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On the force line representation of radiation fields

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Abstract. Field-representing force line pictures for a number of simple moving-charge radiation problems are discussed.

Lines of force come to be known to many of us when we first study physics at school. Later, we have to deal with them as university students. However, only the lines of force of static fields are usually addressed in text-books. H Hertz scrutinized the evolution of the force line system for an alternating electric dipole [1]. Nevertheless, this classical example is rarely cited and fails to be included in curricula.

Evidently, both the shape and the mutual orientation of force lines in an alternating field vary with time.

This short communication is concerned with the lines of force patterns in the case of a charged particle moving in a vacuum with variable speed.

Let us first examine the lines of force picture for a charge moving with a constant speed. In this case, the electric lines of force are straight. Figure 1 shows the electric fields of charges at rest (Fig. 1a) and in uniform motion (Fig. 1b). The lines of force of the uniformly moving charge show an anisotropic distribution. They are concentrated near the plane which crosses the charge normally to the direction of motion. The concentration is the higher the closer the charge velocity to that of light. We shall be dealing below with the motion of the charge in a vacuum where it can not move at a speed exceeding the phase velocity of light.

In the present communication, we shall consider only the lines of force of the electric field. They are known to have the following main properties. First, the direction of the electric field at a given point coincides with that of tangent to a line of force through the same point. Second, the strength of the field at a given point is characterized by the density of the lines of force in the vicinity of the point. In other words, if there is a unit element of area perpendicular to a line of force at a given point, then the number of lines of force crossing this field is equal to the field strength at the same point. Lines of force of an electric field originate and terminate in electric charges. In the case of a single point electric charge, the lines of force originate at the point and run away to infinity. If the field of an electric charge is represented by a certain number of force lines, this number remains unaltered for arbitrary motion of the charge. Therefore, the number of force lines may serve as

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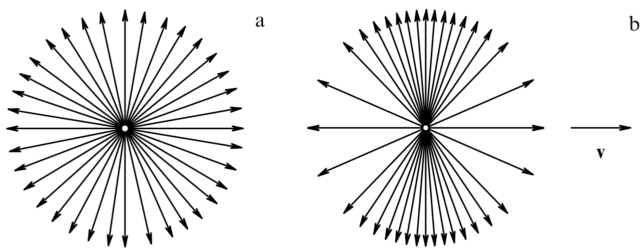

Figure 1. Electric fields of resting (a) and uniformly moving (b) charges; $v = 0.866c$.

the integral of motion. This property of lines of force follows from Gauss theorem. According to this theorem, the electric field flux across any closed surface containing a charge q is $4\pi q$. But the electric field flux in the terms of lines of force is equal to the number of these lines crossing the given surface. For this reason, the number of force lines is a constant independent of the charge motion law.

We shall confine ourselves to representation of the field of a moving charge with the help of lines of force. In the general form, the problem was raised and solved by Harutyunyan [2, 3]. The specific case of the radiation field of a charged particle undergoing instantaneous acceleration was considered by Purcell [4] who also discussed some quantitative aspects of the problem. We shall examine it in more details below.

Let a charged particle lie at the origin of Cartesian coordinates. The particle is at rest before time $t = 0$ and starts moving at $t = 0$ in the positive direction of x axis with speed v. The problem is to find the field of the charged particle for this motion. The assumption of an instantaneous jump of velocity is a sort of idealization. In a real situation, a finite change of velocity occurs over a finite time. However, this assumption is justified if one considers radiation at sufficiently low frequencies. We shall discuss below how the field picture is modified if the assumption of an instantaneous jump of velocity is not in case. Let us encircle the starting point with a sphere of radius $r = ct$. Inside this sphere, the solution of the Maxwell equations gives the field of a charge moving at a constant speed v:

$$E_x = q\left(1 - \frac{v^2}{c^2}\right)(x - vt) \times \left[\left(1 - \frac{v^2}{c^2}\right)(y^2 + z^2) + (x - vt)^2\right]^{-3/2}, \qquad (1)$$

$$E_y = q\left(1 - \frac{v^2}{c^2}\right)y\left[\left(1 - \frac{v^2}{c^2}\right)(y^2 + z^2) + (x - vt)^2\right]^{-3/2}. \qquad (2)$$

Outside the sphere, the field is equal to the Coulomb field of a charge resting at the origin of coordinates. As the sphere $r = ct$ expands, it passes through the observation point. Therefore, at moment $t = r/c$, the Coulomb field of a resting charge is substituted by the field of a uniformly moving charge at point $x = vt$, regardless of the distance of the observer from the starting point. Such a field structure is due to effect of retardation related to the finite velocity of light. When the observer is far from the starting point, the field at the observation point remains the field of a resting charge well after the charge commenced its regular motion. It will be shown below that the field behaves in a similar way after the charge stops. In other words, if the observer is far from the stopping point, the field at the observation point remains that of a moving charge long after the charge stopped.

Figure 2 represents lines of force inside and outside the sphere $r = ct$. It appears that lines are not continuous on the sphere, that is they are ruptured at the surface of the sphere. However, such a disjunction of lines of force at the spherical surface would mean that the surface is charged. But we are considering here only the charge q at point $x = vt$, and no other charge. Hence, the lines of force must be continuous, i.e. each force line inside the sphere must join the one outside it. Such a transition is feasible if the lines of force lie at the surface of the sphere with radius $r = ct$. It is these lines of force at the surface of the expanding sphere that define the radiation in the problem being considered. Indeed, the sphere $r = ct$ expands with the velocity of light. The lines of force at its surface are normal to the direction of expansion. Therefore, the field described by the lines of force lying on the sphere $r = ct$ meets all the requirements which are expected to be satisfied by an electromagnetic wave. It is worthwhile noting that in the instantaneous start problem in question the field inside the sphere does not contain radiation and is therefore the field of a uniformly moving charge. The field outside the sphere does not contain radiation either and represents the field of a resting charge. The radiation field differs from zero only at the sphere $r = ct$.

It has been shown by Purcell [4] that the problem contains one-to-one correspondence between force lines of the fields inside and outside the sphere. If the instantaneous stop problem is considered for certainty, the line of force inside the sphere running at an angle θ to the direction of motion continues as the line of force outside the sphere at an angle θ' to the same direction, with

$$\tan\theta' = \gamma \tan\theta\,, \tag{3}$$

where $\gamma = \left[1 - (v/c)^2\right]^{-1/2}$ is the so-called Lorentz factor. The two lines are linked by a segment of the force line lying on the sphere. Therefore, they can be regarded as different parts of the same line.

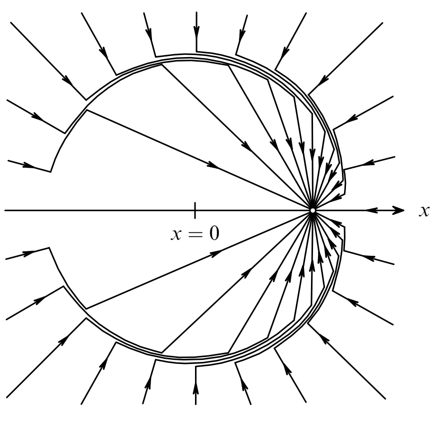

Figure 2. Electric field of a charge instantaneously accelerated at point $x = 0$ and moving with a constant speed.

The strength of the radiation field can be determined in the following way. Let us introduce a spherical system of coordinates with the center at the starting point of the charge and the axis parallel to its velocity. Let us consider a part of the spherical surface at an angle θ to the velocity of the charge. Let us then calculate the electric field flux across this segment. Evidently, this flux is equal to the difference between the internal field flux and external one. The external field flux is easy to find. This flux across the part of the spherical surface corresponding to the solid angle $\mathrm{d}\Omega$ is equal to $q\,\mathrm{d}\Omega$. Let us calculate the internal field flux by assuming $x = r\cos\theta$, $y = r\sin\theta$, $r = ct$ in formulae (1) and (2). This yields the expression for fields E_x, E_y at the sphere:

$$E_x\big|_{r=ct} = \frac{q(1-\beta^2)}{t^2}\,\frac{c\cos\theta - v}{\left[(1-\beta^2)c^2\sin^2\theta + (c\cos\theta - v)^2\right]^{3/2}}\,, \tag{4}$$

$$E_y\big|_{r=ct} = \frac{q(1-\beta^2)}{t^2}\,\frac{c\sin\theta}{\left[(1-\beta^2)c^2\sin^2\theta + (c\cos\theta - v)^2\right]^{3/2}}\,, \tag{5}$$

where $\beta = v/c$.

The flux of the field E across the element $\mathrm{d}S$ of the sphere surface is $E_\mathrm{n}\,\mathrm{d}S$, i.e. $(n_x E_x + n_y E_y)\,\mathrm{d}S$, where n is normal to the surface. Taking into account $n_x = \cos\theta$, $n_y = \sin\theta$, formulas (4) and (5) give the expression for the flux P across the element of the sphere surface corresponding to element of the solid angle $\mathrm{d}\Omega$:

$$\mathrm{d}P = E_x\cos + E_y\sin\theta = q(1-\beta^2)\,\frac{1}{(1-\beta\cos\theta)^2}\,\mathrm{d}\Omega\,. \tag{6}$$

It follows from Eqn (6) that this relation does not include the radius of the sphere, i.e. the field flux is defined only by the element of solid angle $\mathrm{d}\Omega$. This is understandable because the electric field weakens with increasing radius r as r^{-2} while the area of the surface element inside the solid angle $\mathrm{d}\Omega$ is proportional to r^2. Hence, the field of a charge undergoing uniform motion in a vacuum can not be a radiation field. Indeed, a radiation field is characterized by r-independence of the electromagnetic energy flux into the solid angle $\mathrm{d}\Omega$. But such a flux is a bilinear combination of electric and magnetic fields (the Poynting vector $P = 4\pi c\, E \times H$); hence, in this case, the square of the field (rather than its first power) must decrease with increasing r as r^{-2} while the radiation field itself decreases with the distance as r^{-1}. It is not difficult to demonstrate that the electric field flux through the total surface of the sphere $r = ct$ containing the charge is $4\pi q$, in agreement with Gauss theorem. This means that both the internal and external fields can be represented by the same number of force lines.

The difference between the external and internal fluxes across a surface element of the sphere is due to the increased number of lines of force at a given part of the sphere. Let us denote the strength of the radiation field at the sphere as E_θ. It follows from the symmetry of the problem that the radiation field depends only on θ. Then, the condition of conservation of the lines of force number (or the equivalent condition $\operatorname{div} E = 0$) leads to the relation

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(E_\theta \sin\theta) = \frac{q}{r}\left[\frac{1-\beta^2}{(1-\beta\cos\theta)^2} - 1\right]\sin\theta\,. \tag{7}$$

This relation may be regarded as a differential equation for the non-zero tangential component of the electric field E_θ at the expanding light sphere. However, specific features of the field E_θ are apparent even before the solution of this equation. The lines of force of this field lie at the surface of the sphere $r = ct$ with the center at the point from which the charge started. This sphere expands with the velocity of light, i.e. the charge always remains inside it. An important characteristic of the field E_θ is its decrease in proportion to r^{-1} with increasing radius r. In other words, the field E_θ has the property of a radiation field. It propagates at the velocity of light and is at each point perpendicular to the direction of motion.

To avoid misunderstanding, it is worthwhile to note that the field of a regularly moving charge (4), (5) also has a tangential component on the sphere (the corresponding expression is not given here). However, the tangential component of the field (4), (5) decreases as r^{-2}, unlike that of the radiation field E_θ.

The solution of the differential equation (7) satisfying the condition $E_\theta = 0$ at $\theta = 0$ can be written in the form

$$E_\theta = \frac{q}{r}\delta(r-ct)\frac{\beta\sin\theta}{1-\beta\cos\theta}\,, \tag{8}$$

where the delta-function of the argument $r - ct$ takes into account that the field E_θ differs from zero only at the sphere $r = ct$ which expands at the velocity of light. The same expression was obtained in a different way in Ref. [5].

To summarize, in the instantaneous start approximation, a radiation field is formed at the moment of start. Afterwards, the wave packet of the radiation field propagates in accordance with law (7) which gives the expression for the sole non-zero component of the field at the instantaneous start of the charge.

Let us now consider the radiation field arising at a sudden stop of a charge. Let us suppose that a charge q undergoes regular motion in the positive direction of the x axis at a speed v and comes to a stop at point $x = 0$ at time $t = 0$. In this case, the solution of the Maxwell equation within the sphere $r = ct$ with the center at point $x = 0$ gives the Coulomb field of the charge resting at the origin of the coordinates. The field outside the sphere $r = ct$ is the field of a charge undergoing uniform motion with speed v (since the signal of the charge arrest does not reach the points lying outside the light sphere $r = ct$). It is easy to see that the difference between field fluxes inside and outside the sphere has the same magnitude but the opposite sign as compared with that in the case of instantaneous start. It follows that the radiation field E_θ in the case of the stop has the same magnitude that in the case of instantaneous start (8) although the two fields have different signs.

Let us now consider the case of a charged particle resting at the origin of the coordinates until time $t = 0$. At $t = 0$, the particle's velocity changes jump-wise to become v after which the particle moves in the positive direction of the x axis during the time T. At $t = T$, the particle comes to an abrupt stop at the point $x = vt$. Let us find the radiation field for the given motion. This problem can be solved based on the above considerations. Indeed, the radiation field associated with the instantaneous start lies at the sphere $r = ct$ with the center in the origin of the coordinates. The radiation field generated upon the instantaneous stop is on the sphere $|r - vT| = c(t - T)$ with the center at the point $x = vT$. At $v < c$, the light sphere related to the stop is always inside the sphere associated with the start. The space between the spheres forms a certain layer in which the field equals the field of a regularly moving charge. The field inside the layer is the field of a Coulomb charge resting at point $x = vt$ whereas that outside the layer is a Coulomb charge resting at point $x = 0$. The radiation fields situated at the spherical surfaces bounding the layer are responsible for the continuity of lines of force. The radiation field associated with the start of the charge has the form (8). The radiation field associated with the stop may be written as

$$E_\theta^{\mathrm{stop}} = -\frac{q}{r}\frac{\beta\sin\theta}{1-\beta\cos\theta}\delta\big(|r - vT| - c(t-T)\big)\,. \tag{9}$$

To consider radiation fields at distances greater than the distance covered by a charge ($r \gg vT$), it is possible to use the approximate expression $|r - vT| = r - n\cdot vT$, where $n = r/r$ is the unit vector in the direction of observation. Then, the expression for the total field of radiation takes the form

$$E_\theta = \frac{q}{r}\frac{\beta\sin\theta}{1-\beta\cos\theta}\bigg\{\delta(r-ct) - \delta\bigg[r - ct + cT\bigg(1 - \frac{v}{c}\cos\theta\bigg)\bigg]\bigg\}\,. \tag{10}$$

The quantity $cT(1 - v\cos\theta/c)$ is the distance between two radiation pulses examined at an angle θ to the direction of motion. Accordingly, the quantity $T(1 - v\cos\theta/c)$ defines the time-interval between the appearance of the first and second radiation pulses at the observation point. It should be recalled that the first pulse propagates from the starting point and the second from the stopping point. Clearly, the motion time can be large, but the time-interval between two forward radiation pulses ($\theta = 0$) can be much shorter than the duration of motion T provided the particle's speed v is close to the velocity of light. If the interval between two forward radiation pulses is denoted as Δt, then, in the relativistic case when the velocity of a charge is close to that of light c, the following relation holds true:

$$\Delta t = \frac{T}{2\gamma^2}\,. \tag{11}$$

Spectral properties of the radiation field may be obtained by Fourier transformation of the expressions (8) and (9). If the Fourier transformation of a field $E(t)$ is given by the formulae

$$E(t) = \int E_\omega \exp(\mathrm{i}\omega t)\,\mathrm{d}\omega\,, \tag{12}$$

then, for the radiation field in the case of an instantaneous start, it follows from Eqn (8) that

$$E_\omega = \frac{q}{2\pi cr}\,\frac{\beta\sin\theta}{1-\beta\cos\theta}\exp\left(\mathrm{i}\frac{\omega}{c}\,r\right). \qquad (13)$$

Equation (13) was obtained using the representation of the δ-function

$$\delta(t) = \frac{1}{2\pi}\int \exp(\mathrm{i}\omega t)\,\mathrm{d}\omega\,. \qquad (14)$$

The spectral component (13) of the radiation field in the case of an instantaneous start is a spherical wave with frequency ω outgoing from the origin of the coordinates. The amplitude of this wave does not depend on frequency. Therefore, the calculation of the radiation energy at a given frequency followed by integration of the resulting expression over all frequencies leads to a divergent expression. This divergency is due to the assumption of an instantaneous change of velocity. However, the particle's velocity is always a smooth function of time. Therefore, the Fourier component of this velocity or that of the related current (which is the same) undergoes rapid damping at a sufficiently high frequencies [6]. The Fourier component of the field E_ω being proportional to that of the current j_ω in the case of a smooth change of speed, the spectral component of E_ω also undergoes rapid damping at sufficiently high frequencies. Note that if the transition from initial to final velocity is smooth enough and takes finite time, then the radiation field is contained in an expanding layer. This layer divides space into three areas. The field at those points of the space which have not yet been reached by the expanding layer corresponds to the initial state (e.g. to the regular motion with speed v_1). The field at the points of the space which have already been passed by this layer corresponds to the final state (regular motion with speed v_2). The field inside the layer may be represented by lines of force connecting every force line of initial field with corresponding force line of final field. In the case of an instantaneous transition, the layer has zero thickness, hence the lines of force lie on the sphere. Therefore, the entire connecting line at the expanding sphere is perpendicular to the direction of motion and represents the radiation field. Generally speaking, in the case of a smooth transition, the electric field inside the layer has both tangential and normal components with respect to the spherical surfaces bounding the layer. The radiation field is described only by the tangential components.

Let us now consider the spectral components of the radiation field for the case of a uniform charge moving over a finite time T. The coordinate and time dependence of the field is described by formula (10). For simplicity, we shall consider the radiation field far from the charge movement region ($r \gg vT$). The expansion of the radiation field (10) in the Fourier integral over time in accordance with Eqn (12) gives the following expression for the spectral component E_ω:

$$E_\omega = \frac{\mathrm{i}q}{\pi cr}\,\frac{\beta\sin\theta}{1-\beta\cos\theta}\sin\left[\frac{\omega T}{2}(1-\beta\cos\theta)\right] \times \exp\left(\mathrm{i}\,\frac{\omega}{c}\,r\right)\exp\left[-\mathrm{i}\,\frac{\omega T}{2}(1-\beta\cos\theta)\right]. \qquad (15)$$

This expression coincides, up to the phase factor, with that obtained by Tamm [7] when solving the same problem. Tamm derived the expression for the spectral component of the radiation field and did not consider the space – time picture. The appearance of the phase factor $\exp[-i(\omega T/2)(1-\beta\cos\theta)]$ is due to the choice of a different reference point which Tamm placed in the middle of the charge trajectory. In the present work, it coincides with the beginning of the trajectory.

We have until now considered the problem of charge radiation in free space. Similar considerations are applicable to the solution of the problem of transition radiation of a charged particle entering vacuum across a plane boundary of an ideally conductive body (metal). Let half-space $x < 0$ be filled with metal. At the initial moment $t = 0$, the charged particle leaves the metal and moves along the x axis with a constant speed v so that its position is defined by the relation $x = vt$.

The field arising in the half-space $x > 0$ may in this case be represented as the superposition of two moving charged particles, one of which is a real charge q while the other is its image. The charge of the image is equal to that of the outgoing particle but has the opposite sign. The position of the image is defined by the relation $x = -vt$. Evidently, if the plane is drawn through the point $x = 0$ normally to x axis, the lines of force of the combined electric field generated by the charge and the image are also perpendicular to this plane. Thus, the same boundary conditions are satisfied at the plane $x = 0$ and at the metal. Therefore, in this case, the problem of transition radiation is reduced to the problem of radiation emitted upon the instantaneous start of two equal charges having opposite signs which move from one point in opposite directions.

In this case, the field has the following space – time structure. Let us examine a hemisphere with radius $r = ct$ in the half-space $x > 0$ with its center at the start charge's point. Outside this hemisphere, the field is zero but inside, it is the superposition of the moving charge and its image. The radiation field is defined by the lines of force lying at the sphere surface. In the case of an instantaneous start of the charge, this field is defined by the formula (8). The radiation field in the case of an instantaneous start of the charge image can be derived from Eqn (8) by changing the sign in front of $\beta\cos\theta$ in the denominator:

$$E_\theta = \frac{q}{r}\,\delta(r-ct)\left(\frac{\beta\sin\theta}{1-\beta\cos\theta}+\frac{\beta\sin\theta}{1+\beta\cos\theta}\right) = \frac{q}{r}\,\delta(r-ct)\,\frac{2\beta\sin\theta}{1-\beta^2\cos^2\theta}\,. \qquad (16)$$

Formula (16) defines the transition radiation field in space and time. The spectral field composition can be found by expanding Eqn (16) in the Fourier integral. This yields the expression for E_ω coinciding with that obtained by Ginzburg and Frank [8].

One essential point is worth mentioning. The transition radiation field at the sphere $r = ct$ decreases with distance as r^{-1}. The source field inside the expanding sphere weakens with the distance as r^{-2}, that is significantly faster than the transition radiation field. If the detector of radiation is far enough from both the point of charge appearance and the charge trajectory, only the transition radiation field is recorded. If the detector is close enough to the trajectory, the contribution of the source intrinsic field to the readings can be as great as that of the transition radiation or even greater.

In conclusion, paper [9] considers radiation generated upon a sharp change in the charge velocity. The author argues that, at a jump of velocity, the field surrounding the charge is striped off and propagates in the direction of its initial movement. After that, the charged particle for some time moves without any field (the author calls this state a 'bare' or 'half-bare' charge) but gradually acquires the field corresponding to the final velocity of the movement. Similar evolution patterns in the case of a scattering charge have been described in Ref. [10]. The results of the present study appear to be in contradiction with these findings. Indeed, field restructuring upon a change in charge velocity takes some finite time. However, a charge is always surrounded by a field regardless of the evolution rate since the number of lines of force originating in it is an integral of motion. In other words, a transition process, which is described in the present paper, takes place.

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