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Physics of collisionless plasma

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Contents

Abstract. Review of papers dealing with two alternative methods of describing nonequilibrium processes in fully ionized plasma. The example of 'Landau damping' is used to demonstrate that a number of fundamental problems remain unsolved in the kinetic theory of plasma. They arise from inconsistent description of transition from the reversible equations of mechanics of charged particles and field to the irreversible equations of continuous medium in statistical theory of plasma. These difficulties can be overcome through consistently defining the structure of continuous medium and the characteristics of dynamic instability of motion. This leads to generalized irreversible equations which provide the basis for unified description of nonequilibrium processes in plasma on kinetic and hydrodynamic scales.

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1. Introduction

The birthday of the kinetic theory of plasma may well be considered the date of publication of Landau's paper entitled "Kinetic equation in the case of Coulomb interaction" [1]. Landau's equation was based on the kinetic Boltzmann equation for a dilute gas, for which (owing to the short-range interaction between the atoms) a good accuracy is ensured by taking into account only the pairwise interactions between the gas particles.

Landau noted that for a plasma, when the interaction between charged particles is governed by Coulomb's law and therefore falls off very slowly with distance, the collision integral in the Boltzmann equation diverges when the distance between particles is large. This means that the collisions between charged particles are important at large distances, when the change in momentum is small. Therefore, there is good reason for carrying out the expansion in small variations of momentum in the Boltzmann collision integral. As a result, the extremely complicated Boltzmann's expression reduces to a much simpler collision integral which describes diffusion in the momentum space.

This method of constructing the collision integral is, however, not consistent, because a large number of charged particles are involved in simultaneous interactions in a plasma. A manifestation is that Landau's expression contains the integral which diverges (logarithmically) at large, as well as at small, distances. Some help comes from the fact that logarithm has low sensitivity to small changes in the argument when the latter is large. This gives freedom in the selection of integration limits. The upper limit is set equal to the Debye radius, which defines the sphere of interaction of charged particles. Since in a rarefied plasma there are many particles within a sphere of Debye radius, the interaction in plasma is collective, as opposed to the Boltzmann gas. This proves that the choice of the 'starting point' in the construction of the kinetic equation is physically not justified.

We shall see, however, that Landau's collision integral, in spite of this inconsistency, in the best possible way describes dissipation due to redistribution of charged particles in plasma. This justifies our claim that the kinetic description of the 'collision plasma' was born in 1936 – 1937. We put these words in quotes because the interaction of particles in plasma is collective. This term is opposed to 'collisionless plasma', which implies that interactions which are responsible for dissipation, and therefore for relaxation towards equilibrium, are not taken into account.

The statistical theory of collisionless plasma originates from the classical paper of A A Vlasov [2]. It was in this paper that the kinetic equation for the distribution function of electrons in fully ionized plasma was formulated without taking into account those interactions (collisions) which are responsible for dissipation. In this way, all interaction between the charged particles is only defined in terms of the mean field. The rationale (although, as we shall see below, not completely justified) was that the characteristic frequencies of electron plasma in typical cases are much higher than the rate of electron-electron collisions.

In a sense, the approximations of Landau and Vlasov correspond to two extreme cases. Indeed, Landau only takes into account that contribution of the interaction which is responsible for dissipation. The effects due to internal field are disregarded. By contrast, interaction in the Vlasov equation is only through the mean field. Because of this, the Vlasov equations are reversible. However, the question about the role of collisions in the neighbourhood of resonances remains open (this concerns those particles whose velocities are close to the phase velocity of electron waves in plasma). It was only eight years later that the famous paper by Landau [3] was published, devoted to this problem.

Landau starts with the Vlasov equation for the distribution function of electrons and the mean electric field — that is, with the reversible kinetic equation once again. Embarking on the problem of plasma oscillations, Landau writes:

"*These equations have been applied to the study of plasma oscillations by A A Vlasov; however, most of his results are wrong. Vlasov sought solutions of the form* const$\times$ $\exp(-\mathrm{i}\omega t + \mathrm{i}kr)$ *and found the frequency* ω *as function of the wave vector* k. *Vlasov's expression defining this function contained a divergent integral, which by itself points to the mathematical inconsistency of his method. Vlasov dealt with this difficulty by taking the principal value of the integral, for which, however, there are no grounds. In order to obtain a correct solution of the Vlasov equations, the problem must be considered in some particular statement*; *we are going to consider two such statements*."

In this way, it is assumed that the above difficulty, arising from the presence of the divergent integral, is only associated with the insufficiently consistent solution of the reversible Vlasov equations. The question of the modification of equations themselves is not raised.

One of the two problems solved in Landau's paper is the study of time evolution of the initial distribution according to the Vlasov equation linearized with respect to the equilibrium Maxwellian distribution. The Laplace method is used for solving the initial problem. To make sense of the divergent integral, the technique of adiabatic switching on the field (the field is switched on at $t = -\infty$) is used in place of the strictly harmonic oscillation $\exp(-\mathrm{i}\omega t)$. This switching of the field corresponds to adding an infinitesimal positive imaginary part to the real frequency ω — that is, to the replacement $\omega \to \omega + \mathrm{i}\delta$, where $\delta \to +0$. In this way, the rule of detouring around the pole in the divergent integral consists in the replacement $\omega \to \omega + \mathrm{i}0$.

The authors of book [4] write on page 154:

"*It is also possible to approach the justification of the rule of detouring around the pole proposed by Landau for a different standpoint. Namely, one may introduce a small dissipative term* $\nu f_1(r, p, t)$ *into the linearized kinetic Vlasov equation*."

When the kinetic equation is solved by Fourier method, the presence of this dissipative term will result in that the real frequency is replaced by $\omega \to \omega + \mathrm{i}\nu$. Transition to the limit $\nu \to 0$ is performed in the final expressions. This leads to the same expression for the Landau damping coefficient.

So, there are seemingly two different approaches to solving the problem of damping of plasma oscillations. The first is based on the formal mathematical method of regularizing the divergence in Cauchy-type integral. In this case the physical nature of damping is not discussed. The initial equation remains reversible. In the second approach the initial reversible equation is from the outset replaced with the dissipative equation. Although the dissipation is assumed to be small, the transition to the limit $\nu \to 0$ is only carried out in the final formulas. This singles out the resonance contribution, whose width is negligibly small as compared to the width (dispersion) of the velocity distribution. If the transition to the limit $\nu \to 0$ is performed in the kinetic equation itself (which means going back to the reversible Vlasov equation), then it is not possible to obtain the Landau damping.

The second way of introducing the Landau damping seems to be more natural. It shows that, after all, the collisions play a fundamental role. It is this standpoint that is postulated in Section 16 of Chapter 15 in book [5]. We shall see, however, that a more comprehensive solution of this problem can only be based on the generalized kinetic equation formulated below.

Let us return to Landau's paper. The second method of obtaining Landau's result is based on the 'corrupt' Vlasov equation, into which a small dissipative term is introduced. This might seemingly give grounds to assume that Landau damping is a consequence of collisions between charged particles. The authors of [4], however, come to a different conclusion (Section 30, p. 157):

"*Thus, the dissipation already arises in a collisionless plasma; this phenomenon was predicted by L D Landau* (1946), *and it is referred to as Landau damping. Being not* *associated with collisions, it is basically different from dissipation in ordinary absorbing media: the collisionless dissipation is not associated with the entropy increase, and is therefore a thermodynamically reversible process.*''

A similar conclusion is repeated with small variations in almost all monographs and textbooks on plasma theory. We give just one example:

`` *... the most unexpected and important effect, however, related to the physics of Langmuir oscillations, was predicted by L D Landau. He discovered that electron oscillations are attenuated even in the absence of collisions* (*that is, forces of friction*)'' [6], p. 12.

We shall see that the physical nature of Landau damping is not so mysterious. Namely, Landau damping is one of the dissipative processes whose description is impossible without taking collisions into account. In this respect, Landau damping naturally fits in the general scheme of thermodynamics and kinetics of irreversible processes.

In connection with the history of the question, it would be interesting to quote P L Kapitza. This pronouncement can be found in the reminiscences of Kapitza seminars, written by M P Ryutova for the memorial issue of *Physics – Uspekhi* (Vol. 37, No. 12, 1994), dedicated to the 100th birthday of P L Kapitza. On p. 1234 of this issue we read:

``*I would like to draw attention to the interesting study of Landau concerned with the absorption of electrical waves in plasma, — Petr Leonidovich suddenly changed the topic during the defence of a thesis. — This work, which is very important, was done a rather long time ago, in 1946, almost 16 years ago. Landau developed a new type of absorption in plasma.*

Vlasov had begun to work on the theory of plasma earlier, but his findings evoked an active response from the side of theoretical physicists, who started to look for concealed errors. These works, however, resulted in Landau's paper, which nowadays plays such a role.

This is yet another case which proves that wrong results must be published as stimulants of correct results. The worst thing is a work being trivial. The most important in a work is by no means its correctness, it is the presence of a new idea. One must never reject publication of new ideas. Vlasov came with his new ideas to the theory of plasma, some of them could be wrong, but anyway he has made a big step forward, Landau's work would not have been done without him.''

The year 1946 is also marked by the publication of N N Bogolyubov's monograph [7]. Along with the justification of Boltzmann's kinetic theory of gases, it laid the foundation of the statistical basis of the kinetic equations for plasma. Bogolyubov formulated the conditions under which the kinetic equations of Landau and Vlasov can be obtained from the dynamic equations. Moreover, this book lays the groundwork for the statistical theory which allows taking into account the nature of collective interactions in plasma. This paved the way for the construction of kinetic equations incorporating the polarization of plasma. The corresponding equations were independently established in 1960 by R Balescu [8] and A Lenard [9]. One result was the expression for Balescu – Lenard collision integral. It differs from Landau's result by inclusion of the dynamic polarization of plasma — in other words, by a more consistent treatment of the collective interaction between charged particles in the calculation of the dissipative term (the collision integral) in the kinetic equation.

This rounded off a certain stage in the development of the kinetic theory of plasma. Many original papers, reviews, monographs and textbooks are devoted to this theory and its numerous applications. Here we point out just a few monographs and textbooks [4 – 7, 10 – 30], to which we are going to refer in the forthcoming discussion.

Naturally, further development of the kinetic theory of plasma not only expanded the scope of its particular applications, many of which are considered in the works mentioned above, but also led to the solution of a number of fundamental problems of the theory. Let us mention just a few of these.

The statistical theory of both rarefied gases and rarefied plasma was developed initially on the basis of the chain of equations in a hierarchy of distribution functions for coordinates and momenta of charged particles — the Bogolyubov – Born – Green – Kirkwood – Yvon (BBGKY) equations. As shown in Ref. [12], it is possible to construct a kinetic theory of both relativistic and nonrelativistic plasma on the basis of equations in the microscopic phase density of charged particles in the six-dimensional phase space of coordinates and momenta, and the Lorentz equations in the microscopic strengths of electric and magnetic fields. This method was shown to be equivalent to the Bogolyubov (BBGKY) method.

The method of microscopic phase density proved to be very efficient, and is widely used in the studies on the theory of both classical and quantum plasma (see, for example, Refs [5, 18, 19, 21 – 23, 26, 27, 29]. This method also forms the basis of the statistical theory of plasma-molecular systems, exemplified by partly ionized plasma [5, 21, 31].

The kinetic equations of Landau and Balescu and Lenard for a rarefied plasma are constructed in the approximation when the interaction between particles determines the dissipative processes, but does not contribute to the thermodynamic functions. From the standpoint of thermodynamics, these equations only hold for an ideal plasma.

In recent years, the processes in nonideal plasma have been the object of active theoretical and experimental studies [28, 32]. Naturally, at the first stage it was necessary to develop a kinetic theory of slightly nonideal plasma [21].

The kinetic equations of Landau, Vlasov as well as Balescu and Lenard are equations in deterministic (nonrandom) distribution functions. The description of many phenomena requires taking into account the fluctuations of the distribution functions. This called for the development of the kinetic theory of fluctuations [4, 21, 33].

The contemporary theory of nonequilibrium processes in plasma is the basis for description of a large variety of phenomena. This does not imply that all the fundamental problems have been solved. These days, the theory of space – time dissipative structures in plasma containing sources of energy (active plasma) gains in importance. This stimulates the development of the statistical theory of open systems [33], which called for revision of some basic notions and concepts of the kinetic theory of plasma. In particular, it was necessary to give a concrete definition of the structure of continuous medium for which the kinetic description is being developed. Due account for this structure in the kinetic equation gives rise to an additional collision integral, which describes dissipation caused by space diffusion of the distribution function. As a consequence, it becomes possible to give a unified description of nonequilibrium processes at kinetic and hydrodynamic scales without using perturbation theory in Knudsen number.

The use of such generalized kinetic equation allows defining the concepts of collision and collisionless plasmas in a new way. It also becomes possible to treat Landau damping as a dissipative process in collision plasma.

2. Initial equations of statistical theory of plasma

2.1 Microscopic equations for Coulomb plasma

Gas plasma constitutes a multicomponent gas consisting of positively and negatively charged particles and neutrals. The simplest example is the electron – ion plasma, in which the charge of the ion e_i equals the electron charge e. In this paper we shall consider only the *Coulomb plasma*, when we may proceed to the limit $\lim c \to \infty$, where c is the speed of light.

We use the following notation: subscript a denotes the plasma component $(a = \mathrm{e}, \mathrm{i})$; e_a and m_a are the charge and the mass of the particle, respectively; N_a is the total number of particles of type a; $n_a = N/V$ is the mean number density of particles; $N_a = N$, and $n = N/V$. The plasma is electrically neutral, which means that

$$\sum_a e_a N_a = 0 \,, \qquad \text{or} \qquad \sum_a e_a n_a = 0 \,. \tag{2.1}$$

We start with the set of equations in the microscopic phase density of each plasma component in the six-dimensional phase space $x = (r, p)$:

$$N_a(x,t) = \sum_{1 \leqslant i \leqslant N} \delta\big(x - x_{ia}(t)\big) \,, \qquad x = (r, p) \,, \tag{2.2}$$

and the microscopic electric field strength $E^{\mathrm{m}}(r,t)$. The equations for the functions (2.2) and the microscopic field strength have the form [5, 12]

$$\frac{\partial N_a}{\partial t} + v\,\frac{\partial N_a}{\partial r} + e_a E^{\mathrm{m}}(r,t)\,\frac{\partial N_a}{\partial p} = 0 \,, \tag{2.3}$$

$$\operatorname{rot} E^{\mathrm{m}} = 0 \,, \qquad \operatorname{div} E^{\mathrm{m}} = 4\pi \sum_a e_a \int N_a(r,p,t)\,\mathrm{d}p \,. \tag{2.4}$$

Interaction between the charged particles is described by Coulomb's law.

The field equations for Coulomb plasma may also be written in the form different from Eqn (2.4):

$$\operatorname{rot} E^{\mathrm{m}} = 0 \,, \qquad \frac{\partial E^{\mathrm{m}}}{\partial t} = -4\pi \sum_a e_a \int v N_a(r,p,t)\,\mathrm{d}p \,. \tag{2.5}$$

To go over from one form to the other, one must use the continuity equation for the microscopic phase density of electric charge $q^{\mathrm{m}} = \sum_a e_a \int N_a(r,p,t)\,\mathrm{d}p$:

$$\frac{\partial q^{\mathrm{m}}}{\partial t} + \operatorname{div} j^{\mathrm{m}} = 0 \,, \tag{2.6}$$

where $j^{\mathrm{m}} = \sum_a e_a \int v N_a(r,p,t)\,\mathrm{d}p$ is the microscopic current density.

Let us consider now the characteristic parameters of fully ionized plasma as required for further discussion. We introduce them using the most simple examples of processes in plasma, and skip the details which can be readily found in textbooks, for example, in Ref. [25], or Ref. [5], Chapter 5.

2.2 Basic parameters of Coulomb plasma

2.2.1 Natural plasma oscillations. Langmuir frequency. Consider an approximation when thermal motion of charged particles can be neglected. Then the plasma motion can be described by equations in the mean number density of particles n_a, mean velocity u_a for each plasma component, and the mean electric field strength E.

This set of equations has a particular solution, in which the velocities u_a and the field E are zero, and the number densities of electrons and ions are equal, $n_a = n_a^0$. The density of electric charge $q^0 = \sum e_a n_a^0$ is also equal to zero. For a small deviation from this state, the equation for the charge density reduces to the equation of harmonic oscillator

$$\frac{\partial^2 q^1}{\partial t^2} + \omega_{\mathrm{L}}^2 q^1(r,t) = 0 \,. \tag{2.7}$$

The squared eigenfrequency (the Langmuir frequency) is given by

$$\omega_{\mathrm{L}}^2 = \sum_a \frac{4\pi e_a^2 n_a}{m_a} \,. \tag{2.8}$$

Along with the plasma Langmuir frequency ω_{L}, one can also consider eigenfrequencies of the plasma components, the electrons and the ions. The corresponding periods of oscillations are the characteristic time parameters of plasma.

2.2.2 Screening of external field in plasma. Debye length. Consider a semiconfined plasma. Let the electric potential on the boundary be φ_0, and $\varphi(\infty) = 0$ at infinity.

If the plasma is at equilibrium, then the function $n_a(x)$ is given by the Boltzmann distribution. Hence, under the condition $e_a \varphi(x) \ll kT$, from the field equations we get a linear equation for the electric potential

$$\frac{\mathrm{d}^2 \varphi}{\mathrm{d}x^2} - \frac{1}{r_{\mathrm{D}}^2}\,\varphi(x) = 0 \,. \tag{2.9}$$

Under the boundary conditions $\varphi(0) = \varphi_0$ and $\varphi(\infty) = 0$ the solution is written in the form

$$\varphi(x) = \varphi_0 \exp\left(-\frac{x}{r_{\mathrm{D}}}\right) , \tag{2.10}$$

where we use the notation

$$r_{\mathrm{D}}^2 = \frac{kT}{\sum_a 4\pi e_a^2 n_a} \tag{2.11}$$

for the square of characteristic distance at which the electric field is screened by the plasma. This is the so-called *Debye radius* (*or Debye length*) r_{D}.

Along with the radius r_{D}, corresponding lengths are defined also for each of the plasma components. All these quantities are the characteristic parameters of plasma.

2.2.3 Space correlation of charged particles. Correlation radius. We introduce the notation for one-particle f_a and two-particle f_{ab} distribution functions. At equilibrium, in the absence of external field, the one-particle distribution functions are $f_a = 1$, and therefore the two-particle distribution functions f_{ab} are linked with the two-particle correlation functions g_{ab} by equalities

$$f_{ab}(|r-r'|) = 1 + g_{ab}(|r-r'|) . \tag{2.12}$$

Let us compare the relations between the main characteristic lengths and the structure of correlation functions for rarefied gas and rarefied plasma.

Rarefied gas. Assume that the atoms make elastic spheres of diameter r_0. Then the correlation radius is $r_{\text{cor}} \sim r_0$. Another characteristic length is the average distance between the particles $r_{\text{av}} \sim (1/n)^{1/3} \equiv (V/N)^{1/3}$. The third characteristic parameter is the mean free path $l \sim 1/nr_0^2$.

Those gases for which the dimensionless density parameter is small, $\varepsilon = nr_0^3 \ll 1$, are called *rarefied*. In order of magnitude, $r_0 \sim 10^{-8}$ cm. At atmospheric pressure, when $r_{\text{av}} \sim 10^{-6}$ cm, and the mean free path is $l \sim 10^{-4}$ cm, the density parameter is $\varepsilon \sim 10^{-4}$, and the following inequalities hold:

$$r_{\text{cor}} \sim r_0 \ll r_{\text{av}} \ll l . \tag{2.13}$$

Thus, the correlation radius is determined by the smallest characteristic length r_0.

Rarefied plasma. The two-particle correlation function for a rarefied plasma is given by

$$g_{ab}(r) = -\frac{e_a e_b}{kT}\,\frac{1}{r}\exp\left(-\frac{r}{r_{\text{D}}}\right), \qquad g_{ab} \ll 1 , \tag{2.14}$$

from whence it follows that the correlation radius in plasma is $r_{\text{cor}} \sim r_{\text{D}}$.

To define a small parameter for a rarefied plasma, we consider the expression for the radial distribution function

$$4\pi g_{ab}(r) r^2 = -4\pi\,\frac{e_a e_b}{kT}\,r\exp\left(-\frac{r}{r_{\text{D}}}\right). \tag{2.15}$$

The absolute value of this function has a maximum at $r = r_{\text{D}}$:

$$|g_{ab}| \sim \frac{e^2}{kTr_{\text{D}}} \sim \frac{1}{nr_{\text{D}}^3} \sim \frac{1}{N_{\text{D}}} \equiv \mu . \tag{2.16}$$

Here we have introduced the *plasma parameter* μ. For a rarefied plasma,

$$\mu \sim \frac{1}{nr_{\text{D}}^3} \sim \frac{1}{N_{\text{D}}} \ll 1 , \qquad N_{\text{D}} \gg 1 . \tag{2.17}$$

Thus, a large number of particles are involved in a simultaneous interaction in a rarefied plasma — the interaction is collective in character. In this respect the situation here is opposite to that in the case of a rarefied gas.

2.2.4 Relaxation correlation scales in plasma. For the electron–ion plasma there are four characteristic relaxation times τ_{ab} ($a = \text{e}, \text{i}$, $b = \text{e}, \text{i}$), or four collision rates $\nu_{ab} = 1/\tau_{ab}$, and four relaxation lengths — the four mean free paths. For a rarefied plasma ($\mu \ll 1$) the relaxation scales of electron-electron interactions are

$$\tau_{\text{ee}} \sim \frac{1}{\mu\omega_{\text{L}}} \sim \frac{T_{\text{L}}}{\mu} \gg T_{\text{L}} ; \qquad l_{\text{ee}} \sim \frac{r_{\text{D}}}{\mu} \ll r_{\text{D}} . \tag{2.18}$$

From Eqn (2.14) it follows that the correlation radius r_{cor} and the corresponding correlation length l_{cor} are determined by the Debye radius:

$$r_{\text{cor}} \sim l_{\text{cor}} \sim r_{\text{D}} . \tag{2.19}$$

The corresponding relaxation time is

$$\tau_{\text{cor}} \sim \frac{r_{\text{cor}}}{v_T} \sim \frac{r_{\text{D}}}{v_T} \sim \frac{1}{\omega_{\text{L}}} \sim T_{\text{L}} . \tag{2.20}$$

Definitions (2.18)–(2.20) lead to the following inequalities:

$$r_{\text{av}} \ll r_{\text{cor}} \sim r_{\text{D}} \ll l_{\text{ee}} ; \qquad \tau_{\text{av}} \ll \tau_{\text{cor}} \sim T_{\text{L}} \ll \tau_{\text{ee}} \sim \frac{T_{\text{L}}}{\mu} . \tag{2.21}$$

We see that the relationships between the average distance r_{av} and the corresponding correlation lengths (r_0 and r_{D}) for gas and plasma are inverse.

2.3 Physically infinitesimal scales for gas and plasma

2.3.1 Rarefied gas. Kinetic level of description. So then, rarefied gas and rarefied plasma are characterized by dimensionless parameters

$$\varepsilon = nr_0^3 \ll 1 , \qquad \mu \sim \frac{1}{nr_{\text{D}}^3} \ll 1 . \tag{2.22}$$

These parameters determine also the linkage between the physically infinitesimal scales τ_{ph} and l_{ph}, and the collision (relaxation) parameters $\tau \equiv \tau_{\text{rel}}$ and $l \equiv l_{\text{rel}}$ of gas and plasma.

The number of particles within physically infinitesimal volume V_{ph} we denote as $N_{\text{ph}} = nV_{\text{ph}}$. By definition of physically infinitesimal scales, the number of particles within V_{ph} volume is large, and the scales τ_{ph} and l_{ph} are small compared to the characteristic scales T and L (τ and l for the Boltzmann gas and Debye plasma):

$$\tau_{\text{ph}} \ll T , \qquad l_{\text{ph}} \ll L , \qquad N_{\text{ph}} \gg 1 . \tag{2.23}$$

The definition of physically infinitesimal scales is not universal. It depends on the selected level of description of nonequilibrium processes (kinetic, hydrodynamic, or diffusion).

Boltzmann gas. For a rarefied gas we arrive at the following relations between the characteristic lengths:

$$r_0 \ll r_{\text{av}} \ll l , \qquad \varepsilon = nr_0^3 \ll 1 . \tag{2.24}$$

The time of transition to the local distribution with respect to velocities is determined by the collision time τ. The corresponding length scale is the mean free path l. Accordingly, for the kinetic stage of relaxation the following replacements must be carried out in Eqn (2.23):

$$T \to \tau , \qquad L \to l . \tag{2.25}$$

Now we may select the physically infinitesimal scales τ_{ph} and l_{ph} which satisfy inequalities (2.23) with the replacements (2.25).

At first sight, the concept of a rarefied gas as a continuous medium seems paradoxical. To show that this is not the case, we do the following.

Let us divide the time interval τ between two consecutive collisions of any selected particle by the number of particles N_{ph}. The resulting time interval refers to the time between collisions of (any!) one particle within the volume V_{ph}. It would be natural to take this time interval as the definition of τ_{ph}. In the kinetic description we also use the relation $\tau_{ph} = l_{ph}/v_T$. As a result, we get two equations

$$\frac{\tau}{N_{ph}} \sim \frac{\tau}{n l_{ph}^3} = \tau_{ph} , \qquad \tau_{ph} = \frac{l_{ph}}{v_T} , \tag{2.26}$$

from which, using the definitions of $\tau = l/v_T$ and ε, we obtain concrete estimates for the physically infinitesimal scales [34]:

$$\tau_{ph} \sim \sqrt{\varepsilon}\,\tau \ll \tau , \qquad l_{ph} \sim \sqrt{\varepsilon}\,l \ll l , \qquad \text{and} \qquad N_{ph} = \frac{1}{\sqrt{\varepsilon}} \gg 1 . \tag{2.27}$$

For a rarefied gas, when inequalities (2.24) hold, these values satisfy the general conditions (2.23).

2.3.2 Boltzmann gas. Hydrodynamic description [5, 33]. In hydrodynamics, the relaxation times are expressed in terms of the external parameter L and one of the three dissipative coefficients of diffusion D, viscosity ν, and thermal conductivity χ. All these are diffusion-type processes. So we denote by D any of the three coefficients D, ν, or χ. Then the relaxation time is given by

$$\tau_D = \frac{L^2}{D} , \tag{2.28}$$

where $D = D, \nu$ or χ. For diffusive processes the linkage between physically infinitesimal scales is determined (in place of the *kinetic relation* $\tau_{ph} = l_{ph}/v_T$) by the appropriate gasdynamic relation

$$\tau_{ph}^{G} = \frac{\left(l_{ph}^{G}\right)^2}{D} , \tag{2.29}$$

where $D = D, \nu$ or χ. In this way, the 'trace' of the diffusive (hydrodynamic) motion is preserved within the physically infinitesimal volume V_{ph} — that is, at a 'point' of continuous medium.

As a result, using definitions (2.28), (2.29), we get concrete estimates for physically infinitesimal scales within gasdynamic description:

$$\tau_{ph}^{G} \sim \frac{\tau_D}{N^{2/5}} \ll \tau_D , \qquad l_{ph}^{G} \sim \frac{L}{N^{1/5}} \ll L , \qquad N_{ph}^{G} \sim N^{2/5} \gg 1 , \tag{2.30}$$

where we have utilized the definition $nL^3 = N$.

We see that now, as opposed to Eqn (2.27) for the kinetic description, the physically infinitesimal scales are linked with the external scale L. This indicates that the definition of physically infinitesimal scales, and hence the very definition of a 'point' of continuous medium, depends very much on the adopted level of description.

2.3.3 Physical Knudsen number. Thus, we have defined the concept of a 'point' for kinetic and gasdynamic description of nonequilibrium processes. Naturally, the gasdynamic description is more rough than the kinetic description, and the 'point' of continuous medium in gas dynamics is larger, i.e. $V_{ph} \leqslant V_{ph}^{G}$. The transition from the kinetic description to a more rough gasdynamic description is traditionally carried out in the following manner.

A dimensionless parameter, the so-called *Knudsen number*, is introduced as

$$\mathrm{Kn} = \frac{l}{L} . \tag{2.31}$$

The approximate solution of the kinetic equation is sought for using the perturbation theory in small Knudsen number (the Hilbert, Chapman – Enskog and Grad methods). The use of perturbation theory, however, is associated with a number of serious difficulties [33, 35, 36], which arise from the fact that the Knudsen number does not reflect the structure of continuous medium well enough.

Instead of the Knudsen number it is natural to employ another dimensionless parameter which is always small within the approximation of continuous medium. It is the *physical Knudsen number*, which under the kinetic description is defined as

$$\mathrm{K}_{ph} = \frac{l_{ph}}{L} . \tag{2.32}$$

The smallness of this parameter is ensured by the fact that $N_{ph} \gg 1$.

For the domain of free-molecule flow, when the characteristic length L (for example, the diameter of the pipe) is much smaller than the mean free path l, the approximation of continuous medium can be utilized if the following inequalities are satisfied:

$$l_{ph} \ll L \ll l . \tag{2.33}$$

With gasdynamic description, the physically infinitesimal scales are defined by Eqn (2.30), and hence the physical Knudsen number is given by

$$\mathrm{K}_{ph}^{G} = \frac{l_{ph}^{G}}{L} \sim \frac{1}{\left(N_{ph}^{G}\right)^{1/2}} . \tag{2.34}$$

The smallness of this parameter is ensured by the condition $N_{ph}^{G} \gg 1$.

2.3.4 Reconciliation of kinetic and gasdynamic definitions of continuous medium. The relationship between two physically infinitesimal volumes V_{ph} and V_{ph}^{G} can be expressed in terms of the density parameter ε and the Knudsen number Kn:

$$\frac{V_{ph}}{V_{ph}^{G}} \sim \varepsilon^{3/10}\,\mathrm{Kn}^{6/5} \leqslant 1 . \tag{2.35}$$

The equality sign here corresponds to the largest value of the Knudsen number (and, accordingly, to the least value of the external parameter L_{min}), at which a unified kinetic and gasdynamic description of continuous medium is still feasible.

Using definition (2.27), from (2.35) we find that

$$L_{min} \sim \sqrt{N_{ph}}\,l_{ph} \sim \frac{l}{\sqrt{N_{ph}}} , \qquad \mathrm{Kn}_{max} \sim \sqrt{N_{ph}} . \tag{2.36}$$

We see that the minimum length L_{min} (the smallest size of the point for which the trace of diffusive motion is still preserved and the hydrodynamic description of motion is still possible) is smaller than the mean free path l, and greater than the physically infinitesimal scale l_{ph} in the kinetic description.

Thus, a unified description of kinetic and hydrodynamic processes is possible in a broad range of Knudsen numbers without using the perturbation theory in Knudsen number Kn.

The corresponding characteristic time scale is defined by

$$\left(\tau^{G}_{ph}\right)_{min} \sim \frac{L^2_{min}}{D} \sim \sqrt{\varepsilon}\,\tau \sim \tau_{ph}\,, \tag{2.37}$$

and is therefore of the order of the physically infinitesimal time interval in the kinetic description. We shall use this result in the derivation of the generalized kinetic equation for unified description of kinetic and hydrodynamic processes in rarefied gas and rarefied plasma.

So then, it is possible to give a general definition of a point of continuous medium. From the above relations we can estimate the number of particles falling on a point:

$$N_{min} = nL^3_{min} \sim \varepsilon^{-5/4}\,. \tag{2.38}$$

At standard conditions, when $\varepsilon \sim 10^{-4}$, we find that $N_{min} \approx 10^5$.

2.3.5. Rarefied Coulomb plasma. Let us demonstrate now that the feasibility range of the unified description of kinetic and hydrodynamic processes in rarefied plasma is even broader than that for a rarefied gas. This is because of the collective nature of interaction between the charged particles. As a consequence, inequalities (2.24) are replaced by the corresponding inequalities for a plasma:

$$r_{av} \ll r_D \ll l\,; \qquad \tau_{av} \ll \frac{1}{\omega_L} \sim T_L \ll \tau_{ee} \sim \frac{1}{\mu\omega_L} \sim \frac{T_L}{\mu}\,;$$

$$\mu = \frac{1}{nr^3_D} \ll 1\,. \tag{2.39}$$

The number of particles within the Debye sphere is

$$N_D \sim \frac{1}{\mu} = nr^3_D \gg 1\,. \tag{2.40}$$

There are two possible ways of defining the physically infinitesimal scales for rarefied plasma [33, 35]. The first is similar to the kinetic definition (2.26) for rarefied gas. In this case we have

$$l_{rel} = l = \frac{r_D}{\mu}\,, \qquad \tau_{ph} = \frac{\tau_{rel}}{N_{ph}} \sim \frac{\tau_{rel}}{nl^3_{ph}}\,, \qquad \tau_{ph} \sim \frac{l_{ph}}{v_T}\,. \tag{2.41}$$

In this way, we have obtained estimates for the physically infinitesimal scales in rarefied plasma:

$$\tau_{ph} \sim \mu\tau_{ee} \sim \frac{1}{\omega_L} \ll \tau_{ee}\,, \qquad l_{ph} \sim r_D \ll l\,,$$

$$N_{ph} \sim N_D \sim \frac{1}{\mu} \gg 1\,. \tag{2.42}$$

These definitions satisfy the general conditions (2.23). However, they do not reflect well enough the physical difference between the Boltzmann gas and the Debye plasma.

Indeed, the Debye radius defines the distance of interaction between charged particles and the number of particles within the Debye sphere $N_D \gg 1$. *Thus, the nature of interaction between charged particles in plasma is collective*. To take this property of rarefied plasma into account, one must use the parameter r_D for defining the length l_{ph}, and define the physically infinitesimal time interval as the time of diffusion of charged particles within a sphere of Debye radius:

$$l_{ph} \sim r_D \ll l\,, \qquad \tau_{ph} \sim \frac{r^2_D}{D}\,, \qquad D \sim v_T\, l\,. \tag{2.43}$$

Here, like in the case of rarefied gas, we assume by definition that the three kinetic coefficients of diffusion D, viscosity ν, and thermal conductivity χ are the same.

We see that the physically infinitesimal intervals are now linked by the gasdynamic relation

$$\tau_{ph} \sim \frac{l^2_{ph}}{D}\,. \tag{2.44}$$

Thus, for a rarefied plasma there is a broad (broader than in the case of a rarefied gas!) range of parameters wherein a unified description of kinetic and gasdynamic processes is possible. Now the role of the length L_{min} [see Eqn (2.36)] is played by the Debye radius. The physical Knudsen number and the maximum value of the conventional Knudsen number [see Eqns (2.32), (2.31)] are defined for plasma as follows:

$$K_{ph} = \frac{l_{ph}}{L} \sim \frac{r_D}{L}\,, \qquad \mathrm{Kn}_{max} = \frac{l}{r_D} = \frac{1}{\mu} \gg 1\,. \tag{2.45}$$

The last inequality opens up the possibility of unified description of kinetic and hydrodynamic processes in plasma in the range $L > r_D$! Cases are known, however, when there are two Debye radii rather than one. Then inequality (2.45) has to be modified.

For anisothermal plasma, when $T_e > T_i$, one can introduce two Debye radii for electrons and ions:

$$r^2_{De} = \frac{kT_e}{4\pi e^2 n}\,, \qquad r^2_{Di} = \frac{kT_i}{4\pi e^2 n}\,, \qquad r_e \gg r_i\,, \tag{2.46}$$

and define the physically infinitesimal length in terms of the Debye radius for ions.

Now let us summarize our results.

Physically infinitesimal scales for a rarefied gas are expressed via the principal scales of the kinetic theory of gases: the free transit time τ, and the mean free path l. The parameters τ and l, as well as the corresponding physically infinitesimal scales, are linked through the thermal velocity. Upon transition to the gasdynamic description, however, this linkage is changed. Then the principal time and length scales and the corresponding physically infinitesimal scales are linked through the 'diffusion' relations. Because of this, a unified description of kinetic and hydrodynamic processes calls for reconciliation of the kinetic definition of continuous medium with the hydrodynamic one, using equations (2.35) for a unified definition of the size of a point of continuous medium.

In case of a rarefied plasma, along with the principal relaxation parameters τ and l there exist smaller (but still macroscopic) parameters $T_L = 1/\omega_L$ and r_D, which characterize processes in plasma. Like the corresponding relaxation parameters, they are linked through the thermal velocity. As a result, we come to equations (2.41).

It is more natural, however, to express the linkage between the physically infinitesimal scales $\tau_{\rm ph}$ and $l_{\rm ph}$ for plasma in terms of the *diffusion relation*. This definition of the structure of continuous medium opens broad opportunities for a unified description of kinetic and hydrodynamic processes in a rarefied plasma.

Now we collected sufficient information about the structure of rarefied plasma regarded as a continuous medium to embark on the construction of the relevant kinetic equations.

3. Averaging the microscopic equations for plasma

3.1 Approximation of second moments

Let us carry out averaging of the microscopic equations for Coulomb plasma over the Gibbs ensemble. We use the following definitions of the distribution functions and the mean electromagnetic field:

$$\langle N_a(x,t)\rangle = n_a f_a(x,t)\,, \qquad \langle E^{\rm m}(r,t)\rangle = E(r,t)\,, \tag{3.1}$$

and the equation

$$\langle E^{\rm m} N_a\rangle = Enf_a + \langle \delta E \delta N_a\rangle_{x,x',t}\,. \tag{3.2}$$

As a result, we get the equation for the distribution function f_a. Here it will be convenient to represent this equation in the form

$$\frac{\partial f_a}{\partial t} + v\,\frac{\partial f_a}{\partial r} + e_a E(r,t)\,\frac{\partial f_a}{\partial p} = -\frac{1}{n}\frac{\partial}{\partial p}\langle \delta F_a\, \delta N_a\rangle \equiv I_a(r,p,t)\,. \tag{3.3}$$

Averaging the microscopic equations for Coulomb plasma, we arrive at the equations for the mean field:

$$\operatorname{rot} E = 0\,, \qquad \operatorname{div} E = 4\pi \sum_a e_a\, n_a \int f_a(r,p,t)\,{\rm d}p\,. \tag{3.4}$$

The set of equations in averaged functions $f_a(r,p,t)$ and $E(r,t)$ is not closed, because it also includes a correlator — the collision integral

$$-\frac{1}{n}\frac{\partial}{\partial p}\langle \delta F_a\, \delta N_a\rangle \equiv I_a(r,p,t)\,, \tag{3.5}$$

which is determined by correlators of fluctuations

$$\delta N_a = N_a - n_a f_a\,, \qquad \delta E = E^{\rm m} - E\,. \tag{3.6}$$

The term 'collision integral' is used to emphasize the analogy with the kinetic Boltzmann equation for a gas. This analogy, however, is superficial, because each charged particle in a rarefied plasma interacts simultaneously with a large number of surrounding particles. For this reason why Boltzmann's model of pair collisions is not suitable in this case.

Let us now find the equations for fluctuations. The equations for functions $\delta N_a(r,p,t)$ are obtained with the aid of Eqns (2.3), (3.3):

$$\left[\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial r} + e_a \delta E(r,t)\,\frac{\partial}{\partial p}\right]\delta N_a + \delta F_a\,\frac{\partial f_a}{\partial p} = -e_a\,\frac{\partial}{\partial p}\left[\delta E \delta N_a - \langle \delta E \delta N_a\rangle\right]. \tag{3.7}$$

The corresponding equations for the field fluctuations are

$$\operatorname{rot} \delta E = 0\,, \qquad \operatorname{div} \delta E = 4\pi \sum_a e_a \int \delta N_a(r,p,t)\,{\rm d}p\,. \tag{3.8}$$

The calculation of fluctuations is complicated by the fact that equations (3.7) are nonlinear. As in the case of rarefied gas, this gives rise to a chain of coupled equations for the moments of fluctuations. This time, however, the problem is much more complicated, because one has to know not only the particle distribution functions, but also that of the electromagnetic field.

The situation is much simplified for a rarefied plasma. Then, as we already know, it is possible to introduce physically infinitesimal scales of length and time in accordance with the inequalities in Eqns (2.41) – (2.44). The number of particles within the physically infinitesimal volume $V_{\rm ph}$ will be defined in that case by Eqn (2.42). For a rarefied plasma, when the plasma parameter is $\mu \ll 1$, the number of particles is $N_{\rm ph} \gg 1$. By virtue of this condition the fluctuations δN_a may be considered small. Provided this condition is satisfied, the right-hand side of Eqn (3.7) may be set equal to zero. As a result, we get the following equation for fluctuations in Coulomb plasma within the *approximation of second moments*:

$$\left[\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial r} + e_a E(r,v,t)\,\frac{\partial}{\partial p}\right]\delta N_a + e_a n_a \delta E\,\frac{\partial f_a}{\partial p} = 0\,. \tag{3.9}$$

3.2 Approximation of second correlation functions

The approximation considered above is not sufficient for constructing the kinetic equations — that is, equations in the one-particle (or, more precisely, single-point) distribution functions f_a. This is because the approximation of second moments represents the plasma as a continuous medium. In order to take into account the structure of plasma as a system of charged particles, we use the so-called *approximation of second correlation functions*. This enables us to disregard the triple and higher correlation functions, and also to assume the two-particle correlation functions g_{ab} to be small. The last condition implies that

$$g_{ab}(x,x',t) \ll f_a(x,t) f_b(x',t)\,. \tag{3.10}$$

To thrash out the difference between the approximations of second moments and second correlation functions, let us consider the relationship between $\langle \delta N_a \delta N_b\rangle_{x,x',t}$ and $g_{ab}(x,x',t)$. They are linked by the following expression:

$$\langle \delta N_a \delta N_b\rangle_{x,x',t} = n_a n_s g_{ab}(x,x',t) + n_a \delta_{ab}\delta(x-x') f_a(x,t)\,. \tag{3.11}$$

The second term on the right-hand side appears because the two-particle correlation function (by its very nature) characterizes the statistical linkage between different particles. To take this difference into account (thus considering the plasma as a system of charged particles), we represent the fluctuation δN_a as a sum of two terms

$$\delta N_a(x,t) = \delta N_a^{\rm ind}(x,t) + \delta N_a^{\rm (s)}(x,t)\,, \tag{3.12}$$

where superscript 'ind' stands for 'induced', and 's' for 'source'. The second term on the right-hand side $\delta N_a^{\rm (s)}(x,t)$ takes care of the structure of plasma as a system of charged particles. The second one-time moment of these fluctuations (*fluctuations of the source* 's') is defined by the second term on the right-hand side of Eqn (3.11). Thus, we have

$$\left\langle \delta N_a \delta N_b \right\rangle^{(s)}_{x,x',t} = n_a \delta_{ab} \delta(x - x') f_a(x,t) \,. \qquad (3.13)$$

The first term on the right-hand side of Eqn (3.12)

$$\delta N_a^{\mathrm{ind}}(x,t) = \delta N_a(x,t) - \delta N_a^{(s)}(x,t) \qquad (3.14)$$

is defined by the particular solution of the non-homogeneous equation (3.9), which we conveniently represent now as

$$\left[\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial r} + e_a E(r,v,t)\,\frac{\partial}{\partial p}\right]\left[\delta N_a(x,t) - \delta N_a^{(s)}(x,t)\right]$$

$$= -e_a n_a \delta E\,\frac{\partial f_a}{\partial p} = 0 \,. \qquad (3.15)$$

Superscript 'ind' indicates that the right-hand side of the equation is proportional to the field fluctuation δE. Because of this, fluctuations (3.14) in the solution of non-homogeneous equation (3.15) are caused (induced) by fluctuations of the field.

To calculate the fluctuations $\delta N_a^{(s)}(x,t)$ (*the source fluctuations*), one must use the following equation for the two-time correlator of fluctuations:

$$\left[\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial r} + e_a E(r,v,t)\,\frac{\partial}{\partial p}\right]\left\langle \delta N_a \delta N_b \right\rangle^{(s)}_{xt,x',t'} = 0 \,. \qquad (3.16)$$

Expression (3.13) serves as the initial condition (at $t = t'$) for the solution of this equation. Thus, we have

$$\left\langle \delta N_a \delta N_b \right\rangle^{(s)}_{x,x',t,t'}\Big|_{t=t'} = n_a \delta_{ab} \delta(x - x') f_a(x,t) \,. \qquad (3.17)$$

Now we can calculate the fluctuations in fully ionized Coulomb plasma.

4. Two alternative approximations in statistical theory of plasma

4.1 Traditional and nontraditional methods of describing the nonequilibrium processes in Boltzmann gas

Two levels of description of nonequilibrium processes in a rarefied gas are commonly used. One is based on the kinetic Boltzmann equation, and the other on the equations of gas dynamics. The latter are obtained from the approximate solution of the kinetic equation by using perturbation theory in Knudsen number. This results in a closed theory for description of nonequilibrium processes in rarefied gases on kinetic and hydrodynamic scales.

It seems that everything is available for solving particular problems with the aid of the ready-made equations. The situation, however, is not that simple for the following reason. Both the kinetic and the gasdynamic descriptions are given without specifying the structure of continuous medium. At the same time, for kinetic and gasdynamic descriptions the continuous medium is not the same.

As shown in Chapter 13 of Ref. [33], the form of irreversible equations in the statistical theory of nonequilibrium processes is changed considerably when the structure of continuous medium is taken into account. Established was the generalized kinetic equation as a basis for unified description of nonequilibrium processes on both the kinetic and the gasdynamic scales.

Going over from the kinetic equation to gasdynamic equations was accomplished in this case without using the perturbation theory in Knudsen number. This opens up the possibility of providing a unified kinetic description of nonequilibrium processes not only in passive, but also in active media. We shall illustrate it below by the example of Coulomb plasma.

4.2 Smoothing over the volume of a 'point' in nonequilibrium medium

Two levels of description of nonequilibrium processes are also used in the theory of plasma. The first is based on the kinetic equations of Landau, Vlasov, or Balescu and Lenard. The second method is based on the equations of gas dynamics, which are derived from the approximate solution of the kinetic equation by using perturbation theory in small Knudsen number. Both the kinetic and the gasdynamic descriptions are given within the framework of the model of continuous medium. Here also the structure of continuous medium is different for kinetic and hydrodynamic descriptions.

The definition of physically infinitesimal scales for plasma is given by Eqns (2.42), (2.43), which are based on the diffusion relation between the time τ_{ph} and the length l_{ph}. This definition of the structure of continuous medium, as already indicated, opens a broad opportunity for a unified description of kinetic and hydrodynamic processes in a rarefied plasma. The relevant generalized kinetic equations will be presented in the forthcoming sections; they can serve as a basis for unified description of nonequilibrium processes in a rarefied plasma on both the kinetic and the hydrodynamic scales.

The first task in the implementation of this program consists in finding the spectral densities of fluctuations $\delta N_a^{(s)}$ and δE for nonequilibrium states on the basis of the last two equations.

Let us go back to inequalities (2.21), (2.39), (2.41), which characterize the relationships between the basic parameters of the Debye plasma. As with theory of gases, these inequalities allow separating the small-scale (fine-grain) and the large-scale (kinetic, or coarse-grain) fluctuations. The small-scale fluctuations are defined by the following conditions:

$$\tau_{\mathrm{cor}} \leqslant \tau_{\mathrm{ph}} \ll \tau_{\mathrm{rel}} \,, \qquad r_{\mathrm{cor}} \leqslant l_{\mathrm{ph}} \ll l_{\mathrm{rel}} \,. \qquad (4.1)$$

So far the initial equations have been the reversible equations (2.3), (2.4) for the microscopic phase densities $N_a(r,p,t)$ of charged particles and the microscopic electric field strength $E^{\mathrm{m}}(r,t)$. Let us now see how change is made to the irreversible kinetic equation with due regard for the structure of continuous medium.

At the first step in going over to irreversible equations, as in the Boltzmann gas theory (see Eqn (13.3.1) in Ref. [33]), a relaxation term is introduced into Eqn (2.3) for the microscopic phase density of Coulomb plasma (cf. Eqn (13.3.1) in Ref. [33]). This term describes the 'adjustment' of the microscopic phase density of particles $N_a(r,p,t)$ to the corresponding distribution $N_a(r,p,t)$, smoothed over the volume of a point of continuous medium. As a result, in place of (2.3), (2.4) we get the following irreversible equations for the phase density $N_a(r,p,t)$ and the microscopic electric field:

$$\frac{\partial N_a}{\partial t} + v\,\frac{\partial N_a}{\partial r} + e_a E^{\mathrm{m}}\,\frac{\partial N_a}{\partial p} = -\frac{1}{\tau_{\mathrm{ph}}^{(a)}}\left[N_a(r,p,t) - \widetilde{N}_a(r,p,t)\right], \tag{4.2}$$

$$\operatorname{rot} E^{\mathrm{m}} = 0\,, \qquad \operatorname{div} E^{\mathrm{m}} = 4\pi \sum_a e_a \int N_a(r,p,t)\,\mathrm{d}p\,. \tag{4.3}$$

The last term in Eqn (4.3) contains smoothed (over the volume of a point of continuous medium) microscopic phase density

$$\widetilde{N}_a(r,p,t) = \int N_a(r-\rho,\,p,\,t)F_a(\rho)\,\mathrm{d}\rho\,. \tag{4.4}$$

The particular definition of the smoothing function $F_a(\rho)$ depends on a number of factors, which include also the nature of the mean field $E(r,t)$. Here we are going to use the simplest representation in the form of Gaussian distribution with the mean value proportional to the force $e_a E(r,t)$:

$$F_a(\rho) = \frac{1}{\left[2\pi l_{\mathrm{ph}}^{(a)2}\right]^{3/2}} \exp\left[-\frac{\left(\rho - \langle\rho\rangle_a\right)^2}{2l_{\mathrm{ph}}^{(a)2}}\right], \tag{4.5}$$

$$\langle\rho\rangle_a = b_a\,\frac{e_a}{m_a}\,E(r,t)\tau_{\mathrm{ph}}^{(a)}\,.$$

The variance is determined in our case by the size l_{ph} of the point. Coefficient $b = \tau_{\mathrm{rel}}^{(a)}$ characterizes the *mobility* under the action of the mean force. The mobility is determined by the collision time (relaxation time). Finally, $\langle\rho\rangle$ is the corresponding mean displacement in a time τ_{ph}.

Now we carry out averaging over the Gibbs ensemble to obtain the equation for the distribution function $f_a(r,p,t)$ with due regard for the structure of continuous medium. In place of Eqns (4.2), (4.3) we arrive at the following equations for the Coulomb plasma:

$$\frac{\partial f_a}{\partial t} + v\,\frac{\partial f_a}{\partial r} + e_a E\,\frac{\partial f_a}{\partial p} = -\frac{e_a}{n_a}\,\frac{\partial}{\partial p}\langle\delta E\delta N_a\rangle - \frac{1}{\tau_{\mathrm{ph}}^{(a)}}\left[f_a(r,p,t) - \widetilde{f}_a(r,p,t)\right], \tag{4.6}$$

$$\operatorname{rot} E = 0\,, \qquad \operatorname{div} E = 4\pi \sum_a e_a n_a \int f_a(r,p,t)\,\mathrm{d}p\,. \tag{4.7}$$

This set of equations is not closed, because it contains not only the first moments $f_a(r,p,t)$, $E(r,t)$ of the corresponding microscopic functions, but also the correlator $\langle\delta E\delta N_a\rangle$ of fluctuations δN_a, δE. As compared with the traditional equations (see Chapter 15 in Ref. [5]), here we have an additional relaxation term which allows for the structure of continuous medium!

In accordance with the right-hand inequality in Eqn (4.1), when calculating the small-scale fluctuations the distribution functions $f_a(r,p,t)$ and the electric field $E(r,t)$ are slowly varying functions of coordinates and time. The degree of slowness is determined by the following relations

$$\frac{\tau_{\mathrm{ph}}}{\tau_{\mathrm{rel}}} \sim \mu^2 \ll 1\,, \qquad \frac{l_{\mathrm{ph}}}{l_{\mathrm{rel}}} \sim \mu \ll 1\,. \tag{4.8}$$

In the zero approximation with respect to these parameters, one may completely disregard the space and time variations of the functions $f_a(r,p,t)$, $E(r,t)$ in the calculation of small-scale fluctuations. Under this condition, the correlator with respect to variables r, t, r', t' is

$$\big\langle(\delta N_a\delta N_b)\big\rangle_{r,t,r',t'} = \big\langle\delta N_a\delta N_b\big\rangle_{r-r',\,t-t',\,p,\,p'}\,, \tag{4.9}$$

and only depends on the time and coordinate differences. The dependence on r and t is only via the function $f_a(r,p,t)$, whose change is not taken into account in the calculation of small-scale fluctuations.

Equations (4.2) – (4.7) will be used in deriving the generalized kinetic equations for a rarefied plasma. To do this, we must first build a bridge between traditional and nontraditional kinetic theories of plasma. For this purpose we must return to the approximate equations (3.15), (3.16) for fluctuations of the microscopic phase density δN_a.

To separate the fluctuations at the point of continuous medium (the small-scale fluctuations), we introduce the following dissipative terms into equations for spatial Fourier components:

$$-\Delta_a\delta N_a(k,p,t)\,, \qquad -\Delta_a(\delta N_a\delta N_b)_{t-t',\,k,\,p,\,p'}\,; \qquad \Delta_a = \frac{1}{\tau_{\mathrm{ph}}^{(a)}}\,. \tag{4.10}$$

As a result, from Eqns (3.15), (3.16) we get the equations for the relevant Fourier components

$$\left(\frac{\partial}{\partial t} + \Delta_a + \mathrm{i}kv + e_a E\,\frac{\partial}{\partial p}\right)\left[\delta N_a(k,p,t) - \delta N_a^{(\mathrm{s})}(k,p,t)\right] = -e_a n_a\delta E(k,t)\,\frac{\partial f_a}{\partial p} \tag{4.11}$$

and

$$\left(\frac{\partial}{\partial t} + \Delta_a + \mathrm{i}kv + e_a E\,\frac{\partial}{\partial p}\right)(\delta N_a\delta N_b)^{(\mathrm{s})}_{t-t',\,k,\,p,\,p'} = 0\,, \tag{4.12}$$

which are supplemented with the initial conditions

$$(\delta N_a\delta N_b)^{(\mathrm{s})}_{t-t',\,k,\,p,\,p'}\Big|_{t=t'} = n_a\delta_{ab}\delta(p-p')f_a\,. \tag{4.13}$$

Recall that these equations have been obtained in the approximation when the variations of functions $f_a(r,p,t)$, $E(r,t)$ in space and time are completely disregarded in the calculation of small-scale fluctuations. As a result, one arrives at the following expression for the space-time spectral density of fluctuations (see Section 15.5 in Ref. [5]):

$$(\delta N_a\delta N_b)^{(\mathrm{s})}_{\omega,\,k,\,p,\,p'} = n_a\delta_{ab}\delta(p-p')\,\frac{\Delta_a}{(\omega-kv)^2 + \Delta_a^2}\,f_a(r,p,t)\,. \tag{4.14}$$

We have obtained a spectral line whose width is determined by the smallest time interval $\tau_{\mathrm{ph}}^{(a)} = 1/\Delta_a$. This line is much broader than those spectral lines whose widths are determined by the collision rate $1/\tau_{\mathrm{rel}}^{(a)}$ or a diffusion process.

In the zero approximation with respect to $1/\Delta\tau_{\mathrm{rel}}^{(a)}$, the width of this spectral line may be considered infinitely large. This allows us to take advantage of the concept of 'white noise', which is widely used in the theory of Brownian motion in connection with the introduction of the so-called Langevin source. Thus, the spectral density of fluctuations of random source in the zero approximation is the same at all frequencies and corresponds to a white noise.

The corresponding time correlation functions are proportional to $\delta(t - t')$. Of course, the actual width of this time-dependent function is not zero. It is determined by the smallest temporal parameter. In the theory currently under consideration, this parameter is the physically infinitesimal time interval τ_{ph}. Then we have $\delta(t - t')\big|_{t=t'} = 1/\tau_{\text{ph}}$. The corresponding time spectrum is close to a white noise.

From arguments developed above it follows that the results of the theory of Brownian motion can be employed for calculating the fluctuations in plasma and deriving the relevant kinetic equations. In the case of Brownian motion, the source intensity does not depend on the frequency in the zero approximation (white noise). Nevertheless, until recently the development of the theory of nonequilibrium processes in plasma took a different path! Let us explain the essential difference between these two approaches.

4.3 Method of adiabatic switching on the interaction

The traditional theory relies on the formal mathematical technique of regularizing the expressions for spectra in the neighbourhood of resonances. Such regularization is necessary in many problems of mathematics, physics, and both classical and quantum mechanics.

This question arose for the first time when the perturbation theory was applied to the three-body problem. This theory relied heavily on the works of Henri Poincaré, and so we speak today of *Poincaré resonances*. One of the mathematical techniques used to overcome these difficulties is based on the methods of the theory of functions of complex variables, namely, the methods for regularizing the divergence. Regularization is performed by going over into the complex domain and selecting the sign of detouring around the pole in accordance with the causality principle. In physics, this method of regularization is used in the theory of radiation. The sign of the pole detouring is determined by the condition of the wave divergence.

In plasma theory, a similar problem arises, for instance, in connection with the so-called *collisionless Landau damping coefficient*. A physical interpretation of this phenomenon will be given below.

Similar 'difficulties' arise also in quantum mechanics in calculating the probabilities of transition in the presence of variable fields (see, for example, Chapter 6 in Ref. [37]).

Along with the mathematical method, which became known in physics as *Fermi's golden rule*, another method is used for calculating the transition probabilities. It is based on the assumption that the variable field is switched on not at the initial point in time $t = 0$, but at $t = -\infty$. Correspondingly, the field slowly (adiabatically) increases according to the exponential law $\exp(\lambda_a t)$ with positive λ_a. The limit $\lambda_a \to 0$ is reached in the final expressions. In this way, the time interval $1/\lambda_a$ is introduced, which exceeds all the other characteristic scales of the system in question.

The method of adiabatic switching on the interaction is widely employed in the statistical theory of nonequilibrium processes. It is useful when deriving the kinetic equations for most diverse systems. It lies in the basis of a very popular technique, the so-called *method of Green's functions*.

Given all the diversity of the calculation techniques, they have one common feature: a formal change from the reversible equations of classical and quantum mechanics to the irreversible equations of statistical theory. The physical justification of this change, however, remains 'beyond the frame'.

We see that one way around this problem associated with the existence of resonances leads to irreversible equations. In so doing, the dissipation, albeit implicitly, finds its path into the theory. The necessity of including the dissipation in the presence of resonances became obvious a long time ago. For example, L I Mandelshtam in his famous ''Lectures on the theory of oscillations'', read at the Department of Physics of Moscow State University in 1930, said:

''*Hence we see that, when considering a steady oscillation close enough to the resonance, we must take damping into account, however small it may be*.''

And further on:

''*As we have seen, however, the smaller λ_a, the slower the oscillations reach the steady state. When λ_a tends to zero, the amplitude of oscillations goes to infinity, but the steady regime is established in infinitely large time, which means that it is never established*.''

So then, the *method of adiabatic switching on the interaction* is not sufficient for the construction of the theory of nonequilibrium processes, because it does not describe the actual change to irreversible equations.

Indeed, going over to the kinetic equations implies change from the reversible equations of motion to the irreversible equations of continuous medium. This change to a more simple description becomes possible and practically inevitable owing primarily to the *complexity of motion* of the Hamiltonian system under consideration. In particular, this complexity is manifested in the *dynamic instability of motion*, and, as a consequence, in the *mixing of paths in the phase space*. It is the latter that justifies the introduction of physically infinitesimal elements with subsequent smoothing over the physically infinitesimal volume — over the volume of the point of continuous medium. Because of this, the information about the motion through the points of continuous medium is lost, and the corresponding equations of the statistical theory become irreversible [30, 33, 38–41].

Naturally, the physically infinitesimal scales are the smallest among the characteristic scales for equations in the approximation of continuous medium. In this respect the situation is opposite to that encountered in the theory based on the adiabatic switching on the interaction, when the switching on time is larger than any other characteristic parameter of time in equations obtained in this manner.

The nontraditional method of describing the nonequilibrium processes was discussed in detail in Ref. [33] using the example of Boltzmann gas. Here we are going to employ an analogous method when describing the nonequilibrium processes in an essentially different system — in a fully ionized plasma. A similar approach to the construction of statistical theory is also possible for quantum open systems. The discussion of this problem was started in the last chapter of Ref. [33].

Of course, the remarks made above do not deny the usefulness of the traditional statistical theory of nonequilibrium processes in plasma. It would be natural to use a reasonable blend of the *old* and the *new*. From this standpoint we shall first consider in brief the most important results of the traditional kinetic theory of plasma.

5. Kinetic equations for fully ionized plasma. Conventional approximation

5.1 Spectral densities of fluctuations in Coulomb plasma

To go over to the adiabatic approximation, we must change the meaning of the parameter Δ_a in Eqns (4.10)–(4.14). Namely, we replace Δ_a with the parameter of the adiabatic theory λ_a, i.e. $\Delta_a \to \lambda_a$. As a result, Eqn (4.14) takes the form

$$(\delta N_a \delta N_b)^{(s)}_{\omega, k, p, p'} = n_a \delta_{ab} \delta(p - p') \frac{2\lambda_a}{(\omega - kv)^2 + \lambda_a^2} f_a(r, p, t), \quad \lambda_a \to 0 . \tag{5.1}$$

Using the definition of δ-function

$$\lim_{\lambda_a \to 0} \frac{\lambda_a}{(\omega - kv)^2 + \lambda_a^2} = \pi \delta(\omega - kv) , \tag{5.2}$$

we get the final expression which has no explicit dependence on λ_a:

$$(\delta N_a \delta N_b)^{(s)}_{\omega, k, p, p'} = n_a \delta_{ab} \delta(p - p') 2\pi \delta(\omega - kv) f_a(r, p, t) . \tag{5.3}$$

In our current approximation, this is the most general expression for the spectral density of fluctuations conditioned by the molecular structure of the plasma. Making use of the field equations, we find the relationship between the Fourier components of functions δE, δN_a:

$$\delta E(\omega, k) = -\frac{ik}{k^2} \sum_a 4\pi e_a \int \delta N_a(\omega, k, p) \, dp . \tag{5.4}$$

From Eqn (4.11) follows a second relationship between the Fourier components of fluctuations δE, δN_a:

$$\delta N_a(\omega, k, p) = \delta N_a^{(s)}(\omega, k, p) - \frac{ie_a}{\omega - kv + i\lambda_a} \delta E(\omega, k) \frac{\partial (n_a f_a)}{\partial p} , \qquad \lambda_a \to 0 . \tag{5.5}$$

Now we can eliminate the function $\delta N_a(\omega, k, p)$ from the last two equations. It is convenient to represent the resulting equation in the form

$$\varepsilon(\omega, k) \delta E(\omega, k) = -\frac{ik}{k^2} \sum_a 4\pi e_a \int \delta N_a^{(s)}(\omega, k, p) \, dp , \tag{5.6}$$

where

$$\varepsilon(\omega, k) = 1 + \sum_a \frac{4\pi e_a^2}{k^2} n_a \int \frac{k \, \partial f_a / \partial p}{\omega - kv + i\lambda_a} \, dp , \qquad \lambda_a \to 0 \tag{5.7}$$

is the permittivity of Coulomb plasma.

Using these equations, we find the space–time spectral density of the field fluctuations in the *approximation of adiabatic switching on the interaction*:

$$(\delta E \delta E)_{\omega, k} = \frac{1}{|\varepsilon(\omega, k)|^2} \sum_a \frac{(4\pi)^2 e_a^2}{k^2} n_a \int 2\pi \delta(\omega - kv) f_a \, dp . \tag{5.8}$$

From the latter follows the expression for the space spectral density of the field fluctuations in Coulomb plasma:

$$(\delta E \delta E)_k = \sum_a \frac{(4\pi)^2 e_a^2}{k^2} n_a \int \frac{f_a}{|\varepsilon(kv, k)|^2} \, dp , \tag{5.9}$$

which depends on the polarization properties of plasma via $\varepsilon(\omega, k)$.

Observe once again that this calculation of fluctuations has been carried out *in the approximation of adiabatic switching on the interaction*. In this approximation, the dissipation factor λ_a [and, as a consequence, the width of the spectral line of fluctuations (5.1)] tends to zero. As a result, the corresponding correlation time tends to infinity.

Thus, the '*starting point*' *in the traditional theory of fluctuations is the most coherent state*, whose lifetime is longer than any other characteristic time of the system in question.

This approach, however, overlooks the physical essence of the irreversibility origin. Indeed, going over to the irreversible equations of continuous medium is governed by the elimination of particles motion within the physically infinitesimal volumes. In other words, the dissipative terms in the kinetic equations are determined by the small-scale fluctuations, and their correlation time is determined by the least characteristic time interval. So we have *two opposite starting points* for the going over from the reversible equations of particle motion to the irreversible kinetic equations. If we start with equations (4.2)–(4.7), then the first step towards the irreversible kinetic equations consists in the *elimination of the most chaotic (non-coherent) motion within the points* whose size is about $l_{\text{ph}}^{(a)}$. This motion is characterized by the smallest correlation times $\tau_{\text{ph}}^{(a)}$ for the components of rarefied Coulomb plasma.

As we have seen, the conventional kinetic theory starts from the opposite position. The initial state is the most coherent one, *since the corresponding correlation time is larger than any characteristic time of the plasma.*

Of course, these two approaches are not equivalent. To be able to compare the results of two approximations, we must recall the main results of the conventional kinetic theory of plasma.

5.2 Kinetic equations for rarefied Coulomb plasma

Now we can find the expression for the more general spectral density $(\delta N_a \delta E)_{\omega, k}$. The real part of this spectral density determines the form of the 'collision integral' $I_a(r, p, t)$ in the kinetic equation (3.3), since this integral can be represented as

$$I_a(r, p, t) = -\frac{e_a}{n_a} \frac{\partial}{\partial p} \int \text{Re} \, (\delta N_a \delta E)_{\omega, k, r, p, t} \frac{d\omega \, dk}{(2\pi)^4} . \tag{5.10}$$

We shall find the required expression for the spectral density of fluctuations with the aid of Eqns (5.5), (5.6). The dependence on r and t is implicit [via the distribution function $f_a(r, p, t)$]. As a result, we find the collision integral

$$I_a(r, p, t) = \sum_b 2e_a^2 e_b^2 n_b \frac{\partial}{\partial p_i} \int \frac{k_i k_j}{k^4} \frac{\delta(kv - kv')}{|\varepsilon(kv, k)|^2} \times \left[\frac{\partial f_a(r, p, t)}{\partial p_i} f_b(r, p', t) - \frac{\partial f_b(r, p', t)}{\partial p_{ij}} f_a(r, p, t) \right] dk \, dp' . \tag{5.11}$$

This expression was first obtained by Balescu and Lenard, and is known as the *Balescu–Lenard collision integral.*

The permittivity $\varepsilon(\omega, k)$ being dependent on the distribution functions, the Balescu–Lenard collision integral is extremely complicated. It will be expedient therefore to consider the possibilities of simplifying this equation.

5.3 Landau collision integral

The inclusion of polarization into the collision integral restricts the interval of integration with respect to k. At large values of k, however, the integrand in the collision integral is proportional to $1/k$. Then integration with respect to k leads to logarithmic divergence. This is a consequence of violating the applicability condition of perturbation theory. Because of this, the upper limit of integration with respect to k is selected as

$$k < \frac{k_{\mathrm{B}} T}{e^2} = \frac{1}{l_{\mathrm{L}}}\,, \tag{5.12}$$

Now we may proceed further and set $\varepsilon(kv, k) = 1$, but take the polarization into account by imposing the condition $k > 1/r_{\mathrm{D}}$ on the lower limit of integration with respect to k. As a result, the lower and the upper limits of integration with respect to k will be defined by the following conditions

$$k_{\min} = \frac{1}{r_{\mathrm{D}}}\,, \qquad k_{\max} = \frac{1}{l_{\mathrm{L}}}\,, \tag{5.13}$$

and integration with respect to k gives rise to a factor, the so-called *Coulomb logarithm*:

$$L = \ln \frac{r_{\mathrm{D}}}{l_{\mathrm{L}}} \sim \ln \frac{1}{\mu}\,, \qquad \mu \ll 1\,. \tag{5.14}$$

It was this simpler expression that was established in Landau's paper [1] but in a different way. We shall use it later when deriving the generalized kinetic equation for plasma.

5.4 Approximation of first moments. Vlasov equations

Recall that the set of equations in the first moments $\langle N_a(x, t)\rangle = n_a f_a(x, t)$, $\langle E^{\mathrm{m}}(r, t)\rangle = E(r, t)$ of the relevant microscopic functions is nonclosed, since it contains the collision integral I_a. The latter is defined by the correlator of fluctuations of microscopic phase density and microscopic electromagnetic field.

The simplest case, when the set of equations in the first moments is closed, corresponds to the zero approximation with respect to fluctuations. Then the collision integral vanishes, and we arrive at the Vlasov equations (1938). For Coulomb plasma these equations have the form

$$\frac{\partial f_a}{\partial t} + v\,\frac{\partial f_a}{\partial r} + e_a E(r, t)\,\frac{\partial f_a}{\partial p} = 0\,, \tag{5.15}$$

$$\operatorname{rot} E = 0\,, \tag{5.16}$$

$$\operatorname{div} E = 4\pi \sum_a e_a n_a \int f_a(r, p, t)\,\mathrm{d}p\,. \tag{5.17}$$

The Vlasov equations are reversible. This manifests itself in the fact that the entropy of a closed system remains constant in the course of time evolution. The Vlasov equations are identical in form to exact equations in the microscopic functions $N_a(x, t)$, $E^{\mathrm{m}}(r, t)$. They, however, are approximate equations in deterministic (nonrandom) functions $f_a(r, p, t)$, $E(r, t)$ when the *electron-ion plasma is regarded as a continuous medium*.

Like the corresponding microscopic equations, the Vlasov equations are reversible. The reversibility of the Vlasov equations, however, is illusory. As a matter of fact, the values of relaxation time and length for plasma are finite. Going over to the Vlasov equations can only be approximately carried out on condition that the relaxation lengths l_{ee} are much larger than the size of the system L, i.e. $l_{\mathrm{ee}} \gg L$. Hence it follows that the system under consideration is confined, and therefore the Vlasov equations must be supplemented with the boundary conditions. In the case of real systems the latter are always dissipative. Accordingly, the Vlasov equations together with the boundary conditions constitute dissipative equations. Dissipation can be taken into account by introducing the effective collision integral into the reversible Vlasov equations [Eqn (5.16)]. The question now arises how the dissipative term ought to be introduced into the Vlasov equations. This question can be put in a different way: *how should one carry out the regularization of the reversible Vlasov equation in order to take the real dissipation in plasma into account*?

There are two essentially different approaches to such regularization. One was first used in Landau's paper [3] in connection with the introduction of *Landau's collisionless damping*. Landau introduced this dissipation in a formal way, by solving the reversible Vlasov equation with the aid of Laplace transform. The same result can be obtained using an equivalent procedure: namely, we can introduce a small dissipative term, proportional to a certain collision rate ν_a, into the Vlasov equation. In the final (!) results we set $\nu_a \to 0$.

In this way, dissipation is, although formally, introduced to overcome the difficulties associated with the presence of resonances.

Going over from the reversible equations of motion to the kinetic equations, as indicated above, implies the change to the irreversible equations of continuous medium. This requires smoothing over the volume of the points of continuous medium. Naturally, the physically infinitesimal scales are the smallest among the characteristic scales pertinent to the equations of continuous medium.

In this respect, the situation is opposite to that encountered in the traditional theory when the so-called *collisionless approximation* is used. To see this difference, let us recapitulate some results of the conventional theory.

5.5 Waves in collisionless Coulomb plasma. Landau damping

5.5.1 Electric susceptibility of Coulomb plasma. Dispersion equation. Let us distinguish two cases.

(1) The wave properties of unconfined Coulomb plasma are considered under the assumption that the collisions do not play any significant role. This requires that the following conditions must be satisfied:

$$\lambda \ll l_{\mathrm{ee}} \ll L\,, \qquad \omega \gg \nu_{\mathrm{ee}} \gg \frac{1}{T}\,, \tag{5.18}$$

where $\lambda = 1/k$ is the wavelength, ω is the corresponding frequency, L is the smallest characteristic size of the system, and T is the corresponding parameter of time. The zero approximation in λ/l_{ee} and ν_{ee}/ω [*passage to the limit is carried out in the final results* (!)] corresponds to the *collisionless approximation* for plasma.

(2) The relaxation scales, determined by the interaction between the plasma particles, are much greater than the characteristic parameters of the system. Then the following

inequalities hold:

$$\lambda \ll L \ll l_{\rm ee}\,, \qquad \omega \gg \frac{1}{T} \gg \nu_{\rm ee}\,, \tag{5.19}$$

and the Vlasov equations must be supplemented with the boundary conditions, which in the general case are dissipative. Dissipation may be due to various reasons: nonideal conditions of reflection, finite time of flight, etc.

In a certain sense, the case when inequalities (5.18) are satisfied, is more simple. Indeed, in this case the dissipation is determined by the collision integrals, whose general properties are well known. At the same time, it would be difficult to suggest a general form of the effective collision integral which would characterize the dissipative boundary conditions provided that inequalities (5.19) hold.

In the collisionless approximation we may assume that the particular form of the collision integral is not important. This justifies the use of the simplest so-called ν-approximation for the collision integral:

$$I_a = -\nu_a\left[f_a(r,p,t) - f_a^{(0)}(p)\right], \tag{5.20}$$

where $f_a^{(0)}(p)$ is the Maxwellian distribution. As a result, the reversible Vlasov equation (5.15) will be replaced by the irreversible kinetic equation

$$\frac{\partial f_a}{\partial t} + v\,\frac{\partial f_a}{\partial r} + e_a E\,\frac{\partial f_a}{\partial p} = -\nu_a\left[f_a(r,p,t) - f_a^{(0)}(p)\right]. \tag{5.21}$$

This equation (as opposed to the Vlasov equation!) possesses an equilibrium solution in the form of the Maxwellian distribution. The field therewith is $E=0$. Consider a solution $f_a^{(1)}(r,p,t)$, $E^{(1)}(r,t)$, which is close to the equilibrium solution. In the linear approximation, the solution of the relevant equations for the Fourier components culminates in the following equation for the field:

$$\varepsilon(\omega,k)E^{(1)}(\omega,k) = 0\,. \tag{5.22}$$

The permittivity is given by

$$\varepsilon(\omega,k) = 1 + \sum_a \frac{4\pi e_a^2}{k^2}\,n_a \int \frac{k\,\partial f_a^{(0)}/\partial p}{\omega - kv + \mathrm{i}\nu_a}\,\mathrm{d}p\,, \qquad \nu_a \to 0\,. \tag{5.23}$$

The condition of existence of nontrivial solution leads to the *dispersion equation*

$$\varepsilon(\omega,k) = 0\,. \tag{5.24}$$

The wave number for unconfined plasma is real. We represent the complex frequency in the form $\omega = \omega' - \gamma$, and consider the case of slight damping, when $\gamma \ll \omega'$. In the first approximation with respect to γ/ω' we get the following expressions:

$$\mathrm{Re}\,\varepsilon(\omega',k) = 0\,, \qquad \gamma = \mathrm{Im}\,\varepsilon(\omega',k)\left[\frac{\partial \mathrm{Re}\,\varepsilon(\omega',k)}{\partial\omega'}\right]^{-1}. \tag{5.25}$$

The former defines the frequency dispersion of oscillations, i.e. the function $\omega = \omega(k)$, whereas the latter governs the damping coefficient, which is proportional to the imaginary part of permittivity.

The real and imaginary parts of permittivity $\varepsilon(\omega',k)$ are given by the well-known formulas

$$\mathrm{Re}\,\varepsilon(\omega',k) = 1 + \sum_a \frac{4\pi e_a^2}{k^2}\,n_a P\int \frac{k\,\partial f_a^{(0)}/\partial p}{\omega' - kv}\,\mathrm{d}p \tag{5.26}$$

and

$$\mathrm{Im}\,\varepsilon(\omega',k) = -\sum_a \frac{4\pi^2 e_a^2}{k^2}\,n_a \int \delta(\omega' - kv)k\,\frac{\partial f_a^{(0)}}{\partial p}\,\mathrm{d}p\,. \tag{5.27}$$

Symbol P indicates that the principal value of the integral is to be taken.

Let us now consider an example.

5.5.2 Waves in electron plasma. Landau damping. The ground state is characterized by the Maxwellian distribution of the electrons, whereas the ions are assumed to be immobile (cold):

$$f_{\rm e}^{(0)}(p) = \frac{1}{(2\pi m_{\rm e} k_{\rm B} T)^{3/2}}\exp\left(-\frac{p^2}{2m_{\rm e}k_{\rm B}T}\right) \quad \text{and} \quad f_{\rm i}(p) = \delta(p). \tag{5.28}$$

For the phase velocities $\omega'/k \gg v_T$ within the zero approximation in $m_{\rm e}/m_{\rm i}$ we obtain the known results

$$\omega'^2 = \omega_{\rm L\,e}^2(1 + 3r_{\rm D\,e}^2k^2)\,, \tag{5.29}$$

$$\gamma = \frac{\omega_{\rm L\,e}}{2}\,\mathrm{Im}\,\varepsilon(\omega',k) = -\frac{\omega_{\rm L\,e}}{2}\,\frac{4\pi^2 e_a^2}{k^2}\,n_a \int \delta(\omega' - kv)k\,\frac{\partial f_a^{(0)}}{\partial p}\,\mathrm{d}p\,, \tag{5.30}$$

where $\omega_{\rm L\,e}$ is the Langmuir frequency, and $r_{\rm D\,e}$ is the Debye radius for electrons.

Let $\mathbf{k} \parallel x$; then we can integrate the last expression with respect to p_y, p_z, getting

$$\gamma = -\omega_{\rm L\,e}\,\frac{4\pi^2 e_a^2 n_a m_{\rm e}}{2k^2}\,\frac{\partial f_{\rm e}^{(0)}(p_x)}{\partial p_x}\bigg|_{v_x=\omega'/k} = \sqrt{\frac{\pi}{8}}\,\omega_{\rm L\,e}\,\frac{1}{r_{\rm D\,e}^3k^3}\exp\left(-\frac{1}{2r_{\rm D\,e}^2k^2} - \frac{3}{2}\right). \tag{5.31}$$

The expression for γ obtained in this way defines the *Landau damping coefficient*.

The collisional nature of Landau damping is indicated by the fact that γ is zero when the collision rate ν_a is identically equal to zero in the initial equation (5.21). We shall return to this matter in the next section.

5.6 Permittivity and Landau damping in kinetic theory of fluctuations

5.6.1 The kinetic theory. We have described the wave properties of rarefied Coulomb plasma in the collisionless approximation on the basis of the kinetic equation. All the results were obtained by solving the dispersion equation (5.24). Permittivity (5.23) is found by solving Eqn (5.21) and the field equations. Equation (5.21) may be regarded as the result of regularization of the Vlasov equation for the distributions functions $f_a(r,p,t)$, i.e. the first moments of the microscopic phase density $N_a(r,p,t)$.

The term on the right-hand side of kinetic equation (5.21) is a 'caricature' of the collision integral, because it does not possess the general properties of the latter. It is only used for regularizing the solution of the Vlasov equation in the neighbourhood of resonances. This, however, is just a formal mathematical solution to the problem, equivalent to the approach used in the classical work [3]. Now it is timely to clarify the physical content of the problem. Notice once again that the regularization procedure is not unambiguous. From physical standpoint, the various methods of regularization are not equivalent.

Indeed, the quantity ν_a in Eqn (5.21) is unvaried. This means that the time delay and the spatial nonlocality are not taken into account. As a result, the spectral linewidth in Eqn (5.23) is also constant (it does not depend on the frequency ω and the wave number k), and $\nu_a \to 0$ in the final expressions.

One may use expression (5.23) for the dielectric constant also when ν_a is finite — that is, before passing to the limit $\nu_a \to 0$. Then, after integrating it with respect to p_y and p_z, we come to the following expression for the imaginary part of permittivity:

$$\operatorname{Im} \varepsilon(\omega', k) = -\sum_a \frac{4\pi^2 e_a^2}{k^2} n_a \int \frac{\nu_a}{(\omega' - kv_x)^2 + \nu_a^2} k \frac{\partial f_a^{(0)}}{\partial p_x} \, \mathrm{d}p_x \,. \tag{5.32}$$

The integrand is the product of two functions. The former is a Lorentz line with due account for the Doppler shift kv_x. The result of integration with respect to p_x depends on the dimensionless parameter

$$\frac{\nu_a}{kv_T}\,, \quad \text{or} \quad \frac{1}{kl_a}\,, \tag{5.33}$$

where l_a is the mean free path of the charged plasma particles.

In the zero approximation with respect to this parameter we have an infinitesimally narrow Lorentz line, and integration with respect to p_x yields a Doppler contour. In this case the wavelength $1/k$ is much less than l_a. This justifies to some extent the term 'collisionless approximation', which corresponds to Landau's result. If, however, the collision rate ν_a is identically equal to zero, the result also comes to nought, since the Lorentz line itself disappears, and there is nothing to be averaged!

We have paid a lot of attention to this matter because the concept of 'Landau damping' plays an important role in plasma physics.

5.6.2 Theory of fluctuations. All the above results have been obtained on the basis of the kinetic equation for continuous medium. Historically, however, the dielectric constant was first introduced in the theory of small-scale fluctuations. It is these fluctuations that determine the dissipative terms in the kinetic equations. In particular, Eqn (5.5) was obtained for the electric field fluctuations. Let us quote once again the relevant expression of permittivity for fluctuations:

$$\varepsilon(\omega, k) = 1 + \sum_a \frac{4\pi e_a^2}{k^2} n_a \int \frac{k\, \partial f_a / \partial p}{\omega - kv + \mathrm{i}\lambda_a} \, \mathrm{d}p \,, \qquad \lambda_a \to 0 \,. \tag{5.34}$$

Recall also the expression for the space–time spectral density of the source of the electric field fluctuations in the approximation of adiabatic switching on the interaction:

$$(\delta E \delta E)^{(\mathrm{s})}_{\omega,k} = \sum_a \frac{(4\pi)^2 e_a^2}{k^2} n_a \int \frac{2\lambda_a}{(\omega - kv)^2 + \lambda_a^2} f_a \, \mathrm{d}p \,, \qquad \lambda_a \to 0 \,, \tag{5.35}$$

and the expression for the space–time spectral density of the field fluctuations δE

$$(\delta E \delta E)_{\omega,k} = \frac{(\delta E \delta E)^{(\mathrm{s})}_{\omega,k}}{\left|\varepsilon(\omega, k)\right|^2} \,. \tag{5.36}$$

We see that the width of the Lorentz line in Eqns (5.35), (5.36) comes to nothing in the approximation of switching on the interaction adiabatically. This means that the corresponding correlation time of fluctuations is infinite, and therefore we are dealing with a completely coherent state.

Anew, like in the case of the collisionless approximation, in the kinetic theory there exists a contradiction with the natural (but not traditional) concept of the beginning of change from the reversible microscopic equations to the irreversible equations of the kinetic theory. Accordingly, we may once again note that there are two opposite descriptions of the rise of irreversibility. One of these is traditional; then the onset of irreversibility is characterized by coherent fluctuations with infinitely large correlation times (or infinitesimally narrow spectral lines — infinitesimally narrow resonances). In the kinetic theory this corresponds to the collisionless approximation. According to the other description, which is more physical but not traditional, the change to the irreversible equations is inevitable when the model of continuous medium is used. This time, however, the irreversibility is a consequence of a totally chaotic motion of particles within physically infinitesimal volumes — within the points of continuous medium. Thus, the onset of irreversibility and the dissipative terms in the kinetic equations are determined by the noncoherent but small-scale fluctuations. Their correlation time is defined by the smallest characteristic time parameter of the plasma.

From formula (5.36) it follows that equation

$$\varepsilon(\omega, k) = 0 \,, \qquad \lambda_a \to 0 \,, \tag{5.37}$$

which is similar to dispersion equation (5.24), plays an important part in the adiabatic theory of fluctuations as well. Here we can also introduce an analogue of Landau damping. Indeed, from Eqn (5.34) follows the expression for the imaginary part of permittivity

$$\operatorname{Im} \varepsilon(\omega, k) = -\sum_a \frac{4\pi^2 e_a^2}{k^2} n_a \int \frac{\lambda_a}{(\omega - kv_x)^2 + \lambda_a^2} k \frac{\partial f_a}{\partial p_x} \, \mathrm{d}p_x \,, \qquad \lambda_a \to 0 \,, \tag{5.38}$$

which defines the width of the spectral line of field fluctuations $\delta E(\omega, k)$ at a given value of the wave vector k.

From formulas (5.32), (5.38) it follows that the problem of calculation of the imaginary part of permittivity for kinetic equation (5.21) in the collisionless approximation is similar to the problem which arises in connection with solving the corresponding equation in the calculation of the small-scale fluctuations. In both cases the Landau damping arises when dimensionless parameter (5.33) is much less than one:

$$\frac{\nu_a}{kv} \ll 1 \,, \quad \text{or} \quad \frac{1}{kl_a} \ll 1 \,. \tag{5.39}$$

This means that the wavelength $\lambda = 1/k$ is much smaller than the mean free path:

$$\lambda = \frac{1}{k} \ll l_a \,, \tag{5.40}$$

as consistent with the collisionless range of wavelengths.

However, both in the calculation of fluctuations and in the solution of the dissipative kinetic equation within the collisionless approximation, the conclusion concerning the existence of Landau damping is not justified. As a matter of fact, this damping in the theory of fluctuations is a consequence of the replacement of a broad line in the spectrum of fluctuations by an infinitesimally narrow line corresponding to switching on the interaction adiabatically. In other words, the 'white noise' is replaced by an infinitesimally narrow resonance. This corresponds to the collisionless approximation in the calculation of fluctuations.

When kinetic equation (5.21) is used in the collisionless approximation, there is an ambiguity in the selection of the form of the collision integral. The selection made corresponds to a very particular way of regularizing the reversible Vlasov equation. Such regularization eliminates the divergence in the expression for dielectric constant, but, as we shall see below, it does not reflect in full measure the actual importance of dissipation.

All this points to the necessity of (1) a more consistent method for calculating the fluctuations which define the collision integrals in the kinetic equations for plasma, and (2) a more consistent inclusion of dissipation in the study of, for instance, the wave properties of plasma.

We shall see that at the first step towards the irreversible kinetic equations (in deriving the kinetic equations) one must take into account the spatial nonlocality due to the finite size of the points of a continuous medium. As a result, it will be possible to eliminate the ambiguity in the selection of regularization procedure when going over to the irreversible equations of statistical theory of nonequilibrium processes.

6. Nontraditional description of nonequilibrium processes in plasma

6.1 First step to irreversible equations in physics of open systems

Recall that at the first step of going over to the irreversible equations of continuous medium, the microscopic equation (2.3) was supplemented with a term which characterized the relaxation of the exact dynamic distribution $N_a(r, p, t)$ toward the distribution smoothed over the volume of the point of a continuous medium. In this way, the change was made from exact dynamic equations (2.3), (2.4) to approximate equations (4.2), (4.3). The latter are approximate because information about the motion of particles within the points of continuous medium is lost. The smoothing procedure is justified by the path mixing in the phase space because of the dynamic instability of motion. Averaging these equations over the Gibbs ensemble, we obtained the set of equations (4.6), (4.7) for the distribution function $f_a(r, p, t)$ [the first moment of the microscopic function $N_a(r, p, t)$] and the mean electric field. This set of equations is not closed, since it includes the correlator of fluctuations of the phase density and the field, thus making it necessary to calculate the fluctuations. The equations for fluctuations are nonlinear, and we get an infinite chain of coupled equations for the sequence of fluctuation moments. When the model of a continuous medium is used, the number of particles within the point is large. Because of this, we may truncate the chain assuming that the fluctuations are small. As a result, we obtained a linear equation (4.11) for the fluctuations of phase density.

In order to separate the small-scale fluctuations, we introduced a dissipative term into Eqn (4.11), in which, in accordance with Eqn (4.10), the dissipative coefficient Δ_a is inversely proportional to the physically infinitesimal time interval. Of course, it is a quite crude method of selecting the small-scale fluctuations; we only used it to facilitate comparison with the traditional calculation of fluctuations which define the collision integrals in the kinetic equations for plasma. Further on, we shall take the small-scale fluctuations into account in a more consistent fashion when deriving the generalized kinetic equation. The latter is suitable for unified description of nonequilibrium processes on the kinetic and hydrodynamic scales.

But let us continue our 'brief digression into the past'.

Equation (4.11) was used in deriving Eqn (4.14) for the spectral density of the source fluctuations which reflects the molecular structure of continuous medium. The width of this spectral line is inversely proportional to the physically infinitesimal time interval, which is the smallest of all characteristic times for the plasma. In the approximation of continuous medium the linewidth is infinite, and therefore the molecular structure of the point manifests itself as a white noise. This makes the calculation of fluctuations in plasma similar to (although much more complicated than) the calculation of fluctuations in the theory of Brownian motion.

The conventional theory of calculation of fluctuations in plasma, as we have seen, does not pay respect to this circumstance and 'starts' from the opposite ground. Namely, it is based on the method of switching on the interaction adiabatically. As a result, expression (5.1) is used for the spectral density of the source fluctuations in place of (4.14). Accordingly, the linewidth in the ground-floor approximation tends to zero, and we arrive at Eqn (5.3). This means that the initial state is assumed to be completely coherent, since the corresponding correlation time turns out to be infinite. It is on this basis that the above kinetic equations for description of nonequilibrium processes in plasma were constructed.

One might ask, however, whether it is worth making an issue out of the minor blemishes in the formulation of these equations and striving for a more consistent derivation of the main equations when describing the nonequilibrium processes in plasma. Let us show that there are good enough reasons for developing the nontraditional statistical theory of nonequilibrium processes in plasma.

In this connection let us recall that a similar problem has been encountered in the kinetic theory of gases. Section 13.1 of Ref. [33] contains some remarks concerning the expedience of the unified description of nonequilibrium processes on the kinetic and hydrodynamic scales. Taking due account of the structure of continuous medium, it was possible to derive the generalized kinetic equation for a rarefied gas. As compared with the kinetic Boltzmann equation, this equation contains an additional dissipative term (see Eqn (13.3.10) in Ref. [33]). This term allows for the space diffusion of the distribution function. Owing to the diffusion of the distribution function, three dissipative diffusion-like processes appear on the gasdynamic level of description: self-diffusion of matter resulted from occurrence of the density gradient, viscous friction, and heat conduction.

Now let us demonstrate that a similar generalized kinetic description is both possible and necessary in describing a broad scope of phenomena in plasma.

6.2 Generalized kinetic equation for rarefied Coulomb plasma

Let us go back to our discussion of Section 4.2. In this section we listed the modifications which must be made in the initial dynamic equations [Eqns (2.3), (2.4) in the case of Coulomb plasma] for the microscopic phase density of each plasma component and the field strength.

Smoothing of these equations over the point of a continuous medium leads to Eqns (4.2), (4.3). An additional term in the first of these equations describes the 'adjustment' of the microscopic phase density (the dynamic distribution) to corresponding function (4.4) smoothed over the volume of the point. The smoothing function was defined by Gaussian distribution (4.5). The dispersion is determined here by the Debye radius — the size of the point in Coulomb plasma. The adjustment to the smoothed distribution proceeds within physically infinitesimal time interval (2.43) — the time of diffusion across the Debye sphere. A different physical interpretation of the physically infinitesimal time interval for plasma is also possible.

Recall that integration with respect to wave numbers in the expression for Landau collision integral is carried out between the limits [see Eqn (5.13)]

$$k_{\min} = \frac{1}{r_{\mathrm{D}}} \quad \text{and} \quad k_{\max} = \frac{1}{l_{\mathrm{L}}} . \tag{6.1}$$

These definitions indicate that the largest length scale is the same as the Debye radius, and therefore the same as the size of the point of a continuous medium. The smaller scales account for the structure of the point itself, or, in other words, describe the specific character of the Coulomb interaction. The smallest scale is the so-called *Landau length*

$$l_{\mathrm{L}} = \frac{e^2}{k_{\mathrm{B}} T} . \tag{6.2}$$

Since the ratio of the lengths l_{L}, r_{D} is easily proved to be given by the plasma parameter μ, we have the following sequence of the characteristic times for plasma electrons:

$$\tau_{\mathrm{ph}} \sim \frac{r_{\mathrm{D}}^2}{D} \sim \mu \, \frac{1}{\omega_{\mathrm{L}}} \sim \mu \, \frac{r_{\mathrm{D}}}{v_T} \sim \frac{l_{\mathrm{L}}}{v_T} . \tag{6.3}$$

Thus, the physically infinitesimal time interval is μ times smaller than the period of proper oscillations and, accordingly, the time of flight over the length r_{D} which defines the size of the point. It is determined by the smaller time scales: either by the time of diffusion across the point, or by the time of flight over the smallest length scale of interaction (Landau length).

After this digression, let us proceed with the modification of the basic equations.

Recall that after averaging equations (2.3), (2.4) over the Gibbs ensemble we obtained the set of equations (4.6), (4.7) for the one-particle distribution functions and the strength of a mean electric field. This set of equations is not closed: it contains not only the first moments $f_a(r, p, t)$, $E(r, t)$ of the corresponding microscopic functions, but also the correlator $\langle \delta E \, \delta N_a \rangle$ of fluctuations δN_a, δE.

We carry out expansion in terms of the physical Knudsen number of each plasma component in the second term on the right-hand side of Eqn (4.6). As a result, we get the following expression for the dissipative term in the kinetic equation, which is defined by the space diffusion and the mobility of charged plasma particles:

$$I_a^{(r)}(r, p, t) = \frac{\partial}{\partial r} \left(D_{(a)} \, \frac{\partial f_a}{\partial r} \right) - \frac{\partial}{\partial r} \left(\frac{\tau_{\mathrm{rel}}^{(a)}}{m_a} \, e_a E f_a \right) . \tag{6.4}$$

The coefficient of space diffusion is linked with the characteristic relaxation parameters by the following relations:

$$D_{(a)} \sim \frac{l_{\mathrm{rel}}^{(a)^2}}{\tau_{\mathrm{rel}}^{(a)}} \sim v_{T_a} l_{\mathrm{rel}}^{(a)} \sim \tau_{\mathrm{rel}}^{(a)} \, \frac{k_{\mathrm{B}} T}{m_a} . \tag{6.5}$$

This expression is similar to Eqn (13.3.7) in Ref. [33].

So then, we have established the form of one of the dissipative terms in the kinetic equation obtained with due account for the structure of a continuous medium. Now we only have to establish the structure of the second dissipative term on the right-hand side of Eqn (4.6), which is determined by the correlator of the field fluctuations and the phase density. We denote this term by

$$I_a^{(v)}(r, p, t) = -\frac{e_a}{n_a} \, \frac{\partial}{\partial p} \langle \delta E \, \delta N_a \rangle . \tag{6.6}$$

Recall that we have already encountered a similar definition of the collision integral [see the identity on the right-hand side of Eqn (3.3)]. Calculating this correlator within the approximation of the second correlation functions has led us to the expressions for the collision integrals of Balescu and Lenard, and also of Landau.

From our current standpoint the *Balescu – Lenard result is physically not justified*, since it has been derived under the assumption of switching on the interaction adiabatically. This means that a coherent state is selected for the 'starting point'. Landau's expression is more suitable when the most chaotic state serves as the starting point. In the original Landau's paper it was derived from the Boltzmann equation with one important addition. Namely, Landau singled out the principal range of scales (or wave numbers) which determine the collision integral. It is interesting that within the traditional approach Landau's result is less general than the Balescu–Lenard expression: it does not take into account the dynamic polarization of plasma. From the standpoint developed here, however, Landau's expression is more advantageous.

Let us consider this important issue in greater detail.

Indeed, the range of scales (wave numbers) in the Landau integral is defined by conditions (5.13). We see that the largest length corresponds to the Debye radius, and thus to the size of the point of a continuous medium. This ensures elimination of precisely those scales which pertain to the small-scale fluctuations.

Moreover, Landau's expression can be derived from the Langevin equations for charged particles. Then the width of δ-function in the correlator of the Langevin source is determined by the time interval which we have adopted above as the physically infinitesimal time scale. By this means in deducing the Landau collision integral on the basis of Langevin equation the following condition should be met:

$$\tau_{\mathrm{cor}} \sim \tau_{\mathrm{ph}} . \tag{6.7}$$

As a result, the expression for the collision integral pertinent to the Coulomb plasma, which was established in a different way in Landau's classical paper in 1936, defines the second dissipative term in the generalized kinetic equation for Coulomb plasma. The most natural is exactly that form of the collision integral which was proposed in Landau's original work:

$$I_a^{(v)}(r,p,t) = \sum_b C_{ab} \frac{\partial}{\partial p_i} \int V_{ij}(v - v') \times \left(\frac{\partial f_a(r,p,t)}{\partial p_j} f_b(r, p', t) - \frac{\partial f_b(r, p', t)}{\partial p'_j} f_a(r, p, t) \right) \mathrm{d}p' . \tag{6.8}$$

Here we use the notation for the tensor of velocity difference

$$V_{ij}(v - v') = \frac{(v - v')^2 \delta_{ij} - (v - v')_i (v - v')_j}{|v - v'|^3} \tag{6.9}$$

and the constants

$$C_{ab} = 2\pi e_a^2 e_b^2 n_b L , \qquad B_{ab} = n_a C_{ab} . \tag{6.10}$$

We also introduce the notation *for the Coulomb logarithm*

$$L = \ln \frac{r_{\mathrm{D}}}{l_{\mathrm{L}}} \sim \ln \frac{1}{\mu} \qquad \text{for} \qquad \mu \ll 1 . \tag{6.11}$$

The superscript (v) on the collision integral indicates that the Landau integral in the generalized kinetic equation is one of the two collision integrals, which is defined by the redistribution of charged particles with respect to velocities (the Brownian motion).

Let us finally rewrite Eqn (4.6) in the form where the right-hand side is the sum of two collision integrals, viz.

$$\frac{\partial f_a}{\partial t} + v \frac{\partial f_a}{\partial r} + e_a E(r,t) \frac{\partial f_a}{\partial p} = I_a^{(v)}(r,p,t) + I_a^{(r)}(r,p,t) . \tag{6.12}$$

The first integral is given by Landau's expression (6.8); as indicated above, it is responsible for the dissipation due to the redistribution of velocities of the interacting charged particles in Coulomb plasma. The second collision integral is given by Eqn (6.4) and defines the dissipation due to the space diffusion of the distribution function.

Naturally, these kinetic equations must be solved together with the Maxwell equations for Coulomb plasma

$$\operatorname{rot} E = 0 , \qquad \operatorname{div} E = 4\pi \sum_a e_a n_a \int f_a(r,p,t)\, \mathrm{d}p . \tag{6.13}$$

As a result, we have obtained the kinetic equations for rarefied Coulomb plasma with due account for the structure of the continuous medium modelling the system under consideration. As a consequence, the kinetic equation contains an additional dissipative term. This term describes the adjustment of the dynamic and statistical distributions to the distributions smoothed over the volume of the point of a continuous medium. This allows us to call them the *generalized kinetic equations for the Coulomb plasma*.

To end this section, let us mark the following.

We may regard the generalized kinetic equations as an example of equations from the theory of nonlinear Brownian motion [33, 42], and write the appropriate nonlinear Langevin equations. The correlators of Langevin sources will then be characterized by a double set of variables. First of all, these are the correlations on the physically infinitesimal scales. These scales determine the 'widths' of δ-functions in the expressions for the source correlators. The intensities of Langevin sources for plasma will be also determined by the distribution functions $f_a(r,p,t)$, which vary little over the physically infinitesimal space – time scales. The equations of this kind are extremely complicated, which plays down the advantages of the Langevin method as compared with the simpler systems. Because of this, it would be better to concentrate the efforts on solving the kinetic equations themselves.

Now we already may discuss some particular implications and properties of this generalized kinetic equation.

6.3 Properties of generalized kinetic equations

6.3.1 Equilibrium spatially homogeneous distribution of plasma particles. Diffusion in the space of momenta. In the absence of external fields, the set of equations in the functions $f_a(r,p,t)$, $E(r,t)$ has a particular solution, when the charged particles are uniformly distributed over the space, the distribution with respect to momenta is the Maxwellian distribution, and the electric field is zero. To prove the existence of this solution one must use the condition of electric neutrality of plasma (then the charge density is zero) and make sure that the Landau collision integral $I_a^{(v)}(r,p,t)$ vanishes upon substitution of the Maxwellian distribution. This is easily proved using the properties of the tensor of relative velocity components.

We revert to expression (6.8) for the Landau collision integral and rewrite it in the Fokker – Planck form used in the theory of Brownian motion:

$$I_a^{(v)}(r,p,t) = \sum_b C_{ab} \left[\frac{\partial}{\partial p_i} D_{ij}^{(a)}(v) \frac{\partial f_a}{\partial p_j} + \frac{\partial}{\partial p_i} A_i^{(a)}(v) f_a \right] . \tag{6.14}$$

Here we use the following notation for the tensor of diffusion in the space of velocities and the corresponding vector which characterizes the dissipation:

$$D_{ij}^{(a)}(v) = \sum_b C_{ab} \int V_{ij}(v - v') f_b(p')\, \mathrm{d}p' , \tag{6.15}$$

$$A_i^{(a)}(v) = \sum_b C_{ab} \int V_{ij}(v - v') \frac{\partial f_b(p')}{\partial p_j}\, \mathrm{d}p' . \tag{6.16}$$

Let us find the relationship between these coefficients for the state of equilibrium. For this purpose we substitute the Maxwellian distribution into the last two formulas. As a result, with due account for the properties of Landau collision integral, we get

$$D_{ij}^{(a)}(v) v_j = A_i^{(a)}(v) k_{\mathrm{B}} T . \tag{6.17}$$

This is the Einstein relation in the theory of nonlinear Brownian motion [4, 33, 42]. The only difference is that nonlinear are not only the Langevin equations for the particles, but also the kinetic equations. Because of this, both the diffusion tensor and the vector of friction themselves depend on the distribution function.

6.3.2 Equilibrium state in external field. The Boltzmann distribution. Assume that the plasma is in the external electric field, and consider the state of equilibrium. The distribution with respect to velocities is then the Maxwellian distribution, and the distribution with respect to coordinates is to be found. From the kinetic equation we go over to equations in the densities of particles

$$n_a(r,t) = n_a \int f_a(r,p,t)\,\mathrm{d}p\,. \qquad (6.18)$$

For the state of equilibrium we get the following set of equations for the functions $n_a(r)$, $E(r)$:

$$\frac{\partial}{\partial r}\left(D_{(a)}\frac{\partial n_a(r)}{\partial r}\right) - \frac{\partial}{\partial r}\left(\frac{\tau_a}{m_a}\,e_a E n_a(r)\right) = 0\,, \qquad (6.19)$$

$$\operatorname{rot} E = 0\,, \qquad \operatorname{div} E = 4\pi\sum_a e_a n_a(r)\,. \qquad (6.20)$$

From Maxwell equations (6.20) it follows that the electric field is potential: $E(r) = -\operatorname{grad}\varphi(r)$. On this basis the solution of Eqn (6.19) is the Boltzmann distribution

$$n_a(r) = n_a \exp\left[-\frac{e_a\varphi(r)}{k_{\mathrm{B}}T}\right]. \qquad (6.21)$$

6.3.3 Screening of external field by equilibrium plasma. Assume that the potential of the external electrostatic field is held on the boundary of plasma. The field is considered to be weak. This means that the potential energy is much smaller than the kinetic energy, namely

$$e_a\varphi(r) \ll k_{\mathrm{B}}T\,. \qquad (6.22)$$

As a result, we come to the linear equation for the electric potential

$$\Delta\varphi(r) - \frac{1}{r_{\mathrm{D}}^2}\,\varphi(r) = 0\,. \qquad (6.23)$$

Here we used notation (2.11) for the Debye radius. In the one-dimensional case this equation coincides with Eqn (2.9). Its solution (2.10) describes screening of the external field by equilibrium plasma.

This description of the field screening by plasma is more consistent. Indeed, earlier we had to postulate the existence of Boltzmann distribution, whereas now it is the equilibrium solution to the generalized kinetic equation. The presence of two dissipative terms in the latter allows describing the relaxation towards both the equilibrium distribution with respect to velocities (the Maxwellian distribution), and the Boltzmann distribution. The corresponding relaxation times are defined by the collision times τ_{ee}, τ_{ii} for electrons and ions, and by the characteristic times of space diffusion $\tau_{D_a} \sim L^2/D_a$.

Let us consider the relation between these characteristic times, and, as a consequence, the possible description of the time evolution on the basis of diffusion equations.

6.3.4 Coefficients of space diffusion. Ambipolar diffusion in fully ionized plasma. From the above definitions of the characteristic scales for rarefied plasma we find the relationships between the collision time and that of diffusion:

$$\frac{\tau_a}{\tau_{D_a}} \sim \frac{v_{T_a} l_a}{L^2}\,\frac{l_a}{v_{T_a}} \sim \frac{l_a^2}{L^2} \sim \frac{1}{\mu^2}\,\frac{r_{\mathrm{D}}^2}{L^2} \sim \frac{1}{\mu^2}\,\frac{l_{\mathrm{ph}}^2}{L^2}\,. \qquad (6.24)$$

This chain of relations gives rise to important conclusions.

First, the characteristic length in the problem of field screening by equilibrium plasma is the Debye radius, i.e. $L \sim r_{\mathrm{D}}$, and therefore the characteristic diffusion time is smaller than the collision time by the factor of μ^2. In its turn, the diffusion time at $L \sim r_{\mathrm{D}}$ is of the order of the physically infinitesimal time interval.

This implies that the equilibrium Boltzmann distribution on the scale of the order of Debye radius is established much sooner than the Maxwellian distribution. It may seem therefore that there is no reason for describing the time evolution on the basis of equations of space diffusion. It should be recognized, however, that our conclusion does not depend on the electron/ion mass ratio.

A different situation is yet possible, too.

Assume that the distribution with respect to velocities at the initial point in time is the Maxwellian one, and that this distribution is conserved in the course of evolution towards the state of equilibrium (this assumption is justified at least when the linear approximation is used — the approximation of a weak field). Substitution of the Maxwellian distribution into kinetic equation (6.12) reduces to zero the first dissipative term on the right-hand side. As a result, after carrying out integration with respect to moments, we arrive at the time-domain equations for space diffusion:

$$\frac{\partial n_a(r,t)}{\partial t} = \frac{\partial}{\partial r}\left(D_{(a)}\frac{\partial n_a(r,t)}{\partial r}\right) + \frac{\partial}{\partial r}\left(\frac{\tau_{\mathrm{rel}}^{(a)}}{m_a}\,e_a\,\frac{\partial\varphi}{\partial r}\,n_a(r,t)\right). \qquad (6.25)$$

The field equations retain their old form, i.e.

$$\Delta\varphi = 4\pi\sum_a e_a n_a(r,t)\,. \qquad (6.26)$$

We see that the temporal evolution of plasma is described by rather complicated nonlinear equations even in the most simple case. These equations coincide in form with the relevant equations for slightly ionized plasma (see Eqns (25.4)–(25.6) in Ref. [4]). The physical meaning, however, is quite different here.

To wit, the main processes in slightly ionized plasma are the collisions of charged particles with the atoms. The collisions between the charged particles are rare and can be neglected. By contrast, in the fully ionized plasma there are no collisions of charged particles with the atoms. The dissipation is due to the interaction between the charged particles. In the approximation currently under consideration the dissipation depends on the processes of self-diffusion of each of the plasma components. Of course, there is also dissipation because of the interaction between the components of plasma; it does not manifest itself, however, when the distribution of particles with respect to velocities is Maxwellian.

Note that the mean free paths of the electrons and the ions do not depend on the electron/ion mass ratio, because they are expressed in the same way in terms of the Debye radius and the plasma parameter:

$$l_{\mathrm{ee}} \sim l_{\mathrm{ii}} \sim \frac{r_{\mathrm{D}}}{\mu}\,. \qquad (6.27)$$

On the contrary, the ratio of the diffusion coefficients to the corresponding relaxation times depends on the electron/ion mass ratio:

$$\frac{D_{\mathrm{e}}}{D_{\mathrm{i}}} \sim \frac{v_{T_{\mathrm{e}}}}{v_{T_{\mathrm{i}}}} \sim \sqrt{\frac{m_{\mathrm{i}}}{m_{\mathrm{e}}}} \gg 1\,, \qquad (6.28)$$

$$\frac{\tau_{D_{\mathrm{e}}}}{\tau_{D_{\mathrm{i}}}} \sim \frac{D_{\mathrm{i}}}{D_{\mathrm{e}}} \sim \sqrt{\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}} \ll 1\,. \qquad (6.29)$$

We see that the diffusion time of relaxation for the electrons is much smaller than the corresponding time for the ions. This allows us to distinguish two stages in the relaxation towards the equilibrium distribution of particles and the field.

A similar situation is encountered in the theory of weakly ionized plasma, where the small parameter, defined by the electron/ion mass ratio, is used for distinguishing the slower process, related to the diffusion of only the ions. It is the so-called *ambipolar diffusion*. Then the process of relaxation towards the state of equilibrium is described by the linear diffusion equation. Let us show that a similar process is also possible in fully ionized plasma. Owing to the much different times of diffusive relaxation for the electrons and the ions, we may distinguish two stages of relaxation towards the state of equilibrium.

The first stage refers to the relaxation of electrons in the diffusion time towards the distribution which satisfies stationary equation (6.25) with $a = \mathrm{e}$. As a result, the electrons have time to 'adjust' to the ions, and the Boltzmann distribution of the electrons is established. In this way, at the stage of 'fast' relaxation the concentrations of electrons and ions are equalized, and the plasma becomes *quasi-neutral*. The concentrations are given by the following expression

$$n_{\mathrm{e}}(r,t) = n_{\mathrm{i}}(r,t) = n\exp\left[-\frac{e_a\varphi(r,t)}{k_{\mathrm{B}}T}\right]. \tag{6.30}$$

At the same time, the concentration of ions does not keep up with the change. The distribution of ions (and therefore the distribution of electrons) is still nonequilibrium, and is different from the mean concentration n. Because of this, the electric field is yet nonzero. As follows from the last formula, the electric potential is expressed in terms of the function $n_{\mathrm{i}}(r,t)$:

$$e_a\varphi(r,t) = -k_{\mathrm{B}}T\ln\frac{n_{\mathrm{i}}(r,t)}{n}. \tag{6.31}$$

Substituting the latter in diffusion equation (6.25) for the concentration of ions, we come to the conclusion that the evolution of ions at the second stage of relaxation is described by the equation of *ambipolar diffusion in fully ionized plasma*:

$$\frac{\partial n_{\mathrm{i}}(r,t)}{\partial t} = 2D_{\mathrm{i}}\Delta n_{\mathrm{i}}(r,t). \tag{6.32}$$

This equation together with Eqn (6.31) describes the temporal evolution of the distribution of particles and field to the state of equilibrium.

Naturally, the time of relaxation according to this equation depends on the characteristic scale of the problem L ($\tau_{D_{\mathrm{i}}} \sim L^2/D_{\mathrm{i}}$). The smallest value of L is defined by the Debye radius. Even then the process of ambipolar diffusion is 'slow'. This conclusion follows from relationship (6.29) for the diffusion coefficients and the corresponding relaxation times. We see that the characteristic time for the second stage is greater than that for the first stage by the factor of $\sqrt{m_{\mathrm{i}}/m_{\mathrm{e}}}$.

Recall that this calculation has been made under the assumption that the temperature and the Maxwellian distribution with respect to velocities remain constant in the course of temporal evolution. This assumption is not always justified. For example, the calculation of electric conductivity based on the local Maxwellian distribution yields the value which is one-half of the result of the more precise Spitzer calculation. An appropriate refinement can also be made in the calculation of ambipolar diffusion.

6.3.5 Properties of the Landau collision integral. Recall that the properties of the collision integrals of Balescu and Lenard and Landau are similar to those of the Boltzmann collision integral in the kinetic theory of gases.

Consider the integral

$$I(r,t) = \sum_a n_a\int\varphi_a(p)I_a^{(v)}(r,p,t)\,\mathrm{d}p. \tag{6.33}$$

We substitute expression (6.8) into the right-hand side of Eqn (6.33), integrate by parts with respect to p, and symmetrize with respect to the variables (ap) and (bp'). Then we find that

$$I(r,t) = \sum_a n_a\int\varphi_a(p)I_a^{(v)}(r,p,t)\,\mathrm{d}p = 0 \quad \text{for} \quad \varphi_a(p) = 1,\ p,\ \frac{p^2}{2m_a}. \tag{6.34}$$

These properties ensure conservation of matter, total energy and total momentum in a closed system.

6.3.6 *H*-Theorem for spatially homogeneous plasma distribution. Lyapunov functional. If the function $\varphi_a(p)$ is selected in the form

$$\varphi_a(p) = -k_{\mathrm{B}}\ln f_a, \tag{6.35}$$

then, using the notation

$$A = \frac{\partial\ln f_a}{\partial p} - \frac{\partial\ln f_b}{\partial p'}, \tag{6.36}$$

we may rewrite Eqn (6.33) as

$$I(r,t) = k_{\mathrm{B}}\sum_{ab} D_{\mathrm{as}}\int\left\{(v-v')^2A^2 - \left[(v-v')A\right]^2\right\}f_af_b\,\mathrm{d}p\,\mathrm{d}p'. \tag{6.37}$$

Since the integrand is positive for arbitrary distribution functions, the following property holds:

$$I(r,t) \geqslant 0 \quad \text{for} \quad \varphi_a(p) = -k_{\mathrm{B}}\ln f_a. \tag{6.38}$$

The sign of equality corresponds to the equilibrium state, when the collision integral $I_a^{(v)}$ is zero.

Property (6.38) is the core of the proof of Boltzmann's *H*-theorem for rarefied spatially homogeneous plasma. Indeed, using the kinetic equation, in which only one collision integral $I_a^{(v)}(p,t)$ [see Eqn (6.14)] is now nonzero, we find that the total entropy of plasma

$$S(t) = -k_{\mathrm{B}}V\sum_a n_a\int\ln(n_af_a)f_a\,\mathrm{d}p \tag{6.39}$$

for a closed system in the course of temporal evolution remains constant or increases, i.e.

$$\frac{\mathrm{d}S}{\mathrm{d}t} \geqslant 0. \tag{6.40}$$

The sign of equality corresponds to the equilibrium state.

Like in the case of a rarefied gas, the H-theorem for spatially homogeneous plasma can be formulated in terms of the inequalities for the Lyapunov functional, defined as the difference between the entropies of equilibrium and nonequilibrium states:

$$\Lambda_S = S_0 - S(t) \,. \tag{6.41}$$

Given the normalization conditions for the distribution functions and the last property governed by Eqn (6.34), expression (6.41) may be represented as

$$\Lambda_S = S_0 - S(t) = \sum_a n_a \int \ln \frac{f_a(r,p,t)}{f_a^{(0)}(p)} f_a(r,p,t)\,\mathrm{d}r\,\mathrm{d}p \geqslant 0 \,. \tag{6.42}$$

Nonnegativity of the integral is established with the aid of inequality $\ln a \geqslant 1 - 1/a$ at $a = f_a/f_a^{(0)}$.

The quantity Λ_S represents a Lyapunov functional only on condition that the time derivative of Λ_S satisfies the opposite inequality in the course of temporal evolution, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_S = \frac{\mathrm{d}}{\mathrm{d}t}\big(S_0 - S(t)\big) \leqslant 0 \,, \tag{6.43}$$

which is a direct implication of result (6.40) expressing Boltzmann's H-theorem for plasma.

In this way, when the plasma is spatially homogeneous and the mean electric field is zero, the H-theorem may be reduced to the requirement of existence of the Lyapunov functional defined as the difference in the entropies of equilibrium and nonequilibrium states. Such formulation is more general. Indeed, it states not only that the entropy increases in the course of temporal evolution towards the equilibrium state, but also that the latter is stable (Lyapunov's stability). It is important that this formulation is only made possible by condition (6.34), which ensures conservation of the total mean energy of particles in the course of evolution towards the state of equilibrium.

For spatially inhomogeneous plasma in the presence of nonzero mean electric field, the mean energy is not conserved in the course of evolution. Because of this, Boltzmann's H-theorem does not assert, and the Lyapunov functional defined as the difference in the entropies no longer exists. As a consequence the question arises of whether could some other Lyapunov functional be found. A similar problem was also encountered in the kinetic theory of gases. Such a functional does exist; it is, however, determined by the difference in not the entropies but the appropriately defined free energies.

The equation of balance of local entropy can be used for finding the expression for production and flux of entropy, and, in particular, for defining the vector of thermal flux in rarefied electron–ion plasma in terms of the entropy gradient.

We will point out finally that the above properties of the collusion integral are used in deriving the gasdynamic equations for plasma.

Now let us return to the question of the physical nature of Landau damping. We shall see that the generalized kinetic equation formulated above provides a new idea of this most interesting phenomenon in plasma physics.

7. Role of collisions in collisionless plasma

7.1 What is collisionless plasma?

Let us revert to Section 5.4. As before, by L, T we denote the characteristic parameters of the problem. Like we did in the kinetic theory of gases, we consider two extreme cases (see Section 7, Chapter 9 in Ref. [5], and Section 6.6 in Ref. [33]).

One such case corresponds to the approximation of free-molecule flow in the kinetic theory. Flows of this type occur in rarefied fully ionized plasma when the following two-sided inequalities are satisfied:

$$\tau_{\text{ph}}^{(a)} \ll T \ll \tau_a \,; \qquad l_{\text{ph}}^{(a)} \ll L \ll l_a \,. \tag{7.1}$$

The first pair of inequalities allows using the approximation of a continuous medium. The second pair of inequalities justifies using the concept of 'collisionless plasma' in the zero approximation in the appropriate small parameters.

This approximation disregards the relaxation due to collisions: the relaxation scales τ_a, l_a are assumed to be infinite. Accordingly, the collision integrals in the kinetic equations for the functions $f_a(r,p,t)$ are assumed to be equal to zero. As a result, we come to the set of Vlasov equations for a fully ionized Coulomb plasma. In this way, the range of applicability of Vlasov equations is limited by the set of two-sided inequalities.

The reversibility of Vlasov equations is manifested, in particular, in that the entropy of plasma in this approximation remains unchanged. In other words, according to Vlasov equations, the initial uncertainty of the state (the degree of chaoticity) remains the same in the course of evolution (given that the system is closed).

A similar property is displayed by the Euler equation in hydrodynamics, which, as opposed to the more general Navier–Stokes equation, disregards the dissipative processes caused by viscosity and heat conductivity. The formal change to the Euler equation is accomplished by setting $\nu \to 0$, where ν is the viscosity coefficient. This passage to the limit, however, qualitatively alters both the mathematical structure of the equation and its physical content.

The alteration in the mathematical structure is associated with the ill-defined passage to the limit, when the coefficient of the highest derivative (the second derivative in the Navier–Stokes equation) tends to zero. Physically, this passing implies that the Reynolds number $\mathrm{Re} = UL/\nu$ approaches infinity. In this approximation the laminar flows, which are in fact supposed to be described by the Euler equation, are not feasible.

There is also another physical argument against this passage to the limit. Recall that the coefficient of viscosity is connected with thermal velocity and mean free path by the following relation:

$$\nu \sim v_T l \,, \qquad l \sim \frac{1}{n r_0^2} = \frac{r_0}{\varepsilon} \,. \tag{7.2}$$

In this way the zero viscosity conforms to zero mean free path or, equivalently, to infinite density parameter. Naturally, this state fails to fit the concept of rarefied gas.

Of course, one might argue that the Euler equation, in spite of such 'rigorous' restrictions, is widely used for solving numerous problems in hydrodynamics. This is possible, however, for a quite limited range of scales. Going beyond the limits leads to many known paradoxes.

The situation with the Vlasov equation is similar. It is widely used, for instance, in electronics, or when calculating various processes in thermonuclear devices, such as tokamaks. Nevertheless, the use of collisionless approximation for describing nonequilibrium processes in continuous medium on the basis of Vlasov equations is not justified.

As a matter of fact, the reversible equations in all real problems are replaced with the more general irreversible ones. This replacement is concealed behind mathematical formalism. This is what was done in the classical Landau's paper entitled 'On the oscillations of electron plasma' and published in 1946. In the physical papers that followed, the mathematical 'disguise' was replaced with the physical 'disguise'. Landau's damping is treated therewith as a reversible process, as the result of resonance interaction between particles and field in collisionless plasma. We have already noted that in the Introduction. The fundamental fact that the very concept of resonance is only rational in a dissipative system is overlooked in the process. With utmost clarity this has been stated in the above-quoted passage from the "Lectures on the theory of oscillations" delivered by L I Mandelshtam at the Department of Physics of Moscow State University as early as 1930.

We see that the reversible Vlasov equations are not sufficient in describing, for example, the resonance phenomena related to the interaction of charged particles and plasma waves, and it is necessary to use the more general dissipative equations. Notice that two extreme situations are possible, in which the sources of dissipation must be taken into account in different ways. Let us consider them.

7.2 Free-molecule flows in fully ionized confined plasma

Assume that two-sided inequalities (7.1) are satisfied. This brings us to Vlasov equations (5.15). These equations by themselves, however, are not yet sufficient for describing the plasma phenomena. They must be supplemented with initial and boundary conditions.

Let us specify the meaning of the characteristic parameters.

Assume that the state of the plasma is steady, and the plasma is confined. Let it be, for instance, a flow of plasma in a pipe. Then the characteristic parameter is the pipe radius R, so that $L \sim R$. Then the pipe radius is much greater than the Debye length, and therefore much greater than the radius of screening of external field by the plasma. This allows us to consider that the mean electric field in the plasma is zero (given that the flow is steady). As a result, the kinetic equation takes the form

$$v \frac{\partial f_a}{\partial r} = 0 \,, \tag{7.3}$$

which coincides with the relevant equation for a free-molecule gas flow (see Section 7, Chapter 9 in Ref. [5], and Section 6.6 in Ref. [33]). This equation must be supplemented with the boundary condition, which will depend considerably on the nature of interaction between the charged particles and the inner surface of the pipe. Since all real boundary conditions are dissipative, equation (7.3) together with a real boundary condition describes a dissipative process.

Assume that, like in the case of a free-molecule gas flow, the reflection of charged particles from the wall is diffusive. Then the mass flow of plasma in the pipe is given by the expression which formally coincides with the formula for Poiseuille flow. The main difference is that the mean free path in Eqn (7.2) for the viscosity coefficient is replaced with the pipe radius, $l \to R$. As a result, two-sided inequality (7.1) for the length scales is replaced with the one-sided inequality

$$l_{\text{ph}}^{(a)} \ll L \sim R \,, \qquad l_a \to R \,. \tag{7.4}$$

The dissipative boundary conditions for a free-molecule flow of plasma can be taken into account by introducing a certain effective collision integral into the Vlasov kinetic equation. As a result, the problem of description of nonequilibrium processes is also reduced to solving the appropriate dissipative kinetic equation, except that it includes a certain effective collision integral.

Thus, the reversible Vlasov equations are insufficient for describing the nonequilibrium processes in confined plasma even when the mean free path is much greater than the characteristic size of the system. This brings us back again to the dissipative kinetic equation. There is, however, an important difference. Namely, the relaxation length in the effective collision integral is determined not by the mean free path of plasma particles, but rather by the characteristic size of the system.

Open remains the question how useful is the collisionless approximation for describing nonequilibrium processes in unconfined plasma.

7.3 Is collisionless approximation possible for unconfined plasma?

The mean free path and free transit time in unconfined plasma are finite because the density of particles and the temperature have finite values. The characteristic parameters T and L can be the parameters of wave processes in the plasma — for example, the period of proper or forced oscillations ω and the wavelength λ. This offers an opportunity of giving a new definition of the concept of 'collisionless plasma'.

This concept is justified as long as the wavelength λ (the characteristic length of the process under consideration) is much less than the mean free path l_a. One must also take into account the appropriate relationship between the time parameters.

It seemingly would be natural to regard the plasma as collisionless when the collision rate ν_a is much less than the characteristic frequency of plasma oscillations ω (the free transit time τ_a is much greater than the period of oscillations). Accordingly, the approximation of 'collisionless plasma' might seemingly be justified for unconfined plasma as long as the inequalities

$$l_{\text{ph}}^{(a)} \ll \lambda \ll l_a \,, \qquad \tau_{\text{ph}}^{(a)} \ll \frac{1}{\omega} \ll \tau_a \tag{7.5}$$

are satisfied. However, as we have noted more than once, the role of dissipation caused by collisions may be definitive even if strong inequalities (7.1) hold. This is always the case in the neighbourhood of resonances. It is for this reason that, for example, the Landau damping, which is commonly held to be nondissipative and thus not associated with an increase in entropy, actually vanishes when the dissipative terms are set equal to zero in the initial equations. In the forthcoming section we shall consider this fundamental problem of plasma theory in greater detail.

7.4 Collisional nature of Landau damping

Let us pay attention to another important circumstance related to the physical interpretation of Landau damping. This circumstance to some extent justifies the reference to 'collisionless damping' in Landau's formula. Let us revert to Eqn (5.23) for the dielectric constant when the dissipative constant ν_a is finite.

Consider again the case of electron plasma, when the ion mass equals infinity. Let us be $\mathbf{k} \parallel x$, then after carrying out integration with respect to p_x and p_y we obtain the expression for the permittivity of electron plasma in the conditions when the deviation from the state of equilibrium is small. Here we shall only need the imaginary part of this complex function.

The integrand is the product of two functions defining the Lorentz line and the Maxwellian distribution. The widths (dispersions) of these functions for electron plasma depend on the ratio of velocities ν/k and v_T. Landau's result (5.30) follows in the zero approximation in ν/kv_T — that is, when the resonance is infinitesimally narrow, and the Lorentz line is replaced by the δ-function. For the electron plasma the frequency is $\omega = \omega_\mathrm{L}$, and is thus defined by the Langmuir frequency; the damping coefficient is $\gamma \ll \omega_\mathrm{L}$, and hence the wavelength $\lambda = 1/k$ is much greater than the Debye radius r_D. It follows that Landau's formula only holds when the following two strong inequalities are satisfied:

$$r_\mathrm{D} \ll \lambda = \frac{1}{k} \ll \frac{v_T}{\nu} = l\,. \tag{7.6}$$

The left-hand inequality ensures that Landau's damping coefficient is small, and also expresses the condition of applicability of the continuous medium approximation, since the Debye radius defines the size of the point of continuous medium [compare with the left-hand inequality in Eqn (7.5)]. Observe in this connection that another case of extremely short waves, when $kr_\mathrm{D} \gg 1$, was also considered in Ref. [3]. Such scales, however, fall within the size of the point of continuous medium, and therefore do not give any contribution to the mean electric field.

Let us return to the inequalities of Eqn (7.6). The right-hand inequality indicates that the wavelengths under consideration are much less than the mean free path (the relaxation length). This is the reason why Landau's damping may be referred to as 'collisionless damping'.

And one more remark.

To derive Landau's formula, a small dissipative term was introduced into the Vlasov equation, i.e. model collision integral (5.21). It is defined by the constant collision rate ν_a. In the final results we set $\nu_a \to 0$, and thus they do not depend on the collision rate. This is, however, just one of the possible methods of introducing the dissipation into the reversible Vlasov equation — the simplest way to regularize the solution of this equation which contains diverging integrals. Such regularization is not unambiguous. A physically sound procedure can only be selected on the basis of the kinetic equation. Then, as we shall see, the constant ν_a is replaced by a function which depends both on the velocity and the wave number with the use of the generalized kinetic equation. Because of this, the conventional Lorentz line is replaced by a more sophisticated structure, namely

$$\frac{\nu(v,k)}{(\omega - kv)^2 + \nu^2(v,k)}\,. \tag{7.7}$$

Naturally, this will change not only the condition of applicability of the formula for the damping coefficient, but also the physical interpretation of this phenomenon. There will be additional arguments in favour of replacing the term 'collisionless Landau damping' by 'collision Landau damping'. In Section 9 we shall formulate these arguments on the basis of the generalized kinetic equation tailored for the electron plasma.

Before proceeding to this problem, however, we need to consider another question of principle, which does not arise in the conventional kinetic theory but becomes very important when the generalized kinetic equation is used. Namely, we have to modify the definitions of the flows of matter and electric charge when the structure of continuous medium is taken into account.

8. Laws of conservation of matter and charge in Coulomb plasma

8.1 Flow of matter and mean velocity in microscopic theory

Recall that the initial equations in the theory of fully ionized Coulomb plasma were the equations for the microscopic phase density of each component $N_a(r,p,t)$ and the equations for the microscopic electric field strength $E^\mathrm{m}(r,t)$.

Is the system is closed, the flows of matter, momentum and energy across the surface surrounding the system under consideration are zero. The fulfilment of these requirements is ensured by the boundary conditions for the coordinates

$$N_a(r,p,t)\big|_{r_\alpha=\pm\infty} = 0\,, \qquad \alpha = 1,2,3\,, \tag{8.1}$$

and similar boundary conditions in the space of momenta

$$N_a(r,p,t)\big|_{p_\alpha=\pm\infty} = 0\,, \qquad \alpha = 1,2,3\,. \tag{8.2}$$

The microscopic phase density can be used for expressing the simpler microscopic characteristics, like the density of matter $\rho_a^\mathrm{m}(r,t)$ and the flux density $J_a^\mathrm{m}(r,t)$ for each plasma component:

$$\rho_a^\mathrm{m}(r,t) = m_a \int N_a(r,p,t)\,\mathrm{d}p\,, \tag{8.3}$$

$$J_a^\mathrm{m}(r,t) = \rho_a^\mathrm{m}(r,t)u_a^\mathrm{m}(r,t) = m_a \int vN_a(r,p,t)\,\mathrm{d}p\,. \tag{8.4}$$

The second of these expressions may be regarded as the definition of the mean microscopic velocity $u_a^\mathrm{m}(r,t)$ in the space of coordinates. For our future discussion it is very important that we actually gave *two independent definitions of the mean velocity*. The definition expressed by the left-hand equation is in terms of the flux of matter, and will be referred to as the *hydrodynamic* definition. The right-hand equation expresses the mean velocity in terms of the first moment of momentum $p = m_a v$ for the microscopic phase density $N_a(r,p,t)$, and will be referred to as the *statistical* definition of the velocity $u_a^\mathrm{m}(r,t)$. In the microscopic theory these two definitions are, of course, equivalent, both being straightforward implications of the mechanical definition of velocity.

Indeed, the equation of continuity of matter on the microscopic level is written as

$$\frac{\partial \rho_a^\mathrm{m}(r,t)}{\partial t} + \frac{\partial J_a^\mathrm{m}(r,t)}{\partial r} = 0\,. \tag{8.5}$$

At the same time, the equation for the density $\rho_a^\mathrm{m}(r,t)$ can be obtained with the aid of the equation for $N_a(r,p,t)$, and has the form

$$\frac{\partial \rho_a^{\mathrm{m}}(r,t)}{\partial t} + \frac{\partial \rho_a^{\mathrm{m}}(r,t) u_a^{\mathrm{m}}(r,t)}{\partial r} = 0 \,. \tag{8.6}$$

Here we have used the statistical definition of the mean velocity. The equivalence of two above definitions of velocity $u_a^{\mathrm{m}}(r,t)$ follows from comparison of the last two equations.

Let us also recall the microscopic definitions of the densities of electric charge and current:

$$q^{\mathrm{m}}(r,t) = \sum_a e_a \int N_a(r,p,t)\,\mathrm{d}p = \sum_a e_a n_a^{\mathrm{m}}(r,t) \,, \tag{8.7}$$

$$j^{\mathrm{m}}(r,t) = \sum_a e_a \int v N_a(r,p,t)\,\mathrm{d}p = \sum_a e_a n_a^{\mathrm{m}}(r,t) u_a^{\mathrm{m}}(r,t) \,. \tag{8.8}$$

The density of electric current for each component is

$$j_a^{\mathrm{m}}(r,t) = e_a n_a^{\mathrm{m}}(r,t) u_a^{\mathrm{m}}(r,t) \,, \tag{8.9}$$

and is therefore proportional to the product of the microscopic density and the velocity.

Now we can write the microscopic equation of continuity for the density of electric charge. It follows from continuity equation (8.5) and has the form

$$\frac{\partial q^{\mathrm{m}}(r,t)}{\partial t} + \frac{\partial j^{\mathrm{m}}(r,t)}{\partial r} = 0 \,. \tag{8.10}$$

All functions and equations presented in this section follow directly from the equations of motion. The reverse change back to the equations of particles motion is also possible. These equations are invariant with respect to the transformation

$$t \to -t \,, \qquad r_i \to r_i \,, \qquad p_i \to -p_i \,, \tag{8.11}$$

and therefore the motion described by them is reversible. We may reason that the equations of motion display the *time symmetry of reversible processes*.

With respect to transformation (8.11), the microscopic phase density and the microscopic field strength also possess the time symmetry of reversible processes:

$$N_a(r,p,t) = N_a(r,-p,-t) \,; \quad E^{\mathrm{m}}(R,t) = E^{\mathrm{m}}(r,-t) \,. \tag{8.12}$$

Of course, one should remember that the *concept of the 'arrow of time' is more cardinal* than the concepts of 'reversibility' and 'irreversibility'. The replacement $t \to -t$ is not possible in Nature. We may only speak of 'coming back home' — in the time $2(t - t_0)$ the particles will return to their initial positions with reversed velocities if we flip the sign of their velocities at the point in time $t - t_0$. All this considered, Eqn (8.11) must be replaced by

$$N_a(r,p,t_0) = N_a\big(r,-p,2(t-t_0)\big) \,,$$
$$E^{\mathrm{m}}(R,t_0) = E^{\mathrm{m}}\big(r,2(t-t_0)\big) \,. \tag{8.13}$$

The above definitions of the mean velocity seem to be so obvious that they are directly transferred (without good enough reason) to the kinetic theory and hydrodynamics. Thus they are assumed to be suitable for the description of irreversible processes as well.

Before considering the irreversible equations in the averaged microscopic functions, let us check the equivalence of two representations of the microscopic field equations for Coulomb plasma, which follow from the Lorentz equations for the microscopic field strengths. So far we have been using equations (2.4) for the microscopic field $E^{\mathrm{m}}(r,t)$. The equations for the field $E^{\mathrm{m}}(r,t)$ can also be written in a different form

$$\operatorname{rot} E^{\mathrm{m}} = 0 \,, \qquad \frac{\partial E^{\mathrm{m}}}{\partial t} = -4\pi \sum_a e_a \int v N_a(r,p,t)\,\mathrm{d}p \,. \tag{8.14}$$

Both representations are equivalent by virtue of continuity equation (8.10) for the density of charge.

So then, there exist two equivalent representations of the equations for the microscopic phase density $N_a(r,p,t)$ and the microscopic field strength $E^{\mathrm{m}}(r,t)$. Now a natural question is whether this equivalence is preserved when we go over to the kinetic description?

This question is avoided in the conventional kinetic theory of plasma. This becomes possible because the irreversibility does not prevent translating the above two definitions of velocity $u_a^{\mathrm{m}}(r,t)$ to the definition of the mean velocity $u_a(r,t)$ when the Landau kinetic equation is used. The situation is different, however, when we employ the generalized kinetic equation.

8.2 Flow of matter and mean velocity in kinetic theory

The kinetic equations are dissipative. Because of this, the distribution functions $f_a(r,p,t)$ generally do not exhibit the time symmetry of reversible processes. This issue deserves more attention.

Recall that the distribution function $f_a(r,p,t)$ and the mean field strength $E(r,t)$ are linked with the corresponding microscopic functions $N_a(r,p,t)$ and $E^{\mathrm{m}}(r,t)$ by relations (3.1). Let us rewrite them once again:

$$n_a f_a(r,p,t) = \big\langle N_a(r,p,t) \big\rangle \,, \qquad E(r,t) = \big\langle E^{\mathrm{m}}(r,t) \big\rangle \,. \tag{8.15}$$

We see that the distribution function is defined by the value of microscopic phase density averaged over the Gibbs ensemble. This microscopic phase density, as known, displays the time symmetry of a reversible process. Why does averaging destroy this symmetry?

The violation of time symmetry is rooted in the very definition of the *Gibbs ensemble for the irreversible processes*. Recall that J Gibbs used averaging over the ensemble of identical systems only for the state of equilibrium and only for quasistatically reversible processes.

In the statistical theory there are two possible definitions of the Gibbs ensemble for nonequilibrium processes (see Refs [30, 33]).

(1) The microstates of the systems in the ensemble are not the same because of the uncertainty in the initial values of the variables: only the distribution function of the initial conditions for particles and field is defined. The equation for the many-body distribution function itself (the Liouville equation or the equation for the microscopic phase density and the microscopic field) remains unchanged. Then, however, averaging over the initial values preserves the time symmetry of reversible processes. This is the reason why the Gibbs ensemble for nonequilibrium processes is, implicitly as a rule, redefined in the construction of irreversible kinetic equations in the BBGKY theory.

(2) Smoothing over physically infinitesimal volumes (the volumes of the points of continuous medium) is carried out at the first step of going over from the reversible equations to the irreversible equations. Such smoothing is justified by the dynamic instability of particles motion, when even very small uncontrollable external forces have a randomizing effect. The differences in the microstates of systems in the Gibbs ensemble is due to the lack of information about the motion of particles within the points of continuous medium. In this situation the irreversibility of kinetic equations becomes inevitable.

Thus, there is a possibility of defining the Gibbs ensemble for irreversible processes. With such selection of the ensemble, definition (8.15) results in the distribution functions whose time symmetry is different from the time symmetry of reversible processes.

Let us represent the distribution function thus defined as a sum of two terms

$$f_a(r,p,t) = f_a^{(\mathrm{dyn})}(r,p,t) + f_a^{(\mathrm{dis})}(r,p,t)\,, \tag{8.16}$$

where the first dynamic term (dyn) displays the time symmetry of a reversible process,

$$f_a^{(\mathrm{dyn})}(r,p,t_0) = f_a^{(\mathrm{dyn})}\big(r,-p,2(t-t_0)\big)\,, \tag{8.17}$$

while the second dissipative term (dis) does not possess the time symmetry of a reversible process,

$$f_a^{(\mathrm{dis})}(r,p,t_0) \neq f_a^{(\mathrm{dis})}\big(r,-p,2(t-t_0)\big)\,. \tag{8.18}$$

Naturally, the normalization condition applies solely to the total distribution function:

$$n_a \int f_a(r,p,t)\,\mathrm{d}p\,\mathrm{d}r = N\,. \tag{8.19}$$

Let us now discuss the definition of the mean velocity in the kinetic theory.

The density of matter is given by

$$\rho_a(r,t) = \rho_a \int f_a(r,p,t)\,\mathrm{d}p\,, \qquad \rho_a = m_a n_a\,, \tag{8.20}$$

and the mean velocity by

$$\rho_a(r,t)u_a(r,t) = \rho_a \int v f_a(r,p,t)\,\mathrm{d}p\,. \tag{8.21}$$

Let us clarify the meaning of the last expression. Considering the definition of the density, Eqn (8.21) can be rewritten as

$$\rho_a \int \big[v - u_a(r,t)\big] f_a(r,p,t)\,\mathrm{d}p = 0\,. \tag{8.22}$$

Definition (8.21) of the velocity $u_a(r,t)$ implies that this velocity is the first moment of velocity for the distribution function $f_a(r,p,t)$. This allows us to refer to Eqn (8.21) as the *statistical* definition of the mean velocity.

Recall that there exist two equivalent definitions of the velocity $u_a^{\mathrm{m}}(r,t)$ in the microscopic theory, as given by Eqn (8.4). The left-hand equality expresses the velocity in terms of the flux of matter, and the right-hand equality defines the velocity in terms of the microscopic phase density (the microscopic distribution).

In the description of irreversible processes, the definition of the velocity can also be given in terms of the flux of matter:

$$J_a(r,t) = \rho_a(r,t)u_a(r,t)\,. \tag{8.23}$$

This can be referred to as the *hydrodynamic* definition.

The equivalence of these two definitions of the mean velocity in the case of irreversible processes in continuous medium is not obvious. If these definitions are not equivalent, then which of them is more natural from the physical standpoint?

This question has been discussed in Chapters 13, 14 of Ref. [33]. With the aid of the Boltzmann equation it was found that the kinetic and the hydrodynamic definitions of the mean velocity $u_a(r,t)$ are truly equivalent. The same is true in the plasma theory when the kinetic equations of Vlasov, Landau and Balescu and Lenard are used.

When, however, we turn to the generalized kinetic equation, these definitions are no longer equivalent, and the linkage between the flux of matter and the mean velocity is given by a more general relation

$$J_a(r,t) = \rho_a(r,t)u_a(r,t) + J_a^{(\mathrm{dis})}(r,t)\,. \tag{8.24}$$

Superscript 'dis' in the added term indicates that this term is completely defined by the dissipative processes, and vanishes when the macroscopic processes are described by the reversible equations. Such example is the Euler equation in hydrodynamics.

8.3 Continuity equation for plasma

Let us revert to generalized kinetic equation (6.12). As compared with the Landau kinetic equation, it contains the additional dissipative term, which takes care of the structure of continuous medium and corresponds to the 'adjustment' of the microscopic distribution function to the microscopic function smoothed over the volume of the point of continuous medium.

It might seem that this additional term would further complicate the formidable task of solving the kinetic equation. As a matter of fact, however, the inclusion of dissipation caused by the redistribution of particles in space makes the kinetic equation more regular, and, in particular, allows for a more consistent treatment of the irreversibility in the definitions of the flux of matter and the flux of heat.

Recall once again that for accomplishing the change from the kinetic equation to equations of gas dynamics in the theory of gases it sufficed to assume that the distribution function presents a local Maxwellian distribution, which amounts to postulating the existence of local thermodynamic equilibrium.

Admit now that the distribution functions of electrons and ions are given by the local Maxwellian distribution. To obtain the equation for the density of matter, we must multiply kinetic equation (6.12) by the constant $\rho_a = m_a n_a$ and carry out integration with respect to momenta. The contribution from the Landau integral is then equal to zero. Let us find the contributions from the other terms.

The first term gives the time derivative of the density. The second term, given the local Maxwellian distribution, results in $\operatorname{div}\rho_a u_a$ and defines the contribution from the dynamic part of the flux of matter. Finally, the dissipative term defines the dissipative part of the flux of matter. As a result, we arrive at the continuity equation

$$\frac{\partial \rho_a(r,t)}{\partial t} + \operatorname{div} J_a(r,t) = 0\,. \tag{8.25}$$

The flux of matter is defined here as the sum of three terms:

$$J_a(r,t) = \rho_a(r,t)u_a(r,t) - D_a\,\frac{\partial \rho_a(r,t)}{\partial r} + \frac{\tau_a}{m_a}\,e_a E(r,t)\rho_a(r,t)\,. \tag{8.26}$$

The first term is responsible for the convective (dynamic) part of the flux of matter, the second for the flux of matter caused by self-diffusion, and, finally, the third for the flux of matter caused by mobility.

The coefficient of space diffusion D_a is linked with the relaxation time τ_a through the Einstein relation $D_a = \tau_a k_B T/m_a$, and hence it will suffice to define just one of the kinetic coefficients, for example, the coefficient of space diffusion D_a. Recall in this connection that additional dissipative term (6.4) in generalized kinetic equation (6.12) arises because of smoothing over the physically infinitesimal volumes for, respectively, electrons and ions [see Eqns (4.6) and (6.5)]. With this method of smoothing the coefficients of space diffusion D_a $(a = \mathrm{e}, \mathrm{i})$ characterize the self-diffusion of electrons and ions. Accordingly, the relaxation times τ_a are determined by the rates of electron-electron and ion-ion collisions.

The condition of closure of the system is that the total flux of matter across the surface enclosing the system is zero. At the state of equilibrium, which is only possible when the field $E = -\operatorname{grad}\varphi$ is constant, the mean velocity is zero, and the density distribution conforms to the Boltzmann distribution:

$$u_a = 0\,, \qquad \rho_a(r) = \rho_a \exp\left[-\frac{e_a\varphi(r)}{k_B T}\right]. \tag{8.27}$$

The relative contributions of the dynamic and the diffusion processes to the total flux of matter are determined by the diffusion Reynolds number

$$\mathrm{Re}^D = \frac{uL}{D}\,, \tag{8.28}$$

which can be represented as the ratio of two characteristic times: the diffusion time τ_D and the time of passage τ_{pas} taken by the flow to cover the characteristic distance L:

$$\mathrm{Re}^D = \frac{\tau_D}{\tau_{\mathrm{pas}}}\,, \qquad \tau_D = \frac{L^2}{D}\,, \qquad \tau_{\mathrm{pas}} = \frac{L}{u}\,. \tag{8.29}$$

Now we have everything at hand for describing the diffusion and wave processes in electron plasma on the basis of the generalized kinetic equation.

9. Electron plasma

9.1 Generalized kinetic equation for electron plasma

Let us show that the inclusion of space diffusion allows for a more consistent treatment of the conditions when the resonances arising in the solution of the kinetic equation are narrow. This will help us to define the range of existence of the collision Landau damping.

We have already quoted the reasons by which the Landau damping is always a collision process. This is the case even for the confined collisionless plasma, when the characteristic sizes of the system (for example, the diameter of the pipe) are less than the mean free path. Here we shall consider an unbounded plasma, in any case such that the size of the system is much greater than the mean free path. Then there may be wave processes whose wavelength is much greater than the mean free path, but much less than the characteristic size of the system:

$$l \ll \lambda \ll L\,. \tag{9.1}$$

Under these conditions one may introduce, along with the hydrodynamic small parameter (the Knudsen number) $\mathrm{Kn} = l/L$, the wave Knudsen number $\mathrm{K}_\lambda = l/\lambda$, which is also small.

Recall that in gas theory the transition from the generalized kinetic equation to the equations of gas dynamics was carried out without using the perturbation theory in Knudsen number (see Chapters 13, 14 in Ref. [33]). To go over to the equations of gas dynamics it was sufficient to assume that the distribution function accords with the local Maxwellian distribution. Let us consider the corresponding approximation for the electron plasma.

We assume that functions $f_a(r,p,t)$ are defined by the local Maxwellian distributions. In this approximation the integrals of electron-electron and ion-ion collisions are zero.

Consider next the *electron plasma*. The mass of ions is assumed to be infinitely large, and their role consists in creating the positive charge background. This ensures that the plasma as a whole is electrically neutral.

By $f(r,p,t)$ we denote the distribution function for electrons. In our current approximation, kinetic equation (6.12) is simplified and becomes

$$\frac{\partial f}{\partial t} + v\,\frac{\partial f}{\partial r} + eE(r,t)\,\frac{\partial f}{\partial p} = I_{\mathrm{ei}}^{(v)} + I^{(r)}\,. \tag{9.2}$$

The first term on the right-hand side takes care of the electron-ion interaction, and the second term is responsible for the space diffusion of the electron distribution function. Both terms are much simplified for electron plasma in the approximation of local equilibrium.

Since the role of ions is reduced to maintaining the positive charge background, the ion distribution function may be represented by δ-function:

$$f_i(r,p,t) = \delta(p)\,, \tag{9.3}$$

whereas the electron distribution function is given by the local Maxwellian distribution:

$$nf(r,p,t) = \frac{n(r,t)}{(2\pi m k_B T)^{3/2}} \exp\left\{-\frac{\left[p - mu(r,t)\right]^2}{2mk_B T}\right\}. \tag{9.4}$$

Let us return to Landau's collision integral (6.8). In the case of electron plasma, the second term on the right-hand side is zero, since the mass of ions is assumed to be infinite. In the first term [with due account for ion distribution function (9.3)] integration can be carried out with respect to momenta of ions. As a result, we get a relatively simple expression for the integral of electron – ion collisions:

$$I_{\mathrm{ei}}^{(v)} = C_{\mathrm{ei}}\,\frac{\partial}{\partial p_i}\left(V_{ij}(v)\,\frac{\partial f(r,p,t)}{\partial p_j}\right). \tag{9.5}$$

The expressions for the tensor V_{ij} and the constant $C_{\rm ei}$ follow from Eqns (6.9), (6.10), and now are represented as

$$V_{ij}(v) = \frac{v^2\delta_{ij} - v_i v_j}{|v|^3}\,, \qquad V_{ij}v_j = v_i V_{ij} = 0\,, \tag{9.6}$$

$$C_{\rm ei} = 2\pi e^4 nL \qquad (\text{and below } D = nC_{\rm ei})\,. \tag{9.7}$$

The collision integral $I_{\rm ei}^{(v)}$ in this approximation depends linearly on the electron distribution function. The latter is given by local Maxwellian distribution (9.4). This allows for further simplifications of the collision integral.

In the next section we shall use the above kinetic equation for studying the wave processes in electron plasma. It would be natural to start with the linear approximation. For this it will suffice to examine the kinetic equation for small deviations from the state of equilibrium. We mark the equilibrium function with subscript '0', and represent the electron distribution function and the field $E(r,t)$ as

$$f(r,p,t) = f_0(p) + f_1(r,p,t)\,, \qquad f_1 \ll f_0\,, \tag{9.8}$$

$$E(r,t) = E_1(r,t)\,, \qquad E_0 = 0\,. \tag{9.9}$$

Here $f_0(p)$ is the Maxwellian distribution function

$$f_0(p) = \frac{1}{(2\pi m k_{\rm B}T)^{3/2}}\exp\left(-\frac{p^2}{2mk_{\rm B}T}\right). \tag{9.10}$$

The distribution function $f_1(r,p,t)$ is represented as local Maxwellian distribution (9.4). For small deviations from equilibrium, the relevant gasdynamic functions can be reported as

$$n(r,t) = n + n_1(r,t)\,, \qquad T(r,t) = T + T_1(r,t)\,,$$
$$u(r,t) = u_1(r,t)\,, \qquad u_0 = 0\,. \tag{9.11}$$

Hence it follows that the function $f_1(r,p,t)$ is represented as a linear combination of three distribution functions:

$$f_1(r,p,t) = f_1^{(n)}(r,p,t) + f_1^{(u)}(r,p,t) + f_1^{(T)}(r,p,t)\,. \tag{9.12}$$

The terms on the right-hand side are proportional, respectively, to $n_1(r,t)$, $u_1(r,t)$, $T_1(r,t)$, and to the Maxwellian distribution $f_0(p)$. This opens up the possibility of further simplifying the collision integral introduced in (9.5).

Given the properties of the tensor $V_{ij}(v)$, which are expressed by the last equalities in Eqn (9.6), the terms proportional to $n_1(r,t)$, $T_1(r,t)$ give no contribution to collision integral (9.5). Accordingly, we only need to include the contribution proportional to the function $u(r,t)$ (subscript '1' is dropped). Then we can represent the nonequilibrium part of the distribution function as

$$f_1(r,p,t) = \frac{\mathbf{u}(r,t)\mathbf{v}}{v_T^2}\,f_0(v)\,, \tag{9.13}$$

where $\mathbf{u}$ is the vector of hydrodynamic velocity, f_0 is the Maxwellian distribution which depends only on the modulus of velocity.

When distribution function (9.8) is substituted into collision integral (9.5), the contribution of the function $f_0(p)$ comes to nought. Accordingly, it is sufficient to substitute only the function $f_1(r,p,t)$ into Eqn (9.5). Using Eqn (9.13), after some straightforward algebra we get the following expression for the collision integral:

$$I_{\rm ei}(r,p,t) = -\nu_{\rm ei}(v)f_1^{(u)}(r,p,t)\,. \tag{9.14}$$

The collision rate depends therewith on the velocity, and is given by

$$\nu_{\rm ei}(v) = \frac{4\pi e^4 n}{m^2 v^3}\,L\,. \tag{9.15}$$

As before, L is the Coulomb logarithm. We see that the collision rate falls off quickly as the velocity increases. Such dependence is very important for describing the wave properties of electron plasma. This will be demonstrated in the forthcoming section.

Consider the expression for the second dissipative term in the kinetic equation for electron plasma. In the general case it is given by Eqn (6.4), which holds also for the electron plasma (subscript 'a' is dropped):

$$I^{(r)}(r,p,t) = \frac{\partial}{\partial r}\left(D\,\frac{\partial f}{\partial r}\right) - \frac{\partial}{\partial r}\left(\frac{\tau}{m}\,eEf\right). \tag{9.16}$$

In the linear approximation this expression is simplified:

$$I^{(r)}(r,p,t) = \frac{\partial}{\partial r}\left(D\,\frac{\partial f_1}{\partial r}\right) - e\,\frac{\tau}{m}\,\frac{\partial E}{\partial r}\,f_0(p)\,. \tag{9.17}$$

In writing the last term we have taken advantage of the fact that at equilibrium the field is zero — there is no external static field. The coefficient of self-diffusion of electrons is linked with the rate of electron-electron collisions by the Einstein relation $D = \tau k_{\rm B}T/m$ [cf. Eqn (6.5)].

Thus, the generalized kinetic equation for electron plasma in the linear approximation takes the form

$$\frac{\partial f_1}{\partial t} + v\,\frac{\partial f_1}{\partial r} + eE(r,t)\,\frac{\partial f_0}{\partial p} = I_{\rm ei}^{(v)} + I^{(r)}\,. \tag{9.18}$$

The collision integrals are given by formulas (9.14), (9.15) and (9.17). This equation must be supplemented with the equations for the field

$$\operatorname{rot} E = 0\,, \qquad \operatorname{div} E = 4\pi en\int f_1(r,p,t)\,{\rm d}p\,. \tag{9.19}$$

As a result, we have a closed linearized set of equations for the nonequilibrium component of the electron distribution function and electric field for states close to equilibrium.

9.2 Charge continuity equation. Electric current

Now we use a different representation of the field equations for electron plasma:

$$\operatorname{rot} E = 0\,, \qquad \frac{\partial E}{\partial t} = -4\pi\sum_a e_a n_a\int f_a(r,p,t)\,{\rm d}p \equiv 4\pi j(r,t)\,, \tag{9.20}$$

here we have introduced the notation for the vector of electric current density $j(r,t)$; for the electron plasma we derive

$$j(r,t) = q(r,t)u_a(r,t) - D_a\,\frac{\partial q(r,t)}{\partial r} + \frac{\tau_a}{m_a}\,e_a E(r,t)q(r,t)\,. \tag{9.21}$$

Equation (9.21) now defines the current density entering field equation (9.20). It also enters the continuity equation for the electric charge

$$\frac{\partial q(r,t)}{\partial t} + \operatorname{div} j(r,t) = 0\,. \tag{9.22}$$

In this way, the density of electric current is determined not only by the convective transport of charge, but also by the diffusion of the charge density and the mobility of electric charges in the field. Changes in the structure of the expression for the current, as well as in the structure of the kinetic equation itself, are due to the fact that the structure of continuous medium is now taken into account. Let us examine how these changes affect the wave properties of electron plasma. This will also give us an opportunity of making a new physical interpretation of the Landau damping.

We start with the simplest particular cases.

9.3 Self-diffusion in electron plasma

Assume that the mean velocity is zero, and the temperature is constant. Carrying out integration with respect to momenta, from kinetic equation (9.18) we get the equation of self-diffusion for $n_1(r,t)$:

$$\frac{\partial n_1}{\partial t} = D\,\frac{\partial^2 n_1}{\partial r^2} - en\,\frac{\tau}{m}\,\frac{\partial E(r,t)}{\partial r}\,, \tag{9.23}$$

which must be solved together with the field equations

$$\operatorname{rot} E = 0\,, \qquad \operatorname{div} E = 4\pi e n_1(r,t)\,. \tag{9.24}$$

These equations offer a particular case of the more general set of diffusion equations for electron – ion plasma. The corresponding equations for the space – time Fourier components have the form

$$(-\mathrm{i}\omega + Dk^2)n_1(\omega,k) = -\mathrm{i}en\,\frac{\tau}{m}\big(kE(\omega,k)\big)\,, \qquad \tau \equiv \frac{1}{v}\,,$$
$$\mathrm{i}\big(kE(\omega,k)\big) = 4\pi e n_1(\omega,k)\,. \tag{9.25}$$

This set of equations reduces to a single equation for the field

$$\varepsilon(\omega,k)\big(kE(\omega,k)\big) = 0\,. \tag{9.26}$$

Here we have introduced the notation for the complex dielectric constant of a plasma for the diffusion-like relaxation:

$$\varepsilon(\omega,k) = 1 + \frac{4\pi e^2 n}{mv}\,\frac{1}{-\mathrm{i}\omega + Dk^2}\,, \qquad D = \frac{k_{\mathrm{B}}T}{mv}\,. \tag{9.27}$$

Consider the corresponding dispersion equation

$$\varepsilon(\omega,k) = 0\,. \tag{9.28}$$

If $\omega = 0$, we have

$$\operatorname{Re}\varepsilon(\omega=0,k) = 1 + \frac{1}{r_{\mathrm{D_e}}^2 k^2} = 0\,, \qquad \operatorname{Im}\varepsilon(\omega=0,k) = 0\,. \tag{9.29}$$

Here we have used the notation for the Debye radius of electron plasma.

Assume that the field is directed along the x-axis. Then the field that is given on the boundary $x = 0$ is screened at a distance of the order of the Debye radius r_{D}.

9.4 Wave properties of electron plasma. Landau collision damping

Thus, we have considered the simplest model of relaxation when the dissipation is due to the self-diffusion of plasma electrons. The nature of the dissipation does not then depend on the electron distribution with respect to velocities. Now let us try to take this effect into account.

Recall that the solution of the generalized kinetic equation for electron plasma was set as local Maxwellian distribution (9.4). Equations for the three appropriate functions $\rho(r,t)$, $u(r,t)$, $T(r,t)$ (the equations of gas dynamics for plasma electrons) can be obtained from generalized kinetic equation (9.2). In studying the wave properties of plasma, these equations may be replaced with the linearized equations for the functions $\rho_1(r,t)$, $u_1(r,t)$, $T_1(r,t)$, which can be obtained directly from linearized kinetic equation (9.18) with the collision integral in the form of Eqn (9.14).

The kinetic equation, however, allows for a more comprehensive treatment of the electron distribution over velocities, and, in particular, the effects of velocities much higher than the thermal velocity. These 'particulars' fall beyond the scope of the equations of gas dynamics — the equations in the first five moments of the particle distribution with respect to velocities. Recall in this connection that Landau's damping is determined by nothing else but the 'tail' of the Maxwellian distribution.

Let us narrow down the problem still further. Namely, we are going to use linearized kinetic equation (9.18) under the assumption that the local Maxwellian distribution only depends on the function $\rho_1(r,t)$, whereas $u_1(r,t) = 0$ and $T_1(r,t) = 0$. In other words, we suggest that the mean velocity of electrons comes to nothing, and the temperature is constant. As a result, linearized kinetic equation (9.18) is simplified and becomes

$$\frac{\partial f_1}{\partial t} + v\,\frac{\partial f_1}{\partial r} + eE(r,t)\,\frac{\partial f_0}{\partial p} = \frac{\partial}{\partial r}\left(D\,\frac{\partial f_1}{\partial r}\right) - e\,\frac{\tau}{m}\,\frac{\partial E}{\partial r}\,f_0(p)\,. \tag{9.30}$$

Here we have used Eqn (9.17).

Carrying out integration with respect to momenta, we find the linearized continuity equation. Given that $u_1(r,t) = 0$, this equation takes the form

$$\frac{\partial n_1}{\partial t} = D\,\frac{\partial^2 n_1}{\partial r^2} - en\,\frac{\tau}{m}\,\frac{\partial E(r,t)}{\partial r} \tag{9.31}$$

and coincides with diffusion equation (9.23). What is an additional information that is contained in the kinetic equation?

In the linear kinetic equation we carry out the Fourier expansion in terms of time and space variables. As a result, we get a set of equations in the space – time Fourier components of the distribution function and the electric field. The resulting set of equations is written as

$$\big[-\mathrm{i}(\omega - kv) + Dk^2\big]\,f_1(\omega,k,p)$$
$$= -eE(\omega,k)\,\frac{\partial f_0(v)}{\partial p} - \mathrm{i}e\,\frac{\tau}{m}\big(kE(\omega,k)\big)f_0(v)\,,$$
$$\big[kE(\omega,k)\big] = 0\,, \qquad \mathrm{i}\big(kE(\omega,k)\big) = 4\pi en\int f_1(\omega,k,p)\,\mathrm{d}p\,. \tag{9.32}$$

The left-hand side of the first equation in Eqn (9.32) contains all terms with the function $f_1(\omega, k, p)$. The right-hand side contains the terms which are proportional to the distribution function $f_0(v)$ and the electric field strength $E(\omega, k)$.

Using the first equation, we express the distribution function $f_1(\omega, k, p)$ in terms of the electric field strength:

$$f_1(\omega, k, p) = \frac{-\mathrm{i}}{(\omega - kv) + \mathrm{i}Dk^2} \times \left[eE(\omega, k) \frac{\partial f_0(v)}{\partial p} + \mathrm{i}e \frac{\tau}{m} \big(kE(\omega, k)\big) f_0(v) \right]. \quad (9.33)$$

The solution of the set of equations for the field is represented as

$$E(\omega, k) = -\mathrm{i} \frac{\mathbf{k}}{k^2} 4\pi en \int f_1(\omega, k, p)\, \mathrm{d}p. \quad (9.34)$$

Substituting the expression for $f_1(\omega, k, p)$ into the right-hand side of the last equation, we can write the equation for the field $E(\omega, k)$ in the form

$$\varepsilon(\omega, k)\big(kE(\omega, k)\big) = 0. \quad (9.35)$$

Here again we have used the expression for the permittivity of plasma, this time with due account for the distribution of electrons with respect to velocities:

$$\varepsilon(\omega, k) = 1 + \frac{4\pi e^2 n}{k^2} \int k \frac{\partial f_0(v)}{\partial p} \frac{\mathrm{d}p}{(\omega - kv) + \mathrm{i}Dk^2} + \frac{4\pi e^2 n}{mv} \int \frac{\mathrm{i}}{(\omega - kv) + \mathrm{i}Dk^2} f_0(v)\, \mathrm{d}p. \quad (9.36)$$

The relative contribution of the two functions in the integrand is characterized by the dimensionless parameter

$$\frac{v_T}{Dk} \sim \frac{\nu_{\mathrm{ee}}}{Dk^2} \sim \frac{1}{kl} \sim \frac{\lambda}{l}, \quad (9.37)$$

which defines the ratio between 'widths' of the functions in the formula for $\varepsilon(\omega, k)$ — the Maxwellian distribution and the Lorentz spectral line (with respect to velocities, frequencies, lengths). Here, as before, l is the mean free path, and λ is the wavelength.

Let us consider a number of special cases.

9.4.1 Unconfined plasma. Large Knudsen numbers: $\lambda \ll l$. This condition corresponds to the collisionless wave approximation for the selected range of wavelengths (wave numbers).

It is in this approximation that we obtained the expression for the Landau damping coefficient in Section 5.5.2. The width of the Lorentz line was then determined by the collision rate ν_{ee} or by the corresponding velocity ν_{ee}/k. Because of this, the Lorentz line narrowed down with the increasing wave number k (or decreasing wavelength).

Now the situation is reversed: in the zero approximation, the Maxwellian distribution $f_0(v)$ can be replaced with the δ-function:

$$f_0(v) \to \delta(v). \quad (9.38)$$

This allows carrying out integration with respect to momenta in Eqn (9.36). As a result, the expression for permittivity takes the form

$$\varepsilon = 1 + \frac{4\pi e^2 n}{mv} \frac{\mathrm{i}}{\omega + \mathrm{i}Dk^2}, \qquad D = \frac{k_{\mathrm{B}} T}{mv} \quad (9.39)$$

and coincides with Eqn (9.27).

Thus, in the zero approximation in parameter (9.37) — that is, disregarding the motion of plasma electrons — we turn back to the results obtained on the basis of diffusion equation (9.23).

Notice also that the smallness of parameter (9.37) implies that the collision rate in the electron plasma is much less than the diffusion width of the line Dk^2. In this way, the 'old' term 'collisionless approximation' changes now its meaning: for a fixed value of thermal velocity, the diffusion collision rate Dk^2 grows as the wave number k increases. Accordingly, the dissipation is due not to collisions, but rather to self-diffusion.

It is also important that small parameter (9.37) is limited from below. This restriction is related to the structure of continuous medium. Since the size of a point is defined by the Debye radius, then applicability of collisionless approximation in question is limited by the two-sided inequality

$$\frac{r_{\mathrm{D}}}{l} \ll \frac{\lambda}{l} \ll 1. \quad (9.40)$$

We see that the imaginary part in this collisionless approximation is not determined by the Landau damping, but by dissipation caused by diffusion.

Now let us consider another case of Landau damping, which this time is a collision process.

9.4.2 Unconfined plasma. Small Knudsen numbers: $l \ll \lambda$. In this case the width of the Maxwellian distribution is much greater than the width Dk of the Lorentz line (in units of velocity). In the zero approximation in this parameter we may carry out the following replacement in Eqn (9.36):

$$\frac{1}{(\omega - kv) + \mathrm{i}Dk^2} \to P \frac{1}{(\omega - kv)} - \mathrm{i}\pi\delta(\omega - kv). \quad (9.41)$$

As a result, we get the following expression for the imaginary part of permittivity:

$$\mathrm{Re}\, \varepsilon(\omega, k) = 1 + \frac{4\pi e^2 n}{k^2} P \int k \frac{\partial f_0(v)}{\partial p} \frac{\mathrm{d}p}{\omega - kv} + \frac{4\pi^2 e^2 n}{mv} \int \delta(\omega - kv) f_0(v)\, \mathrm{d}p. \quad (9.42)$$

The last term in this expression is exponentially small. Indeed, after integration with respect to p it can be represented in the form

$$\sqrt{\frac{\pi}{2}}\, \omega_{\mathrm{L}} \frac{1}{r_{\mathrm{D}} k} \exp\left(-\frac{1}{2 r_{\mathrm{D}}^2 k^2}\right). \quad (9.43)$$

Since the following two condition hold for electron plasma:

$$\omega \sim \omega_{\mathrm{L}}, \qquad k r_{\mathrm{D}} \ll 1, \quad (9.44)$$

the assumption concerning the exponential smallness of this term is true. Then the expression for the real part of permittivity is simplified, namely

$$\mathrm{Re}\, \varepsilon(\omega, k) = 1 + \frac{4\pi e^2 n}{k^2} P \int k \frac{\partial f_0(v)}{\partial p} \frac{\mathrm{d}p}{\omega - kv}, \quad (9.45)$$

and coincides in the case of electron plasma with our earlier expression (5.26).

Let us now consider the corresponding expression for the imaginary part of dielectric constant in electron plasma:

$$\operatorname{Im}\varepsilon(\omega,k) = -\frac{4\pi e^2 n}{k^2}\int \delta(\omega - kv)\, k\, \frac{\partial f_0(v)}{\partial p}\, \mathrm{d}p + \frac{4\pi e^2 n}{mv}\int \frac{\omega - kv}{(\omega - kv)^2 + (Dk^2)^2}\, f_0(v)\, \mathrm{d}p\,. \quad (9.46)$$

The first term on the right-hand side coincides in the case of electron plasma with the expression for Landau damping coefficient. We denote it by $(\operatorname{Im}\varepsilon)_{\mathrm{L}}$. This time, however, the expression for $\operatorname{Im}\varepsilon(\omega,k)$ contains an additional term. We denote it by $(\operatorname{Im}\varepsilon)_{\mathrm{D}}$, since this term is associated with the presence of the diffusion dissipative term in the kinetic equation. Let us show that this term restricts the range of wave numbers in which the Landau damping is the main dissipative term.

We fix the value of the phase velocity ω/k, and single out the resonance region in the distribution with respect to velocities:

$$|\omega - kv| \sim Dk^2\,. \quad (9.47)$$

The ratio between two contributions to the imaginary part of permittivity is

$$\frac{(\operatorname{Im}\varepsilon)_{\mathrm{L}}}{(\operatorname{Im}\varepsilon)_{\mathrm{D}}} \sim \frac{\lambda^2}{r_{\mathrm{D}} l} \quad (9.48)$$

and is thus defined by the dimensionless combinations of three characteristic lengths. Let the ratio λ/r_{D} be fixed; then the contribution of the Landau damping only depends on the ratio of λ to l. As λ/l decreases (that is, as we move towards the wave collisionless approximation), the contribution of the Landau damping becomes smaller. The situation is opposite to that encountered in the conventional theory of wave processes in plasma.

In this way, in the case of unconfined plasma the Landau damping is overwhelmed by the much stronger diffusive dissipation.

9.4.3 Confined plasma. Landau damping. Let L be the characteristic size of the system. Assume that the flow of plasma is of free-molecule type. This means that the mean free path is much greater than the size of the system, i.e. $l \gg L$.

The calculation of free-molecule flow must be carried out with due account for the dissipative boundary conditions [5, 14, 33]. This is equivalent — at least on the qualitative level — to introducing the effective collision integral into the kinetic equation. One may assume that the structure of this integral remains the same, but the mean free path is replaced by the characteristic size of the system:

$$l \to L\,, \qquad l \gg L\,. \quad (9.49)$$

Let us show that in the case of free-molecule flow there exists a range of wavelengths (wave numbers) for which the contribution from the Landau damping to Eqn (9.46) dominates. From Eqns (9.46), (9.48) we see that this range is defined by the following inequalities:

$$r_{\mathrm{D}} \ll \lambda \ll \sqrt{r_{\mathrm{D}} L}\,, \qquad r_{\mathrm{D}} \ll L \ll l\,. \quad (9.50)$$

Under these conditions, in place of Eqn (9.46) we get a simpler expression for the imaginary part of permittivity of electron plasma

$$\operatorname{Im}\varepsilon(\omega,k) = -\frac{4\pi^2 e^2 n}{k^2}\int \delta(\omega - kv)k\, \frac{\partial f_0(v)}{\partial p}\, \mathrm{d}p\,. \quad (9.51)$$

In the case of electron plasma this expression coincides with Eqn (5.27), and thus leads to expressions (5.30), (5.31) for the Landau damping coefficient.

We see that the use of the generalized kinetic equation for describing the wave processes in electron plasma allows giving physical interpretation of the concept of 'Landau damping', and specifying the conditions under which this damping dominates.

Similar considerations can be applied to other cases — for example, to nonisothermal plasma.

10. Conclusions

Let us make some final remarks.

In the Introduction we gave a historical summary of the kinetic theory of wave processes. Now we see that this history is still in the making. The use of the generalized kinetic equations opens vast possibilities for further studies on nonequilibrium processes in plasma under the most diverse conditions. In particular, they can be useful for describing the nonequilibrium processes in the devices for thermonuclear fusion of light elements.

The plasma in such devices is collisionless in the sense that the mean free path for charged particles is much greater than the size of the container. At the same time, the Debye radius is small. This means that, with a good enough accuracy, the plasma can be regarded as a continuous medium. Because the boundary conditions are dissipative, this is a good opportunity for using the dissipative kinetic equations. The appropriate collision integrals will take care of the dissipative processes on the walls within which the plasma is confined.

Our discussion has only been concerned with the fully ionized plasma. One possible and important generalization of the theory would consist in the extension of the basic microscopic model, which will bring us into the domain of the statistical theory of plasma-molecular open systems [22, 31]. The systems of this kind include the fully ionized plasma and the gas of neutral particles as extreme cases.

The unifying model is the model of partially ionized plasma, which is built up of at least four components: electrons, ions, atoms, and electromagnetic field. Naturally, the analysis of such systems must rely heavily on the quantum theory of open systems.

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