Physics – Uspekhi **39** (6) 573 – 596 (1996)

REVIEWS OF TOPICAL PROBLEMS PACS numbers: 03.65.Bz, 12.20.Ds, 07.60.Dq, 42.50.Dv

The nonclassical light

D N Klyshko

Contents

Abstract. **Properties of the nonclassical light (NCL) have been considered with an emphasis on experimentally-observed features, which are underlain by the well-known Mandel's formula connecting the statistics of photocounts with that of light falling on the detector. A systematic operational approach is presented to study the NCL using two parallel sets of numbers measured: probabilities of photocounts $\{p_m\}$ and normalized factorial moments of counts $\{g_k\}$. Two particular examples are examined in detail: a 'heated' squeezed vacuum and a 'heated' one-photon state. An alternative method is proposed to discover the week nonclassicality using 'generalized' moments $\{a_k(s)\}$. The effect of the linear absorption (amplification) and of the beam-splitting on the NCL, and the relation between the NCL and the absolute calibration of photodetectors are considered. The conditions are elucidated whereat the beam-splitter realizes a mathematical operation of superposition of two one-mode fields useful in studying the NCL.**

D N Klyshko M V Lomonosov Moscow State University,
Physics Department,
Vorob'evy gory, 119899 Moscow
Tel. (7-095) 939-11 04
E-mail: dnk@klyshko.phys.msu.su

Received 22 December 1995
Uspekhi Fizicheskikh Nauk **166** (6) 613–638 (1996)
Translated by K A Postnov, edited by A Radzig

1. Introduction

The concept of the nonclassical light (NCL) draws a great attention in modern quantum optics. By NCL is meant a light whose observed properties cannot be described with customary visualization by considering a light beam as a set of waves. In other terms, the NCL produces effects that have no classical analogies.

The properties of light are usually thought of by analyzing the properties of a photocurrent induced by the incident light due to photoeffect. Traditional optical experiments primarily focused on the mean current intensity proportional to the mean light intensity. Observed deviations of the current from its mean value are of a chaotic, random character. Such *Poissonian* or *shot* noises of the current have been explained by random moments of the photoelectron creation and therefore have not come to the attention.

For the first time, the effect of thermal fluctuations of the light intensity on the photocurrent fluctuations was discovered about 40 years ago in the well-known experiments of Brown and Twiss [1]. They used a light emitted by a mercury lamp or a star and observed additional photocurrent noises which were stronger than the Poissonian ones. (In fact, these experiments revealed not the current fluctuations in one of the detectors, but a *correlation* between fluctuations of currents in two detectors, however these effects are tightly connected to each other.) The excessive, *superpoissonian* noises of the photocurrent have a trivial classical explanation: the amplitude of the light wave varies randomly resulting in the synchronous change in number of photoelectrons generated by the wave. Therefore the light studied in the Brown-Twiss experiments should be related to the category of 'classical light', i.e. those described by the classical theory.

By irony of history, with the Brown – Twiss effect, which can be successfully described in terms of *classical* optics, one usually relates the date of birth of *quantum* optics. At the beginning, this effect even met some difficulties when being treated within the framework of quantum optics. Presently, using visual photon language, it is explained by excessive, superpoissonian fluctuations in the stream of particles (photons), i.e. the phenomenon is thought to appear as a result of some 'bunching' of photons which is inherent in the thermal sources of light.

The first optical experiments with the NCL were described in 1965 – 1967 [2 – 6]. These experiments used the sources emitting photons by two in the form of tight pairs. Such a light, referred to as a *two-photon* light, also produces extra noises of the photocurrent, however their statistical properties are incompatible with classical notion of waves with random amplitude. The two-photon light radiated during two-quanta transitions between three atomic levels was studied in [2, 3]. In 1967, the *spontaneous parametric down-conversion* in transparent, birefringent piezo-crystals was discovered [4 – 6], during which the two-photon light is emitted more efficiently. In 1977, the light with the *antibunching of photons* or, in other words, the *subpoissonian light* was observed for the first time [7], for which the photocurrent fluctuations were smaller than the minimum value allowed by classical theory. Another form of the NCL, the *squeezed light* (also emitted most effectively during nonlinear parametric processes) was observed for the first time in 1985 [8 – 10].

In the squeezed light (more precisely, in the quadrature-squeezed light) the fluctuations of one of the quadratures are suppressed, 'squeezed', and it contains an even number of photons. Actually two types of the NCL, the two-photon-parametric light and squeezed light, are described as limiting cases of the same state (under weak and strong pumping, respectively), which is denoted as a *squeezed vacuum state*. The state with 'squeezed' energy fluctuations corresponds to a subpoissonian light.

The NCL allowed observation of some new optical effects to be made. First of all, we notice the violation of Bell's inequalities [11] (which proved inapplicability of classical models with hidden parameters). This became possible due to discovery of a new type of the intensity interference, called later the *two-photon interference*. It may appear in two forms (polarizational [3, 11] and ordinary [12] ones) and may have almost a 100% visibility. This very last point permitted one to observe the violation of Bell's inequalities in optics. The *subpoissonian shot noise* [7 – 10], *two-photon diffraction* [13], *two-photon image transmission* [14] have been also observed. Using the two-photon light the *absolute photometry* has been practised [15]. One may already speak about the birth of the 'two-photon optics' (both the wave and geometrical [12 – 16] ones).

The NCL results from multiquantum transitions in matter and is tightly connected to the notion of an optical nonlinearity of matter [17]. Strictly speaking, ordinary thermal radiation of a heated body studied by Kirchgoff and Planck may be nonclassical, two-photon in particular, when considering multiquantum transitions [17]. However, practical production of the NCL with a sizeable intensity requires special conditions (which have been realized in [2 – 16]). At present, new experimental methods of NCL generation are successfully being realized with the use, for example, of semiconductor lasers [18 – 20] and cubic nonlinearity of semiconductors [21], as well as during generation of the second harmonics [22 – 24].

The interest to the NCL is primarily due to its possible applications for information transmission and for very precise interferometric measurements. In addition, the NCL allows experimental demonstration of the nonadequacy of classical description of some optical phenomena (see [25]). Its photometric applications are less known.

Different aspects of the NCL have been studied theoretically (see reviews [25 – 35] and references therein, as well as special issues of journals [36 – 38] and monographs [17, 39]). At the same time, it seems that a systematic description of the observable NCL features and of how the quantum efficiency of the detector, linear absorption and amplification affect these features is absent. The present paper tries to fill up this gap.

What is the definition of the NCL? How it differs from a classical light? An opinion is widespread that any kind of light is generally nonclassical as it 'consists of photons'. However, the latter statement requires caution [25] and, in addition, many papers are known justifying the adequacy of classical concepts of optical phenomena. Essentially, the NCL is a light whose statistics admits no description within the framework of classical stochastic optics. The commonly accepted and most general formal definition is as follows: *the light for which P-distribution of Glauber–Soudarshan takes negative values or is an irregular* (*generalized*) *function is referred to as nonclassical* (definition I). (*P*-distribution is a classical analog to classical probability distribution for the field amplitude, see below.)

In the present paper we consider some more specific features of the NCL. Part of them is apparently being studied for the first time. The main attention is given to *operational criteria* of the NCL, i.e. to experimentally examined features (obviously, singularity of the *P*-distribution does not relate to them). These features are based on the well-known Mandel's formula [40] which connects the directly measured quantity, the statistical distribution of current pulses at the detector's output (*the statistics of photocounts*), with the statistics of light incident on the detector. Here a stationary light flux in a free space is assumed to be described by some classic or quantum statistical ensemble. We shall also consider the effect of linear absorption, amplification and beam-splitting on the NCL and relation between the NCL and the absolute (etalon-free) calibration of photodetectors. For simplicity, we consider mainly experimental schemes with one detector, i.e. problems of correlation between counts in two detectors are practically not dealt with. Dynamical (spectral) properties of the field will not be analyzed as well since their quantum description differs insignificantly from the classical one (see [35]).

A special position among many possible states of the field is taken by the *coherent state* [41 – 43] that describes the light of an ideal laser in some approximation. For this state the *P*-distribution takes the form of the δ-function, i.e. it has the simplest singularity not violating the nonnegativity condition. Such a distribution is classically admitted, so the coherent state is assumed to describe the classical light. According to [44], all other pure quantum states exhibit stronger singularities. For example, a quantum state with a given number of photons K is described by the *P*-distribution in the form of the K-th order derivative of the δ-function. Approximation of such a distribution by smooth functions shows that the condition of nonnegativity is violated.

Thus the coherent state is at the interface between the sets of classical and quantum field states. (Notice however that, as shown in [45], it is possible to construct a state that is closer to a classical monochromatic field with a certain amplitude and phase than the coherent state.) Thermal (chaotic, Gaussian) light belongs to a classical light. To the other side of the boundary we find the squeezed light and the light with a certain number of photons. These three types of statistical states of light as well as their superpositions will be the main subject of the present discussion. Different features of nonclassicality will be analyzed using the example of several particular field states with a parameter T allowing continuous transition from the nonclassical to classical light.

The author tried to make the presentation as available as possible for readers not acquainted with quantum optics, so Sections 2 and 3 provides a general information on methods of observation and description of the light statistics. No knowledge of quantum theory is also required in Section 4, where observable appearances of the light nonclassicality are described. In Section 5, the backgrounds of quantum theory of photocounts are presented. In Sections 6 and 7, two typical examples of the NCL demonstrating different manifestations of the nonclassicality are discussed. Sections 8 and 10 are devoted to effects which the linear absorption, amplification and mixing of two light beams by a beam-splitter have on the light statistics. Section 9 considers the role of the detector quantum efficiency and the possibility of its absolute measurement. In the Appendices, we present some mathematical relationships that permit one, in principle, to distinguish classical and nonclassical light in the experiment.

2. Experimental procedure

2.1. Photocounts

In the vast majority of optical experiments information about the field is extracted from the counts on the photoeffect-based light detectors (PEM, semiconductor photodiods, etc.). The most detailed unaveraged information on the statistical properties of a stationary field at some point of space-time is obtained when using a photon counter with a small area and inertial time. Then the number of pulses m is counted periodically at the detector's output (*number of photocounts*) during some fixed small sample interval T. This number fluctuates from one experiment to another. Multiple repetition of this procedure gives a set of numbers $\{m_i\}$. Statistical processing of the array of numbers obtained allows, in principle, the determination of full probability characteristics of the discrete random variable m. The latter may be specified using the probability distribution p_m ($\sum p_m = 1$, $m = 0, 1, 2, \ldots$) or *moments* of this distribution

$$\langle m^k \rangle = \sum_{m=0}^{\infty} m^k p_m , \quad (2.1)$$

where $k = 1, 2, \ldots$

2.2. Antibunching of counts

The simplest two numerical parameters of the photocount statistics are the mean number of counts $\langle m \rangle$ and their variance

$$\langle \Delta m^2 \rangle = \langle m^2 \rangle - \langle m \rangle^2 .$$

From them the *Fano factor* and the *parameter of count bunching* are derived by inspection:

$$F \equiv \frac{\langle \Delta m^2 \rangle}{\langle m \rangle} , \quad g_2 \equiv 1 + \frac{\langle \Delta m^2 \rangle - \langle m \rangle}{\langle m \rangle^2} = 1 + \frac{F - 1}{\langle m \rangle} . \quad (2.2)$$

These parameters characterize the difference of the statistics from the Poissonian one. In the case of Poissonian statistics $\langle \Delta m^2 \rangle = \langle m \rangle$, so $F = g_2 = 1$, therefore when $F < 1$ or $g_2 < 1$ one speaks about *subpoissonian* statistics of photocounts or about subpoissonian nonclassical light (it is equivalently, as a rule, to speak about the effect of *antibunching of counts or photons*; however, one should sometimes distinguish these terms [29, 46]). Relative fluctuations of the number of counts from sample to sample are characterized by the parameter $\langle \Delta m^2 \rangle / \langle m \rangle^2 = F / \langle m \rangle$ that determines the signal-to-noise ratio.

What is unusual in the property of the photoelectron flux to meet $\langle \Delta m^2 \rangle < \langle m \rangle$ (or $g_2 < 1$)? Why this is thought to be a feature of the light nonclassicality? It is trivial by itself. For example, this is characteristic for the electron stream in a vacuum diode without saturation (when the shot noise is suppressed by the space charge). The paradox appears only within some interpretation of the pulses observed at the photodetector output.

2.3. Corpuscular model

The simplest 'corpuscular' model of the photoeffect is based on a visual interpretation of the light beam incident upon the detector cathode in the form of a photon flux. Let us suppose for simplicity that the quantum efficiency of the detector η is 100%, then the arrival of each photon causes the emission of one photoelectron. As a result, the electron flux simply duplicates the photon flux and the condition $\langle \Delta m^2 \rangle \neq \langle m \rangle$ means only that the photon flux exhibits some regularity, the photons being distributed in time in a nonchaotic manner. When $g_2 > 1$, photons have a tendency as to bunch together, and when $g_2 < 1$ – to 'repulse'. (We stress that these terms have nothing to do with intrinsic properties of photons as hypothetical elementary particles; they characterize only the statistical properties of the light sources.)

Thus, the antibunching was observed in experiments during resonance fluorescence of single atoms excited by a laser [7]. After emitting a photon by the excited atom, its repeated excitation occurs only after some finite time interval Δt, so the moments of emission of two subsequent photons are separated by an interval not less than Δt, i.e. close pairs of photons cannot be emitted. This produces the subpoissonian statistics for the counts. (The Poissonian distribution assumes all moments of time to have equal rights, so the consecutive counts may be observed arbitrarily close to each other). In the ideal case of the full antibunching, the photons are emitted regularly at a certain time interval, the number m being nonfluctuated, so that $F = g_2 = \langle \Delta m^2 \rangle = 0$.

Thus the condition $g_2 < 1$ is quite natural within the framework of the corpuscular model of light. This primitive model of photon–balls describes qualitatively many experiments. Within its framework any light is nonclassical and the isolation of a subset of classical statistical states of light has no sense because it is empty. The well-known difficulties, which appear when describing interference and diffraction of light, are connected to this model.

2.4. Quantum model

For precise quantitative description of all the effects observed, it is necessary to use, of course, quantum theory for both the matter and field. Then the number of photons n becomes an operator. It may be expressed through 'base' operators of annihilation a and creation $a^\dagger$ of photons: $\hat{n} = a^\dagger a$. Quantum theory predicts (assuming a certain model for the light source) the distribution of the number of photons p_n and the moments of this distribution $\langle n^k \rangle$. All known experiments agree very well with predictions of quantum optics. In the case of an ideal detector, the observed statistics of counts doubles quantum statistics of photons specified by the density matrix of the field (see Section 5). Within the framework of rigorous quantum theory, the subpoissonian statistics of counts causes no astonishment as well.

However, there is the third way of describing the photocounts' statistics that uses a semiclassical model for photoeffect and does not allow the subpoissonian statistics. It is within the framework of this approach that the concept of the NCL appears.

3. Semiclassical theory of photocounts

In the semiclassical theory, atoms of the detector are considered quantum-mechanically, while the field of the light wave — classically. Then the statistical properties of the field are specified by some probability distribution laws. Quantum theory is much more complicated than the classical stochastic electrodynamics. Apart from having a more complex mathematical apparatus, quantum formalism is distinguished by allowing an interpretation (as a rule, the 'Copenhagen' variant of the interpretation is used) connected with the abandonment of some customary physical concepts (which manifests itself as the complementary principle), so it is natural to try to describe electromagnetic field in a classical way whenever possible, at least in optical and radio wave ranges. It is managed to do in many cases and a lot of attempts were undertaken to 'legitimate' this approach. However, during last decades some optical effects have been discovered that could not been described adequately in such a simplified manner. The term NCL just distinguishes these cases.

3.1. One-mode detector

We use a maximum idealized model. Let the area of the detector A be equal to the coherence area of the light falling on the detector, and the sample interval T be equal to the coherence time τ_{coh} (which is on the order of the reverse spectral bandwidth, $\tau_{\text{coh}} \sim 2\pi/\Delta\omega$; during the coherence time the field amplitude changes insignificantly). Such a detector 'sees' only one mode of the field, i.e. only one independent vibrational degree of freedom (necessary corrections to this idealization will be found in Section 9). Dynamical or statistical description of one mode is equivalent to description of a harmonic oscillator. (Multimode detectors average field fluctuations in time and space, which leads ultimately to a trivial Poissonian statistics of counts regardless of individual properties of the field (see, for example, [47].)

In the case of a stationary quasi-monochromatic light beam, the field on the detector has the form $E_0 \sin(\omega_0 t + \phi)$, where E_0 and ϕ are the random slowly changing functions of time. The characteristic time of changing these quantities is just said to be the coherence time. The probability of appearing the successive photoelectron in the detector is taken to be proportional to the intensity E_0^2 of the beam incident on the detector's cathode at a given moment of time.

It is convenient to go over to a dimensionless quantity n — the field energy fallen within the coherence volume $V_{\text{coh}} = c\tau_{\text{coh}} A_{\text{coh}}$ (which is coincident, according to our suggestion, with the volume of detection $V_{\text{det}} = cTA$) divided by the energy of a photon:

$$n \equiv \frac{E_0^2 V_{\text{coh}}}{8\pi\hbar\omega_0} \,.$$

In the quantum theory, our classical variable n corresponds to the operator of the number of photons in one mode $\hat{n}$ that possesses only discrete spectrum of integer values 0, 1, 2, ... In contrast, here n is simply the field energy expressed in some convenient dimensionless units, which takes on continuous values from zero to infinity. Accordingly, the statistics of photocounts in the quantum theory is determined by a discrete distribution of the field energy p_n, while in classical theory — by a continuous distribution $P(n)$. It is this difference in the type of distributions that makes it possible to introduce an operational (using the statistics of the observed photocounts) definition of the NCL.

Let, for example, the light flux incident upon the detector have the power $I = 10^{-9}$ W, the wavelength $\lambda = 0.5$ μm and the spectral bandwidth $\Delta f = \Delta\omega/2\pi = 10^9$ Hz, then $n = I\lambda/hc\Delta f = 2.5$ photons.

3.2. The Mandel's formula

The probability of appearing the successive photoelectron over the time interval $(t,\ t + \mathrm{d}t)$ is accepted to be $\eta n\,\mathrm{d}t/T$, where the dimensionless proportionality coefficient η is called the quantum efficiency of the detector. The mean number of counts during the sample time T is $\langle m \rangle = \eta \langle n \rangle$. The efficiency of modern detectors approaches 100% and we assume the detector to be ideal, $\eta = 1$ (in Section 9 we will take into account the corrections due to $\eta < 1$).

Let us suppose that the field intensity n is constant and does not fluctuate (like the field of an ideal laser). Here all moments of time are equivalent: a successive photoelectron may appear with the same probability $n\,\mathrm{d}t/T$ within any time interval $\mathrm{d}t$ inside T. The probability theory supposes that this model leads to a Poissonian statistics for the number of counts (see, for example, [48]):

$$p_m = \frac{n^m \exp(-n)}{m!} \,. \qquad (3.1)$$

Thus, even in the case of a light beam of constant amplitude E_0 (ideal laser) the photoelectrons are born at random points in time (chaotically), which results in the Poissonian shot noise. On average, n counts are observed during the time T. In our example $\langle m \rangle = n = 2.5$ photons and $m = 0, 1, 2, 3, 4, 5$ counts will be observed with probabilities $p_m = 0.08$; 0.20; 0.26; 0.21; 0.13; 0.07, respectively. Quantum fluctuations ('photon noise') are introduced into the semiclassical theory by assuming a stochastic character of energy transfer from the field to the detector (see, for example, [35]). Expression (3.1) provides a mapping $n \to p_m$; it expresses a mathematical procedure of 'discretization' and 'stochastization' when a random variable m taking on discrete values $m = 0, 1, 2, \ldots$ with probabilities p_m is put into correspondence to a determined variable n.

It seems obvious that all other, nonlasing sources of light may cause only additional, 'excessive' noises of the photocurrent connected with instability of their intensities n. Excessive fluctuations may be described by assuming the (solely) parameter of the Poissonian distribution n to be a random variable. In classical stochastic electrodynamics, the probability that the intensity of light incident on the detector (expressed in the appropriate units) takes on some value in the interval $(n, n + \mathrm{d}n)$ is $P(n)\,\mathrm{d}n$. The distribution density $P(n)$ must satisfy the axioms of the probability theory: the condition of normalization $\int_0^\infty P(n)\,\mathrm{d}n = 1$ and nonnegativity $P(n) \geqslant 0$. By averaging additionally the Poissonian distribution over the distribution $P(n)$ for the variable n, we arrive at the well-known Mandel's formula [40]

$$p_m = \frac{1}{m!}\int_0^\infty n^m \exp(-n)\,P(n)\,\mathrm{d}n,$$

$$P(n) \geqslant 0, \qquad \int_0^\infty P(n)\,\mathrm{d}n = 1. \tag{3.2}$$

Thus, the distribution p_m of the observable discrete random variable m is connected with the assumed intensity distribution of the incident light $P(n)$ through the 'Poisson transformation'. Expression (3.2) contains a double stochasticity: due to the noises of 'discretization' and due to the light intensity fluctuations. The second component is given the title excessive noise.

Notice that according to (3.2) the quantities $a_m \equiv m!p_m/p_0$ may be considered as moments of the modified distribution $\tilde{P}(n) \equiv P(n)\exp(-n)/p_0$ (the factor $1/p_0$ is a normalizing one) [39].

3.3. Factorial moments

As already mentioned, the results of the experiment on the photons counting may be presented as a set of probabilities of the count distribution p_m ($m = 0, 1, 2, \ldots$) or as moments of this distribution $\langle m^k\rangle$ ($k = 1, 2, \ldots$). These two sets are related by expression (2.1). It is convenient to introduce also the factorial moments of the distribution p_m:

$$G_k \equiv \big\langle m(m-1)\ldots(m-k+1)\big\rangle$$
$$= \sum_{m=k}^\infty \frac{m!}{(m-k)!}\,p_m = \sum_{m=0}^\infty \frac{(m+k)!}{m!}\,p_{m+k}\,. \tag{3.3}$$

They make up linear combinations of ordinary moments $\langle m^k\rangle$, for example, $G_2 = \langle m^2\rangle - \langle m\rangle$. If ordinary moments $\langle m^i\rangle$ are known for $1 \leqslant i \leqslant k$, then one can calculate the factorial moments G_k as well, and vice versa.

It can be shown that the factorial moments of a discrete distribution of counts p_m determined from (3.2) coincide with ordinary moments of the continuous energy distribution $P(n)$:

$$G_k = \langle n^k\rangle \equiv \int n^k P(n)\,\mathrm{d}n\,. \tag{3.4}$$

Thus, owing to properties of the Poisson transformation (3.2), the calculation of ordinary moments $\langle n^k\rangle$ of the original light energy distribution $P(n)$ gives the factorial moments G_k of the count distribution p_m, whereas the calculation of moments of the modified distribution $P(n)\exp(-n)$, 'shortened' by the exponent, provides the probabilities of counts p_m. From the latter, one can form ordinary moments of counts $\langle m^k\rangle$ (see (2.1)).

In the quantum theory, for the case of an ideal detector $p_m = \rho_{nn}$, i.e. G_k represent also the factorial moments of a discrete photon distribution ρ_{nn} (here ρ is the density operator).

It is possible to express p_m inversely through G_k. By expanding the exponent in (3.2) in series, we find with regard to (3.4):

$$p_m = \frac{1}{m!}\sum_{k=0}^\infty \frac{(-1)^k}{k!}\,G_{m+k}\,. \tag{3.4a}$$

From the normalization conditions for p_m it follows

$$\sum_{k=0}^\infty\sum_{m=0}^\infty \frac{(-1)^k}{k!m!}\,G_{m+k} = 1\,. \tag{3.4b}$$

3.4. The amplitude distribution

When describing the field with the help of energy distribution $P(n)$ information on the phase of oscillation ϕ is ignored. The latter may be measured, in principle, using the homodyne detection, i.e. by forming a superposition of the field under study and the reference laser beam with a stable phase. For full statistical description of the mode one should use two-dimensional probability density $P_z(z)$, with $\int \mathrm{d}^2 z P_z(z) = 1$. Here $z = z' + \mathrm{i}z'' = |z|\exp(\mathrm{i}\phi)$ is the complex amplitude of oscillations, which is normalized so that

$$|z|^2 = n \equiv \frac{E_0^2 V_{\mathrm{coh}}}{8\pi\hbar\omega_0}\,;$$

integration is performed over an entire complex plane playing the role of the phase space for the oscillation mode, and $\mathrm{d}^2 z \equiv \mathrm{d}z'\,\mathrm{d}z'' = |z|\,\mathrm{d}|z|\,\mathrm{d}\phi$ is an element of this plane. Now (3.2) takes the form

$$p_m = \frac{1}{m!}\int \mathrm{d}^2 z P_z(z)|z|^{2m}\exp\!\big(-|z|^2\big)\,. \tag{3.5}$$

Let us express the energy distribution $P(n)$ through the complex amplitude distribution $P_z(z) \equiv P_z(z', z'')$. In the general case

$$P(n) = P_z\big(|z|\big)\,\frac{\mathrm{d}|z|}{\mathrm{d}n} = \frac{1}{2}\int_0^{2\pi} \mathrm{d}\phi P_z\big(\sqrt{n}\cos\phi,\ \sqrt{n}\sin\phi\big). \tag{3.6}$$

In the case of a stationary field P_z is independent of the phase and $P(n) = \pi P_z\big(\sqrt{n}\big)$.

Under superposition of two independent oscillations, the distribution of the total field is equal, in accordance with the rules of the probability theory, to the convolution of the original distributions, viz.

$$P_z(z) = \int \mathrm{d}^2 z_1 P_z^{(1)}(z - z_1)\,P_z^{(2)}(z_1)$$
$$= \int \mathrm{d}^2 z_1 P_z^{(1)}(z_1)\,P_z^{(2)}(z - z_1)\,. \tag{3.7}$$

Let the distribution $P^{(2)}(z) = \delta^{(2)}(z - z_0) = \delta(z' - z_0')\times \delta(z'' - z_0'')$ describe a laser light with a certain complex amplitude z_0, then $P_z(z) = P_z^{(1)}(z - z_0)$. Therefore, during the homodyne detection the initial distribution $P_z^{(1)}(z)$ is simply shifted in the z-plane without changing the form. Consequently, the status of the state does not alter: according to the definition I, the NCL remains nonclassical, and classical light — classical as well.

4. Observable features of the NCL

4.1. The Lee's measure

As shown above, the property of being nonclassical is invariant to homodyning with the help of a laser field described by a coherent state with the amplitude distribution $P_z(z) = \delta^{(2)}(z - z_0)$.

This is not the case when the light under study is being superimposed on a thermal field. The latter has an exponential intensity distribution and during the convolution with it, the higher is the thermal field amplitude, the stronger the original distribution is 'smoothed' out. Then singular and negative regions disappear and as a result the NCL may become classical (examples are given below). Thus, the superposition with a coherent field does not change the character of the original field in contrast to the superposition with a thermal field which 'damages' the nonclassicality.

A convenient measure for the nonclassicality was proposed by Lee [49, 50]. Let T be the mean number of photons in one mode of the auxiliary thermal field. The minimum value $T = T_0$ at which $P_z(z)$ remains a nonnegative regular function can be taken, in accordance with definition **I**, as a quantitative measure of the nonclassicality. Below we shall show that T_0 changes from 1 for the most nonclassical (K-photon) states to 0 for the classical ones. Thus, adding a thermal field with the mean number of photons in one mode equal to 1 makes always any light classical. In the case of a squeezed vacuum $T_0 = \sqrt{N} = \Gamma \ll 1$ at low squeezing, and $T_0 = 1/2$ at $\Gamma \gg 1$ (Γ is the amplification factor). The quantity T_0 can, in principle, be measured: for this, one should add to the light involved (using, for example, a beam-splitter (see Section 10)) a thermal beam with intensity that can be tuned. However, the question still remains as to how to determine experimentally the moment of the nonclassicality disappearance with increased T_0?

4.2. Operational determination of the nonclassicality

Let us return to the Mandel's formula (3.2). For a Poissonian distribution it is characteristic to have the variance equal to the mean value: $\langle \Delta m^2 \rangle = \langle m \rangle$. Thus the ideal laser light yields $g_2 = F = 1$. It might appear that all other light sources can only enhance the photocurrent noises due to instability of their intensities. Therefore it seems impossible to observe subpoissonian fluctuations with $g_2 < 1$ within the framework of semiclassical theory.

These qualitative considerations are confirmed by calculations. From (3.2) it follows that $\langle m \rangle = \langle n \rangle$ and $\langle \Delta m^2 \rangle = \langle m \rangle + \langle \Delta n^2 \rangle$. Since $P(n) \geqslant 0$ leads to $\langle \Delta n^2 \rangle \geqslant 0$, then $\langle \Delta m^2 \rangle \geqslant \langle m \rangle$, i.e. the conditions $g_2 \geqslant 1$, $F \geqslant 1$ should always hold. As this takes place

$$g_2 \equiv 1 + \frac{\langle m^2 \rangle - \langle m \rangle}{\langle m \rangle^2} = 1 + \frac{\langle \Delta n^2 \rangle}{\langle n \rangle^2} \, .$$

Thus the value $g_2 = 1$ is a lower boundary in the case of the semiclassical theory of photocounts: $g_{2\,\mathrm{clas}} \geqslant 1$ and the simplest observable feature of the NCL is the inequality $g_{2\,\exp} < 1$, i.e. the photocount antibunching. It will be shown later that in the general case for a light to be nonclassical it is sufficient to fulfill at least one of the infinite set of conditions of the form $D_k < 1$, $k = 1, 2, \ldots$ ($D_1 \equiv g_2$). But even if none of these conditions holds, the light can be nonclassical all the same (see Appendix I).

Notice that in the quantum theory some bounds on the moments also exist (see Appendix I). For example, from $\langle \Delta m^2 \rangle \geqslant 0$ follows $g_2 \geqslant 1 - 1/\langle m \rangle$. The equality here is reached in the case of the states with a definite number of photons.

It is natural to accept the following operational definition of the NCL: *if the statistics of photocounts observed does not agree with the Mandel's semiclassical formula (3.2) at $P(n) \geqslant 0$, i.e. the light falling on the detector cannot be described by some energy distribution $P(n)$, then the light is referred to as nonclassical* (definition II).

In particular, if a light produces the photon antibunching, then it is nonclassical (for instance, if $G_1 = \langle m \rangle = 10$ and $G_2 = \langle m^2 \rangle - \langle m \rangle = 99$).

One may test the light nonclassicality by studying the probabilities of counts p_m. Within the framework of quantum theory, in principle, no restrictions may be imposed on the sets of numbers $\{p_m\}$ observed (apart from, of course, the normalization condition $\Sigma p_m = 1$). Our idealized detector provides us directly with information on the diagonal components of the density matrix ρ for a one-mode field in the Fock representation: $p_m = p_n = \rho_{nn}$. Consequently, if one uses corpuscular concepts and determines the counts' statistics using discrete distribution of the field energy, then any sets of numbers measured $\{p_m\}$ are admissible.

On the other hand, semiclassical formula (3.2), i.e. the Poisson transformation, constrains in a certain manner the sets observed, which permits us, on experimental grounds, to split a possible set of statistical states of the field into two classes, i.e. permits distinction between classical and nonclassical light to be drawn. For example, in Section 4.4 we shall show that the numbers $p_0 = 0.23$, $p_1 = 0.35$, and $p_2 = 0.22$ are inconsistent with (3.2) if $P(n) \geqslant 0$.

In the general case, the problem of applying the definition II to experimental results arises. A straightforward test on being nonclassical includes obviously the attempt to reverse the Mandel's formula (3.2), i.e. determination of the function $P(n)$ through the set of numbers measured $\{p_m\}$ or $\{G_k\}$, and, if it is successful, the subsequent testing of its nonnegativity, $P(n) \geqslant 0$.

4.3. The problem of moments

Different ways of reversing transformation (3.2) were described, for example, in [39, 48]. Generally, this procedure is nontrivial and not uniquely defined, being connected with the well-known mathematical problem of moments (see, for instance, [51 – 54]). Then one has to operate with generalized functions similar to derivatives of the δ-function.

In our case, the problem may be posed as follows. The moments G_k of a nonnegative function $P(x) \geqslant 0$ are known (we change n by x for a while). By definition, $G_k \equiv \int_{-\infty}^{\infty} x^k P(x)\,\mathrm{d}x > 0$ (we assume $P(x) \propto \theta(x)$, where $\theta(x)$ is a step function). Let us go over to the Fourier-representation

$$\tilde{P}(\omega) \equiv \int_{-\infty}^{\infty} P(x) \exp(\mathrm{i}\omega x)\,\mathrm{d}x = \int_0^{\infty} P(x) \exp(\mathrm{i}\omega x)\,\mathrm{d}x \, . \tag{4.1}$$

This function is called the characteristic function. If $\mathrm{i}\omega$ is changed by $-s$ ($s \geqslant 0$), i.e. one uses the Laplace transformation, then we arrive at the generating function

$$C(s) \equiv \int_0^\infty P(x)\exp(-sx)\,\mathrm{d}x = \tilde{P}(\mathrm{i}s)\,. \tag{4.2}$$

Notice that $\tilde{P}(0) = C(0) = 1$ and

$$\frac{\mathrm{d}}{\mathrm{d}s}\,C(s) \equiv -\int_0^\infty xP(x)\exp(-sx)\,\mathrm{d}x < 0\,.$$

Generally, all even derivatives of $C(x)$ are positive while odd ones are negative, and $C(s)$ is a continuously decreasing function from 1 to 0 (such functions are referred to as absolutely monotonic [54]). Violation of these properties can, in principle, serve as one of the features of the NCL.

Differentiation of functions $\tilde{P}(\omega)$ and $C(s)$ allows us to find the moments G_k and probabilities p_m. Expanding exponent in series in the vicinity of zero

$$\tilde{P}(\omega) \equiv \sum_{k=0}^{\infty}\frac{(\mathrm{i}\omega)^k G_k}{2\pi k!}\,, \qquad C(s) \equiv \sum_{k=0}^{\infty}\frac{(-s)^k G_k}{k!} \tag{4.3}$$

shows that the moments G_k of the function $P(x)$ determine coefficients of the power series for its Fourier- and Laplace-images. The reverse Fourier-transformation of $\tilde{P}(\omega)$ yields the formal representation of $P(x)$ as a sum of derivatives of the δ-function

$$\begin{aligned} P(x) &= \int_{-\infty}^{\infty}\tilde{P}(\omega)\exp(-\mathrm{i}\omega x)\,\frac{\mathrm{d}\omega}{2\pi} \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\,G_k\int_{-\infty}^{\infty}(\mathrm{i}\omega)^k\exp(-\mathrm{i}\omega x)\,\frac{\mathrm{d}\omega}{2\pi} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\,G_k\delta^{(k)}(x)\,, \end{aligned} \tag{4.4}$$

here $\delta^{(k)}$ is the k-th order derivative of the δ-function. Assuming $x = n$, we obtain a formal solution to the problem of recovering the energy distribution with factorial moments of photocounts. Let, for example, only one moment be non-zero: $G_k = \delta_{kK}$, then $P(n) = \delta^{(K)}(n)/K!$. In other words, the (generalized) function $\delta^{(K)}(x)$ possesses only one moment. Of course, such a distribution is unacceptable in the classical statistical optics.

On the other hand, as already pointed out, according to the Mandel's formula the probabilities p_m, multiplied by $m!$ and divided by p_0, may be considered as moments of a modified distribution

$$\tilde{P}(n) \equiv P(n)\exp(-n)\frac{1}{p_0}\,,$$

(factor $1/p_0$ serves as a normalizing one)

$$a_m \equiv m!\,\frac{p_m}{p_0} = \int_0^\infty n^m\tilde{P}(n)\,\mathrm{d}n\,. \tag{4.5}$$

Therefore, it is possible to recover $P(n)$ (or $C(s)$) using probabilities of photocounts as well (we remind that the probabilities p_k multiplied by $(-1)^k$ are the coefficients of the $C(s)$ expansion in series near the point $s = 1$). Indeed, the change in (4.4) $P(n) \to p_0\tilde{P}(n)\exp n$, and $G_k \to k!p_k/p_0$ yields

$$P(x) = \exp x\sum_{k=0}^{\infty}(-1)^k p_k\,\delta^{(k)}(x)\,. \tag{4.6}$$

Consider now a pure quantum state $|K\rangle$ with a definite number of photons K. Such a state, in accordance with quantum theory, produces the counts' distribution measured by an ideal detector in the form $p_m = \delta_{mK}$; these counts then do not fluctuate, $m = K$ at all times. According to (4.6) this distribution satisfies (3.2) at

$$P(n) = (-1)^K\delta^{(K)}(n)\exp n\,.$$

Clearly, this singular function cannot describe classical distribution of light intensity. This is obvious when using the representation of the δ-function in the form of the limit of some regular function: derivatives of this function take negative values. Therefore, the K-photon state corresponds to the NCL.

Under specific calculations it is more convenient to evaluate firstly the function $\tilde{P}(\omega)$ (or $C(s)$), and then to find its Fourier- (or Laplace-) image $P(x)$. As an example, let us substitute factorial moments $G_k = k!N^K$ of thermal field into (4.4), then

$$P(x) = \sum_{k=0}^{\infty}(-N)^k\,\delta^{(k)}(x)\,,$$

where N is the mean number of photons. In order to make sure that this series represents a regular function, let us take the Fourier transform according to (4.1):

$$\begin{aligned} \tilde{P}(\omega) &= \sum_{k=0}^{\infty}(-N)^k\int_{-\infty}^{\infty}\mathrm{d}x\exp(\mathrm{i}\omega x)\,\delta^{(k)}(x) \\ &= \sum_{k=0}^{\infty}(\mathrm{i}\omega N)^k = (1-\mathrm{i}\omega N)^{-1}\,. \end{aligned}$$

Here we used the definition of the $\delta^{(k)}$-function according to which the integral is equal to k-th derivative of the function $\exp(\mathrm{i}\omega x)$ at $x = 0$ multiplied by $(-1)^k$, i.e. $(-\mathrm{i}\omega)^k$. The reverse Fourier transformation yields a regular distribution

$$P(x) = \frac{\exp(-x/N)}{N}\,, \qquad x > 0\,.$$

Notice that it is sometimes more convenient to search for $P(x)$ in the form of a series expansion in Laguerre polynomials [39, 48].

In the experiments, we always deal with a limited set of numbers $a_1, \ldots, a_K$, using which it is impossible to fully recover the function $C(x)$ or $P(n)$. Additional difficulties arise from a limited accuracy of statistical data measurements. Nevertheless, successful experiments on the reconstruction of the Wigner's quasi-distribution using a photocurrent statistics have been recently reported [55].

4.4. Relations between the moments

Consider an alternative approach to the problem of experimental discovery of the nonclassicality. The experiment provides us with an array of count numbers $\{m_i\}$. From these numbers we can construct sets of probabilities $\{p_k\}$, factorial moments $\{G_k\}$ or, in the general case, sets of generalized moments $\{a_k(s)\}$ at some values of the parameter s; for example, $a_k(0) = G_k$ and $a_k(1) = k!p_k/p_0$ (see Appendix II). For a fixed s we obtain an ordered set of real numbers $\{a_k\}$. According to classical theory of photocounts, these numbers present moments of some nonnegative function $P(x)\exp(-sx) \geqslant 0$, i.e. $a_k = \int x^k P(x)\exp(-sx)\,\mathrm{d}x$, and, consequently, they must be restricted with a certain number of inequalities (see Appendix I and [51 – 59]).

For example, according to (I.10) for $1 \leqslant m \leqslant n$ it turns out that

$$a_m a_n \leqslant a_{m-1} a_{n+1}\,, \qquad D_m \equiv \frac{a_{m-1} a_{m+1}}{a_m^2} \geqslant 1\,. \tag{4.7}$$

We emphasize that (4.7) make up necessary but not sufficient conditions for a classicality. The inverse inequalities yield sufficient but not necessary features of a nonclassicality.

Inequalities (4.7) have a simple geometrical significance. Let us denote $b_m \equiv \ln a_m$. Then $\Delta_{m-1} \equiv b_m - b_{m-1}$ is a change in the function (of discrete argument) b_m on shifting $m-1 \to m$. Now (4.7) takes the form $\Delta_{m-1} \leqslant \Delta_n$, i.e. change at the point $m-1$ is smaller than or equal to that at the point $n \geqslant m$. This implies that the function $b_m \equiv \ln a_m$ in the case of a classical light is concave everywhere, while in the case of the NCL it possesses convexities. The condition $D_m < 1$ implies a local 'nonclassical' convexity on the plot of the function $b_m \equiv \ln a_m$ at the point m.

Substituting $a_k = k! p_k / p_0$ or $a_k = g_k \equiv G_k / G_1^k$ ($g_0 = g_1 = 1$) into (4.7) we arrive at two infinite series of the sufficient conditions for a nonclassicality:

$$D_k(1) = \frac{(k+1)\, p_{k-1}\, p_{k+1}}{k p_k^2} < 1\,,$$

$$D_k(0) = \frac{g_{k-1}\, g_{k+1}}{g_k^2} < 1\,. \tag{4.8}$$

We shall call these conditions the D_k-criteria. In particular, the condition $D_1(0) = g_1 < 1$ coincides with the most known criterion of nonclassicality — antibunching of photocounts. The condition $D_k(0) < 1$ for $k \geqslant 2$ is sometimes labelled as higher-order antibunching. These quantities may also serve as quantitative measures of the degree of nonclassicality: $D_k = 0$ corresponds to the maximum nonclassicality, and $D_k = 1$ — to the minimum one (however one should remember of the dependence of these quantities upon the choice of the parameter s).

Let us consider some examples of using the D_k-criteria (4.8).

1. An ideal laser light produces the Poisson distribution (3.1) with the parameter $\langle n \rangle = N = |z_0|^2$. Here $a_k = G_k = N^k$, so that $D_k(0) = D_k(1) = 1$. The plot of the function $b_k = \ln a_k = k \ln N$ is a straight line. Therefore, this distribution lies on the boundary between the classical and nonclassical case.

2. A thermal distribution gives $G_k = k! N^k$, $a_k = k! \left[N/(N+1) \right]^k$, so that $D_k(0) = D_k(1) = (k+1) \times 1/k > 1$; plots of the functions $\ln G_k$ and $\ln (p_k / k!)$ are concave.

3. Let the probability for some number of counts k tend to zero: $p_{k-1} \neq 0$, $p_k = 0$ and $p_{k+1} \neq 0$, then $D_{k-1}(1) = D_{k+1}(1) = 0$ and the plot of $\ln (p_k / k!)$ shows two neighbouring 'nonclassical' concavities. Similarly, if $p_{k-1} = 0$, $p_k \neq 0$ and $p_{k+1} = 0$, then $D_k(1) = 0$ and there appears a 'nonclassical' convexity. Any state with a 'truncated' probability distribution of counts, when all probabilities are zero beginning from a certain number, is nonclassical as well.

D_k-criteria considered above are very particular (although the property of being subpoissonian is important for applications). Experimental testing of nonclassicality for a certain mode of light field, strictly speaking, must include measurements of an infinite set of moments or probabilities and testing an infinite set of inequalities composed from them. General necessary and sufficient features of classicality are presented in Appendix I. They use Hankel's matrices H_K and $H_{K'}$ composed from moments $a_0 = 1$, $a_1, \ldots, a_{2K}$. These matrices in the classical case must be positively determined. In practice, of course, a maximal order of measured moments G_k or probabilities p_k is limited (as well as their measurement accuracy), so that the experimenter has in his disposal the matrix H_K of a certain order, and the testing on being nonclassical reduces to the determination of the sign of $\det H_K$ or $K+1$ eigenvalues $\lambda_0, \ldots, \lambda_K$ of this matrix (one may also use the Sylvester's criterion — the positivity of all the angular minors of H_K [60]). If some of these quantities are negative, the light is nonclassical. If all are positive, we should pass to the next-order matrix. One may, in addition, measure $D_k(s)$ at some optimal value of s depending on the state of the light studied.

4.5. Squeezing of quadratures

Let us define two quadratures of the field amplitude in a mode:

$$q \equiv \sqrt{2}\,\mathrm{Re}\,(z), \qquad p \equiv \sqrt{2}\,\mathrm{Im}\,(z)\,. \tag{4.9}$$

A feature of the NCL is often considered as a 'squeezing' of fluctuations of one of the quadratures below the 'vacuum' value, $\langle \Delta q^2 \rangle < 1/2$ (see (6.4)). Quadratures of some weak field can be studied by adding to it the strong coherent field with an adjustable phase (homodyne detection, cf. [35]), i.e. they can be considered as observable quantities. Then fluctuations of the photocurrent induced by the superpositional field are proportional to fluctuations of a certain (depending on the homodyne phase) quadrature of the weak field, i.e. the observed current distribution is coincident with the distribution $P(q)$ for one of the quadratures (disordered, see [35]). The observed features of nonclassicality (antibunching, etc.) prove to be connected with the statistics of the quadrature. Since the Fano factor for the current is equal to twice the variance of a specific quadrature of the weak field, $F = 2\langle \Delta q^2 \rangle$ (see (6.21)), then the conditions of the quadrature squeezing and the count antibunching ($F < 1$) are the same. However, the squeezing condition by itself $\langle \Delta q^2 \rangle < 1/2$ does not contradict classical theory, in which there are no restrictions on $P(q, p)$ and $\langle \Delta q^2 \rangle$. Only within the framework of quantum theory this condition is connected with the irregularity of the P_z-distribution (see Section 6).

5. Quantum theory of photocounts

Quantum theory of photodetection is presented, for example, in [39, 41, 47], so here we restrict ourselves to giving a short list of necessary results. The Dirac notations in use are explained in detail in [47, 61].

5.1. Discrete representation

In quantum theory, statistics of photocounts (as well as all other observable quantities) is determined by the density operator ρ for a free field which falls on the detector.

When using the discrete Fock's basis, the diagonal matrix element in the case of our ideal detector immediately gives us the probability of observation of n counts

$$p_n = \rho_{nn} \equiv \langle n | \rho | n \rangle\,. \tag{5.1}$$

In the case of a pure state $\rho = |\psi\rangle\langle\psi|$, so that $p_n = \left|\langle n|\psi\rangle\right|^2$.

The (ordinary) moments are defined as

$$\langle n^k\rangle = \left\langle (a^\dagger a)^k \right\rangle = \mathrm{tr}\,(\rho n^k) = \sum_n \rho_{nn} n^k\,. \quad (5.2)$$

Normal (normally-ordered) moments of the general form are calculated in the discrete representation as follows:

$$G_{mn} \equiv \left\langle (a^\dagger)^m a^n \right\rangle = \sum_{k=0}^{\infty} \frac{\sqrt{(m+k)!(n+k)!}}{k!}\,\rho_{m+k,\,n+k}\,. \quad (5.3)$$

In the stationary case $\rho_{mn} = \rho_{nn}\delta_{mn}$, so that $G_{mn} = G_{mm}\delta \equiv G_m\delta_{mn}$. When $m = n$, normal moments are coincident with factorial ones:

$$G_m \equiv G_{mm} = \left\langle :n^k: \right\rangle = \left\langle n(n-1)\ldots(n-m+1)\right\rangle = \sum_{k=0}^{\infty} \frac{(m+k)!}{k!}\,\rho_{m+k,\,m+k}\,. \quad (5.4)$$

Here $n \equiv a^\dagger a$ and colons denote the operation of permutation of operators $a^\dagger$ to the left of operators a (under the permutation one does not need to take into account non-commutativity, for instance, $:n^2:\equiv (a^\dagger)^2 a^2 = n^2 - n$). Hence, when identifying $p_m = \rho_{nn}$ we obtain classical formula (3.3) for factorial moments. The generating function for counts may also be expressed via ρ :

$$C(s) \equiv \left\langle (1-s)^{\hat{n}} \right\rangle = \sum_{n=0}^{\infty} (1-s)^n \rho_{nn} = \left\langle :\exp(-s\hat{n}): \right\rangle. \quad (5.5)$$

Derivatives of $C(s)$ at the points 0 and 1 determine correspondingly the factorial moments G_k and probabilities of counts p_m.

5.2. Continuous representation

What is in common between quantum expression (5.1) and classical Mandel's formula (3.5) for probabilities of photocounts? It turns out that the relationship $p_n = \rho_{nn}$ may be transformed to the form like (3.5). For this one should use a continuous representation of vectors and operators by coherent states $|z\rangle$ [39, 41 – 43, 47]. Then the density operator is being mapped by some function $P_z(z)$ called the Glauber-Sudarshan representation [41 – 43]. This function determines the distribution of counts via the Poisson transformation which is coincident in form with the classical Mandel's formula (3.2) or (3.5). The only difference is that the function $P_z(z)$ or $P(n) \equiv \pi P_z(\sqrt{n})$ is now defined via the density operator and thus can be negative and irregular (therefore it is called the 'quasi-distribution'). With the use of $P_z(z)$ it is convenient to calculate normal operators, for example,

$$p_m = \frac{\left\langle :\hat{n}^m \exp(-\hat{n}): \right\rangle}{m!} = \frac{1}{m!}\int \mathrm{d}^2 z P_z(z)|z|^{2m} \exp\left(-|z|^2\right),$$

$$G_k = \left\langle :\hat{n}^k: \right\rangle = \int \mathrm{d}^2 z P_z(z)|z|^{2k}\,,$$

$$C(s) = \left\langle :\exp(-s\hat{n}): \right\rangle = \int \mathrm{d}^2 z P_z(z)\exp(-s|z|^2)\,. \quad (5.6)$$

Fourier-image of the function $P_z(z)$ is designated the normal (normally-ordered) characteristic function

$$\chi(w) = \left\langle \exp(wa^\dagger)\exp(-w^* a)\right\rangle = \left\langle :\exp(wa^\dagger - w^* a): \right\rangle = \int \mathrm{d}^2 z P_z(z)\exp(wz^* - w^* z)\,. \quad (5.7)$$

The derivatives of this function at a zero point determine the normal moments of the general form

$$G_{mn} = (-1)^n \frac{\partial^m}{\partial w^m}\frac{\partial^n}{\partial w^{*n}}\,\chi(w,\, w^*)\Big|_{w=w^*=0} \quad (5.8)$$

(the arguments w and w^* are considered as independent).

5.3. Smoothed-out functions

In the case of the NCL, the function $P_z(z)$ may refer to the class of generalized functions. In order not to deal with generalized functions, it is worthwhile to use 'smoothed-out', regular modifications of the function $P_z(z)$. For this we define a new characteristic function as follows:

$$\chi(w, T) \equiv \exp(-Tww^*)\,\chi(w)\,, \qquad \chi(w) \equiv \chi(w, 0)\,. \quad (5.9)$$

We also define the Fourier-image of this function:

$$\tilde{\chi}(z,\, T) = \pi^{-2}\int \mathrm{d}^2 w \exp(zw^* - z^* w)\,\chi(w, T) = \pi^{-2}\int \mathrm{d}w'\,\mathrm{d}w''\exp\left[2\mathrm{i}(z''w' - z'w'') - T(w'^2 + w''^2)\right]\chi(w', w'')\,. \quad (5.10)$$

The parameter $T \geqslant 0$ plays a double part: firstly, it limits the characteristic function at infinity, which leads to the possibility of its Fourier transformation; secondly, $\chi(w, T)$ describes the superposition between the original field specified by the function $\chi(w, 0) \equiv \chi(w)$, and an independent thermal field with the mean number of photons T. (We recall that under the composition of two independent random processes their characteristic functions are multiplied.)

Let us introduce the following notations:

$$\chi(w, 0) = \chi(w), \qquad \tilde{\chi}(z, 0) = P_z(z)\,,$$

$$\chi\left(w, \frac{1}{2}\right) = \chi_{\mathrm{sym}}(w)\,, \qquad \tilde{\chi}\left(z, \frac{1}{2}\right) = W(z)\,,$$

$$\chi(w, 1) = \chi_{\mathrm{A}}(w)\,, \qquad \tilde{\chi}(z, 1) = Q(z)\,. \quad (5.11)$$

At $T = 0$, the function $\chi(w, T)$ is a normal characteristic function, its derivatives at a zero point determine the normal moments of the form $G_{kl} \equiv \left\langle (a^\dagger)^k a^l \right\rangle$ (see (5.3)). The Fourier transform of $\chi(w, 0)$ yields the Glauber–Sudarshan representation $P_z(z)$ for the density operator. The function $P_z(z)$ plays the part of (quasi-)distribution for normal operators.

When $T = 1/2$, the function $\chi(w, T)$ possesses a symmetrized characteristic function. Its Fourier-image $W(z)$ is called the Wigner function. The latter is always regular (but may takes negative values), it plays the part of the quasi-distribution for operators symmetrical with respect to the permutation of a and $a^\dagger$ operators.

Finally, at $T = 1$ the function $\chi(w, T)$ determines an antinormal characteristic function; its Fourier-image $Q(z) = \left\langle z|\rho|z\right\rangle$ is always nonnegative and plays the part of the distribution for antinormal operators like $\left\langle a^l (a^\dagger)^k \right\rangle$.

The product of the characteristic functions corresponds to the convolution of their Fourier-images. Therefore, the Wigner function $W(z)$ is equal to the convolution of $P_z(z)$ with the function $\exp(-|z|^2/2)$, whereas the function $Q(z)$ does that of $P_z(z)$ with the function $\exp(-|z|^2)$. These mathematical operations, which smooth-out peculiarities of $P_z(z)$, can be realized by adding a thermal field with $T = 1/2$ or $T = 1$ [49, 50].

5.4. The superposition with a thermal field

In what follows we assume T to be a mean number of photons in the additional thermal field, which is used as a measure of nonclassicality of the original field according to Lee [49, 50]. Then the initial state behaves as if it is 'heated'. For example, the Wigner function of a total field has the form $W(z,T) = \tilde{\chi}(z, T+1/2)$ and is determined from (5.11). With the use of the Wigner function the distribution of counts and generating function are determined according to the following formulas [62, 63]:

$$p_m = \frac{2(-1)^m}{m!}\int d^2z W(z,T)\, L_m(4|z|^2)\exp(-2|z|^2)\,, \quad (5.12)$$

$$C(s) = \frac{2}{2-s}\int d^2z W(z,T)\exp\left(-\frac{2s|z|^2}{2-s}\right). \quad (5.13)$$

Here $L_K(x)$ are the Laguerre polynomials,

$$L_K(x) = K!\sum_{m=0}^{K}\frac{(-x)^m}{(m!)^2(K-m)!}\,,$$
$$L_1(x) = 1-x\,, \qquad L_2(x) = 1-2x+\frac{x^2}{2}\,. \quad (5.14)$$

In contrast with (5.6), the Wigner function is used here which is regular for the NCL.

In the case of a stationary field, $\rho_{nm} = \rho_{nn}\delta_{nm}$ and the relationship takes place:

$$\chi(w,T) \equiv \exp(-Tww^*)\sum_n \rho_{nn}L_n(|w|^2)\,. \quad (5.15)$$

5.5. Examples

1. The field of an ideal laser is described by the coherent state

$$P_z(z) = \delta^{(2)}(z-z_0) = \delta(z'-z_0')\,\delta(z''-z_0'')\,. \quad (5.16)$$

The Fourier-transform of this function yields

$$\chi(w) \equiv \exp(wz_0^* - w^*z_0)\,. \quad (5.17)$$

Here the photocounts form a Poissonian process: $p_m = n_0^m\exp(-n_0)/m!$ with the parameter $n_0 = |z_0|^2$, for which

$$C(s) = \sum(1-s)^n p^n = \exp(-sn_0), \qquad G_k = n_0^k\,. \quad (5.18)$$

2. One mode of the field or a harmonic oscillator in the state of thermal equilibrium are described by a Gaussian distribution for the amplitude and an exponential one for the energy:

$$P(n) = \pi P_z(\sqrt{n}) = \frac{1}{T}\exp\left(-\frac{n}{T}\right). \quad (5.19)$$

Here T is the mean energy divided by the energy of photon. Two-dimensional Fourier-image of the function $P_z(z)$ takes the form

$$\chi(w) = \exp(-Tww^*)\,. \quad (5.20)$$

The density matrix therewith is diagonal

$$\rho_{mm} = p_m = \frac{T^m}{(1+T)^{m+1}} = p_0\exp(-m\beta),$$
$$\exp(-\beta) \equiv \frac{T}{1+T}\,. \quad (5.21)$$

Hence it is easy to find

$$C(s) = \sum(1-s)^n\frac{T^n}{(1+T)^{n+1}} = \frac{1}{1+sT}\,,$$
$$G_k = k!T^k\,. \quad (5.22)$$

3. The superposition of the coherent and thermal fields is obtained from (5.19) by translating over vector $\mathbf{z}_0$ (in accordance with (3.7) and (5.16)):

$$P_z(z) = \frac{1}{\pi T}\exp\left(-\frac{|z-z_0|^2}{T}\right). \quad (5.23)$$

The characteristic function equals the product (5.17) and (5.20)

$$\chi(w) = \exp(-Tww^* + wz_0^* - w^*z_0)\,. \quad (5.24)$$

4. The Scully–Lamb distribution [64]

$$p_m = c\,\frac{a^m}{(b+m)!} \approx \frac{a^{b+m}\exp(-a)}{(b+m)!}\,,$$
$$c^{-1} = \sum_{m=0}^{\infty}\frac{a^m}{(b+m)!} \approx a^{-b}e^a \quad (5.25)$$

describes the statistics of laser radiation below the excitation threshold ($a < b$), close to the threshold ($a = b$), and above it ($a > b$). The normalization factor c determines the probability of the count absence: $p_0 = c/b!$. Approximate expressions are valid under a significant excess above the threshold, when $a \gg b$ and $c \ll 1$. The mean number of photons has the form $N = a - b(1-p_0) \approx a - b$. In the limit $b \to \infty$ this is a thermal distribution, while at $b \to 0$ — a Poissonian one. The plots of the function $\ln(m!p_m)$ are concave, i.e. they show a 'classical' character. The generating function takes the form

$$C(s) = c\sum_{m=0}^{\infty}\frac{a^m}{(b+m)!}(1-s)^m \approx \frac{\exp(-as)}{(1-s)^b} \quad (5.26)$$

(the approximate formula is valid at $s \ll 1$).

Other examples will be considered in Sections 6 and 7. In Fig. 1 we present a scheme explaining the relations between the introduced above functions allowing the determination of the photocount statistics.

6. Heated squeezed vacuum

6.1. Light with an even number of photons

Whether the antibunching condition $g_2 < 1$ (g_2-criterion) exhausts all the cases of the light nonclassicality, i.e. whether is it sufficient? Consider as a simple counterexample the noise emission of a degenerate parametric amplifier. This effect is referred to as the parametric scattering or frequency down-conversion [4 – 6, 17]; the corresponding state of the field is called the squeezed vacuum. According to quantum theory $\langle m\rangle = \sinh^2\Gamma$ and $\langle\Delta m^2\rangle = 2\langle m\rangle(\langle m\rangle + 1)$ (Γ is the parametric amplification factor). Hence $D_1(0) = g_2 = 3 + \langle m\rangle^{-1} > 3$, i.e. a superpoissonian statistics of counts takes place and according to the g_2-criterion the light is classical. (Moreover, in the experiments g_2 is usually of order 10^8, i.e. a 'superbunching' occurs.)

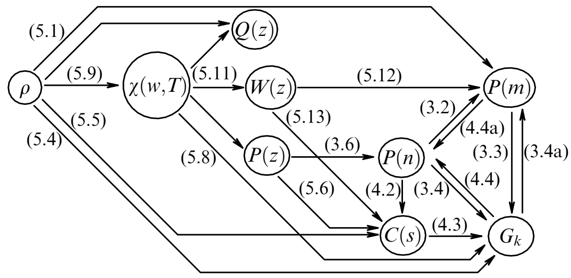

Figure 1. Relations between different functions used to calculate photocount statistics; formula numbers are shownin parentheses. The initial theoretical quantity is the density operator ρ, measured quantities involve count probabilities p_m and factorial moments G_k; $\chi(w,\ T)$ is the characteristic function, $W(z)$ is the Wigner function, $P(z)$ is the Glauber–Sudarshan amplitude distribution, $P(n)$ is the energy distribution, $G(s)$ is its Laplace-image (generating function), the function $Q(z) = \langle z|\rho|z\rangle$ defines antinormally-ordered moments. For the nonclassical state the function $W(z)$ takes negative values, and the function $P(z)$ also becomes negative or does not exist at all as a regular function.

Why the squeezed vacuum is related to the category of NCL? Theoretically this follows from definition I: we shall show below that no regular P_z-distribution exists [66] in the squeezed vacuum (excluding generalized functions from the consideration). It is natural to try to find operational, observable features of the nonclassicality for the squeezed vacuum.

The density matrix calculation for the squeezed vacuum yields $\rho_{nn} = 0$ for odd n, i.e. from the corpuscular point of view the squeezed vacuum consists of an even number of photons. In the ideal case, odd numbers of photocounts should not be observed at all, $p_{2k+1} = 0$. Intuition suggests that such sharp 'dips' in the probability distribution contradict the Mandel's formula (3.2), according to which the 'adjacent' probabilities p_{m-1}, p_m, and p_{m+1} must apparently have comparable values. This is confirmed by condition (4.8): for even m we have $D_m(1) = p_{m-1}p_{m+1}/p_m^2 = 0$, so that the intrinsic noise of the parametric amplifier is nonclassical despite the absence of antibunching. (In addition, we shall verify that at small Γ the inequality $D_{2n}(0) < 1$ is satisfied.)

In the limiting case of weak pumping ($\Gamma \ll 1$) the parametric amplifier emits two-photon light. Then $p_2 \ll p_0$ and other probabilities are negligibly small. Therefore, the photons are either absent or emitted in pairs. Consequently, $D_2(1) = 0$ and the two-photon light (which is the limiting case of a weakly-squeezed vacuum) is essentially nonclassical.

Notice that the parametric two-photon light in the non-degenerate case is used to demonstrate another, more general nonclassicality of light. This type of nonclassicality is based upon opposing not to a classical stochastic optics, but to some very general probabilistic model of Bell [67]. The Bell's inequalities, which are composed of some combination of the counts observed in two detectors therewith break down, [11, 68, see also 26, 34, 69].

6.2. *P*-distribution

As already noted, the squeezed vacuum produces bunching of photons: $g_2 = 3 + 1/N$ ($N \equiv \langle m\rangle$). For $N \ll 1$ this is a super-bunching, $g_2 \gg 1$. To obtain the antibunching ($g_2 < 1$), it is necessary to add a coherent component z_0 with an appropriate phase to the squeezed vacuum. Further, to estimate the degree ('depth') of nonclassicality T_0, one should also add a thermal field with a known mean number of photons T. According to Lee [49, 50], T_0 is a minimum value of T destroying the nonclassicality (after definition I). One may try to evaluate T_0 experimentally, for example, using the condition $g_2(T_0) = 1$. Thus, the coherent field z_0 creates the antibunching, whereas the thermal field T destroys it.

The statistics of the superposition may be conveniently determined through characteristic functions of initial fields, which are then simply multiplied with each other. The characteristic function of the squeezed vacuum can be found on the basis of a quantum model for the parametric amplifier [17, 47, 61, 66]:

$$\chi(w, w^*) = \exp\left[-Nww^* + \frac{1}{2} M(w^2 + w^{*2})\right]. \quad (6.1)$$

Here

$$N = \langle a^\dagger a\rangle = \sinh^2(\Gamma), \quad (6.2)$$

$$M = \langle a^2\rangle = \sqrt{N(N+1)} = \sinh(\Gamma)\cosh(\Gamma) \quad (6.3)$$

are the mean photon number and 'anomalous' second moment, respectively. The parameter M is assumed to be real and positive, which corresponds to the extension of the first quadrature and squeezing of the secondary quadrature:

$$\langle \Delta q^2\rangle = N + M + \frac{1}{2} = \frac{1}{2}\exp(2\Gamma),$$

$$\langle \Delta p^2\rangle = N - M + \frac{1}{2} = \frac{1}{2}\exp(-2\Gamma). \quad (6.4)$$

Here the quadratures are defined as follows:

$$q \equiv \frac{(a + a^\dagger)}{\sqrt{2}}, \quad p \equiv \frac{(a - a^\dagger)}{\mathrm{i}\sqrt{2}}.$$

Under superposition of the squeezed vacuum with a coherent field of the amplitude z_0 and a thermal field with the mean number of photons T, function (6.1) is multiplied by the corresponding characteristic functions, which yields

$$\chi(w, w^*) = \exp\left[-N'ww^* + \frac{1}{2} M(w^2 + w^{*2}) + z_0^* w - z_0 w^*\right], \quad (6.5)$$

where $N' \equiv N + T$. The state described by (6.5) with independent parameters N', M, z_0 may be termed the generalized Gaussian or quasi-Gaussian state [17]. Such a state may be produced by mixing the output intrinsic noise of the degenerate parametric amplifier with the coherent and thermal light through the use of a beam-splitter (see Section 10).

Another way is to supply the coherent and thermal light beams to the amplifier input; in so doing the following change of parameters in (6.5) should be made [17]:

$$T \to T_{\mathrm{in}}(\cosh^2\Gamma + \sinh^2\Gamma) = T_{\mathrm{in}}(1 + 2N),$$

$$M \to (1 + 2T_{\mathrm{in}})\cosh\Gamma\sinh\Gamma = (2 + 2T_{\mathrm{in}})\sqrt{N(N+1)},$$

$$z_0 \to z_{\mathrm{in}}\cosh\Gamma + z_{\mathrm{in}}^*\sinh\Gamma.$$

Here N is the mean number of photons at the amplifier output in the absence of additional fields, T_{in} is the mean number of photons of the thermal field at the input, and z_{in} is the amplitude of the coherent field at the input. The state of the amplifier's output field is obtained then from the thermal state at the input by using operations of squeezing and shifting [63]. At $z_{\rm in} = 0$ this state is spoken of as the squeezed thermal state [70]. If $T_{\rm in} \gg 1/2$, quantum effects may be neglected and the output emission of the amplifier may be termed the classical squeezed light [71,72].

In classical theory the absolute value of the mean square of complex amplitude is less than or equal to the mean intensity

$$\langle a^2\rangle = \int d^2z P_z(z)\, z^2 \leqslant \langle |a|^2\rangle = \int d^2z P_z(z)\, |z|^2\,,$$

i.e. $M \leqslant N$. At the same time, according to (6.2) and (6.3), $M > N$. Hence we immediately find the lower boundary

$$T_0 \geqslant M - N = \sqrt{N(N+1)} - N = \frac{1}{2}\big(1 - \exp(-2\Gamma)\big).$$

Notice that the smaller the amplification factor Γ, the larger the nonclassicality parameter $|M|/N = \coth\Gamma$ of the squeezed vacuum; in typical experiments $|M|/N \sim \sqrt{N} \sim 10^4$. However this nonclassicality, which has been noted in [73] (see also [56, 57]), is not directly observed since M presents the unobservable quantity (see though (6.21)). The same relates to the squeezing coefficient of ellipse s defined via symmetrized variances (6.4) characterizing the width of the Wigner distributions

$$s^2 \equiv \frac{\Delta q^2}{\Delta p^2} = \frac{N' + M + 1/2}{N' - M + 1/2} = \exp 4\Gamma\,. \tag{6.6}$$

In the latter equality $T = 0$ is assumed.

Notice that in the case of the nondegenerate squeezed vacuum the two modes are described by two complex amplitudes a, b, and classical Cauchy–Schwartz-like inequalities take place [56, 57, 74]:

$$\langle |a|^2|b|^2\rangle \leqslant \langle |a|^2\rangle\langle |b|^2\rangle\,, \qquad \big|\langle ab\rangle\big|^2 \leqslant \langle |a|^2\rangle\langle |b|^2\rangle\,. \tag{6.6a}$$

The two-photon light yields strong inequalities of the reverse meaning [56, 57, 73], which leads to a 'superclassical' visibility of the two-photon interference that is needed to demonstrate the Bell's inequality violations [34, 72].

In fact, the parameter M is always complex: $M \sim \exp(i\phi)$, where $\phi(t)$ is the inevitably drifting phase of the parametric amplifier pumping (which has a double frequency), so for observing the stationary effect it is necessary that the pumping and the coherent field z_0 were originated from a common driving laser. Here the phase z_0 is treated as an adjustable phase shift introduced into the homodyne tract, and for antibunching of counts the condition $\mathrm{Re}(z_0^2) < 0$ is necessary. In what follows we assume the coherent field phase to be optimal: $z_0 = i\sqrt{n}$, this condition provides the maximum photon antibunching and $g_2 = \min$.

The P-distribution represents the Fourier-image of the characteristic function (6.5)

$$\begin{aligned} P_z(z) &= \pi^{-2}\int d^2w \exp(zw^* - z^*w)\chi(w) \\ &= \pi^{-2}\int dw'\, dw'' \exp\big[2i(z'' - z_0'')\,w' - 2i(z' - z_0')\,w'' \\ &\quad - N'(w'^2 + w''^2) + M(w'^2 - w''^2)\big] \\ &= \frac{1}{\pi\sqrt{ab}}\exp\left[-\frac{(z' - z_0')^2}{a} - \frac{(z'' - z_0'')^2}{b}\right], \end{aligned} \tag{6.7}$$

$$a \equiv N' + M = N + T + \sqrt{N(N+1)}\,,$$
$$b \equiv N' - M = N + T - \sqrt{N(N+1)} = T - T_0(N)\,.$$

This function at $N' > M$ is the product of two Gaussian distributions centred at the point (z_0', z_0'') and unequal variances $N' \pm M = T + [\exp(\pm 2\Gamma) - 1]/2$. At $M > 0$ the uncertainty ellipse is squeezed along the vertical axis z''.

At $M > N'$ the characteristic function (6.5), due to the term $(M - N')\,w'^2$ in the exponent, is an unlimited function and has no regular (not generalized) Fourier-image. Consequently, according to definition I, function (6.5) at $M > N'$ describes the NCL. From here we find the Lee's measure for the squeezed vacuum

$$T_0(N) = M - N = \sqrt{N(N+1)} - N = \frac{1}{2}\big(1 - \exp(-2\Gamma)\big). \tag{6.8}$$

At a low squeezing $T_0 \sim \sqrt{N} \sim \Gamma \ll 1$, while at a high squeezing $T_0 \to 1/2$. Notice that T_0 is independent of z_0. In contrast to the K-photon-state case, adding of an arbitrarily weak thermal radiation does not render the P-distribution of the squeezed vacuum regular.

When substituting $T \to T + 1/2$, function (6.5) becomes characteristic for the symmetrized moments and, correspondingly, function (6.7) becomes a function of the Wigner distribution. Similarly, the change $T \to T + 1$ in (6.5) and (6.7) yields the characteristic function for antinormally-ordered moments and its Fourier-image, the so-called Q-distribution, which is the diagonal matrix element of the density matrix in the coherent representation: $Q(z) = \langle z|\rho|z\rangle \geqslant 0$.

6.3. Direct detection

Consider at first the properties of the heated squeezed vacuum without shifting when $z_0 = 0$. This corresponds to the direct detection (in contrast to the homodyne detection). Let us find the energy distribution $P(n)$ using the function $P_z(z) \equiv P_z(z', z'')$. Then according to (3.6)

$$\begin{aligned} P(n) &= \frac{1}{\pi\sqrt{ab}}\int_0^{2\pi} d\phi \exp\left[-\frac{n\cos^2\phi}{a} - \frac{n\sin^2\phi}{b}\right] \\ &= \frac{1}{\pi\sqrt{c}}\exp\left(-\frac{N'n}{c}\right) I_0\left(\frac{Mn}{c}\right), \\ c &\equiv ab = T^2 + 2TN - N\,, \end{aligned} \tag{6.9}$$

where we utilized

$$\int_0^{2\pi} d\phi \exp(x\cos^2\phi) = 2\pi\exp\left(\frac{x}{2}\right) I_0\left(\frac{x}{2}\right)$$

($I_0(x)$ is the modified Bessel function).

The generating function for probabilities and factorial moments of the state (6.5) counts has been found in the direct form in [63]. At $z_0 = 0$

$$C(s) = \big[(1 + as)(1 + bs)\big]^{-1/2}\,. \tag{6.10}$$

At $T = 0$ it follows herefrom that $C(s) = \big[1 + Ns(2 - s)\big]^{-1/2}$. Plots of these functions are presented in Fig. 2. It is characteristic that in the case of the NCL (at $T < T_0$) the function $C(s)$ has a pole at the point $s_\infty = -1/b = [T_0 - T]^{-1}$ and exceeds 1 for $s_\infty > s > s_1 = 2N'/(M^2 - N'^2)$, i.e. it is unacceptable as a classical generating function defined as the Laplace-image of $P(n)$. At the same time $C(s)$ continues to define moments and probabilities of counts according to its 'discrete' definition (5.5). At the point $s = s_1/2$ even derivatives of $C(s)$ are positive, while the negative ones are zero (see Fig. 2).

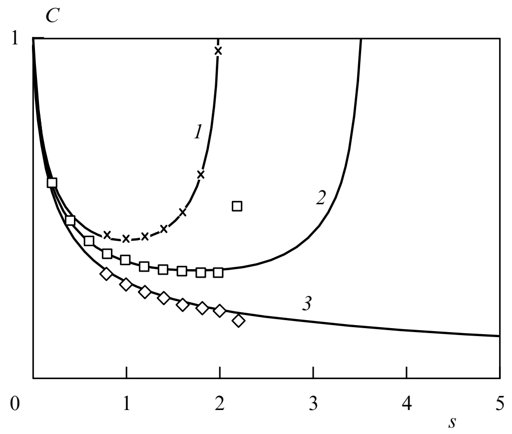

Figure 2. The generating function of the heated squeezed vacuum for $N = 5$ and thermal photon numbers T: *1* — 0; *2* — 0.2; *3* — 0.5. The plots (*1*) and (*2*) evidence for the statistics' nonclassicality. The points were obtained in a numerical experiment consisting of 10^4 random count numbers distributed in accordance with (6.15).

Consider now the g-criteria of nonclassicality. Differentiating repeatedly (6.5) or (6.10) one may find the factorial moments G_k. The general expression for them has been found in [76]:

$$G_k = N'^k \sum_{p=0}^{k} \left(\frac{k!}{p!}\right)^2 \frac{x^p}{2^p (k-p)!} \left| H_p\left(\frac{\mathrm{i} z_0}{\sqrt{2M}}\right)\right|^2 ,$$

$$x \equiv \frac{\sqrt{N(N+1}}{N+T} = \frac{N + T_0(N)}{N+T} . \quad (6.11)$$

Here $H_j(x)$ are the Hermite polynomials. At $z_0 = 0$, $H_{2m+1} = 0$ and

$$H_{2m} = (-1)^m 2 \frac{(2m-1)!}{(m-1)!} ,$$

so that (6.11) takes the form [77]

$$g_k = \sum_{p=0} \left(\frac{k!}{p!}\right)^2 \frac{x^{2p}}{2^{2p}(k-2p)!} = k! F\left(-\frac{k}{2}, -\frac{k-1}{2}; 1; x^2\right) , \quad (6.12)$$

where F is the hypergeometric function, and $g_k \equiv G_k/G_1^k$ are the normalized factorial moments ($G_1 = N + T$). Hence at $T = 0$ in the case of low squeezing ($N = \Gamma^2 \ll 1$) it follows that $g_k \approx \left[(k-1)!!\right]^2/\Gamma^k$ (even k) and $g_k \approx (k!!)^2/\Gamma^{k+1}$ (odd k) [77], whereas at $N \gg 1$, $g_k \approx (2k-1)!!$ [78].

The first several moments, according to (6.12), have the form

$$g_2 = 2 + x^2, \quad g_3 = 6 + 9x^2,$$
$$g_4 = 24 + 72x^2 + 9x^4, \quad (6.13)$$

where $x^2 = 1 + 1/N$ at $T = 0$. At low N and T, the factorial moments reveal a nonmonotonic dependence on the number, which testifies to 'nonclassical' convexities (Fig. 3). This effect, of course, reflects the property of $p_{2k+1} = 0$: from (3.4a) it follows that $G_k \approx k! p_k$ under the conditions considered. According to (6.13), the parameter $D_2(0)$ at $T = 0$ has the form

$$D_2 = 3 \frac{5 + 3N^{-1}}{3 + N^{-1}} . \quad (6.14)$$

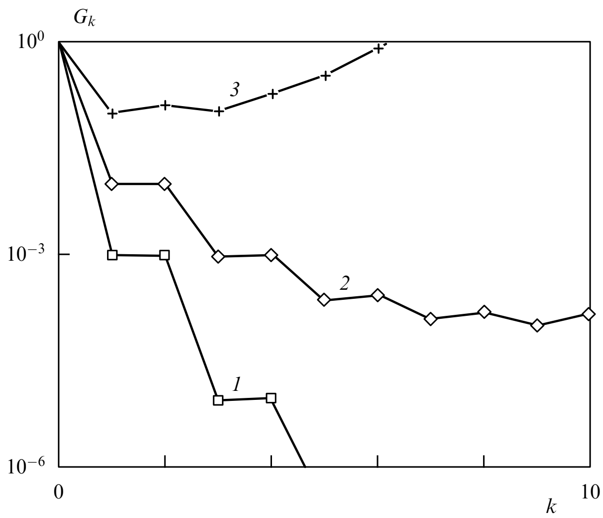

Figure 3. Factorial moments of the squeezed vacuum G_k for different N: *1* — 10^{-3}; *2* — 10^{-2}; *3* — 10^{-1}.

This function is less than 1 (which is a feature of the NCL) at

$$N < \frac{\sqrt{33}}{12} - \frac{1}{4} = 0.229 .$$

Now let us consider probabilities of counts at $z_0 = 0$. According to [63]

$$p_m = \frac{f_1^m}{f_2^{m+1}} P_m\left(\frac{f_3}{f_1 f_2}\right) , \quad (6.15)$$

where

$$f_1^2 \equiv N'^2 - M^2 = ab,$$
$$f_2^2 \equiv (N'+1)^2 - M^2 = ab + a + b + 1 ,$$
$$f_3^2 \equiv N'(N'+1) - M^2 = ab + \frac{1}{2}(a+b)$$

and $P_m(x)$ are the Legendre polynomials. Plots of this function, which have nonclassical convexities at $T < T_0$, are shown in Fig. 4.

At $T = 0$, the argument of Legendre functions in (6.15) vanishes. With regard to $P_{2m}(0) = (-1)^m (2m)!/2^{2m}(m!)^2$,

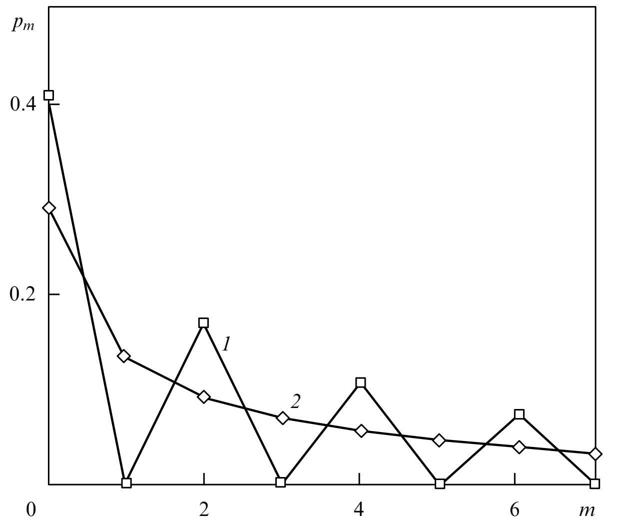

Figure 4. The count probability distribution: *1* — for squeezed vacuum; *2* — for heated squeezed vacuum at $T = T_0(N) = 0.477$. The squeezed vacuum intensity N is equal to 5 photons per mode.

$P_{2m+1}(0) = 0$ we arrive at [39]

$$p_{2m} = \frac{(2m)!}{2^{2m}(m!)^2}\,\frac{N^m}{(N+1)^{m+1/2}}\,, \quad p_{2m+1} = 0\,. \tag{6.16}$$

At $T = 0$, the 'convexity' parameters $D_m(1)$ are zero for $m = 2, 4, \ldots$, but all of them reach unity at $T \sim T_0/2$.

Consider now more sensitive tests on being nonclassical which are based on Hankel's matrices $H_K^{(n)}(s)$ of the order $K + 1 > 2$ (see Appendices). In the case of the squeezed vacuum, only matrices $H_K^{(2n+1)}(s)$ 'feel' the nonclassicality. For a given degree of nonclassicality, determined through the parameter T, $\det H_K^{(1)}(s)$ become negative beginning only from some order $K_{\min} + 1$ (Fig. 5). As it follows from the plots, the necessary order of the matrix increases drastically as

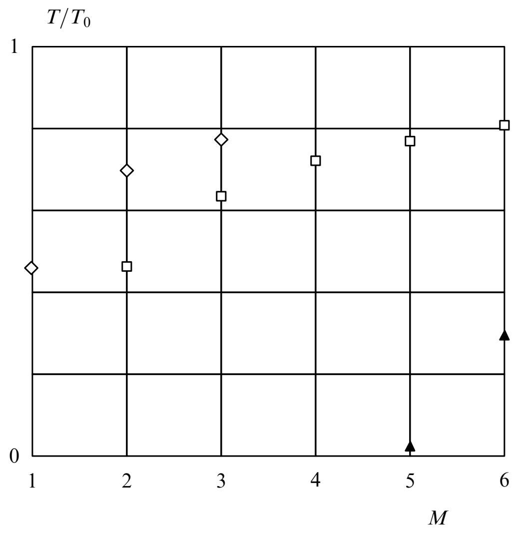

Figure 5. Dependence of the reduced parameter T/T_0 in the case of heated squeezed vacuum with $N = 1$ on the Hankel's matrix order $M = K + 1$, which follows from the condition $\det h_K^{(1)}(s, T) = 0$ in accordance to (I.5): ▲ — $s = 0$; □ — $s = 1$; ◇ — $s = 2$.

T approaches the nonclassicality boundary T_0 (equal to 0.41 for $N = 1$); in so doing the necessary number of probabilities ($s = 1$) is notably smaller than the number of moments ($s = 0$), i.e. p-criteria of nonclassicality are more sensitive. Even more sensible are matrices composed of the generalized moments at $s = 2$. The effect of the s parameter on the nonclassicality is illustrated in Figs 5 – 7.

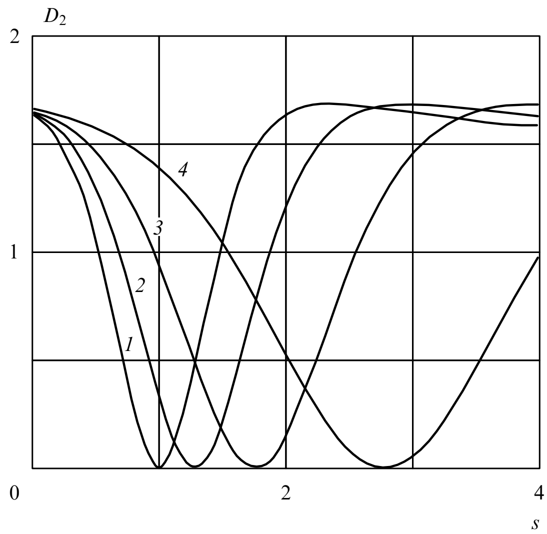

Figure 6. The function $D_2(s)$ for the squeezed vacuum at $N = 5$: *1* — $T = 0$; *2* — $T = 0.1$; *3* — $T = 0.2$; *4* — $T = 0.3$. The light is nonclassical if at least at one s the function $D_2(s)$ is less than 1, so that at $T = 0.3$ the parameters $D_2(0)$ and $D_2(1)$ do not reveal nonclassicality; at the same time it is sufficient to measure, for example, $D_2(2)$.

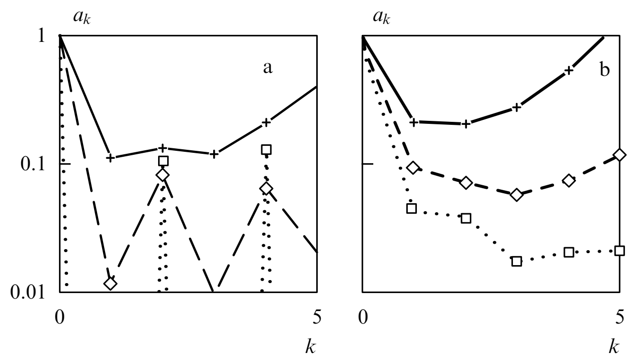

Figure 7. Dependence of the generalized moments $a_k(s)$ on the number k: + — $s = 0$, ◇ — $a = 1$, □ — $s = 2$ in the case of the heated vacuum at $N = 0.1$: (a) $T_0/T = 0.05$ and all three types of moments exhibit nonclassical convexities; (b) $T_0/T = 0.5$ and only the moment $a_k(2)$ shows the convexities.

6.4. Homodyne detection

Now we turn to the effect of squeezed vacuum shifting on the antibunching. According to (6.11), at $z_0 \neq 0$ the first two moments take the form

$$G_1 = N + T + n_0 \equiv N' + n_0\,,$$

$$G_2 = 2N'^2 + 4N'n_0 + n_0^2 + M^2 + 2M\,\mathrm{Re}(z_0^2)\,. \tag{6.17}$$

Here $n_0 \equiv |z_0|^2$. The last term is negative at $\mathrm{Re}(z_0^2) < 0$ (this is the necessary condition for antibunching). Below we assume

$z_0 = \mathrm{i}\sqrt{n_0}$. Then the uncertainty ellipse for the squeezed vacuum is shifted along its small axis. Since T_0 is independent of z_0, we verify once again that the antibunching is not a necessary feature of the NCL. From (6.17) we find the normalized moment

$$g_2 = \frac{G_2}{G_1^2} = 1 + \frac{1 + x^2 + 2y(1-x)}{(1+y)^2}\,, \tag{6.18}$$

where $x = M/N'$, $y = n_0/N'$. According to (6.18), for the given mean energies of the squeezed vacuum N and thermal field T there exists a minimum energy of the coherent component $n_{\min}$, which is necessary for antibunching onset, $g_2 < 1$ (Fig. 8):

$$y_{\min} = \frac{x^2+1}{2(x-1)}\,. \tag{6.19}$$

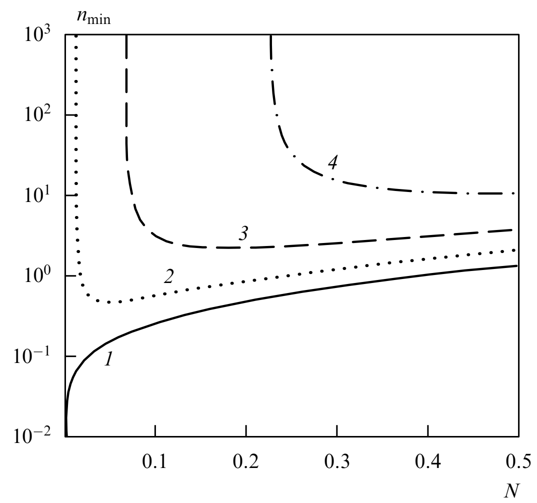

Figure 8. Dependence of the minimal energy of the coherent component $n_{\min}$ needed for the count antibunching on the squeezed vacuum energy N for the thermal energy T equal to: *1* — 0; *2* — 0.1; *3* — 0.2; *4* — 0.3.

We draw our attention to a nonmonotonic character of the $n_{\min}$ dependence on N at fixed T, which has a minimum: $n_{\min}$ increases both for small N when T approaches the critical value T_0 destroying the nonclassicality, and for large N when the strong coherent component is needed for antibunching irrespective of T.

According to (6.18), the function $g_2(n)$ reaches minimum at $y_{\text{opt}} = x(x+1)/(x-1)$. Then

$$g_{2\,\min} = 1 + \frac{(x-1)^2}{(x-1)^2 - 2x^2}\,. \tag{6.20}$$

It follows herefrom that at low squeezing ($N \ll 1$) one may attain strong antibunching ($g_2 = 4\sqrt{N}$): it is necessary that $n_0 = \sqrt{N}$ and $T \ll N$. In this case $G_2 \approx 0$ or $\langle m^2\rangle \approx \langle m\rangle$. This means that only $m = 0$ and $m = 1$ counts are observed, i.e. $p_0 = 1 - n_0$, $p_1 = n_0$, $p_2 = 0$ — the probability of detecting two or more photons is suppressed. In typical experiments $N \approx 10^{-8}$ and a 'superbunching' takes place without shifting, $g_2 \approx 10^8$, whereas an addition of the coherent field with $n_0 \approx 10^{-4}$ yields antibunching with $g_2 \approx 4 \times 10^{-4}$.

However, practically important parameters, such as Fano factor $F = 1 + (N' + n_0)(g_2 - 1)$ and the signal-to-noise ratio $\langle \Delta m^2\rangle / \langle m\rangle^2 = F/\langle m\rangle$, take small values only at large shifting n_0 and strong squeezing. Let $n_0 \gg N \gg T$, then

$$F = 1 + 2(N - M) = 2\langle \Delta p^2\rangle = \exp(-2\Gamma)\,. \tag{6.21}$$

Thus, for a strong homodyne field the Fano factor is inversely proportional to the squeezing of the uncertainty ellipse s (see (6.6)). According to (6.21), the addition of a strong coherent field allows measurements of the variance of the studied low-field quadratures to be made. In so doing the condition of 'squeezing' for one of the quadratures (for example, $\langle \Delta p^2\rangle < 1/2$) at an appropriate phase z_0 implies $F < 1$. The notion of higher orders squeezing is also in use [79].

7. Heated one-photon state

Consider the superposition between a one-photon field and a chaotic thermal field with the mean number of photons T. The mean number of photons of the resulting field $\langle n\rangle$ is obviously $T + 1$. At $T = 0$ we have a pure one-photon state, which, as already noted, according to a $D_k(1)$-criterion (by convexity of the plot of function $\ln(k!p_k)$) is clearly nonclassical (according to [44], any pure state, apart from coherent, is nonclassical).

From a naive corpuscular point of view the photon (and count) distribution in the total field is determined elementary: an addition of one photon simply shifts the geometrical thermal distribution by unity to the right. Then $p_0 = 0$, the minimum number of counts is unity, $D_1(1) = 2p_0p_2/p_1^2 = 0$, and the state is nonclassical independently of T. (States of such type have been considered in [59], their nonclassicality being derived with matrix $H_2(0)$, see (I.5).) However, this consideration is correct only when considering the total number of counts in two independent modes, it does not take into account interference between two components of the original fields.

7.1. *P*-distribution

When calculating consecutively the composition of two independent random fields, both with quantum and classical approaches, the total field distribution $P_z(z)$ is made equal to the convolution of two initial P_z-distributions.

Let us find the characteristic function of the one-photon state:

$$\chi(w) = \langle 1|\exp(wa^\dagger)\exp(-w^*a)|1\rangle\,. \tag{7.1}$$

For any w we have

$$\exp(-w^*a)|1\rangle = \left(1 - w^*a + \frac{w^{*2}a^2}{2} - \ldots\right)\Big|1\Big\rangle = |1\rangle - w^*|0\rangle\,.$$

Hence

$$\chi(w) = 1 - ww^*\,. \tag{7.2}$$

Now we add a thermal field. When composing independent states, the characteristic functions are multiplied with each other:

$$\chi(w) \equiv (1 - ww^*)\exp(-Tww^*) = \left(1 + \frac{\partial}{\partial T}\right)\exp(-Tww^*)\,. \tag{7.3}$$

P_z-distribution for the thermal field takes the form

$$P_z^T(z) = \frac{1}{\pi^2}\int \mathrm{d}w' \exp(2\mathrm{i}z''w' - Tw'^2)$$
$$\times \int \mathrm{d}w'' \exp(2\mathrm{i}z'w'' - Tw''^2) = \frac{1}{\pi T}\exp\left(-\frac{|z|^2}{T}\right). \quad (7.4)$$

By differentiating this expression with respect to T and in accordance with (7.3) we arrive at

$$P_z(z) = \left(1 + \frac{\partial}{\partial T}\right) P_z^{(T)}(z)$$
$$= \frac{1}{\pi T}\exp\left(-\frac{|z|^2}{T}\right)\left(1 - \frac{1}{T} + \frac{|z|^2}{T^2}\right). \quad (7.5)$$

In a similar way one may find the P-distribution for the superposition of a thermal field with a K-photon state [50]:

$$P(n) = \frac{(T-1)^K}{T^{K+1}}\exp\left(-\frac{n}{T}\right) L_K\left(\frac{n}{T(1-T)}\right). \quad (7.6)$$

In particular, for $K = 2$ we have a 'smoothed-out' second derivative of the δ-function

$$P(n) = \frac{1}{T}\exp\left(-\frac{n}{T}\right)\left[-\left(\frac{T-1}{T}\right)^2 + 2\,\frac{T-1}{T^3}\,n + \frac{n^3}{2T^4}\right]. \quad (7.7)$$

According to [50], for $T \to 0$ functions (7.6) represent derivatives of the δ-function, and for $T \geqslant 1$ these functions are nonnegative and may be considered as classical distribution functions.

Let us turn back now to the one-photon state. From (7.5) it follows

$$P(n) = \pi P_z\left(\sqrt{|z|}\right) = \frac{1}{T}\exp\left(-\frac{n}{T}\right)\left(1 - \frac{1}{T} + \frac{n}{T^2}\right). \quad (7.8)$$

Figure 9 displays the plots of function (7.8), which represents a 'smoothed-out' first derivative of the δ-function, and of function (7.7) as well.

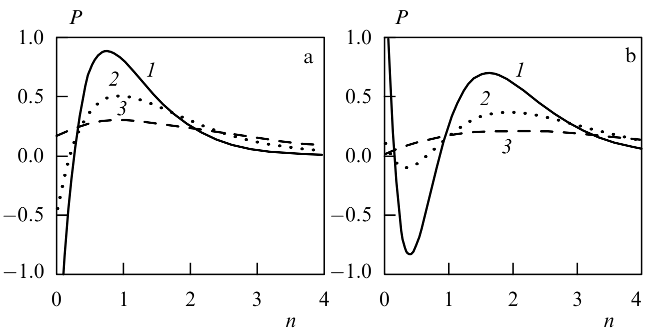

Figure 9. The energy quasi-distribution $P(n)$ for the heated one-photon (a) and two-photon (b) states. The mean number of thermal photons T is: *1* — 0.5; *2* — 0.75; *3* — 1.25. At $T < 1$, the functions $P(n)$ become negative, which is a feature of the NCL by definition. The solid curves also show the Wigner distribution for pure one- and two-photon states at $T = 0$.

7.2. Measures of nonclassicality

As a quantitative measure of nonclassicality, it is natural to choose a parameter connected with areas $S_\pm$ enclosed by the function $P(n)$ within intervals where $P(n) > 0$ and $P(n) < 0$, respectively (see Fig. 9). These areas have the form

$$S_\pm \equiv \int_0^\infty P(n)\,\theta\big(\pm P(n)\big)\,\mathrm{d}n\,, \quad (7.9)$$

where $\theta(x)$ is the step function. The normalization conditions imply $S_+ + S_- = 1$. In the case of a classical light $S_- = 0$, while at most nonclassical light yields $S_+ = 1 - S_- \gg 1$. If we denote $S \equiv 1/S_+$, then by going over from the maximum nonclassical light to a classical one the parameter S changes monotonically over the interval (0,1).

The Poisson transformation (3.2) of the distribution (7.8) produces the following count probabilities (Fig. 10):

$$p_m = \frac{T^m}{(1+T)^{m+1}}\left[1 - \frac{1}{T} + \frac{1+m}{T(1+T)}\right]. \quad (7.10)$$

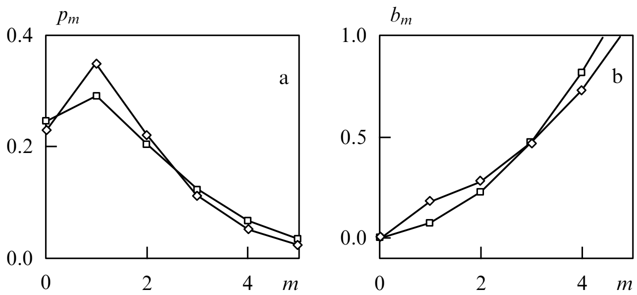

Figure 10. Count probabilities p_m (a) and quantities $b_m - \log(m!p_m/p_0)$ (b) in the case of a heated one-photon state: ◇ — $T = 0.55$; □ — $T = 0.75$. When passing from $T = 0.75$ to $T = 0.55$ in the case (b), a convexity appears at the point $m = 1$ clearly demonstrating the field's nonclassicality according to the $D_1(1)$-criterion. At the same time, in the case (a) the field nonclassicality does not manifest itself clearly, although the set of adjacent probabilities 0.23; 0.35 and 0.22 at $m = 0, 1$ and 2 is inadmissible in the classical variant of the Mandel's formula.

Factorial moments of this distribution are equal to the ordinary moments of the distribution (7.8): $G_k = k!T^{k-1}\times(k + T)$. The normalized factorial moments have the form

$$g_k(T) \equiv \frac{G_k(T)}{\langle m\rangle^k} = k!T^{k-1}\,\frac{k+T}{(1+T)^k}\,. \quad (7.11)$$

In particular,

$$g_2 = 2\left[1 - (1+T)^{-2}\right]. \quad (7.12)$$

Different parameters of nonclassicality are compared in Fig. 11.

From (7.6) we obtain the generating functions for the superposition of K-photon and thermal states (Fig. 12):

$$C_K(s) = \frac{\left[1 + s\,(T-1)\right]^K}{(1+sT)^{K+1}}\,. \quad (7.13)$$

Unlike the case of the heated squeezed vacuum, these functions are real for any real s.

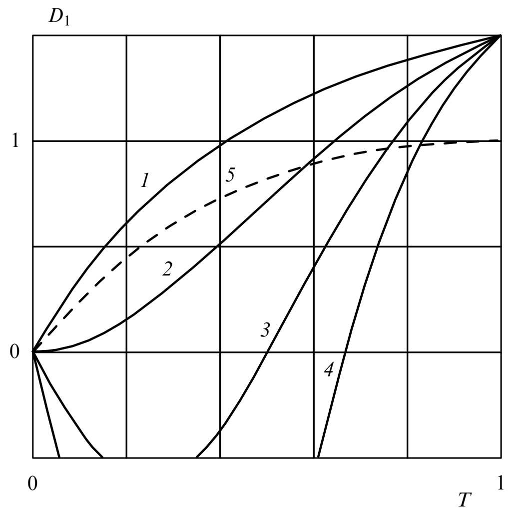

Figure 11. Dependence of the parameter $D_1(s)$ on the mean number of thermal photons T in the case of superposition of one-photon and thermal states: *1* — $s = 0$; *2* — $s = 1$; *3* — $s = 2$; *4* — $s = 3$, and with the parameter S defined by (7.9). The inequality $D_1(s) < 1$ is a feature of light nonclassicality. With increasing s the sensitivity to nonclassicality grows (bunching parameter $g_2 = D_1(0)$ is least sensitive).

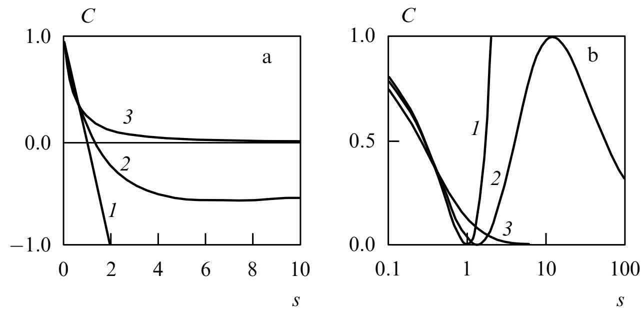

Figure 12. Generating functions for heated one-photon (a) and two-photon (b) states: *1* — $T = 0$; *2* — $T = 0.25$; *3* — $T = 1$. Only functions at $T = 1$ are acceptable in the classical theory.

8. Effect of absorption and amplification on the NCL

Let the light under study pass before detection through a linear absorber with the intensity transmission coefficient η. In classical theory, the absorber simply diminishes the intensity of the passing light: $n \to n' = \eta n$. The appropriate change in the distribution has obviously the form

$$P(n) \to P'(n) = \frac{1}{\eta} P\left(\frac{n}{\eta}\right) . \tag{8.1}$$

Thus, linear absorption does not modify the distribution shape and change its scale solely by 'pinching' it into the low energy region.

The possibility of such a simple description of the absorption holds in the quantum theory as well, provided it is applied to normal operators only (see, for example, [80]). For such operators the contribution of quantum noises is excluded and consequently they behave as the appropriate classical quantities in the course of linear transformations. In particular, one may consider $P(n)$ in (8.1) as a quantum quasi-distribution. Correspondingly, the characteristic and generating functions also change merely their scales:

$$\chi'(w, w^*) = \chi\left(\sqrt{\eta} w, \sqrt{\eta} w^*\right), \tag{8.2}$$

$$C'(s) = C(\eta s) .$$

Factorial moments of the count distribution are simply multiplied by the appropriate power of η: $G_k' = \eta^k G_k$, and normalized factorial moments just prove to be invariants of the transformation (8.2): $g_k' \equiv G_k'/(G_1')^k = g_k$. The generating function for the normalized factorial moments $\tilde{C}(s) \equiv C\left(s/\langle m\rangle\right)$ is invariant with respect to linear absorption: $\tilde{C}'(s) = \tilde{C}(s)$.

From here it follows that the fact of light nonclassicality determined by irregularity and sign of function $P(n)$ (or by unboundedness of the characteristic function $\chi(w)$, and by nonabsolute monotonicity of the generating function $C(s)$) is not affected by the absorption: if the function $P(n)$ has taken negative values, the same is done by $P(n/\eta)$. Thus, the NCL after an arbitrarily strong absorption remains nonclassical. Similarly, linear absorption leaves a classical light classical (a paradoxical exception to this rule was considered in [35]). On the other hand, nonlinear absorption may be used to turn laser light into the NCL [32, 65, 81].

Therefore, quantitative measures of nonclassicality determined through normalized factorial moments g_k, also do not alter under absorption. This relates to a Lee's measure T_0 as well. At the same time, the parameters determined through unnormalized moments G_k and count probabilities p_m, are varied. For example, the Fano factor is transformed as

$$F \to F' = 1 + \langle m\rangle'(g_2 - 1)$$

$$= 1 + \eta\langle m\rangle(g_2 - 1) = 1 + \eta(F - 1) . \tag{8.3}$$

For an ideal initial antibunching $g_2 = 1 - 1/\langle m\rangle$ and $F = 0$; after absorption $F' = 1 - \eta$. Under a sufficiently strong absorption, when $\eta \to 0$, $F \to 1$ and the statistics becomes Poissonian, even if before the absorption the ideal antibunching has taken place, $F = 0$, or inversely, a strong bunching, $F \gg 1$, has occurred. In this sense, the absorption 'Poissonizes' any light. The relative noise at the absorber output is determined by the parameter $\langle\Delta m^2\rangle'/\langle m\rangle'^2 = F'/\langle m\rangle'$, which increases on absorption. In the ideal case, $\langle\Delta m^2\rangle'/\langle m\rangle'^2 = (1 - \eta)/\langle m\rangle'$.

Consider now the change of the count probabilities under absorption. First, from (8.2), (II.2) we find the transformation of the generalized moments

$$a'_k(s) = (-1)^k \frac{\mathrm{d}^k C'(s)}{\mathrm{d}s^k} = \eta^k a_k(\eta s) . \tag{8.4}$$

Assuming $s = 0$, we get $G_k' = \eta^k G_k$. Further, using (II.3) we arrive at

$$a_k'(s) = \eta^k \sum_{m=0}^{\infty} \frac{(s - \eta s)^m}{m!} a_{k+m}(s)$$

$$= \eta^k \sum_{n=k}^{\infty} \frac{(s - \eta s)^{n-k}}{(n - k)!} a_n(s) . \tag{8.5}$$

At $s = 1$ and $a_k(1) = k! p_k$ we obtain herefrom the well-known transformation for probabilities [82]

$$p'_m = \sum_{k=m}^{\infty} C_k^m \eta^m (1-\eta)^{k-m} p_k \,. \qquad (8.6)$$

Substitution of $1/\eta$ for η yields the reverse transformation $p' \to p$ (this follows from $a_k(s) = \eta^{-k} a'_k(s/\eta)$), by means of which we may reconstruct the original distribution p_k using the absorption-changed probabilities p'_k. Notice a limitation from above on $1/\eta$ that results from the condition $p_k \geqslant 0$ in the case of the NCL: $a'_k(1/\eta) \geqslant 0$ (in general, from $G_k \geqslant 0$, $p_k \geqslant 0$ it follows $a_k(0) \geqslant 0$, $a_k(1) \geqslant 0$).

In a general way, transformation (8.2) changes the distribution shape considerably. However, there exist some types of distributions which are 'stable' against this transformation. As this takes place, only numerical parameters of the distribution vary, while its functional form remains unchanged. For example, in the case of Poissonian and thermal distributions the solely parameter of the distribution $\langle m \rangle$ is replaced by $\eta\langle m \rangle$. At the same time, there appear unstable distributions whose shape alters notably. The well-known example is provided by the state with a definite number of photons $|K\rangle$. Then $p_m = \delta_{Km}$ and from (8.6) it follows

$$p'_m = C_K^m \eta^m (1-\eta)^{K-m} \,. \qquad (8.7)$$

Similarly, dips for odd number of counts in the case of the squeezed vacuum (see Fig. 4) are smoothed out in the course of absorption. In the next Section we shall show that a light with unstable statistics may be used for photometrical purposes.

Above we considered a cool absorber. In a general way, its own thermal radiation should be taken into account. Then, according to [17, 80], the field statistics at the detector output is described by the following characteristic function (cf. (8.2)):

$$\chi'(w, w^*) = \chi\big(\sqrt{\eta}w,\ \sqrt{\eta}w^*\big) \exp\big[-ww^*T(1-\eta)\big] \,. \qquad (8.8)$$

Here $T \equiv (\exp\beta - 1)^{-1}$ is the mean number of photons in the equilibrium field with an absorber's temperature $\hbar\omega\kappa\beta$. Therefore, the influence of thermal radiation of a heated absorber is equivalent to the superposition with a thermal field having the mean photon number $\tilde{T} = T(1-\eta)$. The latter relationship is analogous to the Kirchhoff's law for a thermal radiation; $1-\eta$ plays a part of the 'absorption capability' of matter.

Transformation (8.8) holds also in the case of the field amplification by a quantum amplifier with an effective (spin) temperature $\hbar\omega/\kappa\beta < 0$ and the gain $\eta > 1$ [17, 80]. For full inversion of populations in the amplifier $\hbar\omega/\kappa\beta = -0$ and $T = -1$, so that the number of photons in the equivalent superposition field equals $\tilde{T} = \eta - 1$ (this is a Kirchhoff's law for negative temperatures, which describes intrinsic noises of the quantum amplifier). But earlier we convinced ourselves that the superposition with a thermal field having a mean photon number equal or greater than 1 always makes any field classical. Therefore, an amplifier with $\eta > 2$ (i.e. with the gain larger than 3 dB) 'dequantizes' surely any NCL.

In contrast to the cold absorber case, the normalized factorial moments now change as well. For example, in the one-photon state case, by substituting $T = \eta - 1$ into (7.12) we obtain $g'_2 = 2[1 - \eta^{-2}]$. Antibunching disappears after the amplification by the factor $\sqrt{2}$.

9. Allowance for the detector's nonideality and absolute photometry

We assumed above that the detection volume $V_{\rm det}$ and coherence volume $V_{\rm coh}$ are the same. In fact, however, in order to obtain unaveraged information on the field's statistics it is essential usually that a stronger condition $V_{\rm det} \ll V_{\rm coh}$ is fulfilled. Then the energy the detector 'sees' decreases by $V_{\rm det}/V_{\rm coh}$ times, and the count statistics changes as if the light underwent the absorption with $\eta = V_{\rm det}/V_{\rm coh}$.

Similarly, the quantum efficiency of the detector $\eta_{\rm det} < 1$ also leads to effective absorption of the field energy. Both these factors can be taken into account assuming the light to undergo a general additional absorption with $\eta = \eta_{\rm det} V_{\rm det}/V_{\rm coh}$. Thus, these factors taken into consideration, the count statistics is described by equations from Section 8 with a parameter η changed by $\eta_{\rm det} V_{\rm det}/V_{\rm coh}$.

Justification is required for using the one-mode description of a stationary light beam in a free space. According to classical picture, an ideal broad-band detector at $T \ll \tau_{\rm coh}$ measures time fluctuations of the energy flux $R(t)$ (divided by $\hbar\omega_0$) through its surface; then $\langle m \rangle = \eta_{\rm det}\langle R \rangle T$ (here $\langle m \rangle$ is the quantity averaged over a large number of counts $\{m_i\}$ measured for a total time much longer than $\tau_{\rm coh}$, and $\langle R \rangle$ is the mean in time flux). A question arises as to whether one may consider the R fluctuations as those of energy over an ensemble describing *one* equivalent classical or quantum oscillator?

The beam with one transverse mode contains many longitudinal modes (Fourier-components) whose statistics can be specified through a set of Glauber correlation functions $G_n(t_1, \ldots, t_n)$ [41 – 43]. Factorial moments of the detector's counts take the form $G'_n = (\eta_{\rm det}T)^n G_n(0, \ldots, 0)$ at $T \ll \tau_n$, where τ_n is the characteristic time of change of the function $G_n(t_1, \ldots, t_n)$. The dimension of $G_n(t_1, \ldots, t_n)$ is c^{-n}, and $G_1(t_1) \equiv \langle R \rangle$ has a meaning of the mean photon flux (independent of time in a stationary beam). In the general case different coherence times τ_n correspond to functions $G_n(t_1, \ldots, t_n)$ and a one-mode description is impossible. Let, however, all τ_n be the same, $\tau_n = \tau_{\rm coh}$ (the beam is passed, for example, through a filter with the reverse frequency band $\tau_{\rm coh} \gg \tau_n$; another possibility is considered in [90]), then $G_n(0, \ldots, 0) = (\tau_{\rm coh})^{-n} G_n$, where $G_n = \langle :\hat{n}_0^n: \rangle$ are the factorial moments of the number of photons in some effective mode. As a result, $G'_n = \eta^n G_n$, where $\eta = \eta_{\rm det} T/\tau_{\rm coh}$. Thus the observed count statistics after the reverse transformation ('amplification' by the factor η^{-1}) coincides with the count statistics for a single effective mode.

Photocount theory allows absolute (i.e. etalon-free) measuring of the light intensity and the detector's quantum efficiency [15, 33, 65, 83]. For example, according to (6.13) at $T = 0$ the relationship takes place

$$g_2 = 3 + \frac{1}{\langle n \rangle} \,, \qquad (9.1)$$

where $\langle n \rangle$ is the mean number of photons in a single mode of the light incident on the detector, which is connected with the light intensity I and effective frequency band $\Delta\omega$ (with one transverse mode): $I = \hbar\omega\Delta\omega\langle n \rangle / 2\pi$. The quantum efficiency of the detector (as well as absorption) does not affect the normalized factorial moments of counts, so in order to measure the parameter g_2 one needs not to know η and from

(9.1) can determine $\langle n\rangle$:

$$\langle n\rangle = (g_2-3)^{-1}, \qquad g_2 = \frac{\langle m^2\rangle - \langle m\rangle}{\langle m\rangle^2}\,. \tag{9.2}$$

It follows herefrom that

$$\eta = \frac{\langle m\rangle}{\langle n\rangle} = \langle m\rangle(g_2-3)\,. \tag{9.3}$$

All the parameters in the last part of expression (9.3) are determined by photocounts, i.e. they are measured without using any calibrated devices.

For practical application of this general principle of absolute photometry a nondegenerate parametric scattering (when photons in a pair are emitted in different directions and/or with different polarizations) and two-detector scheme of photocounts coincidence [15] are used.

According to (7.12), one may also use a 'heated' one-photon light. In so doing $\langle n\rangle = 1+T$ and $g_2 = 2\big(1-\langle n\rangle^{-2}\big)$, so that the parameter T is excluded and

$$\langle n\rangle = \left(1-\frac{g_2}{2}\right)^{-1/2}, \qquad \eta = \langle m\rangle\left(1-\frac{g_2}{2}\right)^{1/2}. \tag{9.4}$$

In the case of Scully–Lamb distribution (5.21) it is sufficient to measure g_2 and g_3 [65].

Generally, for absolute calibration of detectors it is necessary that normalized factorial moments be dependent upon distribution's parameters, and this dependence should be reversible and known [65]. Then the discrete statistics of photons (and counts) alters its functional form as a result of absorption (or detection), in common, for example, with the case of a K-photon light (see (8.7)). At the same time the coherent or thermal light conserves the shape of distribution in the course of absorption. The generating function for normalized factorial moments $\tilde{C}(s) \equiv C(s/\langle m\rangle)$ is not varied on absorption (detection) and it contains full information on the light one can obtain using a detector with unknown efficiency. Consequently, for absolute photometry it is necessary that the function $\tilde{C}(s)$ be dependent on $\langle m\rangle$ and other parameters of the distribution. But this is not so in the case of 'ordinary' light sources.

In an ideal case the measured function $C'(s)$ is 'squeezed' by the transformation $s \to ks$ $(k>1)$ until it coincides with a given function $C(s)$; then $\eta = 1/k$. An equivalent probability transformation is determined from (8.6) by replacement $p_m \leftrightarrow p'_m$, $\eta \to k$ (the latter transformation in the case of the NCL may lead to negative p_m).

A question arises as to whether the NCL is necessary for absolute detector calibration? The relationship (9.4) is valid at $T>1$ as well, i.e. when one-mode field state is not nonclassical. Similarly, in the case of 'heated' squeezed vacuum equation (6.13) allows measurement of the parameter x to be made using count statistics independently of the nonclassicality parameter T_0 (to transit from x to N and T, one may repeat measurements by decreasing T, for example, twofold). Therefore, the 'calibrating' light for absolute photometry should not obligatorily relate to a nonclassical class. In [65], some examples of such light preparation have been considered. In principle, it is possible to use a laser radiation close to the threshold provided that its statistics is described by the Scully-Lamb distribution (5.18) [64], as well as an ordinary light, Poissonian or thermal, after nonlinear (two-photon or with saturation effect) absorption.

10. Effect of a beam-splitter on the NCL

When accounting for the phase shift in the absorber bulk one needs to introduce an amplitude transmission coefficient $t \equiv \sqrt{\eta}\,\exp(\mathrm{i}\phi)$; then (cf. (8.1), (8.2)) it follows

$$P'_z(z) = t^{-1}P_z(t^{-1}z), \qquad \chi'(w,w^*) = \chi(tw, t^*w^*)\,. \tag{10.1}$$

Similar in form relations are correct under an arbitrary linear transformation of a multimode field [80, 84].

An important example is provided by mixing two light beams with the aid of a beam-splitter — a semitransparent mirror or polarization prism. A lot of papers has been devoted to quantum theory of light beam-splitting (see [32, 80, 85–89]). We shall follow here the paper [80]. In a classical optics, the beam-splitter's action is described by the following transformation of amplitudes of two beams:

$$a' = ta + rb, \qquad b' = -r^*a + t^*b\,. \tag{10.2}$$

Here t, r are the phenomenological coefficients of transmission and reflection, respectively. In the absence of losses $|t|^2+|r|^2=1$. Quantities with primes relate to output beams.

When passing to quantum theory, one should treat a, b as the photon annihilation operators, and a^*, b^* — as photon creation operators $a^\dagger$, $b^\dagger$ (here $a^{\dagger\prime} = t^*a^\dagger + r^*b^\dagger$, $b^{\dagger\prime} = -ra^\dagger + tb^\dagger$), and (10.2) also may be applied, but only after transforming the operator functions of a, b, $a^\dagger$, $b^\dagger$ to a normal (normally-ordered) form [80]. This rule ensures invariance of commutation relations between operators.

Let us express the input amplitudes a, b through output a', b' using the transformation reversal to (10.2):

$$a = t^*a' - rb', \qquad b = r^*a' + tb'\,. \tag{10.3}$$

Let the input beams be independent, then their joint statistics is determined by a product of the original characteristic functions

$$\chi(u,v) = \chi_a(u)\,\chi_b(v)\,. \tag{10.4}$$

As shown in [80], the normal characteristic function of the transformed (output) field coincides with (10.4) after the right-hand-side arguments u, v have been replaced by $u = t^*u' - rv'$, $v = r^*u' + tv'$ in accordance with (10.3). As a result we find

$$\chi'(u',v') = \chi_a(t^*u' - rv')\,\chi_b(r^*u' + tv')$$

or, by omitting primes on the arguments,

$$\chi'(u,v) = \chi_a(t^*u - rv)\,\chi_b(r^*u + tv)\,. \tag{10.5}$$

Here u, v arguments relate to output beams a, b, respectively. The output beams are, of course, not independent, their statistics are 'mixed'.

Let we are interested in light statistics in one output beam only, say a, then it is determined from (10.5) at $v=0$ as follows:

$$\chi(u) \equiv \chi'_a(u) = \chi_0(t^*u)\,\chi_b(r^*u)\,. \tag{10.5a}$$

At the same time, in the case of superposition of two independent fields $\chi(u) = \chi_a(u)\,\chi_b(u)$. A comparison with

(10.5a) shows that the superposition of two initial fields already subjected to amplitude absorption t^* and r^*, respectively, is formed at the beam-splitter output. It is clear that the realization of the undistorted superposition using a beam-splitter is only possible under three conditions: at least one of the field ('homodyne') must have a stable against absorption field's statistics (as, for example, a coherent or thermal field), $t \approx 1$ is required and correspondingly the stable field intensity **b** should be $|r|^{-2} \gg 1$ times increased.

According to (5.4), moments of the superposition $\chi(u) = \chi_a(u)\,\chi_b(u)$ are expressed through the initial fields' moments in accordance with the rule of differentiation of the product of two functions:

$$G_{mn} = (-1)^n \frac{\partial^m}{\partial u^m}\frac{\partial^n}{\partial u^{*n}}(\chi_a\chi_b)\Big|_{u=u^*=0} = \sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l} G_{kl}^{(a)} G_{m-k,\,n-l}^{(b)}\,, \tag{10.6}$$

where $\binom{m}{k}$ are the binomial coefficients of m elements taken k at a time.

In the stationary case

$$G_m = \sum_{k=0}^{m}\binom{m}{k}^2 G_k^{(a)} G_{m-k}^{(b)}\,. \tag{10.7}$$

According to (10.5) we substitute $G_{kl}^{(a)}$, $G_{m-k,\,n-l}^{(b)}$ in (10.6) by

$$\tilde G_{kl}^{(a)} \equiv (t^*)^k\, t^l G_{kl}^{(a)}\,,$$
$$\tilde G_{m-k,\,n-l}^{(b)} \equiv (r^*)^{m-k} r^{n-l} G_{m-k,\,n-l}^{(b)}\,. \tag{10.8}$$

Here $\tilde G_{kl}^{(a)}$, $\tilde G_{kl}^{(b)}$ are the moments of the two initial fields that underwent effective amplitude absorption t and r, respectively. As a result, we express the moments in one output beam through moments of two input light beams:

$$G_{mn} = \sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}\tilde G_{kl}^{(a)}\tilde G_{m-k,\,n-l}^{(b)}\,. \tag{10.9}$$

Consider some particular cases.

1. Vacuum at the input b, $G_{m-k,\,n-l}^{(b)} = \delta_{mk}\delta_{nl}$, so that $G'_{mn} = (t^*)^m t^n G_{mn}^{(a)}$ and $G_m = \eta^m G_m^{(a)}$ (here $\eta \equiv |t|^2$). This is a particular case of absorption when the beam-splitter acts as an absorber. True dissipation is absent in this case, the 'absorbed' energy goes into the second, unobservable, channel.

2. In the stationary case

$$G'_m = \sum_{k=0}^{m}\binom{m}{k}^2 \eta^k (1-\eta)^{m-k} G_k^{(a)} G_{m-k}^{(b)}\,. \tag{10.10}$$

Let states with a definite number of photons N_a and N_b be at the input, then

$$G_k^{(a)} = N_a(N_a-1)\ldots(N_a-k+1)\,,$$
$$G_{m-k}^{(b)} = N_b(N_b-1)\ldots(N_b-m+k+1)\,. \tag{10.11}$$

Hence

$$G'_m = \sum_{k=0}^{m}\binom{m}{k}^2 \frac{\eta^k N_a!}{(N_a-k)!}\,\frac{(1-\eta)^k N_b!}{(N_a-n+k)!}\,. \tag{10.12}$$

In particular,

$$G'_1 = \eta N_a + (1-\eta)\,N_b\,,$$
$$G'_2 = N_a(N_a-1)\,\eta^2 + 4\eta(1-\eta)\,N_a N_b + N_b(N_b-1)(1-\eta)^2\,. \tag{10.13}$$

As a result, we find the Fano factor at the output

$$F' = 1 - \frac{\eta^2 N_a - 2\eta(1-\eta)\,N_a N_b + (1-\eta)^2 N_b}{\eta N_a + (1-\eta)\,N_b}\,. \tag{10.14}$$

At $N_b = 0$ we have $F' = 1 - \eta$. It is characteristic that energy fluctuations $F' \neq 0$ are present at the output despite the 'noiseless' input. This can be explained by the mutual influence of 'noises of splitting' and effective absorption.

3. b-Field is coherent, $G_{m-k,\,n-l}^{(b)} = (z_0^*)^{m-k} z_0^{n-l}$. From (10.8) it follows

$$G'_{mn} = \sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}(t^*)^k t^l G_{kl}^{(a)}(r^* z_0^*)^{m-k}(r z_0)^{n-l}\,. \tag{10.15}$$

In a general way this expression does not describe a shifted state a. But if $t \approx 1$, then we obtain (with some limited accuracy) the result desired. In doing so the initial amplitude of the coherent field must be $1/r \gg 1$ times increased.

4. b-Field is thermal, $G_{m-k}^{(b)} = (m-k)!T^{m-k}$. From (10.10) one finds

$$G'_m = \sum_{k=0}^{m}\binom{m}{k}^2 \eta^k G_k^{(a)}(m-k)!\big[(1-\eta)\,T\big]^{m-k}\,. \tag{10.16}$$

Again, in order to realize the undistorted superposition, $t \approx 1$, $r \approx 0$ should be necessary and the corresponding increase of the intensity T of the thermal filed is required.

11. Conclusions

Thus, the semiclassical Mandel's formula for the photocount statistics gives rise to some observable features of the NCL, i.e. the light which cannot be considered as a sum of waves of random intensity n with some nonnegative distribution $P(n)$. These features are directly connected with the well-known Stieltjes' mathematical problem of moments. Many of them have been observed with a high reliability degree over last years.

Let us track out once again the initial reason for inconsistency of the quantum and semiclassical descriptions of the photodetection process. In quantum models, the process of energy transfer from an excited to unexcited system is determined by normally-ordered moments of the transmitting system and antinormally-ordered moments of the receiving system (see, for example, [47]); the normal ordering provides, in particular, the lack of contribution from vacuum fluctuations. The normally-ordered moments are not the 'true' moments of some nonnegative distribution and therefore they do not obey usual relationships of Cauchy-Schwartz type, in contrast to the classical moments. It is this difference that allows one to isolate a class of the field states having no classical analogs.

In this review we tried to systematize an operational approach to the NCL on the ground of two alternative sets of measurable parameters: photocount probabilities $\{p_m\}$ and normalized factorial moments of counts $\{g_k\}$. Within the framework of classical optics, both sets must be the moment like, i.e. they must be determined through a nonnegative distribution function $P(n)$ for the light intensity; the corresponding generating function $C(s)$ must decrease monotonically from 1 to 0. This leads to an infinite set of inequalities between functions of p_m and g_k. Accordingly, experimental criteria of the nonclassicality are subdivided into two classes: p- and g-criteria. Apparently, p-criteria are considered here for the first time. In some cases they proved to be more sensitive.

The simplest classical inequalities possess a clear geometrical image — concavity of $\ln(g_k)$ or $\ln(m!p_m)$ plots. In particular, convexity of the $\ln(g_k)$ plot at the point $k = 1$ yields the mostly known and practically important g_2-criterion of the NCL — *antibunching* of the number of counts.

We also introduced the generalized moments $a_k(s)$ with an additional parameter s, which allows one to combine and generalize these approaches ($a_0(s) = C(s)$, $a_k(0) = G_k$, $a_k(1) = k!p_k$). Under the appropriate choice of s, conditions $a_k(s) > 0$ allow the discovery of weak nonclassicalities.

All the observable features of the NCL follow from the conditions of negativeness for the Hankel's matrices $H_K^{(n)}(s)$ (K, $n = 1, 2, \ldots$) composed from $a_k(s)$ at a fixed s. It is sufficient to use matrices of two main types: $H_K^{(0)} \equiv H_K$ or $H_K^{(1)} \equiv H'_K$, so that the observable features of the nonclassicality may be subdivided into two classes according to the type of the Hankel's matrix in use: H or H'.

The nonclassicality measure T suggested by Lee [49] (see also [91]) allows tracking for the continuous transition of light from being maximum nonclassical ($T = 0$) to classical (T is the mean number of photons of the auxiliary thermal radiation, which is added to the light under study). For each quantum state there exists a minimum value $T = T_0 \leqslant 1$ at which it appears a classical energy distribution for the superpositional state. A comparison of T_0 with T value at which the loss of some NCL feature occurs, permits one to compare the sensitivity of different observational criteria of the nonclassicality.

Two particular examples — a 'heated' squeezed vacuum (I) and a 'heated' one-photon state (II) — were analyzed in more detail. These examples revealed the existence of two types of nonclassical states: in the case I, the regular P-distribution exists only at $T > T_0$, while in the case II, the regular P-distribution may be found at arbitrarily small T (but it is negative within the interval [0, T_0]). Apart from this, these cases differ from each other by the type of the Hankel's matrix which is sensitive to the nonclassicality: H' in the case I, and H in the case II. Many other types of the NCL are known (both realistic and not as much), for which a similar analysis would be interesting.

The analysis performed also demonstrated that the condition of the light nonclassicality is not obligatory for absolute measurements of the detectors efficiency using the photocount statistics. In this connection it seems important to search for types of light states which are optimal for photometry and methods of light beam preparation.

Finally, the conditions were found whereupon the beam-splitter realizes the mathematical operation of superpositing the two one-mode fields, which is frequently used to study the NCL.

Only one-point statistics of counts was considered for the case with a single one-mode photon counter. The study on dynamics of the field and its spectral properties is of great significance for practical applications of the NCL, however it has not to change our conclusions substantially. Multimode detectors provide no new information about the light, they only average the field's statistics over the time-space detection volume.

Correlation measurements with two (or more) detectors are more interesting. Some features of the nonclassicality are known for this case as well (see (6.6a), (I.12 – I.14) and [27, 41, 56, 57, 74, 92]), however, more systematic studies, in particular p-criteria examinations, are apparently of interest. In a general way we have n modes and the same number of one-mode detectors. Experiments result in sets of joint probabilities $p(m_1, \ldots, m_n)$ or factorial moments. In classical optics they are the moment sets, i.e. the latter are determined via a nonnegative joint distribution of n-mode amplitudes. As a result, to determine operationally the nonclassicality of multimode light, it is necessary to consider the problem of moments for n variables. Notice that a nontrivial multimode statistics leading, in particular, to intensity interference with a 100% visibility and violation of Bell's inequalities assumes a nonfactorized vector of the field state; such states are referred to as entangled states (see [93]).

To conclude, we note some paradox — the quantum nature of a light leads to shot (photon) noises under its detection, but it also suggests a method for eliminating them with the help of subpoissonian nonclassical light. However, this possibility is still to be used in practice and presently the only technical application of the NCL is apparently the absolute photometry.

One more concluding remark: it seems nontrivial that simple models of quantum optics with a small number of phenomenological parameters describe perfectly all known-to-date optical experiments with macrodevices consisting of $\sim 10^{23}$ atoms and separated sometimes by a distance of some hundreds of meters. Probably, this is one more example of a general regularity: all macroevents described by phenomenological dynamical equations obey the corresponding quantum laws at sufficiently low temperatures.

12. Appendices

I. Constraints on the moments

The existence conditions for the problem of moments are formulated by means of a set of Hankel's matrices [39,51-54]. Let us compose a $K + 1$-order matrix $H \equiv H_K^{(0)}$ from the moments $a_0 = 1, a_1, \ldots, a_{2K}$ according to the rule

$$H_{ij} = a_{i+j} = \int_0^\infty \mathrm{d}x\, P(x)\, x^{i+j}\,, \tag{I.1}$$

i.e.

$$H = \begin{bmatrix} 1 & a_1 & a_2 & \ldots & a_K \\ a_1 & a_2 & a_3 & \ldots & a_{K+1} \\ a_2 & a_3 & a_4 & \ldots & a_{K+2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_K & a_{K+1} & a_{K+2} & \ldots & a_{2K} \end{bmatrix}. \tag{I.2}$$

This matrix determines the quadratic form $Q(u) = \sum_{i,j} H_{ij} u_i u_j$ — a function of $K + 1$ variables $u \equiv (u_0, \ldots, u_K)$. Substituting here the definition of moments a_i through $P(x) \geqslant 0$, we arrive at the inequality

$$Q(u) = \int_0^\infty \mathrm{d}x\, P(x) \sum_{i=0}^K \sum_{j=0}^K u_i u_j x^i x^j = \int_0^\infty \mathrm{d}x\, P(x) \left(\sum_{i=0}^K u_i x^i \right)^2 \geqslant 0\,, \tag{I.3}$$

which is valid for any u. The form $Q(u)$ (as well as the matrix H_{ij} corresponding to it) satisfying this inequality is spoken of as nonnegatively determined. Further, this condition is equivalent to the requirement of the determinant nonnegativeness $\det H \geqslant 0$ [60].

As experimental moments are determined with a finite accuracy, one may ignore the case of the equality in (I.3) (and in the inequalities below) and restrict oneself to considering the condition of positive definiteness of the matrix H, i.e. $\det H > 0$.

Consequently, in the case of classical light all the eigenvalues of the moment matrix H_{ij} must be positive: $\lambda_i > 0$, $i = 0, 1, \ldots, K$. This forms a necessary condition that the set $a_0, \ldots, a_{2K}$ is a moment like, i.e. it is determined through some nonnegative distribution $P(x)$. It can be shown that passing to normalized moments $a'_k = g_k \equiv G_k/G_1^k$ reduces to the multiplication of $\det H$ by $G_1^{K(K+1)}$ and does not change the condition $\det H > 0$.

An equivalent condition consists in the positiveness of all angular (adjacent to the upper left corner of the matrix H) minors (Sylvester's criterion [60]), i.e. all determinants of the Hankel's matrices of preceding orders, $K' = 1, 2, \ldots, K-1$.

Let us consider also matrices $H_K^{(n)} \equiv (a_{i+j+n})$ 'shifted' to the right by n steps. For them, similar to (I.3), we find

$$Q(u) = \int_0^\infty \mathrm{d}x\, P(x) \sum_{i=0}^K \sum_{j=0}^K u_i\, u_j\, x^{i+j+n} = \int_0^\infty \mathrm{d}x\, P(x)\, x^n \left(\sum_{i=0}^K u_i\, x^i\right)^2 \geqslant 0\,. \quad \text{(I.4)}$$

Thus, necessary conditions of the light classicality take the form:

$$\det H_1^{(0)} \equiv \begin{vmatrix} 1 & a_1 \\ a_1 & a_2 \end{vmatrix} \geqslant 0\,, \qquad \det H_1^{(1)} \equiv \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \geqslant 0\,,$$

$$\det H_1^{(2)} \equiv \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} \geqslant 0\,, \quad \text{(I.5)}$$

$$\det H_2^{(0)} \equiv \begin{vmatrix} 1 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} \geqslant 0\,,$$

$$\det H_2^{(1)} \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \geqslant 0\,, \ldots\,, \quad \text{(I.6)}$$

where $a_k = G_k$ or $a_k = k!p_k/p_0$. The reverse inequalities make up sufficient conditions of the nonclassicality. For a coherent state yielding a Poissonian statistics, all these determinants vanish. The solution to the problem of moments exists (i.e. the light is classical) if and only if the determinants of all matrices behave as $\det H_K \equiv \det H_K^{(0)} > 0$ and $\det H'_K \equiv \det H_K^{(1)} > 0$ $(K = 1, 2, \ldots)$ [53].

The $\det H_K > 0$ criteria at $a_k = G_k$ were used in [59], where some field state was considered, for which $\det H_1 > 1$ (i.e. no antibunching occurred) but $\det H_2 < 1$. To obtain a quantitative measure of the nonclassicality, a normalization of $\det H_K$ was introduced on the determinants of Hankel's matrices composed from ordinary (not factorial) moments $\langle n^k \rangle$. These matrices, as will be shown below, are always nonnegative.

From (I.5), the simplest necessary conditions of the classicality are written in the form (cf. (4.7))

$$D_k \equiv \frac{\det H_1^{(k-1)}}{a_k^2} + 1 = \frac{a_{k-1}\, a_{k+1}}{a_k^2} \geqslant 1\,, \quad k = 1, 2, \ldots \quad \text{(I.7)}$$

For example,

$$a_2 \geqslant a_1^2\,, \qquad a_1 a_3 \geqslant a_2^2\,. \quad \text{(I.8)}$$

In a quantum theory similar constraints on (ordinary) moments $\langle n^k \rangle$ occur and they follow from the density matrix nonnegativeness. If one considers a_k as $\langle n^k \rangle$ and substitutes in (I.1), (I.3) the classical averaging by the quantum one, then all inequalities given above remain valid, now regardless the field state. For example, $D_1 \geqslant 1$ now means $\langle n^2 \rangle \geqslant \langle n \rangle^2$, i.e. $G_2 \geqslant \langle n \rangle (\langle n \rangle - 1)$.

The conditions (I.7) can easily be derived in a direct way. For this, let us consider the product of two moments of the distribution $P(x) \geqslant 0$:

$$a_k a_l \equiv \langle x^k \rangle \langle y^l \rangle = \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, P(x)\, P(y)\, x^k y^l \equiv \langle x^k y^l \rangle\,. \quad \text{(I.9)}$$

Here $k, l = 0, 1, 2, \ldots$. Let $l - k + 1 \equiv m \geqslant 0$. There is a need to prove the inequality [58]

$$a_k a_l \leqslant a_{k-1} a_{l+1}\,, \quad \text{(I.10)}$$

(at $k = l$ we obtain (I.7); a more general case is considered in [58]). This inequality may be represented as $\langle x^k y^l \rangle \leqslant \langle x^{k-1} y^{l+1} \rangle$ or

$$\langle x^k y^l + y^k x^l \rangle \leqslant \langle x^{k-1} y^{l+1} + y^{k-1} x^{l+1} \rangle\,.$$

Since $P(x)P(y) \geqslant 0$, it is sufficient to prove that for all $x, y \geqslant 0$ the inequality

$$x^k y^l + y^k x^l \leqslant x^{k-1} y^{l+1} + y^{k-1} x^{l+1}$$

holds.

At $x, y = 0$ this inequality is valid, so without loss of generality we may consider $x, y > 0$. After dividing the both sides by $x^{k-1} y^{l+1}$ and denoting $x/y \equiv \varepsilon > 0$, we obtain $\varepsilon + \varepsilon^m \leqslant 1 + \varepsilon^{m+1}$ or $\varepsilon^m - 1 \leqslant \varepsilon(\varepsilon^m - 1)$. At $\varepsilon = 1$ this inequality holds. Let us divide the both sides by $\varepsilon^m - 1$. If $\varepsilon > 1$, then $\varepsilon^m - 1 > 0$ and on dividing the inequality its sign does not change, which yields the result $1 < \varepsilon$ consistent with the original assumption; at $\varepsilon < 1$ we divide the inequality by $\varepsilon^m - 1 < 0$ with the sign changing, which again yields the correct result $1 > \varepsilon$.

Some constraints on the moments follow immediately from Cauchy–Schwartz inequalities [60]

$$\left|\langle fg \rangle\right| \leqslant \sqrt{\langle f^* f \rangle \langle g^* g \rangle} \leqslant \frac{1}{2}\left[\langle f^* f \rangle + \langle g^* g \rangle\right]\,, \quad \text{(I.11)}$$

where $f \equiv f(x, y, \ldots)$, $g \equiv g(x, y, \ldots)$ and $x, y, \ldots$ are the random variables. Assuming $f = x^{(m-k)/2}$, $g = x^{(n+k)/2}$ we obtain (for even $(m+n)/2$)

$$\left[a_{(m+n)/2}\right]^2 \leqslant a_{m-k} a_{n+k}\,. \quad \text{(I.12)}$$

In the case of two modes we have two random intensities x, y and the moments

$$a_{m,n} \equiv \langle x^m y^n \rangle \equiv \iint \mathrm{d}x\, \mathrm{d}y P(x, y)\, x^m y^n$$

(cf. (I.9)). Assuming in (I.11)

$$f^2 = x^m y^n\,, \qquad g^2 = x^{2k-m} y^{2l-n}\,,$$

we get

$$(a_{k,l})^2 \leqslant a_{mn}\, a_{2k-m,\,2l-n}\,. \tag{I.13}$$

In addition, similarly to (I.10) we may prove the inequality [92]

$$a_{k,l} + a_{l,k} \leqslant a_{k+1,l-1} + a_{l+1,k-1} \quad (k \geqslant l = 1,2,\ldots)\,. \tag{I.14}$$

Assuming $a_{mn} = G_{mn}$ or $a_{mn} = m!n!p_{mn}$, we reach some constraints on the factorial moments G_{mn} and joint probabilities of counts in two detectors p_{mn} (the latter are determined by a two-dimensional Poisson transformation). For example,

$$(p_{11})^2 \leqslant 4p_{00}p_{22}, \qquad (G_{11})^2 \leqslant G_{22},$$
$$(p_{11})^2 \leqslant 4p_{02}p_{20}, \qquad (G_{11})^2 \leqslant G_{02}G_{20}\,. \tag{I.15}$$

The high visibility of intensity interference when using a two-photon light [3, 12, 13] evidences for a violation of two latter inequalities.

II. Generalized moments

The examples of states considered in Sections 6 and 7 showed that the experimental checks for the nonclassicality based on the probabilities p_m and factorial moments G_k possess different sensitivity. It is natural to try to find an optimal method for processing the counts array $\{m_i\}$ providing a maximum sensitivity to the light nonclassicality.

Let us determine the generalized moments as follows:

$$a_k(s) \equiv \sum_{m=k}^{\infty} p_m (1-s)^{m-k}\,\frac{m!}{(m-k)!}$$
$$= \int_0^{\infty} x^k \exp(-sx) P(x)\,\mathrm{d}x = \left\langle :\hat{n}^k \exp(-s\hat{n}): \right\rangle \tag{II.1}$$
$$(s \geqslant 0)\,.$$

Hence $a_k(s) > 0$ in the classical theory. At $s=0$ (II.1) yields ordinary factorial moments G_k, and at $s=1$ — the probabilities p_k multiplied by $k!$. According to (II.1) and (4.2), the moment $a_k(s)$ is equal to the k-th derivative of the generating function $C(s)$ multiplied by $(-1)^k$:

$$a_0(s) = C(s), \qquad a_k(s) = (-1)^k\,\frac{\mathrm{d}^k C(s)}{\mathrm{d}s^k}\,,$$
$$\frac{\mathrm{d}a_k(s)}{\mathrm{d}s} = -a_{k+1}(s)\,. \tag{II.2}$$

After multiplying (II.1) by the factor $1-\exp(-s'x)\times \sum (s'x)^m/m!$, we arrive at

$$a_k(s) = \sum_{m=0}^{\infty} \frac{(s'-s)^m}{m!}\,a_{k+m}(s')\,. \tag{II.3}$$

Relationships (3.3) and (3.4a) follow herefrom as particular cases at (s,s') equal to (0,1) and (1,0), respectively. The normalization condition $\sum p_m = 1$ yields

$$\sum_{k,m=0}^{\infty} \frac{(s-1)^m}{k!m!}\,a_{k+m}(s) = 1\,. \tag{II.4}$$

The operational definition by means of the array $\{m_i\}$ has the form

$$a_k(s) = \frac{1}{M}\sum_{i=1}^{M} (1-s)^{m_i-k}\, m_i\,(m_i-1)\ldots(m_i-k+1) \tag{II.5}$$
$$(m_i \geqslant k)\,.$$

At $s \to 1$ only terms with $m_i = k$ are nonzero, i.e. $a_k(1)/k!$ coincides with the empirical definition of the probability p_k.

The convergence radius $s_{\max}$ of the power series for the functions $a_k(s)$ depends on the shape of the p_k distribution.

From (II.1) it follows that in the classical case the generalized moments must satisfy all the inequalities for factorial moments and probabilities considered above; now, however, we have an additional parameter s in our disposal.

Already the inequalities $a_0(s) > 1$, $a_k(s) < 0$ at any s may serve as a sufficient feature of the NCL. The first condition in the case of the heated squeezed vacuum holds, according to (6.10), at

$$s > s_1 \equiv \frac{2(N+T)}{N(N+1)-(N+T)^2}\,. \tag{II.6}$$

At $s = s_1/2$, the odd moments vanish, which yields $D_{2k+1}(s_1) = 0$.

In the case of the heated one-photon state with the mean number of photons $1+T$, the moment $a_0(s)$ is negative according to (7.14), i.e. it reveals the nonclassicality at $s > 1/(1-T)$. For example, in order to discover the nonclassicality at $t = 0.5$, one needs to find $a_0(2) = \left\langle (-1)^m \right\rangle = M^{-1}\sum_i (-1)^{m_i}$ (i is the ordinal number, and M is the number of trials). This procedure may prove to be more precise than the calculation of high-order Hankel's matrix determinants. It is clear from Fig. 9 why the function $a_0(s)$ happens to be negative for sufficiently large arguments: the exponential factor in (II.1) stresses the contribution from the initial portions of the distribution $P(n)$, where it takes negative values (for $T < 1$).

According to (II.2), the 'shift' of the Hankel's matrices is equivalent to their differentiating with respect to the parameter s:

$$H_K^{(n+1)}(s) = \frac{-\,\mathrm{d}H_K^{(n)}(s)}{\mathrm{d}s}\,. \tag{II.7}$$

One may also define the complex moments

$$c_k(\omega) \equiv \int_0^{\infty} \exp(\mathrm{i}k\omega n)\,P(n)\,\mathrm{d}n = \langle:\exp(\mathrm{i}k\omega\hat{n}):\rangle,$$
$$c_k(\omega) = c_{-k}(\omega)^* = c_k(-\omega)^* \tag{II.8}$$

at $k = 0, \pm 1, \ldots, N$. These quantities coincide with $\tilde{P}(k\omega) = C(-\mathrm{i}k\omega)$, where $\tilde{P}(\omega)$ and $C(s)$ are the characteristic and generating functions of the number of counts (see (4.1), (4.2)). Let us compose Hermitian matrices $\left[H_N(\omega)\right]_{kl} \equiv c_{k-l}(\omega)$, $N = 1,2,\ldots$ and the corresponding quadratic forms

$$Q(u) \equiv \sum_{k,l}^{N} c_{k-l}(\omega)\,u_k u_l^* = \int \mathrm{d}nP(n)\left|\sum_k^N \exp(\mathrm{i}k\omega n)\,u_k\right|^2 \tag{II.9}$$

(at $\omega = 2\pi$ they are termed the Toeplitz's forms [52]). In classical theory $P(n) \geqslant 0$, so that matrices $H_N(\omega)$ are nonnegatively determined. Thus, the conditions $\det H_N(\omega) = \det\left\{\tilde{P}\left[(k-l)\,\omega\right]\right\} \geqslant 0$ may also be employed as tests on the nonclassicality. For example, for $N = 1$ and 2 we obtain

$$1 - \left|\tilde{P}(\omega)\right|^2 \geqslant 0,$$

$$1 + 2\mathrm{Re}\left[\tilde{P}(2\omega)\,\tilde{P}(\omega)^{2*}\right] - \left|\tilde{P}(2\omega)\right|^2 - 2\left|\tilde{P}(\omega)\right|^2 \geqslant 0\,. \quad \text{(II.10)}$$

In the case of the heated one-photon state, these conditions reveal the nonclassicality only at $T < 0.41$ and $T < 0.55$, respectively, i.e. they have no advantages with respect to the tests that are based on the real moments $a_k(s)$.

References

1. Brown R H, Twiss R Q *Nature* (London) **177** 27 (1956)
2. Lipeles M, Novick R,Tolk N *Phys. Rev. Lett.* **15** 690 (1965); Kaul R D *J. Opt. Soc. Am.* **56** 1262 (1966)
3. Kocher C A, Commins E D *Phys. Rev. Lett.* **18** 575 (1967)
4. Akhmanov S A et al. *Pis'ma Zh. Eksp. Teor. Fiz.* **6** 575 (1967) [*JETP Lett.* **6** 85 (1967)
5. Harris S E, Oshman M K, Byer R L *Phys. Rev. Lett.* **18** 732 (1967)
6. Magde D, Mahr H *Phys. Rev. Lett.* **18** 905 (1967)
7. Kimble H J, Dagenais M, Mandel L *Phys. Rev. Lett.* **39** 691 (1977)
8. Slusher R E et al. *Phys. Rev. Lett.* **55** 2409 (1985)
9. Shelby R M et al. *Phys. Rev. Lett.* **57** 691 (1986)
10. Wu L- A et al. *Phys. Rev. Lett.* **57** 2520 (1986)
11. Freedman S J, Clauser J F *Phys. Rev. Lett.* **28** 938 (1972)
12. Ghosh R, Mandel L *Phys. Rev. Lett.* **59** 1903 (1987)
13. Strekalov D V et al. *Phys. Rev. Lett.* **74** 3600 (1995)
14. Pittman T B et al. *Phys. Rev. A* **52** *R* 3429 (1995)
15. Kitaeva G Kh et al. *DAN SSSR* **247** 586 (1979); Malygin A A, Penin A N, Sergienko A V *Pis'ma Zh. Eksp. Teor. Fiz.* **33** 493 (1981) [*JETP Lett.* **33** 477 (1981)]
16. Belinskiĭ A V, Klyshko D N *Zh. Eksp. Teor. Fiz.* **105** 487 (1994) [*JETP* **78** 259 (1994)]
17. Klyshko D N *Photon Nonlinear Optics* (New York: Gordon and Breach, 1988)
18. Machida S, Yamamoto Y, Itaya Y *Phys. Rev. Lett.* **58** 1000 (1987)
19. Richardson W H, Shelby R M *Phys. Rev. Lett.* **64** 400 (1990)
20. Wang H, Freeman M J, Steel D G *Phys. Rev. Lett.* **71** 3951 (1993)
21. Fox A M et al. *Phys. Rev. Lett.* **74** 1728 (1995)
22. Pereira S F et al. *Phys. Rev. A* **38** 4931 (1988)
23. Paschotta R et al. *Phys. Rev. Lett.* **72** 3807 (1994)
24. Ralph T C et al. *Opt. Lett.* **20** 1316 (1995)
25. Klyshko D N *Usp. Fiz. Nauk* **164** 1187 (1994) [*Phys. Usp.* **37** 1097 (1994)]
26. Clauser J F, Shimony A *Rep. Prog. Phys.* (GB) **41** 1881 (1978)
27. Loudon R *Rep. Prog. Phys.* (GB) **43** 913 (1980)
28. Loudon R, Knight P L *J. Mod. Optics* **34** 709 (1987)
29. Smirnov D F, Troshin A S *Usp. Fiz. Nauk* **153** 233 (1987) [*Sov. Phys. Usp.* **30** 851 (1987)]
30. Teich M C, Saleh B E *A Quant. Opt.* **1** 153 (1989); *Usp. Fiz. Nauk* **161** 101 (1991)
31. Fabre C *Phys. Rep.* **219** 215 (1992); Kimble H J *Phys. Rep.* **219** 227 (1992)
32. Paul H *Rev. Mod. Phys.* **54** 1061 (1982)
33. Klyshko D N, Penin A N *Usp. Fiz. Nauk* **152** 653 (1987) [*Sov. Phys. Usp.* **30** 716 (1987)]; Penin A N, Sergienko A V *Appl. Opt.* **30** 3582 (1991)
34. Belinskiĭ A V, Klyshko D N *Usp. Fiz. Nauk* **163** (8) 1 (1993) [*Phys. Usp.* **36** 653 (1993)]
35. Klyshko D N, Masalov A V *Usp. Fiz. Nauk* **165** 1249 (1995) [*Phys. Usp.* **38** 1203 (1995)]
36. *J. Mod. Optics* **34** 709 (1987) (special issue)
37. *J. Opt. Soc. Am. B* **4** 1715 (1987) (special issue)
38. *Laser Physics* **4** 4 (1994) (special issue)
39. Perina J *Quantum Statistics of Linear and Nonlinear Optical Phenomena* (Dordrecht: D. Reidel, 1984) [Translated into Russian (Moscow: Mir, 1987)]
40. Mandel L *Proc. Phys. Soc.* (London) **72** 1037 (1958)
41. Glauber R "Quantum Optics and Electronics", in: Lectures delivered at Les Houches during the 1964 Session of Summer School on Theoretical Physics) (New York: Gordon and Breach, 1965)
42. Glauber R J *Phys. Rev.* **131** 2766 (1963)
43. Sudarshan E C G *Phys. Rev. Lett.* **10** 277 (1963)
44. Hillery M *Phys. Lett. A* **111** 409 (1985); *Phys. Rev. A* **39** 2994 (1989)
45. Elyutin P V *Vestn. Mosk. Univ. Ser. Fiz. Astron.* **32** 87 (1991)
46. Teich M C, Saleh B E A, Stoler D *Opt. Commun.* **46** 244 (1983)
47. Klyshko D N *Fizicheskie Osnovy Kvantovoi Elektroniki* (Physical Bases of Quantum Electronics) (Moscow: Nauka, 1986)
48. Akhmanov S A, Djakov Yu E, Chirkin A S *Vvedenie v Statisticheskuyu Radiofiziku i Optiku* (Introduction to Statistical Radiophysics and Optics) (Moscow: Nauka, 1981)
49. Lee C T *Phys. Rev. A* **44** R2775 (1991)
50. Lee C T *Phys. Rev. A* **45** 6586 (1992)
51. Akhiezer N I *Klassicheskaya Problema Momentov* (Classical Problem of Moments) (Moscow: Fizmatgiz, 1961)
52. Bellman R *Introduction to Matrix Analysis* (New York: McGraw–Hill, 1960) [Translated into Russian (Moscow: Nauka, 1969)]
53. Jones W B, Thron W J *Continued Fractions* (Addison–Wesley, 1980) [Translated into Russian (Moscow: Mir, 1985)]
54. Feller W *Introduction to Probability Theory* (New York: Wiley & Sons, 1971) [Translated into Russian (Moscow: Mir, 1984)]
55. Smithey D T, Beck M, Raymer M G *Phys. Rev. Lett.* **70** 1244 (1993)
56. Graham R *Phys. Rev. Lett.* **52** 117 (1984)
57. Reid M D, Walls D F *Phys. Rev. A* **34** 1260 (1986)
58. Lee C T *Phys. Rev. A* **41** 1721 (1990)
59. Agarwal G S, Tara K *Phys. Rev. A* **46** 485 (1992)
60. Korn G, Korn T *Mathematical Handbook* (New York: McGraw–Hill, 1961) [Translated into Russian (Moscow: Nauka, 1970)]
61. Louisell W *Radiation and Noise in Quantum Electronics* (New York: McGraw–Hill, 1964) [Translated into Russian (Moscow: Nauka, 1972)]
62. Agarwal G S, Adam G *Phys. Rev. A* **38** 750 (1988)
63. Chaturvedi S, Srinivasan V *Phys. Rev. A* **40** 6095 (1989)
64. Scully M O, Lamb W E Jr *Phys. Rev.* **159** 208 (1967)
65. Klyshko D N *Zh. Eksp. Teor. Fiz.* **90** 1172 (1986) [*Sov. Phys. JETP* **63** 682 (1986)]
66. Mollow B R, Glauber R J *Phys. Rev.* **160** 1076 (1967)
67. Bell J S *Physics* **1** 195 (1964)
68. Clauser J F et al. *Phys. Rev. Lett.* **23** 880 (1969)
69. Evdokimov N V et al. *Usp. Fiz. Nauk* **166** 91 (1996) [*Phys. Usp.* **39** 83 (1996)]
70. Kim M S, de Oliveira F A M, Knight P L *Phys. Rev. A* **40** 2494 (1989)
71. Akhmanov S A, Belinskiĭ A V, Chirkin A S *Kvantovaya Elektron.* (Moscow) **15** 873 (1988)
72. Klyshko D N *Phys. Lett. A* **146** 93 (1990)
73. Zel'dovich B Ya, Klyshko D N *Pis'ma Zh. Eksp. Teor. Fiz.* **69** (1969) [*JETP Lett.* **9** 40 (1969)]
74. Zubairy M S *Phys. Lett. A* **87** 162 (1982)
75. Klyshko D N *Phys. Lett. A* **132** 299 (1988)
76. Janszky J, Adam P, Yushin Y *Opt. Commun.* **93** 191 (1992); Adam P *J. Mod. Opt.* **42** 1311 (1995)
77. Marian P *Phys. Rev. A* **44** 3325 (1991)
78. Janszky J, Yushin Y *Phys. Rev. A* **36** 1288 (1987)
79. Hong C K, Mandel L *Phys. Rev. Lett.* **54** 323 (1985)
80. Klyshko D N *Phys. Lett. A* **137** 334 (1989)
81. Bandilla A *Opt. Commun.* **23** 299 (1977)
82. Scully M O, Lamb W E Jr *Phys. Rev.* **179** 368 (1969)
83. Klyshko D N *Kvantovaya Elektron.* (Moscow) **4** 1056 (1977) [*Sov. J. Quantum Electron.* **7** 591 (1977)]
84. Belinsky A V, Klyshko D N *Phys. Lett. A* **166** 303 (1992)
85. Fearn H, Loudon R *Opt. Commun.* **64** 485 (1987)
86. Ou Z Y, Hong C K, Mandel L *Opt. Commun.* **63** 118 (1987)
87. Huttner B, Ben-Aryeh Y *Phys. Rev. A* **38** 204 (1988)
88. Bashkanskiĭ É G, Mityugov V V *Teor. Mat. Fizika* **7** 348 (1971)
89. Plank R W, Suttorp L G *Opt. Commun.* **112** 145 (1994)
90. Tatarskii V I *Phys. Lett. A* **144** 491 (1990)
91. Lutkenhaus N, Barnett S M *Phys. Rev. A* **51** 3340 (1995)
92. Lee C T *Phys. Rev. A* **41** 1569; *A* **42** 1608 (1990)
93. Ekert A, Knight P L *Am. J. Phys.* **63** 415 (1995)