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Magnons, magnetic polaritons, magnetostatic waves

M I Kaganov, N B Pustyl'nik, T N Shalaeva

Contents

Abstract. **Electrodynamical properties of magnetically ordered media are analysed theoretically. Two types of magnetic materials, namely ferromagnets and antiferromagnets, are considered and the effect of the magnetic subsystem on the properties of the host metal is discussed. Various types of elementary excitations, such as magnetic polaritons, magnetostatic waves, and magnons, are examined within the conventional quasiparticle framework. The dispersion of the quasiparticles as a function of the relation between the characteristic frequencies of electrical and magnetic nature is described over a wide range of frequencies. Particular attention is given to the statistical thermodynamics and kinetics of magnetic materials. The possibility of Bose – Einstein condensation in a magnon gas is analysed and the kinetic theory of electron-assisted magnetostatic wave damping in a ferromagnetic metal is developed. The dispersion of surface polaritons is examined in detail. Examples of solutions of spatially non-uniform problems are discussed with allowance made for spatial dispersion, which show this latter to be of particular importance in antiferromagnets.**

M I Kaganov, N B Pustyl'nik, T I Shalaeva M V Lomonosov Moscow State University, Vorob'evy Gory, 119899 Moscow, Russia
Tel. (7-095) 939 48 11
E-mail: kaganov@kapitza.ras.ru, natasha@ljar17.jinr.dubna.su

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1. Introduction

Magnetically ordered media are known to possess specific electromagnetic properties. The physical basis of these properties is ferro- and antiferromagnetic resonance which accounts for the induction of specific elementary excitations (spin waves or magnons) by an electromagnetic wave.

In the majority of cases resonant frequencies of magnetic nature lie in the radio-frequency range. If dissipative processes and spatial dispersion are neglected, magnetic permeability at a resonant frequency turns into infinity and, due to this, the length of the electromagnetic wave vanishes. At first sight, this makes equations of electrodynamics of continuous media inapplicable. However, taking into account dissipative processes and spatial dispersion of magnetic permeability, unessential far from resonance, results in that the wavelength remains finite and much longer as compared with the interatomic distance even though it is significantly smaller than in a vacuum. This means that the electromagnetic properties of magnets can and must be described by equations of macroscopic electrodynamics:

$$\operatorname{rot}\mathbf{H} = \frac{4\pi}{c}\,\mathbf{j} + \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}\,, \qquad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\,. \tag{1.1}$$

Equations of electrodynamics are written here in the form that makes them applicable to both dielectrics ($\mathbf{j} = 0$) and metals ($\mathbf{D} = 0$); the conventional notations are used.

In an infinite medium, the role of material equations which make the system (1.1) complete is played by the linear relation† between the fields and their inductions written for the Fourier components:

$$D_i = \varepsilon_{ik}(\omega, \mathbf{k})E_k\,, \qquad j_i = \sigma_{ik}(\omega, \mathbf{k})E_k\,, \qquad B_i = \mu_{ik}(\omega, \mathbf{k})H_k\,. \tag{1.2}$$

Instead of two equations (for current density $\mathbf{j}$ and induction $\mathbf{D}$), it is possible to use the generalized induction $\widetilde{\mathbf{D}}$ related to the electric field $\mathbf{E}$ by the effective dielectric permittivity $\hat{\varepsilon}^{\rm eff}$:

$$\widetilde{D}_i = \varepsilon^{\rm eff}_{ik}(\omega, \mathbf{k})E_k\,, \qquad \varepsilon^{\rm eff}_{ik} = \varepsilon_{ik}(\omega, k) + \frac{4\pi {\rm i}\sigma_{ik}(\omega, k)}{\omega}\,. \tag{1.3}$$

In the case of metals, displacement current responsible for polarization of the ion lattice [item $(1/c)(\partial\mathbf{D}/\partial t)$] may be neglected with a good accuracy; then,

$$\varepsilon^{\rm eff}_{ik} = \frac{4\pi {\rm i}\sigma_{ik}(\omega, k)}{\omega}\,. \tag{1.4}$$

An objective of the microscopic theory is to compute tensors of dielectric permittivity ε_{ik}, electric conductivity σ_{ik}, and magnetic permeability μ_{ik} using more or less realistic models. Naturally, the existence of ferro- or antiferromagnetic resonance is apparent due to the specific (resonant) dependence of tensor μ_{ik} components on the frequency ω.

This review is focused on 'electromagnetic consequences' of the resonant frequency dependence of magnetic permeability rather than on the model-based computation of the above tensors (as a rule, the simplest and most patent models will be used for the purpose). The symmetry of electric and magnetic fields in an infinite medium is practically unbroken, and the difference between resonant properties for which dielectric permittivity and magnetic permeability are responsible is normally reduced to different frequency bands. Resonant frequencies of $\hat{\varepsilon}$ are confined to the optical range (sometimes, in ion crystals, to the infrared one) whereas those of $\hat{\mu}$ are in the radio-frequency range (occasionally, in antiferromagnets, in the submillimeter range). This difference accounts for striking modification of both the experimental technique and the mode of the description of theoretical and experimental results, even if it does not substantially affect the theory of phenomena.

The symmetry between fields is broken in finite media. Even in the simplest case of an isotropic non-gyrotropic medium, dielectric permittivity and magnetic permeability occur in formulas not only as the refractive index $n = \sqrt{\varepsilon\mu}$ but in a different form too.

In order to demonstrate how the non-uniformity of the problem reveals itself (halfspace, plate), we shall consider a few simple examples from electrodynamics of continuous media, on the assumption that

$$\mathbf{D} = \varepsilon(\omega, k)\mathbf{E}\,, \qquad \mathbf{B} = \mu(\omega, k)\mathbf{H}\,. \tag{1.5}$$

† In this review, we confine ourselves to the linear approximation. Moreover, the basic assumption is formulated as $ak \ll 1$, where a is the interatomic distance, $\mathbf{k}$ is the wave vector, and $2\pi/k = \lambda$ is the wavelength. In other words, we examine here macroscopic oscillations and waves.

Taking into account spatial dispersion (dependences of ε and μ on the wave vector k) and dissipation (the existence of ε'' and μ'') is necessary only when their neglect may result in the loss of physical sense.

Let us begin with the simplest case when $\varepsilon = \varepsilon(\omega)$, $\mu = \mu(\omega)$ are frequency functions containing resonance denominators and having ordinary properties: $\varepsilon(0), \mu(0) > 1$ and $\varepsilon(\infty) = \mu(\infty) = 1$. The principal difference between functions $\varepsilon = \varepsilon(\omega)$ and $\mu = \mu(\omega)$ consists in that they have different resonant frequencies: $\omega_{\rm RE} \gg \omega_{\rm RM}$, where $\omega_{\rm RE}(\omega_{\rm RM})$ is the resonant frequency of dielectric permittivity (magnetic permeability) (Fig. 1).

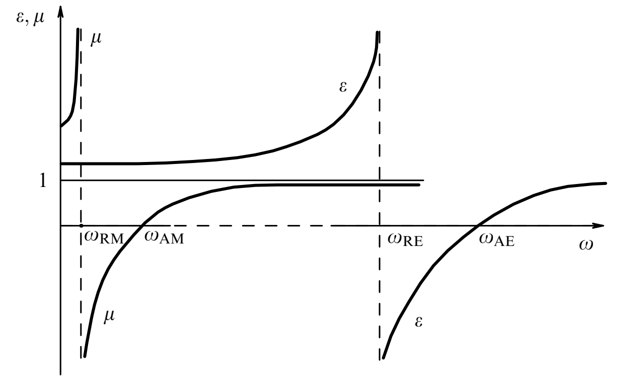

Figure 1. Schematic dependences of dielectric permittivity and magnetic permeability on frequency ω. The break in the axis of abscissas emphasizes that $\omega_{\rm RM}, \omega_{\rm AM}$, and $\omega_{\rm RE}, \omega_{\rm AE}$ occur in different frequency bands. Under real conditions each function can have several resonance sites.

Elementary excitation (quasiparticle) in the form of a photon interacting with oscillations of polarization and magnetization is commonly called polariton. The polariton dispersion law is actually the solution of the equation

$$k^2 = \frac{\omega^2}{c^2}\,\varepsilon(\omega)\mu(\omega) \tag{1.6}$$

in the form of functions $\omega = \omega(k)$. In the case of a simplest resonant dependence (one resonance for permittivity and one for permeability), the dispersion law has tree branches. An exotic case when the characteristic frequencies coincide is discussed in Section 6.

Dissipation is known to eliminate the infinite discontinuity of resonance and introduce the imaginary part of permittivities [e.g. ${\rm Im}\,\varepsilon = \varepsilon''$; function $\varepsilon'' = \varepsilon''(\omega)$ is close to the δ-function, see Fig. 2; the dependence $\mu = \mu(\omega)$ has a similar form].

Spatial dispersion which is essential near resonance frequencies at small dissipation characterizes the dependence of these frequencies on the wave vector:

$$\omega_{\rm RE} = \omega^0_{\rm RE} + \alpha_{\rm E}k^2\,, \qquad \omega_{\rm RM} = \omega^0_{\rm RM} + \alpha_{\rm M}k^2\,. \tag{1.7}$$

Spatial dispersion results from the ability of mechanical oscillations (polarization and/or magnetization) to propagate in a crystal due to internal forces of interaction. Coefficients $\alpha_{\rm E}$ and $\alpha_{\rm M}$ always contain a^2 (squared interatomic distance) as a multiplier, i.e. $\alpha_{\rm E} = \tilde{\omega}_{\rm E}a^2$, $\alpha_{\rm M} = \omega_{\rm ex}a^2$, while $|\tilde{\omega}_{\rm E}| \sim \omega^0_{\rm RE}$

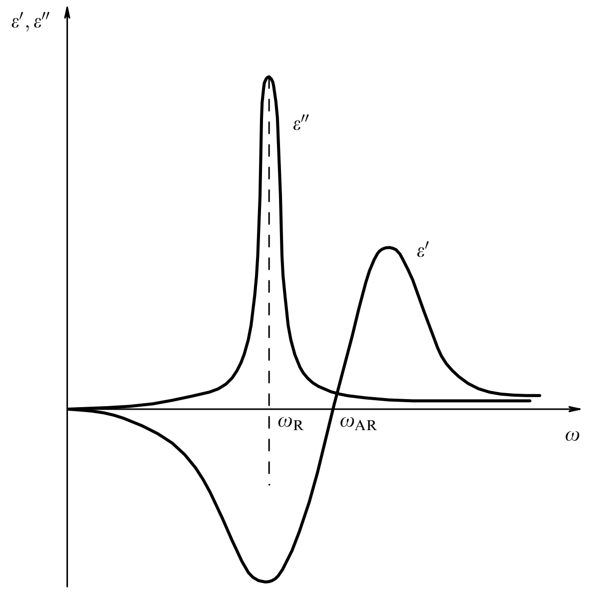

Figure 2. Schematic resonance dependence of dielectric permittivity with consideration for dissipation ($\varepsilon' = \mathrm{Re}\,\varepsilon$, $\varepsilon'' = \mathrm{Im}\,\varepsilon$).

and ω_{ex} greatly exceeds ω^0_{RM} ($\omega_{\mathrm{ex}} \gg \omega^0_{\mathrm{RM}}$) due to the exchange interaction between atomic moments.†

Thus,

$$\omega_{\mathrm{RE}} = \omega^0_{\mathrm{RE}} + \tilde{\omega}_{\mathrm{E}}(ak)^2\,, \qquad \omega_{\mathrm{RM}} = \omega^0_{\mathrm{RM}} + \omega_{\mathrm{ex}}(ak)^2\,. \quad (1.8)$$

The term $\omega_{\mathrm{ex}}(ak)^2$ may exceed ω^0_{RM} even at $ak \ll 1$. Based on formulas (1.6)–(1.8), it is easy to modify the dispersion law for the polariton by transforming the quasistatic limit ($kc \to \infty$, $\omega \not\to \infty$) to a quasistatic wave. In fact, there are two quasistatic limits in the case of (1.6) (due to ε and μ respectively). If spatial dispersion is taken into account, they 'turn' into dispersion laws for mechanical excitons (this terminology is used in optics). The importance of such modification achieved by taking into consideration spatial dispersion can be attributed to the fact that, spatial dispersion being neglected, the group velocity of polariton at $\omega \to \omega^0_{\mathrm{RE}}$, ω^0_{RM} tends to zero (i.e. it does not transfer energy). Spatial dispersion is responsible for the energy transfer by polariton in the quasistatic limit ($c = \infty$). This is natural if one bears in mind that spatial dispersion is a sequel of the existence of quasiparticles whereas polariton arises from the relationship between quasiparticles and electromagnetic oscillations.

Also, it is worth noting that spatial dispersion increases the power of Eqn (1.6) when the wave vector k is a variable. This is essential for the solution of non-uniform problems.

There are two equations like (1.6), for two transversal polarizations. Moreover, two longitudinal excitons can propagate in a medium with isotropic permittivity and permeability: the electric one, with the dispersion law defined by the equation

$$\varepsilon(\omega, k) = 0\,, \quad (1.9)$$

and the magnetic one, with the dispersion law described by the equation

$$\mu(\omega, k) = 0\,. \quad (1.10)$$

It will be shown below (see Section 5) that there are real situations in which a magnetic exciton defined by Eqn (1.10) should exist.

Let us now turn to the halfspace (neglecting spatial dispersion). An important characteristic of electrodynamic (e.g. reflecting) properties of a sample is the surface impedance. In a halfspace with dielectric permittivity ε and magnetic permeability μ, the impedance for a wave falling down perpendicular to the body surface is

$$\zeta = \sqrt{\frac{\mu}{\varepsilon}}\,, \qquad \mathrm{Re}\,\zeta > 0\,. \quad (1.11)$$

This is the first formula showing that ε and μ enter the dependence in a different way. In the case of the electric resonance ($\varepsilon \to \infty$), the impedance vanishes while in the magnetic resonance ($\mu \to \infty$) it turns into infinity. Coefficient of reflection R, i.e. the ratio of reflected to incident wave amplitudes, is related to impedance by the following equation:

$$R = -\frac{1-\zeta}{1+\zeta}\,. \quad (1.12)$$

In both cases, $|R| = 1$, that is the medium resists penetration of a resonant electromagnetic wave. The penetration is possible only due to dissipation, and the calculation of resonance characteristics is impossible without taking it into consideration.

Surface waves can propagate along the halfspace boundary. Their properties have recently become a matter of growing interest. They are an important issue of the present review.

The surface wave is a wave whose amplitude falls exponentially with the distance from the surface. Since electromagnetic waves exist in both a medium and a vacuum, the waves in question undergo exponential decay on either side of the boundary. The logarithmic decrement of attenuation in a vacuum is

$$\gamma_0 = \sqrt{k^2 - \frac{\omega^2}{c^2}}\,, \quad (1.13)$$

where k is the two-dimensional wave vector with components k_x and k_y; the z-axis is perpendicular to the surface. The magnet occupies 'positive' halfspace $z > 0$, which accounts for the wave amplitude being proportional to $\exp(\gamma_0 z)$ in a vacuum and to $\exp(-\gamma z)$ in a body, where

$$\gamma = \sqrt{k^2 - \frac{\omega^2}{c^2}\,\varepsilon\mu}\,. \quad (1.14)$$

Here and henceforward, the root values are positive. It is easy to see that the case being examined involves two different surface waves, one having non-zero components‡ E_x, E_z, H_y and the other non-zero components E_y, H_x, H_z. The former wave is an electric one (E-wave) while the latter is of the

† In the order of magnitude, $\omega_{\mathrm{ex}} = I/\hbar$, where I is the exchange integral. The exchange interaction being responsible for the spontaneous ordering of magnetic moments, $T_{\mathrm{C(N)}} \sim I$, where $T_{\mathrm{C(N)}}$ is the Curie (Neel) temperature. This estimate is very loose since it does not take into account many real factors (e.g. atomic spin, the number of immediate neighbours, etc.).

‡ The x-axis is parallel to vector $\mathbf{k}$ ($k_x = k, k_y = k_z = 0$).

magnetic type (H-wave). Their dispersion equations ensue from the boundary conditions (continuity of the tangential components of vectors $\mathbf{E}$ and $\mathbf{H}$):

$$\sqrt{\frac{k^2 - \varepsilon\mu\omega^2/c^2}{k^2 - \omega^2/c^2}} + \varepsilon = 0 \quad (E\text{-wave}),$$

$$\sqrt{\frac{k^2 - \varepsilon\mu\omega^2/c^2}{k^2 - \omega^2/c^2}} + \mu = 0 \quad (H\text{-wave}). \tag{1.15}$$

In the quasistatic limit ($kc \to \infty$), dispersion equations are essentially simplified:

$$\varepsilon(\omega) + 1 = 0 \quad (E\text{-wave}),$$

$$\mu(\omega) + 1 = 0 \quad (H\text{-wave}) \tag{1.16}$$

There is no surface H-wave in non-magnetic media. The region where surface waves can occur is limited by the natural inequalities:

$$k^2 > \frac{\omega^2}{c^2}, \quad k^2 > \frac{\omega^2}{c^2}\,\varepsilon(\omega)\mu(\omega). \tag{1.17}$$

Which of the two is stronger depends on both the value and the sign of the squared refractive index $n^2 = \varepsilon(\omega)\mu(\omega)$. At $n^2 > 1$, the frequency and wave vector region is restricted by the latter of the two conditions (1.17).

Let us now consider electromagnetic waves in a plate. A wave travels along the plate and decays exponentially on either side of it with logarithmic damping decrement (1.13). If the origin of the coordinates (z-axis) is in the middle of the plate which occupies the band $|z| < 2d$, the problem is symmetric to the change $z \to -z$. This allows the solutions to be categorized into symmetric (s) and asymmetric (a) ones (see Table 1).

Table 1.

E-wave	H-wave	s, a
$H_y(-z) = h_y(z)$ $E_z(-z) = H_y(z)$ $E_x(-z) = -E_x(z)$	$E_y(-z) = E_y(z)$ $H_x(-z) = -E_y(z)$ $H_z(-z) = -H_z(z)$	s
$H_y(-z) = -H_y(z)$ $E_z(-z) = -H_y(z)$ $E_x(-z) = E_x(z)$	$E_y(-z) = -E_y(z)$ $H_x(-z) = H_x(z)$ $H_z(-z) = -H_z(z)$	a

Dispersion equations for all four types of waves can be obtained from the boundary conditions at $z = \pm d$.

For an E-wave

$$\frac{\gamma}{\gamma_0}\tanh(\gamma d) + \varepsilon = 0, \quad (\text{s})$$

$$\frac{\gamma}{\gamma_0}\coth(\gamma d) + \varepsilon = 0. \quad (\text{a})$$

For an H-wave

$$\frac{\gamma}{\gamma_0}\tanh(\gamma d) + \mu = 0, \quad (\text{s})$$

$$\frac{\gamma}{\gamma_0}\coth(\gamma d) + \mu = 0. \quad (\text{a})$$

In the case of waves propagating along a plate, the constraint $k^2 > \varepsilon\mu\omega^2/c^2$ is absent although the condition $k^2 > \omega^2/c^2$ is retained. At

$$k^2 < \frac{\omega^2}{c^2}\,\varepsilon(\omega)\mu(\omega),$$

the fields in the plate are described by trigonometric functions (see Sections 13 and 14 for details). Much attention in this review will be given to solutions described by hyperbolic functions ($k^2 > \varepsilon\mu\omega^2/c^2$). At $\gamma d \gg 1$, they are true surface waves whose amplitudes are large near the plate surface.

Till now, magnets have been regarded as dielectrics in terms of electric properties. This review is concerned with some properties of metal magnets as well. In the Introduction, it appears appropriate only to mention how reciprocated effect of conductivity and magnetic properties is realized, that is how the dispersion of magnetic susceptibility affects the electric properties of conductors.

In the first place, an electromagnetic wave is essentially inhomogeneous due to the skin effect. Naturally, this influences properties of magnets in resonance. The influence of magnetic permeability on the skin-effect is equally interesting. It is easy to see that the skin-layer depth is

$$\delta = \frac{c}{\sqrt{2\pi\sigma\mu(\omega)\omega}}, \tag{1.18}$$

when the magnetic permeability μ and metal conductivity σ are real quantities.

The growth of $\mu(\omega)$ is responsible for a decrease in the thickness of the skin-layer and enhanced inhomogeneity of the alternating magnetic field which excites the spin subsystem, while a decrease in $\mu(\omega)$ makes the electromagnetic field in a metal more homogeneous. Figure 1 demonstrates the existence of antiresonant frequency value $\omega = \omega_{\mathrm{AM}}$ at which the depth of the skin-layer turns into infinity. This phenomenon (ferromagnetic antiresonance, FMAR) is described in Section 18. Its most prominent manifestation is selective transmittance of ferromagnetic metal plates at the antiresonant frequency ω_{AM} ($\mu(\omega_{\mathrm{AM}}) = 0$).

The above considerations concern two types of magnets, i.e. ferromagnets and two-sublattice antiferromagnets of the 'easy axis' (EA) type. In either case, the tensor of magnetic permeability has the following structure:

$$\hat{\mu} = \begin{pmatrix} \mu_1 & \mathrm{i}\mu' & 0 \\ -\mathrm{i}\mu' & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}. \tag{1.19}$$

For a ferromagnet, when an external magnetic field $\mathbf{H}$ is parallel to the axis of anisotropy,

$$\mu_1 = \mu_2 = \mu = 1 + \frac{\omega_0\omega_{\mathrm{M}}}{\omega_0^2 - \omega^2}, \quad \mu' = \frac{\omega\omega_{\mathrm{M}}}{\omega_0^2 - \omega^2}, \quad \mu_3 = 1 \tag{1.20}$$

is true.

Axis 3 is parallel to the axis along which the equilibrium magnetic moment $\mathbf{M}$ is directed,

$$\omega_0 = gH_{\mathrm{eff}}, \quad \omega_{\mathrm{M}} = 4\pi gM, \tag{1.21}$$

where g is the gyromagnetic ratio and $H_{\mathrm{eff}} = H + \beta M$, with β being the anisotropy constant. The index 'eff' will be omitted.

The situation with antiferromagnets is more complicated even in the simplest case of a magnetic field directed along the anisotropy axis because both the position of magnetic moments and the values of permeability components depend on the strength of the magnetic field.

The coverage of several sections is broader than macroscopic electrodynamics. They are devoted to the problems of statistical thermodynamics and kinetics. The titles of these Sections are asterisked.

The concept of quasiparticles in magnetically ordered media has not undergone any substantial modification during the last decades. Basic ideas about the nature of elementary excitations of the spin system (spin waves, magnons) remain essentially unaltered. Spin waves are oscillations of magnetic moments of magnetic sublattices which can be well described by the Landau – Lifshitz equations. Our ideology resembles that of Akhiezer et al. [1]. In the Foreword to their recent book, Gurevich and Melkov [2] state: "This book considers electromagnetic oscillations and waves in magnetically ordered substances: ferro-, antiferro-, and ferrimagnets." We might just as well open this review with the same phrase barring the word 'ferrimagnets'. Nevertheless, the book [2] and the present review do not seem to have much in common.

Other publications worthy of note are the book "Magnetostatic Waves in Superhigh Frequency Electrodynamics" by Vashkovskiĭ, Stel'makh and Sharaevskiĭ [3] and the monograph by Blank and Kaganov on metal electrodynamics under ferromagnetic resonance conditions [4]. Both publications include comprehensive lists of references. We presume the readers will also have to address selected volumes of the Course of Theoretical Physics by Landau and Lifshitz [5 – 8]. The present review largely covers recently published papers. References to early ones are made only when appropriate for better understanding. We apologize to those authors whose works are not cited here: some slips are inevitable in a review like this.

2. Magnetic dielectric: magnetic polaritons, magnetostatic waves, spin waves (magnons)

In this section, we confine ourselves to the assumption that the tensor μ_{ik} has the structure (1.19) at $\mu_1 = \mu_2 = \mu$ and dielectric permittivity is isotropic (in a cubic crystal, $\varepsilon_{ik} \equiv \varepsilon\delta_{ik}$).

We shall look for the solution of the system of Maxwell's equations (1.1) in the form of a plane monochromatic wave. It is not difficult to obtain the dispersion equation relating the frequency ω to the wave vector $\mathbf{k}$:

$$\left(\frac{\omega^2}{c^2}\,\varepsilon\right)^2(\mu'^2-\mu^2)+\frac{\omega^2}{c^2}\,\varepsilon\left[k_\perp^2\left(\mu(\mu+1)-\mu'^2\right)+2k_z^2\mu\right] - k^2(\mu k_\perp^2+k_z^2)=0\,,$$
$$k^2=k_\perp^2+k_z^2\,,\qquad k_\perp^2=k_x^2+k_y^2\,. \tag{2.1}$$

It has been emphasized in the Introduction that we are interested only in longwave oscillations, assuming that

$$ak \ll 1\,. \tag{2.2}$$

This inequality is a condition of the macroscopic approach. Let $\omega \ll \omega_{\mathrm{E}}$, where ω_{E} is the characteristic dispersion frequency of dielectric permittivity (e.g. $\omega_{\mathrm{E}} = \omega_{\mathrm{RE}}$). As a rule, the frequency ω_{E} belongs to the optical range. This allows the dispersion of dielectric permittivity to be ignored when considering magnetic oscillations, on the assumption that ε is a constant equal to its value at $\omega = 0$.

Within the framework of the macroscopic approach, it is possible to take into account spatial dispersion of tensor μ_{ik}, that is, consider μ and μ' to be the functions of both the frequency ω and the wave vector $\mathbf{k}$. Naturally, such an assumption makes Eqn (2.1) much more complicated even though its form is preserved.

In different limiting cases, elementary excitations described by Eqn (2.1) have different names. When spatial dispersion is neglected, the solutions of Eqn (2.1) (at $\varepsilon \equiv \mathrm{const}$) are called magnetic polaritons. The quasistatic limit of Eqn (2.1) describes magnetostatic waves (MSW) if spatial dispersion is disregarded. By introducing spatial dispersion, a MSW is 'converted' into a spin wave (magnon). All these terms will be used in further discussion.

Unless dispersion of the components of the magnetic permeability tensor is known, Eqn (2.1) is purely formal. Nevertheless, some conclusions are appropriate. First ('forgetting' that μ and μ' are functions of ω and $\mathbf{k}$ respectively), let us take advantage of the fact that, at constant ε, μ, and μ' and a given value of $\mathbf{k}$, Eqn (2.1) is a quadratic equation with respect to $\mu^{\mathrm{eff}} = k^2c^2/\omega^2\varepsilon$:

$$(\mu^{\mathrm{eff}})^2-\frac{\sin^2\theta(\mu^2-\mu'^2+\mu)+2\mu\cos^2\theta}{\mu\sin^2\theta+\cos^2\theta}\,\mu^{\mathrm{eff}} +\frac{\mu^2-\mu'^2}{\mu\sin^2\theta+\cos^2\theta}=0\,, \tag{2.3}$$

where θ is the angle between the wave vector $\mathbf{k}$ and the z-axis.

Hence,

$$k_\pm^2=\frac{\omega^2}{c^2}\,\varepsilon\mu_\pm^{\mathrm{eff}}\,,$$
$$\mu_\pm^{\mathrm{eff}}=\left[\mu^2(1+\cos^2\theta)+(\mu^2-\mu'^2)\sin^2\theta \pm\sqrt{(\mu^2-\mu'^2-\mu)^2\sin^4\theta+4\mu'^4}\,\right] \times\left[2(\mu\sin^2\theta+\cos^2\theta)\right]^{-1}. \tag{2.4}$$

Since our objective is to elucidate dispersion laws for quasiparticles (magnetic polaritons, magnons), we are interested in real solutions of Eqns (2.4).

As the frequency grows, $\mu \to 1$, $\mu' \to 0$, and $\mu_\pm^{\mathrm{eff}} \to 1$. Therefore, for waves of both types,

$$k^2=\frac{\omega^2}{c^2}\,\varepsilon\,,\qquad \omega_{\mathrm{E}} \gg \omega \gg \omega_{\mathrm{M}}\,, \tag{2.5}$$

where ω_{M} is the characteristic frequency of the magnetic subsystem (e.g. $\omega_{\mathrm{M}} = \omega_{\mathrm{RM}}$). Corrections to formula (2.5) can be derived from general considerations. The diagonal element of the matrix μ_{ik} is the even function of frequency, and its gyrotropic element is the odd function. Hence, the introduction of two parameters with dimension of frequency results in

$$\mu(\omega)\simeq 1-\frac{\omega_\infty^2}{\omega^2}\,,\qquad \mu'\simeq\frac{\tilde\omega_\infty}{\omega}\,, \tag{2.6}$$

$$\omega\simeq\frac{kc}{\sqrt{\varepsilon}}\left(1\mp\frac{|\tilde\omega_\infty\cos\theta|\sqrt{\varepsilon}}{2kc}\right)=\frac{kc}{\sqrt{\varepsilon}}\mp\frac{1}{2}\,|\tilde\omega_\infty\cos\theta|\,. \tag{2.7}$$

For $\theta = \pi/2$, a separate expansion of the '+'-branch is needed. At $\theta \neq \pi/2$, the deviation from formula (2.5) is determined by gyrotropy.

At an arbitrary angle θ, Eqns (2.4) are very complicated although the analysis is still possible. They are significantly simplified in two limiting cases:

$$\text{at} \quad \theta = 0\,, \qquad k^2 = \frac{\omega^2}{c^2}\,\varepsilon\mu_\pm\,, \qquad \mu_\pm = \mu \pm |\mu'|\,, \tag{2.8}$$

$$\text{at} \quad \theta = \frac{\pi}{2}\,, \qquad k_+^2 = \frac{\omega^2}{c^2}\,\varepsilon\,\frac{\mu^2 - \mu'^2}{\mu}\,, \qquad k_-^2 = \frac{\omega^2}{c^2}\,\varepsilon\,. \tag{2.9}$$

Notations for the branches at $\theta = \pi/2$ are arbitrary. The signs in (2.4) and (2.9) coincide only if $(\mu^2 - \mu'^2)/\mu - 1 > 0$. At $(\mu^2 - \mu'^2)/\mu - 1 < 0$, the '+'-wave in (2.9) is the '−'-wave in (2.4).

The inequality $(\mu^2 - \mu'^2)/\mu - 1 < 0$ does not imply that wave propagation is impossible. At $0 < (\mu^2 - \mu'^2)/\mu < 1$, the '+'-wave is a true wave in the range of parameters where the first equation in (2.9) has a real solution for the frequency ω at a real wave vector k [similar to (2.4), (2.9) is an equation, not a solution].

It is clear from the case where $\theta = \pi/2$ how important is the role of wave polarization: one of the waves is totally independent of magnetic characteristics.

Turning back to formula (2.7) and using the asymptotic values of (2.6), along with the first formula of (2.9), one has

$$\omega \simeq \frac{kc}{\sqrt{\varepsilon}} + \frac{(\omega_\infty^2 + \tilde{\omega}_\infty^2)\sqrt{\varepsilon}}{2kc}\,, \qquad \theta = \frac{\pi}{2}\,. \tag{2.7$'$}$$

One of the values of $\mu_\pm^{\text{eff}}$ in (2.4) vanishes at $\mu^2 = \mu'^2$. According to (2.4), this naturally results in vanishing of the wave vector k of one of the waves. The number of zeros in the equation

$$\mu^2(\omega, k = 0) = \mu'^2(\omega, k = 0) \tag{2.10}$$

is determined by the concrete dispersion dependence of the components of magnetic permeability. The roots of Eqn (2.10) are called antiresonant frequencies (we denote them with ω_{AM}). It should be emphasized that a specific feature of frequencies ω_{AM} is their independence of the direction in which the waves propagate. The dispersion is quadratic near the frequency ω_{AM}:

$$\omega - \omega_{\text{AM}}^{\nu} \simeq \beta_{\text{AM}}^{\nu} k^2\,, \qquad \nu = 1, 2, \ldots, \tag{2.11}$$

and coefficients β_{AM}^{ν} are the sums of two items of the order $c^2/\omega_{\text{AM}}\varepsilon$ and $\omega_{\text{ex}}a^2$ respectively. As a rule, $c^2/\omega_{\text{AM}}\varepsilon \gg \omega_{\text{ex}}a^2$ and the wave dispersion at $\omega \simeq \omega_{\text{AM}}$ is determined by retardation (electrodynamics).

It follows from (2.4) that, spatial dispersion being neglected, the wave vector k of at least one of the waves turns to infinity at

$$\mu(\omega)\sin^2\theta + \cos^2\theta = 0\,, \qquad \mu(\omega) = \mu(\omega, k = 0) \tag{2.12}$$

Equation (2.12) defines frequencies of magnetic resonance ω_{RM} or the MSW spectrum.

Equation (2.12) can be derived in a different way, without neglecting spatial dispersion. It is possible to obtain this equation by means of the limiting transition $kc \to \infty$, i.e. transition to quasistatics. Then, the equation

$$\mu(\omega, k)\sin^2\theta + \cos^2\theta = 0 \tag{2.13}$$

defines the spectrum of magnons.

Equation (2.13) can be derived directly from equations of magnetostatics:†

$$\operatorname{rot}\mathbf{h} = 0\,, \qquad \operatorname{div}\mathbf{b} = 0\,, \qquad b_i = \mu_{ik}(\omega, \mathbf{k})h_k\,. \tag{2.14}$$

At $\theta = \pi/2$, Eqn (2.13) has the form

$$\mu(\omega, \mathbf{k}) = 0 \tag{2.15}$$

and defines the dispersion law for the longitudinal ($\mathbf{h} \parallel \mathbf{k}$) magnetic (mechanical) exciton. At $\mu' \equiv 0$, (2.15) is the exact equation. Gyrotropy of the magnetic system may be lacking only in antiferromagnets (see Section 5).

In magnetostatics, (2.13) is the exact equation; however it leads to a non-analytical dependence of frequencies ω on the components of vector k which is easy to see if $\cos^2\theta$ and $\sin^2\theta$ are expressed through k_z^2/k^2 and $k_\perp^2/k^2$ ($k_\perp^2 = k_x^2 + k_y^2$) respectively. Physically, this can be accounted for by long-range magnetic interactions apparent as the effect of magnetic field oscillations caused by irregular magnetization. The dependence of frequency ω on the angle θ (even at $k \to 0$) may be regarded as ambiguity: there is a frequency band at $k \to 0$, instead of one definite frequency. In the case of finite magnets, the ambiguity is eliminated by taking into consideration the boundary conditions. This will be demonstrated below. Also, the ambiguity is obviated if finiteness of light velocity (retardation) is taken into account [see formulas (2.16)].

If both the wave vector k and the frequency ω tend to zero and the 'origin' of the curves (2.4) needs to be determined, it is natural to assume that μ is equal to μ_0 (static limit of magnetic permeability $\mu_0 > 1$) and $\mu' = 0$.‡

Based on (2.4), it is easy to obtain

$$\omega_\pm = \begin{cases} \dfrac{kc}{\sqrt{\varepsilon}}\left(\dfrac{\cos^2\theta}{\mu_0} + \sin^2\theta\right)^{1/2}, \\[2ex] \dfrac{kc}{\sqrt{\varepsilon\mu_0}}\,. \end{cases} \tag{2.16}$$

As the wave vector grows, it is necessary to take into account spatial dispersion of the components of the magnetic permeability tensor. Moreover, beyond the assumption adopted here (at $k \sim 1/a$), the geometrical structure of a crystal should be expected to manifest itself in the periodic $\mathbf{k}$ dependence of the elementary excitation law. This discussion is limited to a small region of the $\mathbf{k}$-space surrounding the origin of the coordinates, i.e. the centre of the first Brillouin zone.

Using equations of macroscopic electrodynamics, it is impossible to find all branches of low-frequency elementary excitations. For example, we do not consider phonon branches because equations of the elasticity theory are needed to find them (extended to longwave optical oscillations, for the case of complex polyatomic crystals). Even if the

† Small Latin letters denote variable parts of the magnetic field $\mathbf{h}$ and magnetic induction $\mathbf{b}$.

‡ A corollary of the condition that the tensor μ_{ik} is Hermitian is the relation $\mu'(-\omega) = -\mu'(\omega)$. Hence $\mu'(0) = 0$

consideration is limited to small oscillations of magnetic moments in sublattices, one cannot be sure that Eqns (2.4) describe all branches of magnetic oscillations because such oscillations are not always accompanied by a change in the magnetic moment of the unit body volume. In other words, there are oscillations of magnetic moments which do not excite electromagnetic oscillations; such oscillations cannot be described by macroscopic electromagnetic equations.

It should be emphasized, when discussing the above formulas describing dispersion laws for elementary excitations, that these laws are specific for magnetic systems in the frequency range of $\omega \sim \omega_{\mathrm{RM}}, \omega_{\mathrm{AM}}$. In other frequency bands, polariton is hardly distinguishable from photon in a non-magnetic medium. However, the frequency of an elementary excitation (even the one resembling photon) depends on the magnetic field due to the presence of magnetic permeability in (2.4). Therefore, a quasiparticle corresponding to such an excitation must have the moment.

According to the general laws of quantum mechanics, the magnetic moment of a system is equal to the derivative with the inverse sign of the system's energy with respect to the magnetic field. Hence, the magnetic moment of a quasiparticle is

$$\boldsymbol{\beta} = -\hbar \frac{\partial w}{\partial \mathbf{H}} . \tag{2.17}$$

This formula is applicable to both polaritons and magnons. It is hoped that the use of the letter β to designate the magnetic moment of a quasiparticle and anisotropy constant will not lead to confusion.

3. Ferrodielectric: magnetic polaritons, magnetostatic waves, spin waves (magnons)

This section is complementary to the previous one illustrating its formulas and conclusions by the simplest example of an one-sublattice ferromagnet magnetized parallel to the anisotropy axis. Such a magnet is described by formulas (1.20) and (1.21).

Let us begin with the calculation of the antiresonant frequency ω_{AM} (the only one in this case). According to (1.20), (1.21) and (2.10),

$$\omega_{\mathrm{AM}} = \omega_0 + \omega_{\mathrm{M}} , \tag{3.1}$$

or

$$\omega_{\mathrm{AM}} = gB_{\mathrm{eff}} , \qquad B_{\mathrm{eff}} = H_{\mathrm{eff}} + 4\pi M , \tag{3.2}$$

while H_{eff} differs from H in the term βM (see the Introduction).

There is no sense in rewriting Eqns (2.4) by substituting (1.20) because they remain cumbersome any way. At the same time, formulas (2.8) and (2.9) become rather compact:

$$k_{\pm}^2 = \frac{\omega^2}{c^2}\, \varepsilon\, \frac{\omega_{\mathrm{AM}} \pm \omega}{\omega_{\mathrm{RM}} \pm \omega} , \qquad \omega_{\mathrm{RM}} = \omega_0 , \qquad \theta = 0 ; \tag{3.3}$$

$$k_{-}^2 = \frac{\omega^2}{c^2}\, \varepsilon\, \frac{\omega_{\mathrm{AM}}^2 - \omega^2}{\omega_{\mathrm{RM}}^2 - \omega^2} , \qquad k_{+}^2 = \frac{\omega^2}{c^2}\, \varepsilon ; \tag{3.4}$$

$$\omega_{\mathrm{RM}} = \sqrt{\omega_0(\omega_0 + \omega_{\mathrm{M}})} , \qquad \theta = \frac{\pi}{2} .$$

MSWs and spin waves (magnons) differing from MSWs in that they take into account spatial dispersion [i.e. in substitution of $\omega_0 + \omega_{\mathrm{ex}}(ak)^2$ for ω_0] are described by the following formula [see (3.13) and (2.20)]:

$$\omega = \sqrt{\omega_0(\omega_0 + \omega_{\mathrm{M}} \sin^2\theta)} . \tag{3.5}$$

Finally, the lowest-frequency part of the spectrum can be described by formulas (2.16) provided the static magnetic permeability μ_0 is substituted by its value in this model:

$$\mu_0 = 1 + \frac{\omega_{\mathrm{M}}}{\omega_0} . \tag{3.6}$$

Formula (3.3) is convenient to demonstrate magnetization effect on the non-magnetized (photonic) part of the spectrum. Its dispersion law is the solution of Eqn (3.3) with respect to frequency. Here, spatial dispersion is neglected. The measure of magnetization effect may be non-linearity in the dependence $\omega = \omega(k)$ which is apparent as the difference of the group to phase velocity ratio $v_{\mathrm{gr}}/v_{\mathrm{ph}} = (\mathrm{d}\omega/\mathrm{d}k)/(\omega/k)$ from unity. According to (3.3)

$$\frac{v_{\mathrm{gr}}}{v_{\mathrm{ph}}} = \frac{(\omega_0 + \omega)(\omega_0 + \omega_{\mathrm{M}} + \omega)}{(\omega_0 + \omega)^2 + \omega_{\mathrm{M}}(\omega_0 + \omega/2)} .$$

It has the maximum at $\omega = \sqrt{5}\,\omega_0$:

$$\left(\frac{v_{\mathrm{gr}}}{v_{\mathrm{ph}}}\right)_{\max} = \frac{(\sqrt{5}+1)^2 + (\sqrt{5}+1)\omega_{\mathrm{M}}/\omega_0}{(\sqrt{5}+1)^2 + (\sqrt{5}+1/2)\omega_{\mathrm{M}}/\omega_0}$$

which further increases with growing $\omega_{\mathrm{M}}/\omega_0$. At $\omega_{\mathrm{M}}/\omega_0 \gg 1$,

$$\left(\frac{v_{\mathrm{gr}}}{v_{\mathrm{ph}}}\right)_{\max} \simeq \frac{\sqrt{5}+1}{\sqrt{5}+1/2} \simeq 1.25 .$$

The deviation from unity is insignificant yet well-apparent.

Spin waves are normally considered to be specific elementary excitations of ferromagnets. In most cases, a formula in which the term $\omega_{\mathrm{M}} \sin^2\theta$ is absent (i.e. magneto-dipole interaction is neglected) is used instead of (3.5):

$$\omega(k) = \omega_0 + \omega_{\mathrm{ex}}(ak)^2 . \tag{3.7}$$

At $\omega_{\mathrm{ex}}(ak^2) \gg \omega_0$, the dispersion law is especially simple and resembles that for a non-relativistic free particle:

$$\omega = \omega_{\mathrm{ex}}(ak)^2 , \qquad E = \frac{p^2}{2m^*} , \qquad \frac{1}{m^*} = \frac{2\omega_{\mathrm{ex}} a^2}{\hbar} , \tag{3.7$'$}$$

where m^* is the effective magnon mass.

An important role in thermodynamics and kinetics of magnon gases at low temperature is played by the spectrum structure near the bottom of the energy zone ($p = 0$; $\theta = 0$ and $\theta = \pi$). According to (3.5), the lowest magnon energy is $E_0 = \hbar\omega_0$ (if retardation is neglected) while at $E \geqslant E_0$ and $E - E_0 \ll \hbar\omega_{\mathrm{M}}$,

$$E = E_0 + \frac{p^2}{2m^*} + \frac{1}{2}\,\hbar\omega_{\mathrm{M}}\theta^2 . \tag{3.8}$$

Here, θ is counted starting from zero and/or from π.

The presence of the term $\hbar\omega_{\mathrm{M}}\theta^2/2$ (a result of magneto-dipole interaction) along with the quadratic form (the term containing $p^2 = p_x^2 + p_y^2 + p_z^2$) is of great interest and can be described as a rise in the effective dimension of the magnon momentum space $D_{\rm eff}$ if this dimension is defined by the formula which relates the volume $\Omega(E)$ of the quasiparticle's isoenergy surface to its energy E. Normally, if the volume element in a D-dimensional space is $d^D p_i$ $(i = 1, 2, \ldots, D)$,

$$E - E_0 = \sum_{i=1}^{D} \frac{p_i^2}{2m_i^*}$$

(**p**-space anisotropy is apparent as the difference between main values of the effective mass tensor), then

$$\Omega(E) \sim (E - E_0)^{D/2}\,. \tag{3.9}$$

In our case, when formula (3.10) is applicable, the volume element contains factor $\sin\theta\,d\theta \simeq \theta\,d\theta$; therefore, $E \to E_0$,

$$\Omega(E) \sim (E - E_0)^{5/2}\,, \qquad D_{\rm eff} = 5\,. \tag{3.10}$$

Enlarged effective dimension is naturally reflected in the properties of magnon gas (see Section 4). The influence of the growing effective dimension due to dipole – dipole interaction on phase transition from the paramagnetic to ferromagnetic state has been studied by Larkin and Khmel'nitskiĭ. Naturally, it can be noticed in the properties of the magnon gas.

Turning back to formula (3.5) and neglecting the boundaries, it may be said that the formula describes the dispersion law for MSW:

$$\omega = \omega(\mathbf{k})\,, \qquad \omega(\mathbf{k}) = \sqrt{\omega_0\left(\omega_0 + \omega_{\rm M}\frac{k_\perp^2}{k_\perp^2 + k_\parallel^2}\right)}\,, \tag{3.11}$$

where $k_\perp^2 = k^2 - k_\parallel^2$, and $k_\parallel$ is the projection of the wave vector $\mathbf{k}$ on the direction of the magnetic field $\mathbf{H}$ (magnetization $\mathbf{M} \parallel \mathbf{H}$). It should be emphasized that the dependence $\omega = \omega(\mathbf{k})$ is very unusual in this case: it describes the anomalous dispersion along the magnetic field (magnetization) and the normal one at the plane perpendicular to the field (magnetization).

Given the explicit dependence of quasiparticle energy (i.e. that of frequency ω_0 on the magnetic field $\mathbf{H}$), it is possible to calculate magnetic moments of magnons, MSW, and magnetic polaritons based on the expression (2.17). Since we have considered only the magnetic field normal to the anisotropy axis (easy magnetization axis), only the z-th projection of the magnon magnetic moment β_z can be calculated (subscript z is henceforth omitted).

In the microscopic approach which takes into account only exchange and Zeeman energies, the 'elementary disorder' is a wave of the minimal (permitted by spatial quantization) deviation of the atomic spin from equilibrium. The magnetic moment of such a wave naturally coincides with $g\hbar$ $(\beta = g\hbar)$. The situation is more complicated if both the retardation and the dipole – dipole interaction are taken into account: the wave describes not only the motion of a deflected spin but also the associated magnetic or electromagnetic waves. We find it impossible to explain in simple terms how large the magnon magnetic moment 'must' be at any value of the wave vector $\mathbf{k}$. When formula (3.7) is valid, the computation using formula (2.17) yields the 'conventional' value

$$\beta = -g\hbar\,. \tag{3.12}$$

When formulas (2.16) supplemented with (3.6) are valid,

$$\beta_+ = -\frac{1}{2}\,\frac{\omega_+(\mathbf{k})\omega_{\rm M}}{(\omega_0 + \omega_{\rm M})^2}\,\frac{\cos^2\theta}{\left[(\cos^2\theta)/\mu_0 + \sin^2\theta\right]^2}\,g\hbar\,,$$
$$\beta_- = -\frac{1}{2}\,\frac{\omega_-(\mathbf{k})\omega_{\rm M}}{(\omega_0 + \omega_{\rm M})\omega_0}\,g\hbar\,. \tag{3.13}$$

Finally, when formula (3.5) holds,

$$\beta = -\frac{1 + (\omega_{\rm M}/2\omega_0)\sin^2\theta}{1 + (\omega_{\rm M}/\omega_0)\sin^2\theta}\,g\hbar\,. \tag{3.14}$$

Transition to formula (3.11) occurs at $\theta = 0$ and also at $\omega_0 \gg \omega_{\rm M}$, if the angle θ is arbitrary. When $\omega_{\rm M} \gg \omega_0$,

$$\beta = -\frac{1}{2}\,g\hbar \tag{3.15}$$

in a wide range of angles $\sin\theta \gg (2\omega_0/\omega_{\rm M})^{1/2}$.

4. Statistical thermodynamics of ferromagnetic dielectrics*

The main feature of statistical thermodynamics of magnetic dielectrics [9] is the presence of low-frequency quasiparticles (magnons) responsible for the magnetic disorder in crystals. At temperatures much lower than the Curie (Neel) temperature $T_{\rm C}$, magnons are an almost ideal boson gas; therefore, free energy of a magnet has an additional (magnetic) term $F_{\rm M}$:

$$F_{\rm M} = T\,\frac{V}{(2\pi)^3}\sum_\nu \int \ln\left\{1 - \exp\left[-\frac{\hbar\omega_\nu(\mathbf{k})}{T}\right]\right\} d^3k\,. \tag{4.1}$$

Here, ν is the magnetic oscillation branch number [solutions of Eqns (2.1) or Eqn (2.4)], V is the magnetic volume; temperature in formulas and estimates is always in energy units and kelvins respectively; the condition $T \ll T_{\rm C}$ allows us to extend integration to the entire $\mathbf{k}$-space and regard the formulas obtained in the previous sections $(ak \ll 1)$ as valid. This is justified in view of the exponentially small contribution of magnons with $ak \gg 1$ at $T \ll T_{\rm C}$. In agreement with formula (2.17), the magnetic moment of a magnon gas responsible for the temperature dependence of the magnetic moment of the body's unit volume is

$$\Delta M(T) = \frac{1}{(2\pi)^3}\sum_\nu \int \beta_\nu(\mathbf{k})\left\{\exp\left[\frac{\hbar\omega_\nu(\mathbf{k})}{T}\right] - 1\right\}^{-1} d^3k \tag{4.2}$$

(naturally, this is true for the z-th projection of the magnetic moment $\mathbf{M}$). This formula appears to be of paramount importance in the theory of low-temperature magnetism. In the majority of cases, the consideration is restricted to the case† of $T \gg \hbar\omega_0$ which allows for the formula (3.7′) to be used. Then, $\Delta M \propto T^{3/2}$ is valid with high accuracy.

We shall return to the formula (4.2) after examining the total number of quasiparticles (magnons) in the unit volume $n_\nu = \sum_\nu n_\nu$:

$$n_\nu = \frac{1}{(2\pi)^3}\int\left\{\exp\left[\frac{\hbar\omega_\nu(\mathbf{k})}{T}\right] - 1\right\}^{-1} d^3k\,. \tag{4.3}$$

† Recall that $\hbar\omega_0 \sim 1$ K at $H_{\rm eff} \sim 10^4$ Oe.

At $T \ll T_C$, magnon interactions with one another are in many cases more probable than with other quasiparticles (e.g. phonons). Moreover, the interaction processes during which the number of magnons remains unaltered (due to exchange interaction) are more likely to occur than those in which magnon numbers vary (relativistic interactions) (see Ref. [4]). If processes in which the number of particles is not conserved are neglected, the magnon gas resembles a gas of ordinary particles. However, there is a very important difference. In equilibrium, the chemical potential of magnon gas is $\zeta_M = 0$ while that of true particles is $\zeta_p \neq 0$, being unambiguously determined by particle density and temperature. In the case of a degenerated Bose-gas of particles, ζ_p vanishes ($\zeta(T_0) = 0$) at T_0; below this temperature, $\zeta_p \equiv 0$. This phenomenon is called the Bose–Einstein condensation. It is believed that the system must experience phase transition of the third order at $T = T_0$, but it has never been observed because conventional gases undergo condensation at $T > T_0$.

The possibility of Bose–Einstein condensation in magnon gases was discussed by many authors, but the definitive conclusion remains to be found. We would like to draw attention to two issues:

(1) Only an artificial non-equilibrium situation is worth considering. At very low temperature, a system may contain non-equilibrated magnon gas of density n in which the equilibrium sets in due to internal exchange interactions; this equilibrium is described by the Bose function with the chemical potential $\zeta_M \neq 0$.

(2) The similarity between the Bose–Einstein condensation in magnon gases and gases of conventional particles is feasible only if formula (3.7′) holds, but its applicability is known to be restricted.

Suppose that quasiparticles in an artificial magnon gas at temperatures much below $E_0 = \hbar\omega_0$ have energies approximating the threshold energy† E_0, that is

$$E \geqslant E_0 \,, \qquad E - E_0 \ll \hbar\omega_M \,. \tag{4.4}$$

This means that, the processes with the varying number of quasiparticles being neglected, the interaction between magnons results in a quasi-equilibrium gas with density $n = N/V$ and temperature T different from body temperature. In this case, the magnon dispersion law is described by formula (3.8). The question is whether Bose–Einstein condensation occurs in such a system and what form it may have.

In the first place, it is worthwhile to note that both macroscopic parameters describing magnon gases (n and T) depend on the experimental conditions; the phrase 'temperature changes' implies changes in the condition for spin wave (magnon) excitation.

The possibility of Bose–Einstein condensation is determined by the solution of the equation for the condensation temperature T_0:

$$n = \frac{1}{(2\pi\hbar)^3} \int \left\{ \exp\left[\frac{E(\mathbf{p})}{T_0}\right] - 1 \right\}^{-1} \mathrm{d}^3 p \,,$$

i.e

$$\zeta = \zeta(T_0) - E_0 = 0 \,. \tag{4.5}$$

† Here, we neglect retardation ($c \to \infty$). In this approximation, $E_0 = \hbar\omega_0$ is the threshold energy.

The chemical potential of magnon gas ζ cannot be higher than E_0; otherwise, the normalizing integral diverges.

Using expression (3.8), we readily obtain

$$T_0 = T_{\mathrm{tr}}^{3/5} \left(\frac{\hbar\omega_M}{\gamma}\right)^{2/5} , \tag{4.6}$$

$$T_{\mathrm{tr}} = \frac{(2\pi\hbar)^2 n^{2/3}}{2m^*} \,, \tag{4.7}$$

$$\gamma = \int_0^\infty \sqrt{u} \exp(-u) \big|\ln[1 - \exp(-u)]\big| \, \mathrm{d}u = \frac{\sqrt{\pi}}{4} \,,$$

where the parameter T_{tr} is of the same order of magnitude as the usual (trivial) temperature of Bose–Einstein condensation.

The transition is described by the above formulas if its temperature T_0 satisfies the assumption that $T_0 \ll \hbar\omega_M$. Hence, it is easy to obtain the condition for magnon gas density [see (3.7)]:

$$na \ll \frac{\sqrt{\pi}}{2^{7/2}} \left(\frac{\omega_M}{\omega_{\mathrm{ex}}}\right)^{3/2} . \tag{4.8}$$

It remains to find which law is involved in the vanishing of $\zeta = \zeta(T)$ at $T \to T_0$. To this effect, the following equation is suitable:

$$n - n_0(T) = \frac{4\pi}{(2\pi\hbar)^3} \int_0^\infty p^2 \, \mathrm{d}p \times \int_0^\pi \theta \, \mathrm{d}\theta \left\{ \left[\exp\left(\frac{E(p,\theta) + |\zeta|}{T}\right) - 1\right]^{-1} - \left[\exp\left(\frac{E(p,\theta)}{T}\right) - 1\right]^{-1} \right\} , \tag{4.9}$$

where

$$n_0 = n \left(\frac{T}{T_0}\right)^{5/2} \tag{4.10}$$

is the density of 'travelling' magnons.

Since the chemical potential $\zeta \to 0$ at $T \to T_0$, the expansion of the right-hand side in $|\zeta|$ may be used to obtain

$$n - n_0(T) \simeq 2\pi \, \frac{(2m^* T_0)^{3/2}}{(2\pi\hbar)^3} \, \Gamma\left(\frac{3}{2}\right) \zeta\left(\frac{3}{2}\right) \frac{\zeta}{\hbar\omega_M} \,, \tag{4.11}$$

whence

$$\zeta \simeq -\frac{5}{4\pi\Gamma(3/2)\zeta(3/2)} \, \frac{\hbar\omega_M}{T_0} \left(\frac{T_{\mathrm{tr}}}{T_0}\right)^{3/2} (T - T_0) \,, \qquad T \geqslant T_0 \,. \tag{4.12}$$

In appears that in this case, the Bose–Einstein condensation is the second rather than third-order phase transition [the chemical potential ζ is the first derivative of the thermodynamic potential Φ over the particle number whereas the first derivative of ζ with respect to T has a jump, in accordance with (4.11); this means that the second (mixed) derivative of the thermodynamic potential also has a jump (!)]. Normally, a third-order phase transition indicates the presence of a jump in a derivative of thermal capacity. Here, the thermal capacity itself has a jump, in conformity with the general theory of second-order phase transitions. It should be recalled that we deal with the magnon ('artificial') temperature which does not coincide with the lattice (phonon) temperature of the specimen.

A most spectacular feature of a magnon gas is its magnetic moment. In the present case, formula (4.2) needs to be replaced by a different one containing the chemical potential ζ:

$$\Delta\widetilde{M} = \frac{V}{(2\pi\hbar)^3}\int\left[\exp\left(\frac{E+|\zeta|}{T}\right)-1\right]^{-1}\beta(\mathbf{p})\,\mathrm{d}^3p\,, \qquad (4.13)$$

with

$$\beta(\mathbf{p}) = -g\hbar\,\frac{1+(\hbar\omega_{\mathrm{M}}/2E_0)\sin^2\theta}{1+(\hbar\omega_{\mathrm{M}}/E_0)\sin^2\theta} \simeq -g\hbar\left(1-\frac{\hbar\omega_{\mathrm{M}}}{2E_0}\,\theta^2\right) \qquad (4.14)$$

according to (3.13). Hence†, taking into account the existence of two values of the angle ($\theta = 0$, $\theta = \pi$) at which $E = E_0$, we obtain

$$\frac{\partial}{\partial T}\left(\frac{\Delta\widetilde{M}(T)}{V}\right) = \frac{\beta}{(2\pi\hbar)^3}\,\frac{\hbar\omega_{\mathrm{M}}}{E_0} \times \frac{\partial}{\partial T}\int_0^{\infty}\int_0^{\pi}\left\{\exp\left[\left(\frac{p^2}{2m^*}+\hbar\omega_{\mathrm{M}}\theta^2+|\zeta|\right)\frac{1}{T}\right]-1\right\}^{-1} \times \theta^3p^2\,\mathrm{d}p\,\mathrm{d}\theta\,, \qquad \beta = g\hbar\,. \qquad (4.15)$$

A jump of the derivative $\partial(\Delta\widetilde{M}/V)/\partial T$ results from the vanishing of the chemical potential ζ, in accordance with the linear law [see (4.12)]:

$$\frac{\partial\widetilde{M}}{\partial T}\,\frac{1}{\widetilde{M}} = \frac{-5}{12}\,\frac{\zeta(5/2)}{\zeta(3/2)}\,\frac{\Gamma(5/2)}{\zeta(3/2)}\,\frac{1}{E_0} \simeq -\frac{0.3}{E_0}\,. \qquad (4.16)$$

The jump is positive because $\widetilde{M} < 0$ due to the magnetic moment $\beta(\mathbf{p})$.

We have interpreted formula (3.8) as a rise in the effective dimension of **p**-space. Let us now check if the transition patterns correspond to $D_{\mathrm{eff}} = 5$. In an isotropic D-dimensional space at

$$E = \sum_{i=1}^{D}\frac{p_i^2}{2m^*}\,, \qquad i = 1,\ldots,D\,,$$

the transition temperature is

$$T_D = \frac{n_D^{2/D}(2\pi\hbar)^2}{A(D)\Gamma(3/2)\zeta(3/2)\cdot 2^{(D-1)/2}m^*}\,, \qquad (4.17)$$

where

$$A(D) = \frac{(2\pi)^{D/2}}{\Gamma(D/2-2)}$$

is the area of an unit sphere in D-dimensional space while particle density n_D has the dimension cm^{-D}. It can be seen that the formula for T_D at $D = 5$ is very similar to (4.6). Substituting T_{tr} into (4.6), we find

$$T_0 = \left(n^{2/5}\,\frac{\hbar^2}{m^*}\right)\left(\frac{\omega_{\mathrm{M}}m^*}{\hbar}\right)^{2/5}.$$

up to a numerical multiplier. The factor $(\omega_{\mathrm{M}}m^*/\hbar)^{2/5}$ ensures the adequate dimension. Let us now clarify the character of the transition. In analogy with the previously described approach [see (4.9)], the following equation can be found:

$$n_D - n_{D0}(T) = \frac{A(D)}{(2\pi\hbar)^D}\int p^{D-1}\,\mathrm{d}p\left\{\left[\exp\left(\frac{E(p)+|\zeta|}{T}\right)-1\right]^{-1} - \left[\exp\left(\frac{E(p)}{T}\right)-1\right]^{-1}\right\}. \qquad (4.18)$$

This expression indicates that, starting from $D = 5$ ($D \geqslant 5$), the chemical potential ζ exhibits linear dependence on $T - T_D$ at $T \to T_D$. Hence, the Bose – Einstein condensation is a second-order transition.

The comparison between the Bose – Einstein condensation in a five-dimensional space and magnon condensation shows that the introduction of the effective dimension $D_{\mathrm{eff}} = 5$ for magnons with the dispersion law (3.8) is quite opportune.

Let us now turn back to formula (4.2) supplemented with the magnon magnetic moment $\beta_{\nu}(\mathbf{k})$ [see formulas (3.11) – (3.14)] and calculate the temperature dependence of magnetization at $T \ll E_0$, omitting the conventional temperature region ($E_0 \ll T \ll T_{\mathrm{C}}$). It has been shown in Section 3 that the temperature interval between zero and E_0 is split up into two. Formulas (3.5) are valid in one of the resulting intervals while formulas (2.16), (3.4), and (3.12) in the other (in the immediate proximity to absolute zero). The limits of the intervals T_{lim} will be defined below [see (4.25)].

At $T_{\mathrm{lim}} \ll T \ll E_0$, it is possible to use the approximate dependence of magnon energy on its momentum (3.8) and the approximate value of the magnetic moment ($\beta = -g\hbar$). Here, as at a higher temperature, the temperature portion of magnetization is proportional to the magnon density

$$\Delta\widetilde{M} = -g\hbar n(T)\,,$$
$$n(T) = \frac{2}{(2\pi\hbar)^3}\int\left\{\exp\left[\frac{E(\mathbf{p})}{T}\right]-1\right\}^{-1}\mathrm{d}^3p\,, \qquad (4.19)$$
$$E(\mathbf{p}) = E_0 + \frac{p^2}{2m^*} + \frac{1}{2}\,\hbar\omega_{\mathrm{M}}\theta^2\,,$$

with the factor 2 being introduced to take into account that energy has the minimal angular value at $\theta = 0$ and $\theta = \pi$.

Hence,

$$\Delta\widetilde{M}(T) = -\frac{g\hbar}{8\pi^2\hbar^3}\,\frac{T}{\hbar\omega_{\mathrm{M}}}\,(2m^*T)^{3/2}\exp\left(-\frac{E_0}{T}\right)\gamma\,,$$
$$\gamma = \int_0^{\infty}\sqrt{z}\,\exp(-z)\,\mathrm{d}z = \frac{\sqrt{\pi}}{2}\,. \qquad (4.20)$$

The 'extra' power of the temperature in this expression reflects five-dimensional behaviour of magnons in this limit.

† It is worthwhile to note that in this case, the magnetic moment of magnon gas is independent of temperature if its dependence on the angle θ is not taken into consideration: it is determined by the number of magnons which is assumed to be unaltered.

In accordance with what has been said before, in the immediate proximity to absolute zero,

$$\Delta\widetilde{M} = -\frac{g\hbar}{2}\frac{1}{(2\pi)^3}\left\{\int \frac{\omega_+(\mathbf{k})\omega_M}{(\omega_0+\omega_M)^2}\frac{\cos^2\theta}{\left[(\cos^2\theta)/\mu_0+\sin^2\theta\right]^2} \times\left[\exp\left(\frac{\hbar\omega_+(\mathbf{k})}{T}\right)-1\right]^{-1}\mathrm{d}^3k + \int\frac{\omega_-(\mathbf{k})\omega_M}{(\omega_0+\omega_M)\omega_0}\left[\exp\left(\frac{\hbar\omega_-(\mathbf{k})}{T}\right)-1\right]^{-1}\mathrm{d}^3k\right\}. \quad (4.21)$$

It is appropriate to recall that the dispersion laws $\omega = \omega_\pm(k)$ are given by formulas (2.16) and steady state magnetic susceptibility by (3.6). It is convenient to write the result of integration in the form

$$\frac{\Delta\widetilde{M}(T)}{M_0} \simeq -\frac{T}{\hbar\omega_0}\left(\frac{T}{T_M}\right)^3 I(\mu_0)\,, \qquad M_0 = \frac{g\hbar}{a^3}\,,$$
$$T_M = \frac{2\pi\hbar\tilde{c}}{a}\,, \qquad \tilde{c} = \frac{c}{\sqrt{\varepsilon}}\,. \quad (4.22)$$

Here, a is the parameter of the order of the cell size so that M_0 is the magnetic moment of saturation, and

$$I = \frac{(2\pi)^5}{480}\frac{(\mu_0-1)^2}{\mu_0^2} \times\int_0^\pi\left[\cos^2\theta\left(\frac{\cos^2\theta}{\mu_0}+\sin^2\theta\right)^{-7/2}+\frac{1}{\sqrt{\mu_0}}\right]\sin^2\theta\,\mathrm{d}\theta\,. \quad (4.23)$$

The temperature dependence ($\Delta\widetilde{M} \propto T^4$) is standard and is determined by the density of states and the fact that, at $kc \to \infty$, the polariton magnetic moment tends to zero in proportion to k. We are interested in the dependence on the magnetic field. To find this dependence, we should first determine $I = I(\mu_0)$:

$$I(\mu_0) \simeq \begin{cases} \dfrac{8\pi^5}{45}(\mu_0-1)^2\,, & \mu_0 \geqslant 1\,,\\[2mm] \dfrac{2\pi^5}{75}\mu_0^{5/2}\,, & \mu_0 \gg 1\,,\end{cases}$$

or [see (3.6)]

$$I(H_0) = \begin{cases} \dfrac{2\pi^5}{75}\left(\dfrac{4\pi M}{H_0}\right)^{5/2}, & H_0 \ll 4\pi M\,,\\[2mm] \dfrac{8\pi^5}{45}\left(\dfrac{4\pi M}{H_0}\right)^2, & H_0 \gg 4\pi M\,.\end{cases}$$

Formulas (4.22) and (4.20) allow the threshold temperature $T_{\lim}$ to be estimated. However, this requires that (4.20) be rewritten by introducing the temperature $T_C \gg E_0$ in accordance with the formula $T_C = \hbar\omega_{ex}$ and using the definition of the effective mass (3.7′):

$$\frac{\Delta\widetilde{M}}{M_0} = -\frac{1}{8\pi^{3/2}}\frac{T}{\hbar\omega_M}\left(\frac{T}{T_C}\right)^{3/2}\exp\left(-\frac{E_0}{T}\right). \quad (4.24)$$

Hence and from (4.23) (at $\mu_0 \sim 1$, i.e. at $\omega_0 \sim \omega_M$):

$$T_{\lim} \simeq E_0\left|\ln\frac{T_M^3}{(\hbar\omega_{ex})^{3/2}E_0^{3/2}}\right|^{-1} \ll E_0\,, \quad (4.25)$$

since estimated $T_M \sim 4\times10^7$ K, $\hbar\omega_{ex} \sim 10^2-10^3$ K, $E_0 \sim 1$ K.

The last formulas are important for two reasons. First, the progress in the practical use of the low-temperature region is very rapid, and it may be hoped that the temperature dependence of magnetization will be thoroughly measured, which needs these formulas to be taken into consideration. Second, they illustrate the general statement according to which the resonance interaction between magnons and quasiparticles leads to the ‘elimination’ of the gap in the magnon spectrum and to the power dependence of thermodynamic characteristics on temperature at $T \to 0$ (even such specific magnetic characteristics as the magnetic moment), the Goldstone energy being dependent on the magnetic field due to the interaction with magnon (the Goldstone acquires magnetic moment).

5. Uniaxial antiferromagnet

The objective of the present section is to demonstrate how the magnetic nature and structure of the ground state of a body are manifested in the properties of low-frequency oscillations (see [10, 11] for more details).

The magnetic permeability tensor μ_{ik} of a uniaxial (two-sublattice) antiferromagmet has the structure (1.19) if the external magnetic field **H** is directed parallel to a selected axis. For the two-sublattice antiferromagnet of the ‘easy axis’ (EA) type being examined, the selected axis is the direction along which the magnetic moments of sublattices align themselves antiparallelly at $H < F_{SF}$. At $H = H_{SF}$, the reorientational first-order (spin-flop) transition occurs, and magnetic moments become symmetric at an angle of ψ to the axis, with $\cos\psi = H/H_E$. At $H = H_E$ (as a rule, $H_E \gg H_{SF}$), the second-order (spin-flip) transition takes place. This description of the reorientational phase transition is somewhat simplified. The magnetic fields H_{SF} and H_E together with absolute magnetization values for the sublattices at $T = 0$ may serve as a complete set of quantities characterizing the antiferromagnet. The fields H_{SF} and H_E can be expressed through the exchange constant δ and anisotropy constants β, β'. Equilibrium configurations of the magnetic moments of an EA-antiferromagnet are depicted in Fig. 3. Table 2 shows components of the tensor μ_{ik} and effective

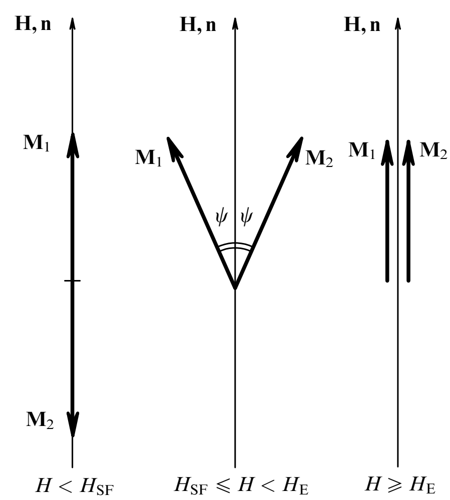

Figure 3. Equilibrium states of EA type antiferromagnets.

Table 2.

$H=0$	$\mu_1=\mu_2=\mu=1+\dfrac{4\pi}{\delta}\dfrac{\Omega_{\rm SF}^2}{\Omega_{\rm SF}^2-\omega^2}$,	$\mu_{\rm eff}=1+\dfrac{4\pi}{\delta}\dfrac{\Omega_{\rm SF}^2}{\Omega_{\rm SF}^2-\omega^2}$
	$\mu'\equiv 0$	
$0<H<H_{\rm SF}$	$\mu_1=\mu_2=\mu=1+\dfrac{2\pi}{\delta}\Omega_{\rm SF}^2\left[\dfrac{1}{\Omega_{\rm SF}^2-(\Omega-\omega)^2}+\dfrac{1}{\Omega_{\rm SF}^2-(\Omega+\omega)^2}\right]$,	$\mu_{\rm eff}=\dfrac{\left[(1+4\pi/\delta)\Omega_{\rm SF}^2-\Omega^2-\omega^2\right]^2-4\Omega^2\omega^2}{(\Omega_{\rm SF}^2-\Omega^2-\omega^2)\left[\Omega_{\rm SF}(1+4\pi/\delta)-\Omega^2-\omega^2\right]-4\omega^2\Omega^2}$
	$\mu'=\dfrac{2\pi}{\delta}\Omega_{\rm SF}^2\left[\dfrac{1}{\Omega_{\rm SF}^2-(\Omega-\omega)^2}-\dfrac{1}{\Omega_{\rm SF}^2-(\Omega+\omega)^2}\right]$	
$H_{\rm SF}\leqslant H<H_{\rm E}$	$\mu_1=1+\dfrac{\delta}{4\pi}\dfrac{\Omega_{\rm M}^2\cos^2\psi}{\Omega_\psi^2-\omega^2}$,	$\mu_{\rm eff}=\dfrac{\Omega_\psi^2(1+4\pi/\delta)+\Omega_{\rm M}^2\cos^2\psi(1+4\pi/\delta)-\omega^2}{\Omega_\psi^2+(4\pi/\delta)\Omega_{\rm M}^2\cos^2\psi-\omega^2}$
	$\mu_2=1+\dfrac{4\pi}{\delta}\dfrac{\Omega_\psi^2}{\Omega_\psi^2-\omega^2}$,	
	$\mu'=\dfrac{\Omega_{\rm M}\omega\cos\psi}{\Omega_\psi^2-\omega^2}$	
$H\geqslant H_{\rm E}$	$\mu_1=\mu_2=\mu=1+\dfrac{\Omega_{\rm M}\Omega}{\Omega^2-\omega^2}$,	$\mu_{\rm eff}=\dfrac{(\Omega+\Omega_{\rm M})^2-\omega^2}{\Omega^2+\Omega\Omega_{\rm M}-\omega^2}$
	$\mu'=\dfrac{\Omega_{\rm M}\omega}{\Omega^2-\omega^2}$	

magnetic permeabilities. In this table,

$$H_{\rm SF}=M\sqrt{2\delta(\beta-\beta')}\,,\quad H_{\rm E}=2\delta M\,,\quad \Omega_{\rm SF}=gH_{\rm SF}\,,$$
$$\Omega=gH\,,\quad \Omega_{\rm M}=8\pi gM\,,$$
$$\Omega_\psi^2=(gM)^2\left[4\delta^2\cos^2\psi-2\delta(\beta-\beta')\sin^2\psi\right],$$
$$\cos\psi=\frac{H}{H_{\rm E}}\,,\quad \mu_0=1+\frac{4\pi}{\delta}\,\delta\gg\beta\,,$$
$$\beta'\sim 1\,,\quad \beta-\beta'>0\,.$$

In the first place, it follows from Table 2 that the number of resonant frequencies of magnetic permeability changes upon transition to the zero magnetic field: there are two resonant frequencies at $H\neq 0$ and one at $H=0$. At $H=H_{\rm E}$, the components of the tensor μ_{ik} are continuous. Assuming that $H_{\rm SF}\ll H_{\rm E}$, anisotropy of the tensor μ_{ik} in the basic plane at $H_{\rm SF}\leqslant H<H_{\rm E}$ is apparent only in the immediate proximity to the spin-flop transition. Since

$$\Omega_\psi=g\sqrt{H^2\left(1+\frac{H_{\rm SF}^2}{H_{\rm E}^2}\right)-H_{\rm SF}^2}$$

(see Table 2),

$$\mu_1=1+\frac{\Omega_{\rm M}}{\Omega_{\rm E}}\frac{\Omega_{\rm SF}^2}{\Omega_{\rm SF}^4/\Omega_{\rm E}^2-\omega^2}\,,$$

$$\mu_2=1+\frac{\Omega_{\rm M}}{\Omega_{\rm E}}\frac{\Omega_{\rm SF}^2}{\Omega_{\rm E}^2}\frac{\Omega_{\rm SF}^2}{\Omega_{\rm SF}^4/\Omega_{\rm E}^2-\omega^2}\,, \tag{5.1}$$

at $H=H_{\rm SF}$. This means that polarizations along different directions at the plane normal to the EA are significantly different:

$$\frac{\mu_1-1}{\mu_2-1}=\left(\frac{H_{\rm E}}{H_{\rm SF}}\right)^2, \tag{5.2}$$

but at $H_{\rm E}>H\gg H_{\rm SF}$, the frequency $\Omega_\psi\simeq\Omega$, and anisotropy at the basic plane is low

$$\mu_1\simeq\mu_2\simeq 1+\frac{\Omega_{\rm M}}{\Omega_{\rm E}}\frac{\Omega^2}{\Omega^2-\omega^2}\,. \tag{5.3}$$

It follows from the previous sections that the value of the magnetic permeability tensor allows the dispersion law for a dimensional magnetic polariton to be calculated.

Let us start from $H=0$ (the first line in Table 2). This case is interesting in that a non-magnetized antiferromagnet has no gyrotropy in the presence of low resonant frequency of magnetic origin. Another important fact is that both characteristic frequencies: resonant frequency in which $\mu=\infty$ and antiresonant frequency $\mu=0$, have similar values due to the smallness of the ratio $\Omega_{\rm M}/\Omega_{\rm E}=4\pi/\delta$.

The dispersion law of one of the electromagnetic waves propagating perpendicular to the selected axis ($\theta=\pi/2$) contains magnetic permeability

$$k^2=\frac{\omega^2}{c^2}\,\varepsilon\,\frac{\Omega_{\rm SF}^2(1+\Omega_{\rm M}/\Omega_{\rm E})-\omega^2}{\Omega_{\rm SF}^2-\omega^2}\,. \tag{5.4}$$

This is the dispersion law of magnetic polariton. The dependence of frequency ω on the wave vector k is schematically represented in Fig. 4. The similarity between resonant and antiresonant frequencies is also reflected in that the velocity of light $\tilde c$ at $\omega\ll\Omega_{\rm SF}$ is close to that at $\omega\gg\Omega_{\rm SF}$:

$$\tilde c=\frac{c}{\sqrt{\varepsilon}}\left(1+\frac{\Omega_{\rm M}}{\Omega_{\rm E}}\right)\simeq\frac{c}{\sqrt{\varepsilon}}\,. \tag{5.5}$$

The quasistatic limit ($k\to\infty$, $\omega\to$ const) corresponds to $\mu=\infty$, i.e.

$$\omega=\Omega_{\rm SF}\,. \tag{5.6}$$

However, in this case ($\mu'\equiv 0$), the quasistatic limit obtained from Maxwell's equations does not coincide with the solution

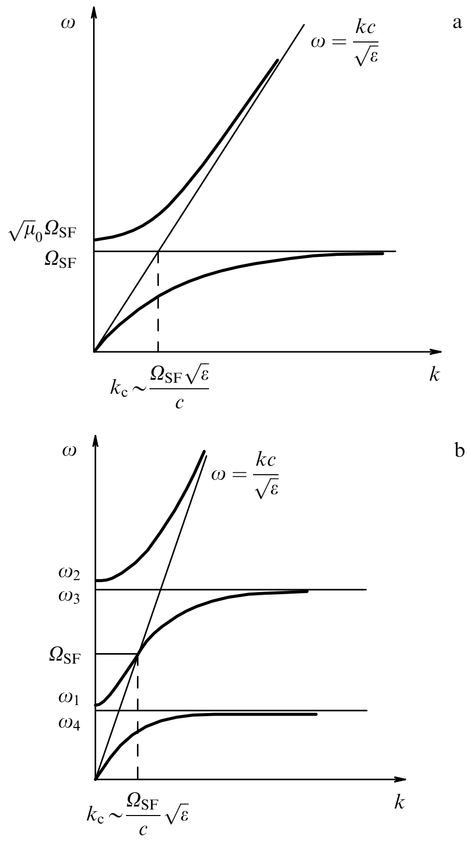

Figure 4. Dispersion law of magnetic polariton in an EA-antiferromagnet at $H = 0$ (a) and $0 < H < H_{SF}$ (b). Characteristic frequencies $\omega_{1,2} = \Omega_{SF}(1 + 2\pi/\delta) \mp \Omega$, $\omega_{3,4} = \Omega_{SF}(1 + \pi/\delta) \pm \Omega$, $\Omega = gH$, $\Omega_{SF} = gH_{SF}$.

of the magnetostatic problem (see the Introduction). Therefore, there is a longitudinal oscillation at $\theta = \pi/2$ whose frequency is the root of the equation $\mu(\omega) = 0$, i.e.

$$\omega = \Omega_{SF}\left(1 + \frac{\Omega_M}{\Omega_E}\right). \tag{5.7}$$

The longitudinal oscillation branch has been shown [10] to exist only when the wave propagates normal to EA (here, $\mu_3 = 1$, i.e. the crystal is anisotropic!). Indeed, when a wave travels at an angle θ to EA, there are two magnetic polaritons of different polarization with the dispersion laws

$$k^2 = \frac{\omega^2}{c^2}\,\varepsilon\mu \quad \text{and} \quad k^2 = \frac{\omega^2}{c^2}\,\varepsilon\,\frac{\mu}{\mu\sin^2\theta + \cos^2\theta}\,, \tag{5.8}$$

It follows from the magnetostatic equations that

$$\mu\sin^2\theta + \cos^2\theta = 0\,. \tag{5.9}$$

At $\theta \neq \pi/2$, the dispersion law for an MSW coincides with the limiting ($kc \to \infty$, $\omega \not\to \infty$) dispersion law for one of the magnetic polaritons, but there is only one magnetic polariton [the first equation in (5.8)] at $\theta = \pi/2$, while an MSW [Eqn (5.9)] 'splits' from photon and Eqn (5.9) turns into the exact one

$$\mu = 0\,. \tag{5.10}$$

We have written [see (5.6) and (5.7)] only resonant and antiresonant frequencies. 'Conversion' of these expressions into dispersion laws for spin waves (magnons) naturally requires that spatial dispersion of magnetic permeability, e.g. non-uniform exchange interaction, be taken into consideration.

Table 2 contains expressions for μ_{eff} obtained from the following representation of the dispersion law for magnetic polariton

$$k^2 = \frac{\omega^2}{c^2}\,\varepsilon\mu_{\text{eff}}(\omega)\,. \tag{5.11}$$

Taken together with μ_1, μ_2, and μ', these expressions give the possibility to define the frequency dependence, hence the dispersion law for magnetic polaritons. Note that the magnetic polariton dispersion law in an antiferromagnet is qualitatively different (in terms of branch numbers) from the dispersion law in a ferromagnet only at $0 \neq H < H_{SF}$. At $H = H_E$, an antiferromagnet actually undergoes 'conversion' to a ferromagnet with the magnetization of $2M$; therefore, its high-frequency properties are indistinguishable from those of ferromagnet. At $H_{SF} < H < H_E$, the above formulas do not describe rotations of the magnetic moments round EA; hence, they do not describe one oscillation branch. This issue is discussed below using the simplest example.

Spatial dispersion of magnetic permeability which will be analysed at $H < H_{SF}$ is characterized by two tensors of non-uniform exchange interactions, α_{ik} and α'_{ik} (cf. Section 3). The former describes interactions inside each sublattice while the latter does the same between sublattices. In a cubic crystal, both tensors are degenerated to scalars ($\alpha_{ik} \to \alpha\delta_{ik}$; $\alpha'_{ik} \to \alpha'\delta_{ik}$). Using the Landau–Lifshitz equation, it is possible to demonstrate that taking into account spatial dispersion leads to renormalization of anisotropy constants

$$\beta - \beta' \to \beta_{\text{eff}} = \beta - \beta' + (\alpha_{ik} - \alpha'_{ik})k_i k_k \tag{5.12}$$

and the exchange constant

$$\delta \to \delta + \alpha'_{ik}k_i k_k\,. \tag{5.13}$$

Matrix $\alpha_{ik} - \alpha'_{ik}$ must be positively definite to ensure stability of the antiferromagnetic state [3].

The order of magnitude of the exchange constant can be estimated from the phase transition temperature (Neel temperature). If $\delta \sim \Theta_N/\mu M$ ($\mu = g\hbar$ is Bohr's magneton), then $\alpha, \alpha' \sim a^2\Theta_N/\mu M$. It follows that for $ak \ll 1$ one does not need to take into account renormalization of the exchange constant whereas renormalization of the anisotropy constant must be taken into consideration because it is likely to lead to a qualitative spectral change. For example, in the absence of a magnetic field ($H = 0$) at $(ak)^2 \gg (\beta - \beta')\mu M/\Theta_N$, the quasi-static limit (5.6) is 'converted' into a spin wave with the linear dispersion law. This limiting case has been thoroughly examined in the literature. It is assumed in the forthcoming discussion that only H_{SF} contains the dependence on the wave vector $\mathbf{k}$:

$$H_{SF} = \sqrt{(H^0_{SF})^2 + \widetilde{H}^2_E(ak)^2}\,, \tag{5.14}$$

where H^0_{SF} is the transition field without regard for spatial dispersion.

When solving magnetostatic equations (2.14), it is not difficult to find the dispersion law for spin waves:

$$\omega^2 = \Omega_{\rm SF}^2\left(1+\frac{2\pi\sin^2\theta}{\delta}\right)+\Omega^2 \pm\sqrt{4\Omega^2\Omega_{\rm SF}^2+\frac{8\pi\sin^2\theta}{\delta}\,\Omega^2\Omega_{\rm SF}^2+\Omega_{\rm SF}^4\left(\frac{2\pi\sin^2\theta}{\delta}\right)^2}\,. \tag{5.15}$$

At $H=0$, it is possible to obtain two solutions from (5.15):

$$\omega_-^2=\Omega_{\rm SF}^2\,,\qquad \omega_+^2=\Omega_{\rm SF}^2\left(1+\frac{4\pi\sin^2\theta}{\delta}\right). \tag{5.16}$$

The first solution (ω_-) is absent if the expression for μ from the first line of Table 2 is used. It should not be discarded, for continuity reason. The second solution (ω_+) contains dependence on the direction of the wave vector. The corresponding spin wave energy ($E=\hbar\omega_+$) may be written in the following form [see (5.16) and (5.12)]:

$$E^2=(\mu H_{\rm SF})^2+(\mu H_{\rm A})^2\sin^2\theta+I_{\rm ex}^2(ak)^2\,, \tag{5.17}$$

$H_{\rm A}=4\pi M\left[2(\beta-\beta')\right]^{1/2}$ is the anisotropy field and $I_{\rm ex}$ is the quantity of the order of the exchange integral between atoms ($I_{\rm ex}\sim\Theta_{\rm N}$). This formula indicates that a dipole–dipole interaction results in anisotropy of the gap in the magnon spectrum. True, it is small, because $H_{\rm A}^2/H_{\rm SF}^2\sim 1/\delta\ll 1$. Nonanalytical dependence on the direction of vector $\mathbf{k}$ [dependence $E(\theta)$ at $k=0$] is responsible for the enhanced dimension of the magnon $\mathbf{k}$-space described in Section 4.

Let us now examine manifestations of the dipole–dipole interaction when the magnetic field is $H_{\rm SF}$ (the upper boundary of the collinear regime). In the approximation used here, one of the oscillation branches (near ω_-) has no gap at $H=H_{\rm SF}$, and the width of the hysteresis loop during the spin-flop transition vanishes while $H=H_{\rm SF}$ coincides with the lability field (cf. [12]). Interestingly, taking into account the dipole–dipole interaction does not alter the situation, that is the gap remains zero at all angles $\theta\big(\omega_-(k=0)=0\big)$, and the magnon energy is

$$E_-=\frac{I_{\rm ex}ak}{2\mu H_{\rm SF}}\sqrt{I_{\rm ex}^2(ak)^2+(\mu H_{\rm A})^2\sin^2\theta}\,, \tag{5.18}$$

where $\mu H_{\rm SF}=g\hbar\Omega_{\rm SF}\sim\mu\sqrt{H_{\rm E}H_{\rm A}}$.

The present review is not concerned with statistical thermodynamic and kinetic properties of antiferromagnets. However, it is worthwhile to emphasize specific temperature dependence of the magnon component of antiferromagnet's thermal capacity directly related to the presence of the dipole–dipole energy in the magnon spectrum (5.18).

At low temperatures ($T\to 0$), it is appropriate to deal with that branch of spin waves which has no gap in our approximation. Therefore, we shall consider only the '–'-branch.

Omitting the second item of the root in (5.18) yields the magnon dispersion law reminiscent of the dispersion law for a relativistic particle with the effective mass

$$m^*=\frac{\hbar^2}{a^2}\,\frac{\mu H_{\rm SF}}{I_{\rm ex}^2}\sim\frac{\hbar^2}{a^2}\sqrt{\frac{H_{\rm A}}{H_{\rm E}}}\,\frac{1}{I_{\rm ex}} \tag{5.19}$$

and hence, with thermal capacity proportional to $T^{3/2}$. The effective magnon mass in an antiferromagnet is small, in conformity with the smallness of the $H_{\rm A}/H_{\rm E}$ ratio.

A simplest model of antiferromagnet may be used to demonstrate the role of the 'lost' branch of magnetic moment oscillations. Let us consider, following [11], an isotropic antiferromagnet devoid of anisotropy (which means that only exchange terms and the Zeeman energy are retained in the expression for its energy density). Unlike the previous consideration, the present one takes into account the nonuniform exchange interaction.

Thus,

$$W=\delta\mathbf{M}_1\cdot\mathbf{M}_2+\frac{\alpha}{2}\left(\frac{\partial\mathbf{M}_1}{\partial x_i}\cdot\frac{\partial\mathbf{M}_1}{\partial x_i}+\frac{\partial\mathbf{M}_2}{\partial x_i}\cdot\frac{\partial\mathbf{M}_2}{\partial x_i}\right) +\alpha'\frac{\partial\mathbf{M}_1}{\partial x_i}\cdot\frac{\partial\mathbf{M}_2}{\partial x_i}-(\mathbf{M}_1+\mathbf{M}_2)\cdot\mathbf{H}\,. \tag{5.20}$$

The exchange constant $\delta\gg 1$ and $|\alpha|,|\alpha'|\sim\delta a^2$. It follows from (5.20) that magnetic moments $\mathbf{M}_1$ and $\mathbf{M}_2$ at $H<2\delta M\equiv H_{\rm E}$ are symmetric relative to the magnetic field $\mathbf{H}$ (at an angle ψ, see Fig. 3):

$$\cos\psi=\frac{H}{H_{\rm E}}\,,\qquad H\leqslant H_{\rm E}=2M\delta\,. \tag{5.21}$$

Considering small oscillations of magnetic moments in an alternating magnetic field $\mathbf{h}=\mathbf{h}_0\exp\left[-\mathrm{i}(\omega t-\mathbf{kr})\right]$, it is easy to calculate the magnetic susceptibility tensor $\hat{\chi}$. If $z\parallel\mathbf{H}$ and the magnetic moments $\mathbf{M}_1$ and $\mathbf{M}_2$ are at the xz-plane, the components of the tensor are

$$\chi_{xx}=\frac{2g^2M(H+M\alpha_+k^2\cos\psi)\cos\psi}{\omega_\perp^2(k)-\omega^2}\,,$$

$$\chi_{yy}=\chi_{xx}+\frac{1}{\delta}\,\frac{\omega_\parallel^2(k)}{\omega_\perp^2(k)-\omega^2}\,,$$

$$\chi_{zz}=\frac{1}{\delta}\,\frac{\omega_\parallel^2}{\omega_\parallel^2(k)-\omega^2}\,,\qquad \chi'=\frac{2\omega gM\cos\psi}{\omega_\perp^2-\omega^2}\,, \tag{5.22}$$

$$\omega_\perp^2(k)=g^2(H+M\alpha_+k^2\cos\psi)^2+2(gM\sin\psi)^2\delta\alpha_-k^2\,,$$

$$\omega_\parallel^2(k)=2(gM\sin\psi)^2\delta\alpha_-k^2\,. \tag{5.23}$$

Finally,

$$\alpha_\pm=\alpha\pm\alpha'\,. \tag{5.24}$$

The stability condition for this model requires that δ and $\alpha-\alpha'$ be positive.

The model does not describe spin-flop transitions but can describe a spin-flip transition (second-order phase transition), i.e. the collapse of magnetic moments $\mathbf{M}_1$ and $\mathbf{M}_2$ at $H=H_{\rm E}$. When $H=H_{\rm E}$, the frequency of longitudinal oscillations is equal to zero, while the frequency of transverse components and magnitudes of transverse components of the tensor χ_{ik} have the form typical of a ferromagnet (with the magnetic moment density being equal to $2M$):

$$\omega_\perp(k)=g(H+M\alpha_+k^2)\,,$$

$$\chi_{xx}=\chi_{yy}=\frac{2gM\omega_\perp(k)}{\omega_\perp^2(k)-\omega^2}\,,$$

$$\chi'=\frac{2\omega gM}{\omega_\perp^2(k)-\omega^2}\,,\qquad H\geqslant H_{\rm E}\,. \tag{5.25}$$

Now, let us consider the case of $H < H_E$, assuming that the value of H is not very close to H_E. It will be clear from the forthcoming discussion that closeness to H_E depends on the parameter δ, i.e.

$$1 - \frac{H}{H_E} \gg \frac{1}{2\delta}\,. \tag{5.26}$$

In this case a reasonable approximation is the expression in which only antiferromagnetic mechanism is responsible for dispersion. Then, the formulas are simplified:

$$\omega_\perp^2 \simeq (gH)^2 + v_a^2(H)k^2\,, \qquad \omega_\parallel^2 = v_a^2(H)k^2\,,$$
$$v_a^2 = 2(gM)^2\left(1 - \frac{H^2}{H_E^2}\right)\delta\alpha_-\,. \tag{5.27}$$

In the same approximation

$$\chi_{xx} = \frac{1}{\delta}\,\frac{(gH)^2}{\omega_\perp^2(k) - \omega^2}\,, \qquad \chi_{yy} = \frac{1}{\delta}\,\frac{\omega_\perp^2(k)}{\omega_\perp^2(k) - \omega^2}\,,$$
$$\chi_{zz} = \frac{1}{\delta}\,\frac{\omega_\parallel^2(k)}{\omega_\parallel^2(k) - \omega^2}\,, \qquad \chi' = \frac{1}{\delta}\,\frac{gH\omega}{\omega_\perp^2(k) - \omega^2}\,, \qquad H \leqslant H_E\,. \tag{5.28}$$

The most important parameter for further discussion, which enters (5.7), is the velocity

$$v_a(H = 0) = gM\sqrt{2\alpha_-\delta} \equiv v_0\,. \tag{5.29}$$

At normal values of H_E and μ ($H_E \sim 10^6$ Oe, $\mu \sim 10^{-20}$ erg G^{-1}), the velocity is $v_0 \sim 10^5$ cm s^{-1}, i.e. significantly lower than the velocity of light.

Taking into account dispersion 'turned' the zero oscillation frequency into Goldstone with the linear dispersion law and 'included' the Goldstone in magnetic electrodynamics by virtue of difference of χ_{zz} from zero.

Let us analyse magnetic polariton dispersion laws to understand how the Goldstone manifests itself in the high-frequency properties of a magnet.

We shall first consider the case of $H = 0$. Magnetic permeability has no gyrotropic terms:

$$\mu_{xx} = 1\,, \qquad \mu_{yy} = \mu_{zz} = \tilde{\mu} = 1 + \frac{4\pi}{\delta}\,\frac{(v_0k)^2}{(v_0k)^2 - \omega^2}\,. \tag{5.30}$$

Let the wave vector $\mathbf{k}$ be directed at an angle ψ to the x-axis. The magnet contains polaritons with different polarizations.

At $h_z \neq 0$ and $h_x = h_y = 0$,

$$k^2 = \frac{\omega^2}{c^2}\,\varepsilon\tilde{\mu}(\omega, k)\,. \tag{5.31}$$

At $h_z = 0$ and $h_x, h_y \neq 0$,

$$k^2 = \frac{\omega^2}{c^2}\,\varepsilon\,\frac{\tilde{\mu}(\omega, k)}{\tilde{\mu}(\omega, k)\sin^2\theta + \cos^2\theta}\,. \tag{5.32}$$

It should be emphasized that polariton (5.31) 'turns' into photon at $\theta = \pi/2$, its dispersion law being independent of the magnetic properties of the medium. This polariton is not considered here. $\theta = 0$ is associated with degeneracy: the dispersion laws coincide. One polarization results in two polaritons, each having the linear dispersion law:

$$\omega = v_{1,2}k\,, \tag{5.33}$$

and v_1 and v_2 are the roots of the biquadratic equation

$$\frac{v_c^2}{v^2} = \frac{v_0^2(1 + 4\pi/\delta) - v^2}{v_0^2\left[1 + (4\pi/\delta)\sin^2\theta\right] - v^2}\,, \qquad v_c = \frac{c}{\sqrt{\varepsilon}}\,. \tag{5.34}$$

Equation (5.34) always has two real solutions (Fig. 5). Since $v_c \gg v_0$ and $\delta \gg 1$,

$$v_{1,2}^2 \simeq \begin{cases} v_c^2\left(1 + \dfrac{4\pi}{\delta}\cos^2\theta\,\dfrac{v_0^2}{v_c^2}\right), \\ v_0^2\left(1 + \dfrac{4\pi}{\delta}\sin^2\theta\right)\left(1 + \dfrac{4\pi}{\delta}\cos^2\theta\,\dfrac{v_0^2}{v_c^2}\right). \end{cases} \tag{5.35}$$

It is worthy of note that two Goldstones (photon and magnon) remain the same when interacting, although the velocity of the slow one (magnon) decreases while that of the fast one (photon) increases. Reference [11] reports the difficulty encountered in the theory which is due to the fact that the velocity of the fast Goldstone seemingly tends to exceed the velocity of light: the theory in question is lacking in a relevant relativistic limitation.

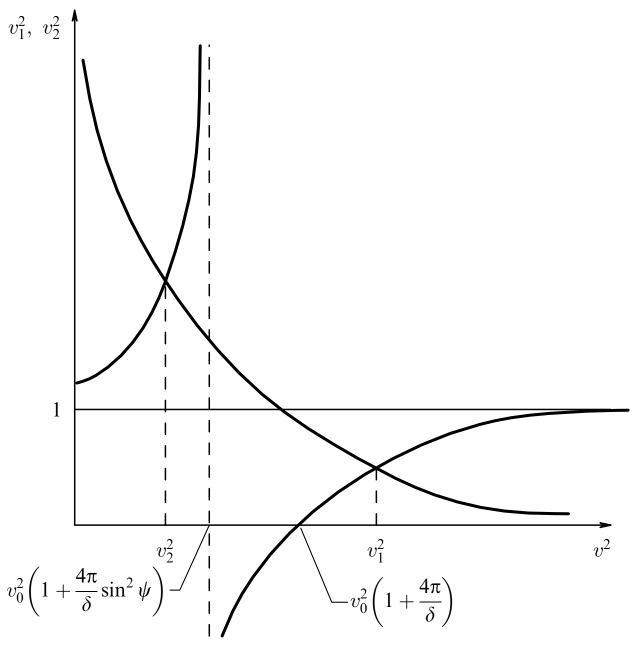

Figure 5. The graphic solution of Eqn (5.34) at $\psi \neq \pi/2$.

Let us now turn to the case of $H \neq 0$. At $h_x = h_y = 0$, $h_z \neq 0$ the dispersion law for polariton with the wave vector along the x-axis (here, we confine ourselves to this specific case) coincides with the dispersion law for polariton of the same polarization at $H = 0$, provided v_0 is substituted by $v_0(H)$ [see (5.29)].

The dispersion law for polariton of a different polarization ($h_x, h_y \neq 0$, $h_z = 0$) is the solution of the equation

$$k^2 = \frac{\omega^2}{c^2}\,\varepsilon\mu_{\text{eff}}\,, \qquad \mu_{\text{eff}} = \mu_{yy} - \frac{\mu'^2}{\mu_{xx}}\,. \tag{5.36}$$

Using (5.28), it is possible to represent this equation as

$$\left[\omega^2 - (v_c k)^2\right]\left[\omega^2 - \omega_\perp^2(k)\right] = \frac{4\pi}{\delta}\,\omega_\perp^2(k)\omega^2\,. \tag{5.37}$$

It follows from the latter equation that polariton is a result of spin wave merging with photon. We do not analyse the solution of Eqn (5.36) because it has been described in Ref. [11].

6. Resonance polaritons

Although we have already noted that characteristic frequencies of dielectric permittivity and magnetic permeability normally lie far from each other, it is easy to see exceptions to this rule. The existence of large magnetic fields governing resonant frequencies of magnetic subsystems of a solid body sometimes makes it expedient to consider coincidence of electric and magnetic frequencies.

In this Section, we shall demonstrate, based on the results of Ref. [13], how the dispersion law of polariton is modified in the case of coincidence (resonance) between the characteristic frequencies of $\varepsilon(\omega)$ and $\mu(\omega)$. We proceed from the dispersion law written in the form [cf. (2.4)]

$$k^2 = \frac{\omega^2}{c^2}\,\varepsilon(\omega)\mu_{\text{eff}}(\omega)\,. \tag{6.1}$$

When speaking of the coincidence between characteristic frequencies, we certainly mean the equality of two frequencies, one characteristic of the electric subsystem and the other of the magnetic one. If damping is neglected, those frequencies should be regarded as characteristic at which ε and μ turn into zero and infinity. In order to demonstrate the dependence of the polariton dispersion law on the relationship between characteristic frequencies of different nature in a broad frequency range, we shall take advantage of the simplest dispersion formulas for $\varepsilon = \varepsilon(\omega)$ and $\mu_{\text{eff}} = \mu_{\text{eff}}(\omega)$ and totally disregard attenuation. Let

$$\varepsilon(\omega) = \frac{\omega_{\text{AE}}^2 - \omega^2}{\omega_{\text{RE}}^2 - \omega^2}\,, \qquad \mu_{\text{eff}}(\omega) = \frac{\omega_{\text{AM}}^2 - \omega^2}{\omega_{\text{RM}}^2 - \omega^2}\,. \tag{6.2}$$

Let us consider the following cases:

(1) $\omega_{\text{RM}} < \omega_{\text{AM}} < \omega_{\text{RE}} < \omega_{\text{AE}}$ (normal case).
(2) $\omega_{\text{AM}} = \omega_{\text{RE}}$, $\omega_{\text{RM}} < \omega_{\text{AE}}$.
(3) $\omega_{\text{RM}} = \omega_{\text{RE}} = \omega_{\text{R}}$, $\omega_{\text{AM}} < \omega_{\text{AE}}$.
(4) $\omega_{\text{AM}} = \omega_{\text{AE}} = \omega_{\text{A}}$, $\omega_{\text{RM}} < \omega_{\text{AE}}$.
(5) $\omega_{\text{RM}} = \omega_{\text{AE}}$, $\omega_{\text{RE}} < \omega_{\text{AE}} < \omega_{\text{AM}}$.

Resonance polaritons are virtually those included in cases 3 and 4. Their dispersion laws are shown in Fig. 6. It should be emphasized that in both cases, there are three polariton branches one of which is sure to possess anomalous dispersion.

Let us write down the dispersion law for the resonance polariton in the vicinity of specific (overlapping) frequencies (zones inside the dashed circumferences in Fig. 6a, b). Near the resonant frequency $\omega = \omega_{\text{R}}$ common for the dielectric permittivity and magnetic permeability,

$$\omega - \omega_{\text{R}} \simeq \pm\frac{\sqrt{(\omega_{\text{AE}}^2 - \omega_{\text{R}}^2)(\omega_{\text{AM}}^2 - \omega_{\text{R}}^2)}}{2kc}\,, \qquad k \to \infty\,,$$

$$v_{\text{gr}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} \simeq \pm\frac{2c(\omega - \omega_{\text{R}})^2}{\sqrt{(\omega_{\text{AE}}^2 - \omega_{\text{R}}^2)(\omega_{\text{AM}}^2 - \omega_{\text{R}}^2)}}\,, \qquad \omega \simeq \omega_{\text{R}}\,. \tag{6.3}$$

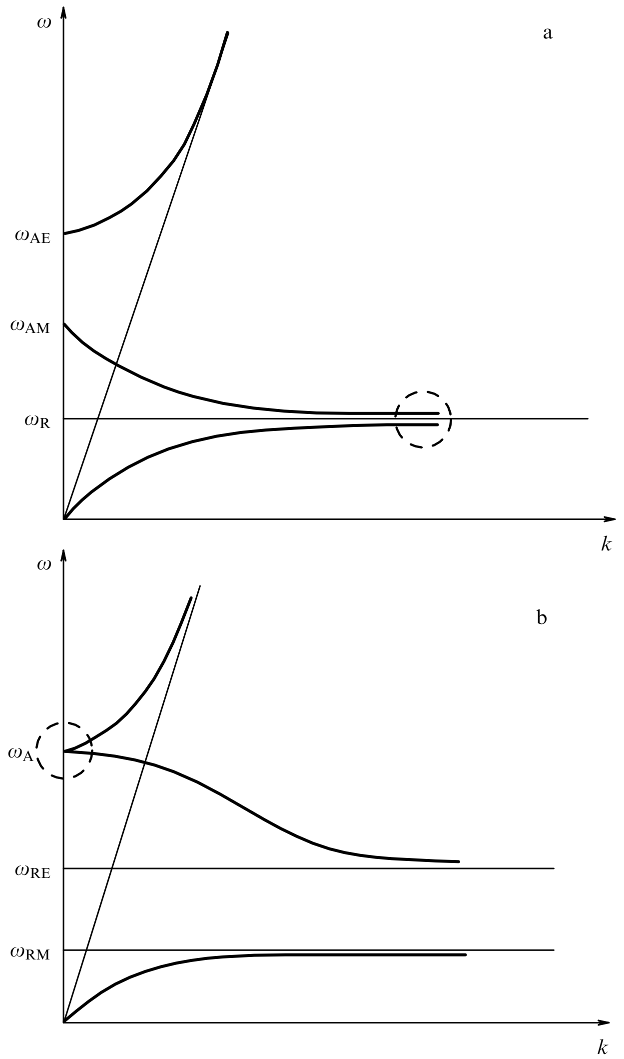

Figure 6. Dispersion laws for resonance polaritons: (a) $\omega_{\text{RM}} = \omega_{\text{RE}} = \omega_{\text{R}}$, $\omega_{\text{AM}} < \omega_{\text{RE}}$, (b) $\omega_{\text{AM}} = \omega_{\text{AE}} = \omega_{\text{A}}$, $\omega_{\text{RM}} < \omega_{\text{RE}}$.

Near the antiresonant frequency $\omega = \omega_{\text{A}}$,

$$\omega - \omega_{\text{A}} \simeq \pm v_{\text{A}} k\,, \qquad v_{\text{A}} = \frac{c}{2\omega_{\text{A}}^2}\sqrt{(\omega_{\text{A}}^2 - \omega_{\text{RM}}^2)(\omega_{\text{A}}^2 - \omega_{\text{RE}}^2)}\,. \tag{6.4}$$

The latter formula satisfies the natural condition $v_{\text{A}} < c$. These dispersion laws are unusual in that they show linear dependence of the deviation $\Delta\omega = \omega - \omega_{\text{R}}$ (or $\omega - \omega_{\text{A}}$) on $1/k$ (or k). For ordinary non-resonance polaritons, $\Delta\omega \sim 1/k^2$ or $\Delta\omega \sim k^2$. Besides, ω_{R} and ω_{A} are usually boundaries of the opacity window $(\omega_{\text{R}}, \omega_{\text{A}})$. This is not true in the present case.

7. Interaction with phonons (kinematics)*

Magnons are not the sole quasiparticles in ferromagnetic dielectrics which are known to invariably contain phonons. Non-interacting ideal magnon and phonon gases fairly well describe equilibrium (thermodynamic) properties at low temperatures. Their kinetic properties cannot be described without regard for interactions between quasiparticles.

Kinetic properties of ferromagnetic dielectrics attributable to the interaction between quasiparticles are well-known (see Refs [1, 8]). This Section is focused on kinematics† of magnon – phonon interactions while remaining in the framework of one-phonon processes. There are two such processes:

— creation (absorption) of phonon by magnon and

— decay of a phonon into two magnons (fusion of two magnons giving rise to one emitted phonon).

The number of magnons in the former process is conserved unlike that in the latter. This means that this process may result from the exchange interaction whereas the other — only from relativistic interactions. However, this difference is immaterial in the context of the present Section because here we are interested only in kinematics of the two processes, that is the possibility to satisfy the laws of conservation of energy and quasimomentum. Assuming the quasiwave vector (quasimomentum) of both magnons and phonons to be small ($ak, af \ll 1$), we ignore probable umklapp processes even though there are kinetic processes (e.g. heat conductivity of defect-free samples) which cannot be described unless umklapp processes are taken into account.

We shall start from the phonon creation by magnon. In order to demonstrate the role of the dispersion law, suffice it to consider the creation of a longwave phonon whose momentum is small compared with magnon momentum. Then, the right-hand side of the equality

$$\varepsilon(\mathbf{p}) + \hbar\omega(\mathbf{f}) = \varepsilon(\mathbf{p} + \hbar\mathbf{f}) , \tag{7.1}$$

describing the laws of conservation of energy and momentum can be expanded in powers of $\hbar\mathbf{f}$. This yields

$$\mathbf{vf} = \omega(\mathbf{f}) , \qquad \mathbf{v} = \frac{\delta\varepsilon(\mathbf{p})}{\delta\mathbf{p}} . \tag{7.2}$$

Hence, the phonon emission (absorption) condition immediately follows:

$$v(\mathbf{p}) > s , \tag{7.3}$$

where s is acoustic velocity ($\omega/f = s$). This condition coincides with the condition of the Cherenkov emission (certainly of sound, not light!) in terms of both the form and the essence. Magnon velocity varies considerably in different regions of quasimomentum magnitude. The condition (7.3) is most readily fulfilled when the dispersion law for magnon is similar to that for photon [this is polariton, $v \sim c$, see (2.16), (2.8), (2.9), (3.3), and (3.4)]. However, it is necessary to verify if the condition $\hbar k \ll p$ is satisfied at $p \to 0$. To this effect, the photon frequency ω_{ph} must be small: $\omega_{\text{ph}} \ll \omega_0 s/\tilde{c}$, where $\tilde{c}$ is the magnetic polariton velocity at $p \to 0$ [see (2.16)]. When $\tilde{c} \sim 10^{10}$ cm s^{-1}, one must have $\omega_{\text{ph}} \ll 10^{-5}\omega_0$, i.e. $\omega_{\text{ph}} \ll 10^5$ at $\omega_0 \sim 10^{10}$ s^{-1}. If the value of quasimomentum $\hbar k$ is not neglected, the phonon absorption (emission) conditions have the form

$$\frac{\tilde{c}}{s} > \frac{\hbar\omega\tilde{c}}{2ps^2} > 1 - \frac{\tilde{c}}{s} . \tag{7.4}$$

Evidently, there is an additional constraint: the double polariton momentum p must exceed the phonon momentum ($p > \hbar f/2$). In order to remain within the limits of applicability of formulas (2.16), it is still necessary that phonon frequency be sufficiently low ($\omega_{\text{ph}} \ll \omega_0 s/\tilde{c}$).

In the limiting low-frequency region, both branches of electromagnetic oscillations are in a way equitable. It needs to be clarified whether a process similar to the Cherenkov one, that is emission (absorption) of a longwave phonon, may occur, with a concomitant change in the magnon type. To be certain, let us examine phonon emission by a more active phonon [note that $\omega_+(\mathbf{k}) > \omega_-(\mathbf{k})$, in agreement with (2.16)]. In the classical limit, it follows from the conservation laws

$$\varepsilon_+(\mathbf{p}) - \hbar\omega(\mathbf{f}) = \varepsilon_-(\mathbf{p} - \hbar\mathbf{f}) \tag{7.5}$$

that

$$\omega_+(\mathbf{k}) - \omega_-(\mathbf{k}) = \omega(\mathbf{f}) - \mathbf{v}_-(\mathbf{p})\mathbf{f} , \tag{7.6}$$

whence the emission condition in the following form ensues (if we take into account that $s \ll c/\sqrt{\varepsilon}$):

$$\frac{k}{f}\left[(\cos^2\theta + \mu_0 \sin^2\theta)^{1/2} - 1\right] < 1 , \qquad \mathbf{k} = \frac{\mathbf{p}}{\hbar} . \tag{7.7}$$

Involved in the emission are magnons of the '+' and '−' branches with small momenta. The limitation upon the quantity k is lifted at $\theta \to 0$.

If the magnon resembles a conventional relativistic particle [see (3.7′)], the condition for the Cherenkov acoustic emission is usually

$$\frac{p}{m^*} > s \qquad \text{or} \qquad E(p) > \frac{m^* s^2}{2} . \tag{7.8}$$

The peculiarity of the magnon dispersion law is especially pronounced at those quasimomentum values which dictate the necessity to apply the formula (3.7), on the assumption that $\omega_0 = \omega_0(k)$ includes the energy of non-uniform exchange interaction [see (3.10)]. The expression for magnon velocity may be written as

$$\mathbf{v} = \frac{\omega_0(k)}{\omega(\mathbf{k})}\,\mathbf{v}^{\text{ex}} + \frac{\omega_{\text{M}}}{2\omega(\mathbf{k})}\left[\mathbf{v}^{\text{ex}}\sin^2\theta + \omega_0(k)\frac{\partial}{\partial\mathbf{k}}\frac{k_\perp^2}{k_\parallel^2 + k_\perp^2}\right] \tag{7.9}$$

Here, the frequency $\omega(\mathbf{k})$ is defined by the formula (3.5), with ω_0 substituted by $\omega_0(k) = \omega_0 + \omega_{\text{ex}}(ak)^2$

$$\mathbf{v}^{\text{ex}} = \frac{\partial\omega_0}{\partial\mathbf{k}} = 2\omega_{\text{ex}}a^2\mathbf{k} = \frac{\mathbf{p}}{m^*}$$

[cf. (3.7′)]. It follows from (7.9) at $p = \hbar k \to 0$ that

$$v_\perp = \frac{\omega_0 + 0.5\omega_{\text{M}}\sin^2\theta}{\sqrt{\omega_0(\omega_0 + \omega_{\text{M}}\sin^2\theta)}}\,v_\perp^{\text{ex}} + \frac{\omega_0\omega_{\text{M}}\cos^2\theta}{\sqrt{\omega_0(\omega_0 + \omega_{\text{M}}\sin^2\theta)}}\,\frac{k_\perp}{k^2} ,$$

$$v_\parallel = \frac{\omega_0 + 0.5\omega_{\text{M}}\sin^2\theta}{\sqrt{\omega_0(\omega_0 + \omega_{\text{M}}\sin^2\theta)}}\,v_\parallel^{\text{ex}} - \frac{\omega_0\omega_{\text{M}}\sin^2\theta}{\sqrt{\omega_0(\omega_0 + \omega_{\text{M}}\sin^2\theta)}}\,\frac{k_\parallel}{k^2} . \tag{7.10}$$

† In 1969, Kopylov published a small book entitled "Just Kinematics" (reviewed by myself in *Soviet Physics Uspekhi* [14]). I was interested as early as that in collecting together odd data on kinematic properties of quasiparticles essentially different from those of particles. In a way, this interest of mine was satisfied in part by my investigating electronic properties of metals with complex Fermi surfaces. The present section is another, up-dated, review of Kopylov's book. The magnon spectrum essentially variable in different wave vector regions provides an ample opportunity to study kinematic processes of interaction between quasiparticles. (*Note by MIK*).

Subscripts $\perp$ and $\|$ are defined with respect to the direction of magnetization $\mathbf{M}$ (label $\|$ denotes the component parallel to $\mathbf{M}$ and label $\perp$ denotes the one normal to $\mathbf{M}$).

It is clear that magnon velocity at $p^2/m^* \ll \hbar\omega_M$ depends on dipole forces. Neglecting the exchange item ($\sim v_{\|}^{\text{ex}}$), we have

$$v = \sqrt{\frac{\omega_0}{\omega_0 + \omega_M \sin^2\theta}} \frac{\omega_M}{k} \sin\theta\,|\cos\theta|\,, \tag{7.11}$$

i.e. the magneto-dipole part of the velocity vanishes at $\theta = 0$ and $\theta = \pi/2$. When the value of angle θ is arbitrary, formula (7.11) has the applicability region bounded from below: the retardation must be taken into account (finiteness of the velocity of light). One should pass from formula (3.7) to (2.16), (3.3) and (3.4) (see the beginning of this section). Figure 7 is a plot of magnon velocity vs. momentum $v = |\partial E/\partial \mathbf{p}|$ at $\theta \neq 0, \pi/2$. The condition for the creation (absorption) of a longwave phonon by a magnon has the following form (if formula (7.11) is applicable):

$$\sqrt{\frac{\omega_0}{\omega_0 + \omega_M \sin^2\theta}} \frac{\hbar\omega_M}{s} \sin\theta\,|\cos\theta| > p\,, \qquad p = \hbar k\,. \tag{7.12}$$

It follows from the comparison of (7.8) and (7.12) that at the minimum magnon velocity

$$v_{\min} = 2a\sqrt{2\omega_M\omega_{\text{ex}}} > s\,,$$

there is no restriction on the momentum of a magnon emitting (absorbing) a longwave phonon. According to (7.11) and (7.12), the limitation is 'transferred' onto the angles: magnons spreading along 'good' directions ($\theta = 0$ and $\theta = \pi/2$) must have momentum

$$p > m^* s = \frac{1}{2}\frac{\hbar}{a}\frac{s}{\omega_{\text{ex}} a}$$

to be able to emit (absorb) a phonon [see (2.7′)]. Note that $\omega_{\text{ex}} a \sim 10^6$ cm s^{-1} if the Curie temperature is $\sim 10^3$ K.

Let us now consider the other one-phonon process, i.e. decay of a phonon into two magnons or fusion of two magnons giving rise to one phonon. The simplest, even if bulky, situation is that where formula (3.7) can be used for the dispersion law of magnons. The laws of conservation of energy and momentum (flip-over processes being neglected, as before) lead to the following expression for angle θ_f (the angle between the direction of phonon propagation and the momentum of one of the magnons):

$$\cos\theta_f = F(f,k)\,, \qquad F(f,k) = \frac{\omega_{\text{ex}} a^2 f^2 - sf + 2\omega(k)}{2\omega_{\text{ex}} a^2 k f}\,,$$

$$\omega(k) = \omega_0 + \omega_{\text{ex}} a^2 k^2\,. \tag{7.13}$$

In this section we are interested only in the threshold of the process, that is the value of the phonon quasiwave vector crucial for its decomposition into two magnons. If the threshold is denoted as f_{th}, the threshold frequency is $\omega_{\text{th}} = s f_{\text{th}}$. Now, what is the value of f_{th}?

The function $F(f,k)$ at $f \to 0$ and $f \to \infty$ tends to infinity. Naturally, only those phonons for which $F(f,k) \leqslant 1$ can give rise to two magnons. The condition of $F(f,k)$ conversion into unity leads to the following phonon momentum:

$$f(k) = k + k_s + \sqrt{k_s^2 - \frac{\omega_0}{\omega_{\text{ex}} a^2} + 2kk_s}\,, \qquad k_s = \frac{s}{\omega_{\text{ex}} a^2}\,. \tag{7.14}$$

If $k_s^2 - \omega_0/\omega_{\text{ex}} a^2 > 0$, the breakdown process is feasible at any magnon momentum (k) and the threshold is defined by the minimum value of $f(k)$, i.e. its value at $k = 0$:

$$f_{\text{th}} = k_s + \sqrt{k_s\left(k_s - \frac{\omega_0}{s}\right)}\,,$$

$$k_s > \frac{\omega_0}{s} \quad \text{or} \quad \sqrt{\omega_0\omega_{\text{ex}}} < \frac{s}{a}\,. \tag{7.15}$$

When $k_s^2 - \omega_0/\omega_{\text{ex}} a^2 < 0$, only those magnons are involved in the breakdown process whose momenta satisfy the condition

$$2kk_s \geqslant \frac{\omega_0}{\omega_{\text{ex}} a^2} - k_s^2\,,$$

and the threshold depends on the k value which turns the expression under the root in (7.14) into zero

$$f_{\text{th}} = \frac{1}{2}\left(k_s + \frac{\omega_0}{s}\right),$$

$$k_s < \frac{\omega_0}{s} \quad \text{or} \quad \sqrt{\omega_0\omega_{\text{ex}}} > \frac{s}{a}\,. \tag{7.16}$$

The frequency ω_0 being a function of the magnetic field H, the above expression can be verified experimentally. Interestingly, f_{th} as a function of ω_0 (hence, of H), is a non-monotonic function having a root singularity at $\omega_0 = k_s s - 0$ (Fig. 8).

Evidently, the threshold is determined by the dispersion laws of quasiparticles involved in the process. As a rule, the singularity of a corresponding kinetic characteristic (e.g. phonon lifetime) also depends on dispersion laws (see Section 8).

When a quasiparticle has anisotropic dispersion laws, the calculation of the threshold value of phonon momentum becomes a difficult problem which can hardly have the exact solution if no numerical method is used. True, it is possible to

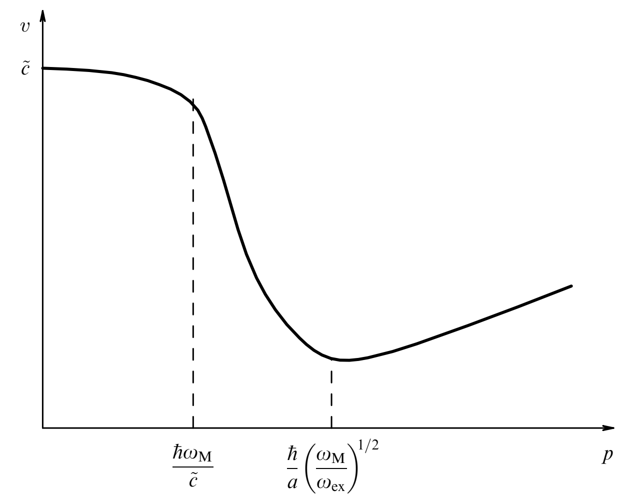

Figure 7. Schematic representation of the dependence of magnon velocity $v = |\partial E/\partial \mathbf{p}|$ on the momentum at $\theta \neq \pi/2, 0$; $\tilde{c} \sim c/\sqrt{\varepsilon}$.

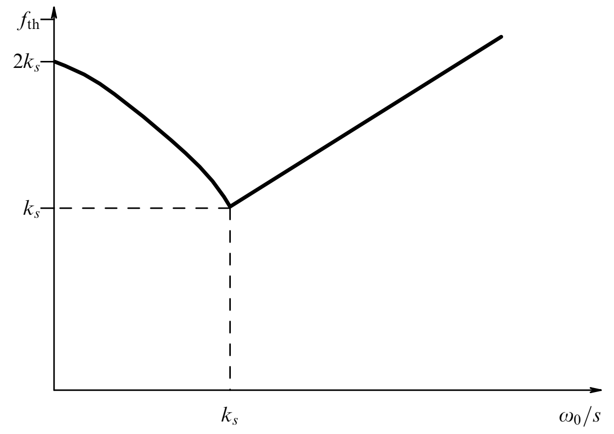

Figure 8. Graphic representation of the dependence of $f_{\rm th}$ on ω_0; $k_s = s/(\omega_{\rm ex} a^2)$.

formulate the algorithm for a search of threshold momentum **f** in the general case. The conservation laws lead to the following equality:

$$S(\mathbf{k},\mathbf{f}) \equiv \omega(\mathbf{k}) + \omega(\mathbf{f}-\mathbf{k}) - s(\mathbf{n})f = 0 \tag{7.17}$$

(the velocity of sound is anisotropic: $\mathbf{n} = \mathbf{f}/f$). On the one hand, the existence of the threshold means that the function $S = S(\mathbf{k},\mathbf{f})$ starting from $f = f_{\rm th}$ (either at $f > f_{\rm th}$ or $f < f_{\rm th}$) has definite sign at all **k**; on the other hand, there are such values of **k** at which $S(\mathbf{k},\mathbf{f}) = 0$. We are interested in the threshold value of f at a given direction of sound propagation. In order to find $f_{\rm th} = f_{\rm th}(\mathbf{n})$ in the general case, it is necessary to solve the following geometrical problem: to find a point in the **k**-space (denoted here as $\mathbf{k}_{\rm c}$) at which

$$S\big(\mathbf{k}_{\rm c}, f_{\rm th}(\mathbf{n})\big) = 0 \quad \text{and} \quad \left(\frac{\partial S}{\partial \mathbf{k}}\right)_{\mathbf{k}=\mathbf{k}_{\rm c}} = 0 \tag{7.18}$$

at the selected value of $f = f_{\rm th}(\mathbf{n})$.

If the problem (7.18) has the solution, the function $S(\mathbf{k}, f)$ near $\mathbf{k} = \mathbf{k}_{\rm c}$ and $f = f_{\rm th}$ can be expanded in powers of deviations from $\mathbf{k}_{\rm c}$ and $f_{\rm th}$; then, the equality (6.17) is rewritten as

$$\left(\frac{\partial^2 S}{\partial k^i\,\partial k^k}\right)_{\mathbf{k}_{\rm c}, f_{\rm th}} (k^i_{\rm c} - k^i)(k^k_{\rm c} - k^k) + \left(\frac{\partial S}{\partial f}\right)_{\mathbf{k}_{\rm c}, f_{\rm th}} (f - f_{\rm th}) = 0\,. \tag{7.19}$$

Thus, the existence of the threshold implies that the quadratic form entering (7.19) has definite sign, that is main values of the diagonal tensor $\left[\partial^2 S/\partial k^i\,\partial k^k\right]_{\mathbf{k}_{\rm c}, f_{\rm th}}$ must be of the same sign. This is an additional condition for the selection of solutions for Eqn (7.18). The problem can have several solutions even if this limitation is imposed. Specifically, there are two thresholds if the breakdown is possible in the finite interval of values of momentum f.

8. Lifetimes*

This section is focused on calculations for the case when specific features of magnon spectra near the bottom of the magnon zone are essential,† i.e. when the dependence of the spin wave frequency (magnon energy) on the direction of propagation (from angle θ, see Sections 3 and 7) is important.

The existence of a finite width ΔE of the magnon energy zone at zero magnon momentum ($p = 0$, Fig. 9) is crucial in a study of the linewidth of ferromagnetic resonance in samples of different shape. Excitation of spin waves (magnons) with different frequencies from the interval $\Delta E/\hbar$ is governed by the shape of the sample. The excited wave (uniform oscillation) turns into a magnon with a finite momentum as a result of elastic scattering from impurities or rough surfaces (see Fig. 9); the lifetime of such magnons depends on their interaction with one another and with phonons. The subsequent relaxation phase is of no special importance because the magnon lifetime at $p \neq 0$ is significantly shorter than that at $p = 0$, due to elastic scattering.

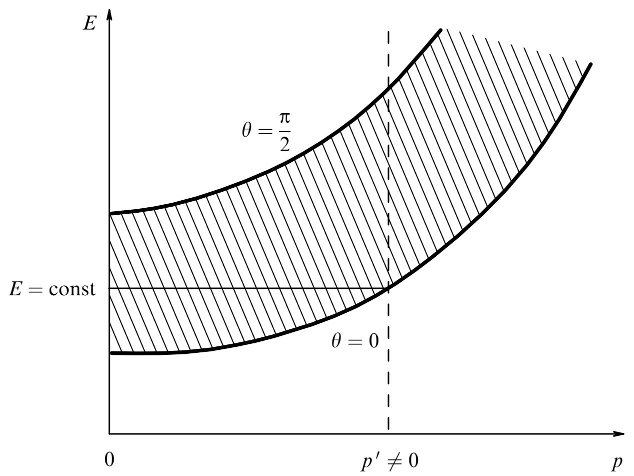

Figure 9. Magnon with momentum $p = 0$ and $\theta \neq 0$ may turn into magnon with $p = p' \neq 0$ as a result of elastic scattering. The band with $0 \leqslant \theta \leqslant \pi/2$ is shaded.

The natural lifetime of a magnon with $p = 0$ is longer than that of a magnon with $p \neq 0$ but differs from infinity. Specific features of intrinsic kinetic processes in magnons with $p = 0$ appear to have been first reported in Ref. [15].

The laws of conservation of energy and momentum forbid that a spin wave with the minimal energy participates in elastic scattering and triple processes normally playing an important role in lifetime calculations. Therefore, an analysis of other (more complicated) dissipative processes is indispensable. An analysis of Hamiltonian of magnon–magnon interactions has revealed two key mechanisms responsible for the finite lifetime of magnons:

(1) relativistic scattering in the first order of the theory of perturbations (fusion of a magnon with $\mathbf{p} = 0$ and $\theta = 0$ and another magnon giving rise to two magnons‡),

(2) many-particle process in which a magnon with $\mathbf{p} = 0$ fuses with a magnon having non-zero momentum to trigger

† We neglect here the effect of retardation ($c \to \infty$).

‡ It should be remembered that the terms quartic in operators of creation and annihilation which describe scattering of a magnon at another magnon arise from both its exchangeable (non-relativistic) and anisotropic (relativistic) parts. The amplitude (with the product of Bose-operators) in the term which is due to the exchange interaction is normally (when all $\mathbf{p} \neq 0$) much higher than the amplitude in the term arising from relativistic scattering. However, if the particle being scattered has zero momentum, the exchange scattering amplitude also vanishes.

another process with $\mathbf{p} \neq 0$. This process is essentially relativistic and is forbidden by the conservation laws, as stated in a previous paragraph. This limitation is removed by the uncertainty of both the momentum and the energy of the magnon with $\mathbf{p} \neq 0$ due to its interaction with other magnons. Therefore, this process is referred to as many-particle one. The relative smallness of energy uncertainty $\Delta\Sigma_{\mathbf{p}}$ of a magnon involved in the collision ($\Delta\Sigma_{\mathbf{p}} \ll \hbar\omega_0$) at low temperatures enables one to be confined to the second approximation of the theory of perturbations which facilitates lifetime computation.

That energy uncertainties of interacting quasiparticles need to be taken into consideration to lift the kinematic limitation was first proposed by Simons [16] who undertook the calculation of longwave phonon lifetimes which is a more complicated case than that of magnons because phonon energy tends to zero together with its momentum (the small parameter used in Ref. [15] to calculate the lifetime of magnons with $p = 0$ is absent). It is worthwhile to note that the above mechanisms are responsible for the exponential temperature dependence of the inverse lifetime

$$\frac{1}{\tau_{\mathrm{MM}}} \propto \exp\left(-\frac{\hbar\omega_0}{T}\right), \qquad T \ll \hbar\omega_0 \,. \tag{8.1}$$

Index MM indicates that the lifetime depends on the magnon-magnon interaction. Formula (8.1) reflects the exponentially small number of phonons at $T \ll \hbar\omega_0$ and implies that the processes being examined occur only when the magnon collides with another magnon from the thermal pool.

Exponential smallness of the probability of magnon–magnon interaction makes it necessary to look for dissipative processes the probabilities of which have no such smallness. Naturally, magnon–phonon interaction especially spontaneous emission of phonons (the process of Cherenkov's type) are the most interesting ones. Kinematics of such processes (see Section 7) implies [9] that a MSW (magnon with the minimal energy of $\hbar\omega_0$, i.e. with $k = 0$) is unable to emit a phonon. On the contrary, a magnon with an arbitrarily small $k_\perp$ emits a phonon, and its inverse lifetime does not vanish even at $T = 0$:

$$\frac{1}{\tau^0_{\mathrm{MPh}}} \simeq \frac{\gamma^2\hbar\omega^3_{\mathrm{M}}}{2^{10}\pi^3\rho s^3}\,\frac{k^4_\perp}{k^2}\,, \qquad k_\perp \ll \sqrt{\frac{2s}{\omega_{\mathrm{M}}}}\,k^{3/2}\,, \tag{8.2}$$

where ρ is the mass density, s is the acoustic velocity, and γ is the dimensionless magnetostatic constant.

It is worthy of note that for small $k_\perp$ such that

$$\frac{k^2_\perp}{k^2} \simeq \frac{2(\omega-\omega_0)}{\omega_{\mathrm{M}}}\,,$$

formula (7.2) can be rewritten in the form

$$\frac{1}{\tau^0_{\mathrm{MPh}}} \simeq \frac{\gamma^2\hbar\omega_{\mathrm{M}}}{2^8\pi^3\rho s^3}\,k^2(\omega-\omega_0)^2\,. \tag{8.2$'$}$$

Evidently, the MSW lifetime depends not only on its frequency but also on the wave vector, longwave oscillations having longer lifetimes than shortwave ones.

The effect of temperature on the lifetime of a phonon-emitting magnon depends on the relationship between T and $\hbar(\omega-\omega_0)$. At $T \ll \hbar(\omega-\omega_0)$, the mechanism of MSW relaxation is actually the Cherenkov emission of acoustic waves (the characteristic value of the wave vector of emitted phonons tends to zero at $T \to 0$), and it occurs only when the MSW velocity exceeds that of sound. Using formula (7.11) at $(\omega-\omega_0)/\omega_0 \ll 1$, it is easy to write the last condition in the following form:

$$v \simeq \frac{2(\omega-\omega_0)}{k} \geqslant s\,. \tag{8.3}$$

Then,

$$\frac{1}{\tau_{\mathrm{MPh}}} = \frac{1}{\tau^{(0)}_{\mathrm{MPh}}} + \frac{1}{\tau^{\mathrm{em}}_{\mathrm{MPh}}}\,, \qquad \frac{1}{\tau^{\mathrm{em}}_{\mathrm{MPh}}} \ll \frac{1}{\tau^0_{\mathrm{MPh}}}\,, \qquad T \ll \frac{\hbar\omega_{\mathrm{M}}}{2}\,\frac{k^2_\perp}{k^2}\,,$$

$$\frac{1}{\tau^{\mathrm{em}}_{\mathrm{MPh}}} = \frac{\pi^2\gamma^2\omega_{\mathrm{M}}}{2^6\cdot 15}\,\frac{\beta M}{m_i s v}\left(\frac{T}{\Theta_{\mathrm{D}}}\right)^3, \qquad v = \left|\frac{\partial\omega(k)}{\partial\mathbf{k}}\right| \geqslant s\,, \tag{8.4}$$

where m_i is the atomic mass in the lattice cell of a magnet ($m_i = \rho a^3$).

Moreover, both phonon absorption and collision of two MSWs resulting in phonon emission [$\tau_1^{-1}(\omega_0)$ and $\tau_2^{-1}(\omega_0)$ respectively] contribute to the inverse lifetime of a magnon at $T \neq 0$:

$$\frac{1}{\tau_1(\omega_0)} = \frac{\pi\gamma^2\omega_{\mathrm{M}}}{2^7\cdot 15}\,\frac{T^4}{m_i s\Theta^3_{\mathrm{D}}}\,,$$

$$\frac{1}{\tau_2(\omega_0)} = \frac{\gamma^2\omega_{\mathrm{M}}}{2^5\pi^3}\,\frac{(\hbar\omega_0)^3 T}{m_i s^2\Theta^2_{\mathrm{D}}}\exp\left(-\frac{\hbar\omega_0}{T}\right). \tag{8.5}$$

These formulas are useful to account for the nature of the ferromagnetic resonance linewidth when excited oscillations have the lowest possible frequencies ($\omega = \omega_0$).

Let us now estimate the phonon lifetime. A longwave phonon ($sf < 2\hbar\omega_0$) cannot produce two magnons. The predominant dissipative process involving magnons (MSWs) is absorption (creation) of a phonon by a MSW. The inverse lifetime of a phonon with the frequency ω conditioned by such a process shows exponential temperature dependence since this process needs a thermal magnon

$$\frac{1}{\omega\tau_f} = \frac{\gamma^2\hbar\omega_{\mathrm{M}}}{4\pi m_i s^2}\exp\left(-\frac{\hbar\omega_0}{T}\right)$$
$$\times\begin{cases} \dfrac{1}{2^5\pi^{7/2}}\left(\dfrac{\sqrt{T\hbar\omega_{\mathrm{M}}}}{\Theta_{\mathrm{D}}}\right)^3\sin^3\Psi_f\,, & \tan\Psi_f \gg \sqrt{\dfrac{T}{\hbar\omega_{\mathrm{M}}}}\,,\\[2ex] \dfrac{3}{16\pi^3}\left(\dfrac{T}{\Theta_{\mathrm{D}}}\right)^3\cos^3\Psi_f\,, & \tan\Psi_f \ll \sqrt{\dfrac{T}{\hbar\omega}} \end{cases} \tag{8.6}$$

to occur. Here, Ψ_f is the angle between vector $\mathbf{f}$ and magnetization $\mathbf{M}$, with $T \ll \hbar\omega_{\mathrm{M}}, f < 2\omega_0\hbar/s$.

Formula (8.6) gives the right order of magnitude almost in the entire angle range. However, the exact compact expression for the arbitrary angle Ψ_f remains to be found. At the same time, the upper line in the expression (8.6) holds at all Ψ_f values, since $T/\hbar\omega_{\mathrm{M}} \ll 1$.

Besides the above applicability conditions for the formulas obtained, it should be borne in mind that they were derived without regard for the non-uniform exchange in the magnon dispersion law. This required additional conditions

$$T \ll \hbar\omega_0, \hbar\left[\sqrt{\omega_0(\omega_0+\omega_{\mathrm{M}})}-\omega_0\right], \qquad I \ll \frac{\Theta_{\mathrm{D}}^2}{2\hbar\omega_0}, \frac{\Theta_D^2}{2\hbar\omega_{\mathrm{M}}} \tag{8.7}$$

to be fulfilled. It is worth noting that $I = \hbar\omega_{\mathrm{ex}}$ is the non-uniform exchange interaction constant.

Ordinary sound and even ultrasound satisfy the condition $sf < 2\hbar\omega_0$. Therefore, τ_f^{-1} is the contribution to the sound damping coefficient for which phonon–magnon interaction is responsible. It is very small but can be identified (by virtue of its exponential dependence on the magnetic field [$\exp(-\hbar\omega_0/T)$, $\omega_0 \sim H$] and specific anisotropy [see (8.6)] against the practically isotropic damping coefficient which is due to the interaction between an acoustic wave and thermal (Debay) phonons.

As the frequency grows, the limitation on the phonon breakdown into two magnons is lifted. This becomes possible as soon as phonon momentum exceeds the threshold value

$$f_{\mathrm{th}} = \frac{2\omega_0}{s - \sqrt{2\omega_{\mathrm{ex}}\omega_{\mathrm{M}}}\, a}, \tag{8.8}$$

The lifetime of a decaying phonon is impossible to calculate if the term associated with the non-uniform exchange interaction is omitted from the magnon dispersion law. Therefore, it is taken into account in the calculation of f_{th} [the last condition (8.7) ensures that the denominator in (8.8) is positive].

On the assumption of elastic isotropy of a magnet, the probability that a phonon moving along magnetization ($\mathbf{f} \parallel \mathbf{M}$) breaks down is zero. Therefore, we confine ourselves to the formula for the inverse lifetime $1/\tau_f$ determined by the breakdown process [only for a phonon travelling perpendicular to magnetization ($\mathbf{f} \perp \mathbf{M}$)]. The phonon momentum exceeds threshold but insignificantly. Then,

$$\frac{1}{\tau_f^{\mathrm{th}}} = \begin{cases} 0, & f < f_{\mathrm{th}}, \\ \dfrac{\gamma^2\hbar}{2^9\pi}\dfrac{\omega_{\mathrm{M}}\omega_0^2}{m_i s^2\omega_{\mathrm{ex}}}\sqrt{\dfrac{2\omega_{\mathrm{M}}\omega_0}{\omega_{\mathrm{D}}^2}}\sqrt{\dfrac{f-f_{\mathrm{th}}}{f_{\mathrm{th}}}}, & \dfrac{f-f_{\mathrm{th}}}{f_{\mathrm{th}}} \ll 1, \end{cases}$$

$$\mathbf{f} \perp \mathbf{M}, \qquad \omega_{\mathrm{D}} = \frac{s}{a}. \tag{8.9}$$

This formula demonstrates a major feature of the threshold process, i.e. the existence of root dependence on the difference $f - f_{\mathrm{th}}$ (or frequency difference $\omega - \omega_{\mathrm{th}}$) when the threshold is overcome. A threshold for the phonon decomposition into two magnons is neither casual nor rare phenomenon: kinematics of interaction between quasiparticles often restricts the values of momenta and energies involved in the process of interest. The presence of the threshold in the inverse lifetime dependence on momentum (energy) is apparent in the general case as a characteristic non-analiticity $[1/\tau \propto (f - f_{\mathrm{th}})^{1/2}]$. The root, non-analytical dependence of quasiparticles' lifetime means singularity of the imaginary part of their energy, $\mathrm{Im}\,E = \hbar/\tau$. This part is coupled to the real one $\mathrm{Re}\,E$ by dispersion relations of the Kramers–Kroning type. The singularity of $\mathrm{Im}\,E$ inevitably leads to the singularity of $\mathrm{Re}\,E$, i.e. to a singularity in the dispersion law of quasiparticle. This singularity can be conveniently examined using the dispersion relation rather than the expression for $\mathrm{Re}\,E$ because this expression is an integral difficult to calculate in terms of the main value [17].

9. Surface magnetic polaritons in magnetic dielectrics

The boundary between a vacuum and a halfspace occupied by a magnet creates several types of problems. We shall consider some of them.

It is natural to examine the incidence of an electromagnetic wave from vacuum onto the halfspace and investigate its reflection from the interface (the general case) and refraction in the transparent region. Problems of this type remain in the framework of electrodynamics of continuous media unless spatial dispersion of permittivities is taken into consideration. Conversely, taking into account spatial dispersion of magnetic permeability (i.e. the existence of spin waves or magnons), even in the solution of the wave reflection problem, means going out beyond the limits of standard electrodynamics of continuous media; specifically, additional boundary conditions need to be formulated.

Specific surface waves (surface polaritons) are known to be able to propagate along the flat vacuum–halfspace interface, with their amplitudes exponentially decreasing on either side. A distinctive property of surface polaritons in gyrotropic media ($\mu' \neq 0$) is the lack of reciprocity, i.e. $\omega(-\mathbf{k}_{\mathrm{t}}) \neq \omega(\mathbf{k}_{\mathrm{t}})$, where $\mathbf{k}_{\mathrm{t}}$ is the two-dimensional wave vector at the boundary plane and ω is the polariton frequency. All non-reciprocity cases considered in this review are of the same nature: the presence of gyrotropy means that the problem contains a pseudovector (magnetic field $\mathbf{H}$, magnetization $\mathbf{M}$) the sign of which changes following the substitution of $-t$ for t, while the boundary is characterized by the normal vector $\mathbf{n}$. Therefore, if the vector responsible for gyrotropy (e.g. vector $\mathbf{M}$) has a constituent in the boundary plane, then the problem contains the flat vector $\mathbf{n} \times \mathbf{M}_{\mathrm{t}}$ which eliminates the inversion centre in the boundary plane.

However, the lack of the inversion centre is insufficient for non-reciprocity to arise. Reciprocity (the inversion centre in $\mathbf{k}$-space) is a result of invariance of wave equations (regardless of dissipation) with respect to time† (upon substitution of $-t$ for t). However, in our case, the vector changes the sign if t is substituted by $-t$. Taken together with the presence of the vector $\mathbf{n} \times \mathbf{M}_{\mathrm{t}}$, this accounts for the lack of reciprocity in the surface polariton.

It has been shown earlier that the dispersion law for bulk polaritons is changed dramatically if spatial dispersion of magnetic permeability is taken into account. Surface polaritons are even more susceptible to spatial dispersion. A most striking example is the transformation of a surface wave to a leakage wave subject to non-dissipative annihilation; the surface wave excites dimensional spin waves which take away a part of its energy (see Section 10).

Skin-effect in metallic magnets is of paramount importance as regards their wave properties for which magnetic permeability is responsible. This has already been noted in the Introduction and will be illustrated below by several examples (see Sections 11 and 12).

Let us first consider a ferromagnet and analyse the dispersion law for a surface magnetic polariton in the simplest, but probably the most interesting, case when the magnetic field $\mathbf{H}$ and magnetization $\mathbf{M}$ are parallel to the sample surface while the wave propagates normally to $\mathbf{H}$ and

† When invariance is with respect to time inversion in the absence of the inversion centre, reciprocity is largely maintained by branch pairs of the spectrum for which $\omega_+(-\mathbf{k}) = \omega_-(\mathbf{k})$.

M. Using the natural boundary conditions (continuity of tangential constituents of electric and magnetic fields), in addition to Maxwell's equations in a medium and vacuum, it is not difficult to derive the dispersion equation [18]

$$\sqrt{k^2 - \frac{\omega^2}{c^2}\,\varepsilon\mu_{\mathrm{eff}}} - \frac{\mu'}{\mu}\,k + \mu_{\mathrm{eff}}\sqrt{k^2 - \frac{\omega^2}{c^2}} = 0\,, \qquad (9.1)$$

$$\mu_{\mathrm{eff}} = \frac{\mu^2 - \mu'^2}{\mu}\,.$$

See formula (1.20) for μ and μ'.

The presence of the term containing the first degree of the wave vector k in Eqn (9.1) suggests non-reciprocity of the wave. The dependence $\omega(-k) \neq \omega(k)$ is graphically represented in Fig. 10. The surface wave exists in finite frequency intervals at both $k < 0$ and $k > 0$. When $k > 0$, the curve $\omega = \omega(k)$ is in the interval $([\omega_0(\omega_0 + \omega_{\mathrm{M}})]^{1/2}, \omega_{\mathrm{DE}})$, where $\omega_{\mathrm{DE}} = \omega_0 + \omega_{\mathrm{M}}/2$ is the Damon–Eshbach (DE) wave frequency [19].

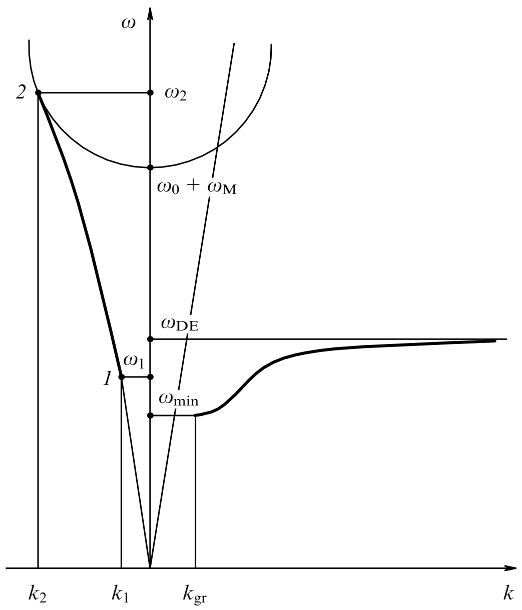

Figure 10. Schematic dependence of the dispersion law for surface polariton, $\omega_{\min} = [\omega_0(\omega_0 + \omega_{\mathrm{M}})]^{1/2}$.

Near the boundary frequencies,

$$\omega \simeq \sqrt{\omega_0(\omega_0 + \omega_{\mathrm{M}})} + \frac{\omega_{\mathrm{M}}^2\sqrt{\omega_0 + \omega_{\mathrm{M}}}}{2\varepsilon\omega_0\sqrt{\omega_0}}\,\frac{(k - k_{\mathrm{gr}})^2}{k_{\mathrm{gr}}^2}\,,$$

$$0 < k - k_{\mathrm{gr}} \leqslant k_{\mathrm{gr}}\,,$$

$$\omega \simeq \omega_{\mathrm{DE}} - \frac{\omega_{\mathrm{M}}\omega_{\mathrm{DE}}^2(1+\varepsilon)}{8c^2k^2}\,, \qquad k \gg \frac{\omega_{\mathrm{DE}}}{c}\,,$$

$$ck_{\mathrm{gr}} = (\omega_0 + \omega_{\mathrm{M}})\sqrt{\frac{\omega_0}{\omega_{\mathrm{M}}}}\,. \qquad (9.2)$$

At $k = k_{\mathrm{gr}}$, the frequency $\omega = [\omega_0(\omega_0 + \omega_{\mathrm{M}})]^{1/2}$, and the depth of the surface wave penetration into the ferromagnetic vanishes. At first sight, this should mean that the macroscopic approach is inapplicable. However, it is possible to use if spatial dispersion is taken into account as in the case of magnetostatic oscillations (see Section 10).

At $k < 0$ and boundary frequencies ω_1 and ω_2, logarithmic damping decrements γ_0 and γ vanish at points (k_1, ω_1) and (k_2, ω_2) respectively:

$$\omega_1^2 = \omega_0(\omega_0 + \omega_{\mathrm{M}}) + \frac{\omega_0\omega_{\mathrm{M}}}{\varepsilon - 1}\,,$$

$$k_1 = -\frac{\omega_1}{c}\,, \qquad k_2 = -\frac{\omega_2}{c}\,\sqrt{\varepsilon\mu_{\mathrm{eff}}}\,. \qquad (9.3)$$

The explicit expression for ω_2 can be obtained in limiting cases:

$$\omega_2 \simeq \begin{cases} \dfrac{\omega_{\mathrm{M}}}{2} + \sqrt{(\omega_0 + \omega_{\mathrm{M}})^2 + \dfrac{\omega_{\mathrm{M}}^2}{4}}\,, & \varepsilon \gg 1\,, \\ \sqrt{\dfrac{\omega_{\mathrm{M}}(\omega_0 + 2\omega_{\mathrm{M}})}{\varepsilon - 1}}\,, & \varepsilon - 1 \ll 1\,. \end{cases} \qquad (9.4)$$

The surface MSW (DE wave) obtainable from Eqn (9.1) by means of natural limiting transition $kc \to \infty$, $\omega \not\to \infty$ exists only at $k > 0$. Taking into account retardation results in the presence of $v_{\mathrm{gr}} \neq 0$ in the MSW

$$v_{\mathrm{gr}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = 2^{5/2}\,\frac{(\omega_{\mathrm{DE}} - \omega)^{3/2}c}{\sqrt{\omega_{\mathrm{M}}}\,\omega_{\mathrm{DE}}\sqrt{1+\varepsilon}} \ll c\,, \qquad \omega \leqslant \omega_{\mathrm{DE}}\,. \quad (9.5)$$

At $k < 0$, the group velocity is close to the velocity of light within the entire frequency range in which magnetic polaritons exist.

Taking into account retardation leads not only to the dependence of DE wave frequency on the wave vector (hence, to non-zero group velocity) but also to an additional (non-magnetic) suppression mechanism of such waves. If dielectric permittivity is complex ($\varepsilon = \varepsilon' + \mathrm{i}\varepsilon''$), the dispersion equation yields

$$\mathrm{Im}\,\omega = -\frac{\omega_{\mathrm{M}}\omega_{\mathrm{DE}}^2}{8(kc)^2}\,\varepsilon''\,, \qquad \mathrm{Re}\,\omega \simeq \omega_{\mathrm{DE}}\,. \qquad (9.6)$$

A characteristic feature of electrical losses is the dependence of $\mathrm{Im}\,\omega$ on the wave vector $\mathbf{k}$ ($\mathrm{Im}\,\omega \to 0$ at $kc \to \infty$). The magnetic losses proper are insensitive to the wavelength:

$$\mathrm{Im}\,\omega = -\frac{1}{2\tau_{\mathrm{M}}}\,\frac{\omega_{\mathrm{DE}}(4\omega_0 + \omega_{\mathrm{M}})}{\omega_{\mathrm{M}}(\omega_0 + \omega_{\mathrm{M}})}\,, \qquad (9.7)$$

where τ_{M} is the magnetic relaxation constant (with dimension of time) which arises from taking into account the relaxation terms in the Landau–Lifshitz equation and appears in the expressions for μ and μ'. The relaxation rate $1/\tau_{\mathrm{M}}$ is the sum of inverted transverse (τ_{tr}) and longitudinal (τ_{l}) relaxation times, i.e.

$$\frac{1}{\tau_{\mathrm{M}}} = \frac{1}{\tau_{\mathrm{tr}}} + \frac{1}{\tau_{\mathrm{l}}}\,.$$

The analysis has shown that there is no surface wave when the wave vector $\mathbf{k}$ is parallel to $\mathbf{H}(\mathbf{M})$, and the corresponding dispersion equation has no solution.

Equation (9.1) remains formally valid for an antiferromagnet if the proper values of the components of the magnetic permeability tensor are employed. Figure 11 shows the dependence of the frequency of a surface magnetic polariton on the wave vector in magnetic fields of different

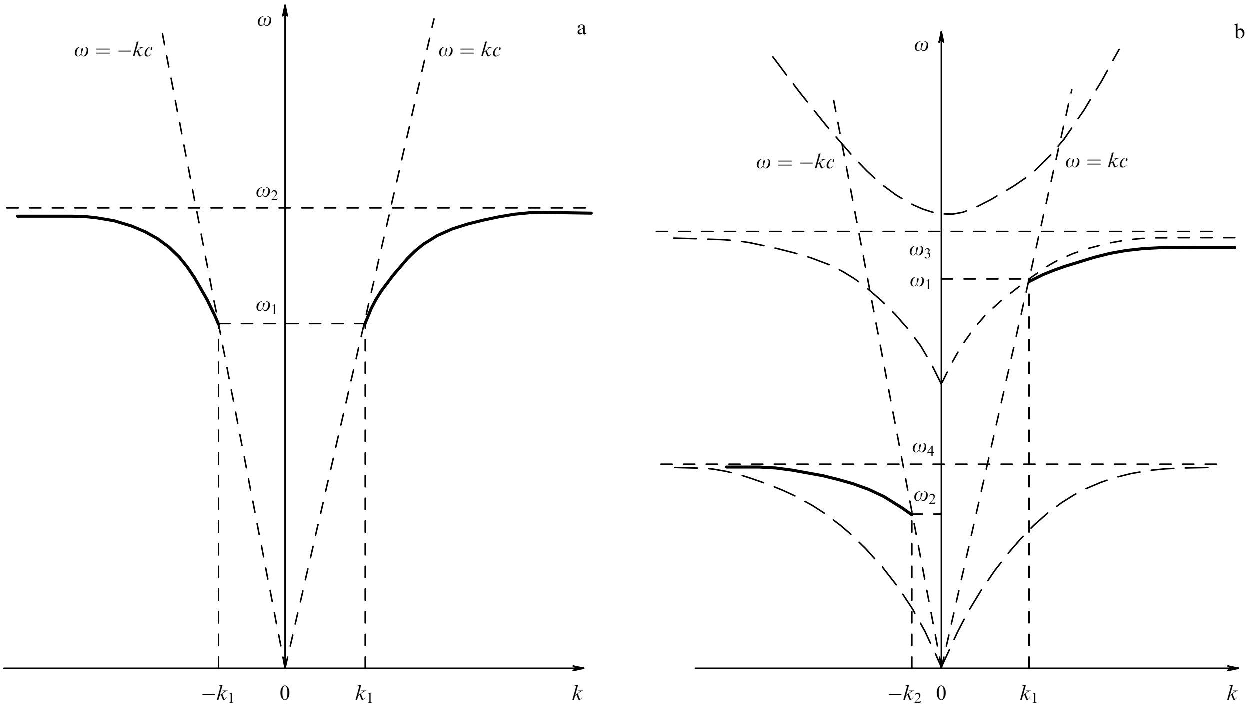

Figure 11. Schematic dependence of the dispersion law for surface polariton in an antiferromagnet: (a) $H = 0$, (b) $0 \leqslant H < H_{\mathrm{SF}}$.

strength. It is important that at $H = 0$, gyrotropy is absent and the wave exhibits reciprocity. Reciprocity exists at $H < H_{\mathrm{SF}}$ too, even if it is not so prominent as at $H > H_{\mathrm{E}}$, when antiferromagnets are hardly different from ferromagnets in terms of high-frequency properties.

It has been shown in Ref. [18] that the presence of nonreciprocal waves centered near the magnetic surface suggests equilibrium (!) between quasiparticle fluxes $\mathbf{j}$ spreading round the magnet and their energy $\mathbf{q}$, each corresponding to nonreciprocal waves because for them,

$$\mathbf{j} = \int \frac{\mathbf{v}}{\exp(\hbar\omega/T) - 1} \frac{\mathrm{d}^2 k}{(2\pi)^2} \neq 0 \,,$$
$$\mathbf{q} = \int \frac{\mathbf{v}\hbar\omega(k)}{\exp(\hbar\omega/T) - 1} \frac{\mathrm{d}^2 k}{(2\pi)^2} \neq 0 \,, \qquad (9.8)$$

where $\mathbf{v} = \partial\omega/\partial\mathbf{k}$.

This implies the existence of oppositely directed macroscopic quasiparticle and energy fluxes on either side of a thick plate (see Sections 13 and 14); also, these fluxes must spread round a cylinder magnetized parallel to the axis. It is not clear how these fluxes could be identified, nor their dependence on dissipative processes is known. Seemingly, the role of dissipation cannot be very important at $\mathrm{Re}\,\omega \gg \mathrm{Im}\,\omega$.

10. Surface magnetostatic waves

Let us first consider a MSW travelling at an angle of θ to the magnetic field and the magnetization vector along the boundary of a halfspace occupied by a ferromagnet. Proceeding from magnetostatic equations, it is not difficult to obtain the expression for the surface wave frequency ω and its logarithmic damping decrement (deep into the sample) γ by taking into account the explicit form of frequency dependences of μ and μ':

$$\omega(\theta) = \frac{\omega_0 + (\omega_0 + \omega_{\mathrm{M}}) \sin^2\theta}{2 \sin\theta} \,,$$
$$\gamma = |k| \, \frac{\omega_{\mathrm{M}} \sin^2\theta + \omega_0 \cos^2\theta}{\omega_{\mathrm{M}} \sin^2\theta - \omega_0 \cos^2\theta} \,. \qquad (10.1)$$

A change in the frequency sign (i.e. the absence of the wave) following the substitution of θ by $-\theta$ reflects the lack of reciprocity in the wave.

The well-known value for the frequency of the DE wave [19]

$$\omega\left(\frac{\pi}{2}\right) \equiv \omega_{\mathrm{DE}} = \omega_0 + \frac{\omega_{\mathrm{M}}}{2}$$

follows at $\theta = \pi/2$, when the wave vector is normal to $\mathbf{H}$. The DE wave travels only in the positive direction of the x-axis and when $\gamma = k > 0$. The lack of dispersion indicates that the wave carries no energy and its group velocity is zero. The critical angle derived from (10.1) is

$$\theta_{\mathrm{c}} = \arcsin \sqrt{\frac{\omega_0}{\omega_0 + \omega_{\mathrm{M}}}} \,. \qquad (10.2)$$

The penetration depth γ^{-1} vanishes at $\theta = \theta_{\mathrm{c}}$ and is negative (i.e. no surface wave) at $\theta < \theta_{\mathrm{c}}$. The frequency ω at $\theta = \theta_{\mathrm{c}}$ is minimal ($\omega = [\omega_0(\omega_0 + \omega_{\mathrm{M}})]^{1/2}$).

Conversion of $\gamma = \gamma(\theta)$ into infinity at $\theta = \theta_{\mathrm{c}}$, when penetration depth γ^{-1} vanishes, makes it necessary to consider processes removing the divergence of $\gamma = \gamma(\theta)$, of which spatial dispersion is the most important one [18].

We have noted before that spatial dispersion is taken into consideration by means of the formal substitution $\omega_0 \to \omega_0 + \omega_{\mathrm{ex}}(ak)^2$. This makes the equation for γ bicubic. Selecting solutions with $\mathrm{Re}\,\gamma > 0$, we have the following structure of magnetic potential at $y < 0$:

$$\varphi \propto \exp(\mathrm{i}\mathbf{k}\cdot\boldsymbol{\rho})\big[A\exp(\gamma_1 y) + B\exp(\gamma_2 y) + C\exp(\gamma_3 y)\big]\,,$$

$$\mathrm{Re}\,\gamma_i > 0\,, \qquad i = 1, 2, 3, \ldots\,, \tag{10.3}$$

where $\boldsymbol{\rho}$ and $\mathbf{k}$ are two-dimensional vectors: $\boldsymbol{\rho} \equiv (x, z)$, $\mathbf{k} \equiv (k_x, k_y)$.

An increase in the number of solutions requires additional boundary conditions (for defining coefficients A, B, and C). To derive them, it is necessary to consider the motion of the magnetic moment at the sample's boundary, with due regard for both the difference between surface and dimensional anisotropy energies and the absence of magnetic atoms at $y < 0$. However, the Landau – Lifshitz equation describing the motion of the magnetic moment in the longwave limit is a second-order equation which enables us to be confined to the phenomenological boundary condition, introducing one constant d with the dimension of length

$$\mathbf{m}_{y=0} + d\left(\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}y}\right)_{y=0} = 0\,. \tag{10.4}$$

Two limiting cases $d = 0$ and $d = \infty$ have clear physical sense in this consideration: at $d \to \infty$, surface anisotropy is unessential (free magnetic moment at the boundary) and at $d \to 0$, surface anisotropy fixes the direction of the magnetic moment at the boundary. Surface anisotropy may result in a more complicated boundary condition (10.4), that is in the substitution of the scalar d by the second-rank tensor $d_{\beta,\beta'}$, where $\beta, \beta' = x, z$. It is appropriate to note that neither the sign of d nor the sign of the components $d_{\beta,\beta'}$ are defined. The sign may be both positive and negative; moreover, the main directions of the tensor $d_{\beta,\beta'}$ do not necessarily coincide with the main crystallographic directions of the sample.

Let us consider the simplest case of $\theta = \pi/2$ which allows logarithmic damping decrement to be accurately calculated:

$$\gamma_1 = k\,, \qquad \gamma_{2,3}^2 = k^2 + \frac{\omega_{\mathrm{DE}} \pm (\omega_{\mathrm{M}}^2/4 + \omega^2)^{1/2}}{\omega_{\mathrm{M}}\alpha}\,. \tag{10.5}$$

and

$$\alpha = \frac{I}{\beta M}\,a^2\,. \tag{10.6}$$

The neglect of spatial dispersion means that $\alpha \to 0$. It follows from (10.5) that $\gamma_{2,3} \to \infty$ at $\alpha \to 0$. The marked difference between $|\gamma_2|$, $|\gamma_3|$ and $|\gamma_1|$ facilitates the analysis of the dispersion equation and the derivation of a compact expression for the DE wave dispersion law which takes into account spatial dispersion.

Using the continuity of the proper components of vectors $\mathbf{h}$ and $\mathbf{b}$ in conjunction with the additional boundary condition (10.4), we obtain the equation which defines the wave dispersion law in the form of zero determinant:

$$0 = \begin{vmatrix} 2\,\dfrac{\omega_{\mathrm{DE}} - \omega}{\omega_{\mathrm{M}}(kd-1)} & k\Big(\dfrac{\omega}{\omega_+} + 1\Big) & k\Big(\dfrac{\omega}{\omega_-} + 1\Big) \\ 1 & \Big(\dfrac{k\omega}{\omega_+} - \gamma_2\Big)(d\gamma_2 - 1) & \Big(\dfrac{k\omega}{\omega_-} - \gamma_3\Big)(d\gamma_3 - 1) \\ 1 & \Big(\dfrac{\gamma_2\omega}{\omega_+} - k\Big)(d\gamma_2 - 1) & \Big(\dfrac{\gamma_3\omega}{\omega_-} - k\Big)(d\gamma_3 - 1) \end{vmatrix}. \tag{10.7}$$

Here, $\omega_\pm = \omega_{\mathrm{M}}/2 \pm (\omega_{\mathrm{M}}^2/4 + \omega^2)^{1/2}$. It readily follows from (10.7) that

$$\omega = \omega_{\mathrm{DE}} + \frac{\omega_{\mathrm{M}}^2 k(1-dk)}{4(d\gamma_2 - 1)(d\gamma_3 - 1)} \times \frac{(1-d\gamma_2)(k-\gamma_2) - (1-d\gamma_3)(k-\gamma_3)}{k\omega(\gamma_2 - \gamma_3) + (\omega^2 + \omega_{\mathrm{M}}^2/4)^{1/2}(\gamma_2\gamma_3 - k^2)}\,. \tag{10.8}$$

It is clear from (10.5) and (10.6) that the inequality

$$ak \ll \sqrt{\frac{\beta M}{I}} \ll 1 \tag{10.9}$$

must be satisfied in order that the surface magnetic wave retained the sense of macroscopic longwave surface excitation. If $\gamma_2, \gamma_3 \gg k$ are real values and the condition (10.9) is fulfilled, the exchange interaction modifies the field structure only in the immediate vicinity of the surface. When $\mathrm{Im}\,\gamma_2, \mathrm{Im}\,\gamma_3 \neq 0$, the field displays oscillatory dependence on the coordinate with the wavelength along y significantly smaller that the length and penetration depth γ_1^{-1} of the surface wave.

It is necessary to distinguish between the boundaries of a ferromagnet with

$$d \gg \sqrt{\frac{I}{\beta M}}\,a \quad \text{and} \quad d \ll \sqrt{\frac{I}{\beta M}}\,a\,.$$

The relation between the wavelength $1/k$ and the width of plate d influences both suppression and dispersion of the wave.

To begin with, let us consider the case of

$$d \gg \sqrt{\frac{I}{\beta M}}\,a\,,$$

which admits transition to the 'free' boundary at which

$$\left.\frac{\partial\mathbf{m}}{\partial y}\right|_{y=0} = 0\,.$$

It follows from (10.8) that

$$\mathrm{Re}\,(\omega - \omega_{\mathrm{DE}}) = \begin{cases} \dfrac{I}{\beta M}\,\dfrac{a}{d}\,\omega_{\mathrm{M}}ak\,, & kd \ll 1\,, \\ 2\,\dfrac{I}{\beta M}\,\omega_{\mathrm{M}}(ak)^2\,, & kd \gg 1\,, \end{cases}$$

$$-\mathrm{Im}\,\omega = \begin{cases} \left(\dfrac{I}{\beta M}\right)^{3/2}\left(\dfrac{a}{d}\right)^2\omega_{\mathrm{M}}S_1 ak\,, & kd \ll 1\,, \\ \left(\dfrac{I}{\beta M}\right)^{3/2}\omega_{\mathrm{M}}S_2(ak)^3\,, & kd \gg 1\,. \end{cases} \tag{10.10}$$

Here, S_1 and S_2 are rather cumbersome functions of ω_0, ω_{M}, and, in limiting cases,

$$S_1 \simeq S_2 \simeq 2\sqrt{\frac{2\omega_0}{\omega_{\mathrm{M}}}}\,, \qquad \omega_{\mathrm{M}} \ll \omega_0\,,$$

$$S_1 \simeq 1.9\,, \qquad S_2 \simeq 0.5\,, \qquad \omega_{\mathrm{M}} \gg \omega_0\,. \tag{10.11}$$

If the magnetic moment is strongly fixed at the boundary ($d \ll (I/\beta M)^{1/2} a$), and the condition (10.9) is satisfied, then automatically $kd \ll 1$ and

$$\mathrm{Re}\,(\omega - \omega_{\mathrm{DE}}) = \sqrt{\frac{I}{\beta M}}\,\omega_{\mathrm{M}} S_{-} ak\,,$$

$$-\,\mathrm{Im}\,\omega = \sqrt{\frac{I}{\beta M}}\,\omega_{\mathrm{M}} S_{+} ak\,, \qquad (10.12)$$

where

$$S_{\pm} = \frac{1}{4}\left[\frac{(\omega_{\mathrm{DE}}/\omega_{\mathrm{M}})^2 + 1/4 \mp (\omega_{\mathrm{DE}}/\omega_{\mathrm{M}})}{(\omega_{\mathrm{DE}}/\omega_{\mathrm{M}})^2 + 1/4}\right]^{1/2}. \qquad (10.13)$$

By generalizing boundary conditions (transition from the scalar d to the tensor $d_{\beta,\beta'}$), it is possible to consider the mixed boundary conditions

$$\left.\frac{\partial m_y}{\partial y}\right|_{y=0} = 0\,, \qquad m_z\Big|_{y=0} = 0\,, \qquad (10.14)$$

or

$$\left.\frac{\partial m_z}{\partial y}\right|_{y=0} = 0\,, \qquad m_y\Big|_{y=0} = 0\,. \qquad (10.14')$$

The analysis has shown that at least one component of the magnetic moment must be fixed at the boundary for $\mathrm{Re}\,(\omega - \omega_{\mathrm{DE}})$ and $\mathrm{Im}\,\omega$ to be quantities of the same order and to show the linear dependence on the wave vector.

The formulas for $\theta \neq \pi/2$ are very complicated, yet their qualitative analysis is possible. At

$$\theta - \theta_{\mathrm{c}} \gg \frac{I}{\beta M}(ak)^2\,,$$

two of the three γ values are still large as compared with one (as at $\theta = \pi/2$). This suggests that the character of the wave remains unaltered while the relation between $\mathrm{Re}\,(\omega - \omega_{\mathrm{DE}}(\theta))$ and $-\mathrm{Im}\,\omega$ is determined by the parameter d. When $d \to \infty$, dispersion prevails over annihilation, similar to the case of $\theta = \pi/2$, whereas at $d \to 0$, they are of the same order. The situation changes dramatically closer to the critical angle $\theta \sim \theta_{\mathrm{c}}$. It has been shown before that at $\theta \to \theta_{\mathrm{c}}$, logarithmic damping decrement of MSW (γ) calculated regardless of spatial dispersion tends to infinity. Hence, the most important property of the surface MSW, i.e. its greater damping length as compared with spin wave characteristics, is lost. With allowance made for the non-uniform exchange interaction, none of the three γ_i values tends to infinity at $\theta \to \theta_{\mathrm{c}}$. One of these values is imaginary, that is the wave is in the strict sense not a surface one but undergoes weak attenuation.

At $d = \infty$,

$$\mathrm{Re}\left[\omega - \sqrt{\omega_0(\omega_0 + \omega_{\mathrm{M}})}\right] \sim -\frac{I}{\beta M}\,\omega_{\mathrm{M}}(ak)^2\,,$$

$$-\,\mathrm{Im}\,\omega \sim \left(\frac{I}{\beta M}\right)^{5/4} \omega_{\mathrm{M}}(ak)^{5/2}\,. \qquad (10.15)$$

At $d = 0$,

$$\mathrm{Re}\left[\omega - \sqrt{\omega_0(\omega_0 + \omega_{\mathrm{M}})}\right] \sim -\sqrt{\frac{I}{\beta M}}\,\omega_{\mathrm{M}} ak\,,$$

$$-\,\mathrm{Im}\,\omega \sim \left(\sqrt{\frac{I}{\beta M}}\,ak\right)^{3/2} \omega_{\mathrm{M}}\,. \qquad (10.16)$$

In both cases, $|\mathrm{Im}\,\omega| \ll \left|\mathrm{Re}\,[\omega - \sqrt{\omega_0(\omega_0 + \omega_{\mathrm{M}})}]\right|$, and the wave shows anomalous dispersion. The comparison between the dispersion laws for an MSW at $\theta \neq \theta_{\mathrm{c}}$ and $\theta = \theta_{\mathrm{c}}$ indicates that the spectrum is subject to pronounced restructuring in the narrow angle range

$$|\theta - \theta_{\mathrm{c}}| \leqslant \frac{I}{\beta M}(ak)^2 \ll 1$$

[see (10.9)]. This does not only result in a change of the k-dependence of $\mathrm{Re}\,\omega$ and $\mathrm{Im}\,\omega$ on k but also causes the dispersion to switch from normal (far from θ_{c}) to abnormal patterns.

In case of an antiferromagnet (see Ref. [10]), in terms of Section 5 the equation defining the frequency of a surface MSW propagating perpendicular to the EA has the form

$$1 + \mathrm{sgn}\,(k)\,\mu'(\omega) + \sqrt{\mu_1(\omega)\mu_2(\omega)} = 0\,, \qquad (10.17)$$

and the logarithmic damping decrement is

$$\gamma = |k|\sqrt{\frac{\mu_1(\omega)}{\mu_2(\omega)}} > 0\,. \qquad (10.18)$$

A surface wave may exist in the frequency range where $\mu_1(\omega)/\mu_2(\omega) > 0$. There is no such limitation in the case of isotropy in the basis plane ($\mu_1 = \mu_2 = \mu$, see Table 2) because $\gamma = |k|$, and Eqn (10.17) is simplified:

$$1 + \mathrm{sgn}\,(k)\,\mu'(\omega) + \mu(\omega) = 0\,. \qquad (10.19)$$

At $\mu' \neq 0$ (gyrotropy), the wave is lacking in reciprocity. Non-reciprocity appears at $H \neq 0$ (as is the case with polariton). There is no gyrotropy at $H = 0$ ($\mu' \equiv 0$), and the wave frequency (ω_{s}) is

$$\omega_{\mathrm{s}} = \sqrt{\frac{1 + \mu_0}{2}}\,\Omega_{\mathrm{SF}}\,, \qquad \mu_0 = 1 + \frac{4\pi}{\delta} \qquad (10.20)$$

regardless of the direction (sign of k). Non-reciprocity also arises at $0 \leqslant H \leqslant H_{\mathrm{SF}}$:

$$\omega_{\mathrm{s}} = \sqrt{\frac{1 + \mu_0}{2}}\,\Omega_{\mathrm{SF}} \pm \Omega\,\mathrm{sgn}\,(k)\,. \qquad (10.21)$$

The problem is further complicated by anisotropy of magnetic susceptibility: at $H_{\mathrm{SF}} < H \leqslant H_{\mathrm{E}}$, Eqn (10.17) is cubic with respect to ω. We shall not analyse this equation, although it is worthwhile to note that anisotropy may be neglected at $H_{\mathrm{SF}} \leqslant H < H_{\mathrm{E}}$, and

$$\mu_1 \simeq \mu_2 \simeq \frac{\Omega^2\mu_0 - \omega^2}{\Omega^2 - \omega^2}\,, \qquad \mu' = (\mu_0 - 1)\frac{\Omega\omega}{\omega^2 - \Omega^2}\,. \qquad (10.22)$$

Hence, in accordance with (10.19), $\omega_{\mathrm{s}} \simeq \Omega$ and appears to exist at both $k > 0$ and $k < 0$. A more accurate analysis shows that the exact Eqn (10.17) has no solution at $k < 0$. Moreover, it is possible to demonstrate that the frequency ω_s is continuous at the point $H = H_E$.

Finally, the surface MSW at $H \geqslant H_E$ also exists only if $k > 0$ and

$$\omega_s = \frac{1}{2}\,\Omega_M + \Omega\,. \tag{10.23}$$

This is an ordinary DE wave in a ferromagnet with the magnetic moment of the unit volume $2M$. Figure 12 is the graphic representation of the dependence of the surface wave frequency ω_s on the magnetic field H.

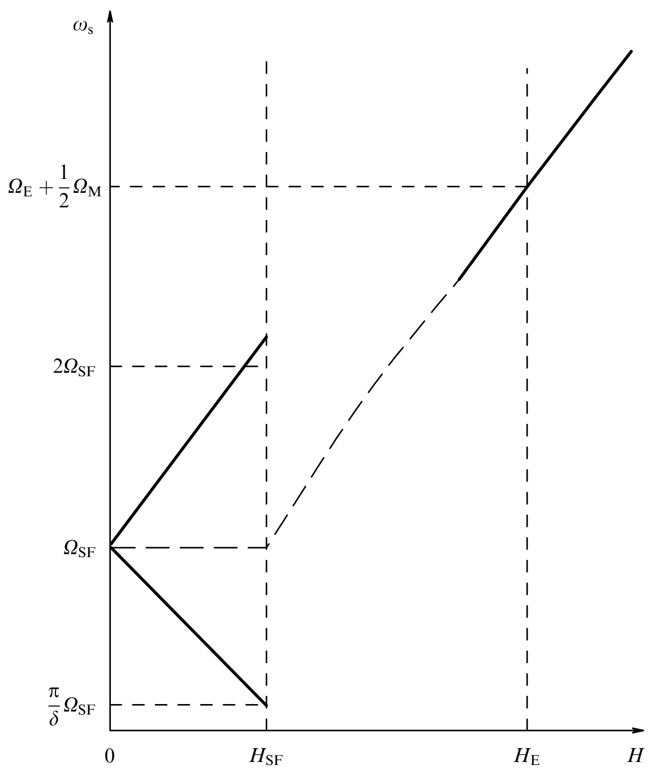

Figure 12. Dependence of the surface MSW frequency on the magnetic field.

11. Ferromagnetic metal. Electronic mechanism of Damon – Eshbach wave attenuation

The presence of conduction electrons in magnetic conductors gives rise to many problems which hamper the progress in the understanding of magnetic properties of metals. In the first place, it is necessary to elucidate the role of conduction electrons in magnetization. Conduction electrons constitute a major source of dissipation for magnons: the interaction of magnons and conduction electrons in conjunction with their interaction with phonons and lattice irregularities determines the lifetime of magnons. Electron scattering at magnons is responsible for the contribution to the mean probability of conduction electron scattering, which in turn accounts for the finite length of their free paths, i.e. constitutes a mechanism of resistance in electron conductors.

Any model approach to the problem solution in macroscopic electrodynamics requires, in some way or other, that material equations be formulated to relate the magnetic induction $\mathbf{B}$ (with the magnetic field $\mathbf{H}$ and the conduction electron current density $\mathbf{j}$) to the electric field strength $\mathbf{E}$; these equations must include results of microscopic model studies [see Eqns (1.1) – (1.4)]. An old review [4] examines various plasma effects. These effects are stipulated by the presence of normal and/or abnormal skin-effect responsible for structural changes of the electromagnetic field. At ferromagnetic resonance, the structure of the field should be taken into account in the calculation of sample parameters, e.g. surface impedance.

This section is focused on the electronic mechanism of DE wave annihilation in metals [20]. We proceed from the assumption that the material equations are available and there is no need to calculate the quantities involved in the Landau – Lifshitz equation. The same refers to the characteristics of conduction electrons.

If conduction electrons are supposed 'to be unable to feel' the presence of magnetization, the penetration depth to be large compared with the path length, and the frequency to satisfy $\omega_{DE} \ll 1/\tau_e$, where τ_e is electron relaxation time, it is possible to use formula (9.6) bearing in mind that the effective dielectric permittivity of a metal in such cases equals $4\pi i\sigma/\omega$ and is a purely imaginary quantity:

$$\operatorname{Im}\omega = -\frac{\pi}{2}\,\frac{\omega_M\omega_{DE}}{(ck)^2}\,\sigma\,, \qquad \operatorname{Re}\omega \simeq \omega_{DE} - \frac{\omega_M\omega_{DE}^2}{8(ck)^2}\,. \tag{11.1}$$

A weakly damped surface wave ($\operatorname{Re}\omega \gg \operatorname{Im}\omega$) exists if rather a relatively severe condition

$$k\delta \gg \frac{1}{2}\left(\frac{\omega_M}{\omega_{DE}}\right)^{1/2} \tag{11.2}$$

is satisfied, where $\delta = c(2\pi\sigma\omega_{DE})^{-1/2}$ is the skin-layer depth at the DE wave frequency.

At low temperatures, electron relaxation rate τ_e^{-1} in highly-purified metals may turn out to be below ω_{DE}. When $\omega\tau_e \gg 1$,

$$\varepsilon' \simeq -\frac{\omega_L^2}{\omega^2}\,, \qquad \varepsilon'' \simeq \frac{\omega_L^2}{\omega^3\tau_e}\,, \tag{11.3}$$

where ω_L^2 is the squared plasma frequency of conduction electrons

$$\omega_L^2 = \frac{4\pi n_e e^2}{m^*}$$

(the notations are conventional: n_e is the density of conduction electrons, m^* is their effective mass). It follows from the results of Section 9 that

$$\operatorname{Re}\omega = \omega_{DE}\left(1 + \frac{\omega_M\omega_L^2}{8c^2k^2\omega_{DE}}\right),$$
$$\operatorname{Im}\omega'' = -\frac{\omega_M\omega_L^2}{8c^2k^2}\,\frac{1}{\omega_{DE}\tau_e}\,, \qquad \left|\operatorname{Im}\omega''\right| \ll \operatorname{Re}(\omega - \omega_{DE})\,. \tag{11.4}$$

It can be deduced from the first formula that

$$v_{gr} = -\frac{c\omega_M\omega_L^2}{4(ck)^3} = -4\sqrt{2}c\,\frac{(\omega - \omega_{DE})^{3/2}}{\omega_M^{1/2}\omega_L}\,, \qquad \omega > \omega_{DE}\,. \tag{11.5}$$

Formulas (11.3)–(11.5) are interesting in that they describe magnetic quasistatic waves in a medium with negative dielectric permittivity: a DE wave shows anomalous dispersion and transfers energy against the direction of the wave vector (it should be remembered that such a wave is lacking in reciprocity). Formulas (11.3)–(11.5) have been obtained without regard for the Lorentz force acting on conduction electrons. The reason for this is that a DE wave in the geometry being considered excites only the electric field constituent parallel to the magnetic field **H** (and magnetization **M**). Therefore, σ must be regarded as $\sigma_{\parallel}$, i.e. the longitudinal constituent of the conduction tensor known to be weakly dependent on the magnetic field [21] (in contrast, the constituents of the tensor $\hat{\sigma}$ transverse relative to **H** undergo marked variation in the magnetic field).

Anomalous skin-effect readily arises in metals at low temperatures when the macroscopic material equation relating the current density $\mathbf{j}_e$ to the electric field strength **E** becomes invalid because spatial dispersion of conductivity has to be taken into consideration [see (1.2)].

Whenever the macroscopic material equation is deemed unnecessary, a kinetic theory needs to be constructed, which is the main objective of the present Section.

Proceeding to the development of the kinetic theory for electron damping of MSW in a ferromagnetic metal, we shall in the first place neglect the effect of magnetic field on electrons (with due regard for what has previously been said about $\sigma_{\parallel}$). Then, the complete system of equations for the problem of interest is

$$\frac{\mathrm{d}^2 e_z}{\mathrm{d}y^2} - k^2 e_z = \frac{4\pi\mathrm{i}\omega}{c^2}\,\frac{\mu'^2-\mu^2}{\mu}\,j_z\,, \qquad y<0\,, \tag{11.6}$$

$$\frac{\mathrm{d}^2 e_z^{\mathrm{v}}}{\mathrm{d}y^2} - \left(k^2 - \frac{\omega^2}{c^2}\right)e_z^{\mathrm{v}} = 0\,, \qquad y>0\,. \tag{11.7}$$

$$v_y\frac{\mathrm{d}f_1}{\mathrm{d}y} + \left(\mathrm{i}kv_x + \frac{1}{\tau_\mathrm{e}}\right)f_1 = -e\,\frac{\partial f_\mathrm{F}}{\partial\varepsilon}\,v_z e_z(y)\,, \qquad y<0\,. \tag{11.8}$$

Here, e_z (e_z^{v}) is the electric field strength in a metal (vacuum), f_1 is the linear (with respect to the electric field) addition to the Fermi distribution function f_F,

$$j_z = \frac{2e}{(2\pi\hbar)^3}\int v_z f_1\,\mathrm{d}^3p\,, \tag{11.9}$$

$\mathbf{v} = \mathbf{p}/m^*$ is the electron velocity; integration is over the entire **p**-space, and the electron gas is degenerate, so that

$$-\frac{\partial f_\mathrm{F}}{\partial\varepsilon} = \delta(\varepsilon-\varepsilon_\mathrm{F})\,.$$

We have written the kinetic equation in the τ approximation because the relaxation time is absent in the solution in the most interesting case of $kl \gg 1$ (l is the length of the electron free path) while the DE wave damping coefficient at $kl \ll 1$ is written in macroscopic terms [see (11.1)].

In the kinetic equation (11.8), the term

$$\frac{\partial f_1}{\partial t} = -\mathrm{i}\omega f_1\,,$$

is omitted since we believe that $\omega\tau_\mathrm{e} \ll 1$ [the case of $\omega\tau_\mathrm{e} \gg 1$ has been described earlier, see (11.3)–(11.5)]. Finally, it is accepted that all the functions of interest are proportional to $\exp(\mathrm{i}kx)$, where k is the wave vector of a DE wave. Conventional electrodynamic boundary conditions should be supplemented with the boundary condition for the electron distribution function f_1. We shall confine ourselves to the case of mirror reflection of conduction electrons from the surface:

$$f_1\Big|_{y=0,\,v_y<0} = f_1\Big|_{y=0,\,v_y>0}\,, \qquad f_1\Big|_{y\to-\infty,\,v_y>0} = 0\,. \tag{11.10}$$

The second equality ensures electron equilibration in the depth of the metal. According to (11.9) as well as (11.8) and (11.10),

$$j_x(y) = \int_{-\infty}^{0} K(y,y')e_z(y')\,\mathrm{d}y'\,,$$

$$K(y,y') = \frac{2e^2}{(2\pi\hbar)^3}\int_{v_y>0}\left(-\frac{\partial f_\mathrm{F}}{\partial\varepsilon}\right)\frac{v_z^2}{v_y} \times\left\{\exp\left[-\frac{|y-y'|}{v_y}\left(\frac{1}{\tau_\mathrm{e}}+\mathrm{i}kv_x\right)\right]+ +\exp\left[-\frac{|y+y'|}{v_y}\left(\frac{1}{\tau_\mathrm{e}}+\mathrm{i}kv_x\right)\right]\right\}\mathrm{d}^3p\,. \tag{11.11}$$

Let us formally extend $e_x(y)$ to the region $y>0$ in order to be able to use the Fourier transform along the y-axis (this leads to the difference kernel):

$$j_z(y) = \int_{-\infty}^{\infty} K(y-y')e_z(y')\,\mathrm{d}y'\,,$$

$$K(y-y') = \frac{2e^2}{(2\pi\hbar)^3}\int_{v_y>0}\left(-\frac{\partial f_\mathrm{F}}{\partial\varepsilon}\right)\frac{v_z^2}{v_y} \times\exp\left[-\frac{(y-y')}{v_y}\left(\frac{1}{\tau_\mathrm{e}}+\mathrm{i}kv_x\right)\right]\mathrm{d}^3p\,. \tag{11.12}$$

This allows Eqn (11.6) to be considered formally true at $y>0$. Let us now apply the Fourier transform to this equation bearing in mind that the function $e_z(y)$ has an inflection:

$$\left[\frac{4\pi\mathrm{i}\omega}{c^2}\,\frac{\mu'^2-\mu^2}{\mu}\,K_k(q)+k^2+q^2\right]e(q) = -2\left(\frac{\mathrm{d}e_z}{\mathrm{d}y}\right)_{y=0}\,. \tag{11.13}$$

Here,

$$K_k(q) = \frac{2e^2\tau_\mathrm{e}}{(2\pi\hbar)^3}\oint\frac{v_z^2}{v}\,\mathrm{d}s\,\frac{1+\mathrm{i}k\tau_\mathrm{e}v_x}{(1+\mathrm{i}k\tau_\mathrm{e}v_x)^2+\tau_\mathrm{e}^2v_x^2q^2} \tag{11.14}$$

is the kernel of the conduction operator in the (k,q) representation;

$$K_0(0) = \sigma = \frac{Ne^2\tau_\mathrm{e}}{m}$$

is the static metal conductivity.

It follows from Maxwell's equations and boundary conditions that

$$\frac{\mathrm{d}e_z}{\mathrm{d}y}\bigg|_{y=0} = -k_x\frac{\mu'}{\mu}\,e_z(0) - \frac{\mathrm{i}\omega}{c}\,\frac{\mu'^2-\mu^2}{\mu}\,h_x(0)\,,$$

$$h_x(0) = \mathrm{i}\,\frac{c}{\omega}\sqrt{k^2-\frac{\omega^2}{c^2}}\,e_z(0)\,. \tag{11.15}$$

On the other hand, it is possible to determine $e(q)$ from Eqn (11.13) and $e_z(y)$ by means of the inverse Fourier transform which allows $(\mathrm{d}e_z/\mathrm{d}y)_{y=0}$ to be excluded from the system (11.15) and leads to the system of two linear equations with respect to unknown $h_x(0)$ and $e_z(0)$. By equating the determinant of this system to zero, we have the equation relating the frequency ω to the wave vector k, i.e. the dispersion law for the DE wave:

$$\mu = \left[\mu' k + (\mu'^2 - \mu^2)\sqrt{k^2 - \frac{\omega^2}{c^2}}\right] \times \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}q \left[q^2 + k^2 + \frac{4\pi i\omega}{c^2}\frac{\mu'^2 - \mu^2}{\mu} K_k(q)\right]^{-1}, \quad (11.16)$$

where the functions $\mu(\omega)$ and $\mu'(\omega)$ are given by the expressions (1.20).

The integral over q in the right-hand side of Eqn (11.16) indicates that spatial dispersion of conductivity has been taken into account. Spatial dispersion of magnetic permeability is disregarded, which imposes limitation on the value of the wave vector k:

$$k \ll \frac{1}{a}\sqrt{\frac{\beta M}{I}}, \ \omega \sim \omega_{\mathrm{DE}}, \omega_0, \omega_{\mathrm{M}} \quad (11.17)$$

(in all estimates, ω_0 and ω_{M} are quantities of the same order).

When k tends formally to infinity (at $\omega \not\to \infty$), it naturally follows from (11.16) that $\omega = \omega_{\mathrm{DE}}$. At $kl \ll 1$, when the local relationship between $\mathbf{e}$ and $\mathbf{j}$ is valid, there is an opportunity to introduce effective dielectric permittivity, hence the formulas (11.1).

The opposite case analogous to the anomalous skin-effect ($kl \gg 1$) is of special interest. In this case,

$$k \gg \frac{\omega}{c}, \quad (k\delta_0)^2 k \gg \frac{\omega}{v_{\mathrm{F}}}, \quad \delta_0 = \frac{c}{\omega_{\mathrm{L}}}, \quad \omega_{\mathrm{L}}^2 = \frac{4\pi N e^2}{m^*}. \quad (11.18)$$

Magnetic frequencies $(\omega_0, \omega_{\mathrm{M}}, \omega_{\mathrm{DE}})$ are so low that

$$\frac{\omega v_{\mathrm{F}}}{\omega_{\mathrm{L}} c} \ll 1.$$

The second condition is even more important than the first one. The inequalities allow the asymptotic value of $K_k(q)$ proportional to $(k^2+q^2)^{-1/2}$ to be used [see (11.14)]. As a result, Eqn (11.16) yields

$$\mathrm{Re}\,\omega \simeq \omega_{\mathrm{DE}} - \frac{\omega_{\mathrm{M}}\omega_{\mathrm{DE}}^2}{8(ck)^2},$$

$$\mathrm{Im}\,\omega \simeq -\frac{\omega_{\mathrm{M}}\omega_{\mathrm{DE}}\omega_{\mathrm{L}}^2}{3\pi^2 c^2 v_{\mathrm{F}} k^3} = \frac{4\pi\sigma\omega_{\mathrm{M}}\omega_{\mathrm{DE}}}{(ck)^2 kl}. \quad (11.19)$$

The comparison of the last equation and (11.1) indicates that only the kl-th part of conduction electrons is involved in the interaction with a DE wave. Spatial dispersion is responsible for the faster tendency of the dissipative term (k^{-3}) to zero as compared with the dispersive one (k^{-2}). At

$$k \gg \frac{8}{3\pi^2}\frac{\omega_{\mathrm{L}}^2}{v_{\mathrm{F}}\omega_{\mathrm{DE}}} \quad (11.20)$$

the DE wave is a weakly damped one. It is to be remembered that the condition (11.20) must not be in conflict with (11.17). It is necessary that

$$\frac{a}{\delta_0} \ll \sqrt{\frac{\beta M}{I}}\left(\frac{v_{\mathrm{F}}}{c}\right)^{1/3}\left(\frac{\omega_{\mathrm{L}}}{\omega_{\mathrm{DE}}}\right)^{1/3}. \quad (11.21)$$

This condition is easy to satisfy. The non-local relationship between current density and electric field strength is responsible for the non-exponential dependence of electromagnetic field and magnetization components on the coordinate y. Also, it gives rise to 'slowly' decaying terms proportional to $y^{3/2}\exp(-ky)$ but having low amplitude $\sim (k\delta_0)^{-2}\omega(kv_{\mathrm{F}})^{-1}$ which are likely to be apparent only far from the metal surface.

It has been noted above that a magnetic field has no effect on static conduction. Moreover, we simply neglect the Shubnikov–de Haas quantum effects and consider the Fermi surface to be spherical. The situation changes if spatial dispersion is taken into account; specifically, wave vector dependence shows concurrently with dependence on the magnetic field. Therefore, formulas in previous sections hold if additional conditions are imposed on the averaged magnetic field B and the wave vector k:

$$kr_B \gg 1 \quad \text{or} \quad kl \ll 1, \quad (11.22)$$

where $r_B = cp_{\mathrm{F}}/eB$ is the radius of the electron orbit in a magnetic field ($p_{\mathrm{F}} = m^* v_{\mathrm{F}}$). Magnetic field effects will be consistently discussed below.

Here is an estimate true for the maximally strong magnetic field where $r_B \ll l$, $kr_B \ll 1$, i.e. the electron orbit radius is the smallest parameter of length dimension. The relationship between l and $1/k$ is arbitrary, but the longitudinal conduction only slightly differs from its macroscopic value $\left(\sigma \to \sigma\left[1 - (1/6\pi)(kr_B)^2\right]\right)$ as the analysis shows, and

$$\omega \simeq \omega_{\mathrm{DE}} - \frac{i\pi}{2}\frac{\omega_{\mathrm{M}}\omega_{\mathrm{DE}}}{c^2k^2}\sigma\left[1 - \frac{1}{6\pi}(kr_B)^2\right]. \quad (11.23)$$

Evidently, the effect of the strong magnetic field is restricted to a small decrease in the intensity of DE wave annihilation.

A most interesting case is that of intermediate fields influenced by limiting anomalous skin-effect (see [22]) when the electron orbit radius r_B is much bigger than the length of a DE wave $\lambda = 1/k$ but significantly smaller than the free path length l:

$$l \gg r_B \gg \lambda. \quad (11.24)$$

In the situation described by (11.24) the interaction between conduction electrons and a plane wave (e.g. acoustic or spin wave) propagating perpendicular to the magnetic field gives rise to a specific geometric resonance effect referred to as the Pippard oscillations (A B Pippard was the first to notice the potential of this effect for spectroscopy). Oscillations (periodic dependence of wave velocity and damping coefficient on the reversed magnetic field) are of simple (kinematic) nature: the situation reiterates each time the inside of an electron orbit in the magnetic field hosts an integer number of waves. It immediately follows from the condition of

$$\frac{cD_{\mathrm{F}}}{eB} = N\lambda$$

($N \gg 1$ is the integer and D_F is the maximum diameter of the Fermi surface; for a sphere $D_F = 2p_F$) that the period is

$$\Delta \frac{1}{B} = \frac{2\pi e}{kcD_F} . \tag{11.25}$$

The DE wave being examined is not plane. However, it appears from the analysis [20] that the dependence of the wave amplitude on the coordinate normal to the magnetic surface does not eliminate the oscillatory dependence of its characteristics.

12. Electromagnetic waves in a gyroanisotropic medium

A magnetic crystal is a gyroanisotropic medium the electrodynamic properties of which are defined by two tensors, those of dielectric permittivity (ε_{ik}) and magnetic permeability (μ_{ik}). Both tensors (especially the latter one) are very sensitive to the external magnetic field and their components may have very different values due to the resonance dependence on frequency. Therefore, there is no reason to maintain that tensors ε_{ik} and μ_{ik} have similar structures. Moreover, rotation of magnetic moments makes both tensors gyrotropic which means that they are not subject to diagonalization by a turn of the Cartesian coordinates. Gyrotropy of tensor ε_{ik}, unlike that of tensor μ_{ik}, is very small in the general case. Properties of electromagnetic waves travelling in a medium are determined by a combination of tensors ε_{ik} and μ_{ik}. Evidently, such media must exhibit an infinite variety of high-frequency properties.

Let us see how unusual electromagnetic fields look in a gyrotropic medium [23]. To begin with, there is a simplest case in which the medium is considered to be non-dissipative, which implies that the tensors ε_{ik} and μ_{ik} are Hermitian:

$$\varepsilon^*_{ik} = \varepsilon_{ki} , \qquad \mu^*_{ik} = \mu_{ki} . \tag{12.1}$$

Secondly, the direction in which the wave propagates (z-axis in the notations used in this section) is such that

$$\varepsilon_{z\alpha} = \varepsilon_{\alpha z} = \mu_{z\alpha} = \mu_{\alpha z} = 0 .$$

Then,

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} \varepsilon_{xx} & \mathrm{i}\varepsilon' \\ -\mathrm{i}\varepsilon' & \varepsilon_{yy} \end{pmatrix}, \qquad \mu_{\alpha\beta} = \begin{pmatrix} \mu_{xx} & \mathrm{i}\mu' \\ -\mathrm{i}\mu' & \mu_{yy} \end{pmatrix}. \tag{12.2}$$

Possibly, $|\varepsilon'| \ll 1$. The expressions for μ_{xx}, μ_{yy}, and μ' can be found in previous sections, but they will not be used here.

Maxwell's equations for a monochromatic wave may be written as follows, using the concrete forms of tensors $\varepsilon_{\alpha\beta}$ and $\mu_{\alpha\beta}$:

$$v_{\alpha\beta} E_k = n^2 E_\alpha . \tag{12.3}$$

Here, n is the refractive index and

$$v_{\alpha\beta} = \mu_{\gamma\gamma}\varepsilon_{\alpha\beta} - \mu_{\gamma\alpha}\varepsilon_{\gamma\beta} . \tag{12.4}$$

The dispersion equation naturally takes the form

$$|n^2\delta_{\alpha\beta} - v_{\alpha\beta}| = 0 . \tag{12.5}$$

The asymmetry of the matrix $v_{\alpha\beta}$ relative to $\varepsilon_{\alpha\beta}$ and $\mu_{\alpha\beta}$ can be accounted for by the fact that Eqn (12.4) was derived by the exclusion of magnetic field. Removal of the electric field would result in the dispersion equation with a matrix $\tilde{v}_{\alpha\beta}$ obtained by exchanging positions of $\varepsilon_{\alpha\beta}$ and $\mu_{\alpha\beta}$ in $v_{\alpha\beta}$. Naturally eigenvalues of both matrices coincide.

In the present case, the matrix $v_{\alpha\beta}$ has the form

$$v_{\alpha\beta} = \begin{pmatrix} \mu_{yy}\varepsilon_{xx} + \mu'\varepsilon' & \mathrm{i}(\mu_{yy}\varepsilon' + \mu'\varepsilon_{yy}) \\ -\mathrm{i}(\mu_{xx}\varepsilon' + \mu'\varepsilon_{xx}) & \mu_{xx}\varepsilon_{yy} + \mu'\varepsilon' \end{pmatrix}. \tag{12.6}$$

The matrix $v_{\alpha\beta}$ is non-Hermitian because the medium simultaneously displays anisotropy and gyrotropy of electric and magnetic properties. It is non-Hermitian character of the matrix $v_{\alpha\beta}$ that is responsible for the specific distribution patterns of electromagnetic waves in such media.

Two refractive indices can be found from Eqn (12.6):

$$n^2_{1,2} = \frac{1}{2}\left[v_{xx} + v_{yy} \pm \sqrt{(v_{xx} - v_{yy})^2 + 4v_{xy}v_{yx}} \right] . \tag{12.7}$$

Eigenwaves in a gyroanisotropic medium undergo characteristic polarization, and in the general case, the former and the latter waves are non-orthogonal to each other. It follows from (12.3)–(12.7) that

$$\gamma^E = \frac{E_x}{E_y} = \frac{n^2 - v_{yy}}{v_{yx}} , \qquad \gamma^H = \frac{H_x}{H_y} = \frac{n^2 - \tilde{v}_{yy}}{\tilde{v}_{yx}} ,$$
$$E_z = H_z = 0 . \tag{12.8}$$

The relationship between electric and magnetic field components has the form

$$\mathbf{H} = n\hat{\mu}^{-1}\mathbf{E} \times \mathbf{s} , \qquad \mathbf{s} = (0, 0, 1) . \tag{12.9}$$

We have omitted subscripts 1 and 2 which denote the wave type.

Let us analyse an interesting case of degeneracy, that is root coincidence in the dispersion equation (12.5). According to (12.7), the roots coalesce if the equality

$$(v_{xx} - v_{yy})^2 = -4v_{xy}v_{yx} \tag{12.10}$$

is fulfilled. For the Hermitian matrix $v_{xy} = v_{yx}$, degeneracy is feasible only if $v_{xx} = v_{yy} = v_0$, $v_{xy} = v_{yx} = 0$ i.e. when $v_{\alpha\beta} = v_0\delta_{\alpha\beta}$. These conditions being satisfied, anisotropy of electromagnetic properties of the medium in the plane normal to the direction of propagation is absent and the choice of basis directions is arbitrary. As before, the system of equations (12.3) has two linearly independent solutions, i.e. there are two linearly independent polarizations $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$, while refractive indices of the two waves coincide.

In the case of the non-Hermitian matrix $v_{\alpha\beta}$, root coincidence in Eqn (12.5) results in the merger of eigenvectors. The matrix is devoid of the complete set of axes, i.e. has only one eigenvector; hence, only plane waves with one polarization can propagate in the medium:

$$\frac{E_x}{E_y} = \sqrt{-\frac{v_{xy}}{v_{yx}}} . \tag{12.11}$$

This observation is not new (see Ref. [24]), but non-Hermitian character of the matrix v_{ik} is normally associated with decay. Then, root coincidence requires that the solutions like

$$(a + bz)\exp(\mathrm{i}kz) \tag{12.12}$$

be introduced since plane waves by themselves are unable to create the complete system.

In the present case, the medium is a non-dissipative one, and the solutions of the (12.12) type are unlikely to arise because the field strengths tend to infinity at $z \to \infty$ (the wave vector k is real). If the expression under the root sign in (12.7) vanishes at ω_0, the square of the refractive index is complex at $\omega < \omega_0$.

Let us consider the frequency interval near $\omega_0 (\omega \geqslant \omega_0)$. Of the four roots in Eqn (12.6) we choose those two for the refractive index n which vanish at infinity following the introduction of infinitely small damping. We assume here, that the wave propagates in the direction of positive z. Let us consider absorption only in $\varepsilon_{\alpha\beta}$:

$$\varepsilon_{xx} = \varepsilon'_{xx} + \mathrm{i}\delta\,, \qquad \varepsilon_{yy} = \varepsilon'_{yy} + \mathrm{i}\delta\,.$$

Due to the smallness of absorption δ, the squared refractive index n^2 can be written in the following form

$$n^2 = s + \mathrm{i}\delta \pm \left(\sqrt{p} + \frac{\mathrm{i}}{2}\frac{q}{\sqrt{p}}\right).$$

Here, p and q are the real and imaginary parts of the expression under the root sign in (12.7), respectively:

$$p = \mathrm{Re}\left[\frac{1}{4}(v_{xx} - v_{yy})^2 + v_{xy}v_{yx}\right],$$

$$q = \mathrm{Im}\left[\frac{1}{4}(v_{xx} - v_{yy})^2 + v_{xy}v_{yx}\right],$$

and the value $s = (v_{xx} + v_{yy})/2$ is naturally regarded as positive. At $s < 0$, the waves do not propagate at all near zero of the expression under the root sign. The imaginary part q is a small quantity of the order of the imaginary part of dielectric permittivity δ while p is small in proportion to the proximity to the root coincidence point $p \sim (\omega - \omega_0)/\omega_0$.

At $\delta \leqslant p \leqslant 1$, we have

$$\mathrm{Im}\, n^2 = \delta(1 + p^{-1/2}) \simeq \delta p^{-1/2}\,,$$

for the imaginary part n_2, while the solutions satisfying the conditions at infinity ($\mathrm{Im}\, n > 0$) are

$$n_1 = (s + \sqrt{p})^{1/2} \exp\left[\frac{\mathrm{i}}{2}\arctan\frac{\delta p^{-1/2}}{s + \sqrt{p}}\right],$$

$$n_2 = (s - \sqrt{p})^{1/2} \exp\left[\mathrm{i}\left(\pi - \frac{1}{2}\arctan\frac{\delta p^{-1/2}}{s - \sqrt{p}}\right)\right]. \quad (12.13)$$

Passing δ to zero, we find

$$n_1 = (s + \sqrt{p})^{1/2}\,, \qquad n_2 = -(s - \sqrt{p})^{1/2}\,. \quad (12.14)$$

Therefore, the solution has the form

$$\mathbf{E}^{(1)} \exp\left(\mathrm{i}\,\frac{\omega}{c}\,n_1 z\right) + \mathbf{E}^{(2)} \exp\left(\mathrm{i}\,\frac{\omega}{c}\,n_2 z\right). \quad (12.15)$$

It should be emphasized that n_1 is positive and n_2 is negative. This means that the phase in the first wave moves in the positive direction (normal dispersion) while that in the second wave in the negative direction (anomalous dispersion). Certainly, energy propagates in the positive z-direction in both waves.

When calculating characteristics of a medium near the frequency ω_0, one must also bear in mind the opposite limiting case ($p \leqslant \delta \leqslant 1$), that is examine the immediate vicinity of the point $\omega = \omega_0$. It is easy to see that real parts of the refractive index also have different signs. A specific feature of this region of the spectrum is the root dependence of the imaginary part of the refractive index on weak damping which must also be apparent as the frequency dependence of both the impedance and reflection coefficient (e.g. the root approximation should be employed when estimating quantities vanishing at $\delta = 0$).

The structural analysis of an electromagnetic field in a medium has shown that it is possible to consider the incidence of an electromagnetic wave onto a halfspace occupied by a gyroanisotropic medium without introducing solutions of the (12.12) type. Let us assume for simplicity that the incidence is normal, the incident wave is polarized along the x-axis ($E_y^{(i)} = 0$), and its amplitude is equal to unity ($E_x^i = H_y^i = 1$). The boundary conditions have the form

$$1 + E_x^{\mathrm{r}} = E_x^{(1)} + E_x^{(2)}\,, \qquad 1 + H_y^{\mathrm{r}} = H_y^{(1)} + H_y^{(2)}\,,$$
$$E_y^{\mathrm{r}} = E_y^{(1)} + E_y^{(2)}\,, \qquad H_x^{\mathrm{r}} = H_x^{(1)} + H_x^{(2)}\,. \quad (12.16)$$

Using Eqns (12.8) and (12.9) together with the relations $E_x^{\mathrm{r}} = -H_y^{\mathrm{r}}$, $E_y^{\mathrm{r}} = H_x^{\mathrm{r}}$, we shall find for a reflected wave

$$E_x^{\mathrm{r}} = \frac{(1 - \beta_1\gamma_1^H n_1)(\gamma_2^E - \beta_2 n_2) - (1 - \beta_2\gamma_2^H n_2)(\gamma_1^E - \beta_1 n_1)}{(1 - \beta_1\gamma_1^H n_1)(\gamma_2^E + \beta_2 n_2) - (1 - \beta_2\gamma_2^H n_2)(\gamma_1^E + \beta_1 n_1)}\,,$$

$$E_y^{\mathrm{r}} = \frac{2(\beta_2\gamma_2^H n_2 - \beta_1\gamma_1^H n_1)}{(1 - \beta_1\gamma_1^H n_1)(\gamma_2^E + \beta_2 n_2) - (1 - \beta_2\gamma_2^H n_2)(\gamma_1^E + \beta_1 n_1)}\,. \quad (12.17)$$

Here

$$\beta_{1,2} = -\frac{\mu_{xx}\gamma_{1,2}^E + \mu_{yx}}{|\mu_{ik}|}\,.$$

The coefficient of reflection $R = |E_x^{\mathrm{r}}|^2 + |E_y^{\mathrm{r}}|^2$ is not presented here. R turns into unity at the degeneracy point. Indeed, at this point, $\gamma_1^E = \gamma_2^E = \mathrm{i}\gamma$, $\gamma_1^H = \gamma_2^H = -\mathrm{i}\gamma$, $\beta_1 = \beta_2 = \mathrm{i}\beta$, and

$$|E_x^{\mathrm{r}}|^2 + |E_y^{\mathrm{r}}|^2 = 1\,. \quad (12.18)$$

Although the coefficient of reflection is unity, the field in the medium does not decrease exponentially. Rather, it is a superposition of two waves running in the opposite directions, with wave vectors equal to $k = \pm(\omega/c)\, n_0$ respectively; i.e. it is a standing wave. The strength of the electric field is proportional to that of the magnetic field. Naturally, the energy flux is zero.

At the frequencies at which the refractive index has the negative imaginary part, the field in the medium is either a superposition of two damped waves or a standing wave with the amplitude exponentially decreasing into the depth of the medium. The strengths of the electric and magnetic fields are parallel to each other.

In the region where the expression under the root sign in (12.7) is positive ($\omega > \omega_0$), γ^E and γ^H are often purely imaginary, which means that both waves are elliptically polarized.

An interesting specific case of the gyroanisotropic medium is a metallic magnet in a strong magnetic field with the magnetic susceptibility similar to that in the formula (12.2) and the tensor of dielectric permittivity containing no diagonal elements:

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & \dfrac{4\pi i\sigma_{xy}}{\omega} \\ -\dfrac{4\pi i\sigma_{xy}}{\omega} & 0 \end{pmatrix}, \quad (12.19)$$

where

$$\sigma_{xy} = \frac{(n_e - n_h)ec}{B}$$

is the Hall conduction component and $n_{e(h)}$ is electron (hole) density in the metal. The diagonal (dissipative) elements are omitted because they are $\omega_c\tau_e$-times smaller at $\omega_c\tau_e \gg 1$ (ω_c is the cyclotron frequency). The condition $\omega_c\tau_e \gg 1$ defines the requirement for the strength of the average magnetic field B.

Since the elements ε_{xx} and ε_{yy} are zero, the matrices $v_{\alpha\beta}$ and $\tilde{v}_{\alpha\beta}$ coincide:

$$v_{\alpha\beta} = \tilde{v}_{\alpha\beta} = \begin{pmatrix} \varepsilon'\mu' & i\mu_{xx}\varepsilon' \\ -i\mu_{yy}\varepsilon' & \varepsilon'\mu' \end{pmatrix}, \quad \varepsilon' = \frac{4\pi\sigma_{xy}}{\omega}. \quad (12.20)$$

It is worthwhile to note, that the matrix $v_{\alpha\beta}$ remains a non-Hermitian one. Polarizations γ^E and γ^H also coincide:

$$\gamma^{E,H} = \pm\sqrt{\frac{\mu_{yy}}{\mu_{xx}}}.$$

The refractive index is defined by the expression

$$n_{1,2}^2 = \left(\mu' \pm \sqrt{\mu_{xx}\mu_{yy}}\right)\frac{4\pi\sigma_{xy}}{\omega}. \quad (12.21)$$

Degeneracy is feasible if either μ_{xx} or μ_{yy} vanishes. Let $\mu_{yy} = 0$ and $\mu_{xx} \neq 0$; then

$$n^2 = \frac{4\pi\sigma_{xy}}{\omega}\mu', \qquad \gamma_1 = \gamma_2 = 0. \quad (12.22)$$

This means that $E_x = H_x = 0$ while E_y and H_y are other than zero. Eigenwaves are linearly polarized. The zero energy flux is especially conspicuous.

The above example is of special interest, largely because it demonstrates how great is the difference between an electromagnetic wave in a gyroanisotropic medium and an ordinary wave in a vacuum or an isotropic body. It should be borne in mind that vectors **E**, **H**, and **k** in an ordinary wave are mutually orthogonal.

13. Surface magnetic polaritons in a plate magnetized parallel to the surfaces

Investigations of high-frequency magnetic materials were greatly promoted by a study of Walker [25] who demonstrated that magneto-dipole interactions lead to the existence of non-uniform eigensolutions of magnetostatic equations (certainly with frequency-dependent magnetic permeability) and the corresponding frequency spectrum of non-uniform resonance.

Let us formulate two important qualitative results of early studies on magneto-dipole modes in plates:

(1) Taking into account boundary conditions eliminates non-analiticity of the dependence of frequency ω on the wave vector k (at $k \to 0$) in magneto-dipole modes.

(2) The discrete Walker modes in a plate (rod) undergo transformation to waves propagating along the plate (rod), with the dispersion (k-dependence of ω) being a result of interference effects if the non-uniform exchange interaction associated with magnetostatic oscillations is disregarded. In this case, the characteristic non-dimensional parameter responsible for the dependence $\omega = \omega(\mathbf{k}_t)$ [$\mathbf{k}_t$ is the wave vector directed along the plate (rod)] is $k_t d$ (d is either plate halfwidth or rod radius).

The objective of this section is to examine the dispersion law for surface magnetic polaritons in a ferromagnetic plate with width $2d$ which occupies the layer $|y| < d$ and is magnetized parallel to its surfaces [26]. The role of anisotropy can be estimated by observing waves travelling perpendicular to the magnetic field **H** and magnetization **M** (assuming that $\mathbf{M} \parallel \mathbf{H}$, as before) and along **H** and **M**.

Let a magnetic polariton propagate perpendicular to the magnetic field and magnetization. Alternating fields are concentrated inside the plate and near it and decrease exponentially as the distance from the plate increases (along the y-axis), with the logarithmic damping decrement

$$\gamma_0 = \sqrt{k^2 - \frac{\omega^2}{c^2}}, \qquad ck > \omega, \qquad k \equiv k_x. \quad (13.1)$$

If the y-th component of the wave vector is denoted by q, it follows from Maxwell's equations that

$$q^2 = -k^2 + \frac{\omega^2}{c^2}\,\varepsilon\mu_{\text{eff}}(\omega), \quad (13.2)$$

with μ_{eff} being given by formula (2.4) at $\theta = \pi/2$. The field structure in the plate depends on the sign of q^2. When $q^2 > 0$ (case A), the alternating fields in the plate are actually a superposition of trigonometric functions. At $q^2 = -\gamma^2$ and $\gamma^2 > 0$, (case B), the alternating fields are a superposition of hyperbolic functions. The boundary conditions lead to the following dispersion equations:

$$\text{A.}\ 2\left[\omega_0(\omega_0 + \omega_M) - \omega^2\right]\gamma_0 q\cot(2qd) = \frac{\omega^2}{c^2}\left[(\omega_0 + \omega_M)^2 + \varepsilon(\omega_0 + \omega_M)\omega_0 - (1+\varepsilon)\omega^2\right] - k^2\left[\omega_0^2 + (\omega_0 + \omega_M)^2 - 2\omega^2\right],$$

$$\text{B.}\ 2\left[\omega^2 - \omega_0(\omega_0 + \omega_M)\right]\gamma_0\gamma\cot(2\gamma d) = \frac{\omega^2}{c^2}\left[(1+\varepsilon)\omega^2 - (\omega_0 + \omega_M)^2 - \varepsilon\omega_0(\omega_0 + \omega_M)\right] + k^2\left[\omega_0^2 + (\omega_0 + \omega_M)^2 - 2\omega^2\right].$$

Multivalued cotangent in case A accounts for an infinite number of spectral branches ($\omega = \omega_n(k)$) which are all localized outside the frequency range $\left(\left[\omega_0(\omega_0 + \omega_M)\right]^{1/2}, \omega_0 + \omega_M/2\right)$ and originate at the straight line $\omega = ck$. The frequency of the lowest branch $\omega = \omega_{(0)}(k)$ at $k \to 0$ vanishes and tends to

$$\omega = \omega_{\lim} = \sqrt{\omega_0(\omega_0 + \omega_M)}$$

at $k \to \infty$. The starting points of all the branches $\omega = \omega_n(k)$, $n > 0$ lie at the straight line $\omega = ck$ ($\gamma_0 = 0$). There are two groups of branches: one, $\omega = \omega_n^{(-)}$, is below $\omega_{\lim}$ and condenses towards $\omega_{\rm lim}$, the other, $\omega = \omega_n^{(+)}$, is above $\omega_0 + \omega_{\rm M}/2$ and asymptotically approaches $\omega = ck/\varepsilon^{1/2}$.

At $\varepsilon - 1 \ll 1$, the initial frequencies of the branches

$$\omega_n^{(+)} \simeq c\sqrt{\frac{\left[\pi(n-1)/2d\right]^2 + \omega_{\rm M}(\omega_0+\omega_{\rm M})/c^2}{\varepsilon - 1}}\,,$$

$$\omega_n^{(-)} \simeq \sqrt{\frac{\omega_0(\omega_0+\omega_{\rm M})}{1+\omega_{\rm M}(\omega_0+\omega_{\rm M})\left[2d/\pi(n-1)c\right]^2}}\,.$$

Interestingly, all upper branches at $\varepsilon \to 1$ tend to infinity (they are totally absent at $\varepsilon = 1$!). At the initial segments of the dispersion curves, group velocities are slightly different from c. When the wave vectors are large ($\omega \to \omega_{\rm lim}$), group velocities of the lower branches tend to zero while the upper ones are not so markedly altered: they tend to $c/\varepsilon^{1/2}$ at $\omega \to \infty$.

In case B, there is one (specific) oscillation branch $\omega = \omega_{\rm sp}(k)$ which occupies the interval $\omega_{\rm lim} < \omega < \omega_0 + \omega_{\rm M}/2$. At $kd \gg 1$ and $c \to \infty$, it 'turns' into a DE wave with the frequency $\omega = \omega_{\rm DE}$.

At $\omega \to \omega_{\rm lim}$,

$$\omega_{\rm sp}(k) - \omega_{\rm lim} \simeq \frac{c^2\omega_{\rm M}^3}{2\varepsilon\omega_0^{5/2}(\omega_0+\omega_{\rm M})^{3/2}}(k-k_0)^2\,,$$

$$k - k_0 \ll k_0\,, \qquad k_0 = \frac{\omega_0}{\omega_{\rm M}}\,\frac{(\omega_0+\omega_{\rm M})^2}{c^2}\,.$$

The group velocity of the specific wave at $\omega \to \omega_{\rm lim}$ is

$$v_{\rm sp} = c\sqrt{\frac{2\omega_{\rm M}^3(\omega - \omega_{\rm lim})}{\varepsilon\omega_0\omega_{\rm lim}^3}}\,.$$

At $\omega \to \omega_{\rm DE}$,

$$\frac{\omega_{\rm sp}(k) - \omega_{\rm DE}}{\omega_{\rm DE}} \simeq -\frac{(1+\varepsilon)\omega_{\rm DE}\omega_{\rm M}}{8c^2k^2}\,,$$

$$v_{\rm sp} = c\sqrt{\frac{2\omega_{\rm DE}}{(1+\varepsilon)\omega_{\rm M}}}\left(\frac{\omega_{\rm DE}-\omega}{\omega_{\rm DE}}\right)^{3/2}.$$

The velocity of specific wave vanishes at the ends of the frequency interval, and its maximum is close to the velocity of light.

It should be noted that the penetration depth at $\omega = \omega_{\rm lim}$ is zero ($\gamma \to \infty$). If $\omega \to \omega_{\rm DE}$, it tends to infinity, together with k and γ. In such cases, spatial dispersion of magnetic permeability must be taken into consideration, as was observed before. Moreover, dissipation processes impose some restrictions. In order to be able to distinguish between branches near the condensation point, the following condition must be satisfied (at $n \gg 1$):

$$k^2 + \left(\frac{\pi n}{2d}\right)^2 < \frac{\pi}{2d}\sqrt{\varepsilon}\,\frac{\omega_{\rm M}\left[\omega_0(\omega_0+\omega_{\rm M})^3\right]^{1/4}}{c\sqrt{\omega_{\rm DE}}}\sqrt{\tau n}\,. \quad (13.3)$$

This is not the sole constraint on k and n (since we did not take into consideration the non-uniform exchange interaction):

$$ak \ll \sqrt{\frac{\hbar\omega_{\rm M}}{I}}\,, \qquad aq \ll \sqrt{\frac{\hbar\omega_{\rm M}}{I}}\,.$$

To fulfill the last inequality, it is necessary that the plate be sufficiently thick and the number of n not very large:

$$\frac{d}{n} \gg a\sqrt{\frac{I}{\hbar\omega_{\rm M}}}\,.$$

Figure 13a presents magnetic polariton spectra described above.

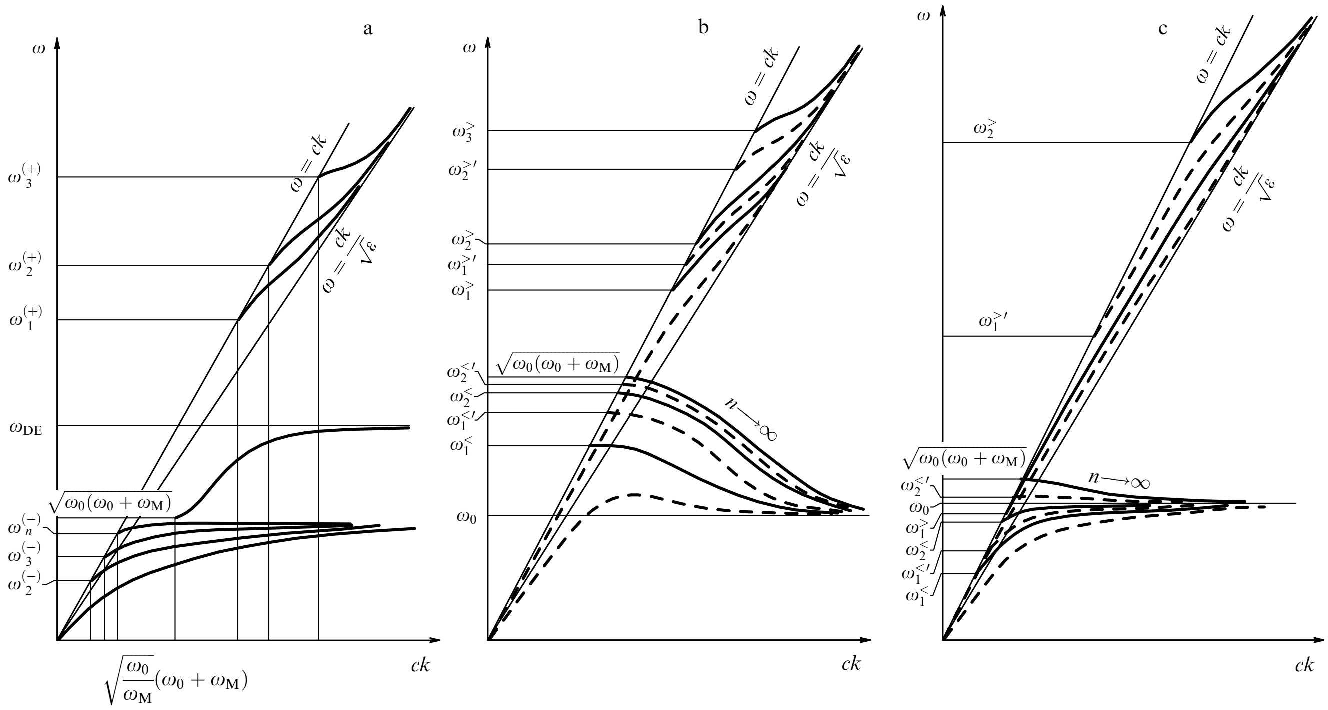

Figure 13. Schematic dependence of the dispersion law for magnetic polaritons in a plate. The magnetic field **H** and the magnetic moment **M** are parallel to the plate surfaces. The waves propagates (a) normal to **M**, (b) parallel to **M**; $\zeta > 1$, (c) parallel to **M**; $\zeta < 1$.

If the plate is thick, the electromagnetic field associated with the specific wave (which appears to be especially interesting to consider) is concentrated at one side, depending on the direction of propagation, and exponentially decreases towards the other.

Let us now consider a magnetic polariton travelling along the magnetic field in a ferromagnetic plate (see Ref. [27]). There are two types of solutions in this case, q_1 and q_2, defined by the formula

$$q_{1,2}^2 = k^2\left[\omega^2 - \omega_0\left(\omega_0 + \frac{\omega_M}{2}\right)\right] - \frac{\omega^2\varepsilon}{c^2}\left[\omega^2 - (\omega_0+\omega_M)\left(\omega_0 + \frac{\omega_M}{2}\right)\right] \pm \frac{\omega_M}{2\left[\omega_0(\omega_0+\omega_M)-\omega^2\right]} \times \sqrt{\left[\frac{\omega^2\varepsilon}{c^2}(\omega_0+\omega_M) - k^2\omega_0\right]^2 + 4k^2\frac{\omega^2\varepsilon}{c^2}\omega^2}\,, \qquad (13.4)$$

while the dispersion equation (zero determinant of the system of equations describing boundary conditions) splits up into two (which may reflect unknown intrinsic symmetry of the problem):

$$\frac{\mu\omega^2\varepsilon/c^2 - k^2 - q_1^2}{\omega^2\varepsilon/c^2 - q_1^2}\left[\cos(q_1 d) + \gamma_0\frac{\sin(q_1 d)}{q_1}\right] \times \left[\cos(q_2 d) + \gamma_0\varepsilon\frac{\sin(q_2 d)}{q_2}\right] - \frac{\mu\omega^2\varepsilon/c^2 - k^2 - q_2^2}{\omega^2\varepsilon/c^2 - q_2^2} \times \left[\cos(q_1 d) + \gamma_0\varepsilon\frac{\sin(q_1 d)}{q_1}\right]\left[\cos(q_2 d) + \gamma_0\frac{\sin(q_2 d)}{q_2}\right] = 0\,, \qquad (13.5)$$

$$\frac{\mu\omega^2\varepsilon/c^2 - k^2 - q_1^2}{\omega^2\varepsilon/c^2 - q_1^2}\left[\gamma_0\cos(q_1 d) - q_1\sin(q_1 d)\right] \times \left[\gamma_0\varepsilon\cos(q_2 d) - q_2\sin(q_2 d)\right] - \frac{\mu\omega^2\varepsilon/c^2 - k^2 - q_2^2}{\omega^2\varepsilon/c^2 - q_2^2}\left[\gamma_0\varepsilon\cos(q_1 d) - q_1\sin(q_1 d)\right] \times \left[\gamma_0\cos(q_2 d) - q_2\sin(q_2 d)\right] = 0\,. \qquad (13.6)$$

The dispersion dependence is essentially related to $\xi = \pi c/2\sqrt{2}\omega_0 d$. Its schematic representations at $\xi > 1$ and $\xi < 1$ are shown in Figs 13b and 13c respectively. The number of branches starting below $\omega = \omega_0$ and intersecting other branches is the greater the lower ξ. There are three types of waves: fast ones with normal dispersion, slow ones with anomalous dispersion turning in the limit into a known MSW type, and specific waves. At $\xi > 1$, there are two specific waves; if $k \to 0$, they are very close to each other and to the straight line $\omega = ck$:

$$ck - \omega_0^{\gtrless\prime}(k) \simeq \frac{ck}{2}(kd)^2\begin{cases}\left(\dfrac{\omega_M}{\omega_0+\omega_M}\right)^2, \\ \left(\dfrac{\omega_M}{\omega_0}\right)^2. \end{cases} \qquad (13.7)$$

At $k \to \infty$, the frequency of one wave tends to its limiting value from above:

$$\omega_0^{<\prime}(k) \simeq \omega_0 + \frac{\omega_M}{2(ck)^2}\left[\left(\frac{\pi c}{2d}\right)^2 - 2\omega_0^2\right], \qquad k \to \infty\,. \qquad (13.8)$$

This occurs only at $\xi > 1$. Hence, the curve $\omega_0^{<\prime}(k)$ intersects the straight line $\omega = \omega_0$. The branch $\omega_0^{<\prime}(k)$ intersects all curves of the lower branch group and asymptotically approaches the straight line $\omega = ck/\varepsilon^{1/2}$ when $k \to \infty$, similar to all the curves of the upper group. When $\xi < 1$ (even at an arbitrarily small ξ), the number of branches displaying specific behaviour is limited, but an infinite number of branches $\omega_n^<$ and $\omega_n^{<\prime}$ come to ω_0 from above, and an infinite number of branches $\omega_n^>$ and $\omega_n^{>\prime}$ lie above $\omega = \left[\omega_0(\omega_0+\omega_M)\right]^{1/2}$. They do not intersect other branches.

In the waves of the upper group, electric and magnetic fields are superpositions of trigonometric and hyperbolic functions at any k whereas other waves have finite values of k at which mixed (trigonometric/hyperbolic) waves 'turn' into the superposition of trigonometric waves only.

Here are some more frequency values at $\varepsilon = 1$ (this case considerably simplifies calculations). At the straight line $\omega = ck$,

$$(\omega^<)^2 \simeq \omega_0(\omega_0+\omega_M)\left(1 - \frac{2\omega_0+\omega_M}{A_n}\right),$$

$$A_n \simeq \left(\frac{\pi c n}{\omega_M d}\right)^2 \quad \text{at} \quad n \to \infty\,.$$

At $\xi > 1$,

$$\omega_n^<(k) \simeq \omega_0 + \frac{\omega_M}{2k^2}\left[\left(\frac{\pi n}{d}\right)^2 - \frac{2\omega_0^2}{c^2}\right], \quad n = 1, 2, \ldots, k \to \infty\,.$$

For $\omega_n^{<\prime}(k)$, $n - 1/2$ must be substituted for n.

Spectra of magnons propagating along and across **M** are significantly different. This rises the problem of transition from one wave type to another and the problem of the critical value of angle θ at which the specific wave exists (see Fig. 13a). It proved impossible to analytically consider wave propagation at an arbitrary angle to **M** without using numerical methods. A distinctive feature of the wave spectrum at $\mathbf{k} \parallel \mathbf{M}$ is the existence of a wave group with anomalous dispersion in a broad range of wave vectors k, with their frequency interval $\left(\omega_0, \left[\omega_0(\omega_0+\omega_M)\right]^{1/2}\right)$ being readily governed by both magnetic field and temperature.

14. Magnetostatic waves in a plate

Certainly, the dispersion law of MSW can be derived from equations describing magnetic polaritons by means of the limiting transition $k \to \infty$ at $\omega \not\to \infty$. However, the formulation of the dispersion equation for MSW directly from magnetostatic equations allows more general cases to be considered than in the solution of the magnetic polariton problem.

Assuming that the tensor of magnetic permeabilities has the pattern (1.19), the plate occupies interval $-d < y < d$, and the axis 3 is parallel to its surface, the equation for finding γ has the form (here, we confine ourselves to the case of 'hyperbolic' fields in the plate)

$$\mu_1 k_x^2 - \mu_2\gamma^2 + \mu_3 k_z^2 = 0\,. \qquad (14.1)$$

From the boundary conditions, we obtain the dispersion equation to find the MSW frequency at an arbitrary angle θ ($\tan\theta = k_x/k_z$), if MSW does exist.

$$-2\mu_2 k\gamma\coth(2\gamma d) = k_x^2(\mu_1\mu_2 - \mu'^2 + 1) + k_z^2(\mu_2\mu_3 + 1)\,. \tag{14.2}$$

In the case of a ferromagnet, formulas (1.20), (14.1), and (14.2) taken together yield

$$\frac{\gamma^2}{k^2} = \frac{\omega_0(\omega_0 + \omega_M\sin^2\theta) - \omega^2}{\omega_0(\omega_0+\omega_M) - \omega^2}\,,$$

$$\frac{\gamma^2}{k^2}(\omega_M\sin^2\theta - \omega_0\cos^2\theta) - 2\,\frac{\gamma}{k}\coth(2\gamma d)\omega_0\cos^2\theta - (\omega_M\sin^2\theta + \omega_0\cos^2\theta) = 0\,, \tag{14.3}$$

with $\theta \neq \pi/2$. At $\theta = \pi/2$, the first equation is automatically satisfied (the equality $\gamma = k$ ensues from Eqn (14.1) at $\mu_1 = \mu_2$ [see (1.20)], and it follows from (14.2) that

$$\omega^2 = \omega_0(\omega_0+\omega_M) + \frac{\omega_M^2}{2\left[\coth(2|k|d) + 1\right]}\,. \tag{14.4}$$

An analysis [27] demonstrated that the second Eqn (14.3) has the solution at an arbitrary plate width only if

$$\theta > \theta_c = \arcsin\sqrt{\frac{\omega_0}{\omega_0+\omega_M}}\,, \tag{14.5}$$

while there are no solutions at $\theta < \theta_c$. Therefore, the angle θ_c (14.5) is the critical MSW angle, as in a halfspace (see Section 10). A change of the sign at $\sin\theta$ has no effect on frequency: the wave is reciprocal. At $kd \gg 1$, it follows from the first equation (14.3) that formally there are two solutions

$$\omega = \pm\frac{\omega_0 + (\omega_0+\omega_M)\sin^2\theta}{2\sin\theta}\,, \qquad \sin\theta \gtrless 0\,, \tag{14.6}$$

or

$$\omega = \frac{\omega_0 + (\omega_0+\omega_M)\sin^2\theta}{2|\sin\theta|}\,.$$

The waves with $\sin\theta > 0$ are concentrated at one side of the plate and those with $\sin\theta < 0$ on the other. We have taken advantage of this fact to analyse the mechanism of generation of irreversible surface waves at the halfspace boundary and especially their damping mechanism associated with the presence of bulk spin waves (see Section 10).

The field and magnetization in a plate can be both trigonometric and hyperbolic functions. Then, instead of (14.2),

$$-2\mu_2 kq\cot(2qd) = k_x^2(\mu_1\mu_2 - \mu'^2 + 1) + k_z^2(\mu_2\mu_3 + 1)\,, \tag{14.7}$$

where q is the y-th component of the wave vector determined from Eqn (14.1) by substituting $+q^2$ for $-\gamma^2$.

'Hyperbolic' and 'trigonometric' waves appear to possess complementary properties. At any rate, there are no 'trigonometric' waves in a ferromagnet at $\theta = \pi/2$ (indeed, $q^2 > 0$ at $k_z = 0$ only if $\mu_3 < 0$ while in ferromagnets $\mu_3 \equiv 1$). 'Trigonometric' waves do occur when $\theta = 0$ (the waves propagate parallel to the magnetic field), and

$$\tan\left(\frac{2kd}{\sqrt{|\mu|}}\right) = \frac{2\sqrt{|\mu|}}{1-|\mu|}\,, \tag{14.8}$$

the solutions to be sought for in the frequency interval where $\mu < 0$.

In the limiting cases of long and short waves,

$$\omega \simeq \begin{cases} \omega_0(\omega_0+\omega_M) - \omega_0\omega_M\left(\dfrac{2kd}{\pi n}\right)^2, & kd \ll 1\,, \\ \omega_0 + \dfrac{\omega_M}{2}\left(\dfrac{\pi n}{2kd}\right)^2, & kd \gg 1\,, \end{cases} \tag{14.9}$$

$$n = 1, 2, \ldots$$

Figure 14a schematically represents a few MSW branches. Dashed and solid lines denote branches which coincide (at $kd \gg 1$) with the corresponding branches of the magnetic polariton discussed in Section 13 [$\omega_n^<(k)$ and $\omega_n^{<\prime}(k)$].

An important feature of the MSW spectrum is the presence of concentration points, $k = 0$, $\omega = \left[\omega_0(\omega_0+\omega_M)\right]^{1/2}$ and $k \to \infty$, $\omega = \omega_0$. It is worthy of note that retardation eliminates the concentration point at $\omega = \left[\omega_0(\omega_0+\omega_M)\right]^{1/2}$ [in accordance with Maxwell's equations, there are no waves with a finite frequency at $k \to 0$ (cf. Fig. 13b and 13c)].

Till now, we considered the magnetic field **H** and magnetization **M** to be parallel to the surface, both in a

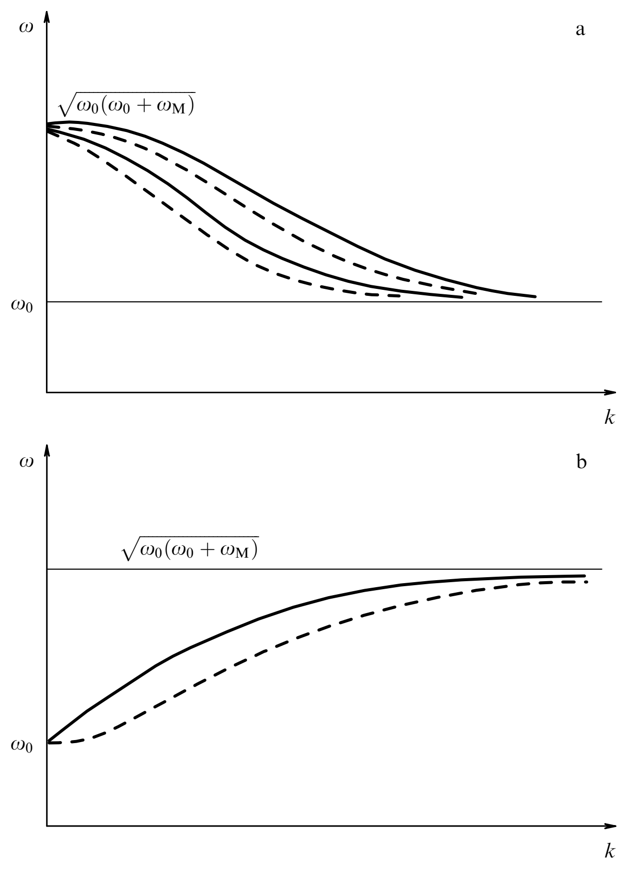

Figure 14. Graphic representation of the dispersion law for an MSW in a ferromagnetic plate magnetized parallel (a) and perpendicular (b) to the surface.

plate and a halfspace. In order to demonstrate the importance of **H** and **M** positions relative to the surface, we shall consider MSWs in a ferromagnetic plate ($|z| < 2d$) magnetized perpendicular to the surface. In this case, the dispersion equation splits up into two, one for an MSW whose potential is symmetric with respect to the plane $z = 0$ [$\propto \cos(q^{s}z)$] and the other for an MSW with antisymmetric potential [$\propto \sin(q^{a}z)$]. These equations have the form

$$k = q^{s}\tan(q^{s}d)\,, \tag{14.10}$$

$$-k = q^{a}\cot(q^{a}d)\,, \tag{14.10$'$}$$

$$\omega \equiv \omega_n(k) = \sqrt{\omega_0^2 + \omega_0\omega_M \frac{k^2}{k^2 + q_n^2}}\,. \tag{14.11}$$

The dispersion dependence is shown in Fig. 14b.

It is clear that the MSW spectrum contains two concentration points and differs from the previous case largely in terms of dispersion patterns (cf. Fig. 14a and 14b): MSWs exhibit normal dispersion in a plate magnetized perpendicular to the surfaces, similar to 'hyperbolic' waves (14.4). The dispersion pattern governs the direction of the wave vector **k** relative to **H**(**M**).

The qualitative difference between symmetric (4.10) and antisymmetric waves (14.10$'$) consists in that the former have finite velocity:

$$\omega^{s}_{n=0} \simeq \omega_0 + \frac{\omega_M}{2}kd\,, \qquad v^{s}_0 = \frac{\omega_M}{2}d \tag{14.12}$$

at $n = 0$ and $\omega \to \omega_0$ and

$$\omega^{s}_n \simeq \omega_0 + \frac{\omega_M}{2(\pi n)^2}(kd)^2\,, \qquad n = 1, 2, \ldots \tag{14.13}$$

at $n \neq 0$. At $kd \gg \pi(n + 1/2)$,

$$\omega^{s}_n \simeq \sqrt{\omega_0(\omega_0 + \omega_M)}\left[1 - \frac{\pi^2}{2}\frac{\omega_M}{\omega_0 + \omega_M}\left(\frac{n + 1/2}{kd}\right)^2\right],$$

$$n = 0, 1, \ldots, \qquad kd \gg \pi\left(n + \frac{1}{2}\right). \tag{14.13$'$}$$

For antisymmetric branches,

$$\omega^{a}_n \simeq \begin{cases} \omega_0 + \frac{\omega_M}{2}\left[\frac{kd}{\pi(n+1/2)}\right]^2, & kd \ll 1\,, \\ \sqrt{\omega_0(\omega_0 + \omega_M)}\left[1 - \frac{\pi^2}{2}\frac{\omega_M}{\omega_0+\omega_M}\left(\frac{n+1}{kd}\right)^2\right], & kd \gg \pi(n+1)\,, \end{cases}$$

$$n = 0, 1, \ldots \tag{14.14}$$

There is specific anomalous dependence on the mode number: the bigger n the lower frequency. It should be remembered that the mode number has real physical sense for it gives the number of waves that the plate is able to host.

In our opinion, the magnetostatic spectrum of antiferromagnetic plates remains to be investigated in more detail. One may use formulas (14.1) and (14.2) to elucidate the dispersion law for the MSW spectrum of a uniaxial antiferromagnet substituting into them the values of μ_1, μ_2, and μ' from Table 2 (Section 5).

To conclude the present section, here is a formula valid at $\mu_1 = \mu_2 = \mu(\omega)$, $\mu_3 = 1$, and $\mu'(\omega) \neq 0$ for the 'hyperbolic' wave propagating perpendicular to axis 3:

$$\frac{(1 + \mu)^2 - \mu'^2}{(1 - \mu)^2 - \mu'^2} = \exp\bigl(-4|k|d\bigr)\,. \tag{14.15}$$

This dispersion equation describes the reversible wave. At $|k|d \to \infty$, it gives rise to two equations

$$1 + \mu + \mu' = 0\,, \qquad 1 + \mu - \mu' = 0\,, \tag{14.16}$$

each describing the surface wave on either side of the plate. It is easy to see that these waves propagate in opposite directions.

The dispersion equation is especially simple at $H = 0$, when for an antiferromagnet $\mu' = 0$:

$$\mu(\omega) = -\sinh\bigl(|k|d\bigr)\,. \tag{14.17}$$

Hence, according to Table 2 and Section 10,

$$\omega^2 = \Omega_{SF}^2\left[1 + \frac{4\pi}{\delta}\frac{1}{1 + \sinh\bigl(|k|d\bigr)}\right]. \tag{14.18}$$

It is clear that as before, the wave frequency in an antiferromagnetic plate is in the immediate proximity to the uniform oscillation frequency of the magnetic moment, if $\delta \gg 1$. The wave shows anomalous dispersion. Its group velocity is low and varies from $(-\pi/\delta)\Omega_{SF}d$ at $k = 0$ to zero at $k \to \infty$. The dependence $\omega = \omega(k)$ for this wave is presented in Fig. 15.

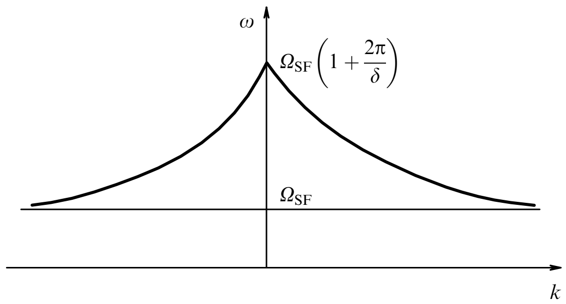

Figure 15. Schematic representation of the frequency $\omega = \omega(k)$ for an antiferromagnetic plate magnetized parallel to the surface, at $H < H_{SF}$.

15. A plate magnetized perpendicular to the surfaces. Magnetostatic waves. Taking into account non-uniform exchange interaction

The purpose of this section is to carry out a qualitative analysis of the outcome of interaction between two processes accounting for wave dispersion: the interference due to the boundary effect (see the previous section) and non-uniform exchange interaction [28].

Let us introduce the exchange interaction parameter l with dimension of length. Then, the natural dimensionless parameter of the problem is

$$\frac{l}{d} = \sqrt{\frac{I}{\mu M}}\,\frac{a}{d}\,. \tag{15.1}$$

We shall confine ourselves to considering the simplest case of a plate which occupies the layer $|z| < d/2$ (in this section, the plate width is d) and is magnetized along the z-axis. Taking into account the non-uniform exchange interaction leads to spatial dispersion of magnetic permeability and formally means the substitution of frequency ω_0 in formulas (1.20) by the function $\Omega = \omega_0 + \omega_{\rm ex}$, where $\omega_{\rm ex} = gM(kl)^2$. Increasing power of the dispersion equation requires additional boundary conditions (see Section 10). We shall analyse the case of 'free' magnetic moment:

$$\left.\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}z}\right|_{z=\pm d/2} = 0\,. \tag{15.2}$$

It has been observed in the previous section that the geometry of the problem admits of a search for a solution in the form of symmetric (in z) and asymmetric functions:

$$\varphi \sim \exp(\mathrm{i}\mathbf{k}\cdot\boldsymbol{\rho})\begin{cases}\cos(qz)\,,\\ \sin(qz)\,,\end{cases} \quad |z| < \frac{d}{2}\,,$$

$$\varphi \simeq \exp(\mathrm{i}\mathbf{k}\cdot\boldsymbol{\rho})\exp(-kz)\,, \qquad z > \frac{d}{2}\,,$$

$$\varphi \sim \pm\exp(\mathrm{i}\mathbf{k}\cdot\boldsymbol{\rho})\exp(kz)\,, \qquad z < -\frac{d}{2}\,, \tag{15.3}$$

where φ is the magnetic field potential ($-\nabla\varphi = \mathbf{h}$), $\mathbf{k}$ and $\boldsymbol{\rho}$ are the two-dimensional vectors, $\mathbf{k} = (k_x, k_y, 0)$, $\boldsymbol{\rho} = (x, y, 0)$, and q is one of the roots in the bicubic equation

$$q^2 - k^2\left(1 + \frac{\omega_{\rm M}\Omega}{\Omega^2 - \omega^2}\right) = 0\,, \tag{15.4}$$

where $\Omega = \omega_0 + \omega_{\rm ex}(k, q)$, and $\omega_{\rm ex}(k, q) = gMl^2(k^2 + q^2)$.

Let us write the dispersion equation satisfying the above boundary conditions and having symmetric

$$\sum_{i=1}^{3} F_i\left[1 - \frac{k}{q_i}\tan\left(\frac{q_i d}{2}\right)\right] = 0\,, \tag{15.5}$$

and antisymmetric

$$\sum_{i=1}^{3} F_i\left[1 + \frac{k}{q_i}\cot\left(\frac{q_i d}{2}\right)\right] = 0\,. \tag{15.6}$$

solutions. Here,

$$F_1 = \frac{q_2^2 - q_3^2}{\omega_{\rm ex}^{(1)}}\left(\omega_0 + \omega_{\rm ex}^{(1)}\right), \qquad F_2 = \frac{q_3^2 - q_1^2}{\omega_{\rm ex}^{(2)}}\left(\omega_0 + \omega_{\rm ex}^{(2)}\right),$$

$$F_3 = \frac{q_1^2 - q_2^2}{\omega_{\rm ex}^{(3)}}\left(\omega_0 + \omega_{\rm ex}^{(3)}\right), \qquad \omega_{\rm ex}^{(i)} = \omega_{\rm ex}(k, q_i)\,. \tag{15.7}$$

Equations (15.5) – (15.7) determine the spin wave spectrum in the plate:

$$\omega = \omega_n^{\rm s,\,a}(k)\,; \tag{15.8}$$

$n = 1, 2, \ldots$ is the solution number, subscripts 's' or 'a' denote the symmetry of the solution.

Neglecting the exchange interaction, we achieve maximum simplification of the spectrum:

$$\omega = \omega_0 + gMl^2\left[k^2 + \left(\frac{\pi n}{d}\right)^2\right]. \tag{15.9}$$

Its structure is shown in Fig. 16.

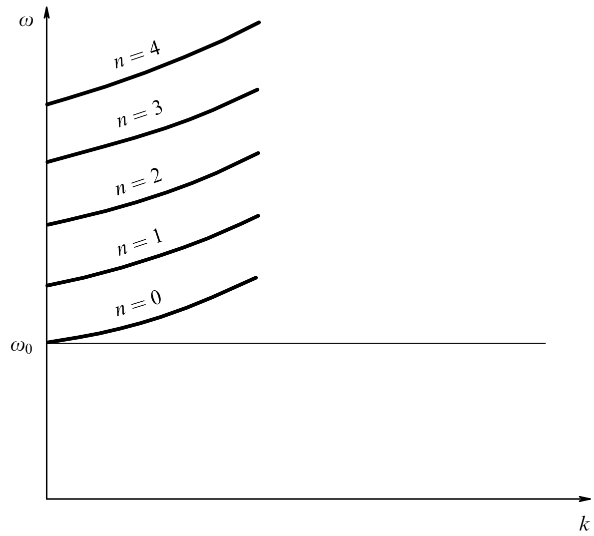

Figure 16. The dependence of the first four exchange frequencies on the wave vector **k**.

If the exchange interaction ($l = 0$) is neglected, the magnetostatic spectrum can be obtained [see (14.10) – (14.14) and Fig. 14b].

A marked difference in the spectral patterns is revealed in their interaction. A thorough analysis of the spectrum can be found in Ref. [28].

When the distance between branches of the exchange spectrum in a thin plate exceeds the width of the interval of magneto-dipole frequencies $\left(\left[\omega_0(\omega_0 + \omega_{\rm M})\right]^{1/2} - \omega_0\right)$, no substantial restructuring of the exchange spectrum occurs.

The role of the magneto-dipole interaction is restricted to the specification of the minimal frequency values in each subzone. True, the lowest exchange subzone is an exception. Due to the magneto-dipole interaction, a wave with the frequency ω which is close to ω_0 at $k \to 0$ has the finite velocity

$$\omega_{n=0}^{\rm s}(k) \simeq \omega_0 + \frac{1}{8}\,\omega_{\rm M} kd\left[1 + \sqrt{1 + \frac{16}{\pi}\left(\frac{l}{d}\right)^2}\,\right]. \tag{15.10}$$

Curiously, this expression allows for the limiting transition to $d \to 0$:

$$\omega^{\rm s}(k) \simeq \omega_0 + \frac{\omega_{\rm M}}{2\sqrt{\pi}}\,kl\,, \qquad d \ll l\,. \tag{15.11}$$

16. Interaction between magnetostatic waves and phonons in a plate (kinematics)*

Dissipative processes in ferromagnetic plates have not until recently attracted much attention. This is first of all true of dissipative processes in the region where MSW spectral characteristics dictated by boundary conditions need to be taken into account (see Section 14).

The present section largely based on the results obtained in Ref. [29] focuses on the kinematics of interactions between MSWs and phonons. We have chosen those one-phonon processes (emission and absorption) which satisfy two conditions:

(1) the process is not accompanied by changes in the MSW symmetry,

(2) the translational vector of a phonon involved in the process is uniform along the plate normal, i.e. the quasiwave vector of the phonon has two non-zero components (f_x and f_y). Therefore the phonon dispersion law is

$$\Omega = sf\,, \qquad \mathbf{f} = (f_x, f_y, 0)\,. \tag{16.1}$$

It has been shown that the interaction with such phonons makes the major contribution to the probability of MSW suppression (at least at limiting parameter values). Here, we are especially interested in kinematics of the interaction process. Conceptually, this section overlaps Section 7.

In order to be confined to the magneto-dipole approximation, i.e. to ignore non-uniform exchange interactions, we assume that the following inequality is satisfied:

$$\omega_0 \sim \omega_{\mathrm{M}} \ll \frac{s^2}{2\omega_{\mathrm{ex}} a^2} \tag{16.2}$$

(the notations are as above).

16.1 Phonon emission

The laws of conservation of energy and momentum are:

$$\mathbf{k} = \mathbf{k}' + \mathbf{f}\,, \qquad \omega_n(k) = \omega_{n'}(k') + \Omega(f)\,. \tag{16.3}$$

The MSW dispersion law is given by formula (4.11) supplemented with Eqns (4.10) and (4.10′).

Let $n = n' = 0$ and $kd \ll 1$.† Then, the MSW dispersion law is very simple [see (4.12)] and (16.3) yields

$$(k^2 + k'^2 - 2kk'\cos\varphi)^{1/2} = \alpha(k - k')\,,$$

$$\alpha = \frac{\omega_{\mathrm{M}} d}{2s}\,, \qquad k > k'\,, \tag{16.4}$$

where φ is the angle between two-dimensional vectors $\mathbf{k}$ and $\mathbf{k}'$. It follows from (16.4) that

$$-1 \leqslant \cos\varphi = \frac{(k^2 + k'^2)(1-\alpha^2) + 2\alpha kk'}{2kk'} \leqslant 1\,, \qquad k' < k\,. \tag{16.5}$$

The equation has no solutions at $\alpha < 1$ whereas at $\alpha > 1$ emission creates an MSW with the wave vector k' from the interval $((\alpha-1)k/(\alpha+1), k)$; according to (16.4), two angle values, $\varphi(k')$ and $2\pi - \varphi(k')$, correspond to each k'. Figure 17 presents possible values of $k' = k'(k, \varphi)$. The ends of the vectors $\mathbf{k}'$ are located at the curve K between circumferences with radii k and $(\alpha-1)k/(\alpha+1)$. At $\alpha \to 1$, the curve K contracts to a straight line $(0, k)$ at the axis $\varphi = 0$ whereas Eqn (16.4) undergoes degeneration to the equality $\cos\varphi = 1$, i.e. $\varphi = 0$ and $\varphi = 2\pi$.

It follows from the formulas in the next section that the probability of phonon creation at $\alpha = 1$ turns into infinity as a result of specific resonance: at $\alpha = 1$, MSW and phonon velocities are identical. However, one should recall that the dispersion law (14.12) results from approximation. Resonance situations need to be considered more carefully. According to (4.10), the corrected dispersion law for an

† This case is probably the most interesting one because the magneto-dipole patterns of the spectrum are especially prominent in the linear $|\mathbf{k}|$-dependence of ω at $|\mathbf{k}| \to 0$.

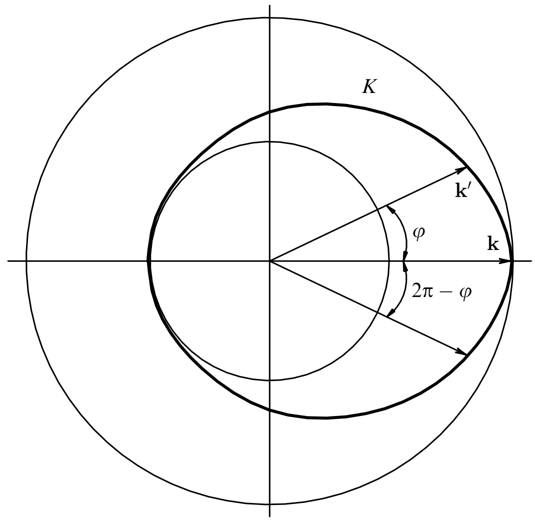

Figure 17. K-curve: the dependence of k' on the angle φ for phonon emission. The vector $\mathbf{k}$ is fixed, $n = n' = 0$, $k'(\varphi = 0) = k$, $k'(\varphi = \pi) = (\alpha - 1)k/(\alpha + 1)$, $kd \ll 1$.

MSW has the form

$$\omega_{n=0} \simeq \omega_0 \left[1 + \frac{\omega_{\mathrm{M}}}{2\omega_0}\, kd - \frac{\omega_{\mathrm{M}}}{4\omega_0}\left(1 + \frac{\omega_{\mathrm{M}}}{2\omega_0}\right)(kd)^2 + \ldots\right]. \tag{16.6}$$

Then, we have

$$(k^2 + k'^2 - 2kk'\cos\varphi)^{1/2} - (k - k')$$
$$= (\alpha - 1)(k - k') - \left(1 + \frac{\omega_{\mathrm{M}}}{2\omega_0}\right)\frac{d}{2}(k^2 - k'^2) \tag{16.7}$$

instead of (16.4). This equation has the solution at $0 < \alpha - 1 \ll 1$ if the following condition is fulfilled:

$$\frac{1}{2}\left(1 + \frac{\omega_0}{2\omega_{\mathrm{M}}}\right) kd < \alpha - 1\,. \tag{16.8}$$

It turned out that the process being examined has the threshold at $\alpha \geqslant 1$: it is necessary that the wave vector of MSW (k) be smaller than k_{th}

$$k_{\mathrm{th}} = \frac{2(\alpha - 1)}{d}\left(1 + \frac{\omega_0}{2\omega_{\mathrm{M}}}\right)^{-1}. \tag{16.9}$$

Since $k_{\mathrm{th}} d \ll 1$, the expansions used are valid.

The estimated probability of phonon emission indicates that the specification of the MSW dispersion law leads to the elimination of infinity: the inverse lifetime shows the peak in the dependence on the wave vector k without turning into infinity (see Section 17).

The existence of the threshold at $kd \ll 1$ raises the question: is there a threshold value of k when the parameter α is significantly different from unity? To answer this question, it is necessary to analyse the conservation laws without expanding the dispersion law $\omega_{n=0}(k) \equiv \omega_0(k)$ in powers of kd. From (16.3),

$$\cos\varphi = \frac{k^2 + k'^2 - \left[\omega_0(k) - \omega_0(k')\right]^2 / s^2}{2kk'} \tag{16.10}$$

is readily obtained, and the condition for the existence of the solution has the form

$$s(k - k') < \omega_0(k) - \omega_0(k')\,. \tag{16.11}$$

The graphic analysis shows that the condition (16.11) is satisfied if $k < k_{\rm th}$; naturally, $k_{\rm th}$ has the former value (16.9) at $a \simeq 1$, and at $a \gg 1$,

$$k_{\rm th} \simeq \frac{\omega_0 \omega_{\rm M}}{s\left\{\omega_0 + \left[\omega_0(\omega_0 + \omega_{\rm M})\right]^{1/2}\right\}} \,. \qquad (16.12)$$

(The inequality $\alpha \gg 1$ does not contradict (16.2) provided the condition $d \gg a\Theta_{\rm C}/\Theta_{\rm D}$, $\Theta_{\rm D} = \pi s/a$ is satisfied). It turned out impossible to obtain a non-analytical expression at an arbitrary value of α. Writing down the expression (16.12) in terms of α and assuming that ω_0 and $\omega_{\rm M}$ are of the same order, we come to the conclusion that in the given case $k_{\rm th} d \sim \alpha \gg 1$ (which is certainly not at variance with $ka \ll 1$).

At $k \leqslant k_{\rm th}$, the range of quasimomentum values of arising MSWs is very narrow and further shrinks to a point at $k \to k_{\rm th}$:

$$k'_{\min} \leqslant k' \leqslant k'_{\max} \,,$$

$$k'_{\min} = (k_{\rm th} - k)\left\{s - \left[\frac{\partial \omega_0(k)}{\partial k}\right]_{k=k_{\rm th}}\right\}\frac{1}{s(\alpha+1)} \,,$$

$$k'_{\max} = (k_{\rm th} - k)\left\{s - \left[\frac{\partial \omega_0(k)}{\partial k}\right]_{k=k_{\rm th}}\right\}\frac{1}{s(\alpha-1)} \,. \qquad (16.13)$$

It can be shown that the angle φ at the ends of the interval equals π and 0.

A kinematic analysis of phonon creation is necessary to calculate MSW lifetimes (Section 17). It is important to emphasize that magnon dispersion due to the non-uniform exchange interaction may also account for the existence of the threshold (see Section 7):

$$k_{\rm th}^{\rm ex} = \frac{s}{2\omega_{\rm ex} a^2} \,.$$

The described threshold is possible to observe if $k_{\rm th}$ is smaller than $k_{\rm th}^{\rm ex}$, which is the case when the condition (16.2) is met.

The anomalous dependence of the MSW frequency on n explains why phonon emission by an MSW with $n = 0$ may give rise to an MSW with $n' > 0$.

Let $kd \ll 1$. The same line of reasoning as before leads to the conclusion that when an MSW with $n = 0$ decomposes into a phonon and an MSW with $n' \neq 0$, the following relation for k' [analogous to (16.4)] is satisfied:

$$(k^2 + k'^2 - 2kk'\cos\varphi)^{1/2} = \alpha k \,. \qquad (16.14)$$

This equation does not include n' because the term $(k'/n')^2$ is omitted, being negligibly small as compared with other terms. Equation (16.4) has the solution at arbitrary α. At $\alpha < 1$, the angle φ is subject to the constraint

$$|\sin\varphi| < \alpha \,. \qquad (16.15)$$

Possible values of k' are presented in Fig. 18a. There are two solutions — two functions of $k'(\varphi)$. For one,

$$\min k'(\varphi = 0) = k(1 - \alpha) \,,$$

for the other,

$$\max\, k'(\varphi = 0) = k(1 + \alpha) \,.$$

At the intersection point,

$$k'(\varphi = \arcsin\alpha) = k\sqrt{1 - \alpha^2} \,, \qquad \alpha < 1 \,.$$

Let us now examine phonon emission by an MSW with $n \neq 0$, assuming that $kd \ll 1$, as before. The emission is possible only if $q_{n'} > q_n$ (see Section 14). The conservation laws in the first non-vanishing order in kd lead to the following equation:

$$(k^2 + k'^2 - 2kk'\cos\varphi)^{1/2} = \frac{\alpha d}{\pi^2}\left(\frac{k^2}{n^2} - \frac{k'^2}{n'^2}\right) . \qquad (16.16)$$

For the solutions to exist, the following inequalities should be fulfilled :

$$|\sin\varphi| < \frac{\alpha}{\pi^2}\left(\frac{1}{n^2} - \frac{1}{n'^2}\right) kd \ll 1 \,, \qquad (16.17)$$

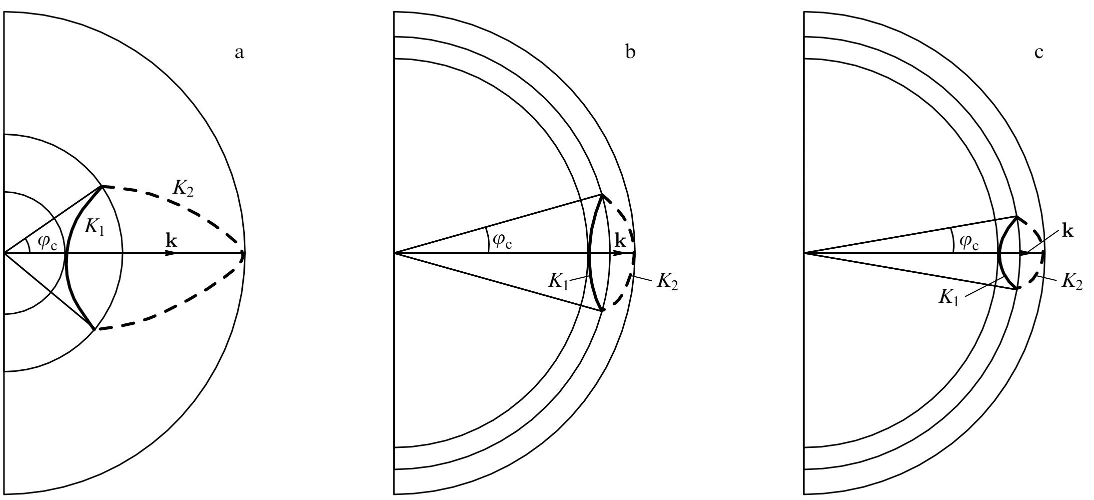

Figure 18. Curves K_1 and K_2: the dependence of k' on the angle φ. Possible k' values upon phonon emission for $n = 0$, $n' > 0$, $kd \ll 1$ (a), $n > 0$, $n' > n$, $kd \ll 1$ (b) and for $n > 0$, $n' > n$, $kd \gg 1$ (c) (see the text).

i.e. the interval of possible angles φ is very small. An MSW does not virtually change its direction upon emission of a phonon. Figure 18b shows possible values of $\mathbf{k}'$. There are two solutions for $k'(k)$ (lines K_1 and K_2). For one of them,

$$\max k'(\varphi = 0) = k\left[1 + \frac{\alpha}{\pi^2}\left(\frac{1}{n^2} - \frac{1}{n'^2}\right)kd\right],$$

for the other,

$$\min k'(\varphi = 0) = k\left[1 - \frac{\alpha}{\pi^2}\left(\frac{1}{n^2} - \frac{1}{n'^2}\right)kd\right],$$

and for both,

$$k'(\varphi_c) = k\left\{1 - \left[\frac{\alpha}{\pi^2}\left(\frac{1}{n^2} - \frac{1}{n'^2}\right)kd\right]^2\right\}^{1/2}.$$

The φ_c value is very small.

$$\sin\varphi_c = \frac{\alpha}{\pi^2}\left(\frac{1}{n^2} - \frac{1}{n'^2}\right)kd.$$

Small kd (long waves) are characterized by closeness to the point of condensation. At $kd \gg 1$, there is also a condensation point. Let us consider phonon emission at $kd \gg 1$. If phonon emission occurs without a change in the MSW mode number ($n' = n$), the wave with $n = 0$ is in no way singled out. Always $q_{n'} > q_n$ and $k'd \gg 1$ together with $kd \gg 1$. It follows from (16.3) that

$$(k^2 + k'^2 - 2kk'\cos\varphi)^{1/2} = \frac{\pi^2}{2s}\,\frac{\omega_0\omega_M(n'^2 + 2n' - n^2 - 2n)}{\left[\omega_0(\omega_0 + \omega_M)\right]^{1/2}(kd)^2}, \qquad n' \geqslant n. \quad (16.18)$$

In accordance with (16.8), the range of possible angles φ is very small, of the order of $(kd)^{-3}$. Figure 18c shows possible values of vector $\mathbf{k}'$. There are two solutions for $k'(\varphi)$ (lines K_1 and K_2). For K_2,

$$\max k'(\varphi = 0) = k\left(1 + \frac{B}{k^3}\right).$$

For K_1,

$$\min k'(\varphi = 0) = k\left(1 - \frac{B}{k^3}\right).$$

At the intersection point,

$$k'(\varphi_c) = k\left[1 - \left(\frac{B}{k^3}\right)^2\right]^{1/2}, \qquad \sin\varphi_c = \frac{B}{k^3},$$

$$B = \frac{\omega_0\omega_M}{2\left[\omega_0(\omega_0 + \omega_M)\right]^{1/2}}\,\frac{\pi^2(n'^2 + 2n' - n^2 - 2n)}{sd^2}.$$

16.2 Phonon absorption

The contribution of this process to the inverse time of an MSW with $n = 0$ at $kd \ll 1$ is much smaller than that of emission. However, this contribution becomes critical for an MSW with $n \neq 0$. Therefore, we shall confine ourselves to the case of $n \neq 0$.

Let us suppose n' to be zero since the probability of processes with all other values of $n' \neq 0$ is significantly smaller. It follows from the conservation laws that

$$(k^2 + k'^2 - 2kk'\cos\varphi)^{1/2} = \alpha k', \qquad k' > k. \quad (16.19)$$

There are no restrictions on the angle at $\alpha > 1$, and k' values lie in the region

$$\left(\frac{k}{\alpha + 1}, \frac{k}{\alpha - 1}\right), \quad (16.20)$$

and

$$\max k'(\varphi = \pi) = \frac{k}{\alpha - 1}, \qquad \min k'(\varphi = 0) = \frac{k}{\alpha + 1}.$$

There are restrictions at $\alpha < 1$:

$$\sin^2\varphi < \alpha^2. \quad (16.21)$$

Also, there are two values of k' for each k, φ [29]. For one of them

$$\max k'(\varphi = 0) = \frac{k}{1 - \alpha},$$

for the other

$$\min k'(\varphi = 0) = \frac{k}{1 + \alpha},$$

and for both

$$k'(\varphi = \arcsin\alpha) = \frac{k}{(1 - \alpha^2)^{1/2}}.$$

To avoid misunderstanding, it should be noted that the emission and the absorption are not reciprocally inverted processes. Either has its own inverse process. Concurrently, $\mathbf{k}$ and $\mathbf{k}'$ change places. The direct and inverse processes are identical in terms of kinematics.

17. Lifetimes of magnetostatic waves and phonons*

This section deals with MSW lifetimes [29] during the processes kinematically characterized in Section 16.

Quantization is performed by the standard procedure using the Holstein–Primakov expansion. The interaction Hamiltonian is limited by a simple invariant (as in Section 8)

$$H_{\text{int}} = \gamma\int M_i M_k u_{ik}\,\mathrm{d}v, \quad (17.1)$$

where γ is the magnetoelastic constant and u_{ik} are the components of the strain tensor. We are interested only in one of the four types of acoustic oscillations present in the plate which makes a major contribution to MSW decay. It has been noted in Section 16 that the translational vector in this wave is homogeneous in the z-coordinate.

The MSW lifetime $\tau_n^{-1}(k)$ is constituted by the contributions of three one-phonon processes two of which, phonon emission and absorption by an MSW, were described earlier while the probability of the third one (phonon creation by the fusion of two MSWs) contains the exponentially small factor $\exp(-\hbar\omega/T)$. Therefore its contribution in $\tau_n^{-1}(k)$ may be neglected.

Let us first consider phonon emission by a symmetric MSW at $n = n' = 0$, $kd \ll 1$ and $\alpha > 1$ (there is no such process at $\alpha < 1$). It gives rise to two new parameters k_{M} and k_{th} (from α and d). Let $k_{\mathrm{th}} d \ll 1$ [see (16.9) and (16.12)]. Then, $\left[\tau^{00}(k)\right]^{-1} \neq 0$ in the wave vector range from $k = 0$ to $k = k_{\mathrm{th}}$. The dependence of the inverse MSW lifetime shows maximum at a certain value of the wave vector $k = k_{\mathrm{M}}$ (Fig. 19). In the peak of the curve at $\alpha \geqslant 1$,

$$\frac{1}{\tau^{00}_{\max}} \sim \frac{\gamma^2 k^3_{\mathrm{th}}}{2^{15}\rho(\alpha-1)^{1/2} d} \begin{cases} \dfrac{T}{s}, & T \gg \hbar\omega_{\mathrm{M}} k_{\mathrm{th}} d, \\ \dfrac{\hbar k_{\mathrm{th}}}{s}, & T \ll \hbar\omega_{\mathrm{M}} k_{\mathrm{th}} d. \end{cases} \tag{17.2}$$

At $\alpha \gg 1$,

$$k_{\mathrm{M}} \simeq \frac{1}{d}\left[\frac{2^7}{\pi\alpha}\left(\frac{\omega_0}{\omega_0+\omega_{\mathrm{M}}}\right)^{1/3}\right]^{1/4},$$

$$\left[\frac{1}{\tau^{00}(k)}\right]_{\max} \simeq \frac{(\gamma g M)^2 \hbar}{2^5\pi^2\rho\omega_0^2 d^5}\sqrt{\frac{\omega_0}{\omega_0+\omega_{\mathrm{M}}}}\,\alpha^2. \tag{17.3}$$

In the beginning of the curve at $kd \ll T/\hbar\omega_{\mathrm{M}}$,

$$\frac{1}{\tau^{00}(k)} = \frac{(\gamma g M)^2 T (kd)^2 k^2}{2^{10}\pi\rho d^3\omega_0^2 s\sqrt{2(\alpha-1)k_{\mathrm{th}} k}}. \tag{17.4}$$

The inverse lifetime $\left[\tau^{00}(k)\right]^{-1}$ vanishes according to a linear law at $k \to k_{\mathrm{th}}$.

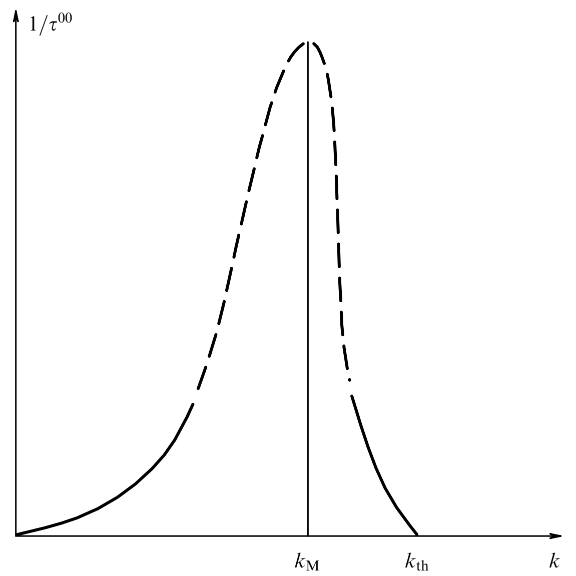

Figure 19. The dependence of $[\tau^{00}(k)]^{-1}$ on k upon phonon emission ($\alpha \geqslant 1$; $kd \ll 1$); $k = 4(\alpha-1)/d(2+\omega_{\mathrm{M}}/\omega_0)$.

When values of k are close to k_{th} and the parameter α is not close to unity,

$$\frac{1}{\tau^{00}_{\mathrm{th}}} = \frac{(\gamma g M)^2\hbar}{2^6\rho d^3}\,\frac{\alpha k^3_{\mathrm{th}}}{(\alpha^2-1)^{3/2}}\,\frac{1-\left(\partial\omega_0(k)/\partial k\right)_{\mathrm{th}} s^{-1}}{\omega_0(\omega_0+s k_{\mathrm{th}})}$$
$$\times\left[\frac{\sin(q_0 d)}{q_0}\right]^2_{\mathrm{th}}(k_{\mathrm{th}}-k)\begin{cases} 1, & T \ll \hbar s k_{\mathrm{th}}, \\ \dfrac{T}{\hbar s k_{\mathrm{th}}}, & T \gg \hbar s k_{\mathrm{th}}. \end{cases} \tag{17.5}$$

Here, q_0 is the zero solution of Eqn (14.10).

As α grows from 1 to $\alpha \gg 1$, the curve in Fig. 19 retains its shape but the peak is 'shifted' to the right, i.e. towards higher k.

The probability of phonon emission at $n = 0$, as the MSW passes towards the state with $n' > 0$ at $kd \ll 1$ (i.e. of the process with a change in the mode number), is significantly smaller than that of the process described above (without a change in the mode number). Therefore, we present the expression for $\left[\tau^{0n'}(k)\right]^{-1}$ only for the case of $\alpha < 1$ in which the process without a change of the mode is impossible:

$$\frac{1}{\tau^{0n'}(k)} = \frac{1}{2n'^4}\left(\frac{\gamma}{8\pi}\right)^2\frac{\omega_{\mathrm{M}}^2}{\omega_0^3}\,\frac{\alpha^4(\alpha^2+1)}{\rho d^5}$$
$$\times\begin{cases} (kd)^5 T, & kd \ll \dfrac{T}{\hbar\omega_0} \ll 1, \\ (kd)^6\hbar\omega_0, & \dfrac{T}{\hbar\omega_0} \ll kd \ll 1. \end{cases} \tag{17.6}$$

The summarized probability $\left[\tau^{0}(k)\right]^{-1}$ that an MSW emits a phonon of the zero mode with all n' values is approximately equal to $\left[\tau^{00}(k)\right]^{-1}$ at $\alpha > 1$ while at $\alpha < 1$,

$$\frac{1}{\tau^{0}(k)} \simeq \frac{1}{2}\left(\frac{\gamma}{8\pi}\right)^2\frac{\omega_{\mathrm{M}}^2}{\omega_0^3}\,\frac{\alpha^4(\alpha^2+1)}{\rho d^5}$$
$$\times\begin{cases} (kd)^5 T, & kd \ll \dfrac{T}{\hbar\omega_0} \ll 1, \\ (kd)^6\hbar\omega_0, & \dfrac{T}{\hbar\omega_0} \ll kd \ll 1. \end{cases} \tag{17.7}$$

However, the probability $\left[\tau^{0n'}(k)\right]^{-1}$ at $\alpha > 1$ (not presented here) differs from zero and hence the summarized probability of phonon creation at $k > k_{\mathrm{th}}$ is also other than zero. The presence of threshold is suggested by the inflection of the curve which describes the dependence of the inverse MSW lifetime on the wave vector. For the phonon emission by an MSW with $n \neq 0$ at $kd \ll 1$, the following formula is valid:

$$\frac{1}{\tau^{nn'}(k)} = \frac{1}{2n^6}\left(\frac{\gamma}{8\pi}\right)^2\frac{\hbar\alpha^4}{\rho d^5}(kd)^{10}\left(1+\frac{n}{n'}\right)^2\left[1+\left(\frac{n}{n'}\right)^2\right]^2,$$

$$\frac{T}{\hbar\omega_0} \ll kd \ll 1. \tag{17.8}$$

We do not present here the formula for $\left[\tau^{nn'}(k)\right]^{-1}$ at $kd \ll T/\hbar\omega_{\mathrm{M}}$ because the major contribution to the suppression of an MSW with $n > 0$ is made by phonon absorption rather than emission.

The MSW spectrum shows two condensation points. The inverse lifetime is of interest only at $kd \gg 1$:

$$\frac{1}{\tau^{nn'}(k)} = \frac{(\gamma g M)^2\hbar}{2^7\pi d\rho s^2}\begin{cases} A^2(kd)^{-6}, & 1 \ll (kd)^2 \ll \dfrac{\hbar s A}{T}, \\ \dfrac{TA}{\hbar s}(kd)^{-4}, & (kd)^2 \gg \dfrac{\hbar s A}{T} \gg 1. \end{cases} \tag{17.9}$$

Here, we use the notation

$$A = \frac{\omega_0\omega_{\mathrm{M}}\pi^2(n'^2+2n'-n^2-2n)}{2\left[\omega_0(\omega_0+\omega_{\mathrm{M}})\right]^{1/3} s}.$$

$(\tau^{nn})^{-1} \equiv 0$ since $A = 0$ at $n = n'$. This formula cannot be used to calculate the total phonon emission time

$$\frac{1}{\tau^n} = \sum_{n'} \frac{1}{\tau^{nn'}}\,.$$

Consideration of limiting cases allows for the estimation of the order of the MSW inverse lifetime resulting from phonon emission. Assuming $\alpha \geqslant 1$ and $\omega_0 \sim \omega_M$, it is easy to see that

$$\frac{1}{\tau_{\text{em}}(k)} \sim \frac{\gamma^2 \hbar}{\rho d^5} f(kd)\,, \tag{17.10}$$

and the function $f(kd)$ the form of which (at $kd \ll 1$ and $kd \gg 1$) is not difficult to deduce from the above formulas has the maximum at $kd \sim 1$, with $f_{\max} \sim 1$. Therefore,

$$\left(\frac{1}{\tau_{\text{em}}}\right)_{\max} \sim \frac{\gamma^2 \hbar}{\rho d^5}\,. \tag{17.11}$$

Of course, it is a very rough estimate because we have omitted all dimensionless multipliers.

The computation of the inverse lifetime dependent on phonon absorption by a symmetric MSW yields

$$\frac{1}{\tau^n(k)} = \frac{(\gamma g M)^2 \alpha^4}{2^6 \pi^3 \omega_0^2 \omega_M \rho} \frac{k^5 T}{n^2}\,, \qquad kd \ll \frac{T}{\hbar\omega_M}\,. \tag{17.12}$$

At $kd \gg 1$, the contribution of phonon absorption to the inverse lifetime is significantly smaller than that of phonon emission (the corresponding formula is not presented).

Summarizing the results obtained for the two MSW decay processes (the contribution of the third process being exponentially small), it is appropriate to present the graphic dependence $[\tau^n(k)]^{-1}$ at $d \gg d_{\text{ex}}$ (Fig. 20). At $\alpha \sim 1$, the maximum of the first curve lies outside the wave vector region $kd \sim 1$ [see (17.10)].

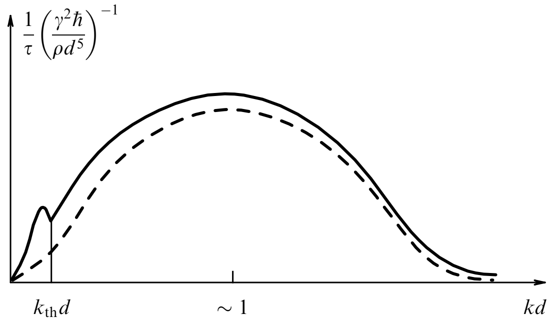

Figure 20. Qualitative wave vector dependence of the decay time of an MSW interacting with phonons. Solid line gives the dependence of $1/\tau^0(k)$; it has an inflection at $k = k_{\text{th}}$. Dashed line plots $1/\tau^n(k)$ at $n > 0$. Maximum inverse lifetime is estimated only up to the order of magnitude, the exact value of $(1/\tau)(\gamma^2\hbar/\rho d^5)^{-1}$ largely depends on the parameters of the system.

Let us briefly discuss suppression of an antisymmetric MSW. The main contribution to it at $kd \ll 1$ is made by phonon emission. All calculations are similar to those for the case of the symmetric wave when $n > 0$ (formally, πn is substituted by $\pi n + \pi/2$). The process is possible only if $n' \geqslant n$:

$$\frac{1}{\tau^n(k)} = \frac{(\gamma g M)^2}{2^{12}\pi^4\omega_0^2} \frac{\hbar a^4 d^5 k^{10}}{\rho(n+1/2)^8} \times \begin{cases} 1\,, & \dfrac{2\pi n^2 T}{\hbar\omega_M} \ll (kd)^2 \ll 1\,, \\ \dfrac{2\pi^2 n^2 T}{(kd)^2\hbar\omega_M}\,, & (kd)^2 \ll \dfrac{2\pi^2 n^2 T}{\hbar\omega_M}\,. \end{cases} \tag{17.13}$$

There is no necessity to specially consider an MSW with $n = 0$ because in this case the frequency $\omega_0(k)$ shows quadratic dependence on k at $k \to 0$.

In order to obtain $[\tau^{nn'}(k)]^{-1}$ at $kd \gg 1$, suffice it to substitute $n + 1/2$ by $n + 1$ and $n' + 1/2$ by $n' + 1$ in (17.9). The lifetime grows with the mode number as in the case of symmetric MSW. It is clear from the comparison of the formulas for symmetric and antisymmetric MSW that the lifetime of the former at the smallest kd is shorter than that of the latter whereas the lifetimes of both are of the same order when kd have maximum values.

It is worthwhile to emphasize two facts:

(1) The lifetime of an MSW in all above cases is virtually dependent on the wavelength (wave vector k). MSW experiments are usually carried out at a fixed frequency ω, with the magnetic field being the parameter to measure. Inverting the dispersion law, it is easy to express k through H (recall that $\omega_0 = gH$ and $H \leqslant \omega/g$) which allows the dependence of MSW lifetime on the magnetic field to be found.

(2) The most interesting of the above findings are the existence of threshold values of the wave vector $k = k_{\text{th}}$ (hence, magnetic field $H = H_{\text{th}} < \omega/g$), quasiresonance peaks of the k (H)-dependence of the lifetime, and certainly inflections at $H = H_{\text{th}}$ when one of the scattering mechanisms proves to be 'out of the game'.

Taking into account the MSW–phonon interaction allows effects of two-magnon processes to be estimated not only on MSW attenuation but also on the suppression of phonons. If phonon energy is smaller than the double magnon energy, phonon decomposition into two MSWs is forbidden, and two-magnon processes lead to the exponentially long (in terms of temperature) phonon lifetime.

Let us consider a phonon with very small wave vector $f \to 0$ and homogeneous in z, taking into account its interaction with a symmetric MSW. Merging of two MSWs and the resulting creation of a phonon are impossible at $f \to 0$ (this may occur only if $sf > 2\omega_0$).

The conservation laws for phonon absorption and emission by an MSW in the case of $n - n' = 0$ are fulfilled only at $\alpha > 1$, and the inverse lifetime of a phonon is

$$\frac{1}{\tau_{\text{ph}}^{00}(f)} = \frac{3!}{2^8\pi^3} \frac{\alpha}{(\alpha^2-1)^{1/3}} \frac{\gamma^2 T}{\rho d^4 \omega_M} \left(\frac{T}{\hbar\omega_M}\right)^2 \exp\left(-\frac{\hbar\omega_0}{T}\right) f\,. \tag{17.14}$$

When $\alpha - 1 \ll 1$, more exact formulas are needed for inverse lifetime computations, in order to eliminate the divergence (17.14) at $\alpha \to 1$:

$$\frac{1}{\tau_{\text{ph}}^{00}(f)} = \frac{(\alpha-1)^{1/2}}{2^{12}\cdot 2^{1/3}\pi^3} \frac{\omega_M}{\omega_0(\omega_0+\omega_M/2)} \frac{(\hbar\omega_M)^2}{\rho d^4 T} \times \exp\left(-\frac{\hbar\omega_0}{T}\right) f\,, \qquad \alpha - 1 \ll fd\,. \tag{17.14$'$}$$

The appearance of the small factor $(\alpha-1)^{1/2}$ is compensated by the large one $(\hbar\omega_{\rm M}/T)^4$ (recall that $T \ll \hbar\omega_{\rm M}, \hbar\omega_0$).

The conservation laws with $n=0$, $n'>0$ are satisfied at any α.

Following summation over $n'>0$, the corresponding contribution to the inverse decay time of phonon is

$$\frac{1}{\tau_{\rm ph}^{0,n'>0}(f)} = \frac{\gamma^2\hbar^2 s^4 f^6}{2^7\pi^4 d^3\rho\omega_{\rm M}\omega_0^2 T}\exp\left(-\frac{\hbar\omega_0}{T}\right). \tag{17.15}$$

The contribution of the terms with $n>0$, $n'>0$ is significantly smaller and may be neglected, along with the contribution from the interaction with an antisymmetric MSW.

To conclude, the inverse lifetime of a phonon in plates with $\alpha<1$ is defined by formula (17.15) and at $(\alpha-1)\gg fd$, by formulas (17.14) and (17.15); note that only formula (17.15) may be used if $T \ll \hbar\Omega(f)(fd)^{1/4}$ and only (17.14) if $T \gg \hbar\Omega(f)(fd)^{1/4}$.

18. Antiresonance. Selective transmittance of ferromagnetic metal plates

Both theorists and experimenters have long paid much attention to the properties of metal plates. It is hardly possible to list all relevant problems attacked by physicists as well as those awaiting solution. Naturally, one focus of interests is metal plate properties responsible for the ability of electrons to reach one side of the plate after starting from another practically without collisions, that is when $ld \gg 1$, where l is the average free path and d is the plate thickness.

In ferromagnetic and other magnetic plates, the magnetic excitation spectrum is significantly different from that in an infinite magnet. If electron and magnon systems are supposed to be quasi-independent, i.e. coupled only due to effects of retardation (see Section 11), conduction electrons must manifest themselves in renormalization of the dispersion law for surface magnetic polaritons (Section 13) responsible for their decay associated with electron dissipation.

In the transition from magnetic polaritons to MSWs (Section 14), it is possible to retain the terms involving conductance, to be able to calculate electron damping coefficient for MSWs. To our knowledge, this procedure has never been examined in detail. It should be mentioned that this problem is very complicated if one considers the dependence of the electron subsystem on collisions not only with plate bounds but also with impurities, phonons, and magnons, along with the influence exerted by the average magnetic field $\mathbf{B}=\mathbf{H}+4\pi\mathbf{M}$ on conduction electron dynamics. True, the problem appears to lend itself to solution on a step-by-step basis since it is difficult to perceive that all factors responsible for its complicated nature are equally important at one time.

This section deals with the modification of electromagnetic properties of metals by the magnetic subsystem or, to be precise, by dispersion of magnetic permeability. To better illustrate the role of magnetic permeability dispersion, we disregard effects of the limited sample size and the magnetic field on conduction electrons. This means that the following conventional relation between the electric field strength $\mathbf{E}$ and current density $\mathbf{j}$ (Ohm's law) holds:

$$\mathbf{j}=\sigma\mathbf{E}, \tag{18.1}$$

where σ is specific metal conductivity. When solving problems of electrodynamics, it is sometimes convenient to use the inverse tensor of magnetic permeability $\hat\nu \equiv \hat\mu^{-1}$ rather than the tensor $\hat\mu$ itself. Then, the magnetic material equation has the form

$$h_\alpha = \nu_{\alpha\beta} b_\beta, \qquad \alpha,\beta = x,y. \tag{18.2}$$

Axis z coincides with the direction of the wave vector $\mathbf{k}$, and the two-dimensional nature of (17.2) is the result of $b_z\equiv 0$ due to the absence of magnetic charges (the induction vector $\mathbf{b}$ is always transverse, $\operatorname{div}\mathbf{b}=0$). For a ferromagnet under the same conditions as before,

$$\hat\nu = \begin{pmatrix} \dfrac{\omega^2-\omega_{\rm AM}(\omega_{\rm AM}-4\pi gM\cos^2\theta)}{\omega^2-\omega_{\rm AM}^2} & \mathrm{i}\,\dfrac{4\pi\omega gM\cos\theta}{\omega^2-\omega_{\rm AM}^2} \\ -\mathrm{i}\,\dfrac{4\pi\omega gM\cos\theta}{\omega^2-\omega_{\rm AM}^2} & \dfrac{\omega^2-\omega_{\rm AM}\omega_0}{\omega^2-\omega_{\rm AM}^2}\end{pmatrix}, \tag{18.3}$$

where

$$\omega_{\rm AM} = \mathrm{g}B = g(H+4\pi M) \tag{18.4}$$

is the antiresonant frequency (see the Introduction) and θ is the angle between z-axis (or $\mathbf{k}$) and $\mathbf{H}$ ($\mathbf{H}$ also includes the anisotropy field). Physical nature of antiresonance is apparent from formula (18.4), where $\omega_{\rm AM}$ is the free precession frequency of vector $\mathbf{h}$ (at $\mathbf{b}=0$). It is noteworthy that the antiresonant frequency, unlike resonant one, does not depend on the direction of wave propagation (on angle θ). This may be due to the absence of effect of the demagnetizing factor at $\mathbf{b}\equiv 0$.

The matrix $\hat\nu$ for elliptically polarized waves is diagonalized:

$$h_\pm = \nu_\pm(\omega) b_\pm, \qquad h_\pm = h_x \pm \mathrm{i}p_\pm h_y,$$

$$p_\pm = \frac{\omega_{\rm AM}\sin^2\theta \pm \sqrt{\omega_{\rm AM}^2\sin^4\theta + 4\omega^2\cos^2\theta}}{2\omega\cos\theta},$$

$$\nu_\pm = \Big\{\omega^2-\omega_{\rm AM}^2 + 2\pi gM\Big[(1+\cos^2\theta)\omega_{\rm AM} \pm \sqrt{(1+\cos^2\theta)^2\omega_{\rm AM}^2 + 4(\omega^2-\omega_{\rm AM}^2)\cos^2\theta}\Big]\Big\} \times (\omega^2-\omega_{\rm AM}^2)^{-1}: \tag{18.5}$$

The component ν_- remains finite in antiresonance (at $\omega=\omega_{\rm AM}$) while ν_+ is converted into infinity. This allows polarization of the freely precessing field $\mathbf{h}$ to be determined ($p_+ = \cos^{-1}\theta$ at $\omega=\omega_{\rm AM}$).

Naturally, we shall be further interested in a wave with polarization p_+ for which values of ν_+ at $\theta=0$ and $\theta=\pi/2$ can be written down:

$$\nu_+ = \begin{cases} \dfrac{\omega-\omega_0}{\omega-\omega_{\rm AM}}, & \theta=0, \\[2ex] \dfrac{\omega^2-\omega_0\omega_{\rm AM}}{\omega^2-\omega_{\rm AM}^2}, & \theta=\dfrac{\pi}{2}. \end{cases} \tag{18.6}$$

At an arbitrary angle θ near resonance,

$$v_+ \simeq \frac{2\pi g M(1+\cos^2\theta)}{\omega - \omega_{AM}}\,, \qquad \omega \simeq \omega_{AM}\,. \tag{18.7}$$

The dispersion equation, a corollary of Maxwell's equations with material equations (18.1) and (18.2) [see also (18.5)], allows the wave vector of a '+'-wave to be determined:

$$k^2 = \frac{4\pi i \omega\sigma}{c^2 v_+(\omega)}\,. \tag{18.8}$$

Near antiresonance,

$$k^2 = \frac{2\pi i \omega_{AM}\sigma(\omega-\omega_{AM})}{gM(1+\cos^2\theta)}\,. \tag{18.7$'$}$$

A few words are in order about the surface impedance of a ferromagnetic halfspace. As is known, the surface impedance ζ in the case of optically dense medium is a sufficiently complete electromagnetic characteristic of the surface. This means that ζ is unrelated to the shape (direction and polarization) of the wave incident onto the surface. Therefore, first, it can be computed based on the simplest assumption of normal wave incidence and, second, may be used to formulate effective boundary conditions (Leontovich – Fock conditions) for the solution of external (with respect to an optically dense body) problems of electrodynamics. The situation is more complicated in the case of a magnet. It has been noted in the Introduction that the refractive index $n = (\varepsilon\mu)^{1/2}$ and impedance $\zeta = (\mu/\varepsilon)^{1/2}$ are not related by the simple expression† ($\zeta \neq 1/n$) and the smallness of the impedance ($|\zeta| \ll 1$) does not automatically imply high optical density (not always $|n| \gg 1$). This accounts for the dependence of the impedance on the wave shape. Indeed, in the case of oblique incidence, the impedance is either

$$\zeta_1 = \frac{\sqrt{\varepsilon\mu - \sin^2\theta}}{\varepsilon} \qquad \text{or} \qquad \zeta_2 = \frac{\mu}{\sqrt{\varepsilon\mu - \sin^2\theta}}\,.$$

Here, θ is the angle between $\mathbf{k}$ and the normal to the surface while subscripts 1 and 2 correspond to polarization of the electric vector at the plane of incidence and in the perpendicular plane. It is clear that ζ_1 and ζ_2 are significantly different even at $|\varepsilon| \gg 1$, and one cannot neglect θ-dependence near antiresonance.

Considering normal incidence of an electromagnet wave with the polarization p_+ onto a metallic halfspace, it is easy to obtain

$$\zeta_+ = \frac{\omega c k}{4\pi i\sigma}\,. \tag{18.9}$$

The wave vector k is defined by (18.7) and (18.7$'$), i.e.

$$\zeta_+ = \sqrt{\frac{\omega}{4\pi i\sigma v_+(\omega)}}\,. \tag{18.10}$$

† For simplicity, this paragraph focuses on an isotropic magnet with dielectric permittivity and magnetic permeability ε and μ respectively (as in the Introduction).

Near antiresonance,

$$\zeta_+ \simeq \sqrt{\frac{\omega_{AM}(\omega-\omega_{AM})}{4\pi i\sigma \cdot 2\pi g M(1+\cos^2\theta)}}\,. \tag{18.11}$$

Thus, the impedance vanishes in the root-like manner at $\omega = \omega_{AM}$. The fact that the spatial dispersion of magnetic permeability necessary for the estimation of the impedance singularity in antiresonance has to be taken into account indicates that the impedance ζ_+ has a small addition near the antiresonance because a part of energy of the electromagnetic wave is carried away by a magnon. By the order of magnitude,

$$\zeta_{ex} \sim \frac{\omega_{AM}}{c k_{ex}}\,, \qquad k_{ex} \sim \frac{1}{a}\left(\frac{\omega_{AM}}{\omega_{ex}}\right)^{1/2}.$$

Strictly speaking, the singularity at $\omega = \omega_{AM}$ should be separated by examining the derivative of $\zeta_+ = \zeta_+(\omega)$ rather than the dependence itself because this derivative goes into infinity at $\omega = \omega_{AM}$ as $(\omega - \omega_{AM})^{-1/2}$.

Finally, let us consider the passage of an electromagnetic wave through a plate as thick as $2d$ (it occupies the layer $|z| \leqslant d$). The observation of selective transmittance of a thick metal plate (far from the antiresonance $\delta \ll d$) led to the discovery of antiresonance in 1969 [30]. Let us suppose that an incident wave has the polarization p_+ (it is justified because the plate shows no anomalous transmittance for a wave with the polarization p_-). If a wave with the unit amplitude falls onto a dielectric plate, the amplitude of the wave β that passed through the plate is given by the following expression:

$$\beta = \frac{i\zeta\exp(-2ik_0 d)}{\left[\cos(kd) - i\zeta\sin(kd)\right]\left[\sin(kd) + i\zeta\cos(kd)\right]}\,, \qquad k_0 = \frac{\omega}{c}\,,$$

$$\zeta = \sqrt{\frac{\mu}{\varepsilon}}, \qquad k = \frac{\omega}{c}\sqrt{\varepsilon\mu}\,, \qquad \mu \equiv \frac{1}{v_+}\,. \tag{18.12}$$

For the dielectric, both ε and μ are real quantities. Transmittivity is

$$|\beta|^2 = \frac{\mu}{\varepsilon}\left[\frac{\mu}{\varepsilon} + \frac{1}{4}\left(1 - \frac{\mu}{\varepsilon}\right)^2 \sin^2(kd)\right]^{-1} \tag{18.13}$$

at $\mu > 0$ or

$$|\beta|^2 = \frac{\mu}{\varepsilon}\left[\frac{\mu}{\varepsilon} + \frac{1}{4}\left(1 - \frac{\mu}{\varepsilon}\right)^2 \sinh^2(kd)\right]^{-1}$$

at $\mu < 0$. In antiresonance ($\omega = \omega_{AM}$, $\mu = 0$),

$$|\beta|^2 = \left[1 + \frac{\omega^2\varepsilon^2 d^2}{c^2}\right]^{-1}. \tag{18.14}$$

For a metal, $\varepsilon = 4\pi i\sigma/\omega$, and if ζ is neglected as compared with unity, we have

$$|\beta|^2 = \frac{c^2}{16\pi^2\sigma^2 d^2}\,\frac{2x^2}{\sinh^2 x + \sin^2 x}\,,$$

$$x = \left|\frac{2d}{\delta}\right|, \qquad \delta = \frac{c}{\sqrt{2\pi\sigma\omega\mu}}\,. \tag{18.15}$$

Maximum transmittance

$$|\beta|^2_{\max} = \frac{c^2}{16\pi^2\sigma^2 d^2} = \left(\frac{d_\sigma}{d}\right)^2, \quad d_\sigma = \frac{c}{4\pi\sigma} \tag{18.16}$$

is not great since $d_\sigma = \delta_L/\omega_L\tau$ is significantly smaller than the interatomic distance ($\delta_L = c/\omega_L$, a $\omega_L^2 = 4\pi ne^2/m^*$, $\tau = l/v_F$). However, transmittivity is many orders of magnitude higher than that for ordinary plates (non-magnetic or magnetic at $\omega \neq \omega_{AM}$) due to the non-exponential thickness dependence. The function of x describes the shape of the transmittance line which narrows as the plate becomes thicker.

Magnetic dissipation was disregarded in this section. Naturally, all the above formulas are valid with good accuracy if $\mu'' \ll 1$ at $\omega \sim \omega_{AM}$. This means that antiresonant frequency must be significantly different from resonant one. In an antiferromagnet of the EA type at $H < H_{SF}$, zero and infinite values of μ_{eff} are very close, and antiresonance is difficult to observe based on selective transmittance of the plates.

Selective transmittance of ferromagnetic plates was discovered 25 years ago (predicted 10 years earlier [31]) but failed to be used for practical purposes (notwithstanding its obvious technological implications). Nor did it become an additional tool for the investigation of magnetic order although it may serve for exciting electromagnetic field in a plate with spatial distribution patterns totally different from those in other cases (e.g. in resonance).

19. Conclusions

We have already mentioned that the concept of magnons (spin waves) first formulated more than half a century ago has not since undergone substantial modification. We are far from the ambitious thought that the present review can drastically change the situation. Yet, we hope that it illustrates a variety of interesting problems in the framework of the magnon concept which deserve attention of theoretical physicists, to say nothing about many intricate properties of magnets that remain to be elucidated.

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A note by M I Kaganov: This review is published in the special issue of *Uspekhi Fizicheskikh Nauk* (*Physics Uspekhi*) celebrating the 80th anniversary of I M Lifshitz' birth. Therefore, a few more lines will be in order.

We largely cited in the review recent papers and other publications which we were unable to discuss with I M Lifshitz. However, he had greatly influenced the development of the macroscopic approach underlying this review. It is tempting to believe that the reader was sensible to the spirit of I M Lifshitz' school. When working on this review, it often grieved me to think that many problems and ideas might have been discussed with I M Lifshitz...

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