Physics – Uspekhi **39** (3) 305 – 311 (1996)

METHODOLOGICAL NOTES PACS numbers: **05.50.+q**, 05.70.Ln, **82.40.–g**, **87.10.+e**

Galilean transformations and evolution of autowave fronts in external fields

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Contents

Abstract. We consider autowave regimes in two-dimensional excitable media in the presence of an external electric field, using Galilean transformations in the reaction – diffusion equations. It is shown that the transformation properties of these equations lead to some general relations for the autowave front and vortex drift velocities, independently of the concrete form of nonlinear terms in the equations. The general field dependence of the critical autowave characteristics is determined. Simple kinematic method discussed in this work is applicable for studying autowave evolution in three-dimensional and multicomponent excitable media.

1. Introduction

Nowadays considerable study is being devoted in Russia and abroad to the phenomena of self-organisation in various nonequilibrium systems, involving the creation and evolution of structures ordered in space and time. These cutting-edge interdisciplinary investigations have received the name synergetics [1 – 3]. An interesting example of systems of this kind is excitable media capable of forming pulses (autowaves) as a response to an external perturbation [4, 5]. There are many examples of excitable media of various nature: physical, chemical, and biological. Among these are, for instance, nervous and muscular tissues [6], colonies of microorganisms [7], a number of chemical solutions and gels [8, 9], magnetic superconductors with current [10], some solid-state systems [11].

Excitable media are a rather new and unorthodox object of investigation. However, their study is important in the creation of new perspective devices for data processing, in the development of methods which increase the effectiveness of technology processes in the chemical industry and even in the search for methods of hazardous disease control.

Autowave structures in a two-dimensional medium, as a rule, have a form of moving excitation fronts. If such front breaks, then special regimes, that is, rotating spiral waves, can occur (see, for example, articles cited in Ref. [12]).

Excitable media are traditionally described by a set of nonlinear parabolic equations of the reaction – diffusion type

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} = F(\mathbf{U}) + \widehat{\mathrm{D}}\,\Delta\mathbf{U}\,, \tag{1}$$

where $\mathbf{U}$ is the state vector of an elementary volume of an excitable medium. For example, in a chemically excited medium the components of vector $\mathbf{U}$ present the concentrations of reagents, matrix $\widehat{\mathrm{D}}$ determines diffusion coefficients, while a nonlinear function $F(\mathbf{U})$ sets the rates of chemical reactions in the elementary volume. In media of other nature the components of vector $\mathbf{U}$ can have the meaning of temperature, potential and so on, while the elements of matrix $\widehat{\mathrm{D}}$ can be given by the coefficients of thermal or electrical conductivities.

Although the real excitable media should be described, as a rule, by a multicomponent vector, numerous studies show that the main features of evolution of autowave structures are well presented by the two-component set [13, 14]:

$$\frac{\partial u}{\partial t} = F(u,g) + D_u\,\Delta u\,,$$
$$\frac{\partial g}{\partial t} = G(u,g) + D_g\,\Delta g\,. \tag{2}$$

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Received 20 September 1995
Uspekhi Fizicheskikh Nauk **166** (3) 327 – 334 (1996)
Translated by G N Chuev; edited by L V Semenova

The variable u is often referred to as 'activator', while g as 'inhibitor'. For excitable media, the zero-isocline $F(u,g)=0$ has a typical *И*-like shape, while the zero-isocline $G(u,g)=0$ can be monotonic or even linear.

General mathematical methods for solving Eqns (1) and (2) have not been developed as yet. Therefore we have to study these equations by numerical or approximated analytical methods. Among them we note the so-called kinematic approach [12, 15, 16], by which many of autowave regimes were studied in two- and three-dimensional nonhomogeneous, nonstationary and anisotropic excitable media.

One of the important problems of the physics of excitable media is to develop methods for effective control over the characteristics of autowave structures, i.e. the shape and velocity of the front motion, rotation frequency of spiral waves, their location on a plane, and so on. These methods include, for example, initiation of resonance drift of spiral waves in nonstationary excitable media [17, 18], and the creation of inhomogeneities in the media. Other methods for autowave control are concerned with the action of an external field, i.e. an electric field, on the excitable medium [19–21].

As a result of the consideration of the effect of a homogeneous electric field on the wave propagation in the right side of the set (1) the terms arise, which are proportional to the gradients of the components of the state vector $\mathbf{U}$ (see Section 2). It is interesting that similar modified equations take place also in the consideration of the motion of curved fronts [13].

We shall show that significant data on flat and curved fronts, propagating in external electric fields and in the absence of the field, can be obtained without the detailed solution of the modified Eqn (1), by using Galilean transformations and some qualitative peculiarities of the autowave motion, revealed in an approximated analytical investigation or by numerical calculations. The simplicity and availability of this method has prompted us to present it as methodological notes. We point out that a number of results were first obtained in this study.

2. Basic equations

2.1 Excitable medium in an electric field

Let us consider a two-component two-dimensional medium, which is described by Eqns (2) modified to include the electric field effect. The results can easily be generalised to the case of three or more components.

The modified set (2) is easily derived in the following way. In the general case the autowave structures in the two-dimensional medium must be described by the set of equations:

$$\frac{\partial u}{\partial t} = F(u,g) - \nabla\cdot\mathbf{J}_u\,,$$

$$\frac{\partial g}{\partial t} = G(u,g) - \nabla\cdot\mathbf{J}_g\,, \tag{3}$$

where $\mathbf{J}_u$ and $\mathbf{J}_g$ are the fluxes of activator and inhibitor respectively. In the absence of external fields these fluxes have the diffusion nature, and we get the 'classic' set (2). The account of the effect of an external electric field $\mathbf{E}$ leads to additional terms in the expressions for the fluxes:

$$\mathbf{J}_u = -D_u\nabla u + \mu_u\mathbf{E}\,u,$$

$$\mathbf{J}_g = -D_g\nabla g + \mu_g\mathbf{E}\,g\,, \tag{4}$$

where μ_u and μ_g are the mobilities of activator and inhibitor. We shall consider the excitable medium to be homogeneous and, hence, diffusion coefficients and mobilities are constant. Substituting Eqn (4) into Eqn (3), we obtain a set of equations describing a two-component excitable medium in an external homogeneous electric field:

$$\frac{\partial u}{\partial t} = F(u,g) + D_u\Delta u - \mu_u\mathbf{E}\cdot\nabla u\,,$$

$$\frac{\partial g}{\partial t} = G(u,g) + D_g\Delta g - \mu_g\mathbf{E}\cdot\nabla g\,. \tag{5}$$

In the one-dimensional case when an autowave with the flat front propagates along the axis x, the set of equations (5) has the form

$$\frac{\partial u}{\partial t} = F(u,g) + D_u\frac{\partial^2 u}{\partial x^2} - \mu_u E\cos\alpha\,\frac{\partial u}{\partial x}\,,$$

$$\frac{\partial g}{\partial t} = G(u,g) + D_g\frac{\partial^2 g}{\partial x^2} - \mu_g E\cos\alpha\,\frac{\partial g}{\partial x}\,, \tag{6}$$

where α is the angle between the direction of the electric field and the axis x.

2.2 Curved wavefront in the absence of the electric field

Let us consider the propagation of a circular front with the centre in the coordinate origin. Turning to the polar coordinate system (r,φ), whose pole is in the centre of the circular front, and taking into account that the dependence on the angle φ vanishes due to polar symmetry, we can write the set of equations (2) as

$$\frac{\partial u}{\partial t} = F(u,g) + D_u\frac{\partial^2 u}{\partial r^2} + \frac{D_u}{r}\frac{\partial u}{\partial r}\,,$$

$$\frac{\partial g}{\partial t} = G(u,g) + D_g\frac{\partial^2 g}{\partial r^2} + \frac{D_g}{r}\frac{\partial g}{\partial r}\,. \tag{7}$$

In almost all the cases when a stable curved autowave can propagate, the curvature radius R of the front far exceeds the front thickness (i.e. dimension of the region where the excitation is localised). This means that $1/r$ in Eqns (7) can be replaced by $1/R = K$, where K is the curvature of the front. As a result, we get the set of equations

$$\frac{\partial u}{\partial t} = F(u,g) + D_u\frac{\partial^2 u}{\partial r^2} + D_u K\frac{\partial u}{\partial r}\,,$$

$$\frac{\partial g}{\partial t} = G(u,g) + D_g\frac{\partial^2 g}{\partial r^2} + D_g K\frac{\partial g}{\partial r}\,, \tag{8}$$

which coincides with Eqns (6), if we replace $r\to x$, $D_u K\to -\mu_u E\cos\alpha$, and $D_g K\to -\mu_g E\cos\alpha$.

Thus, the propagation of flat fronts in an electric field and weakly curved fronts in the absence of the electric field are described by the same set of equations. Note that this set of equations also describes the autowave propagation in a weakly inhomogeneous medium.

All the above examples are particular cases of the following system

$$\frac{\partial u}{\partial t} = F(u,g) + D_u \Delta u - \mathbf{A}\cdot\nabla u\,,$$
$$\frac{\partial g}{\partial t} = G(u,g) + D_g \Delta g - \mathbf{B}\cdot\nabla g\,, \tag{9}$$

where $\mathbf{A}$ and $\mathbf{B}$ are the constant vectors. Below we consider some general properties of the solutions to set (9), and then use them to study the problems of the motion of curved fronts and propagation of autowaves in an electric field.

3. Galilean transformations of coordinates

Let us turn to cartesian coordinate system (x', y') moving with the velocity $\mathbf{w}$ with respect to the initial system (x, y). The corresponding transformation of the spatial coordinates is the Galilean transformation $\mathbf{r} = \mathbf{r}' + \mathbf{w}t$. It is easily checked that in the new coordinate system, Eqns (9) take the form

$$\frac{\partial u}{\partial t} = F(u,g) + D_u \Delta' u + (\mathbf{w}-\mathbf{A})\cdot\nabla' u\,,$$

$$\frac{\partial g}{\partial t} = G(u,g) + D_g \Delta' g + (\mathbf{w}-\mathbf{B})\cdot\nabla' g\,, \tag{10}$$

where Δ' and ∇' are the Laplace operator and gradient expressed in the variables (x', y'). Assume that the set of equations (9) has a solution describing a stationary autowave whose flat front moves with the velocity $\mathbf{V}$. Since the velocity depends parametrically on the vectors $\mathbf{A}$ and $\mathbf{B}$, we denote it as $\mathbf{V}(\mathbf{A},\mathbf{B})$. Apparently, the corresponding solution of Eqns (10) describes a plane wave propagating with the velocity $\mathbf{V}(\mathbf{A}-\mathbf{w},\mathbf{B}-\mathbf{w})$. Note that in both cases we are dealing with the same wave, but consider it in two coordinate systems related to each other by the Galilean transformation. Consequently, the following equality must be fulfilled

$$\mathbf{V}(\mathbf{A}-\mathbf{w},\mathbf{B}-\mathbf{w}) = \mathbf{V}(\mathbf{A},\mathbf{B}) - \mathbf{w}\,, \tag{11}$$

which enables us to make some simple, but important conclusions about the dependence of the velocity $\mathbf{V}$ on the vectors $\mathbf{A}$ and $\mathbf{B}$. For this purpose, it is convenient to consider Galilean transformations for the infinitesimal velocity $\mathbf{w}$. Calculating variation in both sides of Eqn (11) over the projections of the vector $\mathbf{w}$ and setting $\mathbf{w} = 0$, we get the following differential relations for the projections of front velocities as functions of $\mathbf{A}$ and $\mathbf{B}$:

$$\frac{\partial V_x}{\partial A_x} + \frac{\partial V_x}{\partial B_x} = 1\,, \qquad \frac{\partial V_x}{\partial A_y} + \frac{\partial V_x}{\partial B_y} = 0\,,$$
$$\frac{\partial V_y}{\partial A_y} + \frac{\partial V_y}{\partial B_y} = 1\,, \qquad \frac{\partial V_y}{\partial A_x} + \frac{\partial V_y}{\partial B_x} = 0\,. \tag{12}$$

In the general case the functions V_x and V_y satisfying relations (12) are

$$V_x = V_{0x} + \frac{1}{2}(A_x + B_x) + f_x(A_x - B_x,\, A_y - B_y)\,,$$

$$V_y = V_{0y} + \frac{1}{2}(A_y + B_y) + f_y(A_x - B_x,\, A_y - B_y)\,, \tag{13}$$

where V_{0x} and V_{0y} are projections of the velocity vector $\mathbf{V}_0$ of a propagating autowave front at $\mathbf{A} = \mathbf{B} = 0$. As for functions $f_x(\eta_1,\eta_2)$ and $f_y(\eta_1,\eta_2)$, we can only assert them to be projections of a vector (i.e. they are transformed as vectors when the coordinate system is changed), and, in addition, to be equal to zero at $\eta_1 = \eta_2 = 0$.

Relations (13) are valid for any cartesian coordinate system. However, it is convenient to choose the coordinate system so that the x axis should be directed along the vector of the front velocity $\mathbf{V}$. In this case $V_x = V$ and $V_y = 0$, therefore we can consider only the function $f(\eta_1,\eta_2) \equiv f_x(\eta_1,\eta_2)$. This function is easily seen to be independent of the second argument. Actually, to find the velocity and the shape of a stationary autowave moving along the axis x, we should make the replacement $x = Vt + \xi$ in the set of equations (9), and then put $\partial u/\partial t = \partial g/\partial t = 0$ and $\partial u/\partial y = \partial g/\partial y = 0$. This yields

$$D_u \frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} + F(u,g) + (V - A_x)\frac{\mathrm{d}u}{\mathrm{d}\xi} = 0\,,$$
$$D_g \frac{\mathrm{d}^2 g}{\mathrm{d}\xi^2} + G(u,g) + (V - B_x)\frac{\mathrm{d}g}{\mathrm{d}\xi} = 0\,. \tag{14}$$

It is clear that the autowave velocity V calculated as a solution to this set of equations with the corresponding boundary conditions depends parametrically on A_x and B_x, and is independent of A_y and B_y. Thus, the general expression for the velocity of the stationary autowave front can be written as

$$V = V_0 + \frac{1}{2}(A_x + B_x) + f(A_x - B_x)\,, \tag{15}$$

where $f(\eta)$ is some function. Recall that this relation is valid in any coordinate system where the axis x (or, what is the same, the axis ξ) is directed along the vector $\mathbf{V}$. It should be mentioned that V_0 and f can depend on the diffusion coefficients D_u and D_g and the parameters determining the form of the functions F and G in Eqns (14).

4. Evolution of curved fronts in the absence of the electric field

We use the above relations to study the propagation of curved fronts at $\mathbf{E} = 0$. In this case, the transformation of variables $r = Vt + \xi$ reduces the initial system of Eqns (8) to Eqns (14), where

$$A_x = -D_u K\,, \qquad B_x = -D_g K\,. \tag{16}$$

Thus, the general formula (15) for the velocity of the front propagation yields

$$V = V_0 - \frac{1}{2}(D_u + D_g)K + f\big((D_g - D_u)K\big)\,. \tag{17}$$

Recall, that the explicit form of the function $f(\eta)$ can be found by solving the initial reaction–diffusion equations.

For the case when the diffusion coefficients are equal $D_u = D_g = D$, we immediately find the well known result [12]

$$V = V_0 - DK\,. \tag{18}$$

Additional data on the function $f(\eta)$ can be obtained using the results of Ref. [22] (see also [13]). In Ref. [22] the dependence of the front velocity on the curvature has been studied in the

case where there is no diffusion of inhibitor, i.e. $D_g = 0$, and the dependence $V(K)$ was shown to be linear for small curvatures

$$V = V_0 - D_u K . \tag{19}$$

As the curvature rises, the linear dependence (19) changes into a nonlinear dependence. It was found that there is a critical curvature $K_{\rm cr}$ above which a curved front cannot propagate. Note that as the curvature approaches this critical value the front velocity $V_{\rm cr}$ is, generally, not equal to zero.

In the case $D_g = 0$ the general solution of Eqn (17) expresses the front velocity as

$$V = V_0 - \frac{1}{2} D_u K + f(-D_u K) . \tag{20}$$

Comparison of Eqn (19) and Eqn (20) shows that at small η the function $f(\eta)$ is

$$f(\eta) = \frac{1}{2}\eta + f_1(\eta) , \tag{21}$$

where $f_1(\eta)$ is the function, whose expansion in Taylor series in powers of η does not include linear terms. Therefore, from Eqn (20) it follows that

$$V = V_0 - D_u K + f_1(-D_u K) . \tag{22}$$

If we neglect in Eqn (17) a possible dependence of f on D_g, which is an independent parameter, then Eqn (22) is easily generalised to the case where $D_g \neq 0$. Actually, expansion (21) remains valid in this approximation. Using it and taking into account Eqn (17) we find that

$$V = V_0 - D_u K + f_1\big((D_g - D_u)K\big) . \tag{23}$$

Since there are no linear terms in the expansion of f_1 in K, the diffusion coefficient of inhibitor is shown to have no effect on the linear dependence $V(K)$ in the case of small front curvatures. Needless to say that the velocity V_0 of a flat front changes as D_g varies, but the velocity–curvature relationship (i.e. the angle between the tangent to the curve $V(K)$ and the axis K) depends only on value D_u at small K. This general result obtained without solving the reaction–diffusion equations is very important for the study of autowave front kinematics. However, since Eqn (23) has been derived under an additional assumption, it requires detailed experimental and numerical testing.† We can advance an argument in favour of rather weak dependence of f on D_g. It is known that the derivative $\mathrm{d}^2 g/\mathrm{d}\xi^2$ is much less than $\mathrm{d}^2 u/\mathrm{d}\xi^2$ in the region of the autowave front (the inhibitor is even referred to as a 'slow variable'). This means that the inhibitor diffusion has no sufficient effect on the formation of the autowave front, and, hence, the function f in Eqn (17) depends weakly on the parameter D_g. It should be stressed that f depends on D_g via the argument $(D_g - D_u)K$, since it results from 'drift' terms in Eqns (9) rather than from diffusion terms.

†We have received the first independent verification supporting Eqn (23) from V S Zykov. Discussing some results of the present paper, he reported that in his numerical experiments on the reaction–diffusion model he did not actually observe the influence of the inhibitor diffusion on the dependence $V(K)$ in a wide range of values of D_g in the case of small curvatures [23].

5. Flat wavefront in a homogeneous electric field

The study of the evolution of curved fronts made in the previous section can easily be extended to the case of flat fronts moving in an external homogeneous electric field **E**. Let us consider a flat autowave front propagating along the axis x. In this case the stationary motion of the front is described by Eqn (14), where $\xi = x - Vt$ and

$$A_x = \mu_u E_x = \mu_u E \cos\alpha ,$$

$$B_x = \mu_g E_x = \mu_g E \cos\alpha . \tag{24}$$

Further consideration is similar to that in the previous section, and may differ only in notations. Therefore, we write immediately some relations for the front velocity as a function of the external field, which follow from the general formula (15). If the activator and inhibitor have the same mobilities ($\mu_u = \mu_g = \mu$), then the dependence of the front velocity on the electric field is linear in a wide range of E :

$$V = V_0 + \mu E_x . \tag{25}$$

Whether the autowave moves faster or slower, depends on the sign of the projection E_x and sign of the mobility μ. For instance, in the Belousov–Zhabotinsky reaction the electric field decelerates the wave propagation along the field [20].

For models with $D_g = 0$ we should put $\mu_g = 0$, since, according to the Einstein relation, the diffusion coefficient is proportional to mobility. In this case the dependence of the front velocity on the electric field will be

$$V = V_0 + \mu_u E_x + f_1(\mu_u E_x) , \tag{26}$$

the expansion of the function $f_1(\eta)$ in the Taylor series does not include the terms linear with respect to E_x (see Eqn (21)). Repeating the consideration of section 4, we extend the expression (26) to the case $\mu_g \neq 0$. Similarly to (23) we have

$$V = V_0 + \mu_u E_x + f_1\big((\mu_u - \mu_g)E_x\big) . \tag{27}$$

Thus, in a weak electric field the velocity of a flat front depends only on the activator mobility, while its dependence on the inhibitor mobility is manifested only in rather strong fields as nonlinear terms in the expansion of the function f_1 in powers of the external field. This important and not obvious conclusion was verified by numerical experiments [24] in the 'Oregonator' model. The results of these experiments confirmed completely the theoretical perceptions.

Finally note that the existence of the critical electric field intensity $E_{\rm cr}$ follows from the analogy between the equations describing the flat front moving in an electric field and the propagation of a curved front in the absence of the field. If the intensity exceeds $E_{\rm cr}$, then the stable propagation of the autowave is impossible. We shall show that $E_{\rm cr}$ can be expressed by the critical curvature $K_{\rm cr}$ in the absence of the electric field.

We denote the velocity of a curved autowave front as $V_{\rm cr}$ at $K = K_{\rm cr}$ in the absence of the field, and the velocity of a flat autowave front as $\widetilde{V}_{\rm cr}$ at $E_x = E_{\rm cr}$ (note that $E_{\rm cr}$ can be positive or negative depending on signs of mobilities μ_u and μ_g). Since in both cases the stationary regime of the autowave propagation is described by the set of equations (14), where A_x and B_x take the form (16) or (24), then at critical velocities

the following equalities must be fulfilled

$$V_{\rm cr} + D_u K_{\rm cr} = \widetilde{V}_{\rm cr} - \mu_u E_{\rm cr}\,,$$
$$V_{\rm cr} + D_g K_{\rm cr} = \widetilde{V}_{\rm cr} - \mu_g E_{\rm cr}\,. \tag{28}$$

Using Eqn (28) we arrive at the relation between the critical field of the flat front and the critical curvature of the curved front

$$E_{\rm cr} = -K_{\rm cr}\,\frac{D_u - D_g}{\mu_u - \mu_g}\,. \tag{29}$$

Eqns (28) and (29) yield also the relation between critical front velocities in the considered cases

$$\widetilde{V}_{\rm cr} = V_{\rm cr} + K_{\rm cr}\,\frac{\mu_u D_g - \mu_g D_u}{\mu_u - \mu_g}\,. \tag{30}$$

At $\mu_u = \mu_g$ the values of $E_{\rm cr}$ and $\widetilde{V}_{\rm cr}$ become infinite. This result, however, was obvious beforehand, since in the case of equal mobilities of activator and inhibitor, the external field can be eliminated in the initial reaction–diffusion equations by Galilean transformations, so that the velocity of a flat front is determined by Eqn (25) for any field intensities.

We emphasise that $K_{\rm cr}$ and $V_{\rm cr}$ are independent of μ_u and μ_g. Thus Eqns (29) and (30) determine the *universal dependence* of the critical field and the critical velocity of the flat front moving in an electric field on the mobilities of inhibitor and activator for all the excitable media described by the set of equations (9). It should be mentioned, however, that the validity of these formulae is limited by the following condition: the critical curvature radius should far exceed the thickness of the wave front. Otherwise, it is impossible to transform Eqn (7) into Eqn (8).

6. Curved front in an external electric field

To study kinematics of spiral waves and other autowave structures in an external field, we should know the dependence of the front propagation velocity on the local values of curvature and the field. To solve this problem in the case when the curvature radius far exceeds the front thickness and the field intensity does not change rapidly in space, we can use stationary Eqns (14) where A_x and B_x should be taken in the form

$$A_x = \mu_u E_{\rm n} - D_u K\,, \qquad B_x = \mu_g E_{\rm n} - D_g K\,. \tag{31}$$

The projection $E_{\rm n}$ of the electric field which is normal to the front line and the front curvature K may be considered as fixed constants.

The obvious analogy of this problem to the above-considered ones allows us to write down immediately some interesting relations following from formula (15) for the velocity of the stationary motion of the autowave front.

For the models with $D_g = 0$ and $\mu_g = 0$ we have

$$V = V_0 + \mu_u E_{\rm n} - D_u K + f_1(\mu_u E_{\rm n} - D_u K)\,, \tag{32}$$

where $f_1(\eta)$ is the same nonlinear function as in Eqn (22).

In the case when $D_g \neq 0$ and $\mu_g \neq 0$, the velocity of the stationary motion of the front is

$$V = V_0 + \mu_u E_{\rm n} - D_u K + f_1\big((\mu_u - \mu_g)E_{\rm n} - (D_u - D_g)K\big)\,. \tag{33}$$

Note that expressions given in Sections 4 and 5 for the velocity of a flat front in an external field and for the velocity of a curved front in the absence of the external field can be derived from Eqns (32) and (33) as particular cases.

Since in the considered case the velocity of the moving autowave front is affected by both its curvature and the external field, it is clear that the critical curvature of an autowave segment depends on the local value of the normal projection of the field $E_{\rm n}$. This dependence can be found similarly to the derivation of Eqns (29) and (30) in the previous section. We denote the critical front curvature and its velocity as $K_{\rm cr}(E_{\rm n})$ and $V_{\rm cr}(E_{\rm n})$, while the corresponding values in the zero field as $K_{\rm cr}$ and $V_{\rm cr}$. It follows from Eqns (16) and (31) for parameters A_x and B_x that the following relations must be fulfilled in the critical regime of the front motion in both the cases:

$$V_{\rm cr}(E_{\rm n}) + D_u K_{\rm cr}(E_{\rm n}) - \mu_u E_{\rm n} = V_{\rm cr} + D_u K_{\rm cr}\,,$$
$$V_{\rm cr}(E_{\rm n}) + D_g K_{\rm cr}(E_{\rm n}) - \mu_g E_{\rm n} = V_{\rm cr} + D_g K_{\rm cr}\,. \tag{34}$$

Whence we get immediately

$$K_{\rm cr}(E_{\rm n}) = K_{\rm cr} + E_{\rm n}\,\frac{\mu_u - \mu_g}{D_u - D_g}\,. \tag{35}$$

Thus, the critical curvature of the autowave front depends linearly on the normal component of the external field. In the case of equal mobilities of activator and inhibitor, it follows from Eqn (35) that the critical curvature does not depend on the field. However, this could be expected beforehand bearing in mind the properties of the Galilean transformation of the reaction–diffusion equations. We also note that expression (29) can be derived from Eqn (35) for the critical field in the case of the flat front. Actually, assuming $E_{\rm n} = E_{\rm cr}$ and $K_{\rm cr}(E_{\rm n}) = 0$ in Eqn (35), we immediately get Eqn (29).

Another important relation, which follows from Eqn (34), relates the critical front velocities in zero and finite external fields. Taking into account Eqn (35), we write this relation as

$$V_{\rm cr}(E_{\rm n}) = V_{\rm cr} - E_{\rm n}\,\frac{\mu_u D_g - \mu_g D_u}{D_u - D_g}\,. \tag{36}$$

We call attention to an interesting and surprising consequence of Eqns (35) and (36). As seen from Eqn (36), the critical front velocity does not depend on the external field in the systems with $D_g = 0$ and $\mu_g = 0$. At the same time, according to Eqn (35), $K_{\rm cr}(E_{\rm n}) \neq K_{\rm cr}$ if $E_{\rm n} \neq 0$.

7. Drift of a spiral autowave in an electric field

One of the interesting examples of autowave structures in excitable media is spiral waves rotating around the fixed centre with a constant angular velocity. From simple physical considerations one can expect that a weak electric field leads to a drift of the centre of a spiral wave, without changing its front structure. A strong field can so affect the dynamics of activator and inhibitor that the autowave propagation becomes impossible. From the obtained relation (35) it follows that the critical front curvature may be changed significantly in the external field.

It is possible perform a detailed study of spiral autowaves in the external field by numerical solution of the initial reaction–diffusion equations or by using of an approxima-

tion method. The kinematic approach in which the evolution of the wave front is described only in time [12], is rather a simple and universal method. In the framework of this approach the dependence of the local velocity of the wave front on the curvature in linear and nonlinear approximations can be described by Eqns (32) and (33) (in the latter case some additional data on function f_1 are required).

We discuss only some general relationships of drift of spiral waves, which can be found by Galilean transformations in the reaction – diffusion equations without detailed investigations of solutions to these equations. The application of Galilean transformations to the problem of drift of spiral autowaves is considered in greater detail in Ref. [25].

Assume that in the absence of the external field, i.e. for $\mathbf{A} = \mathbf{B} = 0$, the set of equations (9) has the solution $u_0(\mathbf{r}, t)$, $g_0(\mathbf{r}, t)$ describing a stationary autowave rotating around a fixed centre with a constant angular velocity ω. Then for a non-zero field we have

$$\mathbf{A} = \mu_u \mathbf{E}\,, \qquad \mathbf{B} = \mu_g \mathbf{E}\,. \tag{37}$$

Then the solution $u(\mathbf{r} - \mathbf{V}t, t)$, $g(\mathbf{r} - \mathbf{V}t, t)$ of Eqns (9) describes a spiral wave with the centre moving with the constant drift velocity $\mathbf{V}$, which parametrically depends on the vectors $\mathbf{A}$ and $\mathbf{B}$. This velocity can be easily found when the mobilities of inhibitor and activator are equal ($\mu_u = \mu_g = \mu$). Actually, in this case the solution to Eqns (9) takes the form $u_0(\mathbf{r} - \mu\mathbf{E}t, t)$, $g_0(\mathbf{r} - \mu\mathbf{E}t, t)$ in the coordinate system moving with the velocity $\mathbf{w} = \mu\mathbf{E}$. It describes a spiral wave rotating with the same angular velocity ω around the centre which drifts with the velocity

$$\mathbf{V} = \mu\mathbf{E}\,. \tag{38}$$

We consider the case where $\mu_u \neq \mu_g$. For the sake of definiteness, we assume that the external homogeneous electric field is directed along the axis x of the coordinate system. Then, repeating the considerations of Section 3, we arrive at the relations

$$V_x = \frac{1}{2}(\mu_u + \mu_g)E + F_x\big((\mu_u - \mu_g)E\big)\,,$$
$$V_y = F_y\big((\mu_u - \mu_g)E\big)\,, \tag{39}$$

which follow from Eqns (13) at $\mathbf{V}_0 = 0$. The functions $F_x(\eta)$ and $F_y(\eta)$ become zero at $\eta = 0$. Besides, we can state that they are *odd functions*, since the drift velocity must change the sign as the sign of the electric field changes. The explicit form of these functions for concrete models can be found by numerical solution to the reaction – diffusion equations. Detailed numerical calculations of the drift velocities V_x and V_y as well as the functions F_x and F_y are given in Ref. [25], which completely prove the theoretical predictions.

The second relation in Eqns (39) describes the drift of the spiral wave in an electrical field in the direction perpendicular to the field. Note that this effect was observed many times in numerical and real experiments (see, for instance, Ref. [26]).

If we suppose that the electric field is sufficiently small, then in the expansions

$$F_x(\eta) = a_1\eta + O(\eta^3)\,, \qquad F_y(\eta) = a_2\eta + O(\eta^3) \tag{40}$$

we can keep only the linear terms, where a_1 and a_2 are constants depending on the model. Then, denoting the angle between the electric field $\mathbf{E}$ and the autowave drift velocity $\mathbf{V}$ as χ, we find from Eqns (39) that

$$\tan\chi = a_2\,\frac{\mu_u - \mu_g}{(1/2 + a_1)\mu_u + (1/2 - a_1)\mu_g}\,. \tag{41}$$

We emphasise that this formula describes the universal dependence of the drift angle of a spiral wave on the mobilities of inhibitor and activator in a weak electric field. The constants a_1 and a_2 depend on the model of excitable medium.

8. Conclusion

We have presented some results following from Galilean transformations of nonlinear reaction – diffusion equations, which describe the propagation of autowave fronts in an external electric field. This approach enables us to reveal a number of regularities in the behaviour of flat and curved fronts in an external field, by a simple way without solving particular equations. Many of these regularities are so general that they are independent of the form of nonlinear functions in 'microscopic' reaction-diffusion equations. At the same time, they can be verified numerically or experimentally. First and foremost we mean relations (29) and (30) for critical electric field and velocity of a flat front, Eqns (35) and (36) for the critical curvature and velocity depending on the external field, and relation (15) for the autowave velocity.

It should be mentioned that the field of applicability of the method goes far beyond the phenomena considered in this paper. First of all, Galilean transformations can be useful in studying autowave processes described by multicomponent models (with three or more equations). Evidently, the results presented above are easily extended to the case when the 'drift' terms appear only in two equations of the set (for instance, similar equations are used to describe autowave structures in the Belousov – Zhabotinsky reaction in the presence of an electric field). However, when the 'drift' terms appear in three or more equations, some new regularities can occur, which are absent in the two-component system. We intend to investigate this case in detail in later studies.

The results obtained may also be of importance in studying the kinematics of autowave structures in an external field, to investigate their evolution, interaction and methods of control over them. In particular, some relations obtained are necessary to close kinematic equations used in the theory of autowaves [12].

Acknowledgements

The investigations described in the present paper have been carried out under Grant No. 7M000 of the International Science Foundation and Grant No. 7M300 of the International Science Foundation and the Russian Government.

The authors are thankful to V S Zykov, and V Perez-Munuzuri, A P Munuzuri and M Gomez-Gesteira for their useful discussions and interest in this paper.

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