REVIEWS OF TOPICAL PROBLEMS

Self-consistent theory of localization within site and wave approaches

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Contents

Abstract. The mutual complementarity of the Anderson site representation and the Edwards wave approach is considered within the framework of the quantum-statistical theory. The former is used for the description of one-particle excitations. Based on the permutational symmetry analysis of the wave function, it is shown that the symmetry of the Anderson Hamiltonian exceeds that of the space of states. Transition to an extended state is represented within the framework of the quasi-averages theory as a phase transition of order $2 + \delta$, where $\delta \to 0$ is an addition caused by the appearance of a logarithm. A study of the collective mode is possible within the framework of the Edwards wave representation. The examination is reduced to determining the charge distribution autocorrelation function which is expressed in terms of higher correlators of current density and generalised force, using the Mori technique. Dependences of the conductivity and polarizability on the level spread width and Fermi energy are determined. The form of the frequency dependence of the conductivity as well as the spatial dispersion pattern are analyzed.

1. Introduction

It is now quite clear that a comprehensive description of the localization of a quantum particle in a random field can be accomplished in the framework of either field or statistical approaches. The former is based on the Habbard–Stratonovich transformation which allows the exponent of the quadratic form to be expressed through the exponent of the linear one by integration over the effective field which is in fact the subject matter of the subsequent discussion. The field approach has recently been reviewed in [1] so that we shall confine ourselves to the statistical theory arising from the

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initial formulation of the problem by Anderson [2] and bearing in mind that field variables represent a real fermion rather than a fictitious field.

From the very beginning, the Anderson model has been developed along two main lines. The first one is the site (locator) representation made by Anderson [2] while the second is the wave (propagator) representation of Edwards [3]. Accordingly, the role of the initial (disordered) state in the site representation is played by the localized one, and the transition actually consists in the formation of the extended state band. Conversely, Edwards' wave representation assumes the extended state to be the initial one, while the rearrangement of the system is associated with localization.

Methodologically, the theory of localization comprises scaling theories [4 – 6], the replica method [7, 8], the supersymmetrical approach [9, 10], diagram techniques [11 – 15], and the mode-coupling technique [16, 17]. It also examines effects of interaction between electrons on the parameters of a system undergoing the Mott transition (see [1, 15]). The majority of these works considered the behaviour of collective excitations in the Edwards representation. Evidently, this can be accounted for by the fact that the determination of major experimental quantities (e.g., conductivity and polarization) is most naturally achieved in the framework of this approach.

Our reasoning is based on the dualism of the Anderson model which allows for its description in terms of both site and wave representations. The site representation is described in Section 2 while Section 3 is devoted to the discussion of the wave representation. The concluding Section 4 compares the two approaches and considers ways and prospects of their unification. To our knowledge, the problem of localization has not till now been considered in this perspective.

Apart from dualism, a possibility to represent the Anderson transition by a nonlinear σ-model [7, 8] suggests the presence of effective interaction in the one-particle problem with the random potential. Hence, the question of its nature is raised which is possible to address in two ways, that is using the locator representation based on the Anderson scheme or Edwards' propagator representation.

In the framework of the former approach, system perturbation is associated with overlapping of wave functions. It turns out (see Section 2) that the Anderson

Hamiltonian is degenerate with respect to permutational symmetry of the system's site wave function [18]. Removing of this degeneracy reduces the one-particle problem to the site analog of the BCS model in the theory of superconductivity. Accordingly, ergodicity breaking in the Anderson transition is described by analogy with a phase transition of the $2 + \delta$ order, where $\delta \to 0$ is the addition caused by site resonance and corresponding to the logarithmic singularity [19]. It is shown in the end of Section 2 that the locator approach does not only allows one-particle singularities to be represented but also the collective mode of particle diffusion in the random potential to be described. A self-consistent consideration of the two-particle Green function $\varphi(\mathbf{k}, z)$ has demonstrated [14] that in the limit of small values of the wave vector $\mathbf{k}$ and collective excitation frequency z, it has a characteristic singularity $\varphi \propto (z + \mathrm{i} D \mathbf{k}^2)^{-1}$ the pole of which is determined by the diffusion coefficient D.

In the framework of the propagator approach, the random potential plays the role of perturbation. A diagrammatic investigation [11 – 15] of the collective mode both yielded the representation of the $\varphi(\mathbf{k}, z)$ function diffusion pole and provided evidence that zero values of the diffusion coefficient are due to fan-shaped diagrams derived from ladder ones by a turn of the hole propagator rather than to the ladder sequence the terms of which cancel each other. However, the approach [11 – 15] describes only the weak coupling ($W \ll W_c$) limit for systems of different dimensionalities. Examination of the entire W interval for a three-dimensional system is possible in the framework of the equivalent mode-coupling technique [16, 17] discussed in Section 3. This method is based on the Zwanzig – Mori technique where the electron density correlator $\varphi(z)$ is expressed in the form of a second order chain fraction whose kernel describes memory effects. The mode-coupling approximation consists in that the memory function $M(z)$ is expressed through an initial correlator $\varphi(z)$ in the form of the linear relationship $M(z) = \lambda \varphi(z)$, where the effective coupling constant $\lambda \propto W^2$ is determined by the energy level spread width W, the linearity of the relation $M \propto \varphi$ being due to the absence of interaction. A self-consistent study of the system [17] indicates that, in the static limit $z \to 0$, frequency dependences $D(z)$, $g(z)$ of the diffusion coefficient and compressibility factor have the form $D(z) = \mathrm{const} \equiv D$, $g(z) = \mathrm{i}(n/m) D/z$ in the extended state and $g(z) = \mathrm{const} \equiv g$, $D(z) = -\mathrm{i}(m/n) qz$ in the localized one (here, m, n are the particle mass and volume concentration, respectively). Static constants D, g near the critical level spread W_c exhibit root-like behaviour.

2. The Anderson site representation

Let us consider a gas of noninteracting fermions at zero temperature. Let us further assume, for certainty, that it is a three-dimensional gas of electrons with charge $e = 1$, number density n, and mass m.

To begin with, there is a qualitative solution to the problem. Let N_0 impurity atoms be placed in sites $\mathbf{r}_l$ of a regular lattice and have the energy levels ε_i distributed as $P_i = P(\varepsilon_i)$, where the function $P(\varepsilon)$ is dome-shaped, with the width W and a value $\bar{\varepsilon} = N_0^{-1} \sum_i \varepsilon_i e$ in the centre of the cupola. On overlapping site wave functions $\varphi_i^l(\mathbf{r}) = \varphi_{\varepsilon_i}(\mathbf{r} - \mathbf{r}_l)$ with the overlap integral $J_{ij}^{lm} = \int \varphi_i^{l*}(\mathbf{r}) \varphi_j^m(\mathbf{r}) \, \mathrm{d}\mathbf{r}$, the spectrum E_v of the system does not come to the initial set ε_i, and the distribution function $g = g(E_v)$ does not coincide

with the distribution $P_i(\varepsilon)$ of isolated levels. In the absence of overlapping ($J_{ij}^{lm} = 0$), the wave function $\Phi_E(\mathbf{r})$ of the one-electron state $E = E_v$ has the form

$$\Phi_E(\mathbf{r}) = \sum_i a_E^i \varphi_i^l(\mathbf{r}) \tag{1}$$

and the localized site is realized in which an electron initially placed at the site l remains there infinitely long. This means that of all the coefficients in (1), there is only one nonzero coefficient $a_E^i = \delta_{E \varepsilon_i}$. Overlapping of the wave functions $\varphi_i^l(\mathbf{r})$ leads to qualitative rearrangement of the system. To clarify the situation, let us consider two neighbour sites $\mathbf{r}_1$ and $\mathbf{r}_2$ with levels ε_1, ε_2 and the overlap integral $J_{12}^{12} \equiv J$. The states of an electron at these sites are defined by the wave functions

$$
\begin{aligned}
\Phi_0 &= u\varphi_1 - v\varphi_2, \\
\Phi_1 &= u\varphi_2 + v\varphi_1, \\
\left. \begin{matrix} u^2 \\ v^2 \end{matrix} \right\} &= \frac{1}{2}\left(1 \pm \frac{\varepsilon_1 - \varepsilon_2}{\epsilon}\right), \\
\epsilon &\equiv \left[(\varepsilon_1 - \varepsilon_2)^2 + J^2(\varepsilon_1 + \varepsilon_2)^2\right]^{1/2},
\end{aligned}
\tag{2}
$$

with the corresponding energies $E_{0,1} = (1/2)\left[(\varepsilon_1 + \varepsilon_2) \mp \epsilon\right]$ (here, notations $u \equiv a_{E_0}^1 = a_{E_1}^2$, $v \equiv -a_{E_0}^2 = a_{E_0}^1$; $\varphi_1 \equiv \varphi_1^l(\mathbf{r})$, $\varphi_2 \equiv \varphi_2^l(\mathbf{r})$, $l = 1, 2$) are introduced, for brevity). At $|(\varepsilon_1 - \varepsilon_2)/J| \gg 1$, the overlapping of site functions is negligible, and the electronic states on the pair are reduced to the states at isolated sites. In the opposite case of $|(\varepsilon_1 - \varepsilon_2)/J| \leqslant 1$, the site representation has no sense, and one can speak only of states on a pair of bound sites (PBS). For an isolated PBS, the low-energy state E_0 is realized with the wave function Φ_0.

This problem shows that the presence of PBS causes a dramatic change in the system. It leads to ε_i splitting into two levels E_v ($v = 0, 1$) and separation of the corresponding constituent $\Phi_v(E, \mathbf{r})$ from the one-electron function (1), whereas in the absence of PBS, only one ε_i value is valid for each site. Clustering of $\mathcal{N}$ PBS is responsible for the increased number of terms in (1) entering the cluster constituent $\Phi_v(E, \mathbf{r})$ from 2 to $2\mathcal{N}$; for $\mathcal{N} \gg 1$, the site states $\varphi_i^l(\mathbf{r})$ undergo rearrangement to the Bloch states $\varphi_{\mathbf{k}}(\mathbf{r}) \sim \exp(\mathrm{i}\mathbf{k}\mathbf{r})$. As a result, the state wave function takes the form

$$\Phi_E(\mathbf{r}) = \sum_c \sum_{\mathbf{k}} b_{vc}^{\mathbf{k}}(E) \varphi_{\mathbf{k}}(\mathbf{r}) + \sum_i' a_E^i \varphi_i^l(\mathbf{r})$$
$$\equiv \Phi_v(E, \mathbf{r}) + \Phi_E'(\mathbf{r}), \tag{3}$$

where summation over c is performed for N_c clusters while the wave vector $\mathbf{k}$ runs through $\mathcal{N}_c$ values corresponding to the number of PBS in a given cluster c; the prime implies that summation over i is carried out only for unbound sites. In accordance with (3), the distribution function for electronic states $g(E) = \int |\Phi_E(\mathbf{r})|^2 \, \mathrm{d}\mathbf{r}$ is written as

$$g(E) = 2S(E) + AP(\varepsilon - \bar{\varepsilon}), \tag{4}$$

where the dependence $2S(E) = \sum_v \int |\Phi_v(E, \mathbf{r})|^2 \, \mathrm{d}\mathbf{r}$ describes the distribution of PBS, and $P(\varepsilon - \bar{\varepsilon})$ — that of isolated sites; coefficient A and the form of dependence $S(E)$ are determined by the relation between the level spread width and the characteristic value $I \equiv J\bar{\varepsilon}$ of the overlap integral measured in energy units. At $W \gg |I|$, when the number of PBS is small and they are as a rule isolated, $A \approx 1$, $S(E) \approx BP(\varepsilon - \bar{\varepsilon})$,

where the coefficient $B \sim N_c/N_0$ determined by the finite number of PBS N_c is infinitesimal in the macroscopical limit $N_0 \to \infty$. Therefore, at $W \gg |I|$, the state distribution $g(\varepsilon) \approx P(\varepsilon - \bar{\varepsilon})$ is largely given by the superposition of the site states (1); accordingly, the ground state energy is determined by the sum of ε_i levels and is approximately equal to $\bar{\varepsilon} N_0$. The number of PBS increases with decreasing level spreading, and they form bound clusters of which the optimal one contains $\mathcal{N}_c$ PBS, has energy E_c, and is contained in the volume N_c times. The formation of PBS clusters results in a difference between distributions $2S(E)$ and $P(\varepsilon - \bar{\varepsilon})$, that is in the precipitation of maxima $S(E) \sim \mathcal{N}/N_0$, $\mathcal{N} = \mathcal{N}_c N_c$ at $E \sim E_c$, with the ground state energy being reduced by $\sim |I| \mathcal{N}$ which is negligible as compared with the macroscopic value $\bar{\varepsilon} N_0$. As W further decreases, the number $\mathcal{N}_c$ of PBS in a cluster and the number of clusters N_c corresponding to energy E_c increase while the total number of PBS in them $\mathcal{N} = \mathcal{N}_c N_c$ is a finite part of the site number N_0: $2\mathcal{N}/N_0 \to \eta^2 \neq 0$ at $N_0 \to \infty$. At so doing the maximum of dependence $S(E)$ becomes higher and narrower and assumes the form $2S(E) \sim \eta^2 \delta(E - E_c)$ at the critical level spreading W_c. The appearance of the δ-shaped peak suggests the onset of the Bose-Einstein PBS condensation and, consequently, the transition into an extended state, the formation of which results in a decrease of the ground state energy by the macroscopic value $\sim |I| \mathcal{N} = |I| \eta^2 N_0/2$.

The PBS introduced above are the composite Bose-particles formed by two fermions. When level spreading is large and the number of PBS is small they may be regarded as structureless bosons acted upon by repulsive forces (indeed, it is disadvantageous for PBS in the localized state to bind in an infinite cluster which can be interpreted as their effective repulsion). At a finite number of PBS, the behaviour of the system is governed by the optimal cluster (or, in other words, by the most probable Green function used by Anderson [2]). Therefore, in an approach that assumes the localized state to be the initial (disordered) one must use the site Anderson Hamiltonian [2]. It is exactly this version that is presented below.

If the extended state is assumed to be the initial one, it is advantageous for PBS to form an infinite cluster, because of attractive forces between them. However, this does not lead to instability of the system's ground state since PBS density is so high that their structure is exposed and, in agreement with the Pauli principle for fermions whereof PBS consist, the reciprocal repulsion equilibrating 'Bose attraction' between them becomes apparent† Formally, this can be taken into account [7] by substituting anticommuting fields for commuting ones. For this purpose, Edwards' approach [3] based on the wave representation is normally used.

According to [2, 12], the description of the Anderson transition requires summation of infinite series of diverging terms which arise from ergodicity breaking. In the framework of the Edwards approach, this difficulty will be resolved (see Section 3) by the initial alteration of the pole structure of the

relaxation function followed by a procedure of self-consisting based on the mode-coupling technique [16]. This provides, from the very beginning, the possibility to rearrange the system's phase space. There was a similar situation in the theory of superconductivity where the restructuring of the ground state is possible due to the presence of Cooper pairs [20]. This implies that the Anderson Hamiltonian which acts in the space of site wave functions must also be replaced by a certain effective Hamiltonian defined on functions corresponding to the stable ground state. Such a Hamiltonian was first proposed in [18], and the consistent solution was obtained in [19].

The above qualitative picture of the Anderson transition shows that delocalization of the electron state E is associated with the partition (3) of the corresponding wave function $\Phi_E(\mathbf{r})$ into constituents $\Phi_v(E, \mathbf{r})$, $\Phi_E(\mathbf{r})$ of bound and unbound sites, respectively. The former constituent represents the superposition of the quasi-Bloch states $\varphi_{\mathbf{k}}(\mathbf{r})$ and the latter — that of the site states $\varphi_i^l(\mathbf{r})$. At a given number of PBS $\mathcal{N}$, the fraction of the quasi-Bloch states has the order $2\mathcal{N}/N_0 = \eta^2$ while that of the site states, the order $1 - 2\mathcal{N}/N_0 = 1 - \eta^2$. A characteristic feature of the constituent $\Phi'_E(\mathbf{r}) = \sum_i' a_E^l \varphi_i^l(\mathbf{r})$ is the presence of the free (non-summable) index l on the state functions $\varphi_i^l(\mathbf{r})$. From the physical point of view, this means that a set of ε_i levels can be ambiguously distributed over sites $\mathbf{r}_l$, which accounts for the symmetry of an unbound site system relative to the G' group consisting of $N'!$, $N' = N_0 - 2\mathcal{N} = N_0(1 - \eta^2)$ permutations of sites $\mathbf{r}_l$ over levels ε_i. A set of $\Phi'_E(\mathbf{r})$ functions corresponding to different arrangements of $\mathbf{r}_l$ over ε_i provides the basis for group G'. In the absence of overlapping, the group G' is reduced to the complete group G of $N_0!$ permutations while the configuration space of the system is determined by a set of $N_0!$ functions (1). Binding of $2\mathcal{N} = \eta^2 N_0$ sites in a cluster leads to the reduction of the complete group G to its subgroup G' the power of which is given by the parameter η: at $\eta = 0$ in the macroscopic approximation (i.e. up to terms $N_0^{-1} \to 0$), groups G' and G coincide, while at $\eta = 1$ the group G' is reduced to an unit element. Parameter η which takes values of $\eta = 0$ and $0 < \eta \leqslant 1$ in localized and extended states, respectively, corresponds to the usual definition of the order parameter in the theory of phase transitions. The localized state characterized by the complete group G is highly symmetrical and corresponds to a disordered phase ($\eta = 0$), while the low-symmetry extended state given by the subgroup G' corresponds to an ordered phase ($\eta \neq 0$).

Thus, overlapping of site wave functions leads to the decomposition of the entire configuration space $\{\Phi\}$ comprised by $N_0!$ permutations of sites l in (1) into subspaces $\{\Phi_v\}$, $\{\Phi'\}$, which correspond to bound and unbound sites, respectively. Characteristically, in the former subspace, a system may be in states $v = 0, 1$ though the state Φ_0 corresponding to the lowest energy E_0 is more likely. The one-electron Hamiltonian of the problem

$$H = \sum_i \varepsilon_i \psi_l^{+i} \psi_m^i + \sum_{l \neq m} I_{ij}^{lm} \psi_l^{+i} \psi_m^j \equiv H_0 + V \qquad (5)$$

is symmetrical with respect to permutations of the complete group‡ In fact, this is familiar situation in the theory of phase

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† Of course, PBS in extended states should not be unerstood literally, i.e. as certain quasi-molecules composed of two resonance sites. Indeed, the overlap integral for the sites contained in a PBS is of the same order as that for neighbour sites belonging to different PBS whereas in a real molecule the binding energy of its constituents greatly exceeds the energy of interaction between molecules. It is therefore necessary to represent the entire set of bound sites in which they continuosly undergo binding in different combinations to give rise to PBS complexes playing the role of quasi-particles (obeying Bose statistics).

‡ Interestingly, in a one-level term H_0, sites l, m are subject to permutations over fixed levels i, while in the interstice term V, permutations of levels i, j are over sites l, m.

transitions where the symmetry of the configuration space of a system with given boundary conditions is lower than the symmetry of the Hamiltonian. In order to remove such degeneration, one should take into account that the equilibrium state of a system is realized on the subspace $\{\Phi_0\} \otimes \{\Phi'\}$ rather than on the entire space $\{\Phi\}$. Therefore, the effective Hamiltonian defined on the appropriate basis needs to be resolved from (5). In the two-level approximation implying the presence of two possible states Φ_v, $v = 0, 1$, this is accomplished by the introduction of projection operators

$$|\Phi_0\rangle = P|\Phi\rangle, \qquad |\Phi_1\rangle = Q|\Phi\rangle, \qquad P + Q = 1. \qquad (6)$$

Assuming $|I_{ij}^{lm}| \ll W$, the operator V in (5) may be regarded as a perturbation† and then the effective Hamiltonian is expressed in the form of the following series [21]

$$H_{\mathrm{eff}} = P H_0 P + \sum_{n=0}^{\infty} P V \left(\frac{QV}{\mathcal{E}_0 - H_0}\right)^n P, \qquad (7)$$

where $\mathcal{E}_0$ is the energy of the system in the state Φ_0. Hence, taking into account only the overlap of the z nearest neighbours, the following relationship is true [18]

$$H' = \sum_l (\varepsilon_l - \lambda) a_l^+ a_l - \frac{V}{N_0} \sum_{lm} a_l^+ b_{\bar{l}}^+ b_{\bar{m}} a_m \qquad (8)$$

for the Hamiltonian $H' = H_{\mathrm{eff}} - \lambda N_0$ written with regard for the shift $E = \lambda - \bar{\varepsilon}$ from the band centre, with an accuracy up to $(zI/W)^2 \ll 1$. Here, the overlap integral $I \equiv I_{ij}^{lm}$ is assumed to be independent of l, m, i, j; λ is the chemical potential of localized particles (not to be confused with the Fermi energy μ); $V = N_0 (zI)^2/(\mathcal{E}_1 - \mathcal{E}_0)$ is the effective coupling constant; $\mathcal{E}_1$ is the system energy in the state Φ_1; operators a_l^+, a_l of creation and annihilation of particles in the state Φ_0, and operators b_l^+, b_l of the corresponding antiparticles in the state Φ_1 are defined by the equalities

$$a_l^+ = \sum_i P_i \psi_l^{i+}, \qquad a_l = \sum_i \psi_l^i P_i;$$

$$b_l^+ = \sum_i \psi_l^i Q_i, \qquad b_l = \sum_i Q_i \psi_l^{i+}, \qquad (9)$$

where P_i, Q_i are the projection operators at the level i appearing in the form of products in the complete operators P, Q; operators at the 'site' $\bar{l}$ adjacent to the given one l have the form $b_{\bar{l}} = z^{-1} \sum_m b_{l+m}$, where summation is performed over the nearest neighbours.

Therefore, spontaneous symmetry violation with respect to permutations of ε_i levels over sites $\mathbf{r}_i$, which consists in the transition from group G in the localized state to its subgroup G' in the extended one, leads to the problem of an effective system of interacting particles and antiparticles of the Fermi type‡. The nature of this interaction does not depend on the sign of the overlap integral and is given only by the sign of the difference $\Delta\mathcal{E} \equiv \mathcal{E}_1 - \mathcal{E}_0 > 0$ between the system energies in states Φ_1 and Φ_0. Notice that the appearance of the effective

† The mutual complementarity of the Anderson and Edwards representations is apparent from the fact that the role of perturbation in [17] is played by a potential energy of carriers and in (5) — by the hopping energy.

‡ In principle, such a situation is not new: for example, spontaneous symmetry violation in the Higgs model [22] gives rise to vector field mass. In our case, lowered permutational symmetry leads to exchange interaction.

interaction in systems which originally contain only overlapping wave functions is intrinsic not only in the Anderson model but also in the spin glass model. Analysis of the latter model [23] is possible by the replica method in which an effective interaction results from averaging over the random overlap-integral distribution (with the value of interaction being proportional to the square of its dispersion). It is worthwhile to emphasize that in the replica method, an effective interaction is the result of a mathematical trick and has no concrete physical sense. In the approach being considered, such an interaction shows exchange character and results from permutational symmetry violation in the system.

Turning to the formalism based on the effective Hamiltonian (8), it is worth noting that it coincides with the model BCS Hamiltonian in terms of form, with the sole exception that the site representation is used instead of the momentum one and the part of electrons with opposite momenta and spins is played by particles and antiparticles at the 'neighbouring' sites. Similar to [24], it is possible to demonstrate that at $N_0 \to \infty$, the behaviour of the system is asymptotically defined by the approximating Hamiltonian which takes, in the self-consistent field representation, the form

$$\mathcal{H} = \sum_l (\varepsilon_l - \lambda) a_l^+ a_l - \frac{V}{2} \sum_l \left(\eta^* b_{\bar{l}} a_l + \eta a_l^+ b_{\bar{l}}^+\right) + \frac{V}{4} |\eta|^2 N_0. \qquad (10)$$

Anomalous quasi-averages

$$\eta = \frac{2}{N_0} \sum_l \langle b_{\bar{l}} a_l \rangle \equiv \frac{2}{N_0} \sum_{il} \langle Q_i \psi_{\bar{l}}^{i+} \psi_l^i P_i \rangle,$$

$$\eta^* = \frac{2}{N_0} \sum_l \langle a_l^+ b_{\bar{l}}^+ \rangle \equiv \frac{2}{N_0} \sum_{il} \langle P_i \psi_l^{i+} \psi_{\bar{l}}^i Q_i \rangle, \qquad (11)$$

that define the amplitude of particle transition from state Φ_0 at the site l to state Φ_1 at the 'neighbouring' site $\bar{l}$ (and vice versa for η^*) represent the order parameter corresponding to the extended state. Up to the factor $(N_0/2)^{1/2}$, the values of η^*, η coincide with the condensate part of the PBS operators $\Psi_l^+ = a_l^+ b_{\bar{l}}^+$, $\Psi_l = b_{\bar{l}} a_l$, which define the collective excitation mode at a zero quasi-momentum:

$$\langle \Psi(\mathbf{k} = 0) \rangle = \left(\frac{2}{N_0}\right)^{1/2} \sum_l \langle b_{\bar{l}} a_l \rangle,$$

$$\langle \Psi^+(\mathbf{k} = 0) \rangle = \left(\frac{2}{N_0}\right)^{1/2} \sum_l \langle a_l^+ b_{\bar{l}}^+ \rangle. \qquad (12)$$

Hence, the Bose-condensation of $\mathcal{N}$ PBS implying $\langle \Psi(\mathbf{k} = 0) \rangle = \mathcal{N}^{1/2}$ leads to the ordering $\eta \equiv (N_0/2)^{-1/2} \langle \Psi(\mathbf{k} = 0) \rangle = (2\mathcal{N}/N_0)^{1/2}$. The PBS distribution function over quasi-momenta

$$\frac{N_0}{2} S(\mathbf{k}) \equiv \langle \Psi^+(\mathbf{k}) \Psi(\mathbf{k}) \rangle \approx \left|\langle \Psi(\mathbf{k}) \rangle\right|^2 \qquad (13)$$

acquires a δ-shaped peak $S(0) = 2\mathcal{N}/N_0 \equiv \eta^2$ in height. Characteristically, the given singularity specifically arises in a two-particle, but not one-particle, Green function associated with the PBS distribution. This agrees with the results of the scaling theory [25] according to which the critical behaviour near the mobility threshold E_c is exhibited by tensor $Q_{\alpha\beta}^{ik} = \varphi_\alpha^i \varphi_\beta^k$, which is due to spontaneous symmetry violation (transition from the group $O(2n)$ to the non-

compact group $O(n, n)$) rather than by field variables φ_α^i ($i = 1, 2$; $\alpha = 1, \ldots, n$, $n \to 0$ is the replica space size) themselves. Corresponding to this tensor is the definition of the order parameter (11) in the form of the average over a pair of operators a_l, b_l.

Diagonalization of the Hamiltonian (10) is accomplished by the transformation

$$\alpha_{l+} = u_l a_l - v_l b_{\bar{l}}^+ , \qquad \alpha_{l-} = u_l b_l + v_l a_{\bar{l}}^+ , \qquad (14)$$

where $u_l^2 + v_l^2 = 1$. This yields

$$\mathcal{H} = U + \frac{1}{2} \sum_l \epsilon_l \left(\alpha_{l+}^+ \alpha_{l+} + \alpha_{\bar{l}-}^+ \alpha_{\bar{l}-} \right)$$
$$+ \frac{1}{2} \sum_l (\varepsilon_l - \lambda)(\alpha_{l+}^+ \alpha_{l+} - \alpha_{\bar{l}-}^+ \alpha_{\bar{l}-}), \qquad (15)$$

where

$$U = \frac{N_0}{2} \left\{ \frac{V}{2} |\eta|^2 - \frac{1}{N_0} \sum_l [\epsilon_l - (\varepsilon_l - \lambda)] \right\};$$
$$\epsilon_l = \sqrt{(\varepsilon_l - \lambda)^2 + \Delta^2}, \qquad \Delta = V\eta, \qquad \eta = \eta^*;$$
$$\left. \begin{array}{c} u_l^2 \\ v_l^2 \end{array} \right\} = \frac{1}{2} \left(1 \pm \frac{\varepsilon_l - \lambda}{\epsilon_l} \right). \qquad (16)$$

The quantity U represents the energy of the ground state whose wave function is

$$|\Psi_0\rangle = \prod_l (u_l + v_l a_l^+ b_{\bar{l}}^+) |0\rangle, \qquad (17)$$

where $|0\rangle$ is the wave function of Fermi vacuum. When excitations defined by operators α_{l+}, $\alpha_{\bar{l}-}$ do not coincide, they are described by the gapless dispersion laws $\epsilon_l^{(\pm)} = (1/2)[\epsilon_l \pm (\varepsilon_l - \lambda)]$, and the phase transition is impossible. In order to realize it in the framework of the developed scheme, it is necessary to postulate coincident behaviour of elementary excitations in states Φ_0 and Φ_1 (physically, this means that excitation at site l in the state Φ_0 leads to excitation at the 'neighbouring' site $\bar{l}$ in the state Φ_1). Then, energy ϵ_l of elementary excitations defined by the operator $\alpha_l \equiv \alpha_{l+} = \alpha_{\bar{l}-}$ is characterized by the gap width Δ proportional to the order parameter η.

To further develop the theory, it is necessary to specify the averaging procedure $\langle f(\varepsilon_l) \rangle \equiv \sum w_l f(\varepsilon_l)$ by setting probability w_l to have the energy ε_l at the site $\mathbf{r}_l$. In other words, a statistical theory for the ε_l level distribution is needed. It can be built up by analogy with the corresponding theory of thermodynamic systems [26] with the temperature $W/2$.

The Anderson model [2] assumes that levels N_0 of impurity atoms are equidistantly distributed over the W interval so that the distance between them is $D = W/N_0 = $ const. However, dependence $D(N_0)$ ensuing from general statistical considerations (see paragraph 7 in Ref. [26]) must have the form

$$D = W \exp(-sN_0), \qquad (18)$$

where s is the entropy per impurity atom. For $D(N_0)$ to have a realistic form, the equidistance condition must be cancelled. Let us characterize nonuniformity of the level distribution by

constant $W = 2(\mathrm{d} \ln D(\varepsilon) / \mathrm{d}\varepsilon)^{-1}$. Then, level density $P(\varepsilon) \propto 1/D(\varepsilon) \propto \exp(-2\varepsilon/W)$ falls down with increasing ε at $W > 0$ and grows at $W < 0$. Taking the most natural and simple form of the symmetric distribution

$$P(\varepsilon) = \frac{1}{W} \exp\left(-\frac{2|\varepsilon - \bar{\varepsilon}|}{W} \right), \qquad (19)$$

where $W \geqslant 0$ plays the role of the level spread width, one has $P(\varepsilon) = \delta(\varepsilon - \bar{\varepsilon})$ at $W = 0$.

Dependence (18) may be used to find the probability $w_l \propto \exp S(\varepsilon_l)$ to have the energy ε_l ($S = sN_0$ is the total entropy):

$$w_l = \mathrm{const} \cdot \exp\left(-\frac{2\varepsilon_l}{W} \right). \qquad (20)$$

Formally, statistics given by the distribution function (20) coincides with the statistics of a thermodynamic system with the temperature W. Bearing in mind this coincidence, it is possible to develop a statistical theory for the random systems considered by analogy with the corresponding theory of thermodynamic systems, where the dependence (20) plays the role of the Gibbs distribution. Specifically, for the distribution function $v_l = \langle \alpha_l^+ \alpha_l \rangle$ of elementary excitations with the energy $\epsilon_l = [(\varepsilon_l - \lambda)^2 + \Delta^2]^{1/2}$, the following relation can be obtained in the usual way [26]:

$$v_l = \left[1 + \exp \frac{2\epsilon_l}{W} \right]^{-1}. \qquad (21)$$

Equality $W = 0$ corresponds to the ground state where $v_l = 0$. In the opposite case of $W = \infty$, elementary excitations occupy their bands by half ($v_l = 1/2$) and coincide with particles and antiparticles given by operators a_l, $b_{\bar{l}}$. Finally, at small values of $W(\exp(2\Delta/W) \gg 1)$, equality (21) takes the form of the Boltzmann distribution. The distribution functions $n_l = \langle a_l^+ a_l \rangle$, $p_l = \langle b_l^+ b_l \rangle$ of interacting particles and antiparticles are related to v_l by the formula

$$2n_l = 2p_{\bar{l}} = 1 - \frac{\varepsilon_l - \lambda}{\epsilon} (1 - 2v_l) \qquad (22)$$

arising from (14). As expected, the identity

$$\sum_i \langle \psi_l^{i+} \psi_m^i \rangle = \langle a_l^+ a_m \rangle + \langle b_l b_m^+ \rangle + \langle b_l a_m \rangle + \langle a_l^+ b_m^+ \rangle \qquad (23)$$

following from the relation $P_i + Q_i = 1$ yields $N_l \equiv \sum_i \langle \psi_l^{i+} \psi_l^i \rangle = 1$ for the number of levels at a given site. Transition from site functions thus found to corresponding energy distributions is accomplished with the help of multiplication by the level density $P(\varepsilon)$ given by equality (19).

The gap width equation can be obtained by substituting inverse equalities (14) into definition (11):

$$\frac{V}{W} \int_{-\infty}^{\infty} \frac{\tanh\left(\sqrt{(x - e)^2 + d^2}/2 \right) \exp(-|x|)}{\sqrt{(x - e)^2 + d^2}} \, \mathrm{d}x = 1. \qquad (24)$$

Here, the transition $\sum_l \ldots \to N_0 \int \ldots P(\varepsilon) \, \mathrm{d}\varepsilon$ was used, $d \equiv 2\Delta/W$, $e \equiv 2E/W$, $E \equiv \lambda - \bar{\varepsilon}$ being the shift from the band centre. Unlike the situation in the BCS model, the interaction parameter $V = N\Delta_0^2/\Delta\mathcal{E}$, where $\Delta_0 = z|I|$, is not constant, and its value is determined by the energy difference $\Delta\mathcal{E}$ between states Φ_1 and Φ_0 of the system. For an isolated PBS, it equals ϵ_l while in the presence of the macroscopic delocalized phase $\Delta\mathcal{E} = \sum_l \epsilon_l$. Hence, for V it follows

$$V^{-1} = \frac{W}{(2\Delta_0)^2} \int_{-\infty}^{\infty} \sqrt{(x-e)^2 + d^2}\, \exp(-|x|)\, dx. \quad (25)$$

During transition to the extended state, the total energy jump of a system $\Delta F = F_d - F_l$ (an analog of the thermodynamic potential) is described by the formula

$$\Delta F = -\frac{N_0}{4} \int_0^V \frac{\Delta^2}{V^2}\, dV, \quad (26)$$

which follows from equalities $(dF/dV)_{W,\lambda} = \langle \partial\mathcal{H}/\partial V \rangle = \partial U/\partial V$. In compliance with (26), delocalization is associated with a decrease in F, as is usual in a transition to the low-symmetry phase.

The elementary excitation energy $\mathcal{E}_{\mathrm{ex}} = \sum_l \epsilon_l v_l$ has the form

$$\mathcal{E}_{\mathrm{ex}} = \frac{N_0}{4} \int_{-\infty}^{\infty} \frac{\sqrt{(x-e)^2 + d^2}}{1 + \exp\sqrt{(x-e)^2 + d^2}}\, \exp(-|x|)\, dx. \quad (27)$$

The ground state of a system defined by wave function (17) is realized at $W = 0$, when all the levels coincide ($P(\varepsilon) = \delta(\varepsilon - \bar\varepsilon)$), and elementary excitations are absent ($v_l = 0$). At the band centre ($E = 0$), the order parameter and the gap width take maximal values $\eta = 1$, $\Delta = 1$ while the ground state energy and the total energy jump of the system are minimal: $U = \Delta F = -N_0/4$; the interaction parameter is $V = 1$ (here and henceforth, quantities having energy dimension are given in units of $\Delta_0 = z|I|$, if not stated otherwise). The states Φ_0 and Φ_1 are half occupied by particles and antiparticles ($n_l = p_{\bar{l}} = 1/2$). Following a shift from the band centre by E, $\Delta^2 = \eta^2 = 1 - E^2$, $V = 1$, $\Delta F = -(N_0/4) \times (1 - E^2)$, and $n_l = p_{\bar{l}} = (1 + E)/2$. Therefore, with growing E, the order parameter, the gap width, and the absolute value of the change in the total energy of the system decrease monotonically on delocalization and vanish at the boundary value of $E_c = 1$. Accordingly, the number of particles and antiparticles increases to the maximum value $n_l = p_{\bar{l}} = 1$.

It is easy to see that the ground state of a random system as described above corresponds to the usual spread of coincident energy levels $\varepsilon_l = \bar\varepsilon$ into the extended state band as wide as $2E_c = 2z|I|$ (in conventional units). The density of extended states normalized to unity is proportional to the order parameter:

$$g(E) = \frac{2}{\pi}\eta(E) = \frac{2}{\pi}\sqrt{1 - E^2}. \quad (28)$$

Level spreading is accompanied by elementary excitations the number of which is $v_l \sim \exp(-2/W) \ll 1$. Up to the first-order nonvanishing terms in $W \ll 1$, one has

$$\Delta^2 \approx \eta^2 \approx (1 - E^2) - \frac{W^2}{2}, \quad V \approx 1,$$

$$\Delta F \approx -\frac{N_0}{4}\left[(1 - E^2) - \frac{W^2}{2}\right],$$

$$\mathcal{E}_{\mathrm{ex}} \approx N_0 \exp\left(-\frac{2}{W}\right), \quad E_c \approx 1 + \frac{W^2}{4};$$

$$2n_l = 2p_{\bar{l}}$$
$$\approx \begin{cases} (1 + E) - (1 - E^2)(\varepsilon_l - \bar\varepsilon) & \text{at } |\varepsilon_l - \bar\varepsilon| \lesssim \dfrac{W}{2}; \\ 1 + E & \text{at } |\varepsilon_l - \bar\varepsilon| \gg W. \end{cases} \quad (29)$$

The power-like dependence of Δ, η, $|\Delta F|$ on the level spread width W (unlike the exponential one normally observed in phase transitions) is due to the dependence of the interaction parameter V on the excitation spectrum. It follows from (29) that, at small spreading, this dependence is apparent only with growing distance from the band centre ($E \neq 0$). For small shifts ($E < E_{\mathrm{cr}}$), level spreading leads to a decrease in Δ, η, $|\Delta F|$, the difference of the interaction parameter $V(W)$ from unity being responsible for the difference in the boundary values of E_{cr} for the dependences $\eta(W)$ ($E_{\mathrm{cr}} = (5^{1/2} - 1)/2$) and $\Delta(W)$, $\Delta F(W)(E_{\mathrm{cr}} = 2^{-1/2})$. At $E_{\mathrm{cr}} < E < 1$, small spread width results in a square-law increase of Δ, η, $|\Delta F|$ quantities, which becomes linear at the point where $E = 1$. For $E > 1$, η, Δ are nonvanishing only at $W > W_0$, where η, $\Delta \propto \sqrt{W - W_0}$ (for $\eta(W)$, $W_0^2 = 2(1 - E^{-2}) \ll 1$ while for $\Delta(W)$, $W_0^2 = 2(E^2 - 1) \ll 1$); this yields a region with the positive derivative in the dependence $E_c(W)$ (Fig. 1).
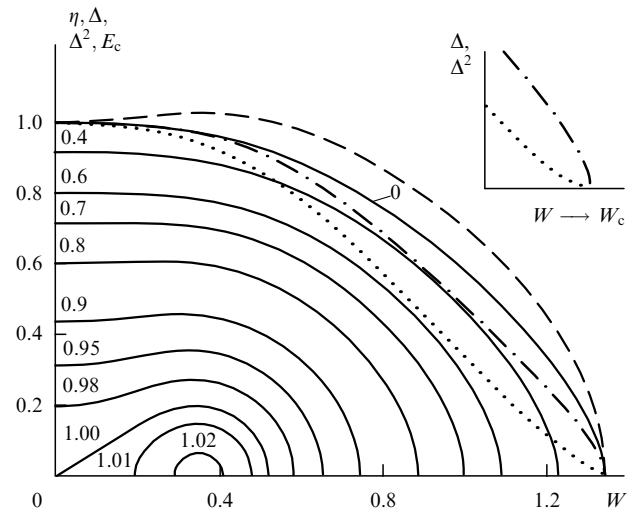
Figure 1. Gap width Δ in the single excitation spectrum (solid lines; figures indicate shifts E from the band centre), maximum shift E_c from the band centre in the extended state (dashed line), order parameter η at $E = 0$ (dot-and-dash line), and squared gap width Δ^2 at $E = 0$ (dotted line) as functions of the level spread width W.

In accordance with (27), the excitation energy $\mathcal{E}_{\mathrm{ex}}$ and the quantity $C_{\mathrm{ex}} = d\mathcal{E}_{\mathrm{ex}}/dW$, which governs its rise with increasing level spreading (an analog of heat capacity in thermodynamics) have the forms

$$\mathcal{E}_{\mathrm{ex}} \approx N_0(1 - E^2)^{-1}\exp\left(-\frac{2}{W}\right),$$

$$C_{\mathrm{ex}} \approx 2N_0(1 - E^2)^{-1}W^{-2}\exp\left(-\frac{2}{W}\right);$$

$$W \ll 1, \quad E < 1. \quad (30)$$

A shift from the band centre leads to a power-like growth of $\mathcal{E}_{\mathrm{ex}}$, C_{ex}, whose exponential smallness is due to the presence of gap.

At arbitrary level spread width W and the shift from the band centre E, both the gap width Δ and the interaction parameter V (hence, the order parameter $\eta = \Delta/V$) may be only found from the numerical solution of Eqns (24), (25). The forms of the resulting dependences are shown in Figs 1, 2. Their characteristic features are the vanishing Δ, η and the

cusp of the dependence $V(W)$ at the critical spread width W_c, the maximum value of which (in conventional units) equals

$$W_{c0} = 2\sqrt{\ln\frac{\pi}{2}}\, z|I| = 1.344 z|I| \qquad (31)$$

and is attained at $E = 0$. The value of $W_{c0}/z|I|$ thus obtained is rather close to 3.3 found in the framework of the Edwards approach [13] and also to 2.4 and 1.8 obtained in the coherent potential [2] and locator [14] approximations, respectively. Unlike temperature dependence of the critical field normally observed in phase transitions, the dependence $E_c(W)$ of the extended state band width on level spreading in the present case (the dashed line in Fig. 1) does not decrease monotonically; rather, the spreading slightly widens the band at $W < W_m \approx 0.35$ (for $W^2 \ll 1$, see (29)) and quickly narrows it at $0 \leqslant (W_{c0} - W)/W_{c0} \ll 1$:

$$E_c \approx 2^{-1/2} W_{c0}^2 \left(1 - \frac{W}{W_{c0}}\right)^{1/2} = 1.277 \left(1 - \frac{W}{W_{c0}}\right)^{1/2}. \quad (32)$$

Nonmonotonical nature of the dependence $E_c(W)$ has been first reported in [27].

It follows from (25) that a rise in the interaction parameter is due to a decrease of elementary excitation energy with increasing level spreading (owing to a fall of $d = 2\Delta/W$); reduction of this parameter is associated with a trivial rise of the integral in the right-hand side of (25) which occurs as the shift from the band centre grows (the former factor prevails at $W < W_c$ and the latter, at $W > W_c$). At the band centre ($E = 0$), the level spreading $W < W_c$ leads to a slow growth of the interaction parameter from $V_0 = 1$. In the case of the finite shift $E < 1$, $V(W)$ first decreases with a power greater than two (see (29)) and increases afterwards (Fig. 2). At $1 < E < E_m$, $E_m \approx 1.02$ and the value of $V = E^{-1}$ is constant in the range $0 \leqslant W < W_0$, where $W_0^2 \approx 2(E^2 - 1)$; while there is a cusp at point $W = W_0$ after which V first shows a slight decrease and then rises. In the vicinity of the critical spreading $(0 \leqslant (W_{c0} - W)/W_{c0} \ll 1,\ (E/W_{c0})^2 \ll 1)$ one finds

$$V \approx \frac{2}{W_{c0}} \left[1 - \frac{W_{c0} - W}{W_{c0}} - 2\left(\frac{E}{W_{c0}}\right)^2\right]. \quad (33)$$
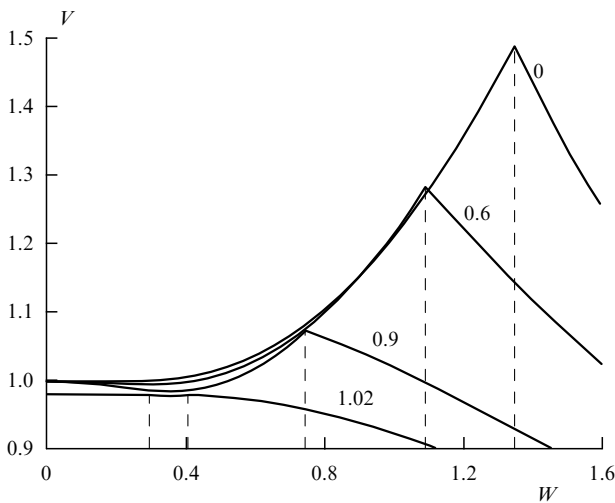
Figure 2. The effective interaction parameter V as a function of level spread width W (figures on the lines indicate shifts E from the band centre).

Above the critical point ($W > W_c$) it follows that

$$V^{-1} = E + \frac{W}{2}\exp\left(-\frac{2E}{W}\right). \qquad (34)$$

A jump of the derivative dV/dW at $W = W_c$ caused by the spectrum rearrangement reaches its maximum $4/W_{c0} = 2.214$ at $E = 0$ but vanishes with growing E.

As is typical of phase transitions, the gap width at the band centre ($E = 0$) decreases monotonically with level spreading W, which plays the role of temperature in our model. However, the initial portion of the dependence $\Delta(W)$ becomes flatter at the shift $E \neq 0$ and, starting from $E_{cr} = 2^{-1/2}$, the gap first widens and then narrows (solid curves in Fig. 1). At greater shifts ($1 < E < E_m$, $E_m \approx 1.02$), the picture is qualitatively different from the usual one: the gap opens at $W_0 \approx 2^{1/2}(E^2 - 1)^{1/2}$ but not due to zero spreading. This difference can be accounted for by the dependence of the interaction parameter on the system's spectrum and is most conspicuous at small W and near the critical point. By excluding parameter V from (24), (25), one has at the band centre, up to the first-order nonvanishing terms in $\Delta/W_{c0} \ll 1$:

$$\left(\frac{W_{c0}}{W}\right)^2 \approx \int_0^\infty \sqrt{x^2 + d^2}\exp(-x)\,dx$$
$$= \frac{\pi}{2}\, d[H_1(d) - Y_1(d)], \qquad (35)$$

where $H_1(d)$ is the Struve function, $Y_1(d)$ is the second-order Bessel function [28]. At $d \ll 1$, $H_1(d) \approx (2/3\pi)d^2$, $Y_1(d) \approx -2/\pi d + (d/\pi)\ln(d/2)$, and it follows from (35) that

$$\left(\frac{\Delta}{W_{c0}}\right)^2 \ln\left(\frac{\Delta}{W}\right) \approx \frac{W_{c0} - W}{W_{c0}}. \qquad (36)$$

Therefore, in the system under consideration, the decrease of Δ near W_c is slower than in an ordinary second-order phase transition without a logarithm in such formula as (36): in the former case, the rate of the $d\Delta^2/dW$ fall is constant, whereas in the latter, $d\Delta^2/dW = W_{c0}/\ln(\Delta/W_{c0})$ and slowly tends to zero at $W \to W_{c0}$. This difference does not affect the graphical pattern of the dependence $\Delta(W)$. For comparison, the dependence $\Delta^2(W)$ corresponding to $E = 0$ is plotted in Fig. 1 (dotted line), which is tangential to the abscissa at $W = W_c$.

The $\eta(W)$ dependence differs from $\Delta(W)$ due to a change in V with level spreading (cf. dot-and-dash and solid curves corresponding to $E = 0$ in Fig. 1). At $W^2 \ll 1$, the order parameter is defined by equality (29); at $E = 0$, in the vicinity of the critical point $(0 < (W_{c0} - W)/W_{c0} \ll 1)$ we arrive at

$$\left(\frac{\eta}{0.903}\right)^2 \ln\left(\frac{\eta}{0.903}\right) \approx \frac{W_{c0} - W}{W_{c0}}. \qquad (37)$$

A change in the total energy $\Delta F = F_d - F_l$ caused by delocalization is minimal $(-(N_0 z/4)|I|$, in conventional units) for a regular system ($W = 0$) at the band centre $E = 0$. The ΔF change being accounted for by the dependence $\Delta(W)$, its value can either increase or decrease with a level spreading and remains negative below the critical point W_c. This reflects the aforementioned instability of the localized state due to the presence of PBS. Near

W_{c0}, one arrives at the expression

$$\Delta F \approx \frac{N}{2}\left(\frac{W_{c0}}{2}\right)^3\left(\frac{\Delta}{W_{c0}}\right)^4 \ln\left(\frac{\Delta}{W_{c0}}\right)$$
$$= 0.228 N_0 \eta^4 \ln(1.107\eta), \qquad (38)$$

which differs from the ordinary one in that it contains a logarithm. This fact is of paramount importance in the sense that the difference between 'spreading capacities' $\Delta C = -W \partial^2 \Delta F/\partial W^2$ vanishes at the critical point according to equality

$$\Delta C \approx -\frac{(N_0/2)(W_{c0}/2)^2}{\ln(\Delta/W_{c0})} = -0.226\frac{N_0}{\ln(1.107\eta)}, \qquad (39)$$

while its derivative with respect to W tends toward $-\infty$. This fact can be conventionally interpreted by relating the transition into the extended state to the $2+\delta$-order, where $\delta \to 0$ is the addition due to logarithm. It arises from the dependence of the effective interaction parameter on the spectrum of the system.

Similar to the transition into a spin glass, the above singularity can be perceived as a cusp of the dependence $C(W)$ at $W = W_c$. However, in spin glasses, a jump of the derivative $\mathrm{d}C/\mathrm{d}T$ at the cusp is finite [23], whereas in our case $\Delta(\mathrm{d}C/\mathrm{d}W) = -\infty$, and in the experiment this cusp may be interpreted as a jump of the quantity itself rather than that of its derivative.

The analytical expression for the total energy of elementary excitations $\mathcal{E}_{ex}$ may be obtained only in the limiting cases. At $W^2 \ll 1$, one has (29), while at $W \geqslant W_{c0}$, $E = 0$, it follows

$$\mathcal{E}_{ex} \approx \frac{\pi^2 - 9}{24}N_0 W = 0.036 N_0 W. \qquad (40)$$

Accordingly, for the 'spreading capacity' $C_{ex} = \mathrm{d}\mathcal{E}_{ex}/\mathrm{d}W$, we get $C_{ex} \approx 0.036 N_0$.

The above discussion has demonstrated that the shift E from the band centre plays the part of a field conjugate to the order parameter, whose nonzero values are realized when E does not exceed the critical value E_c (see the dashed line in Fig. 1). The self-consistent field approximation used here is in line with the mean-field theory, which is known to be essentially reduced to the Landau phenomenological theory. Therefore, the standard phase-transition theory [26] may be used to examine behaviour of physical quantities on delocalization, with the sole difference that the expansion is carried out in $\eta^2 \ln \eta$ rather than in η^2. The comparison with the data obtained allows the microscopical sense of phenomenological parameters to be elucidated.

Thus, the theory of delocalization can be constructed in analogy to the microscopic theory of superconductivity [20], where the role of the order parameter is played by the square root of the ratio of bound to total site numbers and the role of its conjugate field, by a shift from the band centre. A significant difference from the BCS theory is due to the dependence of the effective PBS interaction parameter on the system spectrum. This results in the anomalous behaviour of the order parameter near the critical level spreading which can be expressed by referring the transition into the extended state to the $2+\delta$-order, where $\delta \to 0$ is the addition caused by the presence of the logarithmic multiplier. The total energy of the system is then a nonanalytical function of the order parameter and cannot be expanded in its powers (see (38)).

The major requisites that facilitated the further development of the present theory were taking into consideration the rearrangement of the system ground state and ignoring the tacitly implied constant probability of level occupation. The former allowed the Hamiltonian for the problem to bring to the BCS form, and the latter — to reduce averaging to the procedure of the type used in statistical physics. Certainly, for level distribution density, it is possible to use, instead of (19), any other one-parametric function determining the dome-shaped curve. However, the form (20) of the level occupation probability is essential in the sense that it is (20) that allows for the use of the conventional apparatus of quantum statistics. Our choice of function (19) for the level distribution density which coincides, in terms of the form, with the probability of level occupation (20) can be accounted for by the fact that both these functions are inversely proportional to the distance between levels.

We have used above the simplest method of quasi-averages [24]. Of course, the described picture of one-particle excitations can be obtained in the framework of Green's formalism as well [29]. The advantage of this formalism is in the possibility not only to describe unit excitations but also to represent the behaviour of the collective diffusion mode. Below, there is a brief description of the scheme of such representation using the locator approach [14].

It is known [29] that this scheme is based on the Green function of electron G and the vertex function Γ related to the Green's function of collective excitations φ by the expression $\Gamma = -2\pi\mathrm{i}\varphi$. The former obeys the Dyson equation which has, in the framework of the locator approach, the form $G^{-1} = G_0 - \Delta_0$, where $\Delta_0 = z|I|$, G_0 is the locator at $I = 0$, and the overlap integral I plays a part of perturbation. An essential thing about this approximation is the applicability of the above equation to Green's function $\langle G \rangle$ averaged over level spread and to the average locator $\sigma = \langle G_0 \rangle$: $\langle G \rangle^{-1} = \sigma^{-1} - \Delta_0$. Moreover, an effective interactor $U \equiv \Delta_0 + \Delta_0^2\langle G \rangle$ is introduced which, unlike the previously used potential V, contains the first-order term in the overlap integral I. Also, the Dyson equation $U^{-1} = \Delta_0^{-1} - \sigma$ corresponds to this interactor. The two-particle Green's function [7 – 15]

$$\varphi(E;\mathbf{k},z) = -\frac{1}{2\pi\mathrm{i}}\sum_{\mathbf{k'}\mathbf{k''}}\left\langle G^R(\mathbf{k'_+},\mathbf{k''_+};E+z)\,G^A(\mathbf{k'_-},\mathbf{k''_-};E)\right\rangle$$

$$(41)$$

has, in the ladder approximation, the form [14]

$$\varphi(E;\mathbf{k},z) = -\frac{1}{2\pi\mathrm{i}}\left[\gamma^{-1} - \sum_{\mathbf{k'}}U^R(\mathbf{k'_+},E+z)\,G^A(\mathbf{k'_-},E)\right]^{-1},$$

$$(42)$$

where $\mathbf{k'_\pm} = \mathbf{k'} \pm \mathbf{k}/2$, γ is the irreducible four-tail function, and indices R, A of the retarded and advanced functions correspond to the choice of components $v = 0,1$ from different subspaces of the system's states. This leads (taking into consideration the Dyson equation and the Ward identity [14]

$$\sigma^R_{\mathbf{k_+}}(E+z) - \sigma^A_{\mathbf{k_-}}(E)$$
$$= \gamma\sum_{\mathbf{k'}}\left[U^R(\mathbf{k'_+},E+z) - U^A(\mathbf{k'_-},E)\right] - \tilde{\gamma}z, \qquad (43)$$

where the irreducible vertex $\tilde{\gamma}$ has, unlike γ, two coinciding tails, which correspond to identical sites) to the usual

expression for the two-particle Green's function [7 – 15]

$$\varphi(\mathbf{k}, z) = -\frac{\chi(\mathbf{k})}{z + iD(\mathbf{k}, z)\,\mathbf{k}^2}, \qquad (44)$$

where $\chi(\mathbf{k})$ is the thermodynamic susceptibility equal, in the hydrodynamic limit $\mathbf{k} = 0$, to the state density $g(E)$ at the E level, and $D(\mathbf{k}, z)$ is the dispersing diffusion coefficient which takes, at $\mathbf{k} = 0$, the value of [14]

$$D = \frac{1}{\pi g} \sum_{\mathbf{k}'} \left(\frac{k'}{m} \, \mathrm{Im} \langle G_{\mathbf{k}'}^A \rangle \right)^2, \qquad (45)$$

here k' is the projection of $\mathbf{k}'$ onto $\mathbf{k}$, and m is the particle mass.

3. The Edwards wave representation

In this section, the uniform extended state is assumed to be the initial (disordered) one. Therefore, quantities taken in the wave representation will be considered. A set of such quantities is reducible to the electron density operator

$$\rho_{\mathbf{k}} = \sum_{\mathbf{k}'} \rho_{\mathbf{k}}^{\mathbf{k}'}, \qquad \rho_{\mathbf{k}}^{\mathbf{k}'} \equiv \psi_{\mathbf{k}'-\mathbf{k}/2}^+ \psi_{\mathbf{k}'+\mathbf{k}/2}, \qquad (46)$$

the longitudinal component of the current

$$j_{\mathbf{k}} = \sum_{\mathbf{k}'} j_{\mathbf{k}}^{\mathbf{k}'}, \qquad j_{\mathbf{k}}^{\mathbf{k}'} \equiv \frac{\mathbf{k}\mathbf{k}'}{km} \rho_{\mathbf{k}}^{\mathbf{k}'}, \qquad (47)$$

and the effective force

$$f_{\mathbf{k}} = \sum_{\mathbf{k}'} f_{\mathbf{k}}^{\mathbf{k}'}, \qquad f_{\mathbf{k}}^{\mathbf{k}'} \equiv \sum_{\mathbf{k}''} \varepsilon_{\mathbf{k}-\mathbf{k}''} \left(\rho_{\mathbf{k}''}^{\mathbf{k}'+(\mathbf{k}-\mathbf{k}'')/2} - \rho_{\mathbf{k}''}^{\mathbf{k}'-(\mathbf{k}-\mathbf{k}'')/2} \right), \qquad (48)$$

expressions for which can be obtained from equations of continuity and motion. Here, $\psi_{\mathbf{k}}^+, \psi_{\mathbf{k}}$ are the operators of creation and annihilation for electrons in the state $|\mathbf{k}\rangle$, and $\varepsilon_{\mathbf{k}}$ are the corresponding energy levels (of random nature, in accordance with the Anderson model [2]). The Anderson Hamiltonian written with regard for the condition

$$\sum_{\mathbf{k}} n_{\mathbf{k}} = N_0, \qquad n_{\mathbf{k}} \equiv \langle \psi_{\mathbf{k}}^+ \psi_{\mathbf{k}} \rangle \qquad (49)$$

of conserving the number of electrons N_0 has the form

$$H = \sum_{\mathbf{k}} (I_{\mathbf{k}} - \mu)\, \psi_{\mathbf{k}}^+ \psi_{\mathbf{k}} + \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}}^* \rho_{\mathbf{k}} \equiv H_0 + V, \qquad (50)$$

where $I_{\mathbf{k}}$ is the Fourier transform of the overlap integral for site electronic states, μ is the Lagrange factor taking into account the condition (49). Evidently,

$$I_{\mathbf{k}} = (\mu - \varepsilon_F) + \frac{\mathbf{k}^2}{2m}, \qquad (51)$$

hence, $\mu = \varepsilon_F$, because the overlap integral tends to zero at large distances and, therefore, $\lim_{\mathbf{k} \to 0} I_{\mathbf{k}} = 0$.

Let us follow the Götze's scheme [16,17] based on the Zwanzig – Mori technique [30]. The main object studied in the framework of this scheme is the relaxation function

$$\varphi_{\mathbf{k}}(t) = \langle\langle \rho_{\mathbf{k}}(t) | \rho_{\mathbf{k}} \rangle\rangle \equiv \int_t^{+\infty} \langle [\rho_{\mathbf{k}}^+(t'), \rho_{\mathbf{k}}(0)] \rangle \, dt'. \qquad (52)$$

It is related to the ordinary susceptibility

$$\chi_{\mathbf{k}}(t) = i\theta(t) \langle [\rho_{\mathbf{k}}^+(t), \rho_{\mathbf{k}}(0)] \rangle \equiv -G_{\mathbf{k}}^R(t) \qquad (53)$$

reduced to the retarded Green function $G_{\mathbf{k}}^R(t)$, where $\theta(t) = 1$ at $t > 0$ and $\theta(t) = 0$ at $t < 0$, by the following expression

$$\varphi_{\mathbf{k}}(z) = z^{-1} [\chi_{\mathbf{k}}(z) - \chi_{\mathbf{k}}], \qquad \chi_{\mathbf{k}} \equiv \chi_{\mathbf{k}}(i0). \qquad (54)$$

Here and henceforth, time $\varphi_{\mathbf{k}}(t)$ and frequency $\varphi_{\mathbf{k}}(z)$ dependences are interrelated by the standard Laplace transformation.

Turning to the direct representation of the localization process, we shall first describe, in the zero approximation, the behaviour of free delocalized electrons, for which the random potential $\varepsilon_{\mathbf{k}} = 0$. In the context of the linear response theory, definition (52) leads to [17, 31]

$$\varphi_{\mathbf{k}}^{\mathbf{k}'\mathbf{k}''(0)}(z) \equiv \langle\langle \rho_{\mathbf{k}}^{\mathbf{k}'}(z) | \rho_{\mathbf{k}}^{\mathbf{k}''}(z) \rangle\rangle = -g_{\mathbf{k}}^{\mathbf{k}'\mathbf{k}''(0)} \left(z + \frac{\mathbf{k}\mathbf{k}'}{m} \right)^{-1}, \qquad (55)$$

where the static susceptibility $g_{\mathbf{k}}^{\mathbf{k}'\mathbf{k}''} \equiv \langle \rho_{\mathbf{k}}^{\mathbf{k}'} | \rho_{\mathbf{k}}^{\mathbf{k}''} \rangle = \varphi_{\mathbf{k}}^{\mathbf{k}'\mathbf{k}''}(t = 0)$ is given in the zero approximation by the expression

$$g_{\mathbf{k}}^{\mathbf{k}'\mathbf{k}''(0)} = -\delta_{\mathbf{k}'\mathbf{k}''} \left(\frac{\mathbf{k}\mathbf{k}'}{m} \right)^{-1} \left[n_0 \left(\mathbf{k}' + \frac{\mathbf{k}}{2} \right) - n_0 \left(\mathbf{k}' - \frac{\mathbf{k}}{2} \right) \right], \qquad (56)$$

with $n_0(\mathbf{k}) = \theta(\varepsilon_F - \mathbf{k}^2/2m)$ being the Fermi step. Let us introduce the dimensionless frequency $\zeta = z/\Delta_0$, the wavenumber $\varkappa = k/k_D$, and the Fermi energy $E_F = \varepsilon_F/\Delta_0$ expressed in the units of energy $\Delta_0 \equiv k_D^2/2m$ and the Debye wavenumber k_D. Then, the total compressibility $g_{\mathbf{k}} = \sum_{\mathbf{k}'\mathbf{k}''} g_{\mathbf{k}}^{\mathbf{k}'\mathbf{k}''}$ assumes the canonical form [31]

$$g_{\mathbf{k}}^{(0)} = g_F \chi \left(\frac{\varkappa}{2E_F^{1/2}} \right), \qquad \chi(x) \equiv \frac{1}{2} + \frac{1}{4x}(1 - x^2) \ln \left| \frac{1+x}{1-x} \right|. \qquad (57)$$

Summation of (55) over $\mathbf{k}'$, $\mathbf{k}''$ leads to a (54)-like expression for the zero relaxation function $\varphi_{\mathbf{k}}^{(0)}(z)$, where the role of the dynamic susceptibility $\chi_{\mathbf{k}}(z)$ is played by the generalized compressibility

$$g_{\mathbf{k}}^{(0)}(z) = g_F \chi \left(\frac{\varkappa}{2E_F^{1/2}}, \frac{\zeta}{4E_F} \right), $$

expressed through the Lindhard function [31]

$$\chi(x, y) = \frac{1}{2} + \frac{1}{8x} \left\{ \left[1 - \left(x - \frac{y}{x} \right)^2 \right] \ln \left| \frac{x+1-y/x}{x-1-y/x} \right| + \left[1 - \left(x + \frac{y}{x} \right)^2 \right] \ln \left| \frac{x+1+y/x}{x-1+y/x} \right| \right\}. \qquad (58)$$

Here is the long-wave limit of expression (58) to be used below:

$$\chi(x, y) \approx -\frac{1}{3} \left(\frac{x}{y} \right)^2 \left[1 + \left(\frac{x}{y} \right)^2 \left(\frac{3}{5} + x^2 \right) \right], $$
$$x \equiv \frac{k}{2k_F} \ll 1, \qquad y \equiv \frac{z}{4\varepsilon_F}. \qquad (59)$$

In order to go beyond the framework of the zero approximation, we shall use the projection operator method

of Zwanzig and Mori [30]. According to the definition

$$P = |\rho\rangle \langle \rho | \rho \rangle^{-1} \langle \rho | \equiv 1 - Q, \qquad (60)$$

where arguments $\mathbf{k}$, z are omitted for brevity, the operator P projects all quantities in the space of dynamical variables (46) – (48) onto the charge density 'axis' ρ while the complementary operator Q does the same on the subspace orthogonal to this axis. The equation of motion

$$\dot{\rho}(t) = \mathrm{i}L\rho(t) \qquad (61)$$

written with the help of the Liouville operator L allows the generation of the space of dynamical quantities based on the ρ axis: the current axis $j \propto \dot{\rho} \propto L\rho$ appears as a result of a single action of operator L, the effective force axis $f \propto \ddot{\rho} \propto L^2\rho$ is a result of the double action, etc.

It is easy to see from (61) that the Laplace transform of the correlator (52) has the form

$$\varphi(z) = -\langle \rho | (z - L)^{-1} | \rho \rangle. \qquad (62)$$

Hence, using the operator identity

$$\frac{1}{A+B} = \frac{1}{A} - \frac{1}{B} B \frac{1}{A+B}, \qquad (63)$$

it is easy to obtain relations

$$\varphi(z) = -g\big[z + \Omega + K(z)\big]^{-1}, \quad g \equiv \langle \rho | \rho \rangle, \qquad (64a)$$

$$\Omega = -\omega_0^2 \langle \rho | L | \rho \rangle \equiv \omega_0 \langle \rho | j \rangle, \quad \omega_0^2 g = 1, \qquad (64b)$$

$$K(z) = -\langle j | Q(z - L_\rho)^{-1} Q | j \rangle, \quad j \equiv \mathrm{i}\omega_0 \dot{\rho}, \quad L_\rho \equiv QLQ. \qquad (64c)$$

The former is the initial link in the Mori chain while two others expose the microscopic sense of kernels $\Omega, K(z)$. Making calculi for the latter relation by analogy to those for the initial correlator (62) yields the expression for the second link in the chain

$$K(z) = -\omega_0^2 \big[z + M(z)\big]^{-1}, \qquad (65a)$$

$$M(z) = -\langle f | Q_j(z - L_j)^{-1} Q_j | f \rangle, \quad f = \omega_0^{-1} L_\rho j, \qquad$$

$$L_j = Q_j L_\rho Q_j, \quad Q_j = 1 - |j\rangle \langle j | j \rangle^{-1} \langle j |. \qquad (65b)$$

In contrast to (64a), there is no cross-correlator of the (64b)-type here, and the memory function (65b) is the correlator of the generalized force f. The chain can be lengthened further; in this case, the correlator of the n-th charge density derivatives (46) with respect to time will be expressed in each link through the correlator of corresponding $n + 1$-order derivatives. Such a structure facilitates explicit interpretation of the approximation arising from a break of the Mori chain. If, following Götze [16, 17], only terms written in (64a), (65a) are retained, it is possible to represent memory effects playing a major part in the description of ergodicity breaking in the course of electron localization. Self-consistency of the scheme is ensured by expression of the memory function $M(z)$ through the initial correlator $\varphi(z)$. Moreover, the following boundary conditions must be fulfilled [16]:

$$\lim_{t \to 0} \varphi_{\mathbf{k}}(t) = \lim_{z \to \infty} \big[-z\varphi_{\mathbf{k}}(z)\big] = g_{\mathbf{k}}, \quad \lim_{\mathbf{k} \to 0} g_{\mathbf{k}} = g_{\mathrm{F}}; \qquad (66)$$

$$\lim_{t \to 0} K_{\mathbf{k}}(t) = \lim_{z \to \infty} \big[-zK_{\mathbf{k}}(z)\big] = \chi_{\mathbf{k}}, \quad \lim_{\mathbf{k} \to 0} \chi_{\mathbf{k}} = \frac{n}{m}, \qquad (67)$$

where $g_{\mathbf{k}}$, $\chi_{\mathbf{k}}$ are the compressibility and polarizability of an electron gas, respectively†.

The Götze theory [16, 17] is a variant of the above projection approach. In the construction of (52)-like correlation functions as basic ones, it is convenient to use constituents $\rho_{\mathbf{k}}^{\mathbf{k}'}$ rather than the total density $\rho_{\mathbf{k}}$ (see (46)). Then, $\varphi_{\mathbf{k}}^{\mathbf{k}'\mathbf{k}''}(z) = \langle\langle \rho_{\mathbf{k}}^{\mathbf{k}'}(z) | \rho_{\mathbf{k}}^{\mathbf{k}''}(z) \rangle\rangle$-type correlators acquire a pair of additional indices. Also, it is convenient to present the original equation (64a) in the form

$$\big[z\hat{I}_{\mathbf{k}} + \hat{\Omega}_{\mathbf{k}} + \hat{K}_{\mathbf{k}}(z)\big]\hat{\varphi}_{\mathbf{k}}(z) = -\hat{g}_{\mathbf{k}}. \qquad (68)$$

The top badge indicates a matrix the elements of which are defined by superscripts (e.g., $\varphi_{\mathbf{k}}^{\mathbf{k}'\mathbf{k}''}(z)$ for $\hat{\varphi}_{\mathbf{k}}(z)$), $\hat{I}$ is the unit matrix, $\hat{g}_{\mathbf{k}}$, $\hat{\Omega}_{\mathbf{k}}$ and $\hat{K}_{\mathbf{k}}(z)$ are defined by relations of the (64b), (64c)-type. To solve Eqn (68), its matrices need to be represented in the basis of vectors $\alpha_{\mathbf{k}}^{\mathbf{k}'\alpha}$, $\alpha = 0, 1, ...,$ orthonormalized with respect to metric $g_{\mathbf{k}}^{\mathbf{k}'\mathbf{k}''}$. In this basis, quantities $\rho_{\mathbf{k}} = g_{\mathbf{k}}^{1/2} A_{\mathbf{k}}^0$, $j_{\mathbf{k}} = (n/m)^{1/2} A_{\mathbf{k}}^1$ defined by equalities (46), (47) are expressed through vectors

$$A_{\mathbf{k}}^\alpha = \sum_{\mathbf{k}'} \alpha_{\mathbf{k}}^{\mathbf{k}'\alpha} \rho_{\mathbf{k}}^{\mathbf{k}'}, \quad \alpha = 0, 1, \dots \qquad (69)$$

Accordingly, Eqn (68) takes the form

$$(z\delta^{\alpha\beta} - \omega_{\mathbf{k}}^{\alpha\beta}) \varphi_{\mathbf{k}}^{\beta\gamma}(z) = -\delta^{\alpha\gamma} - m_{\mathbf{k}}^{\alpha\beta}(z) \varphi_{\mathbf{k}}^{\beta\gamma}(z), \qquad (70)$$

where summation is over repeated Greek indices and matrix elements are given by formulas [17]

$$\omega_{\mathbf{k}}^{\alpha\beta} = \sum_{\mathbf{k}'} \alpha_{\mathbf{k}}^{\mathbf{k}'\alpha*} \alpha_{\mathbf{k}}^{\mathbf{k}'\beta} (n_{\mathbf{k}'+\mathbf{k}/2} - n_{\mathbf{k}'-\mathbf{k}/2}), \qquad (71)$$

$$m_{\mathbf{k}}^{\alpha\beta}(z) = \sum_{\mathbf{k}'\mathbf{k}''} \alpha_{\mathbf{k}}^{\mathbf{k}'\alpha*} \alpha_{\mathbf{k}}^{\mathbf{k}''\beta} \langle f_{\mathbf{k}}^{\mathbf{k}'} | (QLQ - z)^{-1} | f_{\mathbf{k}}^{\mathbf{k}''} \rangle. \qquad (72)$$

Characteristically, the memory matrix (72) is built up on the vectors of generalized force

$$f_{\mathbf{k}}^\alpha = \sum_{\mathbf{k}'} \alpha_{\mathbf{k}}^{\mathbf{k}'\alpha} f_{\mathbf{k}}^{\mathbf{k}'}, \qquad (73)$$

whose zero component $f_{\mathbf{k}}^{(0)} = 0$. This follows from the definition (73) provided force components (48) are substituted into it, and from the fact that no force affects an electron in the zero approximation. By virtue of this property, matrix $m_{\mathbf{k}}^{\alpha\beta}(z)$ in the two-component basis (69) has only one nonvanishing component $m_{\mathbf{k}}^{11}(z) \equiv M_{\mathbf{k}}(z)$ of the (65b)-type, which is in fact the correlator of random forces

$$f_{\mathbf{k}} = (mn)^{-1/2} \sum_{\mathbf{k}'} \varepsilon_{\mathbf{k}-\mathbf{k}'} \frac{\mathbf{k}(\mathbf{k} - \mathbf{k}')}{k} \rho_{\mathbf{k}'}. \qquad (74)$$

Assuming fluctuations of quantities $\varepsilon_{\mathbf{k}}$, $\rho_{\mathbf{k}}$ to be independent, the expression for the mass operator [17] is

$$M_{\mathbf{k}}(z) = \frac{1}{mn} \sum_{\mathbf{k}'} \overline{|\varepsilon_{\mathbf{k}-\mathbf{k}'}|^2} \frac{(\mathbf{k} - \mathbf{k}')\mathbf{k}}{k} \varphi_{\mathbf{k}'}(z), \qquad (75)$$

where the bar means averaging over level spreading. Equality (75) represents the self-consistency condition which expresses,

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† Do not confuse polarizability $\chi_{\mathbf{k}}$ representing the static limit of correlator $K_{\mathbf{k}}(z)$ by polarization $P_{\mathbf{k}} = \dot{\rho}_{\mathbf{k}}$ and generalized susceptibility (53).

in terms of the mode-coupling theory [16], the highest correlator $M_{\mathbf{k}}(z)$ through the original one $\varphi_{\mathbf{k}}(z)$. The linear character of the coupling (75) is due to the fact that all basis operators (69) are linear in $\rho_{\mathbf{k}}^{\mathbf{k}'}$.

Bearing in mind the aforementioned property of the matrix $m_{\mathbf{k}}^{\alpha\beta}(z)$, Eqn (70) for the function $\varphi_{\mathbf{k}}(z) = g_{\mathbf{k}}\varphi_{\mathbf{k}}^{00}(z)$ has the solution [17]

$$\varphi_{\mathbf{k}}(z) = \varphi_{\mathbf{k}}^{(0)}\left(z + M_{\mathbf{k}}(z)\right)\left[1 + \frac{\varphi_{\mathbf{k}}^{(0)}\left(z + M_{\mathbf{k}}(z)\right)M_{\mathbf{k}}(z)}{g_{\mathbf{k}}^{(0)}}\right]^{-1},$$
(76)

where the zero function $\varphi_{\mathbf{k}}^{(0)}(z)$ is expressed by the relationship (54) through susceptibilities (57), (58). It is easy to see that (76) can be represented in the form of the first terms (64a), (65a) of the chain fraction. In the long-wave limit, when susceptibility assumes the form (59), equality (76) is reduced to

$$\varphi_{\varkappa}(\zeta) = -g_{\mathrm{F}}\chi\left(\frac{\varkappa}{2E_{\mathrm{F}}^{1/2}}\right)\left[\zeta + \mathrm{i}D_{\varkappa}(\zeta)\,\varkappa^2\right]^{-1}.$$
(77)

Here, a diffusion pole was introduced being expressed in units of $\Delta_0/k_D^2 = (2m)^{-1}$:

$$D_{\varkappa}(\zeta) \approx \frac{4}{3}\,\mathrm{i}\,\frac{E_{\mathrm{F}}}{\chi(\varkappa/2E_{\mathrm{F}}^{1/2})}\left[\zeta + M_{\varkappa}(\zeta)\right]^{-1}$$
$$\times\left\{1 + \left(\frac{2\varkappa E_{\mathrm{F}}^{1/2}}{\zeta} + M_{\varkappa}(\zeta)\right)^2\right.$$
$$\times\left.\left[\frac{3}{5} + \frac{\varkappa^2}{4E_{\mathrm{F}}} - \left(3\chi\left(\frac{\varkappa}{2E_{\mathrm{F}}^{1/2}}\right)\right)^{-1}\right]\right\},$$
(78)

where $M_{\varkappa}(\zeta)$ is expressed in units of Δ_0. The usual diffusion coefficient

$$D = \frac{1}{3}v_{\mathrm{F}}^2\tau, \qquad \tau^{-1} \equiv \Delta_0 \mathrm{Im}\, M_{\varkappa=0}(\zeta = 0),$$
(79)

follows from (78) as the limiting expression $D_{\varkappa=0}(\zeta = 0) \equiv 2mD$. This coefficient is related to static conductivity σ and mobility $\mu \equiv \sigma/n$ by the Einstein relation $\sigma = g_{\mathrm{F}}D$. Hence, the Drude formulas are as follows:

$$\mu = \frac{\tau}{m}, \qquad \sigma = \frac{n\tau}{m}.$$
(80)

Dynamic conductivity defined as the current correlator (47) has the form

$$\sigma(z) = \frac{n}{m}\,D(z), \quad D(z) = -\frac{\mathrm{i}}{z + M(z)},$$
(81)

where $D(z) \equiv -\mathrm{i}\lim_{\mathbf{k}\to 0}\left(K_{\mathbf{k}}(z)/k^2\right), M(z) \equiv M_{\mathbf{k}=0}(z)$. Hence, expressions for dynamic polarizability $\chi(z) = (\mathrm{i}/z)\sigma(z)$, dielectric permeability $\varepsilon(z) = 1 + 4\pi\chi(z)$, and dynamic mobility $\mu(z) = \sigma(z)/n$ could be derived immediately.

In the extended state, the conductivity regime is realized with $\mu \neq 0$. Therefore, one finds here

$$\lim_{z\to 0}\sigma(z) = \sigma \neq 0; \quad \chi(z) = \frac{\mathrm{i}\sigma}{z} \quad \text{at } z \to 0.$$
(82)

Accordingly, in the localized state, there is an insulator with mobility $\mu = 0$ and polarizability $\chi \neq 0$. Here

$$\lim_{z\to 0}\chi(z) = \chi \equiv \delta^{-2}, \qquad \sigma(z) = -\mathrm{i}z\chi \text{ at } z \to 0.$$
(83)

It follows from (82), (83) that the occurrence of conductivity σ under delocalization results in the static pole of generalized polarizability $\chi(z)$ and the localization parameter $\delta = 0$, whereas localization is characterized by finite values of static polarizability σ and parameter δ, along with a linear decrease in dynamic conductivity $\sigma(z) \propto z$. Mathematically, the localized state is convenient to fix basing on the presence of the memory function pole:

$$\delta^2 = -zM(z) \qquad \text{at } z \to 0$$
(84)

rather than the features intrinsic in (83). According to (84), (83), the residue of this pole is a microscopic localization parameter δ^2 inversely proportional to static polarizability χ.

Let us further simplify the model in order to formulate the self-consistent scheme and be able to determine parameter (84). For this purpose, let us neglect the dependence of the memory function on the wave vector: $M_{\mathbf{k}}(z) \approx M_{\mathbf{k}=0}(z) \equiv M(z)$, and then reduce the dependence $\overline{|\varepsilon_{\mathbf{k}}|^2}$ to the stepwise one:

$$\overline{|\varepsilon_{\mathbf{k}}|^2} = W^2\frac{6\pi^2}{k_D^3}\,\theta\left(k_D - k\right).$$
(85)

It ensues from the approximation (85) that random electronic states are uniformly spread over the $\mathbf{k}$ space in a sphere of radius k_D and along the energy axis in the interval of width W. As a result, the self-consistency condition (75) assumes, in dimensionless quantities $\zeta = z/\Delta_0$, $\varkappa = k/k_D$, $E_{\mathrm{F}} = \varepsilon_{\mathrm{F}}/\Delta_0$, and $\lambda = (3/4)(W/\varepsilon_{\mathrm{F}})^2$, the form

$$M(\zeta) = 4\lambda E_{\mathrm{F}}\int_0^1 \varphi_{\varkappa}(\zeta)\varkappa^4\,\mathrm{d}\varkappa,$$
(86)

where $M(\zeta)$ is expressed in units of Δ_0, and $\varphi_{\varkappa}(\zeta)$ — in units of g_{F}/Δ_0.

The system of Eqns (76), (86) yields the self-consistent solution for mobility μ in the range of conductivity and inverse polarizability $\chi^{-1} \equiv \delta^2$ in an insulator. In the former case, it is possible to assume that $M(\zeta) \approx \mathrm{i}v \equiv \mathrm{i}/\tau\Delta_0$; then $\varphi_{\varkappa}^{(0)}(\mathrm{i}v) = \mathrm{i}\varphi_{\varkappa}^{(0)}(v)$, and (86) takes the form

$$v = 4\lambda E_{\mathrm{F}}\int_0^1 F(\varkappa, v)\,\varkappa^4\,\mathrm{d}\varkappa,$$
(87a)

$$F(\varkappa, v) \equiv -\varphi_{\varkappa}^{(0)}(v)\left[1 - \frac{v\varphi_{\varkappa}^{(0)}(v)}{\chi(\varkappa/2E_{\mathrm{F}}^{1/2})}\right]^{-1},$$
(87b)

where function $\chi(x)$ is given by equality (57). In an insulator, where the behaviour in the vicinity of $\zeta = 0$ is important, equality (78) takes the form $D_{\varkappa}(\zeta) \approx -(4/3)\times \mathrm{i}(E_{\mathrm{F}}/\delta^2)\chi^{-1}(\varkappa/2E_{\mathrm{F}}^{1/2})\zeta$, and the relaxation function acquires the pole $\varphi_{\varkappa}(\zeta) = -(\Delta_0/W)\Delta(\varkappa)\zeta^{-1}$ at $\zeta \to 0$, where the macroscopic localization parameter $\Delta(\varkappa)$ and the related microscopic one δ have the form

$$\Delta(\varkappa) = 2\left(\frac{\lambda}{3}\right)^{1/2}E_{\mathrm{F}}\chi\left(\frac{\varkappa}{2E_{\mathrm{F}}^{1/2}}\right)$$
$$\times\left[1 + \frac{4}{3}E_{\mathrm{F}}\left(\frac{\varkappa}{\delta}\right)^2\chi^{-1}\left(\frac{\varkappa}{2E_{\mathrm{F}}^{1/2}}\right)\right]^{-1},$$
(88a)

$$\delta^2 = 2(3\lambda)^{1/2}\int_0^1 \Delta(\varkappa)\,\varkappa^4\,\mathrm{d}\varkappa.$$
(88b)

Pairs of equalities (87), (88) represent transcendental equations for determining mobility $\mu = (m\Delta_0 v)^{-1}$ of charge carriers in a conductor and static polarizability $\chi = \delta^{-2}$ of a dielectric.

At arbitrary values of parameters λ, E_F, Eqns (87), (88) have only numerical solutions. The resultant dependences $\mu(W)$, $\delta(W)$ at different E_F are shown in Figs 3, 4. The analytical form is achieved in a limiting case of small μ and δ values, which are realized in the intermediate region in both the weak ($\lambda \ll 1$) and strong ($\lambda \gg 1$) coupling regimes. Thus, near the critical value W_c, at which $\mu = \delta = 0$, the relationships are true up to μ^2, δ^4 inclusive:

$$\mu^2 B(\mu) \approx 1 - A, \tag{89}$$

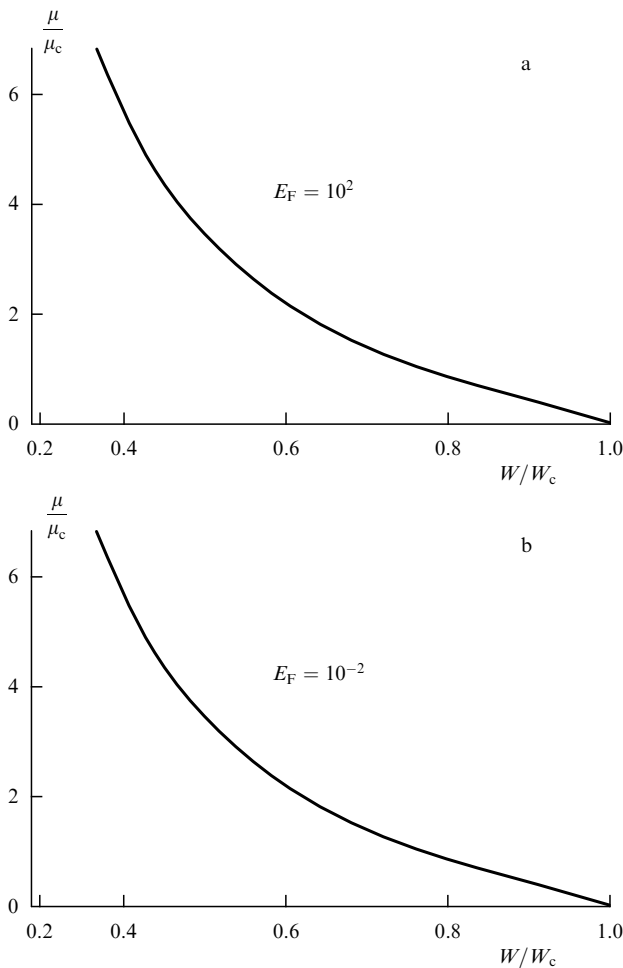$$C\delta^4 \approx (A-1)\,\delta^2, \tag{90}$$
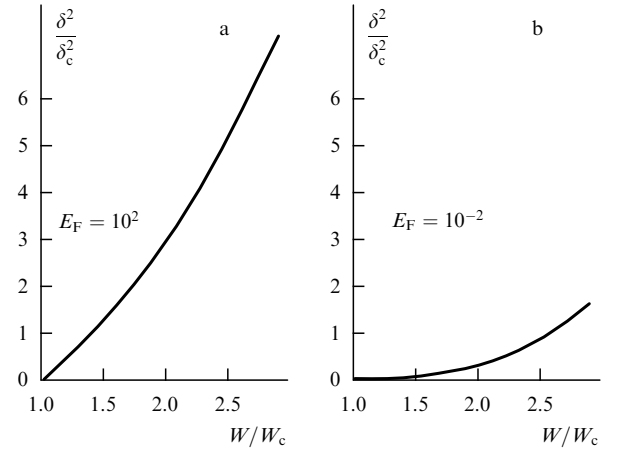
where the following notations are introduced:

$$A = 24\lambda E_F^{3/2} \int_0^\alpha \chi^2(\varkappa)\,\varkappa^2\,\mathrm{d}\varkappa, \qquad \alpha^{-1} \equiv 2E_F^{1/2}, \tag{91}$$

$$B = 384\lambda E_F^{7/2} \int_0^\alpha \left[\frac{3}{5} + \varkappa^2 - \frac{1}{3}\chi^{-1}(\varkappa)\right] \chi^2(\varkappa)\,\varkappa^4\,\mathrm{d}\varkappa, \tag{92}$$

$$C = \frac{9}{2}\lambda E_F^{-1/2} \int_0^\alpha \chi^3(\varkappa)\,\mathrm{d}\varkappa. \tag{93}$$

Figure 3. Mobility as a function of the level spread width in a conductor for long-range (a) and short-range (b) correlations in the level spreading. Values of $\mu_c = \mu^{(0)}(W_c)$, W_c are determined from formulas (101), (98) [17].

Figure 4. Localization parameter as a function of the level spread width in an insulator for long-range (a) and short-range (b) correlations in the level spreading. Values of $\delta_c^2 = \delta_0^2(W_c)$, W_c are determined from formulas (102), (98) [11].

In the limit of large and small E_F values

$$A \approx 6\left(\frac{W}{\Delta_0}\right)^2 \begin{cases} \dfrac{1}{8} E_F^{-2}\left[1 - \dfrac{1}{10}E_F^{-1}\right], \\[2mm] 3E_F^{-1/2}\left[0.329 - \dfrac{2}{9}E_F^{1/2}\right], \end{cases} \tag{94}$$

$$B \approx \frac{12}{25}\left(\frac{W}{\Delta_0}\right)^2 \begin{cases} E_F^{-1}\left[1 + \dfrac{10}{21}E_F^{-1}\right], \\[2mm] \dfrac{80}{3}E_F\left[1 + 0.846E_F^{1/2}\right], \end{cases} \tag{95}$$

$$C \approx \frac{27}{16}\left(\frac{W}{\Delta_0}\right)^2 \begin{cases} E_F^{-3}\left[1 - \dfrac{1}{12}E_F^{-1}\right], \\[2mm] 2E_F^{-5/2}\left[0.709 - \dfrac{32}{135}E_F^{5/2}\right], \end{cases} \tag{96}$$

where the first and second lines correspond to $E_F \gg 1$ and $E_F \ll 1$, respectively.

In accordance with (89), (90), the transition line $W_c(E_F)$ separating extended ($A < 1$) and localized ($A > 1$) states on the phase plane $E_F - W$ is defined by the condition

$$A(W, E_F) = 1. \tag{97}$$

Substitution of (91) results in a phase diagram shown in Fig. 5. Accordingly, the use of (94) yields limiting analytical expressions for the critical level spreading (in units of Δ_0):

$$W_c \approx 1.15 E_F \left(1 + 0.050E_F^{-1} + 0.004E_F^{-2}\right),$$

$$W_c \approx 0.411 E_F^{1/4}\left(1 + 0.338E_F^{1/2} + 0.171E_F\right). \tag{98}$$

Here, the first and second lines refer to large and small values of the parameter $E_F \equiv \varepsilon_F/\Delta_0$, respectively. According to (98), the Anderson transition line near the point $W = E_F = 0$ behaves in variables W, k_F in the same (root-like) way as under usual phase transformations.

Expressions for mobility μ and inverse polarizability δ^2 ensuing from (89), (90) has also, close to the line $W_c(E_F)$, the
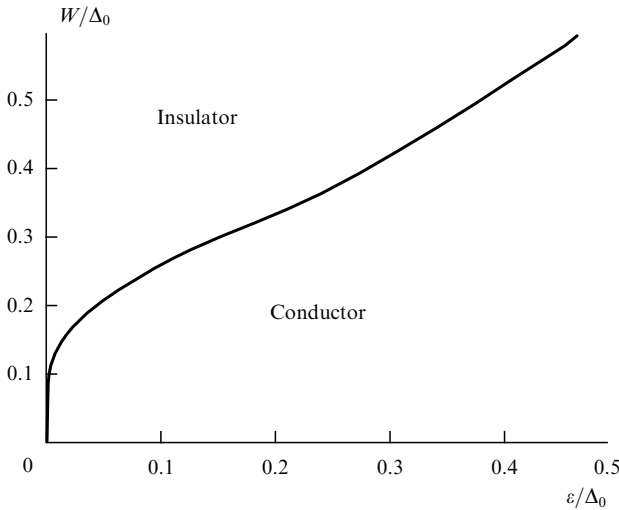
Figure 5. Insulator/conductor phase separation line [17].

root-like form:

$$(m\Delta_0)\mu \approx \left(\frac{2}{B_c}\right)^{1/2}\left(1 - \frac{W}{W_c}\right)^{1/2},\tag{99}$$

$$\delta \approx \left(\frac{2}{C_c}\right)^{1/2}\left(\frac{W}{W_c} - 1\right)^{1/2},\tag{100}$$

where parameters B, C are taken in the critical line.

In the weak coupling ($\lambda \ll 1$) limit, the solution to Eqns (87) is represented by terms $\mu^{(0)} \propto W^{-2}$, $\delta\mu \propto W^0$ [17]:

$$\mu \approx \mu^{(0)} - \delta\mu,$$

$$(m\Delta_0)\mu^{(0)} = \frac{1}{3\pi}\left(\frac{W}{\Delta_0}\right)^{-2} E_F^{-1/2} \max\left(1; 16E_F^2\right),$$

$$(m\Delta_0)\delta\mu$$

$$\approx \begin{cases} \frac{2}{3} E_F^{-1/2}\left\{\frac{\pi}{2}\left[1 + \frac{1}{20} E_F^{-1}\right] - \frac{2}{\pi}\left[1 + \frac{1}{10} E_F^{-1}\right]\right\}, \\ \frac{\pi}{2} E_F^{-1}\left\{0.458 - \frac{8}{3\pi^2} E_F^{1/2}\left[1 + \frac{8}{5} E_F\right]\right\}, \end{cases}\tag{101}$$

where the first line in the expression for $\delta\mu$ corresponds to large, and the second line, to small E_F.

In the inverse limit, expansion of Eqns (88) in $(\delta/W)^2$ leads to the results [17]:

$$\delta^2 \approx \delta_0^2 - \frac{4}{7}\left(\frac{W}{\Delta_0}\right)^2 \delta_0^{-2},$$

$$\delta_0^2 \approx \frac{3}{5}\left(\frac{W}{\Delta_0}\right)^2 \begin{cases} E_F^{-1}\left[1 - \frac{5}{84} E_F^{-1}\right], \\ \left[1 + \frac{12}{5} E_F\right]. \end{cases}\tag{102}$$

Now, let us turn to Anderson transition dynamics [16, 17]. It can be shown that, in three-dimensional space of variables W, ε_F, ω, the critical range is defined by conditions

$$(\varepsilon_F\Delta_0)^{1/2} < |\omega + M(\omega)|,$$

$$\omega < |M(\omega)|, \qquad \omega M(\omega) < \varepsilon_F^2.\tag{103}$$

The first of them is responsible for a decrease in dynamical conductivity (81); the second, for that of inverse polarizability (84), and the third, for the hydrodynamic regime, the only one in which the Anderson transition can be expected to appear. Given conditions (103), the self-consistent system of Eqns (76), (86) is reduced to an equality which generalizes (89), (90):

$$A - BM^{-2}(\zeta) + C\zeta M(\zeta) = 1,\tag{104}$$

where notations (90) – (93) and limiting expressions (77), 78) are used. Let us introduce the quantities

$$\Omega = \left(\frac{B}{C}\right)^{1/3}, \qquad \varepsilon = \frac{(1-A)\Omega^2}{3B},\tag{105}$$

which assume, near the transition line (97), the form

$$\varepsilon \approx \varepsilon_c\left(1 - \frac{W}{W_c}\right), \quad \varepsilon_c \equiv \frac{2\Omega_c^2}{3B_c}, \quad \Omega \approx \Omega_c \equiv \left(\frac{B_c}{C_c}\right)^{1/3}.\tag{106}$$

Therefore, ε is the distance from the transition line, and Ω is the characteristic frequency. The solution of Eqn (104):

$$M^{-1}(\zeta) = -\Omega^{-1}\begin{cases} |\varepsilon|^{1/2} F_\pm\left(|\varepsilon|^{-3/2}\zeta\right), & \varepsilon \neq 0; \\ F_0(\zeta), & \varepsilon = 0, \end{cases}\tag{107}$$

is expressed through functions $F_\pm$, F_0 representing roots of equations

$$F_\pm^3 \mp 3F_\pm + \zeta = 0,$$

$$F_0^3 + \zeta = 0.\tag{108}$$

The form of frequency dependence of these roots (Fig. 6) determines, taking into account (107), (81), conductivity (in units of $n/m\Delta_0$):

$$\mathrm{Re}\ \sigma(\omega) = \frac{M''(\omega)}{[\omega + M'(\omega)]^2 + [M''(\omega)]^2},\tag{109}$$
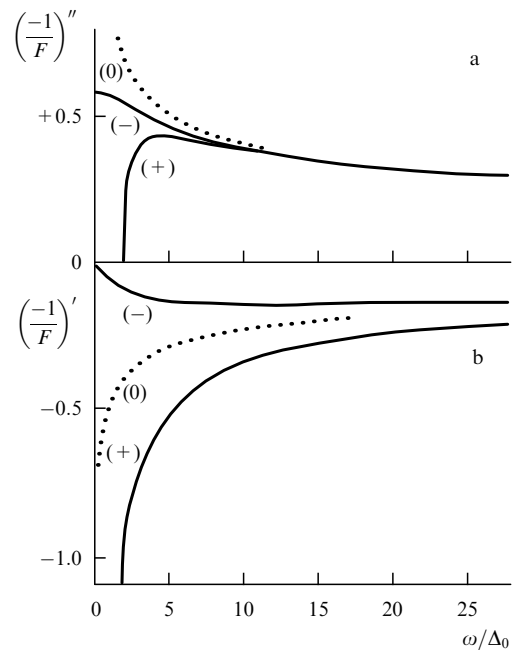

Figure 6. Imaginary part (a) and real part (b) for roots of Eqns (108) as functions of frequency [17].

where M', M'' are the real and imaginary parts of (107), respectively. The dependence (109) presented in Fig. 7 shows the appearance of a maximum at the frequency ω_m. It is seen from Fig. 8 that such character of this dependence is equally preserved outside the critical region (103). Frequency ω_m of the conductivity maximum is shifted as the level spread width W grows, following the curve in Fig. 9. This figure also shows corresponding dependences for the thermal gap $\Delta\varepsilon = \varepsilon_F - \varepsilon_c$ representing the difference between the Fermi level ε_F and the boundary energy ε_c (see Fig. 5), and for the optical gap starting from which finite conductivity becomes distinctive. Characteristically, a nonzero value of ω_m indicating the difference between (109) and the Drude dependence is just apparent below the transition point W_c due to polarization of the system prior to its transition to the insulating state. Conversely, quantities $\Delta\varepsilon(W)$, $\omega^0(W) \neq 0$ appear only at the transition point. It is worthwhile to note that the presence of the optical gap ω^0 is generally speaking an artifact of the method employed, which ignores the tendency to zero of the limiting value ($\varkappa \to \infty$) of the memory function $M_\varkappa(\zeta)$ on its replacement by $M(\zeta) \equiv M_{\varkappa=0}(\zeta)$ [17].
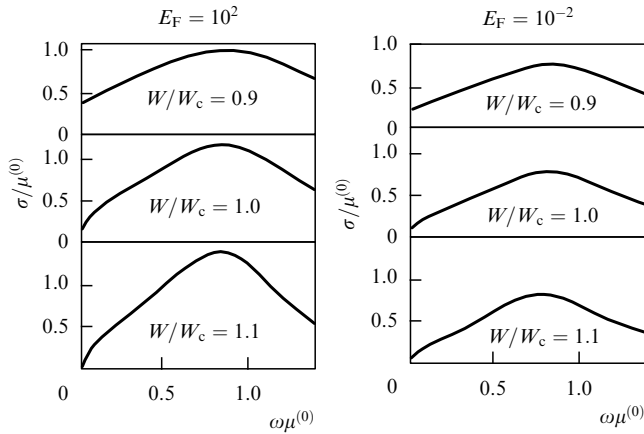
Figure 7. Frequency dependence of conductivity at different level spreading/correlation type combinations. Values of $\mu^{(0)}$, W_c are given by formulas (101), (98) [17].

Figure 8. Real part of the conductivity as a function of level spread width. Values of $\mu_c = \mu^{(0)}(W_c)$, W_c are given by formulas (101), (98) [17].
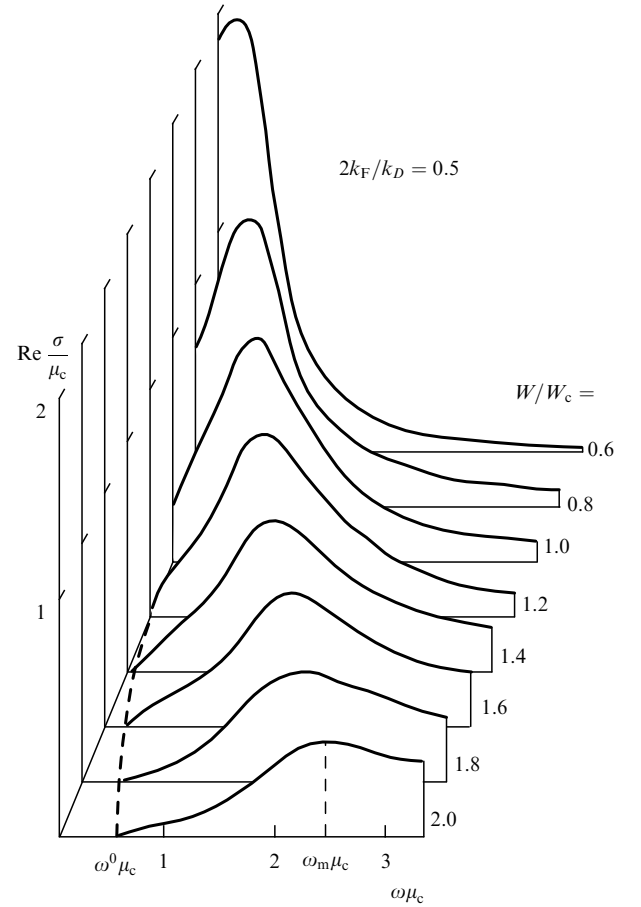
Let us now consider the nature of spatial dispersion. For this purpose, it is convenient to use the long-wave limit of equality (88a). It is easy to see that the macroscopic localization parameter obeys dependence of the Ornstein – Zernike type:

$$\Delta_\mathbf{k} = \frac{W}{\Delta_0}\,\xi^2 k^2 (1 + \xi^2 k^2)^{-1}, \qquad (k_D\xi)^2 \equiv \frac{1}{3}\frac{2k_F}{k_D}\,\delta^{-2} \quad (110)$$
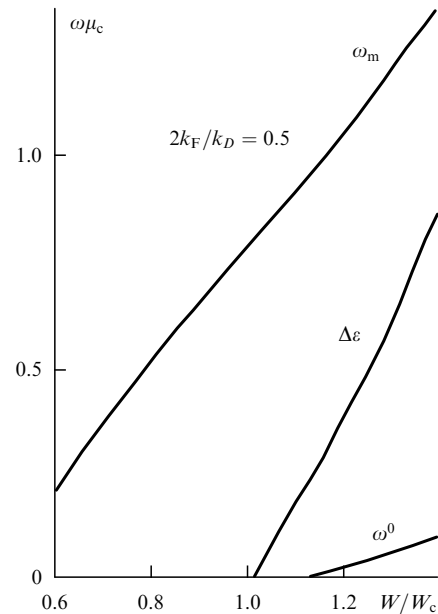
with the correlation length $\xi \propto \delta^{-1}$. Therefore, the greater the macroscopic localization parameter δ, the smaller the region ξ in which $\Delta(\mathbf{r})$ acquires an equilibrium magnitude. In other words, the region of localization of carriers in an insulator decreases with the distance from the transition point $W = W_c$ according to the equality

$$\xi \approx \xi_c \left(\frac{W}{W_c} - 1\right)^{-1/2}, \quad (k_D\xi_c)^2 \equiv \frac{2}{3}\left(\frac{k_F}{k_D}\right)^2 C_c, \qquad (111)$$

where the relation (100) is taken into consideration.

Figure 9. Maximal conductivity frequency ω_m, thermal gap $\Delta\varepsilon = \varepsilon_F - \varepsilon_c$ between the Fermi level and the boundary energy ε_c for the appearance of conductivity (see Fig. 5), and optical gap ω^0 starting from which conductivity appears as a function of the level spread width (see Fig. 8) [17].

The above discussion was concerned with the three-dimensional case for which both extended and localized states are possible depending on quantities W, ζ. The diagram technique [29] was used in [12] to study the behaviour of the vertex part of the relaxation function $\varphi_k(z)$ for an arbitrary dimension d. It has been shown that at $d = 3$, the consideration is reduced to the above self-consistent scheme [17], whereas for the dimensions $d \leqslant 2$, at any values of parameters W, ε_F, the vertex part has the static pole suggesting localization. Also, the absence of the extended state in low-dimensional systems was demonstrated in a renormalization-group study [6]. However, this does not mean that they always represent insulators in the experiment. Indeed, despite the fact that a delocalized cluster is finite at spreading $W \neq 0$, its size may prove so large that it will be apparent as a macroscopic one.

One should also bear in mind another aspect of clustering, specifically the percolation nature of the insulator/conductor transition [32]. Such a problem was considered using the Lorentz model as an example to illustrate the particle propagation in the field of scattering centres [33]. Conditions for the formation of an infinite cluster and particle localization in a finite one have been found to depend on the scattering centre density N. It turned out that formally this problem is not essentially different from that discussed in a previous paragraph. However, the dependence of the particle diffusion coefficient in an infinite cluster on the difference between the scatterer density N and its critical value N_c corresponding to the percolation threshold has the linear form $D \propto N_c - N$ rather than the root-like one as in (99).

At the same time, both dependences were found in the experiment, the root-like one being accompanied by magnetic anomalies, which suggest scattering that violates symmetry with respect to time reversal [34]. This raises the problem of modifying the present scheme in such a way as to obtain root-like and linear singularities, on the one hand, and realize the symmetrical and asymmetrical regimes, on the other hand.

According to [35], solution of this problem requires a self-consisting procedure to be applied for the nonreducible part $\Sigma_k(z)$, which defines $M_k(z)$ by an equality of the (65a)-type rather than at the level of the memory function $M_k(z)$ itself. Therefore, ignoring the quantity dependences on the wave-vector, for simplicity, the following expressions can be written instead of (86):

$$M(z) = \lambda \omega_0^2 \varphi(z) - \delta^2 [z + \Sigma(z)]^{-1}, \tag{112}$$

$$\Sigma(z) = \tau_1^{-1} [i + D_1 \lambda_1 \varphi(z)], \quad \tau_1 \equiv \left(\frac{\omega_0}{\delta}\right)^2 D_1^{-1}, \tag{113}$$

where λ, λ_1 are the coupling constants, ω_0 is the characteristic frequency, τ_1, D_1 are the relaxation time and diffusion coefficient irreducible to the previously used τ, D (see (79)). In the hydrodynamic frequency range $|z|\tau_1 \ll 1$, the current correlator (64c) takes the form

$$K(z) = -\left\{ [\omega_0^{-2} z + \lambda \varphi(z)] + iD_1^{-1} [1 - iD_1 \lambda_1 \varphi(z)]^{-1} \right\}^{-1}. \tag{114}$$

It was assumed throughout the foregoing discussion that $D_1 = \infty$, and the remaining terms of (114) led, at small times, to an even asymptotic of the relaxation function: $\varphi(t \to 0) \approx g_F [1 - (1/2)\omega_0^2 t^2]$. On the contrary, the assumption of $D_1 < \infty$ yields an odd asymptotic $\varphi(t \to 0)$

$\approx g_F(1 - D_1 t)$ even in the absence of the feedback ($\lambda, \lambda_1 = 0$). It may be inferred that a break of the chain (64), (65), (112), ... at different steps affects symmetry with respect to time reversal. Evidently, the coupling constant $\lambda \neq 0$ takes into account even effects that do not violate this symmetry, while the constant $\lambda_1 \neq 0$ takes into consideration odd, asymmetric effects. A survey of the entire λ, λ_1 range may be hoped to clarify the real situation.

Let us introduce for quantitative consideration the dimensionless variables $\zeta = z/D_1$, $\varphi(\zeta) = (D_1/g_F)\varphi(z)$, $K(\zeta) = K(z)/D_1$, $M(\zeta) = M(z)/D_1$ and coupling constants

$$\varepsilon = 1 - \lambda g_F, \qquad \eta = \lambda_1 g_F. \tag{115}$$

Then, it is possible to omit the term $\omega_0^{-2} z$ in (114) in the frequency range $|\zeta| < |M(\zeta)|$, and the self-consistent condition obtained by substituting (114) into (64) takes the form [35]

$$[\zeta + \varepsilon K(\zeta)][\zeta + i\eta + K(\zeta)] = iK(\zeta)[\zeta + K(\zeta)]^2. \tag{116}$$

In a conductor, the limiting value $-iK(\zeta = 0) \equiv K_+ > 0$ is realized. Therefore, it is possible to write down $K_+(\zeta) = iK_+[1 + f_+(\zeta)]$, where

$$K_+ = \frac{1}{2}\left(\varepsilon + \sqrt{\varepsilon^2 + 4\varepsilon\eta}\right),$$

$$f_+(\zeta) = -iA^{-1}\zeta, \quad A \equiv \varepsilon(2K_+ - \varepsilon)\left[1 - \frac{\varepsilon(2K_+ - \varepsilon)}{K_+}\right]. \tag{117}$$

It can be seen that the conducting phase occurs only at $\varepsilon > 0$, that is parameter ε has the former meaning such as (106). The position of the critical point does not depend on the second parameter η, however its value dictates the dependence $D(\varepsilon)$ of the diffusion coefficient $D \equiv D_1 K_+$ on the distance ε from the critical point. Indeed, according to (117), there is a root-like dependence $D \approx D_1(\eta\varepsilon)^{1/2}$ in the immediate vicinity of the critical point ($\varepsilon \ll 4\eta$) and a linear one $D \approx D_1(\eta + \varepsilon)$ far from it ($\varepsilon \gg 4\eta$).

An insulator is characterized by an asymptotic $\lim_{\zeta \to 0}[\zeta^{-1} K(\zeta)] = K_- > 0$ taking which into account yields $K_-(\zeta) = K_- \zeta[1 + f_-(\zeta)]$. Then, it follows from (116) that

$$K_- = -\varepsilon^{-1}, \quad f_-(\zeta) = i\left[\frac{\zeta(1-\varepsilon)}{\varepsilon}\right]^2 [\zeta(1-\varepsilon) - i\eta\varepsilon]. \tag{118}$$

Accordingly, polarizability $\chi(\zeta) = i\zeta^{-1} K_-(\zeta)$ takes the form $\chi(\zeta) = iK_-[1 + f_-(\zeta)]$.

In the analysis of dispersion, three boundary values should be introduced, apart from the soft mode frequency ω_0, viz., the hydrodynamic limit $\omega_\varepsilon = D_1|\varepsilon|\sqrt{\varepsilon^2 + 4|\varepsilon|\eta}$ and critical frequencies $\omega_c \approx D_1\eta^2$ and $\omega_{c\varepsilon} = D_1\eta\varepsilon$. At large values of $\omega \sim \omega_0$, the quantity $K(z)$ in (64a) may be neglected and the spectral density has the limiting form $\varphi''(\zeta) \propto \zeta^{-1}$. In the conducting phase ($\varepsilon > 0$), only the hydrodynamic boundary defined by the condition $|f_+(\zeta)| \ll 1$ is apparent. Within this boundary ($\omega \ll \omega_\varepsilon$), $\varphi''(\omega) = g_F/D$ is constant. The form of dependence $f_-(\zeta)$ in insulator is more complicated and the following limiting relations arise from (118):

$$K''(\omega) \approx D_1 \begin{cases} \left(\dfrac{\omega}{D_1\varepsilon}\right)^4 \eta^{-2}, & \omega \ll \omega_{c\varepsilon}; \\[2ex] \left(\dfrac{\omega}{D_1\varepsilon}\right)^2 \varepsilon^{-1}, & \omega \gg \omega_{c\varepsilon}, \end{cases} \tag{119}$$

the applicability of these relations is restricted by the frequency $\omega_{c\varepsilon} \propto \varepsilon\eta$. According to (119), symmetry violation with respect to time reversal leads to the extension of the region of the anomalous dependence $K''(\omega) \propto \omega^4$ with increasing η. Here, the relaxation function has the form

$$\varphi(\zeta) \approx -\frac{D_1}{W} q\zeta^{-1} - \varepsilon(1-\varepsilon)^{-2} f_-(\zeta)\zeta^{-1}, \quad \zeta \to 0. \quad (120)$$

If the first pole leads to an ordinary δ-peak characterized by the Edwards–Anderson parameter q, the second term stands for the pseudogap/white noise transition:

$$\varphi''(\omega) \approx \frac{g_F}{D_1} \begin{cases} \left(\dfrac{\omega}{D_1\eta}\right)^2 |\varepsilon|^{-3}, & \omega \ll \omega_{c\varepsilon}; \\ |\varepsilon|^{-1}, & \omega \gg \omega_{c\varepsilon}. \end{cases} \quad (121)$$

Therefore, in the $\omega \ll \omega_{c\varepsilon}$ range, time-symmetry violation gives rise to different frequency dependences $\varphi''(\omega)$ for a conductor and insulator. At time reversal ($\omega_{c\varepsilon} \propto \eta = 0$), the relationship $\varphi''(\omega) = $ const holds in both cases.

On the assumption that $\lim_{\zeta\to 0}\left[\zeta K^{-1}(\zeta)\right] = 0$, Eqn (116) in the transition line ($\varepsilon = 0$) takes the form

$$K^3(\zeta) - i\zeta K(\zeta) + \eta\zeta = 0. \quad (122)$$

Hence, in limiting cases given by frequency $\omega_c \propto \eta^2$, it follows

$$K''(\omega) \approx \frac{1}{2} D_1 \begin{cases} 3^{1/2}\left(\dfrac{\eta\omega}{D_1}\right)^{1/3}, & \omega \ll \omega_c; \\ 2^{1/2}\left(\dfrac{\omega}{D_1}\right)^{1/2}, & \omega \gg \omega_c. \end{cases} \quad (123)$$

Therefore, the spectral density $\varphi''(\omega) \propto 1/K''(\omega)$ undergoes transition from the normal asymptotic $\varphi''(\omega) \propto \omega^{-1/2}$ in the $\omega \gg \omega_c$ region to the anomalous one $\varphi''(\omega) \propto \omega^{-1/3}$ at $\omega \ll \omega_c$. Hence, in the time axis, we get

$$\varphi(t) - \frac{q}{W} \propto \begin{cases} t^{-1/2}, & \eta^2 \ll \omega_c t \ll 1; \\ t^{-2/3}, & 1 \ll \omega_c t \ll \dfrac{\omega_c}{\omega_\varepsilon}. \end{cases} \quad (124)$$

At $t \ll \omega_0^{-1}$, $\varphi(t) \approx g_F\left[1 - (1/2)\omega_0^2 t_0^2\right]$, whereas in the inverse limiting case $t \gg \omega_\varepsilon^{-1}$, the Debye fall $\varphi(t) \propto \exp(-\omega_\varepsilon t)$ is involved. In the middle, there is a region of applicability of asymptotics (124).

4. Conclusions

The main objective of this review is to demonstrate the mutual complementarity of the site and wave approaches to the problem of quantum particle localization in the random potential. In so doing, one-particle properties are most naturally represented in the framework of the site approach, whereas the wave approach is useful for representing collective properties. Physically, this situation can be accounted for by the fact that in the former case the role of the initial (disordered) state is played by localized, and in the latter case, by extended states.

It is clear from Section 2 that the theoretical scheme of the site approximation is based on two major premises. In the first place, it should be borne in mind that overlapping of site wave functions leads to the division of the configuration space into isolated and multisite state constituents. Due to this, the Anderson Hamiltonian (5) undergoes transformation to the

expression (8), which differs from the BCS Hamiltonian for the theory of superconductivity only in that it contains the site representation instead of the momentum one and the effective potential of interparticle interaction depends on the energy spectrum of the system. Another essential feature of our approach [18, 19] is the idea to represent quenched disorder in the distribution of a cold variable ε by the distribution (19) reducible to the Gibbs's one. In other words, we propose to represent strongly nonequilibrium statistical disorder as frozen thermodynamic fluctuations characterized by the 'temperature' $W/2$.

At first site, the second proposal is most liable to criticism because the Gibbs distribution describes equilibrium statistical ensembles, whereas the Anderson model defines a strongly nonequilibrium system. This fact may be simply ignored which was exactly what Anderson [2] did by accepting *a priori* the stepwise approximation $P(\varepsilon)$. This line of reasoning leads to considering our distribution (19) as a feasible approximation in a series of models proposed by Anderson, Lloyd, and other authors. However, there is every reason to prefer the exponential distribution to all others not only because it is more customary to deal with but also in principle. Indeed, it has recently been shown (see, for instance, [36]) that the shape of the Gibbs distribution is intrinsic in statistical ensembles just as well as in systems being far from equilibrium. The main conditions for conserving the exponential form of statistical distribution are the stationary state† of a nonequilibrium system and its ergodicity. If these conditions are satisfied, the statistical distribution of a nonequilibrium system will have the form of the Gibbs distribution, where the synergetic potential plays the part of energy and temperature is reduced to noise intensity, which determines the degree of the quenched disorder [36]. It has been shown in Section 2 that our distribution (19) has exactly this sense. As regards stationary and ergodic conditions, the former is due to the infinitely rapid quenching of the system down to zero temperature as implied in the Anderson model. The ergodicity condition does not seem to be met because localization of electrons is none other than ergodicity breaking in terms of their movement in the real space. However, in contrast to hierarchically coordinated systems like spin glasses, the phase space is not splitted into an infinite set of isolated macroregions. More specifically, localization results in the separation of isolated microscopic regions (in the **r** space their scale is restricted to the interatomic one) from the configuration space corresponding to the extended state. These regions being alike, it is possible to pool them into a phase space macroregion corresponding to the localized phase, whereas the remaining part of the space will govern behaviour of the delocalized phase. As a result, the division of the phase space due to localization is represented in the same way as in usual phase transitions. It is this fact that allowed the process of delocalization to be represented in agreement with the standard phase transition scheme.

It has already been mentioned that in the framework of our approach, the order parameter determines a fraction of delocalized electrons, whereas its conjugate field — a shift from the band centre. The width of level spreading plays the role of the state parameter like a temperature. A marked difference of the delocalization scheme described in Section 2 from the standard version of the microscopic phase transition theory is due to the aforementioned division of the system's

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† For example, a fluid flowing with a constant speed is stationary.

configuration space into localized and extended state regions. It turned out that the effective interaction potential depends on the character of this division, that is on external conditions. This is manifested in the appearance of the logarithmic multiplier for the dependence of the 'thermodynamic' potential on the order parameter, which resulted in a change in the order of phase transition from 2 to $2 + \delta$, $\delta \to 0$.

It is worthwhile to note that this observation does not exhaust all causes underlying the difference between the Anderson transition and the mean-field theory for ordinary phase transitions. According to [37], this approximation implies that a new phase is created in regions of similar shape and size. In other words, it is assumed that fluctuations are uniformly distributed in space. Taking into account the deviation from this pattern and especially the fractal nature of fluctuations in the critical region, it can be seen that the values of all critical indices are dissipated within certain limited intervals [37]. For example, the exponent β in the relation $\eta \propto (W_c/W - 1)^{\beta}$ obeys the condition $1/3 \leqslant \beta \leqslant 1/2$. It is noteworthy that the fractal structure is characterized by a certain distribution of β values along the entire length of this interval rather than by a single value.

As far as specific features of the wave approximation are concerned as described in Section 3, it should be in the first place emphasized that it is intrinsically different from the site approach. The latter allows only stationary values to be found (see Section 2), whereas the former is principally designed to derive hydrodynamic expressions for characteristic correlators depending on complex frequency and wave vector. Therefore, the most suitable mathematical tool for the solution of this problem is the Zwanzig – Mori chain fraction, which allows major correlators of the 'density-density' type to be expressed through higher correlation functions. From the physical point of view, such a fraction is convenient because each of these functions has clear physical sense. This makes it easy to understand the character of the approximation that results from a break of a chain fraction.

We used the mode-coupling technique to elaborate a self-consistent scheme for it allows the higher correlator to be expressed through the original one. In the simplest version of the localization theory suggested by Götze [17], effects of ergodicity breaking are made explicit in the memory function, which represents the kernel of the second link in the Mori chain. Taking into consideration the absence of the direct interaction between electrons, Götze assumed that the memory function is proportional to the first power of the original correlator (52). Therefore, the mode-coupling technique which, by definition, implies the presence of higher powers in such approximations, was actually reduced to examining uncoupling modes. This explains the lack of nonlinear feedback in the scheme [17] notwithstanding self-consistent nature of the problem. Also, it was apprehended that the scheme [17] may not contain phase transition since, in accordance with synergic ideology [38], it is positive feedback that gives rise to self-organization. For this reason, I V Koplyk and A A Koloskov, my coworkers, undertook it to check up the numerical solution of self-consisting equations (87), which underlie the Götze scheme [17]. Surprisingly, mobility $\mu \propto v^{-1}$ turned out to be other than zero at all W values, which indicates that these equations do not lead to the Anderson transition even though the dependence $\mu(W)$ shown in Fig. 3 is well reproduced far from the critical point W_c. Since none of Götze's publications mentions this fact, we had to conclude

that allegedly the self-consistent scheme [17] is actually lacking in consistency. In addition to what was said in connection with Eqns (87), this inference ensues from the condition (89) giving the phase diagram of the system: to derive this expression, the long-wave approximation (77) is used, which assumes $\zeta = 0$, whereas in the following expression (78) for the diffusion coefficient, the first cofactor has $\zeta = 0$ and the second, $\zeta = \infty$. Thus, the theory [17] is an approximation scheme in the framework of which the presence of the Anderson transition is due only to artificial selection of terms in the self-consisting equation.

To summarize, a few remarks are in order concerning problems and prospects of further development of the approaches discussed in this review. To begin with, many results (e.g., mobility threshold, state density) must not be sensitive to the type of quantum particle statistics by virtue of the one-particle character of the problem even though the above discussion was concerned with an electron in a random field. The question: 'Is this really so?' remains to be answered.

It is worthy of note that either approach is discussed independently in Sections 2 and 3, respectively. This naturally prompts the idea of their unification. We have recently realized this programme [39] based on the locator approximation in the framework of which one-particle properties of a system are described by the Green's function for fermions taken in the site representation. This function obeys the Dyson equation in which the self-energy function is expressed through the vertex function for the effective interaction potential of fermions. This function represents collective excitations of a system and obeys the Bethe-Salpeter equation, which closes the self-consistent description of both types of excitation. In the framework of such a modality, the mutual complementarity of site-temporal and frequency-wave approaches is manifested in the fact that the Dyson and Bethe – Salpeter equations are in the simplest way represented in terms of the former and the latter approaches, respectively. This is not unexpected bearing in mind that the site approximation in Section 2 was used to mainly describe one-particle properties, and the wave approximation in Section 3 — collective properties.

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