

REVIEWS OF TOPICAL PROBLEMS PACS number: 12.38.Aw

The confinement

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Contents

Abstract. **Various aspects and mechanisms of QCD colour confinement are surveyed. Following the introduction of gauge-invariant order parameters, a field-correlator definition of the phenomenon is given, and a class of correlators responsible for confinement is explicitly separated. It is shown that, in terms of effective Lagrangians, confinement is a dual Meissner effect, and for its quantitative description the popular Abelian projection method is used, which is explained in detail. To determine the field configurations responsible for confinement, classical solutions are analysed using a newly-developed criterion. Finally, all aspects of confinement are illustrated by a simple example of string formation for a quark moving in a field of heavy antiquark.**

1. Introduction

Ten years ago the author delivered lectures on confinement at the XXII LINP Winter School [1]. Different models of confinement were considered, and the idea was stressed that behind the phenomenon of confinement is a disorder or stochasticity of vacuum at large distances.

This phenomenon becomes especially fascinating when one discovers that the disorder is due to topologically nontrivial configurations [2], and the latter could be quantum or classical. In the last case one should look for appropriate classical solutions: instantons, multiinstantons, dyons, etc.

The tedious work of many explores during the last ten years has given an additional support to the stochastic mechanism and has discarded some models popular in the past, e.g. the dielectric vacuum model and the Z_2-flux model. Those are reported in Ref. [1] (see also articles cited in Ref. [1]). At the same time a new and deeper understanding of confinement has resulted.

It was found that on the phenomenological level the stochastic mechanism reveals itself as a dual Meissner effect, which was suggested as a confinement mechanism 20 years ago [3]. In recent years there has appeared a lot of lattice data in favour of the dual Meissner effect and the so-called Abelian projection (AP) method was suggested [4], which helps to quantify the analogy between the QCD string and the Abrikosov – Nielsen – Olesen (ANO) string.

At the same time the explicit form of the confining configuration (whether it is a classical solution or quantum fluctuations) is still unclear, some theories are being considered and an active work is going on.

The present review is based on numerous data obtained from lattice calculations, phenomenology of strong interaction and theoretical studies.

The structure of the review is the following. After formulation of criteria for confining mechanisms in the next section, we describe in Section 3 the general method of vacuum correlators (MVC) [5] to characterise confinement as a property of field correlators. A detailed comparison of superconductivity and confinement in the MVC language and in the effective Lagrangian formalism of Ginzburg –Landau type is given in Section 4.

The AP method is introduced in Section 5 and lattice data are used to establish the similarity of the QCD vacuum and the dual superconductor medium. At the same time the AP method reveals the topological properties of confining configurations. In Section 6 the most probable candidates for such configurations are sought among classical solutions. A new principle is introduced to select solutions which are able to confine when a dilute gas is formed of them.

An interconnection of topology and stochastic properties of the QCD vacuum is discussed in Section 7.

In conclusion, the search for confining configurations is recapitulated and besides a simple picture of the confining vacuum and string formation is given as seen from different points of view. A possible temperature deconfinement scenario is briefly discussed.

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2. Definition of confinement and order parameters

Confinement is understood as the phenomenon of absence in physical spectrum of those particles (fields) which are present in the fundamental Lagrangian. In the case of QCD it means that quarks and gluons and in general all coloured objects cannot exist as separate asymptotic objects.

To be more exact let us note that sometimes the massless particles entering with zero bare masses in the Lagrangian, e.g. photons or gluons, can effectively acquire mass. This is the phenomenon of screening in plasma or in QCD vacuum above deconfinement temperature. Due to the definition of confinement given above, the screened asymptotic colour states cannot exist in the confining phase.

On the other hand in the deconfined phase quarks and gluons can evolve in Euclidean (or Minkowskian) time separately, unconnected by strings. From this fact it follows that the free energy at high temperature t is proportional to t^4 as usual for the Stephan – Boltzmann gas and it is natural to call this phase as the quark – gluon plasma (QGP). It turns out, however, that the QGP dynamics is very far from the dynamics of the ideal gas of quarks and gluons; a new characteristic interaction appears there, called the *magnetic confinement*. It will be discussed in detail in Conclusions, but now we come back to the confinement phase.

It is very important to discuss confinement in gauge-invariant terms. Then the physical contents of the confinement phenomenon can be best understood comparing a gauge-invariant system of an electron and positron (+e, −e) in QED and a system of quark and antiquark ($q\bar{q}$) in QCD, when no other charges are present. When the distance between +e and −e is large the electromagnetic interaction becomes negligible and the wave function (w.f.) of (+e, −e) factorises into a product of individual w.f. [the same is true for the gauge-invariant Green's function of (+e, −e)]. Therefore, the notion of isolated electromagnetic charge and its individual dynamics makes sense. In contrast to this, in the modern picture of confinement in agreement with the experiment and lattice data (see below), quark and antiquark attract each other with the force of approximately 14 ton. Therefore, q and $\bar{q}$ cannot separate and individual dynamics, individual w.f. and Green's function of a quark (antiquark) has in principle no sense: quark (or antiquark) is *confined* by its partner. This statement can be generalised to include all nonzero colour changes, e.g. two gluons are confined and never escape each other (later we shall discuss what happens when additional colour charges come into play).

It is also clear now that the phenomenon of confinement is connected to the formation of the string between colour charges, i.e. the string gives a constant force at large distances and the dynamics of colour charges without the string is inadequate there.

This picture is nicely illustrated by many lattice measurements of potential between static quarks (Fig. 1). One can clearly see in Fig. 1 the linear growth of potential $V(r) = \sigma r$ at large distances, $r \geqslant 0.25$ fm.

We shall formulate this as the first property of confinement.

I. *The linear interaction between coloured objects*.

To give an exact meaning to this statement, making possible to check it by lattice calculations, it is convenient to introduce the so-called Wilson loop [6] and to define through it the potential between colour changes. This is done as follows. Take a heavy quark Q and a heavy antiquark $\bar{Q}$ and consider a process, where the pair $Q\bar{Q}$ is created at some point x, then the pair separates at distance r and after a period of time T the pair annihilates at the point y. It is clear that trajectories of Q and $\bar{Q}$ form a loop C (starting at the point x and passing through the point y and finishing at the point x). The amplitude (Green's function) of such a process according to quantum mechanics, is proportional to the phase (Schwinger) factor $W \propto \exp(ig \int A_\mu j_\mu \, d^4x)$, where A_μ is the total colour vector potential of quarks and vacuum fields, and j_μ is the current of the $Q\bar{Q}$ pair, depicting its motion along the loop C. Taking into account that A_μ is a matrix in colour space ($A_\mu = A_\mu^a T^a$, $a = 1, \ldots, N_c^2 - 1$, and $T^a = \sigma^a/2$ for SU(2) and $T^a = \lambda^a/2$ for SU(3), where N_c is number of colours, σ^a and λ^a are the Pauli and the Gell-Mann matrices respectively) one should insert an ordering operator P, which orders these matrices along the loop, and the trace operator in colour indices, since quarks were created and annihilated in the white state of the object $Q\bar{Q}$. Hence one obtains the Wilson operator for a given field distribution

$$\langle W(C) \rangle = \mathrm{tr}\left[P \exp\left(ig \int_C A_\mu \, dx_\mu\right)\right],$$

where we have used the point-like structure of quarks, $j_\mu(x) \propto \delta^{(3)}\big(x - x(t)\big)\, dx_\mu/dt$.

Now one must take into account, that the vacuum fields $A_\mu(x)$ form a stochastic ensemble, and one should average over it. This is a necessary consequence of the vector character of $A_\mu(x)$ and of the Lorentz-invariance of vacuum, otherwise for any fixed function $A_\mu(x)$ this invariance would be violated contrary to the experiments.

Finally, in the field theory in general and for the vacuum description in particular it is convenient to use the Euclidean space – time for the path integral representation of partition functions, and also for the Monte-Carlo calculations on the lattice. There are at least two reasons for this. The first one is a

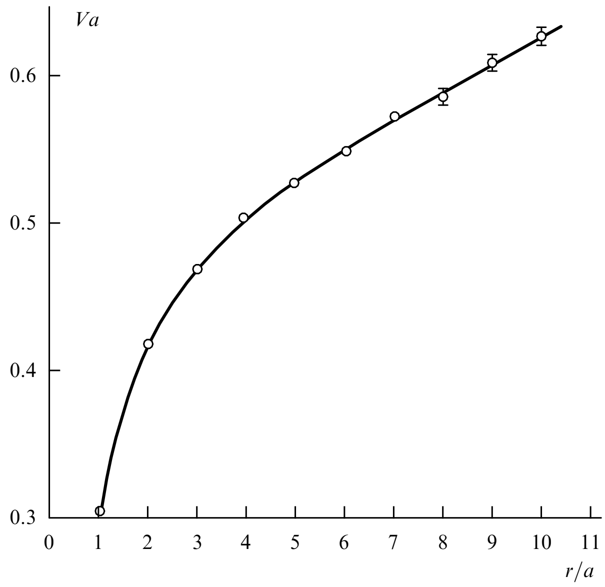

Figure 1. Potential between static quarks in the triplet representation of SU(3), computed in Ref. [10] on the lattice 32^4. The solid line is the fit of the form $C/r + \sigma r + \mathrm{const}$. The potential V and distance r are measured in lattice units a, equal to 0.055 fm for $\beta = 6.9$. Dynamical quarks are absent.

technical reason; the Euclidean path integrals have a real (and positive definite) measure $\exp(-S_E)$, where S_E is the Euclidean action, and the convergence of the path integrals is better founded (but not strictly proven in the continuum).

Another, and possibly deeper reason: until recently all nontrivial classical solutions in QCD (instantons, dyons, etc.) have been of a Euclidean nature, i.e. in the usual Minkowskian space–time they describe some tunnelling processes. As for the covalent bonds between atoms due to the electron tunnelling from one atom to another, the *Euclidean configurations in gluodynamics and QCD yield attraction*, i.e. the lowering of the vacuum energy, and therefore they are advantageous for the nonperturbative vacuum reconstruction (see discussion in Conclusions). Therefore, everywhere in the review we shall use the Euclidean space–time, i.e. we shift from x_0, A_0 to the real Euclidean components $x_4 = \mathrm{i}x_0$, $A_4 = -\mathrm{i}A_0$.

Now we shall return to the Wilson loop and connect it to the potential. To this end we recall that W is an amplitude, or Green's function of the $\mathrm{Q\bar{Q}}$ system, and therefore it can be expressed through the Hamiltonian H, namely, $W = \exp(-HT)$, where T is the Euclidean time. For heavy quarks the kinetic part of H vanishes and only the $\mathrm{Q\bar{Q}}$ potential $V(r)$ is left, so that we finally obtain for the averaged (over all vacuum field configurations denoted by angular brackets) Wilson loop

$$\left\langle W(C)\right\rangle = \left\langle \mathrm{tr}\left[P\exp\left(\mathrm{i}g\int_C A_\mu\,\mathrm{d}x_\mu\right)\right]\right\rangle = \exp\left[-V(r)T\right], \tag{1}$$

where the loop C can be conveniently chosen as a rectangular $r \times T$, and confinement corresponds to the linear dependence of the potential $V(r) = \sigma r$, where σ is called the string tension.

Having said this, one should add a few necessary elaborations. First, what happens when other charges are present, e.g. $\mathrm{q\bar{q}}$ pairs are created from vacuum. In both cases screening occurs, but whereas for $\mathrm{e\bar{e}}$ system nothing crucial happens at large distances, the $\mathrm{q\bar{q}}$ system can be spitted by an additional $\mathrm{q_1\bar{q}_1}$ pair into two neutral systems of $\mathrm{q\bar{q}_1}$ and $\mathrm{q_1\bar{q}}$ which can now separate: the string breaks into two pieces.

The same is always true for a pair of gluons which can be screened by gluon pairs from vacuum.

In practice (i.e. in lattice computations) the pair creation from the vacuum is suppressed even for gluons [7]; one of suppression factors is $1/N_c$ [8], another one is numerical and not yet understood; for $N_c = 3$ the overall factor is around 0.1, as can be seen, e.g., from the ratio of hadron resonance widths over their masses.

This circumstance allows one to see on the lattice the almost constant force between static q and $\mathrm{\bar{q}}$ up to the distance of around 1 fm or larger. This property can be seen in lattice calculations made with the account of dynamical fermions (i.e. additional quark pairs), e.g., in Fig. 2 one can see the persistence of linear confinement in all measured region with accuracy of 10%. Therefore, we formulate the second property of confinement in QCD, which should be obeyed by all realistic theoretical models.

II. *Linear confinement between static quarks persists also in the presence of* $\mathrm{q\bar{q}}$ *pairs in the physical region* (0.3 fm $\leqslant r \leqslant$ 1.5 fm).

Another important comment concerns quark (or gluon) dynamics at small distances. Perturbative interaction dominates there because it is singular; this can be seen in one-

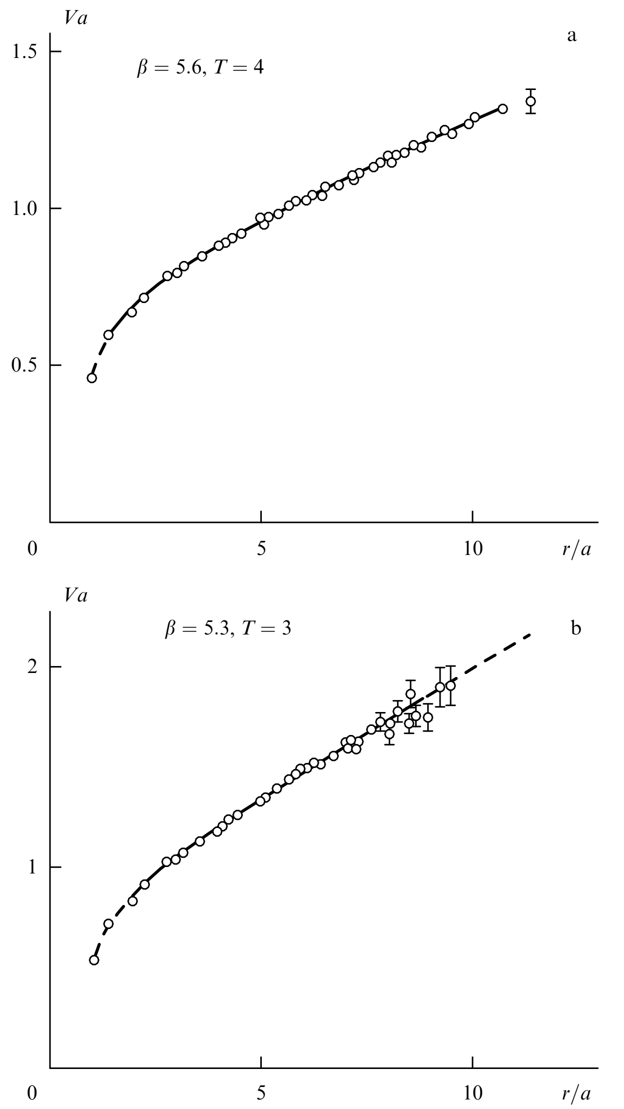

Figure 2. The same potential as in Fig. 1, but with dynamical quarks taken into account in two versions: (a) staggered fermions, $a \approx 0.11$ fm; (b) the Wilson fermions, $a \approx 0.16$ fm; calculations of Heller et al. (second entry of Ref. [10]).

gluon-exchange force of $4\alpha_s/3r^2$. Comparing it with the confinement force mentioned above ($\sigma = 0.2\ \mathrm{GeV}^2 \approx$ 14 ton), one can see that perturbative dynamics dominates for $r < 0.25$ fm.

At these distances quarks and colour charges in general can be considered independently, with essentially perturbative dynamics, which is supported by many successes of perturbative QCD.

Till now only colour charges in fundamental representation of $\mathrm{SU}(N_c)$ (quarks) have been discussed.

Very surprising results have been obtained for interaction of static charges in other representations. For example, adjoint charges, which can easily be screened by gluons from the vacuum, in lattice calculations are linearly confined in the physical region $r \leqslant 1.5$ fm) (Fig. 3).

One can partly understand this property from the point of view of large N_c: the screening part of potential $V_s \propto r^{-1}\exp(-\mu r)$ is suppressed by the factor $1/N_c^2$ as

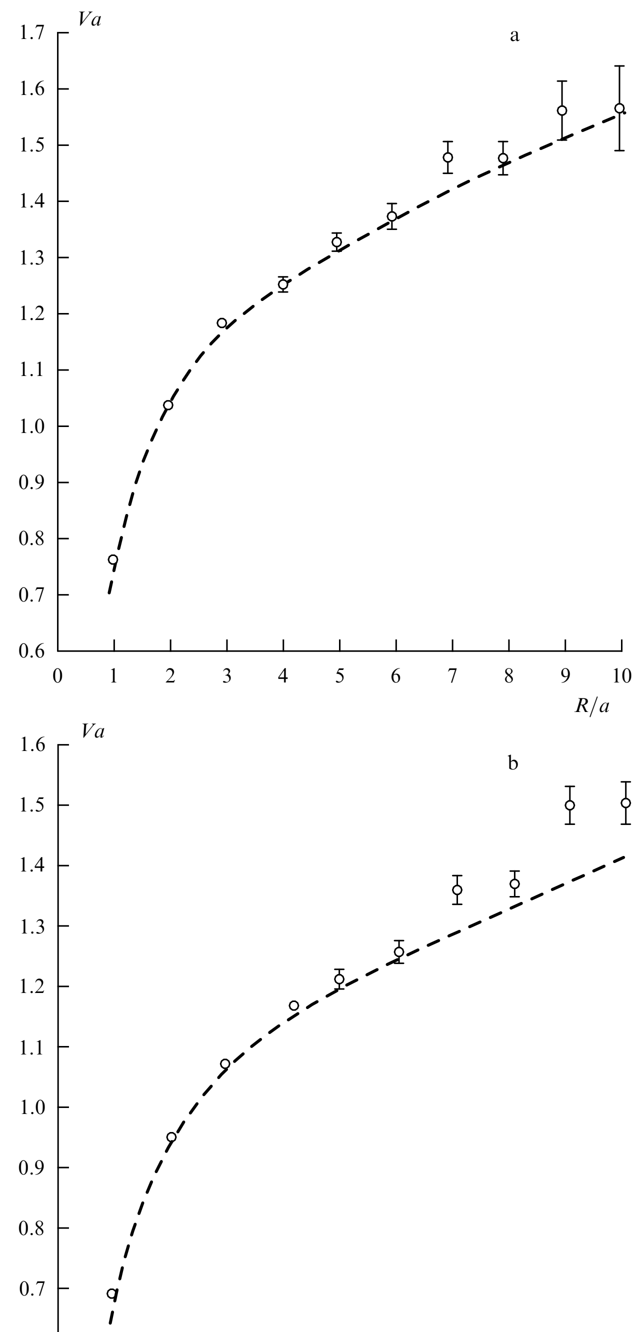

Figure 3. The same potential as in Fig. 1, but for quarks in the sextet (a) and octet (b) representation. Broken line is the triplet potential of Fig. 1 multiplied by the ratio of the Casimir operators, equal to 2.25 for the octet and 2.5 for the sextet.

compared with linear part, but prevails at large distances [7]:

$$\left\langle W_{\mathrm{adj}}(C)\right\rangle = C_1 \exp(-\sigma_{\mathrm{adj}}\, rT) + \frac{C_2}{N_{\mathrm{c}}^2} \exp\left[-V_{\mathrm{s}}(r)T\right]. \quad (2)$$

The same property holds for other charge representations [10], and moreover the string tension $\sigma(j)$ for given representation j satisfies approximate relation [7]

$$\frac{\sigma(j)}{\sigma(\mathrm{fund})} = \frac{C_2(j)}{C_2(\mathrm{fund})}\,, \quad (3)$$

where $C_2(j) \equiv C_2(j, N_{\mathrm{c}})$ is the quadratic Casimir operator,

$$C_2(\mathrm{adj}, N_{\mathrm{c}}) = N_{\mathrm{c}}, \qquad C_2(\mathrm{fund}, N_{\mathrm{c}}) = \frac{N_{\mathrm{c}}^2 - 1}{2N_{\mathrm{c}}}\,.$$

Correspondingly we formulate the third property of confinement.

III. *Adjoint and other charges are effectively confined in the physical region* ($r \leqslant 1.5$ fm) *with string tension satisfying* Eqn (3).

We shall conclude this section with two remarks: one concerning the interrelation of confinement and gauge invariance, another about many erroneous attempts to define confinement through some specific form of nonperturbative gluon and quark propagator.

Firstly, gauge invariance is absolutely necessary to study properly the mechanism of confinement. It requires that any gauge-noninvariant quantity, like quark or gluon propagator, 3-gluon vertex, etc., vanish when averaged over all gauge copies. One can fix the gauge only for the gauge-invariant amplitude, otherwise one lacks important part of dynamics, that of confinement.

As a popular example one may consider the 'nonperturbative' quark propagator, which is suggested in a form without poles at real masses. As stated previously, this propagator is meaningless, since on formal level it is gauge-noninvariant and vanishes upon averaging, and on physical grounds, the propagator of coloured object cannot be considered separately from other coloured partner(s), since it is connected by a string to it, and this string (confinement) dynamics dominates at large distances. The same can be said of the 'nonperturbative' gluon propagator behaving like $1/q^4$ at small q, which in addition imposes wrong singular nonanalytic behaviour of the two-gluon – glueball Green function.

Likewise, the well-known Dyson – Schwinger (DSE) equations for one-particle Green functions cannot be used for QCD in the confined phase, because again they are gauge-noninvariant and they should be a string connected to any of propagators of DSE, omitted there.

More subtle formal, but basically the same is the case of a Bethe – Salpeter equation (BSE) on the fundamental level, i.e, when the kernel of BSE contains coloured gluon or quark exchanges: any finite approximation (i.e. ladder type) of the kernel violates gauge invariance and loses confinement. When averaging in the amplitude over all vacuum fields is made, the resulting effective interaction can be treated approximately in the framework of BSE. Its value if any might lie in phenomenological applications and not in a fundamental understanding of the confinement, which is the primary purpose of the present study.

Let us turn now to the order parameters which define the confining phase. To distinguish between confined and deconfined phases several order parameters are used in absence of dynamical (sea) quarks. One of them is the Wilson loop introduced above [see Eqn (1)]. Confinement is defined as the phase where the area law (linear potential) is valid for large contours, whereas in the deconfining phase the perimeter law appears.

Another and sometimes practically more convenient for nonzero temperature T, is the order parameter called as the Polyakov line:

$$\left\langle \Omega(\mathbf{x})\right\rangle = \frac{1}{n}\left\langle \mathrm{tr}\left[P\exp\left(\mathrm{i}g\int_0^{\beta} A_4\,\mathrm{d}x_4\right)\right]\right\rangle, \qquad \beta = \frac{1}{T}\,. \quad (4)$$

Since $\langle \Omega \rangle$ is connected to the free energy F of isolated colour charge, $\langle \Omega \rangle = \exp(-F/T)$, vanishing of $\langle \Omega \rangle$ at $T < T_c$ means that F is infinite in the confined phase. On the other hand, vanishing of $\langle \Omega \rangle$ is connected to the $Z(N_c)$ symmetry, which is respected in the confining vacuum and broken in the deconfined phase. As the Wilson loops, the Polyakov lines can be defined both for fundamental and adjoint charges; the first vanish rigorously in the confined phase in absence of dynamical quarks, while adjoint lines are very small there. Strictly speaking, both the Wilson loop and the Polyakov line are not the order parameters when dynamical quarks are admitted in the vacuum; there is no area law for the large enough Wilson loops and $\langle \Omega \rangle$ does not vanish in the confined phase. However, for large N_c, and practically even for $N_c = 2, 3$ these quantities can be considered as approximate and useful order parameters even in presence of dynamical quarks, as will be seen in the following sections.

From the dynamical point of view the (approximate) validity of area law (linear potential) for any colour charges even in the presence of dynamical quarks at distances $r \leqslant 1.5$ fm means that strings are formed at these distances and string dynamics defines the behaviour of colour charges in the most physically interesting region.

3. Field correlators and confinement

As can be seen from the area law (1) the phenomenon of confinement necessarily implies the appearance of a new mass parameter in the theory, because the string tension σ has the dimension of $[\text{mass}]^2$.

Since perturbative QCD depends on mass scale only due to renormalisation through Λ_{QCD}, using the asymptotic freedom (and renormalisation group properties) one can express the coupling constant $g(\Lambda)$ at Λ scale through Λ_{QCD} as

$$g^2(\Lambda) = \frac{4\pi}{\beta_0 \ln(\Lambda^2/\Lambda^2_{\text{QCD}})}\,, \qquad \beta_0 = \frac{11}{3}\,N_c - \frac{2}{3}\,n_f\,.$$

And one can write

$$\sigma = \text{const}\,\Lambda^2_{\text{QCD}} \propto \Lambda^2 \exp\left(-\frac{16\pi^2}{\beta_0\, g^2(\Lambda)}\right), \tag{5}$$

where Λ is the cut-off momentum. It is clear from Eqn (5) that σ cannot be obtained from the perturbation series; hence, the source of σ and of the whole confinement phenomenon is purely nonperturbative. Correspondingly one should admit in the QCD vacuum the nonperturbative component and split the total gluonic vector potential

$$A_\mu = B_\mu + a_\mu\,, \tag{6}$$

where B_μ is nonperturbative and a_μ is perturbative part. As for B_μ, it can be

(a) quasiclassical, i.e. consisting of superposition of classical solutions like instantons, multiinstantons, dyons, etc; this possibility will be discussed below in Section 6;

(b) purely quantum (but nonperturbative); the picture of the Gaussian stochastic vacuum gives an example, which is discussed below.

It is important to stress at this point, that the formalism of field correlators, given below in this section, is of general character and allows to discuss both situations (a) and (b), quasiclassical and stochastic. In the first case, however, some modifications are necessary and they will be introduced at the end of the section. As mentioned above, confinement implies the string formation between the colour charges. To understand how the string is related to field correlators, consider a simple example of a nonrelativistic quark moving at a distance r from a heavy antiquark fixed at the origin. As is known from quantum mechanics, the quark Green function is proportional to the phase integral along its trajectory C:

$$G(\mathbf{r}, t) \propto \left\langle \exp\left[\mathrm{i}g\int_C A_\mu\big(\mathbf{r}(t'), t'\big)\,\mathrm{d}x_\mu\right]\right\rangle,$$

where the averaging is over all vacuum configurations. It is convenient to express A_μ though the field strength $F_{\mu\nu}$, since the latter would be the basic stochastic quantities, and this can be done, e.g., by using the Fock – Schwinger gauge

$$A_\mu(x) = \int_0^x F_{\nu\mu}(u)\alpha(u)\,\mathrm{d}u_\nu\,, \qquad \alpha(u) = \frac{u}{x}\,.$$

Hence, in the lowest order one obtains

$$G(\bar{r}, t) \propto 1 - \frac{g^2}{2}\int \mathrm{d}\sigma_{\nu\mu}(u)\,\mathrm{d}\sigma_{\nu'\mu'}(u')\big\langle F_{\nu\mu}(u)F_{\nu'\mu'}(u')\big\rangle + \ldots,$$

where $\mathrm{d}\sigma_{\nu\mu} = \alpha(u)\,\mathrm{d}x_\mu\,\mathrm{d}u_\nu$.

On the other hand one can introduce the potential $V(r)$ acting on the quark

$$G \propto \exp\left[-\int V(r, t')\,\mathrm{d}t'\right] \propto 1 - \int V(r, t')\,\mathrm{d}t'\,.$$

The string formation implies that $V(r, t)$ is proportional to r and this depends, as we see, on the field correlators $\langle FF \rangle$.

For the exact Lorentz-invariant treatment let us introduce gauge-invariant field correlators and express the Wilson loop average through the field correlators. This is done using the non-Abelian Stokes theorem [11]

$$\begin{aligned}\big\langle W(C)\big\rangle &= \left\langle \frac{1}{N_c}\,P\,\mathrm{tr}\left\{\exp\left[\mathrm{i}g\int_C A_\mu\,\mathrm{d}x_\mu\right]\right\}\right\rangle \\ &= \frac{1}{N_c}\left\langle P\,\mathrm{tr}\left\{\exp\left[\mathrm{i}g\int_S \mathrm{d}\sigma_{\mu\nu}F_{\mu\nu}(u, z_0)\right]\right\}\right\rangle,\end{aligned} \tag{7}$$

where we have defined

$$\begin{aligned}&F_{\mu\nu}(u, z_0) = \Phi(z_0, u)F_{\mu\nu}(u)\Phi(u, z_0),\\ &\Phi(x, y) = P\exp\left[\mathrm{i}g\int_y^x A_\mu\,\mathrm{d}z_\mu\right]\end{aligned} \tag{8}$$

and integration in Eqn (7) is over the surface S inside the contour C, while z_0 is an arbitrary point, which $\big\langle W(C)\big\rangle$ does not evidently depend on. In the Abelian case the parallel transporters $\Phi(z_0, u)$ and $\Phi(u, z_0)$ are cancelled and one obtains the usual Stokes theorem.

Note that the non-Abelian Stokes theorem (7) is gauge-invariant even before averaging over all vacuum configurations; the latter is implied by the angular brackets in Eqn (7).

One can now use the cluster expansion theorem [12] to express the right hand side of Eqn (7) in terms of field correlators [5], namely

$$\big\langle W(C)\big\rangle = \frac{1}{N_c}\,\mathrm{tr}\left\{\exp\left[\sum_{n=1}^{\infty}\frac{(\mathrm{i}g)^n}{n!}\int \mathrm{d}\sigma(1)\,\mathrm{d}\sigma(2)\ldots\mathrm{d}\sigma(n) \times \big\langle\big\langle F(1)\ldots F(n)\big\rangle\big\rangle\right]\right\}, \qquad (9)$$

where lower indices of $\mathrm{d}\sigma_{\mu\nu}$ and $F_{\mu\nu}$ are suppressed and $F(k) \equiv F_{\mu_k\nu_k}(u^{(k)}, z_0)$.

Note an important simplification: the averages $\langle\langle F(1)\ldots F(n)\rangle\rangle$ in the colour symmetric vacuum are proportional to the unit matrix in colour space, and the ordering operator P is not needed any more.

Eqn (9) expresses the Wilson loop in the terms of gauge-invariant field correlators (also called cumulants [12]) defined in the terms of field correlators as follows

$$\begin{aligned}
&\big\langle\big\langle F(1)F(2)\big\rangle\big\rangle \equiv \big\langle F(1)F(2)\big\rangle - \big\langle F(1)\big\rangle\big\langle F(2)\big\rangle\,,\\
&\big\langle\big\langle F(1)F(2)F(3)\big\rangle\big\rangle \equiv \big\langle F(1)F(2)F(3)\big\rangle\\
&\quad - \big\langle\big\langle F(1)F(2)\big\rangle\big\rangle\big\langle F(3)\big\rangle - \big\langle F(1)\big\rangle\big\langle\big\langle F(2)F(3)\big\rangle\big\rangle\\
&\quad - \big\langle F(2)\big\rangle\big\langle\big\langle F(1)F(3)\big\rangle\big\rangle - \big\langle F(1)\big\rangle\big\langle F(2)\big\rangle\big\langle F(3)\big\rangle\,. \qquad (10)
\end{aligned}$$

Let us have a look at the lowest cumulant

$$\big\langle F(x)\Phi(x,z_0)\Phi(z_0,y)F(y)\Phi(y,z_0)\Phi(z_0,x)\big\rangle\,. \qquad (11)$$

It depends not only on x, y but also on an arbitrary point z_0. In the case where a classical solution (dyon or instanton) is present, it is convenient to place z_0 at its centre, and then z_0 acquires a clear physical meaning. We shall investigate this case in detail in Section 6, but for now we shall consider the limit of stochastic vacuum, when the expansion (9) is particularly useful. To this end let us consider parameters on which a generic cumulant $\langle\langle F(1)\ldots F(n)\rangle\rangle$ depends. When all coordinates u^k coincide with z_0, one obtains condensate $\langle\langle (F_{\mu_n\nu_n}(0))^n\rangle\rangle$;† we assign to it an order of magnitude F^n. The coordinate dependence can be characterised by the gluon correlation length T_g, which is assumed to be of the same order of magnitude for all cumulants. Then the series in Eqn (9) have the following estimate

$$\big\langle W(C)\big\rangle \approx \frac{1}{N_c}\,\mathrm{tr}\left\{\exp\left[\sum_{n=1}^{\infty}\frac{(\mathrm{i}g)^n}{n!}F^n T_g^{2(n-1)} S\right]\right\}, \qquad (12)$$

where S is the area of the surface inside contour C. To obtain the result (12) we have taken into account that in each cumulant

$$\big\langle\big\langle F(x^{(1)},z_0)F(x^{(2)},z_0)\ldots F(x^{(n)},z_0)\big\rangle\big\rangle\,, \qquad (13)$$

whenever x and y are close to each other,

$$|x-y| \ll |x-z_0|\,, \qquad |y-z_0|\,,$$

the dependence on z_0 drops out; therefore in Eqn (13) in a generic situation when all distances $|x^{(i)}-x^{(j)}| \propto T_g \ll |x^{(i)}-z_0|$, $|x^{(j)}-z_0|$ one can omit the dependence on z_0.

† In the QCD sum rules all the terms of this kind for $n > 2$ usually cancel as they give zero after the so-called vacuum insertion.

The expansion in Eqn (12) is in powers of (FT_g^2) and, when this parameter is small,

$$FT_g^2 \ll 1\,, \qquad (14)$$

one gets the limit of the Gaussian stochastic ensemble, where the lowest (quadratic in $F_{\mu\nu}$) cumulant is dominant.

In the same approximation (e.g., $T_g \to 0$ while $\langle F_{\mu\nu}^2\rangle$ is kept fixed) one can disregard in this cumulant the dependence in z_0, using the equivalent (effective) form of Eqn (11):

$$D_{\mu\nu\lambda\sigma} \equiv \frac{1}{N_c}\,\mathrm{tr}\,\big\langle F_{\mu\nu}(x)\Phi(x,y)F_{\lambda\sigma}(y)\Phi(y,x)\big\rangle\,. \qquad (15)$$

The form (15) has a general decomposition in terms of two Lorentz scalar functions $D(x-y)$ and $D_1(x-y)$ [5]

$$\begin{aligned}
g^2 D_{\mu\nu\lambda\sigma} &= (\delta_{\mu\lambda}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\lambda})D(x-y)\\
&\quad + \frac{1}{2}\,\partial_\mu\Big\{\big[(h_\lambda\delta_{\nu\sigma} - h_\sigma\delta_{\nu\lambda}) + \ldots\big]D_1(x-y)\Big\}. \qquad (16)
\end{aligned}$$

Here the ellipsis implies terms obtained by the permutation of indices. It is important that the second term on the right hand side of Eqn (16) is a full derivative by construction.

Insertion of Eqn (16) into Eqn (9) allows us to obtain the area law of the Wilson loop with the string tension σ

$$\begin{aligned}
&\big\langle W(C)\big\rangle = \exp(-\sigma S_{\min})\,,\\
&\sigma = \frac{1}{2}\int D(x)\,\mathrm{d}^2x\big(1 + O(FT_g^2)\big)\,, \qquad (17)
\end{aligned}$$

where $O(FT_g^2)$ stands for the contribution of higher cumulants, and $S_{\min}$ is the minimal area for contour C.

Note that D_1 does not enter in σ, but gives rise to the perimeter term and to the higher order curvature terms. On the other hand the lowest order of perturbative QCD contributes to D_1 and not to D; namely, the one-gluon-exchange contribution is

$$D_1^{\mathrm{pert}}(x) = \frac{16\alpha_s}{3\pi x^4}\,. \qquad (18)$$

Nonperturbative parts of $D(x)$ and $D_1(x)$ have been computed on the lattice using the cooling method, which suppresses perturbative fluctuations [13], and are shown in Fig. 4. As one can see in Fig. 4, both functions are well described by an exponent in the measured region, and $D_1(x) \approx D(x)/3 \approx \exp(-x/T_g)$, where $T_g \approx 0.2$ fm. The smallness of T_g as compared to hadron size confirms the approximations made before, in particular the stochasticity condition (14). One should also take into account that F in Eqn (14) is an effective field in cumulants which vanish when vacuum insertion is made and therefore can be small as compared with F in the gluonic condensate.

The representation (16) is valid both for Abelian and non-Abelian theories, and it is interesting to understand whether the area law and nonzero string tension obtained in Eqn (17) could be valid also for QED (or U(1) in the lattice version). To check it we shall apply the operator $(1/2)\varepsilon_{\mu\nu\alpha\beta}\,\partial/\partial x_\alpha$ to both sides of Eqns (15), (16) [5].

In the Abelian case, when Φ is cancelled in Eqn (15), one obtains (the term with D_1 drops out)

$$\partial_\alpha\big\langle\tilde F_{\alpha\beta}(x)F_{\lambda\sigma}(y)\big\rangle = \varepsilon_{\lambda\sigma\gamma\beta}\,\partial_\gamma D(x-y)\,, \qquad (19)$$

where $\tilde F_{\alpha\beta} = \varepsilon_{\alpha\beta\mu\nu}F_{\mu\nu}/2$.

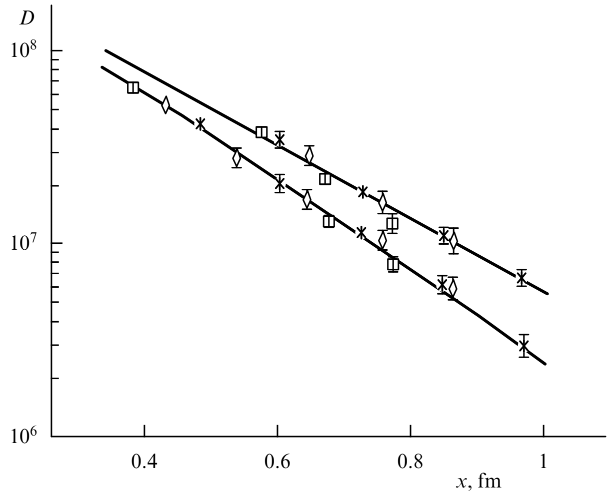

Figure 4. Correlators $D_\parallel(x) = D + D_1 + x^2\,\partial D_1/\partial x^2$ (lower set of points) and $D_\perp(x) = D + D_1$ (upper set of points) as functions of distance x; $\times$ correspond to $\beta = 5.8$; $\diamond$ — to $\beta = 5.9$ and $\square$ — to $\beta = 6.0$. Solid lines are the best fits in the form of independent exponents for $D(x)$ and $D_1(x)$. Computations from Ref. [13].

If magnetic monopoles are present in the Abelian theory (e.g., the Dirac monopoles) with the current $\tilde{j}_\mu$, one has

$$\partial_\alpha \tilde{F}_{\alpha\beta}(x) = \tilde{j}_\beta(x)\,. \tag{20}$$

In the absence of magnetic monopoles (for pure QED) the Abelian Bianchi identity (the second pair of the Maxwell equations) requires that

$$\partial_\alpha \tilde{F}_{\alpha\beta} \equiv 0\,. \tag{21}$$

Thus, for QED (without magnetic monopoles) the function $D(x)$ vanishes due to Eqn (19) and hence confinement is absent, as observed in nature.

In the lattice version of U(1) magnetic monopoles are present (as lattice artefacts) and the lattice formulation of our method would predict the confinement regime with nonzero string tension, as it is observed in Monte-Carlo calculations [14].

The latter can be connected through $D(x)$ to the correlator of magnetic monopole currents. Indeed, multiplying both sides of Eqn (19) with $(1/2)\varepsilon_{\lambda\sigma\gamma\delta}\,\partial/\partial y_\gamma$ one obtains

$$\left\langle \tilde{j}_\beta(x)\tilde{j}_\delta(y)\right\rangle = \left(\frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial y_\alpha}\,\delta_{\beta\delta} - \frac{\partial}{\partial x_\delta}\frac{\partial}{\partial y_\beta}\right) D(x-y)\,. \tag{22}$$

The form of Eqn (22) identically satisfies the monopole current conservation, i.e. applying $\partial/\partial x_\beta$ or $\partial/\partial y_\delta$ on both sides of Eqn (22) gives zero.

It is interesting to note that confinement [nonzero σ and D, see Eqn (17)] in U(1) theory with monopole currents occurs not due to average monopole density $\left\langle \tilde{j}_4(x)\right\rangle$, but rather due to more subtle feature such as correlator of monopole currents (22), which can be nonzero for the configuration where $\left\langle \tilde{j}_4(x)\right\rangle = 0$. The latter is fulfilled for the system with an equal number of monopoles and antimonopoles.

Note that our analysis here is strictly speaking applicable when stochasticity condition (14) is fulfilled. Therefore, the case of monopoles with the Dirac quantization condition is out of the region of Eqn (14) and needs some elaboration to be discussed later in this section.

Let us now turn to the non-Abelian case, again assuming the stochasticity condition (14), so that one can keep only the lowest cumulant [Eqns (15), (16)].

Applying as in the Abelian case the operator $(1/2)\varepsilon_{\mu\nu\alpha\beta}\,\partial/\partial x_\alpha$ to the right hand side of Eqns (15), (16) one obtains [15]

$$\left\langle D_\alpha \tilde{F}_{\alpha\beta}(x)\Phi(x,y)F_{\lambda\sigma}(y)\Phi(y,x)\right\rangle + \Delta_{\beta\lambda\sigma}(x,y) = \varepsilon_{\alpha\beta\lambda\sigma}\,\frac{\partial}{\partial x_\alpha}D(x-y)\,. \tag{23}$$

The term with D_1 in Eqn (23) drops out as in the Abelian case; the first term on the left hand side of Eqn (23) now contains the non-Abelian Bianchi identity term, which should vanish even in the presence of dyons (magnetic monopoles), i.e. classical solutions of the Yang–Mills theory, so that

$$D_\alpha \tilde{F}_{\alpha\beta}(x) = 0\,. \tag{24}$$

It is another question, whether or not in lattice formulation one can violate Eqn (24) in the definition of lattice artefact monopoles, similarly to the Abelian case. We shall discuss this topic when studying lattice results on the Abelian-projected monopoles in Section 5. To conclude the discussion of Eqn (23), one should define $\Delta_{\beta\delta\sigma}$; it appears only in the non-Abelian case due to the shift of the straight-line contour (x,y) of the correlator (15) into the position $(x+\delta x, y)$, which is implied by differentiation $\partial/\partial x_\alpha$. This 'contour differentiation' is well known [16] and leads to the answer:

$$\begin{aligned}\Delta_{\beta\lambda\sigma}(x,y) &= \mathrm{i}g\int_y^x \mathrm{d}z_\rho\,\alpha(z)\\ &\times \left\langle \mathrm{tr}\left[\tilde{F}_{\alpha\beta}(x)\Phi(x,z)F_{\alpha\rho}(z)\Phi(z,y)F_{\lambda\sigma}(y)\Phi(y,x)\right.\right.\\ &\left.\left. - \tilde{F}_{\alpha\beta}(x)\Phi(x,y)F_{\lambda\sigma}(y)\Phi(y,z)F_{\alpha\rho}(z)\Phi(z,x)\right]\right\rangle\,.\end{aligned} \tag{25}$$

An especially simple form of Eqn (23) occurs when using Eqn (24) we tend x to y; one obtains [15]

$$\left.\frac{\mathrm{d}D(z)}{\mathrm{d}z^2}\right|_{z=0} = \frac{g}{8}\,f^{\mathrm{abc}}\left\langle F^a_{\alpha\beta}(0)F^b_{\beta\gamma}(0)F^c_{\gamma\alpha}(0)\right\rangle\,. \tag{26}$$

Thus, confinement (nonzero σ because of nonzero D) occurs in the non-Abelian case due to the purely non-Abelian correlator $\left\langle \mathrm{tr}\,(F_{\alpha\beta}F_{\beta\gamma}F_{\gamma\alpha})\right\rangle = 3\left\langle \mathrm{tr}\,(E_iE_jB_k)\right\rangle\varepsilon_{ijk}$. To see the physical meaning of this correlator, one can visualise magnetic and electric field strength lines (FSL) in the space. Each magnetic monopole is a source of FSL, whether it is a real object (classical solution or external object like the Dirac monopole) or a lattice artefact.

In the non-Abelian theory these lines may form branches, e.g., electric FSL may emit a magnetic FSL at some point, playing the role of magnetic monopole at this point. This is what exactly the nonzero triple correlator $\left\langle \mathrm{tr}\,(FFF)\right\rangle$ implies. Note that this could be a purely quantum effect, and no real magnetic monopoles are necessary for this mechanism of confinement.

So far we have discussed confinement in terms of the lowest cumulant $D(x)$, which is justified when stochasticity condition (14) is fulfilled and $D(x)$ gives a dominant contribution. Let us now turn to other terms in the cluster expansion (9). It is clear that the general structure of higher cumulants is much more complicated than Eqn (16), but a Kronecker-type term $D(x_1, x_2, \ldots, x_n) \prod \delta_{\mu_l \mu_k}$ similar to $D(x-y)$ in Eqn (16) and other terms containing derivatives and coordinate differences like D_1 are always present. The term $D(x_1, x_2, \ldots, x_n)$ contributes to string tension, and application of the same operator $(1/2)\varepsilon_{\mu\nu\alpha\beta}\,\partial/\partial x_\alpha$ again reveals the non-Abelian Bianchi identity term (24) and an analogue of $\Delta_{\beta\lambda\sigma}$ in Eqn (23). This means that the string tension in general case is a sum:

$$\sigma = \sum_{n=2}^{\infty} \sigma^{(n)} ,$$

$$\sigma^{(n)} = g^n \int \langle\langle F(1)F(2)\ldots F(n)\rangle\rangle\, \mathrm{d}\sigma(2)\ldots \mathrm{d}\sigma(n)\,. \qquad (27)$$

When the stochasticity condition (14) holds, the lowest term, $\sigma^{(2)}$, dominates in the sum; in general case all terms in sum (27) are significant. The most important example of such a situation is the case of a quasiclassical vacuum, which we shall now discuss, while postponing a detailed consideration to Section 6.

For a dilute gas of classical solutions the role of vacuum correlation length T_g is played by the size of the solution ρ. For the correlator $\langle F(x_1)\ldots F(x_n)\rangle$ the essential nonzero result occurs when all $x_1, \ldots, x_n$ are inside the radius ρ of the solution (e.g., dyon or instanton).

On the other side the typical F for the solution, e.g., $F = F_{\mu\nu}(0)$, is connected to ρ by the value of topological charge Q; for instanton

$$Q = \frac{g^2}{32\pi^2} \int \mathrm{d}^4 x\, F_{\mu\nu}^2\,. \qquad (28)$$

Since Q is an integer, one immediately obtains that

$$(gF\rho^2)^2 \sim n\,, \qquad n = 1, 2, \ldots \qquad (29)$$

The same estimate holds for the dyon, whose solution is discussed in Section 6.

Hence, the series (12) and (27) have an expansion parameter gFT_g^2 of the order of unity and may not converge. A more detailed analysis of the gas of instantons and magnetic monopoles shows that the string tension series (27) for instantons [for simplicity centres of instantons and monopoles were taken on the plane (*12*) of the Wilson loop] look like [17]

$$\sigma = \rho_0\big(1 - \langle\cos\beta\rangle\big)\,;$$

$$\beta = g\int F_{12}(z)\,\mathrm{d}^2 z = 2\pi\,, \qquad \sigma = 0\,, \qquad (30)$$

while for magnetic monopoles

$$\beta = \pi\,, \qquad \sigma = 2\rho_0 \qquad (31)$$

where ρ_0 is the surface density of instantons (monopoles).

Note that $\sigma^{(2)} = \rho_0\langle\beta^2\rangle/2$ in both cases is positive, and for instantons the total sum for σ vanishes, while for monopoles σ is nonzero. As we shall see below this is in agreement with the statement in Section 6: all topological charges (29) yield flux through the Wilson loop which is equal to $2\pi Q$, and for (multi)instantons with $Q = n = 1, 2, \ldots$ the Wilson loop is $W = \exp(\mathrm{i}2\pi Q) = 1$, and no confinement results for the dilute gas of such solutions. For magnetic monopoles (dyons) topology is different and elementary flux is equal to π, bringing confinement for the dilute gas of monopoles in agreement with Eqn (31).

Thus, the lowest cumulant $\langle F(x)\Phi F(y)\Phi\rangle$ might give a misleading result in the case of quasiclassical vacuum, and one should sum up all the series to get the correct answer as in Eqns (30), (31). Therefore, to treat the vacuum containing topological charges one should separate them and write their contribution explicitly, while the rest, quantum fluctuations with $FT_g^2 \ll 1$, can be considered via the lowest cumulants. An example of such vacuum with instanton gas and confining configurations was studied in Ref. [18] to obtain the breaking of chiral symmetry; this work demonstrates the usefulness of such an approach.

We shall conclude this section with a discussion of confinement for charges in higher representations. As stated in the previous section, our definition of confinement, based on lattice data, requires the linear potential between static charges in any representation, with string tension proportional to the quadratic Casimir operator.

So, let us consider the Wilson loop (1) for the charge in some representation; the latter was not specified above in all equations leading to Eqn (27). One can write in general

$$A_\mu(x) = A_\mu^a T^a\,, \qquad \mathrm{tr}(T^a T^b) = \frac{1}{2}\,\delta^{ab}\,. \qquad (32)$$

Similarly to Eqn (9) one has for the representation $j \equiv (m_1, m_2, \ldots)$ of the SU(N) group with $N(j)$ dimension

$$\langle W(C)\rangle = \frac{1}{N(j)} \times \mathrm{tr}_j\left\{\exp\left[\sum_{n=1}^{\infty}\frac{(\mathrm{i}g)^n}{n!}\int \mathrm{d}\sigma(1)\ldots\mathrm{d}\sigma(n)\langle\langle F(1)\ldots F(n)\rangle\rangle\right]\right\} \qquad (33)$$

and by the usual arguments one has Eqn (27) .

Due to the colour neutrality of the vacuum each cumulant is proportional to the unit matrix in the colour space, e.g., for the lowest cumulant one has

$$\langle F(1)F(2)\rangle_{ab} = \langle F^c(1)F^d(2)\rangle T_{an}^c T_{nb}^d = \langle F^e(1)F^e(2)\rangle\,\frac{1}{N_c^2-1}\,T_{an}^c T_{nb}^c = \Lambda^{(2)} C_2(j)\,\hat{1}_{ab}\,, \qquad (34)$$

where the definition

$$T^c T^c = C_2(j)\,\hat{1}\,, \qquad (35)$$

is used and a constant not depending on representation is introduced:

$$\Lambda^{(2)} \equiv \frac{1}{N_c^2-1}\,\langle F^e(1)F^e(2)\rangle\,, \qquad (36)$$

and also the colour neutrality of vacuum

$$\langle F^c(1)F^d(2)\rangle = \delta_{cd}\,\frac{\langle F^e(1)F^e(2)\rangle}{N_c^2-1} \qquad (37)$$

is used. For the next quartic cumulant one has

$$\langle\langle F(1)F(2)F(3)F(4)\rangle\rangle_{\alpha\varepsilon} = \langle\langle F^{a_1}(1)F^{a_2}(2)F^{a_3}(3)F^{a_4}(4)\rangle\rangle T_{\alpha\beta}^{a_1} T_{\beta\gamma}^{a_2} T_{\gamma\delta}^{a_3} T_{\delta\varepsilon}^{a_4} = \Lambda_1^{(4)}\big(C_2(j)\big)^2\delta_{\alpha\varepsilon} + \Lambda_2^{(4)}\big(T^{a_1}T^{a_2}T^{a_1}T^{a_2}\big)_{\alpha\varepsilon}\,. \qquad (38)$$

Thus, one can see in quartic cumulant the higher orders of the quadratic Casimir operator and the higher Casimir operators.

The string tension for the representation j is the coefficient of the diagonal element in Eqns (34) and (38):

$$\sigma(j) = C_2(j) \int \frac{g^2 \Lambda^{(2)}}{2}\, \mathrm{d}^2 x + O\big(C_2^2(j)\big)\,, \tag{39}$$

where the term $O\big(C_2^2(j)\big)$ contains the higher degrees of $C_2(j)$ and the higher Casimir operators.

Comparing our result (39) with the lattice data [10] (see Fig. 3) one can see that the first quadratic cumulant should be dominant as it ensures proportionality of $\sigma(j)$ to the quadratic Casimir operator.

Another interesting and important check of the dominance of bilocal correlator (of the Gaussian stochasticity) is the calculation of the QCD string profile, done in Ref. [19]. The string profile means the distribution $\rho_{\parallel}$ of the longitudinal component of the colourelectric field as a function of distance $x_{\perp}$ to the string axis. This distribution can be expressed through an integral of functions $D(x)$, $D_1(x)$ [18], and with the measured values of D, D_1 from Ref. [13] one can compute $\rho_{\parallel}(x_{\perp})$ and compare it with independent measurements. This comparison was done in Ref. [19] and is shown in Fig. 5. One can see a good agreement of the computed $\rho_{\parallel}(x_{\perp})$ with 'experimental' values. Thus, MVC gives a good description of data in the simplest (bilocal) approximation, even for such delicate characteristics as field correlators inside the string.

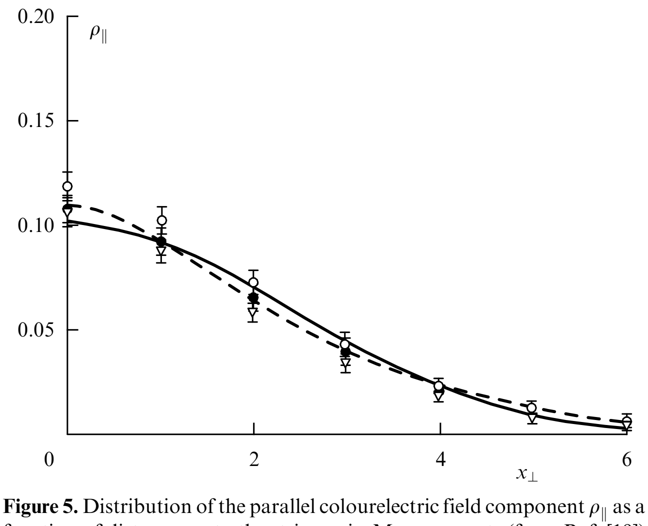

Figure 5. Distribution of the parallel colourelectric field component $\rho_{\parallel}$ as a function of distance $x_{\perp}$ to the string axis. Measurements (from Ref. [19]) are made at different distances $x_{\parallel}$ from the string end: ● — $x_{\parallel} = 3a$; ◇ — $x_{\parallel} = 5a$ and ▽ — $x_{\parallel} = 7a$ (a is the lattice unit). Solid line is the Gaussian fit, dashed line is calculated in Ref. [19] with the help of $D(x)$ taken from Ref. [13].

4. The dual Meissner mechanism, confinement and superconductivity

The physical essence of the confinement phenomenon is the formation of the string between the probing charges introduced into vacuum, which in turn means that the electric field distribution drastically changes from the usual dipole picture (for the empty vacuum) and it focused instead into a string picture in the confining vacuum. From the point of view of macroscopic electrodynamics of media this effect can be described by introducing dielectric function $\varepsilon(x)$ and electric induction $\mathbf{D}(x)$ together with electric field $\mathbf{E}(x)$: $\mathbf{D}(x) = \varepsilon(x)\mathbf{E}(x)$.

One can then adjust $\varepsilon(x)$ or better $\varepsilon(D)$, or $\varepsilon(E)$ to obtain the string formation. Another possibility is to choose the effective action as a function of $F^2_{\mu\nu}$ in such a way as to reproduce string-type distribution. This direction was reviewed in Ref. [1], and the main conclusion reached long ago [20] was that the physical vacuum of QCD considered as a medium could be called a pure dielectric, i.e. $\varepsilon = 0$ far from probing charges [21].

As also shown it is possible to choose $\varepsilon(E)$ in such a way as to obtain a string of constant radius [21] or with radius slowly dependent on length. We shall not follow neither these results, nor results of the so-called dielectric model [1, 21], referring the reader to the mentioned literature.

Instead we shall focus in this and the following sections on another approach, which has proved useful in recent years: confinement as the dual Meissner effect [3] and the AP method ideologically connected to it [4].

The physical idea used by 't Hooft and Mandelstam [3] is the analogy between the Abrikosov string formation in type II superconductor between magnetic poles and the proposed string formation between colour electric charges in QCD.

We shall study this analogy from different points of view:

(1) energetics of vacuum, i.e. minimal free energy of vacuum;

(2) classical equations of motion (Maxwell and London equations);

(3) condensate formation and symmetry breaking;

(4) vacuum correlation functions of fields and currents.

We shall consider in this section the 4d generalisation of the Ginzburg – Landau model of superconductivity, which is called the Abelian Higgs model with the Lagrangian [22]:

$$\mathcal{L} = -\frac{1}{4} F^2_{\mu\nu} - |D_{\mu}\varphi|^2 - \frac{\lambda}{4}\big(|\varphi|^2 - \varphi_0^2\big)^2\,,$$

$$D_{\mu} = \partial_{\mu} - \mathrm{i}eA_{\mu}\,. \tag{40}$$

This model is known to possess classical solutions, namely, the Nilsen – Olesen strings, which are 4d generalisation of the Abrikosov strings, occurring in the type II superconductor. The latter can be described by the Ginzburg – Landau Lagrangian, when coupling constants are chosen correspondingly [23].

The Lagrangian (40) combines two fields: the electromagnetic field (A_{μ}, $F_{\mu\nu}$) which will be an analogue of the gluonic QCD field, and the complex Higgs field $\varphi(x)$, which describes the amplitude of the Cooper-paired electrons in a superconductor. When $\lambda \to \infty$ the wave functional $\Psi\{A_{\mu}, \varphi(x)\}$ has a strong maximum around $\varphi(x) = \varphi_0$, which means formation of the condensate of the Cooper pairs of amplitude φ_0.

Let us look more closely at the model (40) discussing points (1) – (4) successively.

(1) The energy density corresponding to Eqn (40) is

$$\varepsilon = \frac{\mathbf{E}^2 + \mathbf{B}^2}{2} + |\mathbf{D}\varphi|^2 + |D_0\varphi|^2 + \frac{\lambda}{4}\big(|\varphi|^2 - \varphi_0^2\big)^2\,. \tag{41}$$

From Eqn (41) it is clear that in the absence of external sources and for large λ the lowest (vacuum) state corresponds to $\varphi = \varphi_0 = \text{const}$, and $\mathbf{E} = \mathbf{B} = 0$. For future comparison with QCD it is worthwhile to stress that formation of the condensate $\langle F_{\mu\nu}^2 \rangle$ is not advantageous for large λ, since the mixed term $|A_\mu \varphi|^2$ will yield a very large positive contribution. *Condensate of electric charges φ repels and suppresses electromagnetic field $F_{\mu\nu}$ everywhere*. The same situation occurs in the non-Abelian Higgs, i.e. the Georgi – Glashow, model. There appear two phases depending on values of λ, g and for large λ the deconfined phase with $\varphi = \varphi_0$ persists. Here the word deconfined means that external (colour) electric charges are not confined, while magnetic monopoles can be confined. For smaller values of λ one can reach a region where it is advantageous to form the condensate $\langle F_{\mu\nu}^2 \rangle$ and then possibly nonzero $D(x)$. As we discussed in Section 3, the confinement of colourelectric charges is possible in this phase. But let us return to the Abelian Higgs model and to the limit of large λ, which is of primary interest to us.

In this case it is advantageous to form the condensate of electric charges ($\varphi(x) = \varphi_0$); the condensate of electromagnetic field is suppressed and one obtains the ANO strings connecting magnetic poles. This is the confinement of magnetic monopoles and mass generation of (colour) electromagnetic field, leading to the deconfinement of electric charges.

In the dual picture one would assume that the condensate of magnetic charges (monopoles) would help to create strings of electric field, connecting (colour)electric charges, yielding confinement of the latter. Our tentative guess is that the condensate of magnetic monopoles (dyons) in gluodynamics is associated with the gluonic condensate $\langle F_{\mu\nu}^2 \rangle$.

(2) We shall now discuss the structure of the ANO strings from the point of view of classical equations of motion. The Maxwell equations give

$$\operatorname{rot} \mathbf{B} = \mathbf{j}\,, \tag{42}$$

where $\mathbf{j}$ is the microscopic current of electric charges, including the condensate of the Cooper pairs.

To obtain a closed equation for $\mathbf{B}$ one needs the specific feature of a superconductor in the form of the London equation

$$\operatorname{rot} \mathbf{j} = \delta^{-2} \mathbf{B}\,. \tag{43}$$

The latter can be derived from the Ginzburg – Landau-type Lagrangian (40). Indeed, writing the current for the Lagrangian (40) in usual way, we have

$$j_\mu = \mathrm{i}e(\phi^+ \partial_\mu \phi - \partial_\mu \phi^+ \phi) - 2e^2 A_\mu |\phi|^2\,. \tag{44}$$

Assume that there exists a domain where ϕ is already constant and A_μ is still nonzero (we shall define this region later in a better way). Applying the rot operation to both sides of Eqn (44) one obtains the London equations (43) and δ is defined by

$$\delta^{-2} = 2e^2 \phi_0^2\,. \tag{45}$$

Insertion of Eqn (43) into Eqn (42) yields equation for $\mathbf{B}$:

$$\Delta \mathbf{B} - \delta^{-2} \mathbf{B} = 0\,. \tag{46}$$

Solving Eqn (46) one obtains the exponential fall-off of $\mathbf{B}$ away from the centre of the ANO string:

$$B(r) = \text{const}\, K_0\left(\frac{r}{\delta}\right), \quad B(r) \propto \exp\left(-\frac{r}{\delta}\right), \quad r \gg \delta\,. \tag{47}$$

It is clear from Eqn (47) that δ^{-1} is the photon mass generated by the Higgs mechanism. On the other hand, the field ϕ has its own correlation length ξ, connected to the mass of quanta of the field ϕ (the 'Higgs mass'):

$$\xi = \frac{1}{m_\phi}\,, \qquad m_\phi^2 = 2\lambda \phi_0^2\,. \tag{48}$$

As is known [23], the London limit for the type II superconductor corresponds to the relations

$$\delta \gg \xi\,, \qquad \text{or} \qquad e \ll \lambda\,. \tag{49}$$

One can also calculate the string energy for the unit length (the string tension) for the string of the minimal magnetic flux of 2π. Using Eqns (41), (47), (48) one obtains [23]

$$\sigma_{\text{ANO}} = \frac{\pi}{\delta^2} \ln \frac{\delta}{\xi}\,, \qquad \frac{\delta}{\xi} \gg 1\,. \tag{50}$$

Note that the main contribution to σ_{ANO} comes from the term $|\mathbf{D}\phi|^2$ in Eqn (41).

Thus, the physical picture of the ANO string in the London limit implies mass generation of the magnetic quanta ($m = 1/\delta$), which are much less than the Higgs mass $m_\phi = 1/\xi$.

It is instructive to see how the screening of the magnetic field (mass generation) occurs.

At first, magnetic field creates around its flux (field-strength line) the circle of the current $\mathbf{j}$ from the superconducting medium, as described by the London equation (43). Then the Maxwell equation (42) tells that around the induced current $\mathbf{j}$ there appears a circulating magnetic field, directed *opposite* to the original magnetic field $\mathbf{B}$ and proportional to it. As a result, the magnetic field is partly screened in the middle and completely screened far from the centre of the magnetic flux (which is the centre of the ANO string).

The profile of the string, i.e. $B_\parallel(r)$, as a function of the distance from the string axis is shown in Fig. 6, as obtained from Eqns (47).

It is interesting to compare this ANO string profile with the corresponding profile obtained in the gluodynamics. As we discussed in the previous section, the distribution of the parallel component of the colourelectric field was measured in Ref. [24] and it also exponentially decrease far from the axis, as can be seen in Fig. 6. Both profiles of the ANO string and of the QCD string in Fig. 6 are very similar, thus supporting the idea of the dual Meissner mechanism.

To examine more closely the similarity of classical Eqns (42), (43) with the corresponding equations, obtained in the lattice Monte-Carlo simulations, one needs firstly an instrument to recognise the similarity of the effective Lagrangians in model (40) and in QCD. This is discussed in the next section.

We shall conclude point (2) of this section by discussing parameters δ_{QCD} and ξ_{QCD}, which are dual-analogical to δ and ξ [see Eqns (44), (45)]. This is done via the AP method in

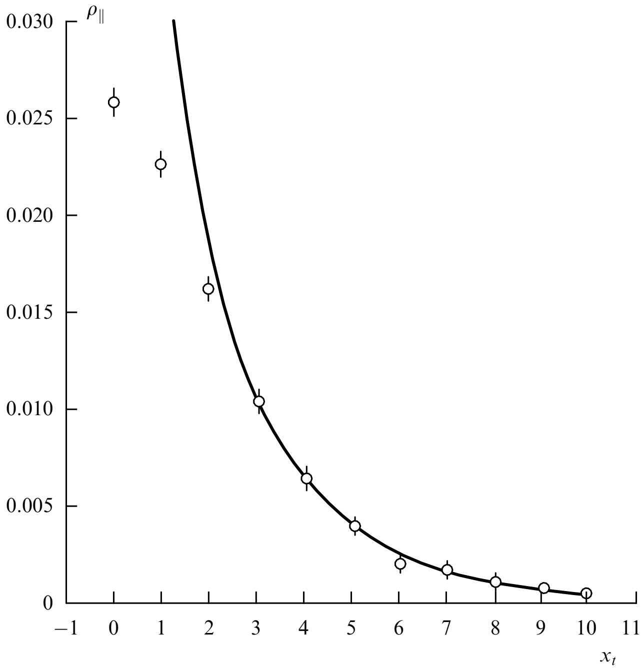

Figure 6. The same distribution $\rho_\parallel$ as in Fig. 5 for SU(2) gluodynamics measured in Ref. [24] on the lattice 24^4 for $\beta = 2.7$, $x_\parallel = 5a$ ($\bigcirc$) as compared to the $B_\parallel$ distribution in the Abrikosov string (solid line). The length of the string is 10a. The growth of the solid line at small x_t is nonphysical and is due to violation of approximations made in Eqn (47).

Ref. [25] with the result (see Fig. 7)

$$\delta_{\mathrm{QCD}} \approx \xi_{\mathrm{QCD}} \approx 0.2 \text{ fm} . \tag{51}$$

Thus in QCD the situation is somewhat in the middle between the type I and type II dual superconductors. The calculation of the effective potential $V(\phi)$ in SU(2) gluodynamics, made recently in Ref. [26], shows a two-well structure, but with a rather shallow well, hence the effective λ_{QCD} in the Lagrangian (40) is not large, again in agreement with Eqn (51). A stimulating discussion of properties of dual monopoles and their measurement on the lattice is contained in Ref. [27].

(3) Condensate formation and symmetry breaking.

The phenomenon of superconductivity is usually associated with the formation of the condensate of the Cooper pairs (although it is not necessary).

The notion of condensate is most clearly understood for the noninteracting Bose–Einstein gas at almost zero temperature, where the phenomenon of the Bose–Einstein condensation takes place. Ideally in quantum-mechanical systems the condensate can be considered as a coherent state.

When interaction is taken into account, the meaning of the condensate is less clear [28].

In quantum field theory one associates condensate with the properties of the wave functional and/or the Fock columns. Again, in the case of noninteraction one can construct the coherent state in the second-quantised formalism (see, e.g., the definition of wave operators in the superfluidity case in Ref. [29]).

One of the properties of such a state is the fixed phase of the wave functional, which means that the U(1) symmetry is violated.

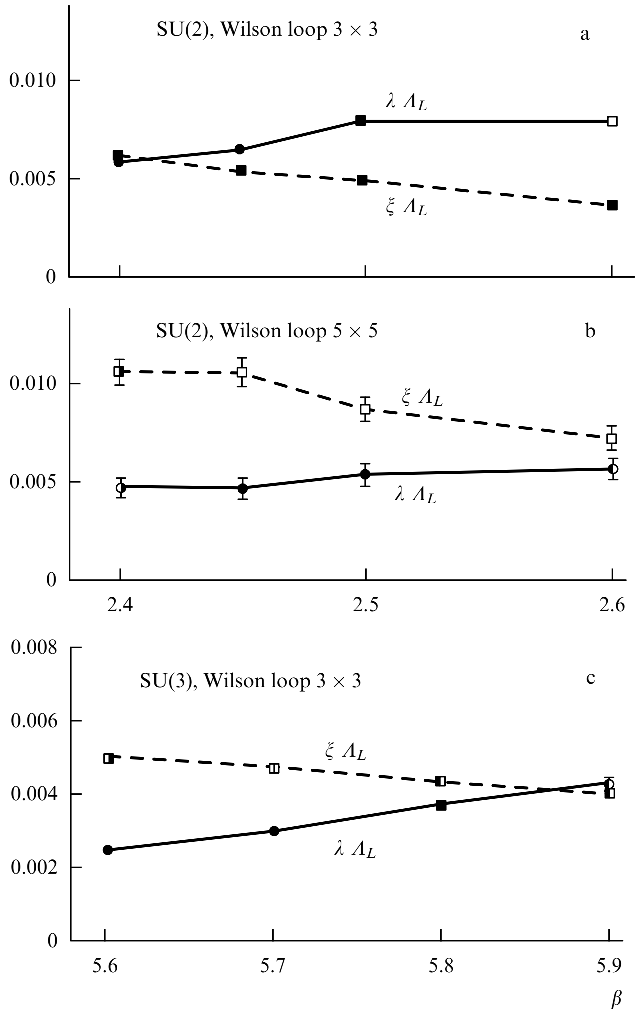

Figure 7. Lattice measurements [25] of the penetration length $\delta \equiv \lambda$ and the coherence length ξ as functions of $\beta = 2N_c/g^2$ for AP configurations: (a) and (b) in SU(2) gluodynamics; (c) in SU(3) theory. The values of δ and ξ are defined by comparison of the field distribution and the AP monopole currents with the solution of the Ginzburg–Landau equations.

The simplest example is given by the Ginzburg–Landau theory (40), where in the approximation when $\lambda \to \infty$, the wave functional $\Psi\{\varphi\}$ can be approximated by the classical solution $\varphi = \varphi_0$, $\Psi\{\varphi\} \to \Psi\{\varphi_0\}$.

The solution $\varphi = \varphi_0$, where φ_0 has a fixed phase, violates U(1) symmetry of the Lagrangian (40), and one has the phenomenon of spontaneous symmetry breaking (SSB) [30]. The easiest way to exemplify SSB is the double-well Higgs potential as in Eqn (40). Therefore, looking for the dual Meissner mechanism in QCD (gluodynamics) one may identify the magnetic monopole condensate $\tilde{\varphi}$, dual to the Cooper pair condensate $\varphi = \varphi_0$, and find the effective potential $V(\tilde{\varphi})$, demonstrating that it has a typical double-well shape. Such an analysis was performed in Ref. [26] using the AP method and will be discussed in the next section.

(4) Finally, in this section we shall examine the dual Meissner mechanism from the point of view of field and current correlators [31]. This will enable us to formulate the mechanism in the most general form, valid both in (quasi) classical and quantum vacuum.

One can use the correlator (16) to study confinement of both magnetic and electric charges (this was discussed in Section 3, see Eqn (19) and subsequent). Let us rewrite Eqn (16) for correlators of electric and magnetic fields separately:

$$\langle E_i(x)E_j(y)\rangle = \delta_{ij}\left(D^E + D_1^E + h_4^2\,\frac{\partial D_1^E}{\partial h^2}\right) + h_i h_j\,\frac{\partial D_1^E}{\partial h^2}\,, \quad (52)$$

$$\langle H_i(x)H_j(y)\rangle = \delta_{ij}\left(D^H + D_1^H + h^2\,\frac{\partial D_1^H}{\partial h^2}\right) - h_i h_j\,\frac{\partial D_1^H}{\partial h^2}\,, \quad (53)$$

where $h_\mu = x_\mu - y_\mu$, $h^2 = h_\mu h_\mu$.

In Eqns (52), (53) we have specified correlators D, D_1 for electric and magnetic fields separately, since in Lorentz-invariant vacuum $D^E = D^H$, $D_1^E = D_1^H$; otherwise, e.g. in the Ginzburg – Landau model, electric and magnetic correlators may differ, as also in any theory for nonzero temperature.

Now let us compare the Wilson loop averages for electric and magnetic charges. In case of electric charges the result is the area law (17) with the function $D \to D^E$ responsible for confinement.

Now we consider a magnetic charge in the contour C in the plane (*14*); the corresponding Wilson loop is

$$\langle \tilde W(C)\rangle = \left\langle \exp\left(\mathrm{i}g\int \tilde F_{14}\,\mathrm{d}\sigma_{14}\right)\right\rangle = \exp(-\sigma^* S_{\min})\,. \quad (54)$$

Here $\tilde F_{\mu\nu}$ is the dual field, $\tilde F_{\mu\nu} = (1/2)e_{\mu\nu\alpha\beta}F_{\alpha\beta}$, and $\tilde F_{14} = H_1$. From Eqn (17) one obtains

$$\sigma^* = \frac{g^2}{2}\int \mathrm{d}^2x\, D_1^H(x)\left[1 + O(T_g^2 H)\right]. \quad (55)$$

Eqn (55) is something of a surprise. For electric charges D^E yields confinement and is nonperturbative, supported by magnetic monopoles [see Eqn (22)], while D_1^E contributes perimeter correction to the area law and contains also perturbative contributions like a Coulomb term.

The same electric charges in the plane (*23*) bring about the area law (17) again with $D \to D^H$. Also D^H is coupled to magnetic monopoles in the vacuum; indeed, taking divergence from both sides of Eqn (53) one gets

$$\langle \operatorname{div} H(x), \operatorname{div} H(y)\rangle = -\partial^2 D^H(x-y)\,. \quad (56)$$

Instead, the confinement of external magnetic charges [Eqn (55)] is coupled to the function D_1^H [or D_1^E if one takes $\langle \tilde W(C)\rangle$ for the loop in the plane (*23*)].

Thus duality of electric-magnetic external charges requires interchange $D^H, D^E \leftrightarrow D_1^E, D_1^H$.

At this point one should be careful and separate out the perturbative interaction, which is contained in D_1, namely, one should replace in Eqn (55) D_1^H by $\tilde D_1^H$, where

$$\tilde D_1^H = D_1^H - \frac{4e^2}{x^4}\,. \quad (57)$$

It is important to stress again that it is the nonperturbative contents of correlators, which may create a new mass parameter like σ^*; the latter should enter therefore in correlators (the perturbative term contributes the usual Coulomb-like interaction, which is technically easier to consider not as a part of D_1^H, but to separate at earlier stage; see, e.g., Ref. [32]).

This mass creation can be visualised in the Ginzburg – Landau model, where D_1 can be computed explicitly from Eqn (40):

$$D_1^{\mathrm{LG}}(x-y) = \left(e^2|\phi|^2 - \partial^2\right)^{-1}_{xy} \approx \exp\left(-m|x-y|\right), \quad (58)$$

where in the asymptotic region $|\phi| = |\phi_0|$ and $m = e|\phi_0| = 1/\delta$.

From the point of view of duality $D_1^H(x)$, Eqn (58) should be compared with the behaviour of $D^E(x)$, which was discussed above in Section 3 and measured in Ref. [13]:

$$D(x) = D^E(x) = \exp\left(-\mu|x|\right), \qquad \mu \approx 1\ \mathrm{GeV}\,. \quad (59)$$

Eqns (58), (59) demonstrate the validity of the dual Meissner mechanism on the level of field correlators.

5. The Abelian Projection method

In the previous section it was shown how the string is formed between magnetic sources in the vacuum described by the Abelian Higgs model.

In the dual Meissner mechanism of confinement [53] it is assumed that the condensate of magnetic monopoles occurs in QCD; it creates strings between colour electric charges. There are two possible ways to proceed from this point. In the first, one may assume the form of Abelian or non-Abelian Higgs model for dual gluonic field and scalar field of magnetic monopoles. This type of approach was pursued in Ref. [33] and phenomenologically is quite successful. The linear confinement appears naturally and even spin-dependent forces are predicted in reasonable agreement with experiment [34]. We shall not go into details of this very interesting approach, referring the reader to the papers cited, since the most fundamental part of the problem, i.e. the derivation of this dual Meissner model from the first principles, namely the QCD Lagrangian, is missing in it.

Instead we turn to another direction, which has been pursued very intensively over the last 8 – 10 years, namely, the AP method dating from the seminal paper by 't Hooft [4].

The main problem is how to recognise configurations with properties of magnetic monopoles, which are responsible for confinement. 't Hooft's suggestion [4] is to choose a specific gauge, where monopole degrees of freedom hidden in a given configuration become evident. The corresponding procedure was elaborated both in continuum and on lattice [35]. The most subsequent efforts have been devoted to the practical separation (the Abelian projection) of lattice configurations and study of the separated degrees of freedom and construction of the effective Lagrangian for them. We start with the formal procedure in continuum for $\mathrm{SU}(N_c)$ gluodynamics following Refs [4, 35].

For any composite field X transforming as an adjoint representation like $F_{\mu\nu}$, e.g.,

$$X \to X' = VXV^{-1}\,, \quad (60)$$

let us find the specific unitary matrix V (the gauge), where X is diagonal:

$$X' = VXV^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)\,. \quad (61)$$

For X from the Lie algebra of $\mathrm{SU}(N_c)$, one can choose $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots \lambda_N$. It is clear that V is determined up to left multiplication by a diagonal $\mathrm{SU}(N_c)$ matrix.

This matrix belongs to the Cartan or the largest Abelian subgroup of $\mathrm{SU}(N_c)$, $\mathrm{U}(1)^{N_c-1} \subset \mathrm{SU}(N_c)$.

Now we transform A_μ to the gauge (61):

$$\tilde{A}_\mu = V\left(A_\mu + \frac{\mathrm{i}}{g}\,\partial_\mu\right)V^{-1} \tag{62}$$

and consider how components of $\tilde{A}_\mu$ transform under $\mathrm{U}(1)^{N-1}$. The diagonal ones

$$a^i_\mu \equiv (\tilde{A}_\mu)_{ii} \tag{63}$$

transform as 'photons':

$$a^i_\mu \to a^i_\mu = a^i_\mu + \frac{1}{g}\,\partial_\mu \alpha_i\,, \tag{64}$$

while nondiagonal, $c^{ij}_\mu \equiv (\tilde{A}_\mu)_{ij}$, transform as charged fields:

$$c'^{ij}_\mu = \exp\left[\mathrm{i}(\alpha_i - \alpha_j)\right]c^{ij}_\mu\,. \tag{65}$$

Here i, j are colour indices, $1 \leqslant i, j \leqslant N_c$.

But as 't Hooft remarks (see Ref. [4]), this is not the whole story. There appear singularities (due to possible coincidence of two or more eigenvalues λ_i) with the properties of magnetic monopoles. To make it explicit we consider the 'photon' field strength as in Ref. [35]:

$$\begin{aligned} f^i_{\mu\nu} &= \partial_\mu a^i_\nu - \partial_\nu a^i_\mu \\ &= VF_{\mu\nu}V^{-1} + \mathrm{i}g\left[V\left(A_\mu + \frac{\mathrm{i}}{g}\,\partial_\mu\right)V^{-1},\, V\left(A_\nu + \frac{\mathrm{i}}{g}\,\partial_\nu\right)V^{-1}\right], \end{aligned} \tag{66}$$

and define the monopole current:

$$K^i_\mu = \frac{1}{8\pi}\,\varepsilon_{\mu\nu\sigma\rho}\,\partial_\nu f^i_{\rho\sigma}\,, \qquad \partial_\mu K^i_\mu = 0\,. \tag{67}$$

Since $F_{\mu\nu}$ is regular, the only singularity giving rise to K^i_μ is the commutator term in Eqn (66), otherwise the smooth part of a^i_μ does not contribute to K^i_μ because of antisymmetric tensor.

Hence, one can define the magnetic charge $m^i(\Omega)$ in the 3d region Ω:

$$m^i(\Omega) = \int_\Omega \mathrm{d}^3\sigma_\mu\, K^i_\mu = \frac{1}{8\pi}\int_{\partial\Omega} \mathrm{d}^2\sigma_{\mu\nu}\,\varepsilon_{\mu\nu\rho\sigma} f^i_{\rho\sigma}\,. \tag{68}$$

Consider now the situation when two eigenvalues of Eqn (61) coincide, e.g. $\lambda_1 = \lambda_2$. This may happen at one 3d point in Ω, $x^{(1)}$, i.e. on the line in 4d, which one can visualise as the magnetic monopole world line. The contribution to $m^i(\Omega)$ comes only from the infinitesimal neighbourhood B_ε of $x^{(1)}$:

$$\begin{aligned} m^i\left(B_\varepsilon(x^{(1)})\right) &= \frac{\mathrm{i}}{4\pi}\int_{S(\varepsilon)} \mathrm{d}^2\sigma_{\mu\nu}\,\varepsilon_{\mu\nu\rho\sigma}\left[V\partial_\rho V^{-1}, V\partial_\sigma V^{-1}\right]_{ii} \\ &= -\frac{\mathrm{i}}{4\pi}\int \mathrm{d}^2\sigma_{\mu\nu}\,\varepsilon_{\mu\nu\rho\sigma}\partial_\rho\left[V\partial_\sigma V^{-1}\right]_{ii}\,. \end{aligned} \tag{69}$$

The term $V\partial_\sigma V^{-1}$ is singular and should be treated with care. To make it explicit, one can write

$$V = W\begin{bmatrix} \cos\frac{1}{2}\theta + \mathrm{i}\boldsymbol{\sigma}\mathbf{e}_\phi \sin\frac{\theta}{2} & 0 \\ 0 & 1 \end{bmatrix}, \tag{70}$$

where W is a smooth $\mathrm{SU}(N)$ function near $x^{(1)}$. Inserting it in Eqn (69) one obtains

$$m^i\left(B_\varepsilon(x^{(1)})\right) = \frac{1}{8\pi}\int_{S(\varepsilon)} \mathrm{d}^2\sigma_{\mu\nu}\,\varepsilon_{\mu\nu\rho\sigma}\partial_\rho(1-\cos\theta)\partial_\sigma\phi[\sigma_3]_{ii}\,, \tag{71}$$

where ϕ and θ are azimuthal and polar anges, respectively.

The integrand in Eqn (71) is a Jacobian displaying a mapping from $S^2_\varepsilon(x^{(1)})$ to $(\theta, \phi) \propto \mathrm{SU}(2)/\mathrm{U}(1)$.

Since

$$\Pi_2\left(\frac{\mathrm{SU}(2)}{\mathrm{U}(1)}\right) = Z\,, \tag{72}$$

the magnetic charge is $m^i = \pm 1/2$.

From the derivation above it is clear that the point $x = x^{(1)}$, where $\lambda_1(x^{(1)}) = \lambda_2(x^{(1)})$, is a singular point of the gauge-transformed $\tilde{A}_\mu$ and a^i_μ, and the latter behaves near $x = x^{(1)}$ as $O\left(|x - x^{(1)}|^{-1}\right)$, while the Abelian projected field strength $f^i_{\mu\nu}$ is $O\left(|x - x^{(1)}|^{-2}\right)$, like the field of a point-like magnetic monopole.

However several points should be stressed now:

(1) the original vector potential A_μ and $F_{\mu\nu}$ are smooth and do not show any singular behaviour;

(2) at large distances $f^i_{\mu\nu}$ is not, generally speaking, monopole-like, i.e. does not decrease as $|x - x^{(1)}|^{-2}$, so that similarity to the magnetic monopole can be seen only in topological properties in the vicinity of the singular point $x^{(1)}$;

(3) the fields A_μ and a^i_μ have in general nothing to do with classical solutions and may be quantum fluctuations; actually almost any field distribution in the vacuum may be the Abelian projected into a^i_μ, $f^i_{\mu\nu}$ and magnetic monopoles can be detected then.

Examples of this statement will be given below, but before doing that we must say a few words about the choice of the field X in Eqn (60) and more generally about the explicit gauge choice.

By now the most popular choice for the adjoint operator X [see Eqn (60)] is the fundamental Polyakov line for nonzero temperature:

$$L_{ij}(\mathbf{x}) = \left[P\exp\left(\mathrm{i}g\int_0^\beta \mathrm{d}x_4\, A_4(x_4, \mathbf{x})\right)\right]_{ij}\,, \tag{73}$$

where $\beta = 1/T$, and T is the temperature, $A_\mu(x_4, \mathbf{x})$ is required to be periodic in x_4, and i, j are fundamental colour indices.

Another widely used gauge is the so-called maximal Abelian gauge (MAG) [35], which in lattice notations can be expressed as a gauge where the following quantity is maximised:

$$R = \sum_{s,\mu} \mathrm{tr}\left(\sigma_3 \tilde{U}_\mu(s)\sigma_3 \tilde{U}^+_\mu(s)\right)\,. \tag{74}$$

Here the link matrix $U_\mu \propto \exp(\mathrm{i}gA^{(s)}_\mu \Delta z_\mu)$, and $\tilde{U}_\mu$ is gauge-transformed (with the help of V) matrix:

$$\tilde{U}_\mu(s) = V(s)U_\mu(s)V^{-1}(s+\mu)\,. \tag{75}$$

In the continuum MAG is characterised by the condition, which in SU(2) case looks most simple:

$$(\partial_\mu \pm \mathrm{i}gA^3_\mu)A^\pm_\mu = 0\,, \qquad A^\pm_\mu = A^1_\mu \pm \mathrm{i}A^2_\mu\,. \tag{76}$$

As will be seen, the choice of gauge in the AP method is crucial, e.g., for the minimal Abelian gauge corresponding to the minimum of R [see Eqn (74)] the Abelian projected monopoles have no influence on confinement [36].

Since the total number of papers on the AP is now enormous, let us discuss shortly the main ideas and results. Most results are obtained by performing the AP on the lattice. (A short introduction and discussion of lattice technique is given in Ref. [27]). The separation of monopole degrees of freedom was done as follows. For an Abelian projected link (75) one can define the U(1) angle θ_μ:

$$\tilde{U}_\mu = \exp(\mathrm{i}\theta_\mu \sigma_3)\,, \qquad -\pi \leqslant \theta_\mu \leqslant \pi\,. \tag{77}$$

Then for the plaquette $\tilde{U}_{\mu\nu} \propto \tilde{U}_\mu \tilde{U}_\nu \propto \sum \exp(\mathrm{i}\theta_{\mu\nu}\sigma_3)$ one can write $-4\pi \leqslant \theta_{\mu\nu} \leqslant 4\pi$ and define the 'Coulomb part' of $\theta_{\mu\nu}$, $\bar{\theta}_{\mu\nu} = \mathrm{mod}_{2\pi}\{\theta_{\mu\nu}\}$, so that

$$\bar{\theta}_{\mu\nu} = \theta_{\mu\nu} + 2\pi n_{\mu\nu} \tag{78}$$

where $n_{\mu\nu}$ counts the number of Dirac strings across the plaquette $\mu\nu$.

Now one can calculate different observables in AP and find the contribution to them separately from the 'Coulomb part' $\bar{\theta}_{\mu\nu}$ and the monopole part $m_{\mu\nu} \equiv 2\pi n_{\mu\nu}$. This has been done in a number of papers (see review in Ref. [37]) and the monopole dominance was demonstrated

(1) for the string tension [38];

(2) for fermion propagators and hadron masses [39];

(3) for topological susceptibility [40].

In Fig. 8 a comparison is made of the total AP contribution to the string tension and of the monopole part, which is seen to be dominant as compared to the Coulomb part.

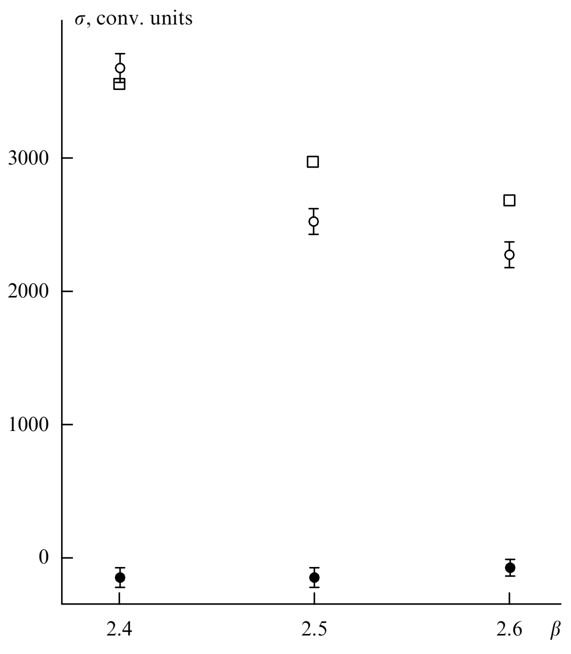

Figure 8. The string tension measured in Ref. [43] for all AP configurations (□) and separately for AP monopoles (○) and 'photons' (●) as functions of $\beta = 4/g^2$ in SU(2) gluodynamics.

Another line of activity in AP is the derivation of the effective Lagrangians for AP degrees of freedom (see, e.g., Ref. [41]). The resulting Lagrangian, however, does not confine AP neutral objects such as 'photons' and the latter should contribute to the spectrum in contradiction to the experiment. Therefore, the Lagrangian is not very useful both phenomenologically and fundamentally. We shall not discuss these Lagrangians but simply refer the reader to the cited literature. Let us come back to the investigation of the confinement mechanism with the help of the AP method.

A direct check of the dual Meissner mechanism of confinement should contain at least two elements: a detection of the dual London current and a check of magnetic monopole condensation. The first was done in Ref. [42]. The dual London equation (43) is

$$\mathbf{E} = \delta^2 \,\mathrm{rot}\,\mathbf{j}_\mathrm{m}\,, \qquad \delta = \frac{1}{m}\,. \tag{79}$$

For the ideal type II superconducting picture, it is necessary that $\delta \gg \xi$, where $\xi = m_\phi^{-1}$ is the Higgs mass (corresponding to the magnetic monopole condensation). In practice (in Ref. [42]) it was found that $\delta \approx \xi$, and therefore the condensate is 'soft', implying that exact solutions of the ANO string are required. Such analysis was performed in Ref. [43] for SU(2) and SU(3) and recently in Ref. [44].

The string profile $\rho(x_\perp)$ was also studied using AP and the resulting $\rho_{\mathrm{AP}}(x_\perp)$ [45] is similar to obtained in the full lattice simulation [46]. Thus, the whole picture of currents and density is compatible with dual superconductivity.

Now we come to the second check, i.e. to search of monopole condensation. Of special interest is the problem of definition of monopoles. In early lattice studies [35] the AP monopoles in the maximally Abelian gauge have been identified through magnetic currents on the links of the dual lattice and the perimeter density of the currents was measured below and above transition temperature T_c, showing a strong decrease of this density at $T > T_\mathrm{c}$.

Later it was realised [47] that monopole density defined in this way may not be a good characteristic of dual superconductivity and monopole condensate; for the latter one needs the creation operator of the magnetic monopole. In this way one can define the dual analogue of the Higgs field φ, and settle the question of condensation. The general mathematical construction of the monopole creation operator was given in Ref. [48]. Several groups used this U(1) construction for the AP monopoles [49].

Another construction was used for the U(1) theory in Ref. [50] and, as in Ref. [49], the condensation of monopoles was also demonstrated. In the SU(2) case, analysis was performed in Refs [51] and [26].

In Ref. [26] an important step was taken in finding an effective potential $V(\varphi)$ for the monopole creation operator φ (defined similarly to the Froelich – Marchetti method [48]; the exact equivalence to Ref. [48] is still not proven in Ref. [26]).

If one is to believe the dual Meissner picture of confinement, then one should expect the two-well structure of $V(\varphi)$, symmetric with respect to the change $\varphi \to -\varphi$. One can see in Fig. 9a the right hand side of $V(\phi)$ (for positive ϕ), which indeed has a minimum at $\phi = \phi_\mathrm{c}$ shifted to the right from $\phi = 0$ for values of $\beta = 4/g^2$ in the region of confinement. In Fig. 9b the same quality $V(\phi)$ is measured in the deconfinement phase and, as one can see, the minimum $\varphi = \varphi_\mathrm{c}$ is at zero, $\phi_\mathrm{c} = 0$. In this way the analysis of Ref. [26] gives the

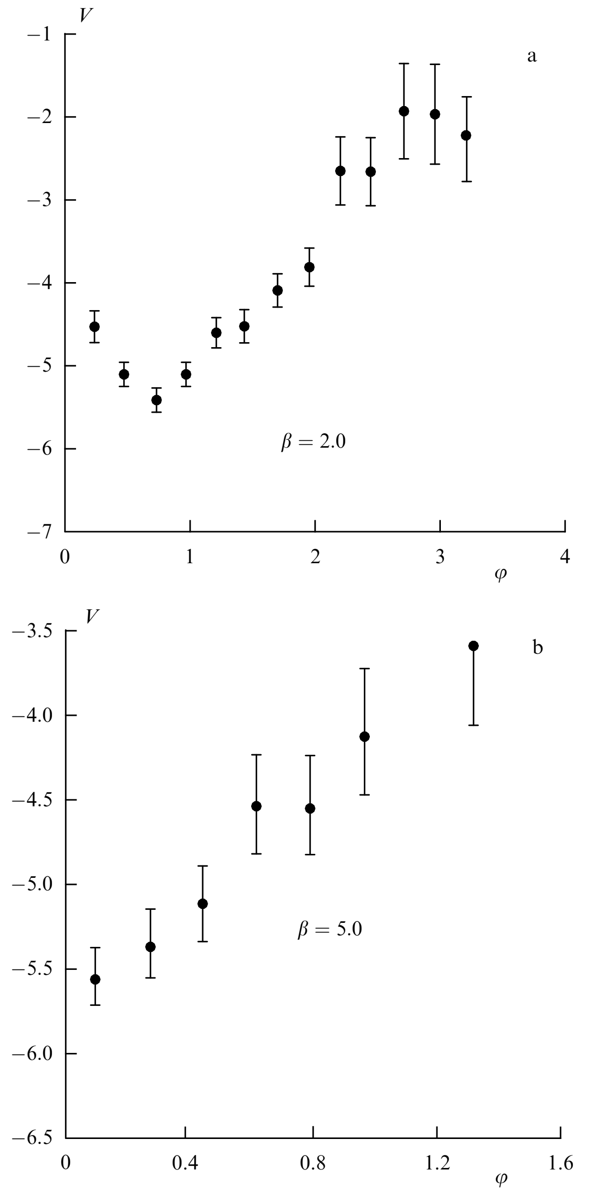

Figure 9. Effective potential of the AP monopole field φ, defined according to Ref. [48], measured in Ref. [26] for two values of β in SU(2) gluodynamics: (a) corresponding to confinement and (b) deconfinement.

evidence of AP monopole condensation. Note, however, that strictly speaking, condensation should be proven in the London limit ($\lambda \to \infty$), otherwise quantum fluctuations of φ might prevail for a shallow well like that in Fig. 9a (it should be recalled that the system has a finite number of degrees of freedom in a finite lattice volume). Until now we have said nothing about the nature of configurations, which due to AP disclose a magnetic monopole structure and ensure confinement. They could be classical configurations or quantum fluctuations (the latter possibility is preferred in most research).

Recently an interesting AP analysis of classical configurations was done [52–54]. We shall mostly examine the first paper [52], which created a branch of further activity. The authors of Ref. [52] make the AP analytically of an isolated instanton, multiinstanton, and the Prasad–Sommerfield monopole and demonstrate in all cases the appearance of a straight-line monopole current. In the first case the current is concentrated at the centre of the instanton and its direction depends on the parametrization chosen. Later (Ref. [53]) the same type of analysis was made numerically with the instanton–antiinstanton gas, and the appearance of monopole-current loops (of the size of an instanton radius and stable with respect to quantum fluctuations) was demonstrated.

Hence, everything looks as if there is a hidden magnetic monopole inside an instanton. This result is extremely surprising from several points of view. Firstly, the flux of magnetic monopole through the Wilson loop is equal to π (this will be shown in the next section), whereas the same flux of instanton is equal to zero (modulo is 2π). This fact helps to explain why the monopole gas may ensure confinement, while the instanton gas does not. Therefore, the identification of monopoles and instantons is not possible. Secondly, confinement in the instanton gas was shown to be absent in several independent calculations [55] and the appearance of rather large monopole loops in this gas looks suspiciously [53]. To understand what happens in the AP method and whether it can generate monopoles where they were originally absent, we turn again to the case of one instanton [52] and take into account that AP contains a singular gauge transformation, which transforms an originally smooth field $F_{\mu\nu}$ into a singular one, namely [52],

$$F_{\mu\nu}(x) = VF_{\mu\nu}(x)V^{-1} + F_{\mu\nu}^{\text{sing}}\,, \tag{80}$$

where

$$F_{\mu}^{\text{sing}}(x) = -\mathrm{i}V(x)\left[\partial_\mu\partial_\nu - \partial_\nu\partial_\mu\right]V^{-1}(x)\,. \tag{81}$$

The matrix $V(x)$ is singular and $F_{\mu\nu}^{\text{sing}}$ therefore does not vanish. Hence, the correlator function $D(x)$ [of the fields (15) and currents (22)] acquires due to this singular gauge transformation a new term:

$$D(x) \to D(x) + D^{\text{sing}}(x)\,, \tag{82}$$

where

$$D^{\text{sing}}(x-y) \propto \left\langle F^{\text{sing}}(x)F^{\text{sing}}(y)\right\rangle\,. \tag{83}$$

The same $F_{\mu\nu}^{\text{sing}}$ causes the appearance of a magnetic monopole current which passes through the centre of an isolated instanton [52] . Therefore the AP method indeed inserts a singularity of magnetic monopole type into practically any configuration and, therefore, cannot be a reliable method for separation of confining configurations. On the other hand, the use of AP for lattice configurations reproduces well the confinement observables, e.g., the string tension [38], which means that confining contributions readily pass the AP test.

From this point of view it is interesting to consider another example, i.e. the dyonic (Prasad–Sommerfield) solution, also studied in Ref. [52]. It appears that the AP monopole current passes exactly through the multiinstanton centres and coincides with the trajectory of the physical dyon, which can be calculated independently, i.e. the AP magnetic monopole current coincides with the total magnetic monopole current. This again supports the idea that the genuine confining configurations are well projected by the AP method.

6. Search for classical solutions. Monopoles, multiinstantons, and dyons

It was noted in Section 5 that the AP method provides no answer to the question of the nature of confining configurations. They can be both classical fields and quantum fluctuations. Nothing about it can be inferred from the field correlators, although lattice measurements of correlators do yield some information on the possible profile of confining configurations.

At the same time interesting information can be obtained from the lattice calculations using the so-called cooling method [56], where, at each step of cooling, quantum fluctuations are suppressed more and more, and configurations evolve in the direction in which the action decreases. Roughly speaking, these results demonstrate that out of tens of thousands of original configurations (which are mostly quantum noise) at some step of cooling only a few (sometimes 15 – 25) are retained, which ensure the same string tension, as the original ('hot') vacuum. With additional cooling the number of configurations drops to several units [they are, mostly, (anti)instantons] and confinement disappears.

Thus, one may assume that confining configurations in the vacuum differ from usual quantum fluctuations, and their action is probably larger than instantonic action or they are less stable.

It is therefore interesting to scrutinise all existing classical solutions and check whether one can build up the confining vacuum from these solutions.

In this section we shall study several classical solutions: instantons, dyons, lattice periodic instantons and give a short discussion of some other objects, such as torons.

After disposing the individual properties of those we specifically concentrate on the contribution of each of the object to the Wilson loop (what we call as the elementary flux of the object) and argue that flux, proportional to π, of dyons and twisted instantons with $Q = 1/2$ may yield confinement, in contrast to the case of instantons with flux equal to 2π.

To prove this one should construct the dilute gas of objects, what we shall do in the most nontrivial example of dyons.

6.1 Classical solutions, i.e. solutions of the equation

$$D_\mu F_{\mu\nu} = 0 \tag{84}$$

can be written in the (anti)selfdual case in the form of the so-called 't Hooft ansatz [57] or in the most general form of the Atiya – Drinfeld – Hitchin – Manin ansatz [58].

In the first (simpler) case one has

$$A_\mu^a = -\frac{1}{g}\,\bar\eta_{\mu\nu}^a\,\partial_\nu \ln W\,, \tag{85}$$

where $\bar\eta_{\mu\nu}^a$ is the 't Hooft symbol:

$$\bar\eta_{\mu\nu}^a = \begin{cases} e_{a\mu\nu}\,, & \mu,\nu = 1,2,3\,,\\ \delta_{a\nu}\,, & \mu = 4\,,\\ -\delta_{a\mu}\,, & \nu = 4\,, \end{cases} \tag{86}$$

and W satisfies equation $\partial^2 W = 0$ with a particular solution:

$$W = 1 + \sum_{n=1}^{N} \frac{\rho_n^2}{(x - x^{(n)})^2}\,. \tag{87}$$

Here $\rho_n, x_\mu^{(n)}, n = 1, \ldots, N$ are real parameters. In the simplest case, $N = 1$, one has the instanton solution [59] with size $\rho_n = \rho$ and position $x_\mu^{(1)}$. For ρ_n finite and N arbitrary Eqns (85), (87) give a multiinstanton solution with topological charge $Q = N$. In particular for $N \to \infty$, $x_4^{(n)} = nb$, $x_i^{(n)} = r_i$ one gets the Harrington – Shepard periodic instanton [60], for which A_μ^a periodically depends on (the Euclidean) time $x_4 \equiv t$.

One specific feature of instantons is their finite size; fields $F_{\mu\nu}$ fall off at large distances from the centre as x^{-4}. This is very different from the case of magnetic monopole, where fields decay as x^{-2}.

Another class of solutions with these properties can be obtained from Eqns (85) and (87) in the limit $\rho_n = \rho \to \infty$.

The case of $N = 1$, when $W = 1/(x - x^{(0)})^2$, yields no solution, since it is a pure gauge.

The next case, $N = 2$, is gauge equivalent to the (anti)-instanton with the position at $(x^{(1)} + x^{(2)})/2$ and the size of $|x^{(1)} - x^{(2)}|/2$.

We shall be interested in the case when $N \to \infty$, $\rho_i = \rho \to \infty$, $x_4^{(n)} = nb$, $\mathbf{x}^{(n)} \equiv \mathbf{R}$, $n = 1, 2, \ldots$ and call this solution a dyon, since, as we demonstrate below, it has both electric and magnetic long-distance field.

In this case W can be written as

$$W \equiv \sum_{n=-\infty}^{\infty} \frac{1}{(\mathbf{x} - \mathbf{R})^2 + (x_4 - nb)^2}\,, \tag{88}$$

and one can rewrite Eqn (88) using variables

$$\gamma|\mathbf{x} - \mathbf{R}| \equiv r\,, \qquad x_4\gamma \equiv t\,, \qquad \gamma = \frac{2\pi}{b}\,, \tag{89}$$

as

$$W = \frac{1}{2r}\,\frac{\sinh r}{\cosh r - \cos t}\,. \tag{90}$$

From Eqn (85) one obtains the vector-potentials:

$$A_{ia} = e_{aik} n_k \left(\frac{1}{r} - \coth r + \frac{\sinh r}{\cosh r - \cos t}\right) - \frac{\delta_{ia}\sin t}{\cosh r - \cos t}\,, \tag{91}$$

$$A_{4a} = n_a \left(\frac{1}{r} - \coth r + \frac{\sinh r}{\cosh r - \cos t}\right), \tag{92}$$

with

$$\mathbf{n} = \frac{\mathbf{x} - \mathbf{R}}{|\mathbf{x} - \mathbf{R}|}\,.$$

Here one notices that the dyonic field in this (singular or 't Hooft) gauge is now long ranged in spatial coordinates

$$A_{\mu a} \propto \frac{1}{r}\,, \qquad F_{\mu\nu} \propto \frac{1}{r^2} \tag{93}$$

and periodic in 'time' t.

Even more similarity to the magnetic monopole field can be seen when one makes the (singular) gauge transformation [61]

$$\tilde A_\mu = U^+\left(A_\mu + \frac{\mathrm{i}}{g}\,\partial_\mu\right)U\,, \qquad U = \exp\left(\frac{\mathrm{i}\boldsymbol{\tau}\mathbf{n}}{2}\,\theta\right), \tag{94}$$

with

$$\tan\theta = W_4\left[\frac{W}{r} + W_r\right]^{-1}, \qquad W_\mu \equiv \partial_\mu W\,.$$

In this gauge, sometimes called the Rossi gauge, $\tilde{A}_\mu$ looks like the Prasad – Sommerfield solution [62]:

$$\tilde{A}_{ia} = f(r) e_{iba} n_b \,, \qquad f(r) = \frac{1}{gr}\left(1 - \frac{r}{\sinh r}\right), \tag{95}$$

$$\tilde{A}_{4a} = \varphi(r) n_a \,, \qquad \varphi(r) = \frac{1}{gr}(r \coth r - 1)\,. \tag{96}$$

Note that $\tilde{A}_{\mu a}$ does not depend on time, it describes a static dyonic solution, since it has both a (colour)electric and a (colour)magnetic field:

$$E_{ka} = B_{ka} = \delta_{ak}\left(-f' - \frac{f}{r}\right) + n_a n_k \left(f' - \frac{f}{r} + g f^2\right). \tag{97}$$

One may additionally gauge rotate E_{ka}, B_{ka} to the quasi-Abelian gauge, where the only long-range component is directed along axis 3:

$$E'_{k3} = B'_{k3}(r \to \infty) \propto -\frac{1}{gr^2}\, n_k\,. \tag{98}$$

Eqn (98) justifies our use of the name dyon for the solution and demonstrates its similarity to the magnetic monopole. Note also that $\tilde{A}_{4a}$ [see Eqn (96)] tends to a constant at spacial infinity like the Higgs field component of the 't Hooft – Polyakov monopole [63].

The total action of the dyon is calculated from Eqns (95), (96) or Eqns (91), (92) to be

$$S = \frac{1}{2}\int \mathrm{d}^3\mathbf{r} \int_0^T \mathrm{d}t\,(B_{ak}^2 + E_{ak}^2) = \frac{8\pi^2}{g^2 b}\, T\,, \tag{99}$$

where T is the length of the 'dyonic string' in terms of the number N of centres in Eqn (87); $T = b(N-1)$. For the given N one also has

$$S(N) = \frac{8\pi^2}{g^2}\, Q(N)\,, \qquad Q(N) = N - 1\,. \tag{100}$$

6.2 This short section is devoted to another type of classical solutions, those depending on boundary conditions (b.c.) and defined in finite volume. Here we consider torons and instantons on torus [64], which obey the twisted b.c. in the box $0 \leqslant x_\mu \leqslant a_\mu$. Periodic b.c. are imposed modulo gauge transformation (twisted b.c.):

$$A_\lambda(x_\mu = a_\mu) = \Omega_\mu\left[A_\lambda(x_\mu = 0) - \mathrm{i}\,\frac{\partial}{\partial x_\lambda}\right]\Omega_\mu^+\,. \tag{101}$$

To ensure self-consistency of A_λ on the lines four functions Ω_μ ($\mu = 1, \ldots, 4$) should satisfy the following conditions

$$\Omega_1(x_2 = a_2)\Omega_2(x_1 = 0) = \Omega_2(x_1 = a_1)\Omega_1(x_2 = 0) Z_{12}\,, \tag{102}$$

and analogous conditions for $1, 2 \to i, j$, where $Z_{12} \in \mathrm{Z}(N)$ is the centre of the group $\mathrm{SU}(N)$:

$$Z_{\mu\nu} = \exp\left(2\pi\mathrm{i}\,\frac{n_{\mu\nu}}{N}\right), \qquad n_{\mu\nu} = -n_{\nu\mu}\,. \tag{103}$$

Here $n_{\mu\nu}$ are integers not depending on coordinates x_μ.

The twisted solutions A_μ (101) contribute to the topological charge

$$\frac{g^2}{16\pi^2}\int_{|x_\mu| \leqslant a_\mu} \mathrm{tr}(F_{\mu\nu}\tilde{F}_{\mu\nu})\,\mathrm{d}^4x = \nu - \frac{\chi}{N}\,, \tag{104}$$

where ν is an integer and $\chi = n_{\mu\nu}\tilde{n}_{\mu\nu}/4 = n_{12}n_{34} + n_{13}n_{42} + n_{14}n_{23}$.

The action in the box is bounded from below

$$\frac{1}{2}\int \mathrm{tr}(F_{\mu\nu}F_{\mu\nu})\,\mathrm{d}^4x \geqslant \frac{8\pi^2}{g^2}\left|\nu - \frac{\chi}{N}\right|. \tag{105}$$

Consider, e.g., the case of $n_{34} = -n_{12} = 1$, all other $n_{\mu\nu} = 0$, $\chi = 1$. Then one has

$$A_\lambda(x) = -\frac{\omega}{g}\sum_\mu \frac{\alpha_{\mu\lambda}x_\mu}{a_\mu a_\lambda}\,, \qquad \alpha_{\mu\lambda} = -\alpha_{\lambda\mu}\,, \tag{106}$$

where $\alpha_{12} = 1/2Nk$; $\alpha_{34} = 1/2Nl$, $k + l = N$,

$$\omega = 2\pi\,\mathrm{diag}(l, \ldots, l, -k, \ldots, -k)\,. \tag{107}$$

The matrix ω has k elements equal to l and l elements equal to $-k$, and a condition is imposed $a_1a_2(a_3a_4)^{-1} = l/k = (N-k)/k$. As a simple example we shall take SU(2) and cubic box, then $k = l = 1$, $\omega = 2\pi\tau_3$ and

$$A_\lambda(x) = -\frac{\tau_3}{a^2}\,\frac{\pi}{2g}\sum_\mu \bar{\alpha}_{\mu\lambda}x_\mu\,, \qquad \bar{\alpha}_{12} = \bar{\alpha}_{34} = 1\,. \tag{108}$$

This solution is self-dual and the following relation holds

$$\mathrm{tr}\,(F_{\mu\nu}F_{\mu\nu}) = \mathrm{tr}\,(F_{\mu\nu}\tilde{F}_{\mu\nu}) = \frac{16\pi^2}{\prod_\mu a_\mu N g^2}\,. \tag{109}$$

From Eqn (109) one can see that toron (108) is a particular case of a self-dual solution with constant field $\bar{F}_{\mu\nu}$:

$$A_\mu(x) = \bar{F}_{\mu\nu}x_\nu\,\frac{\tau_3}{2}\,, \tag{110}$$

where the amplitude of constant field $\bar{F}_{\mu\nu}$ is quantised. For constant (anti)selfdual field the analysis of Leutwyler [65] shows that such solutions are stable with respect to quantum fluctuations.

The flux through the Wilson loop for the solution (108) in the planes (*12*) or (*34*) is

$$P\exp\left(\mathrm{i}g\int_C A_\mu\,\mathrm{d}x_\mu\right) = \exp\left(-\mathrm{i}\pi\,\frac{S}{a^2}\,\tau_3\right), \tag{111}$$

where S is the area, bounded by the contour C.

As we shall see in the next section, the flux equal to π for $S = a^2$ is a property very important for confinement. Another interesting property of torons (not shared by any other solutions) is noteworthy one: its action is proportional to $1/Ng^2$ and therefore stays constant for large N, where $g^2 = g_0^2/N$. We shall return to this property in the Conclusions.

Another type of twisted solutions are twisted instantons [64]. These are solutions with topological charge Q (104) and an integer non zero ν.

These solutions have been seen on the lattice [66], and the profile [distribution of $\mathrm{tr}\,F_{\mu\nu}^2(x)$] is very close to that of the usual instanton.

Unfortunately, the analytic form of twisted instanton is still unknown; the top charge was found to be 1/2 [66], and the extrapolated string tension is probably nonzero. These two facts are not accidental; in the next section it will be shown that half-integer topological charge ensures a flux of π, which in turn may lead to confinement.

6.3 We compute elementary flux inside a Wilson loop for (multi)instanton, dyon, and twisted instanton and connect properties of the elementary flux to confinement in the gas of classical solutions [67].

Consider a circular Wilson loop in the plane (*12*) and take A_μ in the form of the 't Hooft ansatz (85), where N is fixed and $x_i^{(n)}=0$, $x_4^{(n)}=nb$. In this way one can study the case of instanton ($N=1$), periodic Harrington – Shepard instanton ($N\to\infty$, $\rho_i=\rho$ fixed), multiinstanton (N finite, ρ_i finite), and dyon ($\rho_i=\rho\to\infty$, $N\to\infty$).

When the radius of the loop R is much larger than the core of the solution (i.e. $R\gg\rho$ for (multi)instantons or $R\gg b/2\pi$ for dyon), the Wilson loop is

$$W(C_R)=\exp\left(\mathrm{i}\tau_3\pi\,\frac{RW_r}{W}\right)\equiv\exp(\mathrm{i}\tau_3\times\text{flux})\,, \qquad (112)$$

where

$$W_r=\left.\frac{\partial W}{\partial|\mathbf{x}|}\right|_{|\mathbf{x}|=R}\,,\qquad |\mathbf{x}|\equiv r\,.$$

Now for (multi)instanton one has for $R\gg\rho$

$$\left.\frac{RW_r}{W}\right|_{r=R}=\frac{-\sum 2\rho_n^2R/\left[R^2+(x_4-nb)^2\right]^2}{1+\sum\rho_n^2/\left[R^2+(x_4-nb)^2\right]}\to 0\,. \qquad (113)$$

In a nonsingular gauge one would obtain the flux of 2π for (multi)instanton [68] and in all gauges one has

$$W(C_R)=1\,,\qquad \text{(multi)instantons}\,. \qquad (114)$$

Consider now the case of dyons corresponding to the limit $\rho_n\to\infty$ in W_r in Eqn (112).

One can use the form (90) to obtain for dyon

$$\frac{RW_r}{W}=-1\,;\qquad \text{flux}=-\pi\,,\qquad W(C_R)=-1\,. \qquad (115)$$

It is also amusing to consider the intermediate case of so-called τ-monopoles [69], when $\rho_n\to\infty$, but N is fixed, so the length of the chain $L=Nb$ is finite. One can use Eqn (88) to find two limiting cases:

$$R\gg L\,,\qquad \frac{RW_r}{W}=-2\,,\qquad \text{flux}=-2\pi\,;\qquad W(C_R)=1\,; \qquad (116)$$

$$R\ll L\,,\qquad \frac{RW_r}{W}=-1\,,\qquad \text{flux}=-\pi\,;\qquad W(C_R)=-1\,. \qquad (117)$$

Thus, only τ-monopoles long enough (i.e. almost dyons) may ensure the nontrivial Wilson loop, $W(C_R)\neq 1$.

To connect flux values (114) – (117) to confinement one can take a model consideration of stochastic distribution of fluxes in the dilute gas, as was done in Refs [68, 69]. More generally the picture of stochastic fluxes was formulated in the model of stochastic confinement [70] and checked on the lattice in Ref. [71].

We shall come back to the model of stochastic confinement in the next section and for now use simple arguments from Refs [68, 69].

Indeed, consider a thin 3d layer above and below the Wilson loop (of thickness $l\ll R$) and assume that it is filled with gas of (multi)instantons or dyons. If 3d density of the gas is v, so that the average number of objects is $\bar n=vSl$, then the Poisson distribution, which gives the probability of having n objects in the layer around the plane of Wilson loop, is

$$w(n)=\exp(-\bar n)\,\frac{(\bar n)^n}{n!}\,. \qquad (118)$$

If the contribution to the Wilson loop of one object is λ ($\lambda=+1$ and -1 for instantons and dyons respectively) then the total contribution is

$$\begin{aligned}\big\langle W(C_R)\big\rangle&=\sum_n\exp(-\bar n)\,\frac{\lambda^n\bar n^n}{n!}\\&=\exp\left[-\bar n(1-\lambda)\right]=\exp(-\sigma S)\,,\end{aligned} \qquad (119)$$

where

$$\sigma=(1-\lambda)vl\,. \qquad (120)$$

Thus, for instantons ($\lambda=1$) one obtains zero string tension in agreement with other calculations [55], while for dyons ($\lambda=-1$) confinement is present according to the model. Let us now stress which features are important for this conclusion:

(1) the flux is equal to π, so that $W=-1$ for one dyon;

(2) stochastic distribution of fluxes enables us to use the Poisson (or similar) distribution;

(3) existence of finite thickness, i.e. of finite screening length l, so that objects more distant than l completely screen each other and do not contribute to the Wilson loop.

Notice that point (3) is necessary for the area law, otherwise (for large l, e.g., $l=R$) one obtains σ which increases with R, i.e. superconfinement.

The same reasoning is applicable to torons and twisted instantons [66]. Indeed from Eqn (111) one can see that their elementary flux is equal to $-\pi$. Therefore, if one divides all volume into a set of twisted cubic cells and ensures stochasticity of fluxes in the cells, one should obtain the same result (119) with confinement present.

Torons [72] and twisted instantons [66] have been studied from the point of view of confinement both analytically [72] and on the lattice [66]. For torons, the requirement of stochasticity is difficult to implement, since boundary conditions of adjacent cells should ensure continuity of $A_\mu(x)$, and this introduces ordering in the fluxes, and confinement may be lost. In the case of twisted instantons with $Q=1/2$ [66] the field is essentially nonzero around the centre of instanton, and b.c. are much less essential. The authors of Ref. [66] note a possibility of nonzero extrapolated value for the string tension when the box size is increased beyond 1.2 fm. It is still unclear what would be the result when the twisted b.c. are imposed only on the internal boundaries.

6.4 We shall consider below the dyons as the most probable candidates for classical confining configurations. One must study the properties of dyon gas and show that interaction in this gas is weak enough to ensure the validity of the dilute gas approximation. As usual, one assumes the superposition ansatz

$$A_\mu = \sum_{i=1}^{N_+} A_\mu^{+(i)}(x) + \sum_{i=1}^{N_-} A_\mu^{-(i)}(x)\,, \tag{121}$$

where N_+, N_- are numbers of dyons and antidyons respectively. To make the QCD vacuum O(4) invariant, one should take any direction of the dyonic line characterised by the unit vector $\omega_\mu^{(i)}$ ($\omega_\mu^{(i)}\omega_\mu^{(i)} = 1$) and the position vector $R_\mu^{(i)}$ so that

$$A_\mu^{(i)}(x) = \Omega_i^+ (LA)_\mu(r,t)\Omega_i\,. \tag{122}$$

Here Ω_i is the colour orientation matrix and L is the O(4) (Lorentz) rotation matrix, while r and t are

$$r = \left\{(x - R^{(i)})^2 - \left[(x - R^{(i)})_\mu \omega_\mu^{(i)}\right]^2\right\}^{1/2}, \tag{123}$$

$$t = (x - R^{(i)})_\mu \omega_\mu^{(i)}\,. \tag{124}$$

It is now nontrivial to choose the gauge for the solution A_μ in Eqn (122), e.g., one may take a singular gauge solution (91) or a time-independent one (95), (96). The sum (121) is not obtained by gauge transformation from one case to another. Indeed, it appears that the form (95), (96) is not suitable, since the action for the sum (121) in this case diverges (see [67] for details).

The form (91) is adoptable in this sense and we shall consider it in more detail.

Since solutions fall off fast enough [see Eqn (93)], the interaction between dyons defined as S_{int}

$$S(A) = \sum_{i=1}^{N_+ + N_-} S_i(A^{(i)}) + S_{\text{int}} \tag{125}$$

is Coulomb-like at large distances, e.g. for parallel dyon lines one has for two dyons

$$S_{\text{int}}(R^{(1)}, R^{(2)}) = \frac{\text{const} \times T}{|\mathbf{R}^{(1)} - \mathbf{R}^{(2)}|}\,, \tag{126}$$

where $T = Nb$ is the length of dyon lines. A similar estimate can be obtained for nonparallel lines.

As discussed in the previous section, the crucial point for the appearance of the area law of the Wilson loop is the screening phenomenon. To check this, let us consider the field of tightly correlated pair $\mathrm{d}\bar{\mathrm{d}}$. When distance between d and $\bar{\mathrm{d}}$ is zero, the resulting vector potential is obtained using the superposition ansatz (121) and vector potentials (91), (92) for d and $\bar{\mathrm{d}}$ [the corresponding one for $\bar{\mathrm{d}}$ differs from Eqn (91) by the sign of the last term and the total sign in Eqn (92)]:

$$A_{ia}(\mathrm{d}\bar{\mathrm{d}}) = \frac{2}{g}\, e_{aik} n_k \left(\frac{1}{r} - \coth r + \frac{\sinh r}{\cosh r - \cos t}\right), \tag{127}$$

$$A_{4a}(\mathrm{d}\bar{\mathrm{d}}) = 0\,. \tag{128}$$

At long range one has

$$A_{ia}(\mathrm{d}\bar{\mathrm{d}}) = 2e_{aik}\,\frac{1}{gr} + O\big(\exp(-r)\big)\,. \tag{129}$$

Calculation of B_{ka} amounts to insertion of $f = 2/gr$ in Eqn (98), which immediately yields

$$B_{ia}(\mathrm{d}\bar{\mathrm{d}}) = O\big(\exp(-r)\big)\,, \qquad E_{ia} \equiv O\big(\exp(-r)\big)\,. \tag{130}$$

Thus fields of d and $\bar{\mathrm{d}}$ completely screen each other at large distances. Note that this is purely non-Abelian effect, since the cancellation is due to the quadratic term in f [Eqn (98)].

Now let us take distance between d and $\bar{\mathrm{d}}$ equal to $\boldsymbol{\rho} = \mathbf{R}^{(1)} - \mathbf{R}^{(2)}$ and distance between observation point $\mathbf{x}$ and centre of $\mathrm{d}\bar{\mathrm{d}}$ equal to $\mathbf{r}$, $\mathbf{r} = \mathbf{x} - (\mathbf{R}^{(1)} + \mathbf{R}^{(2)})/2$; assume that $r \gg \rho$. Then the field of $\mathrm{d}\bar{\mathrm{d}}$ (averaged over direction of $\boldsymbol{\rho}$) is of the order

$$B_k, E_k = O\left(\frac{\rho^2}{r^4}\right). \tag{131}$$

Hence, the contribution of a distant correlated pair of $\mathrm{d}\bar{\mathrm{d}}$ is unessential and indeed in the Wilson loop calculating one can take into account the distances to the plane of the loop smaller than the correlation length l, which is actually the screening length.

From the dimensional arguments (we have the only parameter in our Coulomb-like system: the average distance between the nearest neighbours, $\nu^{-1/3}$) one has

$$l = c\nu^{-1/3}\,, \tag{132}$$

where c is some numerical constant, and ν is the 3d density of the dyon gas.

Hence, one expects the string tension in the dyonic gas to be of the order of

$$\sigma = c\nu^{2/3}\,. \tag{133}$$

Numerical calculations of $\langle W(C)\rangle$ for the dyonic gas have been done in Ref. [73], but the density used was still much below that, which is necessary for observation of the screening; work is now in progress.

Summarising this section let us discuss perspectives of the classical solutions reported above as candidates for confining configurations. Only two solutions, dyons and twisted instantons, yield the suitable flux through the Wilson loop equal to π, therefore we shall discuss them separately. Dyons can be represented as a coherent chain of instantons of large radius with correlated orientation of colour field. When one goes from the instanton gas to these coherent chains, the action changes a little, but entropy decreases significantly, and confinement occurs. To estimate the advantage or disadvantage of dyonic configurations from the point of view of the minimum of the vacuum free energy, one must perform complicated computations, which are planned in the nearest future.

As for the twisted instantons, they require an internal lattice structure in the vacuum, which may violate the Lorentz invariance in some field correlators.

The problem is ultimately solved, as in dyonic case, by calculating the free energy of the vacuum, because in the nature there should be realised that vacuum structure, which ensures the minimal free energy. The lattice Monte-Carlo calculations satisfy this principle of minimal free energy (up to the finite size effects) and predict the confining vacuum with special nonperturbative configurations responsible for confinement.

While it is possible that these configurations are dyons, it is also probable that they are not at all classical, and it is not to be ruled out, that there exist unknown classical solutions, which account for confinement ultimately.

7. Topology and stochasticity

In the previous section we have used the stochasticity of fluxes to obtain the area law for dyons (magnetic monopoles). We shall begin this section by giving more rigorous treatment of this stochasticity and comparing it to lattice data .

For Abelian theory the magnetic flux through the loop C is defined unambiguously through the Wilson loop

$$W(C) = \exp\left(\mathrm{i}e\oint_C A_\mu\,\mathrm{d}x_\mu\right) = \exp\left(\mathrm{i}e\int_S \mathbf{H}\,\mathrm{d}\boldsymbol{\sigma}\right), \quad (134)$$

and the magnetic flux is

$$\mu = e\int_S \mathbf{H}\,\mathrm{d}\boldsymbol{\sigma}. \quad (135)$$

For $\mathrm{SU}(N_c)$ theory the flux can be defined analogously [70] (we shall omit the word 'magnetic', since it depends on the orientation of the loop).

Consider eigenvalues of the Wilson operator (note the absence of trace in its definition):

$$U(C) = P\exp\left(\mathrm{i}g\oint_C A_\mu\,\mathrm{d}x_\mu\right) \equiv \exp\left[\mathrm{i}\hat{\alpha}(C)\right]. \quad (136)$$

The eigenvalues of the unitary operator $U(C)$ are equal to $\exp\left[\mathrm{i}\hat{\alpha}(C)\right]$, where $\hat{\alpha}(C)$ is a diagonal matrix $N\times N$, depending on A_μ.

In electrodynamics $\alpha_n(C_{12})$ are additive for the contour C_{12}, consisting of two closed contours C_1 and C_2:

$$\alpha_n(C_{12}) = \alpha_n(C_1) + \alpha_n(C_2). \quad (137)$$

In $\mathrm{SU}(N_c)$ theory this is generally not the case. Consider now the spectral density $\rho_s(\alpha)$, i.e. averaged with the weight $\exp\left[-S_0(A)\right]$ the probability of the flux $\alpha(C)$:

$$\rho_s(\alpha) = \frac{\int DA_\mu \exp\left[-S_0(A)\right] N^{-1}\sum_{m=1}^{N}\delta_{2\pi}\left(\alpha-\alpha_m(A_\mu, C)\right)}{\int DA_\mu \exp\left[-S(A)\right]}, \quad (138)$$

where $S_0(A)$ is the standard action of the $\mathrm{SU}(N_c)$ theory.

Now any averaged Wilson operator over contour C, and also those for contour C^n, i.e. contour C, followed n times, can be calculated with the help of $\rho_s(\alpha)$:

$$\left\langle W(C^n)\right\rangle = \int_{-\pi}^{\pi}\mathrm{d}\alpha\exp(\mathrm{i}n\alpha)\rho_s(\alpha). \quad (139)$$

We shall assume that there is confinement in the system, i.e. that the area law holds both for contour C and C^n:

$$\left\langle W(C^n)\right\rangle = \exp(-k_n S). \quad (140)$$

Then the following equality holds [70, 71]:

$$\rho_S^C(\alpha) = \int_{-\pi}^{\pi}\mathrm{d}\alpha_1\ldots\int_{-\pi}^{\pi}\mathrm{d}\alpha_n\,\rho_{S_1}^{C_1}(\alpha_1)\ldots\rho_{S_n}^{C_n}(\alpha_n) \times\delta_{2\pi}(\alpha-\alpha_1-\ldots-\alpha_n), \quad (141)$$

where the contour C with area S is made of contours C_i splitting the area S into pieces S_i. The proof [70, 71] works in both directions: from Eqns (139), (140) to Eqn (141) and back. Sometimes this statement is formulated as a theorem [70]. *The necessary and sufficient condition of confinement is the additivity of random fluxes.*

The randomness is seen in Eqn (141), which has the form of convolution as it should be for the product of probabilities for independent events. Additivity is evident from the argument of the δ-function in Eqn (141).

Density $\rho_s(\alpha)$ was measured in lattice calculations [71], and it was found that $\rho_s(\alpha)$ indeed satisfies Eqn (141) and approximately coincides with the $\rho_s^d(\alpha)$ (the density for $d=2$ chromodynamics) if one renormalizes properly the charge. For $d=2$ the case $\rho_s(\alpha)$ is also known explicitly and satisfies Eqn (141) exactly [71]. In that case confinement exists for trivial reasons.

Let us now consider the non-Abelian Stokes theorem [11], which for the operator (136) looks like

$$U(C) = P\exp\left[\mathrm{i}g\int_S \mathrm{d}\sigma_{\mu\nu}(u)\,F_{\mu\nu}(u,x_0)\right]. \quad (142)$$

We take into account that [under gauge transformation $V(x)$] it turns into

$$U(C)\to V^+(x_0)U(C)V(x_0),$$

$$F_{\mu\nu}(u,x_0)\to V^+(x_0)F_{\mu\nu}V(x_0). \quad (143)$$

Since $U(C)$ can be brought to the form $U(C)=\exp(\mathrm{i}\hat{\alpha})$ by diagonal unitary transformation $\hat{\alpha}$, one can deduce that this is some gauge transformation $V(x_0)$ and, moreover, it is the same which makes $\int\mathrm{d}\sigma_{\mu\nu}F_{\mu\nu}(u,x_0)$ diagonal. Thus, one can define the flux μ similarly to Eqn (135):

$$\hat{\mu} = \mathrm{diag}\left\{\mathrm{i}gV^+(x_0)\int_S\mathrm{d}\sigma_{\mu\nu}(u)\,F_{\mu\nu}(u,x_0)V(x_0)\right\}. \quad (144)$$

Note that the dependence on x_0 present in $U(C)$, is cancelled in $W(C)=\mathrm{tr}\left[U(C)\right]$.

The additivity of fluxes is seen in Eqn (144) explicitly.

Now consider the statistical independence of fluxes, which obtains when one divides the surface S into pieces $S_1,\ldots,S_n$. Using the cluster expansion theorem [12] and discussion in Section 3, one can conclude, that a necessary and sufficient condition for this is the finite correlation length T_g, which appears in correlation functions (cumulants) $\left\langle\left\langle F(1)\ldots F(n)\right\rangle\right\rangle$. In the case, when each piece S_k, $k=1,\ldots,n$ is much larger in size than T_g, the different pieces become statistically independent. Thus, our consideration in the framework of the field correlators in Section 3 is in clear agreement with the idea of stochastic confinement [70, 71]. The MVC in addition contains the quantitative method to calculate all observables in terms of given local correlators, which is absent in the stochastic confinement model [70, 71].

The appearance of new physical quantity T_g and cumulants is a further development of the idea of stochastic vacuum, which gives an exact quantitative characteristics of randomness.

When the size of contours is of the order of T_g, the fluxes are no longer random, and the area law at such distances disappears. There is no area law at small distances, as explained in Section 3.

The lattice measurements in Ref. [71] show that $\rho_s(\alpha)$ is strongly peaked around $\alpha = 0$ for small contours, when there is no area law, which corresponds to the perturbative regime.

Instead, for large contours the measured $\rho_s(\alpha)$ are rather isotropic.

One can compare this fact with our result for fluxes of instantons and dyons. From our definition of fluxes (136) a dyon in the plane of contour C corresponds to [see Eqn (112)]

$$\hat{\alpha} = \begin{pmatrix} \pi & 0 \\ 0 & -\pi \end{pmatrix}, \tag{145}$$

and a dyon with its centre off the plane has smaller eigenvalues α_m. It is clear that instantons with zero flux cannot bring about isotropic distribution of fluxes, while having maximal flux $\pm\pi$ are most effective in creating the isotropic $\rho_s(\alpha)$, when one integrates over all dyons in the layer above and below the plane.

It is instructive now to study the question of fluxes for the adjoint Wilson loop and in general for the Wilson loops of higher representations.

One can keep the definition (136) also in this case, but A_μ and $\hat{\alpha}(C)$ should be expressed through generators of a given representation:

$$A_\mu = \sum_a A_{\mu a} T^a , \qquad \mathrm{tr}\,(T^a T^b) = \frac{1}{2}\,\delta_{ab} . \tag{146}$$

Thus $\hat{\alpha}(C)$ for the Wilson loop in adjoint representation is a matrix $(N_c^2 - 1) \times (N_c^2 - 1)$, e.g. for SU(2) $(T^a)_{bc} = (\mathrm{i}/2) e_{abc}$. To understand how a stochastic vacuum model works for adjoint representation, let us take as an example the flux of one dyon and calculate $\hat{\alpha}_{\mathrm{adj}}(C)$. Repeating our discussion preceding Eqn (112) for large loops in the plane (*12*), one concludes again that only colour index $a = 3$ contributes and one has

$$\hat{\alpha}_{\mathrm{adj}}(C) = \pi\,\mathrm{diag}(T^3) \lim \left(\frac{RW_r}{W} \right) . \tag{147}$$

Since the last factor for dyon is $\lim(RW_r/W) = -1$, and

$$\mathrm{diag}(T^3) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix},$$

one finally obtains

$$\hat{\alpha}_{\mathrm{adj}}(C) = \begin{pmatrix} -\pi & & \\ & \pi & \\ & & 0 \end{pmatrix}. \tag{148}$$

Thus, our conclusion of the elementary flux equal to π holds true also in the adjoint representation (and all higher representations), which gives an argument for confinement of adjoint charges on the same ground as for fundamental charges.

One has

$$\left\langle W_{\mathrm{adj}} \right\rangle = \frac{1}{N_c^2 - 1}\,\mathrm{tr}_{\mathrm{adj}} \left\{ \exp\left[\mathrm{i}\hat{\alpha}_{\mathrm{adj}}(C) \right] \right\} = -1 , \tag{149}$$

as well as $\left\langle \hat{W}_{\mathrm{fund}} \right\rangle = -1$.

So far we have discussed the stochasticity of the vacuum from the point of view of fluxes and have concluded that it shows up as random distribution of fluxes. In Section 3 the vacuum stochasticity was formulated in the language of field correlators. Through the AP method one can connect the latter with the distribution of AP magnetic monopole currents (in the U(1) theory an exact connection holds even without AP). One may wonder why magnetic monopoles or dyons are needed to maintain the stochastic picture of the vacuum?

To answer this question we start with the Abelian theory. Without magnetic monopoles the Bianchi identities $\mathrm{div}\,\mathbf{H} = 0$ are operating, requiring that all magnetic FSL are closed.

This introduces strong ordering in the distribution of magnetic field, and no stochastic picture emerges.

As a result, confinement is not present in the system, as can be seen from Eqn (22). In the presence of magnetic monopoles the magnetic lines can start and end at any place, where a monopole is present, and one can have an actual stochastic distribution. As discussed in Section 3, the non-Abelian dynamics can mimick the effect of monopoles due to triple correlators $\langle E_i E_j B_k \rangle$ thereby ensuring the stochastic distribution of fields.

Thus, as for magnetic monopoles in the Abelian theory, dyons in gluodynamics tend to create disorder in the system.

The same situation occurs in other spin and lattice systems, e.g., in the planar Heisenberg model the Berezinskiĭ–Costerlitz–Thouless vortices create disorder and master the phase transition into the high temperature phase [74] (for details see also Ref. [1]).

All this is a manifestation of the general principle [2]. *Topologically nontrivial field configurations are responsible for creation of disorder and they drive the phase transition order–disorder*.

From the QCD point of view the 'ordered phase' is the perturbative vacuum of QCD with long distance correlations ($D_1(x) \propto 1/x^4$) and flux distribution $\rho_s(\alpha)$ centred at zero, while the 'disordered phase' is the real QCD vacuum with short correlation length T_g and with random fluxes; there is no real phase transition in continuum: the phases coexist on two different scales of distances (or moments). In the lattice version of U(1) theory there are indeed two phases: weak coupling phase, corresponding to the usual QED, and strong coupling phase with magnetic monopoles (lattice artefacts) driving the phase transition.

The dyon is a continuum example of topologically nontrivial configuration. In singular ('t Hooft's) gauge a dyon has a multiinstanton topological number proportional to its length.

The dyon saturates the triple correlator $\langle E_i E_j B_k \rangle$ and may be a source of randomness of field distribution.

The ultimate answer, however, to the question about the nature of confining configurations is still missing. The analysis of the dyonic vacuum as a model of the QCD vacuum is not yet completed, and it is possible that the topologically nontrivial configurations responsible for confinement are dyons, or some other unknown solutions, or else purely quantum fluctuations.

We shall conclude this section with a discussion of the possible connection between confinement and the Anderson localisation [75].

At the base of the similarity between these two phenomena lies the field stochasticity in the vacuum (medium), where quark (electron) propagates. This is where the similarity comes to the end, though.

Namely, for an electron one can discuss the individual Green function, which always (for any density of defects) decays exponentially with distance. [There exists, however, a special correlator, e.g., the direct current conductivity $\sigma_{\rm dc}$, which vanishes for localised states (for large density of localised defects [76]) and is nonzero for the delocalised states].

In the case of a quark in the confining vacuum its Green's function (more precisely the gauge-invariant Green's function of the $q\bar{q}$ system averaged over vacuum configurations) corresponds to the linear potential, i.e. it behaves as $G(r) \propto \exp(-r^{3/2})$, where r is the distance between q and $\bar{q}$. Thus the quark Green's function decays faster than any exponent, in contrast to the Green's function of an electron in the medium, which always decays exponentially. This property of vacuum Green's functions was coined by the author of Ref. [77] as *superlocalization*. If the average potential $\overline{V}$, acting on a quark, had been finite, then quark at some high energy could be freed and get to the detector.

The essence of superlocalization is precisely the fact that the averaged potential $\overline{V}$ grows with distance without limits and therefore quarks are confined at any energy. This is the *absolute confinement*.

It is interesting to follow the mechanism whereby the unbounded growth of $\overline{V}$ occurs. To this end we shall consider, as at the beginning of Section 3, a nonrelativistic quark, moving in the (x, t) plane, while the heavy antiquark is fixed at its origin.

According to quantum mechanics laws [78] the quark Green function is proportional to the phase integral; thus, by using the Fock – Schwinger gauge, one can write

$$G(X,T) = \left\langle \exp\left[\mathrm{i}g \int_0^T A_4(x,t)\,\mathrm{d}t \right] \right\rangle$$

$$\approx 1 - \frac{g^2}{2} \int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' \int_0^X \mathrm{d}u \int_0^X \mathrm{d}u' \big\langle E_1(u,t) E_1(u',t') \big\rangle$$

$$\approx 1 - \overline{V} T\,. \tag{150}$$

Stochasticity of vacuum fields implies the finite correlation length $T_{\rm g}$ for the correlator $\langle E_1(u,t)E_1(u',t')\rangle$, i.e. according to Eqn (16) one has

$$\big\langle E_1(u,t)E_1(u',t') \big\rangle = D(u-u', t-t') + \ldots\,, \tag{151}$$

and for large T and X we obtain

$$\overline{V} \approx \mathrm{const}|X|\,, \qquad |X| \to \infty\,. \tag{152}$$

Thus the linear growth of $\overline{V}$ is a consequence of random distribution of *field strength* $\mathbf{E}(u,t)$ and of the fact, that $\overline{V}$ is a result of the averaging of *vector-potentials* A_μ, which are connected to $F_{\mu\nu}$ by an additional integral. This extra integration causes the linear growth of $\overline{V}$ and from the physical point of view this means the *accumulation* of fluctuations of the field $F_{\mu\nu}$ on the whole distance X from the quark to the antiquark.

This is the essence of the superlocalization phenomenon, which as yet has no analogue in the physics of condensed matter.

8. Conclusions

This study has examined confinement from different angles and described the mechanism of this phenomenon in the language of field correlators, drawing heavily on the phenomenological language of dual superconductivity, i.e. effective classical equations of the Ginzburg – Landau type, on the language of the stochastic flux distributions. Finally we have studied classical configurations, which may account for confinement.

At every step along the way we have stressed that confinement is a string formation between colour charges, the string mostly consisting of a longitudial colourelectric field.

Let us now try to combine different descriptions of confinement derived above from our analysis and show in a simple example what the string looks like.

To this end we shall use the simple picture of the nonrelativistic quark and the heavy antiquark at distance X from them, discussed at the end of Section 7. From the point of view of field correlators, confinement (the string formation) is a consequence of the fact that there exists correlation length $T_{\rm g}$, such that the fields inside this length are coherent and those outside this length are random. This is shown in Fig. 10a, where the strip of size $T_{\rm g}$ is indicated in the plane (*14*) [one could take instead any other plane, i.e. the plane (*12*) or (*13*)]. Inside this strip the field is directed mostly in the same way (i.e. as on the string axis), while outside the strip directions are random. The correlation length $T_{\rm g}$ characterises the string thickness (if the plane (*12*) or (*14*) is chosen). Thus, $T_{\rm g}$ plays a double role. It gives the coherence length, where the string is created, and beyond which stochastic vacuum fields existed also before quark and antiquark, have been inserted in the vacuum.

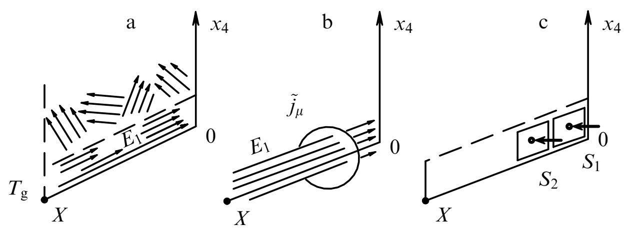

Figure 10. The picture of string formation between a nonrelativistic quark and a heavy antiquark is illustrated in three different approaches: (a) in the formalism of field correlators; (b) in the formalism of dual superconductivity; (c) in the picture of the stochastic flux distribution.

Let us look at the same construction from the point of view of dual superconductivity. One then obtains the picture shown in Fig. 10b. Here the circle depicts the monopole current $\tilde{j}_\mu$, which is caused by the colourelectric field E_x of the string in accordance with the dual London equation $\mathrm{rot}\,\tilde{j} = m^2\mathbf{E}$. The effect of this current is the squeezing of the string field, which prevents the field flux lines from diverging into the space, and as a result E_x decays exponentially away from the string axis $0x$ like $\exp(-m\sqrt{y^2+z^2})$. Thus, m defines the string thickness and one can conclude that $m \sim 1/T_{\rm g}$. And indeed Eqn (22) confirms this conclusion.

Let us turn now to the flux distribution and the stochastic vacuum model (Fig. 10c). In this case the strip, corresponding to the string in Fig. 10a, can be divided into pieces $S_1, S_2, \ldots$ of the size d^2, such that the fluxes inside each piece are coherent and equal to, e.g., $\pm\pi$ for the case of dyons, while two neighbouring pieces are noncoherent: their fluxes are mutually random. The string thickness is now built up due to the size d of the piece S_n, containing a coherent flux. If the surface S_n is penetrated by a monopole or a dyon, then d coincides with the monopole or dyon size.

To clarify this point let us find the minimal size of the loop R, where the dyon flux is equal to the asymptotic value of $-\pi$. To this end we shall use Eqn (115) and insert there Eqn (90), and obtain the result that for $R \gg b/2\pi \equiv \gamma^{-1}$ the flux is equal to $-\pi$ with exponential accuracy; hence, the size of the dyon flux is equal to γ^{-1} and this should be the size d of the piece S_n.

From the point of view of field correlators, d should coincide with T_g, therefore the string thickness is of the order of the typical dyon (or monopole) size (or another classical solutions).

Hence, all our pictures represented in Fig. 10 can be combined under one generalised mechanism of the string formation, which is based on the existence of coherent field domains of the size T_g, while beyond that size the fields are independent and random.

A question arises. Who manages this structure of QCD vacuum and why in the case of QCD and gluodynamics the vacuum is made this way, while in the case of QED and the Weinberg – Salam theory the nonperturbative configurations are probably suppressed and the vacuum structure is different? To be able to answer this question is also to answer the question of phase transition mechanism for the temperature deconfinement, which was observed on the lattice [79]. This topic merits a separate review, since the amount of available information is by now very large. We shall confine ourselves to merely a few remarks on this point.

Firstly, one can connect the density of the (nonperturbative) vacuum energy with the help of the scale anomaly theorem with the magnitude of the nonperturbative gluonic condensate [8]:

$$\varepsilon_{\text{nonpert}} = +\frac{\beta(\alpha_s)}{16\alpha_s}\left\langle F^a_{\mu\nu}(0)F^a_{\mu\nu}(0)\right\rangle . \qquad (153)$$

For small α_s the function $\beta(\alpha_s)$ is negative, in contrast to QED, and if keeps the sign in the whole effective region of α_s, then one can deduce that the nonperturbative vacuum shift (153) is advantageous, since it diminishes vacuum energy (and also free energy at small temperatures).

This conclusion can be considered as an intuitive idea why the nonperturbative vacuum in QCD is advantageous and comes into existence, while in QED it is not advantageous and not realised.

Secondly, let us briefly discuss the phase transition with an increase of temperature in QCD, referring the reader to lattice calculations [79] and original papers [81] for details. The main criterion, which defines the vacuum structure preferred at a given temperature, is the criterion of the minimum of free energy (which is a corollary of the second law of thermodynamics). In the confining phase for $T > 0$ the free energy consists of the term (153) and of the contribution of hadronic excitations (glueballs, mesons and baryons), which slowly grows, up to $T \approx 150$ MeV. Note that the gluonic condensate contains both colourelectric and colourmagnetic fields, but only the first ones have to do with confinement in the true sense of this term.

The deconfinement phase, realised at $T > T_c$, usually was identified as the phase with a perturbative vacuum, where quarks and gluons in the lowest order in g are free [82]. However, from the point of view of the minimum of free energy it is advantageous to keep colourmagnetic fields and the corresponding part of the condensate in the vacuum (153) since quarks in this case remain essentially free, and there is a significant gain in energy [around one half of the amount in Eqn (153)]. This is the 'magnetic confinement' phase [81]. Calculations in Ref. [81] yield T_c in good agreement with lattice data for a number of flavours $n_f = 0, 2, 4$.

The main prediction of 'magnetic confinement' is the area law for $T > T_c$ of the spacial Wilson loops [83] and the phenomenon of 'hadronic screening lengths', i.e. the existence of the hadronic spectra for $T > T_c$ in Green's functions with evolution along space directions [81], which agrees well with the lattice measurements [79].

Hence the picture of the phase transition [81] into the 'magnetic confinement' phase, supported by computations, seems to be well founded. Colourelectric correlators disappear in this picture at $T > T_c$ [more explicitly, the correlators of the type of $D(x)$, which contribute to string tension].†

What happens then to effective or real magnetic monopoles and dyons? In the AP method at $T > T_c$ the monopole density significantly decreases [35], which can be understood as an active annihilation or a close pairing of monopoles and antimonopoles. The same can be said about pairs of dyons and antidyons. Thus, the deconfinement phase of colour charges can be associated with the confinement phase of monopoles (or dyons). However, the 'magnetic confinement' phenomenon imposes definite requirements on the vacuum structure at $T > T_c$; e.g., there should exist magnetic monopole (dyonic) currents along the 4th (Euclidean time) axis, i.e. static or periodic monopoles (dyons). Such currents can confine in spacial planes (what is observed on lattices [83]), but do not participate in the usual confinement [i.e. in the temporal planes ($i4$), $i = 1, 2, 3$]. These points will be elucidated in later studies.

Due to space limitations we have not discussed an important question of the connection between confinement and spontaneous breaking of chiral symmetry, as well as $U_A(1)$, where magnetic monopoles (dyons) can play an important role [84].

We have always maintained that confinement is a property not only of QCD (with quarks present in the vacuum), but also of gluodynamics (without quarks). This conclusion follows from numerous lattice data (see, e.g., Ref. [10]) and also from computations supporting the dual Meissner effect as a basis of confinement, where quarks do not play an important role.

On the other hand there are no existing calculations or experimental data which support the key role of quarks in the confinement mechanism. For this reason we have not discussed above the model of V N Gribov (interesting by itself) and the reader is referred to Ref. [85].

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† Recently this picture has obtained an additional support from the lattice measurements in hep-lat/9603018, where $D^E(x)$ was found to disappear at $T > T_c$, while $D^H(x)$ does not change.

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† Available via, for example, www http://xxx.lanl.gov/archive.