# Collectively fluctuating assets in the presence of arbitrage opportunities, and option pricing 

A N Adamchuk, S E Esipov

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#### Abstract

Methods of functional analysis are applied to describe collectively fluctuating default-free pure discount bonds subject to trading-related noise which generates arbitrage opportunities. Two key elements of the model are: (i) the naturally incorporated fixed bond price at maturity which is achieved by making use of only those fluctuating paths of price motion which terminate at a specified final condition, and (ii) the most attractive arbitrage opportunities between bonds with close maturities, with modeled a local linear approximation. The model can be written in different closed forms as a stochastic partial differential equation. The functional Black - Scholes equation for contingent claims is derived, and a connection with the conventional methods of option valuation is indicated.


## 1. Introduction. Phenomenon of correlated fluctuations. A self-consistent description

Close inspection of the daily, monthly, and annual variations of bond prices reveals different evolving correlations. The conventional method of presenting this information is the term structure or the yield curve, and forward interest rates. Although the collective behavior of (bond) markets is evident, it is difficult to model, partially due to the large volumes of

[^0]statistical information needed to determine multiple effects of cross-correlations. Variations of bond prices lead to arbitrage opportunities. However small these are or are perceived to be, it is of interest to model them explicitly. The presence of arbitrage allows analysis of the region of validity of the wellestablished no-arbitrage [1, 2] or equilibrium models, and provides an approach to analyzing markets where arbitrage is expected.

Equilibrium models of the term structure [3-5] admit that the interest rate may deviate from some prescribed 'mean' value, and indicate that efficient markets have mechanisms of relaxation towards the mean. Since interest rates, in general, are not more tradeable assets, the connection between the arbitrage opportunities and the mechanism of relaxation is implicit. On the other hand, linear relaxation is the natural way to model the evolution of the moderate changes exhibited by term structure and bond prices. We intend to use the same type of linear relaxation below to model arbitrage.

Historically, financial models have incorporated more and more details of real markets explicitly, representing the remaining factors as an 'effective medium' which provides driving noise or specified mean values. The seminal model of Black and Scholes (Black and Scholes, 1973) divides the market into two sub-systems: a single asset, and the remaining market. The market provides the mean interest rate and stochastic noise with a specified volatility, while the asset does not influence the market at all [6]. Two, three, and more assets are usually considered along the same lines, and their mutual influence is modeled by pairwise coefficients of crosscorrelation [7]. New quality comes in focus when the number of assets becomes large or their combined value becomes comparable with the total trading volumes, as occurs place for bond markets.

In the Heath-Jarrow-Morton model [1] the market provides families of drift functions and volatilities for the
evolution of the term structure. Here, the 'effective medium' approximation is employed at full strength. Indeed, we know that for an arbitrage-free market the forward rate, $f(t, T)$, is a complete description, and therefore the mentioned drift functions and volatilities must somehow depend on $f(t, T)$. This self-consistency is replaced in the model by having the drift and volatility functions measured from real market data.

In this article we study a different model of bond prices and term structure, which has the advantage of being complete, i.e. it describes the entire bond market or the fluctuations of the entire yield curve starting from a specified initial condition. In the simplest version there are just two parameters in the model (the volatility, $\sigma$, and a measure of the time needed for arbitrage, $\tau$ ). No 'external' interest rates or mean values are employed. Different terms in the stochastic dynamics of the bond prices in our model are related to the 'Brownian-bridge' modification [8] of the Black - Scholes log-normal process $\dagger$ and the linear arbitrage description of the term structure $[4,5]$.

For our purposes, it is necessary to review a few mathematical techniques which will subsequently allow us to model term structures. This is performed step by step below. In the next section we continue to discuss the difference between stochastic and deterministic interactions of different market prices. In Section 3 we then revisit the linear relaxation approach to arbitrage and use it in the simplest possible complete system of two bonds serving as an illustration of a complete system with arbitrage and market makers. The mathematical techniques needed are given in Section 4 which is devoted to the analysis of the 'Brownianbridge' model, and its connection to conditional probabilities, Fokker-Planck equations, path integrals, and the generation of steady-state distributions, all exposing the Ball-Torous drift term from complementary angles. The Ball-Torous model is known to have divergent behavior at maturity making it problematic to perform changes of variables to the equivalent martingale measure [9]. This issue is discussed at the end of Section 4 where we argue that in the world with one bond the price of this bond is not a martingale.

[^1]Having developed the necessary methods we model the many-bond dynamics by relaxational arbitrage terms in Section 5, and derive a stochastic partial differential equation (SPDE) for the bond market together with its pathintegral counterpart and other related processes. The Black Scholes strategy in the presence of arbitrage opportunities is outlined later in Section 5 where the connection with the parameters used in conventional models is established, and a few analytical solutions are presented.

## 2. Difference between stochastic correlations and deterministic interactions

Economical and financial markets are systems with coexisting stochastic and deterministic (cor)relations between prices, indices, rates, etc. In general, correlated events, may be subject to either deterministic or stochastic modeling, the latter by using observable or phenomenological coefficients of cross-correlation. This property is shared by financial models. Consider the case of the Vasicek model of the term structure:

$$
\begin{equation*}
\mathrm{d} r=\frac{1}{\tau}(b-r) \mathrm{d} t+\sigma \mathrm{d} \Xi \tag{1}
\end{equation*}
$$

where $r(t)$ is the short-term risk-free interest rate at time $t, b$ is a constant which models the mean reversion the rate $\tau$, and $\sigma$ is the interest rate volatility [5]. In this model the tendency of the interest rate to correlate with the mean value $b$ is modeled by a linear relaxational term $(b-r) / \tau$ which is fully deterministic. The 'remaining' market influence is taken into account as a stochastic (Gaussian) influence, $\sigma \xi$, with $\mathrm{d} \Xi=\xi \mathrm{d} t$, similar to the noise acting on the asset log-price in the Black-Scholes equation. The distribution of the noise term $\xi$ is assumed to be Gaussian,

$$
\begin{equation*}
P(\Xi)=N \exp \left(-\frac{1}{2} \int \xi^{2} \mathrm{~d} t\right) \tag{2}
\end{equation*}
$$

where the integration is performed over the entire time interval, and $N$ is the normalization constant. Alternatively, first and second order correlators can be specified for the Gaussian process $\xi(t):\langle\xi(t)\rangle=0,\left\langle\xi\left(t_{1}\right) \xi\left(t_{2}\right)\right\rangle=\delta\left(t_{1}-t_{2}\right)$, where $\delta(x)$ is the delta-function.

## Cross-correlation

In cases where the relations between fluctuating quantities are complex, the other alternative for modeling statistical relations is frequently used. The coefficient of cross-correlation, $\rho$, is a good description when we do not intend to model the underlying mechanism of cross-influence of correlated observables. The classical example of this approach is the concept of the diversified portfolio by Markowitz [10] where the correlation, $\beta$, represents the interaction between the market and a given asset.

Particular examples include Black - Scholes type models for two or more underlying variables which are significantly correlated while, again, we do not attempt to model the cause of this correlation. For example, in the cross-currency futures and options the foreign index and exchange rate follow two different stochastic equations with log-normal noise, while their clearly correlated evolution is modeled by a coefficient of instantaneous correlation ( $\rho$ ) [7]. The same approach is used for relating the bond underlying a bond option and another bond expiring simultaneously with the option [6], and many others.

## 3. Arbitrage opportunities - the mechanism of relaxation

The strategies used by financial institutions are becoming more and more complicated. The time needed to adopt new models, the time needed for computation and to react to exotic market conditions, and the barriers provided by transaction costs and taxes make the arbitrage observable. The more difficult it becomes to take advantage of modern observable arbitrage, the more significant is the contribution of collected historical data with arbitrage opportunities which are used in transactions based on accepted financial models. Many models however are intrinsically arbitrage-free, and the perspective of combining them with arbitrage containing data is undesirable.

### 3.1 Taylor series, linear arbitrage, and arbitrage time

If it were possible to observe the influence of arbitrageurs it would be found that, once created, arbitrage opportunities diminish over time and cease to exist. This process is clearly deterministic, and there is a characteristic time associated with it. One can model such a process by the 'first term' in the Taylor expansion of some unknown function of current price, $B$, and no-arbitrage price, $\bar{B}$, namely by $-(B-\bar{B}) / \tau$ where $\tau$ is a characteristic time. Linear relaxation has been extensively used in financial models [4, 5, 11, 12].

### 3.2 Two correlated assets. Relaxational arguments

Suppose that there are only two trading boards in the world, each engaged in trading identical assets, available at prices, $B_{1}$ and $B_{2}$ at each board, respectively. Arbitrageurs willing to act when $B_{1} \neq B_{2}$ face a finite time, $\tau$, needed to transmit information, directions, and funds between the boards. Thus, the appropriate model for the asset prices would be

$$
\begin{align*}
& \mathrm{d} B_{1}=\left[\mu B_{1}+\frac{1}{\tau_{1}}\left(B_{2}-B_{1}\right)\right] \mathrm{d} t+\sigma_{1} B_{1} \mathrm{~d} \Xi_{1},  \tag{3}\\
& \mathrm{~d} B_{2}=\left[\mu B_{2}+\frac{1}{\tau_{2}}\left(B_{1}-B_{2}\right)\right] \mathrm{d} t+\sigma_{2} B_{2} \mathrm{~d} \Xi_{2},
\end{align*}
$$

where $\mu$ is the common interest rate, $\sigma_{1,2}$ are the price volatilities, and the deterministic relaxational terms model arbitrage opportunities. The logarithms of the prices, $A_{1,2}=\ln B_{1,2}$ satisfy the equations

$$
\begin{aligned}
& d A_{1}=\left\{\mu-\frac{\sigma_{1}^{2}}{2}+\frac{1}{\tau_{1}}\left[\exp \left(A_{2}-A_{1}\right)-1\right]\right\} \mathrm{d} t+\sigma_{1} \mathrm{~d} \Xi_{1}, \\
& \mathrm{~d} A_{2}=\left\{\mu-\frac{\sigma_{2}^{2}}{2}+\frac{1}{\tau_{2}}\left[\exp \left(A_{1}-A_{2}\right)-1\right]\right\} \mathrm{d} t+\sigma_{2} \mathrm{~d} \Xi_{2}
\end{aligned}
$$

which can be linearized for small enough differences $\left|A_{1}-A_{2}\right|$. The system of two assets is presented here only as an illustration of arbitrage relaxation. Our subsequent analysis will be performed with many assets (bonds) at once.

## 4. Path integrals, action, and the 'Brownian-bridge' model

In this section the 'Brownian-bridge' model [8] is revisited, thus explaining our motivation to incorporate the 'Brownianbridge' drift term in our subsequent study of collectively fluctuating bonds. We also indicate a connection to the method of path integrals [13]; their mathematical counter-
parts and examples of financial applications may be found in Sections 21, 22 of Ref. [14]. When computing probabilities and expectation values with stochastic models it is sometimes convenient to present the answer as an integral or sum taken over possible 'trajectories', $B(t)$, followed by the asset price, or over trajectories $A(t)$, followed by its logarithm, log-price. In the method of path integrals [13] one considers all possible realizations of the asset price dependence on time. The underlying space is that of (possibly discontinuous) singlevalued functions of time. The probability of each realization is given by $P[A(t)]=N \exp (-S)$, where $S$ measures the contribution of a given trajectory, $A(t)$, and $N$ is the normalization prefactor [13]. The functional $S[A(t)]$ is traditionally termed the action.

The stochastic differential equation with additive noise

$$
\begin{equation*}
\mathrm{d} A=-v(A, t) \mathrm{d} t+\sigma \mathrm{d} \Xi \tag{5}
\end{equation*}
$$

considered in the time interval $t_{1} \leqslant t \leqslant t_{2}$ generates different realizations, $A(t)$, of the asset log-price as prescribed by the drift term, $v$, and the noise term. The probability of each realization is given by [15]

$$
\begin{align*}
P[A(t)] & =N \exp (-S) \\
& =N \exp \left[-\frac{1}{2 \sigma^{2}} \int_{t_{1}}^{t_{2}} \mathrm{~d} t\left(\frac{\mathrm{~d} A}{\mathrm{~d} t}+v\right)^{2}-\frac{\sigma^{2}}{2} \frac{\partial v}{\partial A}\right] \tag{6}
\end{align*}
$$

Some results of financial models with Wiener (Brownian) noise can be rewritten by using path integrals. As an example we show how to derive the results of the Ball and Torous model [8]. One of the problems arising with evaluating bonds is the deterministic final condition for the bond price, $B(T)=B_{T}$ at maturity, $T$. In order to modify the BlackScholes ansatz in view of this condition, a particular stochastic process, the so-called 'Brownian-bridge' process was considered by Ball and Torous

$$
\begin{equation*}
\mathrm{d} B=\left(\frac{B}{T-t} \ln \frac{B_{T}}{B}\right) \mathrm{d} t+\sigma B \mathrm{~d} \Xi . \tag{7}
\end{equation*}
$$

The divergent denominator of the drift term, $T-t$, represents a 'restoring force' which guarantees the final price to be a root of the numerator, $B(T)=B_{T}$. It is assumed that the interest rate is given by $\mu=\left[\ln B_{T}-\ln B_{0}\right] / T$. We shall postpone the discussion of equivalent martingale measures till the end of this section.

The path integral approach provides a technically independent test of the results by Ball and Torous, and illuminates the meaning of their designated drift term. One considers all possible trajectories of the price, $B(t)$, which begin at the point $B_{0}$ and terminate at the point $B_{T}$. Examples of such trajectories are given in Fig. 1.

Consider a log-normal trajectory, $B(t)$ satisfying an ordinary stochastic differential equation,

$$
\begin{equation*}
\mathrm{d} A=\left(\mu-\frac{\sigma^{2}}{2}\right) \mathrm{d} t+\sigma \mathrm{d} \Xi \tag{8}
\end{equation*}
$$

$A=\ln B$, which is defined in the time interval $t_{1} \leqslant t \leqslant t_{2}$. The probability of a particular realization $A(t)$ is given by

$$
\begin{equation*}
P[A(t)]=N \exp \left[-\frac{1}{2 \sigma^{2}} \int_{t_{1}}^{t_{2}} \mathrm{~d} t\left(\frac{\mathrm{~d} A}{\mathrm{~d} t}-\mu+\frac{\sigma^{2}}{2}\right)^{2}\right] . \tag{9}
\end{equation*}
$$



Figure 1. Three possible price - time trajectories connecting present bond $\log$-price $A_{0}$ and log-price, $A_{T}$, at maturity, $T$. Curves are obtained by Monte Carlo simulations of Eqn (21). Parameters: $t$ is discretized in $N=512$ points; $10^{4}$ steps were made along the artificial time, $\theta ; \sigma^{2}=10.0$; $A_{0}=0 ; A_{T}=1 ; T=1$ (arbitrary units).

The path integral of expression (9) over properly constrained trajectories can be computed explicitly, for example, by dividing the trajectory into a large number of time intervals, and computing the individual integrals, and later taking the limit of infinitely many time intervals [13]. With the specified initial and final conditions, $A\left(t_{1}\right)=A_{1}, A\left(t_{2}\right)=A_{2}$, one arrives at the Gaussian distribution

$$
\begin{align*}
& P_{\mathrm{G}}\left(A_{1}, t_{1} ; A_{2}, t_{2}\right) \\
& =\int_{A\left(t_{1}\right)=A_{1}}^{A\left(t_{2}\right)=A_{2}} \mathcal{D} A(t) \exp \left[-\frac{1}{2 \sigma^{2}} \int_{t_{1}}^{t_{2}} \mathrm{~d} t\left(\frac{\mathrm{~d} A}{\mathrm{~d} t}-\mu+\frac{\sigma^{2}}{2}\right)^{2}\right] \\
& =\frac{1}{\left[2 \pi \sigma^{2}\left(t_{2}-t_{1}\right)\right]^{1 / 2}} \\
& \times \exp \left\{-\frac{\left[A_{2}-A_{1}-\left(\mu-\sigma^{2} / 2\right)\left(t_{2}-t_{1}\right)\right]^{2}}{2 \sigma^{2}\left(t_{2}-t_{1}\right)}\right\} \tag{10}
\end{align*}
$$

as it should be. This is the usual log-normal distribution for $B$. For the constraint conditions of interest, when the initial, $B_{0}$, intermediate, $B$, and final, $B_{T}$, conditions are specified, the calculation of the path integral gives

$$
\begin{align*}
& P_{\mathrm{C}}\left(A_{0}, 0 ; A, t ; A_{T}, T\right)=\frac{1}{\left[2 \pi \sigma^{2} t(1-t / T)\right]^{1 / 2}} \\
& \quad \times \exp \left\{-\frac{\left[A-A_{0}-\left(A_{T}-A_{0}\right)(t / T)\right]^{2}}{2 \sigma^{2} t(1-t / T)}\right\} . \tag{11}
\end{align*}
$$

It can be explicitly verified that the probability $P_{\mathrm{C}}$ satisfies a partial differential equation,

$$
\begin{equation*}
\frac{\partial P_{\mathrm{C}}}{\partial t}=\frac{\partial}{\partial A}\left(\frac{A_{T}-A}{T-t} P_{\mathrm{C}}\right)+\frac{\sigma^{2}}{2} \frac{\partial^{2} P_{\mathrm{C}}}{\partial A^{2}}, \tag{12}
\end{equation*}
$$

in which one can recognize the Fokker - Planck equation for the stochastic process (7). Thus, the Ball-Torous drift is exactly the drift provided by the final constraint (boundary condition), $B(T)=B_{T}$ on any Brownian trajectory, and no artificial drift is employed. Moreover, as can be immediately verified by simple algebra, $P_{\mathrm{C}}$ is just the conditional probability

$$
\begin{equation*}
P_{\mathrm{C}}\left(A_{0}, 0 ; A, t ; A_{T}, T\right)=\frac{P_{\mathrm{G}}\left(A_{0}, 0 ; A, t\right) P_{\mathrm{G}}\left(A, t ; A_{T}, T\right)}{P_{\mathrm{G}}\left(A_{0}, 0 ; A_{T}, T\right)} . \tag{13}
\end{equation*}
$$

Although the path integral may seem excessive at this stage, we believe it provides a useful insight and is generalizable to the case of collectively fluctuating assets, while Eqns (7) and (12) are not.

### 4.1 Forward and backward Fokker - Planck equations

An alternative method to derive Eqn (12) stems from relation (13). Given that the relation is valid for any time, $t$, one can consider the change caused by an infinitesimal increment, $\mathrm{d} t$. In a slightly more general case that we shall explore [c.f. Eqn (5)], the unconditional probabilities entering the right-hand side of Eqn (13) obey forward (subscript $f$ and the plus sign below) and backward (subscript $b$ and the minus sign) Fokker-Planck equations,

$$
\begin{equation*}
\pm \frac{\partial P_{\mathrm{f}, \mathrm{~b}}}{\partial t}=\frac{\partial\left(v P_{\mathrm{f}, \mathrm{~b}}\right)}{\partial A}+\frac{\sigma^{2}}{2} \frac{\partial^{2} P_{\mathrm{f}, \mathrm{~b}}}{\partial A^{2}}, \tag{14}
\end{equation*}
$$

where $v$ is the drift rate ( $v=-\mu+\sigma^{2} / 2$ for the Gaussian case considered above). Forward and backward solutions are specified by the initial, $P_{\mathrm{f}}(A, 0)=\delta\left(A-A_{0}\right)$, and final, $P_{\mathrm{b}}(A, T)=\delta\left(A-A_{T}\right)$, conditions, respectively. Differentiating the relation $P_{\mathrm{C}}=P_{\mathrm{f}} P_{\mathrm{b}}$ with respect to time and using (14) one obtains an expression

$$
\begin{equation*}
\frac{\partial P_{\mathrm{C}}}{\partial t}=v\left(P_{\mathrm{b}} \frac{\partial P_{\mathrm{f}}}{\partial A}-P_{\mathrm{f}} \frac{\partial P_{\mathrm{b}}}{\partial A}\right)+\frac{\sigma^{2}}{2}\left(\frac{\partial^{2} P_{\mathrm{f}}}{\partial A^{2}}-\frac{\partial^{2} P_{\mathrm{b}}}{\partial A^{2}}\right), \tag{15}
\end{equation*}
$$

to be compared with the expected Fokker-Planck form

$$
\begin{equation*}
\frac{\partial P_{\mathrm{C}}}{\partial t}=\frac{\partial\left(V P_{\mathrm{C}}\right)}{\partial A}+\frac{\sigma^{2}}{2} \frac{\partial^{2} P_{\mathrm{C}}}{\partial A^{2}}, \tag{16}
\end{equation*}
$$

with an unknown drift rate $V$. Comparison of the right-hand sides of Eqns (15) and (16) results in an equation for $V$ :

$$
\begin{equation*}
\frac{\partial}{\partial A}\left(V P_{\mathrm{f}} P_{\mathrm{b}}+\sigma^{2} P_{\mathrm{f}} \frac{\partial}{\partial A}\right)=v\left(P_{\mathrm{b}} \frac{\partial P_{\mathrm{f}}}{\partial A}-P_{\mathrm{f}} \frac{\partial P_{\mathrm{b}}}{\partial A}\right) . \tag{17}
\end{equation*}
$$

Integrating over the interval $\left(A, A_{T}\right)$ for a fixed $t$, and assuming $V\left(A_{T}\right)=0$, one gets

$$
\begin{equation*}
V(A)=-\sigma^{2} \frac{\partial \ln P_{\mathrm{b}}}{\partial A}-\frac{1}{P_{\mathrm{f}} P_{\mathrm{b}}} \int_{A}^{A_{T}}\left(P_{\mathrm{b}} \frac{\partial P_{\mathrm{f}}}{\partial A}-P_{\mathrm{f}} \frac{\partial P_{\mathrm{b}}}{\partial A}\right) v \mathrm{~d} A^{\prime} . \tag{18}
\end{equation*}
$$

This is a general formula connecting the drift rate of the conditional probability $P_{\mathrm{C}}$ with the Green function of the unconstrained Fokker-Planck equation $\left[P_{\mathrm{f}}\right.$ and $P_{\mathrm{b}}$ are just two different representations of the Green function, (10)]. It demonstrates the existence of the closed Fokker-Planck equation (16) and corresponding stochastic equation,

$$
\begin{equation*}
\frac{\mathrm{d} A_{\mathrm{C}}}{\mathrm{~d} t}=-V+\sigma \Xi, \tag{19}
\end{equation*}
$$

for the underlying process (5).

### 4.2 Distribution (6) as a steady-state distribution

We shall make use of one more equation related to Eqn (5). Consider a function of two variables, $A(\theta, t)$, satisfying the equation

$$
\begin{align*}
\mathrm{d} A & =-\frac{\delta S}{\delta A} \mathrm{~d} \theta+\mathrm{d} \Xi(\theta, t) \\
& =\frac{1}{\sigma^{2}}\left[\frac{\partial^{2} A}{\partial t^{2}}+\frac{\partial v}{\partial t}-v \frac{\partial v}{\partial A}+\frac{\sigma^{2}}{2} \frac{\partial^{2} v}{\partial A^{2}}\right] \mathrm{d} t+\mathrm{d} \Xi(\theta, t) \tag{20}
\end{align*}
$$

with a $\delta$-correlated two-dimensional Gaussian noise $\Xi(\theta, t)$ having variance unity; $\delta / \delta A$ stands for a functional derivative. Unlike (5) Eqn (20) is second-order in time $t$. It does not discriminate between forward and backward evolutions. These advantages are partially balanced by the extra dimension, $\theta$, and extra terms on the right-hand side. Evolving this SPDE in artificial time, $\theta$, one arrives at a steady-state distribution $P[A(\infty, t)]=N \exp (-S)$, which is exactly the distribution characterizing the related stochastic equation (5). Equation (20) allows us to impose explicitly different constraints and conditions on the trajectories described by the action $S(t)$, [see Eqn (6)]. For the log-normal process (8) Eqn (20) becomes

$$
\begin{equation*}
\mathrm{d} A=\frac{1}{\sigma^{2}} \frac{\partial^{2} A}{\partial t^{2}} \mathrm{~d} t+\mathrm{d} \Xi(\theta, t) . \tag{21}
\end{equation*}
$$

This equation is defined on the strip, $0 \leqslant t \leqslant T$, with the boundary conditions $A(\theta, 0)=A_{0}, \quad A(\theta, T)=A_{T}$. The steady-state distribution $P(A)$ at large $\theta$ is given by Eqn (11). It can be obtained by the Fourier transform (see also Section 5). Equation (21) was used to produce Fig. 1.

### 4.3 Divergent drift of the Ball-Torous model and the equivalent martingale measure

According to Eqn (7) the fixed final bond price, $B(T)=B_{T}$, or log-price, $A(T)=A_{T}$, are enforced by divergent drift for all other values of $B$ or $A$. At the final point, $t=T$, the equation is, in some sense, no longer stochastic: the solution is known in advance. To explore this issue in detail Cheng has shown [9] that the Ball-Torous drift is not square-integrable

$$
\begin{equation*}
\left\langle\int_{0}^{T} \mathrm{~d} t^{\prime}\left[\frac{A\left(t^{\prime}\right)-A_{T}}{T-t^{\prime}}\right]^{2}\right\rangle \geqslant \int_{0}^{T} \mathrm{~d} t^{\prime} \frac{t}{T\left(T-t^{\prime}\right)}=\infty \tag{22}
\end{equation*}
$$

and the Radon-Nikodym derivative corresponding to eliminating drift in Eqn (7) diverges at $t=T$. Then, the weakest known sufficient conditions of Girsanov's theorem are not met. Correspondingly, at $t=T$ it is no longer possible to eliminate the drift by changing variables, and there is no guarantee that the random processes $B(t)$ or $A(t)$ are martingales.

To focus on this issue further let us discuss a world with only one bond [that is Eqn (7)]. In this world there is no knowledge of the interest rates except for what can be extracted from $B$ and $B_{T}$ (that is $[1 /(T-t)] \ln \left(B_{T} / B\right)$ ). There is only one constraint to be met - the final bond price. In this world one cannot form 'expectations' of the present bond price and discount the final price, since there is no additional information available. It is difficult to reduce the question further, since at the next stage one faces a problem to 'solve the equation in such a way as if the solution was already known'. In the world with many bonds one may use neighboring bonds to get a feeling for the pricing of a given bond. However, no-arbitrage models do not impose any constraint on the slope of the yield curve. It is namely the arbitrage considerations of the bonds with close maturities which we now address.

## 5. Term structure - probabilistic description of the forward rate curve and collective fluctuations

We are now in a position to model collectively fluctuating assets. We would like to focus on a particular example of collective behavior provided by the bond market. Bonds maturing at the same time are easy to compare at any instant by computing the forward interest rates. Arbitrage opportunities provided by such bonds relax quickly and vanish from our description of slow arbitrage applicable to a fluctuating term structure.

Let us denote by $B(t, T)$ the current price of a discount bond maturing at time $T$. This is a synthetic value, in reality it is represented by an averaged price of bonds with identical maturities. The curve, $B(t, T)$ versus $T$, is related to the forward rate $\mathcal{R}\left(t, T_{1}, T_{2}\right)=$ $-\left[\ln B\left(t, T_{2}\right)-\ln B\left(t, T_{1}\right)\right] /\left(T_{2}-T_{1}\right)$ as seen at time $t$ for the period from $T_{1}$ to $T_{2}$, and to the instantaneous forward rate, $R(t, T)$, as seen at time $t$ for a bond maturing at time $T$. By definition, $R(t, T)=\mathcal{R}(t, T, T)=-\partial[\ln B(t, T)] / \partial T$. The function $B(t, T)$ can be retrieved from $R(t, T)$ by integration, $B(t, T)=B_{T} \exp \left[-\int_{t}^{T} \mathrm{~d} t^{\prime} R\left(t, t^{\prime}\right)\right]$. In the Heath - Jarrow Morton model any realization of the function $R(t, T)$ is possible, and it does not imply arbitrage opportunities [1].

At the same time a number of financial models incorporate the so-called mean reversion: interest rates appear over time to be pulled back to some long-term average level (see Ref. [7]). Imagine that as a result of intensive trading (possibly attempted by speculators) the prices of bonds with maturities around time $T_{0}$ are noticeably increased. The instantaneous forward rate, $R$, being the log-derivative, acquires an N -shaped profile in the vicinity of the point $T_{0}$. One can argue that such a 'kink' could not continue to exist for a long time in view of the above-mentioned mean reversal. However, the bond market in a complete model is itself the source of our knowledge of the interest rates, and there is no such a thing as a prescribed 'mean interest rate' to which the function $R$ must relax unless the role of market makers, the laws of supply and demand, the role of the Treasury, etc, are taken into account. The time evolution of the term structure suggests high correlations between bond prices with different close maturities rather than global relaxation to a specified mean value.

Let us return to the example of two assets considered above (see Section 3.2). Such an extreme situation shows that there is no 'mean' or preferred asset price other than the price specified by another asset. Bonds are organized here by their maturity dates, regardless of the duration (the same approach was used by Heath et al. [1] (see also Ref. [7]). We assume that bonds with maturities close to $T_{0}$ from above and below will be considered first by arbitrageurs. This arbitrage has a probabilistic component. Nothing would tell us that a kink on the yield curve should vanish if it were not for historical evidence. It is more related to the fact that the kink is generated by the trading itself, not by the expectation of quick changes of the economy in the future, at time $T_{0}$, and the bond market has mechanisms to 'heal' kinks in the forward rates.

We expect that the bonds with maturities $T_{0} \pm \Delta T$, where $\Delta T$ is the smallest existing increment of maturities, will be considered for arbitrage [16], since the change in $B\left(t, T_{0}\right)$ results in severe changes in the forward rates between
$T_{0}-\Delta T, T_{0}$, and $T_{0}+\Delta T$. This leads to relaxational arbitrage-related terms like $(1 / \tau)\left\{B\left(t, T_{0}+\Delta T\right) \times\right.$ $\left.\exp [-R(t, T) \Delta T]-B\left(t, T_{0}\right)\right\} \quad$ and $\quad(1 / \tau)\left\{B\left(t, T_{0}-\Delta T\right) \times\right.$ $\left.\exp [R(t, T) \Delta T]-B\left(t, T_{0}\right)\right\}$ or, in the continuous limit, $v\left[\partial^{2} B / \partial T^{2}-2 R \partial B / \partial T+R^{2} B\right]$, where $v=\Delta T^{2} / \tau$. We now complete the derivation of the stochastic equation for the term structure. Additional arguments in favor of linear local arbitrage are given at the end of this section.

Combining the collective contribution to the drift term given above with the usual log-normal noise term reflecting independent sources of fluctuations (such as trading itself) one obtains

$$
\begin{equation*}
\mathrm{d} B=\left\{\mu B+v\left[\frac{\partial^{2} B}{\partial T^{2}}-2 R \frac{\partial B}{\partial T}+R^{2} B\right]\right\} \mathrm{d} t+\sigma B \mathrm{~d} \Xi . \tag{23}
\end{equation*}
$$

Equation (23) is a SPDE. It is defined on a strip $\{t \geqslant 0$, $\left.T \geqslant t, T \leqslant t+T_{\mathrm{m}}\right\}$ where $T_{\mathrm{m}}$ is the maximal existing maturity. This equation is supplied by initial and boundary conditions. The initial condition $B(0, T)$ is provided by the currently available bond prices. The boundary condition at $T=t$ is the fixed bond price at maturity: $B(T, T)=B_{T}$. The boundary condition at $T=t+T_{\mathrm{m}}$ is the original bond price when the bond is offered for sale. This price can be obtained by extrapolating the instantaneous forward rate from the interval prior to $R\left(t, t+T_{\mathrm{m}}-\Delta T\right)$. Then, $\partial(\ln B) / \partial T=$ $(\ln B) / T_{\mathrm{m}}$. The log-derivative of Eqn (23) gives a (nonclosed) SPDE for the term structure. The latter equation is related to the Burgers equation which has been addressed in the statistical physics of turbulence [17-19] for a different class of initial and boundary conditions. The Burgers equation has been used previously in financial literature in a different context [20].

Note, that we are able to meet the final condition $B(T, T)=B_{T}$ without a designated drift term. This is the consequence of the second-order $T$-derivative. However attractive this capability of the second order SPDE may seem, one still needs to supply the correct drift term reflecting specifically the Brownian character of the collectively interacting assets (see the previous section).

The diffusive-like term $\partial^{2} B / \partial T^{2}$ implies additional assumptions. Firstly, it suggests that the arbitrage-related increase (decrease) of a given bond price is accompanied by a compensating decrease (increase) of the prices of bonds with close maturities. This is essentially equivalent to the assumption that the amount of money involved in the bond market remains approximately unaffected by such arbitrage, ascribing all other changes to the noise term.

Secondly, effective arbitrage is assumed to be 'shortranged' or local over the maturity time, $T$. There certainly exists a 'long-range' or non-local arbitrage, when the comparison of prices of bonds with different distant maturities is involved [16]. Such arbitrage is assumed to have a lower rate, or that the average squared time difference between maturities of sold and bought bonds converges quickly if averaged over the transaction rates and volumes. The 'range' of arbitrage can be short or long depending on the other time intervals involved. For example, arbitrage between bonds with a one-year difference in maturities is non-local regarding the annual structure of the forward rates, and can be approximated as local in comparison to the entire yield curve.

Thirdly, we neglect higher derivatives in $T$. The validity of this diffusion approximation is based on the assumption that
the term structure is a smooth curve. If the corresponding temporal scale of change is $T_{\mathrm{s}}$, the small parameter is $\Delta T / T_{\mathrm{s}} \ll 1$. If the minimal possible difference in bond maturities is one day, $\Delta T=1$, and the time $\tau$ needed for arbitrage with computer trading is a second, then $v \sim 10^{5}$ days. The so-called diffusion length in one year, $\Delta t=300$ is $\sqrt{v \Delta t}=5500$ days, i.e. about fifteen years. Thus, it takes less than a year with local arbitrage to completely forget about the given initial conditions.

Fourthly, it is assumed that a linear axis, $T$, and arbitrage between bond prices, $B$, is preferable to, say, a logarithmic axis, $\ln T$, and arbitrage between different log-prices, $A$. These are 'adjustable' assumptions.

### 5.1 Functional-integral representation and the Fokker - Planck equation

We now reproduce some results of the previous section with respect to the stochastic equation (23). By substituting $A=\ln B$, one finds

$$
\begin{equation*}
\mathrm{d} A=\left\{\mu-\frac{\sigma^{2}}{2}+v\left[\frac{\partial^{2} A}{\partial T^{2}}+4\left(\frac{\partial A}{\partial T}\right)^{2}\right]\right\} \mathrm{d} t+\sigma \mathrm{d} \Xi \tag{24}
\end{equation*}
$$

The action becomes,

$$
\begin{align*}
S & =\int_{t_{1}}^{t_{2}} \mathrm{~d} t \int_{t}^{t+T_{\mathrm{m}}} \mathrm{~d} T \frac{1}{2 \sigma^{2}} \\
& \times\left[\frac{\partial A}{\partial t}-\mu+\frac{\sigma^{2}}{2}-v \frac{\partial^{2} A}{\partial T^{2}}-4 v\left(\frac{\partial A}{\partial T}\right)^{2}\right]^{2}, \tag{25}
\end{align*}
$$

and the corresponding Green function(al) [c.f. Eqn (10)] is

$$
\begin{equation*}
P\left(A_{1}(T), t_{1} ; A_{2}, t_{2}\right)=\int_{A\left(t_{1}, T\right)=A_{1}(T)}^{A\left(t_{2}, T\right)=A_{2}(T)} \mathcal{D} A \exp (-S) \tag{26}
\end{equation*}
$$

When the number of integrations in (25) is larger than one, the realizations $A(t, T)$ are called fields, not paths, while the path integral is termed a functional integral [15]. At this stage a few comments are in order. Unfortunately, the solution of Eqn (24) is unknown, and the integral entering Eqn (26) cannot be computed. Problems of this sort are well-known in statistical physics, and various techniques for analyzing correlation functions emerging from SPDEs and their functional integral counterparts have been proposed [15, 21]. In most cases only the so-called critical indices or exponents become the major objective of research. From a financial viewpoint this would correspond to the observation that, say, the logarithm of an asset price following the Black-Scholes equation fluctuates as the square root of time, $t^{1 / 2}$. The exponent $1 / 2$ would be the corresponding answer.

The benefits of describing a market with a single number are limited. On the other hand, the problem encountered here is the consequence of our attempts to derive a closed SPDE which correctly accounts for the specified final boundary condition $B(T, T)=B_{T}$, similar to the BallTorous equation (7).

Equation (24) corresponds to the log-normal process of Black and Scholes, and the proper drift rate remains to be determined. The (functional) Fokker - Planck equation for the Green functional (26) can be derived by varying the upper, $t_{2}$, or the lower, $t_{1}$, limits of integration in the action $S$ (25) (c.f. Ref. [13]). The time-dependent limits $t$ and $t+T_{\mathrm{m}}$ bounding the $T$-integration do not contribute to the infinite-
simal variation of the double integral. One finds

$$
\begin{align*}
\pm \frac{\partial P_{\mathrm{f}, \mathrm{~b}}}{\partial t} & =\int_{t}^{t+T_{\mathrm{m}}} \mathrm{~d} T\left[\frac{\delta}{\delta A(t, T)} v+\frac{\sigma^{2}}{2} \frac{\delta^{2}}{\delta A(t, T)^{2}}\right] P_{\mathrm{f}, \mathrm{~b}} \\
& -v=\mu-\frac{\sigma^{2}}{2}+v\left[\frac{\partial^{2} A}{\partial T^{2}}+4\left(\frac{\partial A}{\partial T}\right)^{2}\right] \tag{27}
\end{align*}
$$

where $\delta / \delta A$ stands for a functional derivative. Equation (27) is the analog of Eqn (14). We know that the functional integral (26) represents the required solution of Eqn (27) which we cannot compute. Nevertheless, the analog of formula (18),

$$
\begin{align*}
V(A, T) & =-\sigma^{2} \frac{\delta \ln P_{\mathrm{b}}}{\delta A}-\frac{1}{P_{\mathrm{f}} P_{\mathrm{b}}} \int_{A(t, T)}^{A_{T}} \mathrm{~d} A(t, T) v \\
& \times\left(P_{\mathrm{b}} \frac{\delta P_{\mathrm{f}}}{\delta A(t, T)}-P_{\mathrm{f}} \frac{\delta P_{\mathrm{b}}}{\delta A(t, T)}\right), \tag{28}
\end{align*}
$$

represents the required drift rate in terms of the functional integral (26). In the stochastic equation for the bond prices

$$
\begin{equation*}
\mathrm{d} B=\left(B V+v \frac{\partial^{2} B}{\partial T^{2}}-2 R \frac{\partial B}{\partial T}+R^{2} B\right) \mathrm{d} t+\sigma B \mathrm{~d} \Xi \tag{29}
\end{equation*}
$$

the drift rate, $V(A)$, remains unknown in its explicit form, although the analysis presented above guarantees its existence. The next section is based on the observation that the evolution of the Black - Scholes portfolio is not influenced by the detailed structure of the drift term. We then return to the problem of the final boundary condition.

### 5.2 Black - Scholes strategy with linear arbitrage. Bond option pricing

The Black-Scholes strategy assumes perfect markets, and allows one to price contingent claims by arbitrage. Suppose, a portfolio, $\Pi$, is formed consisting of $a$ assets $H, b$ bonds $B$, (expiring simultaneously with the option), and the European call option itself, $F$ [6]. It is assumed that it is possible to maintain a zero-investment position regarding this portfolio $\Pi=a H+b B-F=0$. The differential condition $\partial \Pi / \partial t=0$ together with the stochastic processes defined for $H$ and $B$,

$$
\begin{align*}
& \mathrm{d} H=\mu_{1} H \mathrm{~d} t+\sigma_{1} H \mathrm{~d} \Xi_{1},  \tag{30}\\
& \mathrm{~d} B=\mu_{2} B \mathrm{~d} t+\sigma_{2} B \mathrm{~d} \Xi_{2}, \tag{31}
\end{align*}
$$

leads to an expression where one can cancel stochastic contributions by choosing proper expressions for $a$ and $b$ (see below), and obtain a deterministic Black-Scholes equation.

In what follows we would like to analyse the consequences of relaxing the condition $\Pi=a H+b B-F=0$ together with $\partial \Pi / \partial t=0$, and replacing them by a linear arbitrage relation $\partial \Pi / \partial t=-(1 / \tau) \Pi$, which implies that it takes a time of the order $\tau$ to enforce the desired changes in the portfolio. By using Eqns $(30,31)$ one finds

$$
\begin{align*}
& {\left[a \mu_{1} H+b \mu_{2} B-\frac{\partial f}{\partial t}-\frac{\partial f}{\partial H} \mu_{1} H-\frac{1}{2} \sigma_{1}^{2} H^{2} \frac{\partial^{2} f}{\partial H^{2}}-\frac{\partial f}{\partial B} \mu_{2} B\right.} \\
& \left.\quad-\frac{1}{2} \sigma_{2}^{2} B^{2} \frac{\partial^{2} f}{\partial B^{2}}-\rho \sigma_{1} \sigma_{2} H B \frac{\partial f^{2}}{\partial H \partial B}+\frac{a}{\tau} H+\frac{b}{\tau} B-\frac{f}{\tau}\right] \mathrm{d} t \\
& \quad+\left(a \sigma_{1} H-\frac{\partial f}{\partial H} \sigma_{1} H\right) \mathrm{d} \Xi_{1}+\left(b \sigma_{2} B-\frac{\partial f}{\partial B} \sigma_{2} B\right) \mathrm{d} \Xi_{2}=0 \tag{32}
\end{align*}
$$

where $\rho$ is the coefficient of cross-correlation of the random processes $\Xi_{1}$ and $\Xi_{2}$. The strategies $a, b$ required to cancel stochastic terms are the same as in the no-arbitrage case: $a=\partial f / \partial H, b=\partial f / \partial B$. Note, that even if the values of $a$ and $b$ are computed with a delay, a financial institution still has a time interval of order $\tau$ to complete the adjustment of the portfolio. One is left with a deterministic partial differential equation

$$
\begin{align*}
-\frac{\partial f}{\partial t}-\frac{1}{2} \sigma_{1}^{2} H^{2} \frac{\partial^{2} f}{\partial H^{2}} & -\frac{1}{2} \sigma_{2}^{2} B^{2} \frac{\partial^{2} f}{\partial B^{2}}-\rho \sigma_{1} \sigma_{2} H B \frac{\partial f^{2}}{\partial H \partial B} \\
& +\frac{\partial f}{\partial H} \frac{H}{\tau}+\frac{\partial f}{\partial B} \frac{B}{\tau}-\frac{f}{\tau}=0 . \tag{33}
\end{align*}
$$

This equation is subject to the initial and final conditions

$$
\begin{align*}
& f\left(H=0, B=B_{0}, t=0\right)=0 \\
& f\left(H, B=B_{T}, t=T\right)=\max (0, H-X) \tag{34}
\end{align*}
$$

where $X$ is the strike price [6]. Following Merton [6] one considers a similarity solution

$$
\begin{equation*}
f(S, B, t)=X B h\left(\frac{H}{X B}, t\right), \tag{35}
\end{equation*}
$$

which offers a reduction of arguments to the similarity variable $x=H / X B$. By substituting (35) into Eqn (33) one finds that the similarity ansatz is exact and leads to the closed equation for $h(x, t)$

$$
\begin{equation*}
\frac{\partial h}{\partial t}+\frac{1}{2} D x^{2} \frac{\partial^{2} h}{\partial x^{2}}=0, \quad D=\sigma_{1}^{2}+\sigma_{2}^{2}-2 \rho \sigma_{1} \sigma_{2} \tag{36}
\end{equation*}
$$

subject to the conditions

$$
\begin{align*}
& f(x=0, t=0)=0, \\
& f(x, t=T)=\max \left(0, x-\frac{1}{B_{T}}\right) . \tag{37}
\end{align*}
$$

At this stage the problem no longer depends on $\tau$. The solution (in the case of constant $D$ ) reads

$$
\begin{align*}
h(x, t) & =\frac{1}{2 B_{T}}\left\{x B_{T} \operatorname{erfc}\left[-\frac{\ln x B_{T}+(1 / 2) D(T-t)}{\sqrt{D(T-t)}}\right]\right. \\
& \left.-\operatorname{erfc}\left[-\frac{\ln x B_{T}-(1 / 2) D(T-t)}{\sqrt{D(T-t)}}\right]\right\}, \tag{38}
\end{align*}
$$

and, according to Eqn (35), the option value is

$$
\begin{equation*}
f=X B_{0} h\left(\frac{H_{0}}{X B_{0}}, 0\right), \tag{39}
\end{equation*}
$$

where $H_{0}$, and $B_{0}$ are the present values of the asset and bond, correspondingly. Thus, the linear arbitrage relaxation does not lead to the modification of the Black-Scholes strategy, but rather implies that it has a larger region of validity than is assumed within the no-arbitrage framework. In the derivation presented above the mechanism of arbitrage leading to linear relaxation, or the agent performing it has not been specified. The position of the financial institution does not depend on $\tau$. Let it be required to price a bond option now in view of the collective nature of fluctuating bonds which was
given above. Consider a European call option contingent on $B\left(t, T_{X}\right)$, i.e. $\mathcal{F}\left[B\left(t, T_{X}\right), t\right]$ expiring at the time $t=t_{X}$ with the strike price $X$. One has to focus on the fields $B$ corresponding to the specified bond expiring at $T_{X}$ and all other bonds. These fields should originate on a curve $B_{0}(0, T)$ at $t=0$, $t \leqslant T \leqslant t+T_{\mathrm{m}}$, pass through any of the points $B\left(t_{X}, T_{X}\right) \geqslant X$ at time $t=t_{X}$, the other components remaining unconstrained (otherwise the call will not be exercised), and terminate on the curve $B(T, T)=B_{T}$ at $t=T$ for all $T$. In the absence of trading, the call price, $\mathcal{F}$ is estimated to be the expectation value of $\max \left[\left(B\left(t_{X}, T_{X}\right)-X\right), 0\right]$ with probability distribution (26). The trading strategy based on the Black Scholes portfolio, and applied to Eqn (24), corresponds to a portfolio, $\Pi$, consisting of a written derivative security and $b(t, T)$ bonds having values $B(t, T)$ and maturing at different $T$. The value of the portfolio is

$$
\begin{equation*}
\Pi=-\mathcal{F}+\int_{t}^{t+T_{\mathrm{m}}} b(t, T) B(t, T) \mathrm{d} T \tag{40}
\end{equation*}
$$

As before, to ensure that the noise terms explicitly cancel, one chooses the Black-Scholes hedge, $b=\delta \mathcal{F} / \delta B(t, T)$. Equation (33) becomes a functional equation

$$
\begin{align*}
\frac{\partial \mathcal{F}}{\partial t}+ & \int_{t}^{t+T_{\mathrm{m}}} \\
& \frac{\sigma^{2} B(t, T)^{2}}{2} \frac{\delta^{2} \mathcal{F}}{\delta B^{2}(t, T)} \mathrm{d} T  \tag{41}\\
& +\frac{\mathcal{F}}{\tau}-\frac{1}{\tau} \int_{t}^{t+T_{\mathrm{m}}} \frac{\delta \mathcal{F}}{\delta B(t, T)} B(t, T) \mathrm{d} T=0
\end{align*}
$$

supplied by the final condition $\mathcal{F}\left(t=T_{X}\right)=$ $\max \left[\left(B\left(t_{X}, T_{X}\right)-X\right), 0\right]$ and the initial condition $\mathcal{F}\left(B\left(0, T_{X}\right)=0, t=0\right)=0$ (while all the other bonds existing at $t=0$ have known values). For the logarithm of the bond price, $A=\ln B$, Eqn (41) is a linear infinitely-dimensional diffusion equation which can be solved as it is. It is more convenient, however, to follow the similarity approach of Merton, which we reproduced above, and consider the substitution

$$
\begin{align*}
& \mathcal{F}=X \mathcal{P}^{\prime} h\left(\frac{B\left(t, T_{X}\right)}{X \mathcal{P}^{\prime}}, t\right),  \tag{42}\\
& \ln \mathcal{P}=\int_{t}^{t+T_{\mathrm{m}}} \mathrm{~d} T \ln B(t, T),
\end{align*}
$$

where prime denotes the fact that the contribution of the bond of interest, $B\left(t, T_{X}\right)$ is dismissed in the infinite product. Equation (41) acquires the following form

$$
\begin{align*}
& \frac{\partial h}{\partial t}+\frac{1}{2} D x^{2} \frac{\partial^{2} h}{\partial x^{2}}=\frac{N-1}{\tau} f  \tag{43}\\
& D=\int_{t}^{t+T_{\mathrm{m}}} \sigma^{2} \mathrm{~d} T \tag{44}
\end{align*}
$$

where $N$ is the number of bonds considered. Equation (43) together with the infinite product $\mathcal{P}$ require special care since the limit $N \rightarrow \infty$ is understood in all other expressions. The solution of this equation is the product of $\exp [-(N-1)(T-t) / \tau]$ and the right-hand side of (38). Relation (42) then leads to a pricing formula.

Although it is possible to manage the entire portfolio $\Pi$ (in frictionless ideal markets), it is preferable to restrict oneself to just two bonds, $B\left(t, t_{X}\right)$ and $B\left(t, T_{X}\right)$, as was done in the cited papers $[6,8]$. This is possible due to the fact that the initial
condition on the presently known bond prices can be used partially, i.e. only for the bonds in the portfolio. Then, Eqn (41) is reducible to Eqn (33). The analysis given at the beginning of this subsection is applicable, and formula (39) represents the desired answer (with $\rho=0$ ). As stated by Merton, the validity of Eqn (39) is not based on the participants' opinion regarding the term structure drift. However, the question of cross-correlation and the relevant value of $\rho$ requires additional study. We shall postpone this issue to the end of the paper.

### 5.3 Equation with artificial time

Although Eqn (41) allows the pricing of claims contingent on many bonds simultaneously, and demonstrates that pricing in the presence of arbitrage is reducible to that without arbitrage, at first sight it may seem somewhat discouraging that no novel pricing formulae are immediately obtained. Equation (41) corresponds to what one can write down without the preceding analysis; it is based solely on the insensitivity of the pricing with Brownian processes to the detailed form of the drift rate. However, the underlying process (23) modifies the procedure of parameter acquisition, such as $\sigma(t, T)$, needed to perform pricing.

To perform parameter acquisition from historical market data one needs to establish the explicit form of the drift term $V$. An alternative approach is to use the last method demonstrated in the previous section. Varying the action $S$ given by Eqn (25) one finds

$$
\begin{align*}
\mathrm{d} A=\frac{\mathrm{d} t}{\sigma^{2}} & {\left[\frac{\partial}{\partial t}+v \frac{\partial^{2}}{\partial T^{2}}-2 v\left(\frac{\partial A}{\partial T}\right) \frac{\partial}{\partial T}\right] } \\
& \times\left[\frac{\partial A}{\partial t}-v \frac{\partial^{2} A}{\partial T^{2}}-v\left(\frac{\partial A}{\partial T}\right)^{2}\right]+\mathrm{d} \Xi(\theta, t, T) \tag{45}
\end{align*}
$$

from the required conditions: $A(\theta, 0, T)=A_{0}(T)$ (this is the initial profile of term structure), $A(\theta, t, t)=A_{T}(T)$ (this is the profile of bond log-prices at different maturities), $\partial A\left(\theta, t, t+T_{\mathrm{m}}\right) / \partial T=0$. Performing multiple Monte Carlo simulations of Eqn (45) one can obtain the distribution of $A$ at different $t, T$ and compare it with historical data. An example of solution to Eqn (45) is shown in Fig. 2

The parameters $v, \sigma$ may, in principle, depend on time and maturity. The positive average slope, $s(T)$, of the yield curve is not preserved in the present model. In order to incorporate, for example, the so-called liquidity preference theory (see Ref. [7]), one can add a term $-\gamma[s(T)+\partial A / \partial T]$ to the action (25), with $\gamma$ being the rate of relaxation. SPDEs offer a broad spectrum of phenomena and the analysis of distinct cases is useful. The solution of the linear version of Eqn (45) will be published elsewhere.

We now return to the question of option pricing $[6,8]$, namely to the analysis which makes use of the empirical coefficient of cross-correlation, $\rho$, between the prices of bonds maturing at $t_{X}$ and $T_{X}$. It is this cross-correlation coefficient that we compute in Section 5.4 on the basis of Eqn (23) to demonstrate the relation of our model to those previously used.

### 5.4 Coefficient of cross-correlation between two bonds. The case of a short-term option

In order to investigate the difference between the results of Section 5.2 and the previous work we explicitly compute here the value of $\rho$. In addition to the basic assumptions of


Figure 2. Shaded surface of a possible field, $A(t, T)$, evolving from the present log-prices of bonds with different maturities, $A_{0}(T)$, and satisfying boundary conditions of fixed bond log-price, $A(T, T)=0$ at maturity, and $(\partial A / \partial T)=A / T_{\mathrm{m}}$ for the bonds with largest time to maturity, $T=t+T_{\mathrm{m}}$. Surface is obtained by a Monte Carlo simulation of (linearized in $A$ ) Eqn (45): $\mathrm{d} A=\sigma^{-2}\left[\partial^{2} A / \partial t^{2}-v^{2} \partial^{4} A / \partial T^{4}\right] \mathrm{d} t+\mathrm{d} \Xi(t, T, \theta)$. The following numerical values and parameters were used: $t$ and $T$ are discretized on a grid of $256 \times 256$ points, $3 \times 10^{5}$ steps is performed along the artificial time $\theta ; \sigma^{2}=10, v=100, T_{\mathrm{m}}=1$ (arbitrary units), $A_{0}(T)=T^{1 / 2}$. Anisotropic folds on the $A(t, T)$ surface are due to the large value of $v$ (despite the slow $4^{\text {th }}$ order diffusion process along the $T$ axis).
frictionless markets with no transaction costs, continuous trading, and no restrictions on contract terms [7] used above, we confine ourselves to the case when a short-term option is written on a bond far from maturity and first sale. We also assume that the fluctuations of the bond market are moderate, so that the noise amplitude $\sigma B$ can be approximated by a constant $\varkappa$, reducing the log-normal noise to the normal one. This is the so-called absolute diffusion model (see Ref. [7]); a different but related approximation is to dismiss the nonlinear term in the corresponding equation for the logprice $A=\ln B$.

In the case of normal noise linearized Eqn (23) becomes

$$
\begin{equation*}
\mathrm{d} B=\left(\mu B+v \frac{\partial^{2} B}{\partial T^{2}}\right)+\chi \mathrm{d} \Xi \tag{46}
\end{equation*}
$$

It is supplied by the initial condition, $B(0, T)=B_{0}(T)$, and analyzed over an infinite domain $-\infty<T<\infty$, since we plan to focus on the evolution in the region $(t, T)$, such that $t \ll T \ll T_{\mathrm{m}}$. This linear equation describes a Gaussian process, and can be solved analytically using a Fourier transform in $T$ for the case $\mu(T)=$ const. The evolution of Fourier modes,

$$
\begin{align*}
& b(t, \Omega)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \mathrm{d} T \exp (-\mathrm{i} \Omega T) B(t, T), \\
& \xi(t, \Omega)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \mathrm{d} T \exp (-\mathrm{i} \Omega T) \Xi(t, T), \tag{47}
\end{align*}
$$

is decoupled and follows an Ornstein - Ulenbeck process (c.f. Ref. [22]),

$$
\begin{equation*}
\mathrm{d} b=\left(\mu b-v \Omega^{2} b\right) \mathrm{d} t+\varkappa \mathrm{d} \xi=\bar{\mu} b \mathrm{~d} t+\varkappa \mathrm{d} \xi \tag{48}
\end{equation*}
$$

with the mode-dependent rate $\bar{\mu}=\mu-v \Omega^{2}$.

The corresponding Gaussian Green function is

$$
\begin{align*}
P\left(b_{1}, t_{1} ; b_{2}, t_{2}\right) & =\left(\frac{\pi \varkappa^{2}}{\bar{\mu}}\left\{\exp \left[2 \bar{\mu}\left(t_{2}-t_{1}\right)\right]-1\right\}\right)^{-1 / 2} \\
& \times \exp \left(-\frac{\bar{\mu}\left\{b_{2}-b_{1} \exp \left[\bar{\mu}\left(t_{2}-t_{1}\right)\right]\right\}^{2}}{x^{2}\left\{\exp \left[\bar{\mu}\left(t_{2}-t_{1}\right)\right]-1\right\}}\right) \cdot(4 \tag{49}
\end{align*}
$$

The probability of arriving at a specified profile $B\left(t_{2}, T\right)=B_{2}(T)$ beginning from the specified profile $B\left(t_{1}, T\right)=B_{1}(T)$ is given by

$$
\begin{equation*}
P\left[B_{1}(T), t_{1} ; B_{2}(T), t_{2}\right]=\exp \left[\int_{-\infty}^{\infty} \mathrm{d} \Omega \ln P\left(b_{1}, t_{1} ; b_{2}, t_{2}\right)\right] \tag{50}
\end{equation*}
$$

This allows various correlators to be computed. For example the mean value and single-time pair correlation function can be shown to be

$$
\begin{align*}
& \left\langle B_{2}(T)\right\rangle=\frac{1}{\left[2 \pi v\left(t_{2}-t_{1}\right)\right]^{1 / 2}} \int_{-\infty}^{\infty} \mathrm{d} T^{\prime} B_{1}\left(T^{\prime}\right) \\
& \quad \times \exp \left[-\frac{\left(T-T^{\prime}\right)^{2}}{2 v\left(t_{2}-t_{1}\right)}\right] \\
& \begin{aligned}
\left\langle B_{2}\left(T_{1}\right) B_{2}\left(T_{2}\right)\right\rangle & =\left\langle B_{2}(T)\right\rangle\left\langle B_{2}\left(T_{2}\right)\right\rangle \\
& +\int_{-\infty}^{\infty} \frac{\mathrm{d} \Omega}{8 \pi^{2} v \Omega^{2}}\left\{1-\exp \left[-2 v \Omega^{2}\left(t_{2}-t_{1}\right)\right]\right\} \\
& \quad \times \exp \left[\mathrm{i} \Omega\left(T_{1}+T_{2}\right)\right] .
\end{aligned}
\end{align*}
$$

In the local models without $\partial / \partial T$ derivatives Eqn (51) is replaced by two stochastic ordinary differential equations for bonds maturing at $t_{X}$ and $T_{X}$ [6]. These are equivalent to Eqns (30), (31) with the convention that H represents the $t_{X}$ bond, and $B$ represents the $T_{X}$ bond. Comparing Eqns (30), (31) with Eqn (51) one identifies the terms $\sigma_{1} S \mathrm{~d} z_{1}$ and $\sigma_{2} B \mathrm{~d} z_{2}$ with the term $\left(\partial^{2} B / \partial T^{2}\right) \mathrm{d} t+\chi \mathrm{d} \Xi$ taken at times $t_{X}$ and $T_{X}$, respectively. The coefficient of cross-correlation is given by

$$
\begin{equation*}
\rho\left(t, t_{X}, T_{X}\right)=\left\langle\Xi_{\alpha} \Xi_{\beta}\right\rangle_{\mathrm{c}}=\frac{v}{\sigma_{1} \sigma_{2}} \frac{\partial^{4}}{\partial^{2} t_{X} \partial^{2} T_{X}}\left\langle B_{2}\left(t_{X}\right) B_{2}\left(T_{X}\right)\right\rangle, \tag{52}
\end{equation*}
$$

which can be computed with the help of Eqn (51). Here we denote the bond maturing at $t_{X}$ as $B_{1}$ and the bond maturing at $T_{X}$ as $B_{2}$. One obtains

$$
\begin{gather*}
\rho=\frac{v^{2}}{\sigma_{1}^{2} \sigma_{2}^{2}}\left\{\frac{x^{2}\left[4 v t-\left(t_{X}+T_{X}\right)^{2}\right]}{2^{13 / 2}(\pi v t)^{3 / 2}} \exp \left[-\frac{\left(t_{X}+T_{X}\right)^{2}}{8 v t}\right]\right. \\
\left.+\frac{\partial^{4}}{\partial t_{X}^{2} \partial T_{X}^{2}}\left\langle B_{2}\left(t_{X}\right)\right\rangle\left\langle B_{2}\left(T_{X}\right)\right\rangle\right\} \tag{53}
\end{gather*}
$$

Thus, the 'effective' stochastic processes governing the evolution of both bonds display cross-correlation, and instead of using the pricing formula with $\rho=0$ and parameters appropriate for the model of collectively fluctuating assets (Section 5.2) one can use the usual pricing approach [8]. This flexibility is limited to the case of Gaussian statistics.

## 6. Conclusions

We have indicated several methods of functional analysis which allow a generalization of financial models to the level of describing the probabilities of different realizations of yield curves and forward rate structures. The adaptation of our model of collectively fluctuating assets in the presence of linear arbitrage to realistic yield curves is in progress. Our model also allows us to study the influence of arbitrage occuring at different rates. We hope that the underlying principles will be useful for modeling other collectively fluctuating systems.

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[^0]:    A N Adamchuk The University of Chicago, Chicago, Illinois 60611
    Tel. (312) 464-86 58. E-mail: aadamchu@finmath.uchicago.edu; NAFT CORP. 1642E 56th st., suite 619, Chicago, IL 60637 Fax (773) 684-42 32. E-mail: naftcorp@aol.com S E Esipov 5640 South Ellis Avenue, Department of Physics and James Franck Institute, University of Chicago, Chicago, IL 50637
    E-mail: Sergei_Esipov\%CENTRE-RE@ notes.interliant.com
    Received 3 April 1997
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[^1]:    $\dagger$ On the 14th October 1997 the Nobel laureates in the field of economics were declared in Stockholm. They were 53 year old Robert C Merton (of Harvard Business School) and 56 year old Myron Scholes (from Stanford University). None of the laureates to date have been able to boast that their discovery in economics has been used in real life, but this year has seen the first attempt to change economists into physicists. According to a representative of the Royal Swedish Academy of Sciences, "their method of pricing financial derivatives is, without exaggeration, the most significant contribution to economic science of the last 25 years". At the beginning of the 70s they worked out a formula to price derivatives which subsequently became the basis of the activity of all financial markets. The main condition for successful activity in this field is the correct evaluation of the value of a derivative. Merton, Scholes, and late Fischer Black developed this method of evaluation. Most importantly, Black and Scholes produced the so-called Black-Scholes equation which, since 1973, has been used daily by thousands of traders and investors for the evaluation of options on the world's financial markets. The equation considers a host of financial-economic factors: interest rates, the level of share fluctuations, hedging, etc. The Black - Scholes formula is sometimes used for pricing insurance contracts and guarantees, and to the determination of the efficiency of investment projects. But Merton's method, further developing their theory, allows researchers to penetrate into new areas of science, even beyond the bounds of financial economics. Unfortunately, Black did not get to share the triumph of his colleagues - he died in 1995 at the age of 57. (Editor's note)

