]. At $\varepsilon = 0$ the soliton of Eqn (73) is known to have the form

$$u_{\rm S}(x,t) = 4\arctan\bigl[(x - x_0 - Vt)(1 - V^2)^{-1/2}\bigr]\,. \tag{74}$$

As we have already discussed in Sections 4.1 and 4.2, the adiabatic perturbation theory was used for predicting the critical velocity of capture of a soliton by an attractive impurity ($\varepsilon > 0$). The critical velocity for Eqn (73) was calculated on the basis of such an approach in Ref. [112]; the resulting value was exponentially small:

$$V_{\mathrm{cr}} = 2^{21/8}\pi^{1/4}\varepsilon^{3/8}\exp\left(-\sqrt{\frac{2}{\varepsilon}}\right). \tag{75}$$

As found in Ref. [130], the actual value of the critical velocity is entirely different from the result of the adiabatic theory. The discrepancy is apparently due to the fact that the adiabatic approximation completely ignores the possible transfer of some of the energy into the discrete impurity mode. At low energies, however, this mechanism accounts for most of the energy lost by the soliton. The discrete impurity mode for Eqn (73) has the form

$$u_{\mathrm{disc}}(x,t) = a_0\cos(\Omega t + \theta_0)\exp\left(-\frac{\varepsilon|x|}{2}\right), \tag{76}$$

where $\Omega = (1-\varepsilon^2/4)^{1/2}$. The energy contained in this mode is

$$E_{\mathrm{disc}} = \frac{\Omega^2 a_0^2}{\varepsilon}. \tag{77}$$

Straightforward experimental scattering of a soliton by an impurity above the critical velocity indicates that the discrete impurity mode is actually excited (Fig. 15).

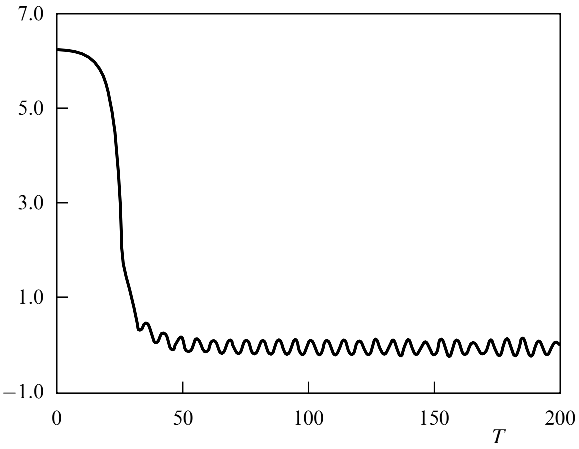

Figure 15. The displacement of the coordinate of the impurity after the passage of the soliton at $V > V_{\mathrm{cr}}$.

Calculation of the critical velocity was based on the method of collective coordinates, discussed earlier in connection with the problem of interaction between a soliton and an impurity in the $\lambda\phi_2^4$ theory [122, 123]. Substitution of the real field function

$$u(x,t) = u_{\mathrm{S}}(x,t) + u_{\mathrm{disc}} = 4\arctan\left[x - X(t)\right] + a(t)\exp\left(-\frac{\varepsilon|x|}{2}\right) \tag{78}$$

into the Lagrangian

$$\mathcal{L} = \int_{-\infty}^{+\infty}\mathrm{d}x\left\{\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \left[1 - \varepsilon\delta(x)\right](1-\cos u)\right\} \tag{79}$$

yields the following expression for the effective Lagrangian:

$$L_{\mathrm{eff}} = 4\dot{X}^2 + \frac{1}{\varepsilon}(\dot{a}^2 - \Omega^2 a^2) - U(X) - aF(X), \tag{80}$$

where $U(X) = -2\varepsilon/\cosh^2 X$ (same as in Ref. [118]), and $F(X) = -2\varepsilon\tanh X/\cosh X$. Calculating the energy lost by a soliton coming down on the impurity from a large distance $\left[X(0) = -7,\ \dot{X}(0) = V_{\mathrm{in}} > 0\right]$ due to the excitation of oscillator $a(t)$, the authors of Ref. [130] found the critical velocity as function of ε, which is entirely different from that predicted by Eqn (75), and is in qualitative agreement with the numerical solution of Eqn (73). Thus, the discrete impurity mode accounts well for the energy lost by the soliton.

4.7 Solitons scattering by impurity for the sine-Gordon equation

As discussed above, a soliton moving at a velocity above critical passes the impurity losing some of its energy on excitation of the discrete mode. Below V_{cr} the pattern is more complicated [131]. Back scattering windows occur at certain velocities, in complete agreement with what was found earlier for the $\lambda\phi_2^4$ theory in Refs [122, 123] and discussed in Section 4.4. By contrast to the GLH equation, however, in the sG-theory we are dealing with the discrete mode of the impurity rather than the discrete mode of the soliton. For example, Eqn (73) with $\varepsilon = 0.7$ gives 11 scattering windows which may be qualified as two-bounce interactions: the soliton collides with the impurity and moves on, exciting oscillations of the discrete mode at the point $x = 0$, then stops and goes back to the impurity. In the scattering windows some of the energy taken away in the first collision is returned to the translational motion of the soliton, and it goes to minus infinity. The basic mechanism of resonant energy exchange is undoubtedly associated with the discrete mode of the impurity. This was proved by noting that, as in case of the $\mathrm{K}\bar{\mathrm{K}}$ interaction, in the $\lambda\phi_2^4$ theory there is a fairly accurate empirical formula which links the time T_{12} between the two bounces in the scattering windows with the number n of periods of small oscillations:

$$T_{12}(V) = nT_{\mathrm{discr}} + \tau, \tag{81}$$

where $T_{\mathrm{discr}} = 2\pi/\Omega$, and τ is the phase constant for a given ε. The numerical results of Refs [130, 131] are discussed in greater detail in Ref. [132], where the interaction of soliton with impurity whose discrete mode has been excited beforehand is also discussed. The importance of the observations made in Refs [130–132] cannot be challenged. At the same time it ought to be noted that the numerical solution of Eqn [73] was obtained with the δ-function being replaced with the rectangular well $\delta(x) \to 1/\Delta x$ on the interval Δx in the neighborhood of $x = 0$. The effect of this replacement on the results of the numerical calculation has not been analyzed. Therefore, one has to be cautious about the precision of the velocity values and the location of the scattering windows.

Observe also that no passage windows have been discovered for the sG-equation. Attention must also be paid to the fact that for $\varepsilon = 0.7$ there were no scattering windows with $n < 6$. At the same time, quasi-resonances were discovered at those velocity values where the scattering windows were anticipated with $n < 6$. This phenomenon consists in that after the second interaction of the soliton with the impurity the soliton moved away to a considerable distance, but failed to leave the sphere of attraction. In other words, the energy returned to the soliton in the case of quasi-resonance was not sufficient for the soliton to break out of the attraction sphere of the impurity.

It may be said that in the sG-equation with an impurity (73), following a similar discovery with the GLH equation, a very beautiful effect of resonant reflection of soliton from an attractive impurity was observed.

4.8 Behavior of kink in the $\lambda\phi_2^4$ theory near an inhomogeneity. Interaction of two discrete modes

Further to Ref. [122], in Ref. [133] the problem of kink scattering by impurity for Eqn (65) was discussed in detail. The cases of $\mu = 0.5$ and $\mu = 0.7$ were considered. Unfortunately, the case of $\mu = 0.3$, treated more thoroughly in Ref. [122], was left out in Ref. [133]. Because of this, it is not possible to compare quantitatively the results of the numerical calculations obtained in Refs. [122] and [133]. The main features of the interaction between kink and impurity, found earlier in Ref. [122], were confirmed in Ref. [133]: the existence of the critical velocity, the scattering windows and the passage (three-bounce) windows.

Greater attention, however, was paid to the problem of the actual path of the kinetic energy of the kink interacting with the impurity. While in Ref. [122] the excitation of the discrete mode of the impurity was not discussed, in Ref. [133] a quantitative analysis was carried out to determine which of the discrete modes is excited more strongly. It was found that, both for $V > V_{\rm cr}$ and for scattering windows, the energy transferred to the discrete mode of kink excitation is greater than the energy used for exciting the discrete mode of the impurity.

In the present case, unlike the sG-equation with an impurity (73), the linear relation for the time between two bounces and the period T of the high-frequency field oscillations at the location of the impurity (81) more or less holds, but is far from establishing a correspondence between the value of T and the period of the discrete mode of the impurity. For example, for $\mu = 0.5$ the empirically found value is $T = 5.30$, which is closer to (and even slightly greater than) the period $T_1 = 2\pi/\omega_1 = 5.13$ of the discrete mode of the kink excitation, while the period of the impurity mode is much lower, $T_{\rm imp} = 2\pi/\tilde{\omega} = 4.75$. On the strength of this result, the authors of Ref. [133] claimed that in the $\lambda\phi_2^4$ theory the discrete mode of kink excitation is more important for the mechanism of the RRE than the discrete mode of the impurity. Unfortunately, the possible causes of T being greater than T_1 were not discussed in Ref. [133].

At the same time, the numerical data indicated quite clearly that in some cases the excitation of the discrete mode of the impurity strongly affects the behavior of the solution in the region of the bounce windows. For illustration let us turn to Fig. 16, where one readily detects the amplitude modulation of the high-frequency field oscillations at $x = 0$ between the two bounces. It seems, however, that the most conspicuous consequence of the interaction between the two modes is the absence of some bounce windows, corresponding to certain values of n, in the spectrum of windows. For example, no bounce windows were found for $\mu = 0.5$ at the anticipated velocity values V_7 and V_8. Instead, quasi-resonances were observed with a large time T_{23} between the second and third bounces, followed by the capture of the kink by the impurity. In addition, it was found that the final

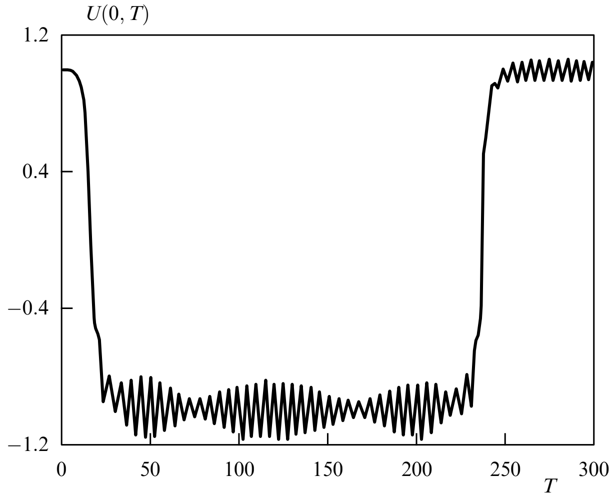

Figure 16. Amplitude modulation of the impurity displacement in a high-order resonance, $V_i = 0.3111$ and $\varepsilon = 0.7$.

velocity in the bounce windows decreases upon approaching the region of quasi-resonances.

This phenomenon may be regarded as evidence in favor of the mechanism of resonant energy transfer into both discrete modes. At the same time, the return of energy into the kinetic mode of the kink at the second bounce is not phased for the two modes, which for certain initial velocities may lead to a situation where the energy acquired by the kink at the second bounce is not sufficient for escaping from the sphere of action of the attractive potential. Observe, however, that the discovered pulsations can only tentatively be associated with the superposition of the two discrete modes, since a spectral analysis of the problem was not performed.

It would be interesting to note also that for $\varepsilon = 0.7$ in Ref. [133] the frequency of the discrete mode $\tilde{\omega}_{\rm disc} = 1.2288$ is rather close to the frequency $\omega_1 = 1.2247$ of the discrete mode of kink excitation. Unfortunately, the most interesting case of coincident frequencies has not been discussed in the existing literature.

Observe finally that the solutions were analyzed in Ref. [133] in the context of the effective Lagrangian which took into account the possible excitation of both discrete modes in addition to the potential interaction between the kink and the impurity. It was possible to reproduce qualitatively the main features of the field problem. In particular, the quasi-resonances discussed for the exact problem were found — that is, the absence of resonance scattering windows for certain values of n.

5. Time-dependent many-dimensional solutions

5.1 Domain walls, bubbles, and such like

As is known, the field-theory models with spontaneous symmetry violation have been used to construct the unified theory of weak and electromagnetic processes [2, 3]. Cosmological problems of spontaneous symmetry violation have been considered by many authors (see Ref. [134]).

The tentative domain structure of a vacuum and its cosmological implications were discussed in Ref. [25]. The theory for the real Higgs field [1] was considered, and the discussion was concerned with what happens with the solutions in the presence of regions belonging to different vacuums ϕ_+ and ϕ_-. As is known, in the one-dimensional case there is a topologically stable solution — a kink which links the vacuums ϕ_+ and ϕ_-. A similar solution in the three-dimensional case, not depending on y and z, is a domain wall. The additional energy associated with the transition from one vacuum to the other is localized on the wall. If, however, one of the vacuum solutions is contained within a restricted domain, the stable stationary solutions are not realized. As a matter of fact, the energy of the domain configuration turns out to be proportional to the area of the domain surface:

$$E = \mu S . \tag{82}$$

As a consequence, the system tends to reduce its volume, and the solution becomes non-stationary. The evolution of a spontaneously formed spherically symmetrical domain was studied in Ref. [25]. Assuming that the wall l is narrow compared with the dimensions of the domain, the following equation of motion for the domain radius $R(t)$ was proposed:

$$\ddot{R} = -\frac{2(1-\dot{R}^2)}{R} , \tag{83}$$

which follows from the effective Lagrangian

$$\mathcal{L}(R) = -4\pi\mu R^2 (1-\dot{R}^2)^{-1/2} \tag{84}$$

(see also Ref. [26], in which this treatment is applied to the case when one of the vacuums is metastable). The analysis of Eqn (83) reveals that the vacuum bubble, whose initial radius is R_0, starts to collapse. The time of collapse is $T \cong 1.3R_0$, which means that the walls of the bubble are moving at nearly the speed of light. The eventual fate of the bubble, however, is not clear, since Eqn (83) no longer holds when $R \sim l$ (the wall thickness).

5.2 Discovery of pulsating solutions

The subsequent fate of the bubble was first investigated in Ref. [135]. Special attention was paid to the fact that the solution of Eqn (82) for the collapsing bubble at large times leads to a periodical solution for the bubble radius $R(t)$:

$$R(t) = R_0 \operatorname{cn}\left(\frac{\sqrt{2}\,t}{R_0}, \frac{1}{2}\right) , \tag{85}$$

where $\operatorname{cn}(z, 1/2)$ is the elliptic cosine modulo 1/2. Accordingly, if Eqn (82) is at least an approximation of the solution of the exact field problem, one could hope that at some point the collapsing bubble could start to expand. In Ref. [135] this hypothesis was checked by studying the Cauchy problem for the equation

$$u_{tt} - u_{rr} - \frac{2}{r}\,u_r = -4u(u^2 - 1) \tag{86}$$

with the initial conditions

$$u(r,0) = \tanh\left[\sqrt{2}(r - R_0)\right] , \qquad u_t(r,0) = 0 . \tag{87}$$

It was found that the collapse of the bubble with an initial radius $R_0 = 5$ if followed by its expansion, is in agreement with the solution (85). The results of Ref. [135], however, could not be regarded as convincing proof of the existence of pulsations, since the energy integral was poorly conserved in the numerical simulation of the problem.

Shortly afterwards, a problem in the same formulation was solved numerically with better accuracy [136]. The pulsation of the bubble, observed in Ref. [135], was generally confirmed. The higher accuracy allowed the studying of both the evolution for longer times, and some important details of the evolution. It was found that the anticipated bounce pattern is realized not for any value of the initial bubble radius R_0. The most regular bounce pattern was observed at $R_0 = 3.875$ (the results are quoted in units adopted in Ref. [135]). This value of R_0 corresponded to five periods of almost elastic pulsations; after the sixth swing, however, the solution started to fall apart rapidly into individual spherical layers. At other initial values of R_0 the energy loss by radiation after the first expansion was much greater. Figure 17 shows the time dependence of the energy localized in the bubble for the radius of maximum expansion of the bubble. At $R_0 = 2.5$, 3.5 and 7.5 no bubbles were observed which pulsated steadily several times.

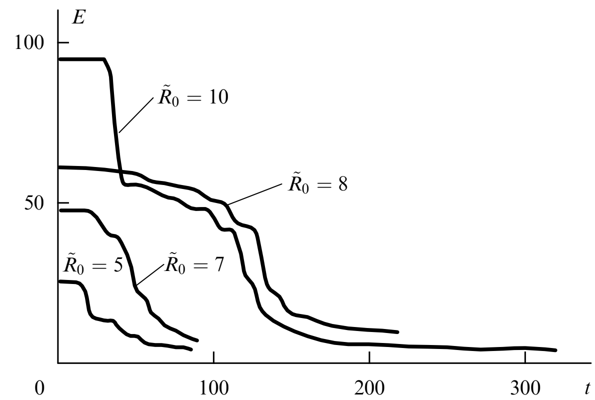

Figure 17. Time dependence of the energy of the bubble for different initial values of $\tilde{R}_0$ ($\tilde{R}_0 = 2R_0$).

At the same time, in Ref. [136] it was found that the collapse of a large-radius bubble is followed by the formation of a pulsating solution in the center of the bubble, with an amplitude of field oscillations of the order of unity, and a lifetime of about a hundred periods of oscillations.

The study of large-radius solutions, started in Refs [135, 136], was continued in Ref. [54]. The range of the initial values of bubble radius where the occurrence of repeated pulsations was likely ($3 \leqslant R_0 \leqslant 5$) was carefully investigated. The elasticity of the collision was found to depend strongly on the initial value of R_0, and the energy loss is the least when $R_0 \cong 3.8$. Then the ratio R_1/R_0 is close to unity, being equal to 0.98. Pulsating solutions of large radius $R \cong 4$ were also studied in Ref. [137].

Finally, the results of detailed investigation of the field function in the most elastic case, $R_0 = 3.875$, were reported in Ref. [138]. The time dependence of the solution indicates clearly, that in addition to the large-radius and large-amplitude pulsations of the bubble, the solution also describes oscillations near $r = 0$ with amplitude of about 0.7 with respect to the lower vacuum $u_- = -1$, whose frequency is several times that of the pulsations of the bubble. This is also confirmed by the time evolution at the origin at $R_0 = 3.875$, shown in Fig. 18. The repeatedly collapsing bubbles are regarded in Ref. [138] as a decaying resonant structure. A similar behavior of large-amplitude pulsating

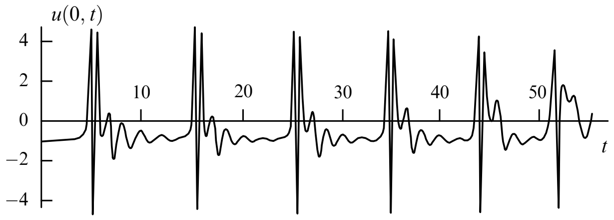

Figure 18. Time dependence of function $u(0, t)$ for $R_0 = 3.875$.

solutions was also discovered for the spherically symmetrical sG-equation [48, 137].

Attention in Ref. [139] was paid mainly to further study of the properties of 'pulsons', associated with the collapsing bubbles of the GLH and sG-equations. Differently from the former statement of the problem, the initial solution for the sG-equation

$$u_{tt} - \Delta u + \sin u = 0 \tag{88}$$

had the form of spherical bion,

$$u(r, 0) = 4 \arctan \frac{\varepsilon}{\omega \cosh \varepsilon r}\,, \qquad \omega = (1 - \varepsilon^2)^{1/2}\,, \tag{89}$$

with $\varepsilon/\omega = 10$. It was noted that three main characteristic stages of time evolution of the solution of the sG-equation can be distinguished. At the first stage ($t \cong 0-200$) the pulson is formed, and half of the energy is lost by radiation. The profile of the function varies quasi-periodically, with a period $T \cong 7.4$. At the second stage ($t \cong 200-630$) the amplitude of the pulson is clearly modulated with a period $T_{\mathrm{mod}} \cong 10T$, and falls off slowly from 2π to $4\pi/3$. The evolution of the full-fledged pulson practically does not depend on the selection of the initial conditions. After $T \cong 630$ the large-amplitude pulson rather rapidly (within $\Delta t \sim 80$) emits a large part of its energy as spherical waves and transforms into another pulson, whose amplitude is small compared to unity.

The dynamic behavior of the pulson of the GLH equation is similar to that of the pulson of the sG-equation, although at the second stage of evolution the shape of the pulson of the GLH equation is asymmetrical, and the modulation of the solution is more pronounced. At the third stage the amplitude decreases rapidly from 1 to 0.2 over a time $\Delta t \sim 300$.

Observe finally that 'heavy pulsons' were found numerically for the sG-equation in Ref. [140]. By contrast to the pulsons of the sG-equation discussed above, whose amplitude is approximately 2π, the range of stability of heavy pulsons corresponds to an amplitude of 4π. These pulsons oscillate for a long time, $t \sim 740$, then their amplitude sharply decreases, and at $t \sim 850$ they turn into pulsons whose amplitude is approximately 2π. This phenomenon is not observed in the GLH theory, where the vacuum is only twice-degenerate.

5.3 Time-dependent cylindrically symmetric solutions

The time evolution of bubbles symmetrical with respect to rotations for the two and three-dimensional sG-equation was studied in Refs [137, 141]. The initial condition was chosen in the form of the soliton of the sG-equation

$$u(r, 0) = 4 \arctan \exp\left[\gamma(V)(r - R)\right]\,, \tag{90}$$

whose initial velocity corresponded to an expansion. The so-called reflection effect was discovered: the initial expansion went on to a certain radius, and then the bubble started to collapse to $R \sim 1$. In Ref. [142] this effect was explained within the framework of the effective equation of motion for the bubble radius $R(t)$, which in the three-dimensional case coincided with Eqn [83] for the radius. In the two-dimensional case the pulson was first discovered in Ref. [143]. It was formed in the course of the evolution of the 2π soliton of the sG-equation with initial radius $R_0 = 3$. Two-dimensional pulsons were discovered in Ref. [144] in a broader range of initial conditions. A most comprehensive study of cylindrically symmetrical two-dimensional pulsons was performed in Ref. [145], where both the GLH and the sG-equations were considered. For the GLH equation

$$\phi_{rr} + \frac{1}{r}\,\phi_r - \phi_{tt} = \phi^3 - \phi\,, \qquad r \geqslant 0\,, \qquad t \geqslant 0 \tag{91}$$

solutions were studied corresponding to the initial conditions

$$\phi(r, 0) = \tanh \frac{\gamma(r - R_0)}{\sqrt{2}}\,, \qquad \gamma = (1 - V^2)^{1/2}\,, \tag{92a}$$

and

$$\phi_t(r, 0) = -\frac{\gamma V}{\sqrt{2}} \cosh^{-2} \frac{\gamma(r - R_0)}{\sqrt{2}} \tag{92b}$$

for different initial values of R_0 and V.

For the case of $V = 0$ the evolution of the solution was analyzed in a broad range of values of R_0: $2 \leqslant R_0 \leqslant 93$. The formation of pulsons is readily observed for small initial radii $R_0 = 2$ and $R_0 = 3$. At these values of R_0 the pulsons have large amplitudes and low modulation. At $R_0 = 3.7$ the formation of pulson is accompanied by a large loss of energy by radiation and considerable amplitude modulation of the pulson. In the range $4 \leqslant R_0 \leqslant 33$ the collapse of the bubble did not result in a pulson, and all the initial energy was lost by radiation. Then, a large-amplitude pulson was again formed at $R_0 = 40$. At $R_0 = 54$ and $R_0 = 93$ the wall was reflected after the collapse down to $R_1^{\mathrm{s}} \cong 8$ and $R_1^{\mathrm{s}} \cong 39$ respectively. Observations reveal that at lower values of R_0 (in case of $R_0 = 54$) the second collapse with $R_1^{\mathrm{s}} \cong 8$ resulted in the generation of waves, whereas the second collapse for $R_0 = 93$ resulted in a large-amplitude pulson in the middle. Thus, double pulsations of bubbles of macroscopic nature were observed in the range of $R_0 \geqslant 54$, similar to the pulsations at $3 < R_0 < 4$ in the three-dimensional case.

The pulson of Eqn (91) was also studied in the arrangement corresponding to the initial conditions

$$\phi(r, 0) = \tanh \frac{r - R_0}{\sqrt{2}} - \tanh \frac{r + R_0}{\sqrt{2}} + 1 \quad \text{and} \quad \phi_t(r, 0) = 0\,. \tag{93}$$

At $R_0 = 2$ the regular pulson region is formed practically from the start of the numerical simulation. The evolution of solutions was monitored up to the time $t = 6000$. Over this time the amplitude of oscillations of the pulson decreased by just 20%. It was found that the rate of amplitude fall-off slows down with time. The main period of pulsations varied slowly from $T = 6.4$ in the beginning to $T = 5.2$ at $t = 6000$. At the same time, the observed period of amplitude modulation increased from $T_{\mathrm{mod}} \cong 12$ at the start to $T_{\mathrm{mod}} \cong 34$ at $t = 6000$.

A similar trend with weak damping was also discovered for the sG-equation.

We see that both equations in the cylindrically symmetrical case give rise to long-lived pulsons, which are similar to the three-dimensional pulsons and the bions of the one-dimensional GLH equation. As regards the macroscopic pulsations of the bubble, the cases of $R_0 = 54$ and $R_0 = 93$ studied for the GLH equation only revealed single reflections of collapsing bubble with a considerable loss of energy.

5.4 The Kosevich – Kovalev approximation for spherically symmetric 'pulsons' and the problem of stability

Cylindrically and spherically symmetrical solutions were discovered in the two-dimensional and three-dimensional cases for the sG- and GLH equations. They may be classified as pulsons — high-frequency large-amplitude field oscillations localized over a short length. The second type of solution is represented by repeated reflections of a collapsing bubble. These solutions are only realized for certain particular initial conditions. It ought to be noted that spherically symmetrical quasi-stable pulsons were discovered and studied numerically for the nonlinear Klein – Gordon equation (NKG) [48]:

$$u_{tt} - \Delta u + u - u^3 = 0 \,. \tag{94}$$

They are localized weakly damped solutions with amplitude $|u| < 1$. In spite of the absence of a degenerate vacuum in the theory in question, such classical solutions may be relevant, for example, to the problem of generation of bubbles from the excited metastable vacuum. The possible explanation of the reflection of the bubble after collapse was discussed in Ref. [54] in terms of potentials. As noted, the considerable distortion of the walls at the time of collapse, and the strong radiation prevent the use of the potential description suitable for the one-dimensional case (see Section 2).

As far as the three-dimensional pulsons are concerned, soon after their discovery for the GLH equation the formalism for finding the small-amplitude pulsons was developed in Ref. [47]. This formalism is a generalization of the asymptotic Kosevich – Kovalev expansion [45] for the three-dimensional case. The equation for the main harmonic of the expansion of the field function (see Section 2) in the three-dimensional case is

$$\Delta_\rho f_1^{(0)} - f_1^{(0)} + \frac{3}{2}\left[f_1^{(0)}\right]^3 = 0 \,, \tag{95}$$

with boundary conditions

$$f_1^{(0)}(\rho) \to 0 \quad \text{at} \quad \rho \to \infty \quad \text{and} \quad \left.\frac{\mathrm{d}f_1^{(0)}}{\mathrm{d}\rho}\right|_{\rho=0} = 0 \,.$$

This equation, by contrast to the one-dimensional case, has a countable number of solutions [50], which differ in the number of nodes $n = 0, 1, 2, \ldots$. In this way, the existence of excited pulsons was predicted in Ref. [47]. Practically simultaneously with Ref. [47], a similar method of description of small-amplitude pulsons was applied in Ref. [48] to pulsons of the NKG equation (94). In Ref. [48] it was demonstrated numerically that pulsons with nodes are indeed the quasi-stable solutions of Eqn (94). Soon the method of asymptotic expansion was applied in Ref. [139] to pulsons of the GLH equation; it was shown that the masses of pulsons of the same amplitude with different number of nodes n relate as

$$m_0 : m_1 : m_2 : \ldots = 1 : 2 : 3 : 4 : 9 \ldots \tag{96}$$

Observe that the problem of stability of pulsons was discussed in Refs [146, 147, 138] within the framework of the exponential stability of solutions for the asymptotic series of functions in the Kosevich – Kovalev expansion.

The analysis of the three-dimensional situation in Ref. [138] revealed the zones of exponential instability which, however, are hard to interpret in terms of the amplitude dependence of the pulson.

In Ref. [54] it was noted that the Kosevich – Kovalev expansion by definition does not include the terms exponentially suppressed with respect to their amplitude ε, like

$$\delta u \sim \exp\left(-\frac{c}{\varepsilon}\right) , \tag{97}$$

where c is a certain constant. Such terms could have been responsible for the emission of waves from the pulson. As is known, in the one-dimensional case [55] the non-exponential corrections to the Kosevich – Kovalev expansions for the bion of the GLH equation were actually found (see Section 2). Shortly afterwards the exponentially damped dependence of radiation was confirmed in the letter [148] by estimates based on the perturbation theory applied to the deviations of the bion from the breather of the sG-equation. To our knowledge, the energy loss by a pulson in the three-dimensional case has not been discussed in this aspect.

5.5 Interpretation of solutions of pulsating bubble type. Resonant structures

Recall that in the one-dimensional case it was possible to give a quite consistent interpretation of the solutions found in the bounce windows, much different from the bions. In the three-dimensional and two-dimensional cases the situation with the resonant reflection of the collapsing bubble is not that clear. An attempt to calculate the radiative energy loss by a collapsing bubble for the sG-equation was made in Ref. [149], treating the term $(2/r)\phi_r$ as a perturbation. This approximation, which holds for the early stages of collapse, is certainly not valid when the dimensions of the bubble are comparable with the size of the wall. Recall that in the numerical experiments the radiation at the time of collapse is the most important.

In Ref. [150] the problem of collapse of the bubble of the GLH theory was considered on the basis of an approximation (linearized with respect to ψ) which characterizes the deviation of the exact solution from the kink:

$$\phi(x, t) = \tanh z + \psi(x, t) \,,$$

where $|\psi| \ll 1$. Attention was paid to the fact that the spectrum of the kink excitation in the $(1+1)$-dimensional case includes a discrete mode. The energy spent on excitation of this mode is not converted into radiation. Because of this, the collapse of the bubble of the $\lambda\phi^4$ theory at the stages of large bubble size must proceed slower than the collapse of the bubble of the sG-equation. Unfortunately, this statement was not proved by comparing the rates of bubble collapse for the two equations.

The study of solution of the $(3+1)$-dimensional GLH equation in Ref. [54] in the region of repeated reflection of the bubble ($R_0 = 3.875$) revealed that the reflection in the region of small values of $r \sim 1-2$ gives rise to a pulsating solution. By analogy with the multiple resonance windows (see Ref. [86]), a conclusion was drawn in Ref. [138] concerning the existence of a resonance relation between the period of oscillations of a bubble of radius R_0 and the period of a pulson of small amplitude ε. The lifetime of this state was estimated in Ref. [151]. Apparently, the formation of the reflection zone in the two-dimensional case for the GLH equation is also associated with the emergence of a pulson as indicated in Ref. [145]. At the same time, the resonant reflection of the collapsing bubble takes place for the spherically symmetrical sG-equation [139], which is an indication that the excitation of the discrete mode of an antikink of the $\lambda\phi^4$ theory is of minor importance.

The situation with many-dimensional pulsating solutions is not as clear as the one-dimensional case. As far as the pulson of small amplitude of the order of unity is concerned, its existence and long lifetime are well established. In all likelihood, the spherically symmetrical pulson is the counterpart of the bion of the GLH equation in the one-dimensional case (see Section 2). At the same time, the observed peculiarities of emission of radiation by pulson at different stages of its evolution are not quite clear. The developed technique of finding asymptotic small-amplitude solutions is not yet capable of explaining the last rather fast stage of the disappearance of the pulson as observed in numerical experiments. The problem of large resonant oscillations of the bubble with the nascent pulson calls for further research.

6. Time-dependent non-topological solutions of equations of charged fields

6.1 Q-balls: soliton solutions of complex scalar field equations

As mentioned above, the existence of conservation laws in addition to the conservation of energy and momentum lifts the ban on the existence of stable solitons which follows from the HD theorem [17, 18]. The first attempt to find stable three-dimensional solitons in the class of complex scalar fields with self-action was made in Ref. [28]. With this purpose, the Lagrangian for a complex massive scalar field with four-boson attraction was considered:

$$\mathcal{L} = \Psi_t \Psi_t^* - \nabla\Psi\nabla\Psi^* - m^2|\Psi|^2 + \frac{\lambda^2}{2}\,|\Psi|^4 . \tag{98}$$

The solutions of the equation of motion for the field Ψ were sought in the form

$$\Psi(\mathbf{r}, t) = \phi(\mathbf{r})\exp(-\mathrm{i}\omega t) , \tag{99}$$

where ω is a certain fixed frequency of the global phase of the field. The Lagrangian being invariant with respect to the U(1) transform $\Psi \to \exp(-\mathrm{i}\theta)\Psi$, the equations of motion conserve the integral of motion (the charge):

$$Q = -\mathrm{i}\int \mathrm{d}^{\mathrm{D}}x\,(\Psi\Psi_t^* - \Psi^*\Psi_t) . \tag{100}$$

Accordingly, the HD theorem does not literally apply to such fields, and therefore the solutions may be stable with respect to variations of the fields as long as the charge is conserved. However, no three-dimensional stable solutions for the Lagrangian (98) were obtained in Ref. [28]. Stable soliton solutions for complex fields were soon found in Ref. [152] for a Lagrangian different from Eqn (98), which included the six-boson repulsion:

$$\delta\mathcal{L} = \frac{\mu}{3}\,|\Psi\Psi^*|^3 . \tag{101}$$

It was found that for the values of the parameter $B = \mu m^2(1 - \tilde{\omega}^2)/\lambda^2$ which fall within the interval

$$0 < B < \frac{3}{16} \tag{102}$$

($\tilde{\omega} = \omega/m$), the corresponding equations of motion admit stable soliton solutions. In this way, the stability of soliton solutions for charged fields is not necessarily realized at all values of the parameters of solitons: the solitons may be stable at some values of charge and energy, and unstable at others.

An important contribution to the study of solitons in systems with charged fields was made in Refs [153–156]. These problems having been treated in numerous reviews (see, for example, Refs [29, 15, 157, 158]), we shall confine ourselves here to just a brief listing of the main results, paying more attention to those issues which are not covered in the above papers.

In the general case, the charge Q in Eqn (100) is linked with the frequency ω by the following relation:

$$Q = 2\omega\int \phi^2(\mathbf{r})\,\mathrm{d}\mathbf{r} . \tag{103}$$

For a sake of definition, we are going to consider theories with a self-action potential $U\big(|\Psi|\big)$ which have the following property:

$$U \to m^2\Psi\Psi^* \quad \text{at} \quad |\Psi| \to 0 , \tag{104}$$

where $m^2 > 0$. In terms of the field $\phi(\mathbf{r})$, the equation of motion takes the form

$$\nabla^2\phi + \omega^2\phi - \phi\,\frac{\mathrm{d}U(\phi^2)}{\mathrm{d}\phi^2} = 0 , \tag{105}$$

which may be interpreted as the equation for the scalar field ϕ with the potential

$$V(\phi) = \frac{1}{2}\,U(\phi^2) - \frac{1}{2}\,\omega^2\phi^2 . \tag{106}$$

The condition of existence of a soliton for this problem may be formulated as follows: if in the space of any dimensionality there is a range of values of the field ϕ such that $U(\phi^2) - m^2\phi^2 < 0$, then a non-topological soliton solutions exist for values of ω in the range

$$\omega_{\min}^2 \leqslant \omega^2 \leqslant m^2 , \tag{107}$$

where $\omega_{\min}$ is found from the condition that the functions $U(\phi)$ and $\omega_{\min}^2\phi^2$ have one point of contact ϕ_0. The soliton is then a spherically symmetrical solution which has maximum at its center and falls off exponentially at large distances:

$$\phi \sim (m^2 - \omega^2)^{1/2}\exp\big[-(m^2 - \omega^2)^{1/2}r\big] . \tag{108}$$

The particular solutions at all distances depend on the form of the potential, and in the three-dimensional cases are found numerically. Analytical solutions for certain potentials are only known for the one-dimensional case. For example, for the potential

$$U(|\Psi|^2) = m^2|\Psi|^2 - \frac{\lambda}{2}|\Psi|^4 + \frac{\mu}{3}|\Psi|^6 \tag{109}$$

the function $\phi(x)$ has the form [29, 159, 160]

$$\phi(x) = \frac{m}{\lambda}(1-\tilde{\omega}^2)^{1/2}\phi_0(\xi)\,, \tag{110}$$

where $\tilde{\omega} = \omega/m$, $0 \leqslant \tilde{\omega} \leqslant 1$, $\xi = kx(1-\tilde{\omega}^2)^{1/2}$, and

$$\phi_0(\xi) = 2\sqrt{3}\,(a\cosh^2\xi - b\sinh^2\xi)^{-1/2}\,, \tag{111}$$

$$a = 3 + (9-48B)^{1/2}\,, \qquad b = 3 - (9-48B)^{1/2}\,,$$

$$B = \frac{\mu m^2}{\lambda}(1-\tilde{\omega}^2)^{1/2}\,, \qquad B < \frac{3}{16}\,.$$

Observe that at frequencies $\omega^2 \to \omega^2_{\min}$ the shape of the soliton becomes trivial and weakly depends on the model of the potential. Indeed, squaring the equation of motion in the stationary case for the field ϕ, we get the equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)^2 - V(\phi)\right] = -\frac{D-1}{r}\left(\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)^2. \tag{112}$$

In the case $D = 1$ this equation may be treated as the motion of a point with coordinate ϕ at time r in the potential $-V(\phi)$. At $D > 1$ the right-hand side may be regarded as friction. Since at $\omega^2 \to \omega^2_{\min}$ the 'potential' $V(\phi)$ at the point ϕ_0 touches the axis of abscissas, the point placed at $\phi(0) \cong \phi_0$ will for a long time remain near its initial position. This means that the field $\phi \cong \phi_0$ inside the soliton is constant, and its size R becomes large. Since a nonzero field $\phi = \phi_0$ inside a sphere corresponds to the constant charge density $2\omega_{\min}\phi^2$, this solution may be regarded as a 'charged' sphere — termed a Q-ball by Coleman [156]. The existence of such a solution as this charged macroscopic ball has many applications which will be mentioned below.

6.2 Stability of Q-balls with respect to large perturbations

Charged solitons are not the only type of solution in the class of fields under consideration. Solutions of the plane wave type also carry a charge. In view of this it would be interesting to find which of the solutions with a given charge has less energy — that is, which of the solutions is stable. In the one-dimensional case the solution normalized to the charge Q in the region of the size L is

$$\Psi = \left(\frac{Q}{2\omega L}\right)^{1/2}\exp(\mathrm{i}kx - \mathrm{i}\omega t)\,. \tag{113}$$

As $L \to \infty$ the nonlinear effects can be disregarded, and in this approximation the dispersion law for the solution (113) is

$$\omega^2 = k^2 + m^2\,. \tag{114}$$

It would be interesting to note that $\omega^2 > m^2$ for plane-wave solutions, but $\omega^2 < m^2$ for solitons. Observe that for the soliton solution the following relation holds:

$$\frac{\mathrm{d}E}{\mathrm{d}Q} = \omega\,. \tag{115}$$

When $\omega^2 \to m^2$, the charge Q of the soliton in the one-dimensional case tends to zero by virtue of the asymptotic relation (108):

$$Q = 2\omega\int\phi^2(x)\,\mathrm{d}x \sim (m^2-\omega^2)^{1/2}\,. \tag{116}$$

By contrast, the charge of the system increases when $\omega^2 \to \omega^2_{\min}$, and up to terms of the order of $O(1)$, the following relation holds:

$$E = \omega_{\min}Q\,, \tag{117}$$

which defines the ultimate slope of the curve $E(Q)$ for the soliton at large values of Q.

Thus, if the theory admits a soliton in the one-dimensional case, the soliton will exist for any value of the charge Q. Its energy is then less than the energy of the plane wave with the same charge Q. In this sense the soliton solution may be considered stable with respect to disintegration into solutions of the wave type.

The situation is not that simple when the dimensionality of the problem is higher. When $\omega^2 \to m^2$, the charge of the system, as different from Eqn (116), is

$$Q \sim \begin{cases} \text{const} & \text{at} \quad D = 2\,, \\ \infty & \text{at} \quad D \geqslant 3\,. \end{cases} \tag{118}$$

The most complicated situation is encountered when the number of dimensions is $D = 3$. Then the energy of the system is a two-valued function of the charge Q. When $\omega^2 \to m^2$, the charge $Q \to \infty$, and the energy $E \approx Qm$. The difference $E - Qm$ is positive, which means that the upper branch of the soliton solution approaches the plane-wave limit $E = Qm$ from above. As ω decreases, the graph $E(Q)$ goes to the point C, which lies above the plane-wave straight line $E = Qm$ by virtue of Eqn (115). As ω continues to decrease, the energy and the charge increase in accordance with the lower branch. At $\omega^2 \to \omega^2_{\min}$ the charge Q again goes to infinity. Then the ultimate slope of the function $E(Q)$ is defined by Eqn (115) with $\omega = \omega_{\min}$. The function $E(Q)$ has two critical points: a cusp-type singularity (point C), and the point of 'absolute' stability S. There are no soliton solutions for $Q < Q_{\mathrm{C}}$. For $Q > Q_{\mathrm{S}}$ the lower state in terms of energy is always a soliton. This state cannot fall apart into waves, and in this sense is absolutely stable. Neither can it split into two solitons with smaller charges (fission) because the curvature of the solution is $\partial^2E/\partial Q^2 < 0$. At the same time, the segment CS of the lower branch is unstable with respect to disintegration into plane waves.

The above analysis of stability is not quite consistent. On the one hand, the analysis reveals that the soliton cannot break down into plane waves if the energy of the plane-wave solution is large. On the other hand, it is not quite clear whether or not the soliton and the plane wave are separated by a barrier in the space of field configurations. Observe also that other charge-carrying field configurations are also possible in principle — for example, a singular solution contracting into a point. Stability with respect to disintegration into such configurations has not been discussed. In this regard the study of stability with respect to small deformations is more consistent.

6.3 Stability of Q-balls with respect to small deformations

Let us first consider the problem of the stability of the soliton solution in the one-dimensional case, using the example of the self-action potential (109) the soliton solutions for which are known in analytical form [28, 152, 159, 160]. The stability of solutions is determined by the time dependence of a small deviation $\delta\Psi(x,t)$ from the soliton solution $\Psi_0(x,t)$:

$$\Psi(x,t) = \Psi_0(x,t) + \delta\Psi(x,t)\,. \tag{119}$$

By assumption, $\delta\Psi(x,t)$ is small — in any case, at the early stage of evolution, and the function $\Psi_0(x,t) = \exp(-\mathrm{i}\omega t)\phi_0(x)$ is given by Eqn (110), (111). It is convenient to represent the function $\delta\Psi(x,t)$ in the form

$$\delta\Psi(x,t) = \eta(x)\exp\left[-\mathrm{i}(\Omega+\omega)t\right] + \chi^*(x)\exp\left[\mathrm{i}(\Omega^*-\omega)t\right], \tag{120}$$

where $\Omega = \Omega_\mathrm{r} + \mathrm{i}\Omega_\mathrm{i}$ is the complex parameter which is to be defined. Linearizing the equation of motion with respect to $\delta\Psi$ and $\delta\Psi^*$, we obtain a set of linear differential equations in the functions $\eta(x)$ and $\chi(x)$, which is conveniently represented in matrix form

$$\left[v^2 I - 2vD - H(\xi)\right]\zeta(\xi) = 0\,, \tag{121}$$

where $v = \widetilde{\Omega}(1-\tilde{\omega}^2)^{1/2}$, $\widetilde{\Omega} = \Omega/m$; I and D are the diagonal and the antidiagonal matrices of differential operators:

$$H_{11}(\xi) = -\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + 1 - 3\phi_0^2(\xi) + 5B\phi_0^4(\xi)\,,$$

$$H_{22}(\xi) = -\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + 1 - \phi_0^2(\xi) + B\phi_0^4(\xi)\,. \tag{122}$$

The eigenfunctions are

$$\zeta = \begin{pmatrix} \tilde{\eta} & +\tilde{\chi} \\ \tilde{\eta} & -\tilde{\chi} \end{pmatrix}, \tag{123}$$

where the functions $\tilde{\eta}$, $\tilde{\chi}$ differ from η, χ in the argument. Now, if $\widetilde{\Omega}_\mathrm{i} \neq 0$, then $\delta\Psi(x,t)$ will grow exponentially with time and destroy the solution $\Psi(x,t)$, which means that $\widetilde{\Omega}_\mathrm{i}^{-1}$ is a measure of the 'disintegration time' of the soliton. As found in Ref. [160], if the solutions $\zeta(\xi)$ are sought in the class of continuous and limited functions, such that $\int \zeta^*\zeta\,\mathrm{d}\xi < \infty$, then the behavior of $\widetilde{\Omega}_\mathrm{i}$ and $\widetilde{\Omega}_\mathrm{r}$ as functions in $\tilde{\omega}$ on the interval $0 < \tilde{\omega} < 1$ is quite unambiguous. For any value of the parameter B in the interval (102) there is a critical frequency $\tilde{\omega}_\mathrm{cr}$, $0 < \tilde{\omega}_\mathrm{cr} < 1$, such that for $\tilde{\omega} > \tilde{\omega}_\mathrm{cr}$ the eigenvalues of $\widetilde{\Omega}$ are entirely real (the corresponding solitons are stable), and entirely imaginary below the critical frequency. The graph of $\widetilde{\Omega}_\mathrm{i}$ and $\widetilde{\Omega}_\mathrm{r}$ versus $\tilde{\omega}$ is plotted in Fig. 19. At $\tilde{\omega} \to 1$ the real values decrease, and $\widetilde{\Omega}_\mathrm{r} \cong 1 - \tilde{\omega}$. The existence of the real frequency $\widetilde{\Omega}_\mathrm{r}$ points to the presence of a discrete mode in the excitation spectrum of the soliton. In the case of $\tilde{\omega} \to 1$, which corresponds to the nonrelativistic limit, this discrete mode apparently corresponds to the known bound two-soliton state for the NSE, found in Ref. [161] for a particular relationship between the parameters of the problem $\eta_1^2 \gg \eta_2^2$ (in the notation of Ref. 161]). A relevant remark can be also found in Ref. [162].

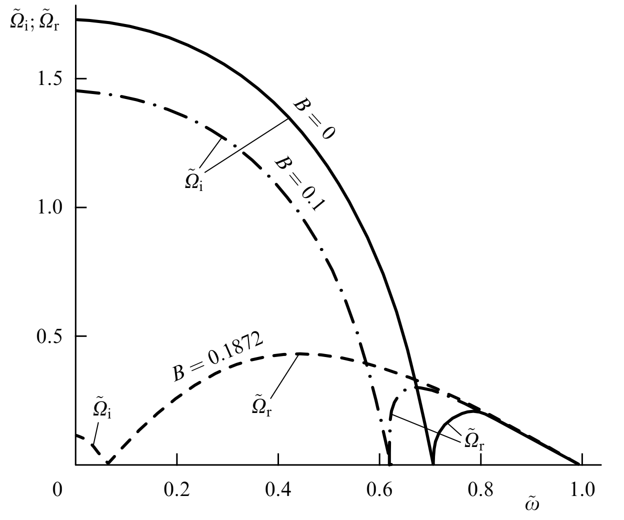

Figure 19. Real and imaginary eigenvalues of problem (121) as function of $\tilde{\omega}$ for different values of B.

The existence of the critical frequency $\tilde{\omega}_\mathrm{cr}$ clearly demonstrates the importance of the infinitesimal analysis of stability. Indeed, global analysis in the one-dimensional case tells us that the soliton is stable with respect to disintegration into plane waves. Now we see that the soliton is exponentially unstable at $\tilde{\omega} < \tilde{\omega}_\mathrm{cr}$.

As follows from Ref. [29], in the three-dimensional case the entire lower branch of the soliton solution is stable with respect to small perturbations. Certain issues of stability of charged solitons in three-dimensional cases have also been discussed in Refs [163, 164].

6.4 Renormalizability of theory and Q-balls

The discovered stable solitons in the three-dimensional cases exist in a much broader range of theories than discussed above. Indeed, the requirement that the potential $V(\phi)$ in Eqn (106) should be negative has made it necessary to consider theories with self-action in at least the sixth degree. In $D = 3$ such theories cannot be renormalized. However, there are examples of renormalizable theories which admit solutions of the Q-ball type. One such example was discussed in Ref. [153]. This study was concerned with a system of interacting scalar fields χ and ϕ, the former being real and the latter complex. The Lagrangian of the theory in question is

$$\mathcal{L} = \partial_\mu\phi^+\,\partial^\mu\phi + \frac{1}{2}\,\chi^\mu\chi_\mu - f^2\chi^2\phi^+\phi - U(\chi)\,, \tag{124}$$

where the self-action of the field χ has the form of the Higgs self-action:

$$U(\chi) = \frac{1}{8}\,g^2(\chi^2 - \chi_\mathrm{vac}^2)^2\,. \tag{125}$$

In the vacuum state, the field χ imparts a mass of $m = f\chi_\mathrm{vac}$ to the field ϕ. The search for soliton solutions is associated with the substitution

$$\chi(\mathbf{r},t) = \chi_\mathrm{vac}A(\boldsymbol{\rho})\,,$$

$$\phi(\mathbf{r},t) = \frac{1}{\sqrt{2}}\,\chi_\mathrm{vac}B(\boldsymbol{\rho})\exp(-\mathrm{i}\omega t)\,, \tag{126}$$

where $\boldsymbol{\rho} \equiv g\chi_{\text{vac}}\mathbf{r}$. Then for the functions A and B we get a system of coupled differential equations

$$\begin{cases} \nabla^2 A - \varkappa^2 B^2 A - \dfrac{1}{2}(A^2 - 1)A = 0\,, \\ \nabla^2 B - \varkappa^2 A^2 B + v^2 B = 0 \end{cases} \tag{127}$$

with the boundary conditions $\mathrm{d}A/\mathrm{d}\rho = \mathrm{d}B/\mathrm{d}\rho = 0$ at $\rho = 0$; $A \to 1$, $B \to 0$ at $\rho \to \infty$. As was defined for the system (127), $v = (\omega/g)\chi_{\text{vac}}$, and $\varkappa = (m/g)\chi_{\text{vac}}$. Numerical analysis indicates that solutions with the specified boundary conditions exist, and the energy of the system as function of the charge behaves like $E(Q)$ in the three-dimensional case.

6.5 Q-lumps

An interesting variety of Q-balls, which displays features of topological solitons, is represented by the so-called Q-lumps discussed in Ref. [165]. They exist for the modified σ-model in $(2+1)$-dimensions.

Consider a triplet $\boldsymbol{\phi} = \{\phi_1, \phi_2, \phi_3\}$, $\boldsymbol{\phi}^2 = 1$ (the field $\boldsymbol{\phi}$ takes on values on the sphere of unit radius). The Lagrangian describing these fields is

$$\mathcal{L} = \frac{1}{4}\,\partial_\mu \boldsymbol{\phi}\,\partial^\mu \boldsymbol{\phi} - \frac{1}{4}\,\alpha^2(1 - \phi_3^2)\,. \tag{128}$$

The finite-energy solutions at large distances must comply with the asymptotics $\phi_3 \to 1$. Thus, a point with the coordinates (x, y) is mapped into the space of fields, $S^2_{\text{space}} \to S^2_{\text{fields}}$.

The solutions are characterized by the degree of mapping N of sphere into sphere (that is, by the topological charge N). In terms of the newly-introduced complex function $u = (\phi_1 + \mathrm{i}\phi_2)/(1 - \phi_3)$ we have the Lagrangian

$$\mathcal{L} = \left[\partial_\mu u\,\partial^\mu \bar{u} + \alpha^2 u\bar{u}\right](1 + u\bar{u})^{-2}\,, \tag{129}$$

and the equations of motion follow from this Lagrangian. In terms of field u the topological charge N and the charge Q are

$$N = \frac{\mathrm{i}}{2\pi}\int \left[\partial_x u\,\partial_y \bar{u} - \partial_y u\,\partial_x \bar{u}\right](1 + u\bar{u})^{-2}\,\mathrm{d}^2 x\,, \tag{130a}$$

$$Q = \mathrm{i}\int \left[\bar{u}\,\partial_t u - u\,\partial_t \bar{u}\right](1 + u\bar{u})^{-2}\,\mathrm{d}^2 x\,. \tag{130b}$$

Using the explicit representation of the energy, N, and Q, one may prove the following inequality:

$$E \geqslant 2\pi|N| + |\alpha Q|\,.$$

For given N and Q, the lowest value of energy is attained when the function u satisfies the following conditions:

$$\partial_i u \pm \mathrm{i}\varepsilon_{iy}\,\partial_j u = 0\,, \tag{131a}$$

$$\partial_t u \pm \mathrm{i}\alpha u = 0 \tag{131b}$$

$(i = x, y)$. The first of these, as in the case of pure O(3)-symmetry, defines the dependence of u on only $z = x + \mathrm{i}y$ or $\bar{z} = x - \mathrm{i}y$ — that is, u is an analytical function. Then the equation defines the time dependence of the solution:

$$u(z, t) = u_0(z)\exp(\pm \mathrm{i}\alpha t)\,. \tag{132}$$

The expression (132) is the general solution. The simplest solution with a given topological charge N has the form

$$u(z, t) = \left(\frac{\lambda}{z}\right)^N \exp(\mathrm{i}\alpha t)\,, \tag{133}$$

and its energy is

$$E = 2\pi N + \frac{2\pi^2\alpha^2\lambda^2}{N^2 \sin N\pi}\,. \tag{134}$$

Observe that the energy is only finite when $N \geqslant 2$.

6.6 Q-ball type solitons interaction

Problems of the interaction of Q-ball type solitons in the one-dimensional case were discussed in Refs [166, 30, 162, 151, 67]. In Ref. [162] the one-dimensional version of the $\lambda|\phi|^n$ theory ($n = 4$) was considered defined by the Lagrangian (98). In this case the theory admits a one-soliton solution

$$\Psi_{\mathrm{S}} = \frac{\sqrt{2}}{\lambda}(m^2 - \omega^2)^{1/2}\exp(-\mathrm{i}\omega t)\cosh^{-1}\left[x(m^2 - \omega^2)^{1/2}\right]\,, \tag{135}$$

which is stable at

$$\omega_{\text{cr}} \leqslant \omega \leqslant m\,, \qquad \omega_{\text{cr}} = \frac{1}{\sqrt{2}}\,. \tag{136}$$

When ω is close to m, $m - \omega \ll m$, the numerically studied scattering of two solitons is almost elastic. This is due to the fact that when $\omega \to m$ the situation is close to nonrelativistic, and the equation of motion for the field Ψ can be reduced to the NSE, integrated by the MISP. The scattering of solitons for the NSE is elastic [12]. It is interesting to note that the interaction of solitons of opposite signs [that is, the interaction between soliton (S) and antisoliton (A)] was also found to be elastic at the collision velocity $V \sim 0.3-0.4$. As demonstrated in Ref. [62], the elasticity of interaction ought to be violated at low velocities, when the time of interaction between the solitons satisfies the condition

$$t_{\text{int}} \sim m(m^2 - \omega^2)^{-1/2} \equiv t_{\text{char}}\,. \tag{137}$$

When $t \ll t_{\text{char}}$, the interaction of solitons may be neglected. In the extreme case of zero velocity, a quasi-stable bound soliton–antisoliton state was found in Ref. [166]. As far as the SS collisions are concerned, the almost elastic collisions at high velocities give way to instability and inelastic collisions at low velocities. Numerical simulation reveals that there is a certain critical velocity V_{cr}, below which the solution is changed dramatically. When solitons collide (in the case of SS collisions [162], or SSS collisions [67]), the function $\left|\Psi(x, t)\right|$ near $x = 0$ exhibits a tall narrow peak. Because of the absence of a positively defined potential of self-action in the case in question, a further increase in the field amplitude at $x = 0$ becomes energetically advantageous, and the two-soliton (or three-soliton) solution becomes singular.

However, in a narrow range of velocities near V_{cr}, a bound long-lived two-soliton state was discovered [162]. A similar phenomenon was observed in Ref. [67] for the SSS system. In case of the SS system with $\omega = 0.95$, the critical velocity was $V_{\text{cr}} \cong 0.240$. SS collisions with $V \cong 0.255$ result in ordinary scattering of solitons. The study of a process with $V = 0.250$ revealed that the solitons do not pass through one another, but stick together. The behavior of the field $\left|\Psi(0, t)\right|$ in this case is shown in Fig. 20. Apparently, the existence of large

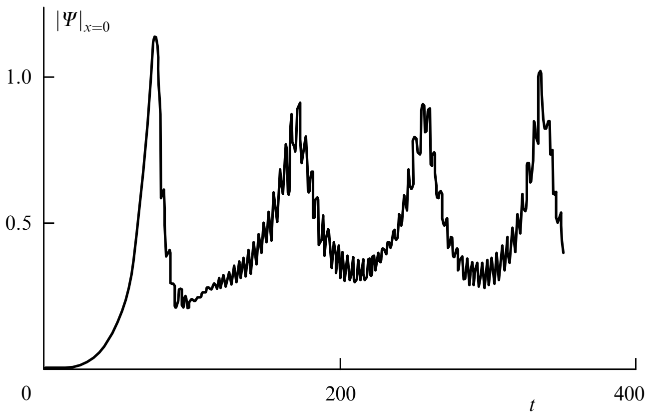

Figure 20. Function $|\Psi(0, t)|$ in the oscillating regime: $\omega_1 = \omega_2 = 0.95$; $V_1 = V_2 = 0.250$.

period oscillations is associated with the solitons having a discrete excitation mode in the region of exponential stability [160].

7. Conclusions

Speaking of the practical applications of the results discussed above, we shall first of all mention the problems of cosmology. In this connection we refer to the studies [25–27, 135, 54], concerned with domain structures in a degenerate vacuum, their interactions and the formation of bubbles in both planar and curved space [167]. As pointed out in Ref. [168], such vacuum bubbles in the theory of the interacting complex scalar field $\lambda|\phi|^6$ and electromagnetic fields with due account for gravity lead to the formation of black holes. In Ref. [169] the interaction of broad domain walls with gravity taken into account resulted in the collapse and formation of black holes. The behavior of solitary soliton-type waves for scalar fields or systems of scalar and complex charged fields was considered in inflational cosmological models [170–172].

In astrophysics, the solutions of Q-ball type are widely used for describing boson stars [173, 174]. The possibility of the unconventional annihilation of such extensive objects upon collision was noted in Ref. [162].

One of the sciences which make use of the phenomena discussed in this review is solid state physics. Here the formation and interaction of domain structures occur quite naturally [92]. The details of interaction of solitary waves in dynamic systems simulating solid-state problems can be found in Ref. [175]. To give an example, we refer the reader to Section 42, Chapter VIII in Ref. [175], where soliton waves in Peierls systems and the pinning of kinks in the $\lambda\phi^4$ theory are considered. The theory of superconductivity in ^{3}He also allows for the existence of nontrivial topological objects (see Chapter V in Ref. [176]).

It ought to be noted that the phenomena under consideration are by themselves a beautiful example of fractal structures (see also review [177]).

Some relevant mathematical aspects are considered in Ref. [178]. The linkage between the nonlinearity of processes with dynamic behavior and information theory was discussed in review [179].

The problem of the domain structure of a vacuum was first formulated in the paper by Ya B Zel'dovich, I Yu Kobzarev and L B Okun' [25], and was discussed at length at the seminars of the Theoretical Department of the Institute for Theoretical and Experimental Physics. In the same paper the evolution of the domain bubble was discussed for the first time. Further research of I Yu Kobzarev and his colleagues was concerned with the detailed study of the evolution of domain bubbles. This review is in homage to the memory of I Yu Kobzarev.

Acknowledgements. For fruitful discussions at different stages of their work the authors wish to express their gratitude to N A Voronov, I L Bogolyubskiĭ, V Zakrzhevskiĭ, N B Konyukhova, L B Okun', A V Smilga, Yu P Rybakov, G Leitwiller, V G Makhan'kov, M I Monastyrskiĭ, I S Shapiro, and V V Okun'kov for technical assistance in the preparation of artwork.

The work was supported by the Russian Foundation for Fundamental Research (Grant No. 95-02-04681A) and in part by Grant INTAS-93-633 to one of the authors (AEK).

Glossary

Kink (loop, twist) — a term used historically (in the meaning of loop) for a special solution for a circular pendulum; in the meaning of twist, the time behavior of the phase of this solution defines the profile of one-soliton solutions for the sG-equation. Subsequently, all stationary solutions of nonlinear equations of a similar profile (with a twist) came to be referred to as kinks. The exception is the sG-equation, where the term 'soliton' is more common.

Q-ball — a spherically symmetrical solution for charged scalar fields. A term coined by Coleman [156]. The solution appears as a sphere of radius R and charge Q. Later this definition was used for a broad class of soliton solutions with the additional conservation of an integral of motion like U(1)-charge.

Q-lump — a topologically nontrivial solution of Q-ball type. Carries simultaneously conserved topological charge (N) and conventional charge (U(1)).

Toda chain [125] — a model of a one-dimensional discrete chain of atoms with an exponential interaction between closest neighbors. Initially used for studying the propagation of shock waves.

Lyapunov index — functional defined on the set of functions $\{f(t)\}$ and given by

$$\lambda[f] = \overline{\lim_{t\to\infty}} \left[\frac{1}{t} \ln|f(t)| \right] .$$

The positive Lyapunov index for $\{f(t)\} = \{\exp \lambda t\}$ implies the exponential divergence of any two initially close points in the phase space of a dynamic system.

Breather — a general name for localized solutions of differential equations with time-periodic amplitude. First used for a special solution of the sG-equation.

Bion — a breather in the $(1+1)$-dimensional problem, arising as a bound state in the kink-antikink system.

Triton (wobbling kink) — an oscillating kink; a localized excited state of a kink, first discovered in collision of three kinks.

Bounce window — a term used for the solution in the region of resonance reflection windows. Bounce windows are distinguished by the number of bounces; for example, a two-bounce window, etc.

Pinning of a kink — the capture (trapping) of a kink (dislocation) by an inhomogeneity.

Quasi-closed orbits — a representation in the phase space of solutions of equations of motion for the effective Lagrangian (the case of bounce windows)

Pulson — a spherically symmetrical breather for $D \geqslant 2$.

Resonant structure — a bound state of bubble and pulson resulting from the first collapse for $D = 2, 3$ at certain initial radii of the bubble ($R \geqslant l$).

Fluxon — a quantum of magnetic flux described by the soliton of the sG-equation.

Abbreviations

KdV — Korteweg – de Vries (equation)
NSE — Nonlinear Schrödinger Equation
sG — sine-Gordon (equation)
GLH — Ginzburg – Landau – Higgs (equation)
MsG — Modified sine-Gordon (equation)
DsG — Double sine-Gordon (equation)
MISP — Method of Inverse Scattering Problem
HD — Hobbard – Derrick (theorem)
$K\bar{K}$ — kink – antikink (interaction)
SS — Soliton – Soliton (collision)
SA — Soliton – Antisoliton (collision)
SSS — three-soliton (collision)

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