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METHODOLOGICAL NOTES

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Geometrical phase effects in neutron optics

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Abstract. The aim of this paper is to study in detail the geometric phase (Berry phase) and to discuss peculiarities of its measurements in the most typical physical case, i.e. the problem of neutron spin evolution in a precessing magnetic field.

1. Introduction

In the classical paper [1] on the passage of neutrons through ferromagnetics, Halpern and Holstein suggested an iteration scheme which enabled one to find the final polarisation vector from the initial one for any considered configuration of a magnetic field in ferromagnetics. Even today this iteration scheme remains basic to processing the data on the passage of neutrons through magnetic fields of various noncollinear configurations.

Can the Halpern and Holstein approach be generalised? Are there any regularities in the behaviour of the polarisation vector unlikely to be revealed in the context of the Halpern-Holstein approach? The search for answers to these questions necessitates a detailed analysis of another no less famous work.

In 1984 Berry published in the "Proceedings of the Royal Society" a paper [2] under the title "Quantal phase factors accompanying adiabatic changes". It is in this paper that Berry fully realised from the standpoint of quantum mechanics that the behaviour of a so-called *nonholonomic* system with time- (or coordinate-) dependent parameters differs qualitatively from the behaviour of systems with these parameters fixed. Berry showed that in the limit when the parameters of a system are adiabatically changed, the total change in the wave function phase can differ from the dynamic one as the quantum-mechanical system reverts to its initial state. The arising phase difference is related by a simple expression to a solid angle traced by the vector-parameter $\mathbf{R} = \mathbf{R}(t)$ of the system (in the case of neutrons, by the vector of the magnetic field intensity $\mathbf{H}(t)$) during its cyclic evolution. This phase difference can be responsible for experimentally seen effects and in the limiting case its value is independent of the evolution time. The ten years that have passed since this remarkable paper came out have been marked by numerous experimental and theoretical studies devoted to various manifestations of the geometric or topological phase.

Having applied a similar approach to classical systems, Hannay [3] found that in mechanical systems with time-variant parameters there is an additional angle displacement, or the so-called Hannay angle. In terms of classical mechanics it is due to the fact that an adiabatic invariant of the action exists in parallel with a nonvariant variable, namely, the angle. Subsequently Berry analyzed [4] how the additional phase of the wave function relates to the Hannay angle in the quasiclassical approximation.

Aharonov and Anandan proved that the geometric phase arises in the case of nonadiabatic evolution as well. Moreover, this reasoning can be generalised to the cases of noncyclic, nonunitary, and non-Abelian evolutions [6].

Having a chance to refer to numerous reviews (see, for example, [7 – 10]) and pursuing the goal to analyze precisely the neutron-optical aspect of the geometric phase manifestations, we only outline in Section 2 a broad spectrum of nonholonomic problems. Moreover, we will not be too general in what follows and use the term 'neutron in a magnetic field' for 'a particle with spin 1/2 in an external field', since it is just the same in essence. In Section 3 we carry out a simple consideration which will require us to introduce the concept of the geometric phase. In Sections 4 and 5 we derive and study the Berry and Aharonov–Anandan phases, respectively, for wave functions of a neutron in a precessing magnetic field. The Berry phases for wave functions with positive and negative spin projections on the quantization axis are adiabatic limits of the more general (nonadiabatic)

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Aharonov–Anandan geometric phases. The latter are obtained and studied by the Aharonov–Anandan method determining the geometric phase indirectly as the difference between the total and the dynamic phases. In Section 6 we briefly run through neutron-optical experiments on measuring the geometric phases. In the Appendix we extend the Berry approach to the direct calculation of the nonadiabatic geometric phases. Considering a precessing field, we show that the extended Berry approach and the Aharonov–Anandan method yield identical values of the nonadiabatic geometric phases.

2. Nonholonomic problems

The history of the phenomenon discussed in Berry's paper dates back to the Foucault pendulum and Sagnac's experiments with a rotating interferometer. Rytov, Vladimirskiĭ and Pancharatnam in their studies [11] on the optics of light guides with anisotropic refraction index analyzed how the light guide geometry influences the results of polarisation experiments. Subsequently experiments by Tomita and Chiao [12] provided support for the idea that a polarisation plane rotates in a spiral-shaped fiber light guide (see Fig. 1).

Neutron optics is best suited to observe the geometric phase effects. Experiments by Bitter, Dubbers [13], Richardson et al. [14], Weinfurter and Badurek [15] confirmed a conceptual possibility to measure geometric phases in neutron optics. The motion of neutrons in a magnetic field accompanied by the spin *evolution* is of theoretical interest as well, since the corresponding Schrödinger equation has an exact solution in an important particular case of a precessing magnetic field [16]. It should be stressed once again that in the context of the geometric phase by evolution we mean the behaviour of a neutron spin given an *implicit* (via parameters) time dependence of the Hamiltonian concerned. This dependence can be realised, for example, in the time-of-flight experiments with the availability of noncollinear (helicoidal) magnetic fields. A simple picture of traditionally considered precession with positive and negative projections of the spin in a uniform and stationary magnetic field converts to a qualitatively distinct in complexity nonholonomic behaviour in the case of spin evolution. We mention in this connection some of the recent papers [17 – 19] analyzing various aspects of nonholonomicity in a two-level system with the SU(2)-symmetric Hamiltonian.

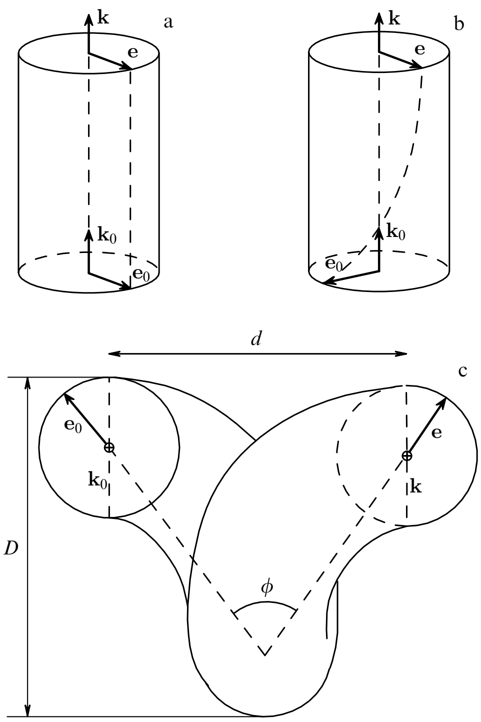

Figure 1. Light guide optics: (a) straight or folded into a plane spiral cylindric light guide do not change the direction of polarization $\mathbf{e}$ of the light beam; (b) light guide with natural or artificial (as a result of torsion deformation) gyrotropicity changes the direction of polarization of the beam passing through a light guide's section (Pancharatnam phase); (c) change in the direction of polarization of light propagating in twisted nonplanar light guides (Rytov–Vladimirskiĭ phase) [10]. In particular, in a helical-shaped light guide with a constant pitch d and the diameter D, the arising Rytov–Vladimirskiĭ phase is equal to the twisting angle $\phi = 2\arcsin(d/2D)\,n$ [11], where n is the rotation speed of the helix. Obviously that as $d \to 2D$ we have $\phi = \pi n$. Further rise in d (up to the maximum value πD) can be performed without changes in the twisting angle.

In the present paper we consider the effects connected with the evolution in *macrofields*, when the neutron spin moves in a magnetic field of an external macroscopic source or in an averaged field of the condensed matter. The evolution of a physical system in *microfields*, or *quantum* evolution, suffice it to say, merits special consideration. We only note here that the geometric phase is bound to arise in the analysis of such quantum phenomena as the Jahn–Teller effect (or the Aharonov–Bohm molecular effect) [20], the Hall quantum effect [21] and the nuclear quadrupole resonance in a rotating crystal with the magnetic anisotropy axis shifted relative to the rotation axis, where a specific 'spreading' of resonance peaks occurs [22].

3. Berry phase. General case

In the context of the perturbation theory, the Hamiltonian of a quantum system can be written as the sum of a nonperturbed Hamiltonian $\hat{H}_0$ and perturbation $\hat{V}$:

$$\hat{H} = \hat{H}_0 + \hat{V}\,.$$

An explicit time dependence of the Hamiltonian is usually associated with the time–dependent perturbation

$$\hat{V} = \hat{V}(t)\,.$$

In neutron-optical time-of-flight experiments we deal with the motion of a neutron through a magnetic field, whose intensity can vary from one point of the trajectory to another both in the magnitude and the direction. This fact has no effect on the character of local interaction (between the neutron spin and the magnetic field) but leads to the implicit time dependence of the Hamiltonian via the components of the magnetic field intensity (i.e. vector-parameter $\mathbf{R}(t)$), such that

$$\hat{H}\big(\mathbf{R}(t)\big) = \hat{H}_0\big(\mathbf{R}(t)\big)\,.$$

In the case of adiabatic evolution, the magnetic field $\mathbf{H}$ (the quantization axis) is assumed to change its direction so slowly that the polarisation vector (precessing around $\mathbf{H}$) conserves its component parallel to the field $\mathbf{P} = \langle\boldsymbol{\sigma}\rangle$. However, the direction of the cyclically–evolved component, perpendicular to the quantization axis, differs generally from the initial one due to the motion of the quantization axis. It is vital to note that this fact can take place even when the time of the vector-parameter evolution is a multiple of the precession period (see Fig. 2).

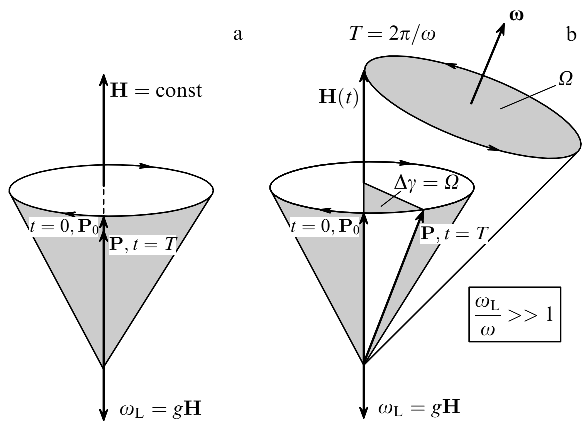

Figure 2. The formation of the revealed geometric Berry phase $\Delta\gamma$ for neutron spin in the precessing magnetic field $\mathbf{H}(t)$. The vector $\mathbf{P}$ of the neutron polarization is shown at moments $t = 0$ and $t = T$, where T is the evolution period of the field $\mathbf{H}$. For simplicity we consider the case when T is a multiple of the Larmore precession period T_L: (a) regular Larmore precession of the vector $\mathbf{P}$ around the vector $\mathbf{H} = \text{const}$ when the initial position of the vector $\mathbf{P}$ coincides with the final one (provided periods T and T_L are the multiples); (b) typical behaviour of the vector $\mathbf{P}$ when the vector $\mathbf{H}$ precesses at the frequency $+\omega$. The initial position of the vector $\mathbf{P}$ does not coincide with the final one due to arising geometric phases on the wave functions level. In this case we imply that the precession cone of the vector $\mathbf{P}$ follows adiabatically the vector $\mathbf{H}$, i.e. the adiabatic condition $\omega_L/\omega = T/T_L \gg 1$ takes place. Angle $\Delta\gamma = (\gamma_- - \gamma_+)$ is the difference between the Berry phases for the spin basis wave functions of the neutron. The main conclusion of the Berry theory for the neutron spin is that $\Delta\gamma = \Omega$, where the solid angle Ω is traced by the field strength vector on the Poincare sphere.

How should we describe the behaviour of a neutron in this case? What is generally known about systems with implicit time dependence?

According to the Ehrenfest adiabatic hypothesis [24], such a quantum system after a cyclic vector–parameter evolution should not differ from that in the initial state providing the evolution is adiabatic. What is the value of Berry's paper? He has pointed out that the wave function of the system under cyclically varied parameters generally acquires an additional nontrivial phase multiplier. Actually, for any quantum system, in particular, for that with time-dependent parameters, there formally raises an eigen-value problem, i.e. the system is described, at least locally in time, by the 'stationary' Schrödinger equation

$$\hat{H}\big(\mathbf{R}(t)\big)\,\psi_n\big(q, \mathbf{R}(t)\big) = E_n\big(\mathbf{R}(t)\big)\,\psi_n\big(q, \mathbf{R}(t)\big)\,.$$

As this takes place, the total wave function

$$\Psi_n\big(q, \mathbf{R}(t), t\big) = \psi_n\big(q, \mathbf{R}(t)\big)\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t E_n(\tau)\,\mathrm{d}\tau\right]$$

must satisfy the time-dependent Schrödinger equation

$$\mathrm{i}\hbar\dot{\Psi}_n = \hat{H}\Psi_n\,.$$

Thus, considering matrix elements we finally arrive at the condition of parallel transport

$$\left(\psi_n\big(q, \mathbf{R}(t)\big),\ \frac{\partial}{\partial t}\,\psi_n\big(q, \mathbf{R}(t)\big)\right) = 0\,.$$

This condition can be met exclusively if one draws on the fact that the wave function phase is yet arbitrary. With the replacement $\psi_n \to \tilde{\psi}_n \exp(\mathrm{i}\,\gamma_n)$ we have

$$\mathrm{i}\hbar\left(\left(\frac{\partial}{\partial t}\,\tilde{\psi}_n\big(q, \mathbf{R}(t)\big)\right)\exp(\mathrm{i}\gamma_n) + \mathrm{i}\left(\frac{\partial}{\partial t}\,\gamma_n\right)\tilde{\psi}_n\exp(\mathrm{i}\,\gamma_n)\right) = 0\,,$$

or

$$\frac{\partial}{\partial t}\,\gamma_n = \mathrm{i}\left(\tilde{\psi}_n\big(q, \mathbf{R}(t)\big),\ \frac{\partial}{\partial t}\,\tilde{\psi}_n\big(q, \mathbf{R}(t)\big)\right). \tag{1}$$

Thus determined γ_n is just the geometric phase or the Berry phase.

Assume that the system adiabatically evolves along a closed contour C in the space of parameters, i.e. $\mathbf{R}(T_C) = \mathbf{R}(0)$, where $T_C \to \infty$ is the evolution time. Then the change in the phase γ_n during the time T_C is given by the expression

$$\begin{aligned}\Delta\gamma_n \to \gamma_n &= \mathrm{i}\oint_0^{T_C}\left(\tilde{\psi}_n\big(q, \mathbf{R}(t)\big),\ \frac{\partial}{\partial t}\,\tilde{\psi}_n\big(q, \mathbf{R}(t)\big)\right)\mathrm{d}t \\ &= \mathrm{i}\oint_C \mathbf{A}_{nn}\,\mathrm{d}\mathbf{R}\,,\end{aligned} \tag{2}$$

where

$$\mathbf{A}_{nn} = \mathrm{i}\left(\tilde{\psi}_n,\ \vec{\nabla}_{\mathbf{R}}\tilde{\psi}_n\right). \tag{3}$$

If the contour has no singularities, then, according to the Stokes theorem, we can write the phase in the form

$$\gamma_n = \int_{\Sigma_C} \mathrm{rot}\,\mathbf{A}_{nn}\,\mathrm{d}\mathbf{s}\,, \tag{4}$$

where $\mathrm{d}\mathbf{s}$ is the element of an oriented surface Σ_C stretched on the C contour. A characteristic feature of the adiabatic evolution is that the change in the phase during the cyclic evolution is independent of the evolution time and depends solely upon the geometry of the parametric space within the closed C contour. In this case

$$\gamma_n \propto \Omega\,,$$

where Ω is the solid angle traced by the vector-parameter $\mathbf{R}$. For the neutron wave function, the phases of two spin states (along the field denoted as $+$ and in the opposite direction as $-$) are expressed via the solid angle traced by the magnetic field vector $\mathbf{H}$ as follows [2]:

$$\gamma_\pm = \mp\frac{1}{2}\,\Omega\,. \tag{5}$$

Therefore, a consistent analysis of the adiabatic limit yields rather a different picture of the cyclic evolution. The system turns back to the state which differs from the initial one by the corresponding geometric phase determined by the trajectory in the parametric space.

However, the assumption that the changes are adiabatic in the system running through a set of the above-mentioned 'stationary' states $\Psi_n\big(q, \mathbf{R}(t), t\big)$, keeping the quantum number n constant during the evolution time, may not be generally valid for any time dependence of the parameters. Therefore, the actual description of such systems involves some additional assumptions of the parameter evolution which enable one, for example, to describe nonadiabatic behaviour by the methods of the nonstationary perturbation theory.

Thus, on the whole, the wave function of a system with an implicitly time-dependent Hamiltonian is not only time-variant, but also changes over the space of parameters. In principle, if the space has geometric singularities (curvature, torsion, poles, or the space is multiply connected) they can influence the behaviour of the system. In this case the system is said to possess a (nontrivial) holonomy. In particular, if we assume that the system evolves only in the direction-variant fields, then the consideration of possible system evolutions is reduced to the analysis of holonomy of corresponding trajectories, traced by the vector-parameter $\mathbf{R}(t)$ on a unit sphere of possible directions of the vector, i.e. the Poincare sphere.

4. Berry phase for a neutron in a precessing magnetic field

It is well known that the solution to the problem of a neutron spin in a uniform magnetic field admits of two eigen states with the spin projection on the chosen axis:

$$\Psi_+^o(\theta,\phi) = \exp\left(-\frac{\mathrm{i}\,\omega_\mathrm{L} t}{2}\right)\begin{pmatrix} \cos\left(\dfrac{\theta}{2}\right) \quad \exp(-\mathrm{i}\,\phi) \\ \sin\left(\dfrac{\theta}{2}\right)\end{pmatrix}, \quad (6)$$

$$\Psi_-^o(\theta,\phi) = \exp\left(\frac{\mathrm{i}\,\omega_\mathrm{L} t}{2}\right)\begin{pmatrix} -\sin\left(\dfrac{\theta}{2}\right) \quad \exp(-\mathrm{i}\,\phi) \\ \cos\left(\dfrac{\theta}{2}\right)\end{pmatrix}, \quad (7)$$

where θ and ϕ are, respectively, the polar and the azimuth angles of the quantization axis ($\mathbf{H}$) in the chosen coordinate system; $\omega_\mathrm{L} = 2\mu H/\hbar$ is the frequency of the Larmore precession; μ is the absolute value of the neutron magnetic momentum, $H = |\mathbf{H}|$.

The neutron spin behaves in such a way that, considering the solutions in the case of a precessing magnetic field instead of a constant field, we exactly solve the corresponding time Schrödinger equation (Pauli equation) [16]. Calculating the wave functions in the problem of precessing field, we can study in detail the formation of the adiabatic geometric Berry phase for the neutron wave function by the above method. In addition, the data on the exact solution enable us to study thoroughly the nonadiabatic case by the Aharonov–Anandan method as well (see Section 5).

Let us consider in terms of the geometric phase the behaviour of the spin of a neutron in a magnetic field, which, being constant in magnitude, rotates evenly around the axis z at the angle velocity ω, making the angle θ with this axis:

$$H_x = H\sin\theta\cos(\omega t + \phi),$$
$$H_y = H\sin\theta\sin(\omega t + \phi),$$
$$H_z = H\cos\theta.$$

The Pauli equation takes the form

$$\mathrm{i}\,\hbar\dot{\Psi} = 2\mu\mathbf{H}\hat{\mathbf{s}}\Psi. \quad (8)$$

Without going into details of the solution, we finally arrive at the following form of the wave function

$$\Psi(t) = C_+\Psi_+(t) + C_-\Psi_-(t), \quad |C_+|^2 + |C_-|^2 = 1, \quad (9)$$

$$\Psi_+(t) = \exp\left[\frac{-\mathrm{i}(\Lambda+\omega)t}{2}\right]\begin{pmatrix} \cos\left(\dfrac{\Theta}{2}\right) \quad \exp(-\mathrm{i}\,\Phi) \\ \sin\left(\dfrac{\Theta}{2}\right)\end{pmatrix}, \quad (10)$$

$$\Psi_-(t) = \exp\left[\frac{\mathrm{i}\,(\Lambda-\omega)t}{2}\right]\begin{pmatrix} -\sin\left(\dfrac{\Theta}{2}\right) \quad \exp(-\mathrm{i}\,\Phi) \\ \cos\left(\dfrac{\Theta}{2}\right)\end{pmatrix} \quad (11)$$

where $\Phi = \omega t + \phi$,

$$\cos\frac{\Theta}{2} = \sqrt{\frac{\Lambda + \omega_\mathrm{L}\cos\theta + \omega}{2\Lambda}}, \quad (12)$$

$$\Lambda = \sqrt{(\omega + \omega_\mathrm{L}\cos\theta)^2 + \omega_\mathrm{L}^2\sin^2\theta}\,. \quad (13)$$

Comparing (10), (11) with (6), (7) we conclude the quantization axis to be determined at present by an effective field $\mathbf{H}_\mathrm{eff}$, directed at the angle Θ (12) to the z axis (Fig. 4a).

Note that the $\Psi_\pm$ states are orthogonal:

$$(\Psi_+, \Psi_-) = 0$$

and correspond to the states with the projection of the spin on the axis z:

$$(\Psi_\pm, s_z\Psi_\pm) = \pm\frac{1}{2}\cos\Theta\,.$$

What is the geometric Berry phase equal to in this problem? Let us use the above calculation method. The components of the vectors $\mathbf{A}_{\pm\pm}$ (3) in spherical coordinates are

$$\mathbf{A}_{++} = \mathrm{i}\,(\psi_+, \vec{\nabla}\psi_+) = \left(0, 0, \frac{1}{2H\sin\theta}\cos^2\frac{\Theta}{2}\right), \quad (14)$$

$$\mathbf{A}_{--} = \mathrm{i}\,(\psi_-, \vec{\nabla}\psi_-) = \left(0, 0, \frac{1}{2H\sin\theta}\sin^2\frac{\Theta}{2}\right). \quad (15)$$

Then, by (4)

$$\gamma_{\pm\pm} = \int_{\Sigma_C} \mathrm{rot}\,(\mathbf{A}_{\pm\pm})\;\mathbf{ds}\,. \quad (16)$$

Having calculated the rotors

$$\mathrm{rot}\,\mathbf{A}_{\pm\pm} = \left(\mp\frac{1}{2H^2}\left(\frac{\sin\Theta}{\sin\theta}\right)\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}, 0, 0\right), \quad (17)$$

we express $\gamma_{\pm\pm}$ as ($\omega_L > \omega$)

$$\gamma_{\pm\pm} = \mp\frac{1}{2}\,\Omega(\Theta)\,, \quad (18)$$

here $\Omega(\Theta) = 2\pi\,(1 - \cos\Theta)$ is the solid angle formed by the cone of angle 2Θ:

$$\Theta = \arccos\left(\frac{\omega_\mathrm{L}\cos\theta + \omega}{\Lambda}\right). \quad (19)$$

If the adiabatic condition is fulfilled, i.e. $\omega_L/\omega \gg 1$, $\gamma_\pm$ coincides with expression (5):

$$\gamma_\pm = \mp \frac{1}{2}\,\Omega(\theta)\,.$$

We note that expression (18) is valid in the case of the locally stationary Shrödinger equation, i.e. at $\omega_L/\omega \gg 1$. However, this expression can be formally generalised to the case when $\omega_L/\omega \leqslant 1$, i.e. when the adiabatic condition is not met. Therefore, it is γ_{++} that should be called in our case the partial nonadiabatic Berry phase in the range $\omega_L/\omega \leqslant 1$. In the Appendix we present a method to calculate the total nonadiabatic geometric phase for arbitrary ω_L/ω on the basis of the Berry approach. However, in citations the term nonadiabatic Berry phase is often used for the total nonadiabatic geometric phase (the Aharonov–Anandan phase) considered below. Let us derive formulas for the total nonadiabatic geometric phases, using the Aharonov–Anandan approach to the problem of a precessing field.

5. Aharonov–Anandan phase for a neutron in a precessing magnetic field

As Aharonov and Anandan showed [5] there exists a method for indirect calculation of the nonadiabatic geometric phase. Let us turn back to the introductory reasoning in Section 3. Instead of the assumption about the neutron passage through a set of ‘stationary’ states we will consider a quantum system described by the cyclic Hamiltonian:

$$\hat{H}(T) = \hat{H}(0).$$

Then at the moment T the wave function, correct to a phase, coincides with the initial wave function:

$$\Psi(T) = \exp(\mathrm{i}\alpha)\,\Psi(0).$$

We can write the wave function in the form

$$\Psi(t) = \exp\left[\mathrm{i}\alpha(t)\right]\tilde{\Psi}(t), \qquad \tilde{\Psi}(T) = \tilde{\Psi}(0) = \Psi(0).$$

Substituting this wave function into the Pauli equation (8) and multiplying scalarly both sides of the equation by Ψ^* we get

$$-\frac{\partial\alpha}{\partial t} = -\mathrm{i}\left(\tilde{\Psi}, \frac{\partial}{\partial t}\tilde{\Psi}\right) + \frac{1}{\hbar}(\Psi, \hat{H}\Psi).$$

After integration from 0 to T we have

$$\alpha(T) = -\frac{1}{\hbar}\int_0^T (\Psi, \hat{H}\Psi)\,\mathrm{d}\tau + \mathrm{i}\int_0^T \left(\tilde{\Psi}, \frac{\partial}{\partial\tau}\tilde{\Psi}\right)\mathrm{d}\tau\,, \tag{20}$$

where

$$-\frac{1}{\hbar}\int_0^T (\Psi, \hat{H}\Psi)\,\mathrm{d}\tau = \beta \tag{21}$$

is the dynamic phase, while

$$\mathrm{i}\int_0^T \left(\tilde{\Psi}, \frac{\partial}{\partial\tau}\tilde{\Psi}\right)\mathrm{d}\tau = \gamma \tag{22}$$

is the Aharonov–Anandan geometric phase. Thus, at moment T the geometric phase is easily seen to be

$$\gamma = \alpha - \beta. \tag{23}$$

The Aharonov–Anandan approach enables one to calculate the geometric phase in the general case of a precession with arbitrary ω_L/ω.

The wave functions (10), (11) are cyclic in our problem. Actually, it is readily seen that

$$\Psi_\pm(T) = \exp\left[\mathrm{i}\pi\left(1 \pm \frac{\Lambda}{\omega}\right)\right]\Psi_\pm(0)\,, \tag{24}$$

where $T = 2\pi/\omega$ is the evolution period. Therefore, the total change in the phase of these functions by the moment T is

$$\alpha_\pm = \pi\left(1 \pm \frac{\Lambda}{\omega}\right). \tag{25}$$

Substituting the Pauli Hamiltonian $\hat{H} = \hbar\,\boldsymbol{\omega}_L\boldsymbol{\sigma}$ in (21), where $\boldsymbol{\sigma}$ are the Pauli matrices, integrating this expression with the wave functions (10), (11), we get the dynamic phases

$$\beta_\pm = \pm\frac{\pi\omega_L}{\Lambda\omega}(\omega\cos\theta + \omega_L) = \pm\frac{\pi}{\omega}(\Lambda - \omega\cos\Theta)\,. \tag{26}$$

It should be mentioned here the essential feature of the nonadiabatic problem: the dynamic phase β does not coincide already with the Larmore angle of rotation in the field $\mathbf{H}$ during the time T. According to (23), the geometric phases are equal to

$$\gamma_+ = \alpha_+ - \beta_+ = \pi(1 + \cos\Theta) = 2\pi - \frac{1}{2}\,\Omega(\Theta), \tag{27}$$

$$\gamma_- = \alpha_- - \beta_- = \pi(1 - \cos\Theta) = \frac{1}{2}\,\Omega(\Theta). \tag{28}$$

The difference $\gamma_- - \gamma_+$, contributing to the observed phases, is equal to

$$\Delta\gamma = \gamma_- - \gamma_+ = -2\pi\cos(\Theta) = \Omega(\Theta) - 2\pi. \tag{29}$$

Within the adiabatic Berry approach the sign of the geometric phase is determined by the direction of tracing around the path in the parametric space. Thus, to put the formulae calculated in the adiabatic limit in correspondence with the Berry consideration, we suppose that

$$\Delta\gamma = \Omega(\Theta) \text{ at } \omega > 0; \quad \Delta\gamma = -\Omega(\Theta) \text{ at } \omega < 0. \tag{30}$$

It does not change principally the formulae calculated, since the new $\Delta\gamma$ differs from (29) by 2π. Fig. 3 plots the dependence of $\Delta\gamma$ on the parameter ω/ω_L.

How does the geometric phase influence the polarisation vector of neutrons passing through a helicoidal cyclic field, which is reduced to the precessing magnetic field in the coordinate system of the moving neutron?

Using the wave functions (10), (11), let us calculate the observable quantum-mechanical averages of operators $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$, which are the components of the final polarisation vector, i.e. the polarisation at the moment $T = 2\pi/\omega$. For the sake of simplicity, we set the combination coefficients of the wave functions in (9) equal to $(C_+, C_-) = (1/\sqrt{2}, \mathrm{i}/\sqrt{2})$ and $\phi = 0$.

Then we have

$$P_x = -\cos\Theta\sin(\Delta\beta + \Delta\gamma), \tag{31}$$

$$P_y = \cos(\Delta\beta + \Delta\gamma), \tag{32}$$

$$P_z = \sin\Theta\sin(\Delta\beta + \Delta\gamma). \tag{33}$$

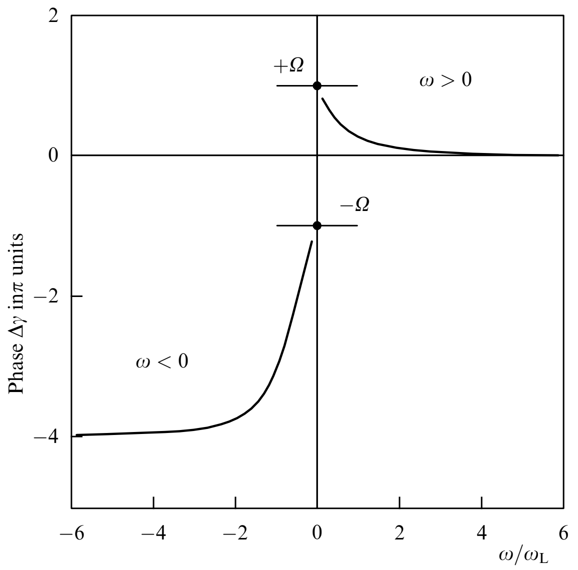

Figure 3. The observable difference geometric phase of the wave function $\Delta\gamma$ calculated for the precessing field by (30) against the parameter ω/ω_L ($\theta = 60°$, $|\mathbf{H}| = 1$ Oe). As $\omega/\omega_L \to 0$ (Berry limit), $\Delta\gamma \to \pm\Omega$, where Ω is the solid angle traced by the vector $\mathbf{H}$ of the magnetic field. The sign of $\Delta\gamma$ is determined by the sign of ω.

At $\omega_L/\omega \gg 1$, the final polarisation vector behaves as follows:

$$P_x \to -\cos\theta \sin\big(T\omega_L + \Omega(\theta)\big), \tag{34}$$

$$P_y \to \cos\big(T\omega_L + \Omega(\theta)\big), \tag{35}$$

$$P_z \to \sin\theta \sin\big(T\omega_L + \Omega(\theta)\big). \tag{36}$$

The observable phase is seen to be the difference in the geometric phases $\Delta\gamma = \gamma_- - \gamma_+ = \Omega$ (see (27), (28)) naturally arising in arguments of the components of the vector $\mathbf{P}$ as an additive constant.

In conclusion to Sections 4 and 5, noteworthy is an important generalisation of the precessing field problem in context of the geometric phases. The form of the wave functions (10) and (11) enables one to introduce an effective magnetic field $\mathbf{H}_{\rm eff}$ so that the precessing field problem formally becomes adiabatic (Fig. 4). Therefore, the adiabatic Berry approach can be used in a space where the intensity of the effective field is a parameter (the construction of the effective field vector is shown in Fig. 4a). Fig. 4b plots the precession of $\mathbf{H}_{\rm eff}$ over cones of various angles 2Θ depending on the parameter ω/ω_L.

Therefore, the effective field traces various solid angles on the Poincare sphere, depending on the sign and magnitude of ω. Varying ω between $-\infty$ and $+\infty$, we can determine all the solid angles $\Omega(\omega/\omega_L)$, i.e. all the values of observed geometric phases $\Delta\gamma = \Omega(\omega/\omega_L)$. The adiabatic Berry approach as well as the nonadiabatic Aharonov–Anandan method are naturally combined in the $\mathbf{H}_{\rm eff}$ space. The adiabatic Berry limit strictly corresponds to the limiting trajectory on the Poincare sphere at $\omega/\omega_L \to 0$. Thus, the proposed line of attack on the precessing field problem does not make difference in the Berry and Aharonov–Anandan approaches and enables one to calculate the total set of the geometric phases $\Delta\gamma(\omega/\omega_L)$. Using this consideration, we can easily explain now the behaviour of the geometric phase $\Delta\gamma$ (see Fig. 3), implying the function $\Delta\gamma$ (ω/ω_L) to be simply the solid angle $\Omega(\omega/\omega_L)$ on the Poincare sphere presented in Fig. 4c.

6. Measurements of the geometric phase by polarised neutrons

Let us examine the problem of geometric phase from the viewpoint of its observing in neutron-optical experiments. First of all, we concentrate our attention on the fact that any wave function phase manifests itself in the values observed as an argument of a certain experimentally obtained function, which can depend on some other parameters besides the phase (for example, on the parameters characterising the

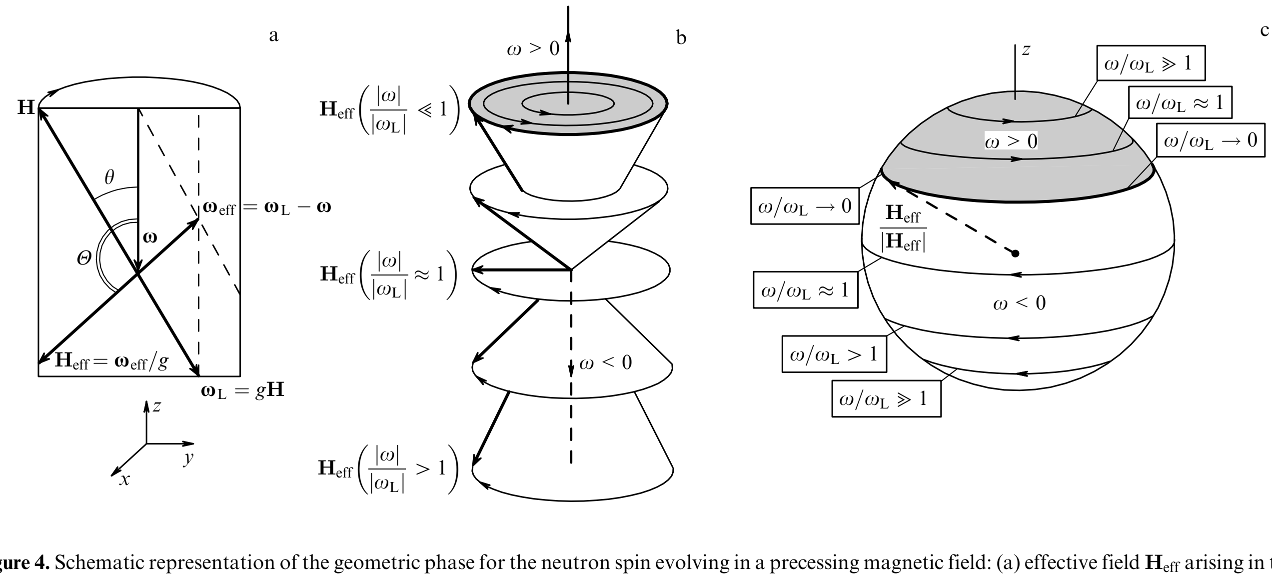

Figure 4. Schematic representation of the geometric phase for the neutron spin evolving in a precessing magnetic field: (a) effective field $\mathbf{H}_{\rm eff}$ arising in the problem of a spin in a precessing magnetic field $\mathbf{H}$. The vectors $\mathbf{H}$ and $\mathbf{H}_{\rm eff}$ are antiparallel to the vectors ω_L and $\omega_{\rm eff}$, respectively, due to negative sign of the gyromagnetic neutron factor g; (b) precession of the effective field over the cones of various angles 2Θ depending on the parameter ω/ω_L; (c) illustration for the dependence of the observable geometric phase $\Delta\gamma = \Omega$ on the parameter ω/ω_L on the Poincare sphere for the vector $\mathbf{H}_{\rm eff}$. The bold line corresponds to the limiting contour traced by the vector $\mathbf{H}_{\rm eff}/|\mathbf{H}_{\rm eff}|$ at $\omega/\omega_L \to 0$ which coincides with the contour of the real field. As absolute value of the parameter ω/ω_L rises, the vector $\mathbf{H}_{\rm eff}/|\mathbf{H}_{\rm eff}|$ begins to approache the sphere poles, depending on the sign of ω.

initial state of the beam polarisation). Therefore, here we can measure the geometric phase only indirectly in so far as it contributes, along with the dynamic phase, to the experimentally derived values. In the experiments with polarised neutrons such type values are components of the final polarisation vector. These experiments are performed by the 'spin-rotation' technique. According to this technique at the initial point of the neutron trajectory, the polarisation vector is directed perpendicular to the magnetic field and at the final point one measures its projection on one of the axes in the plane normal to the field direction and thus determines the total wave function phase.

The division of the total phase into the dynamic and the geometric components is based on some additional reasoning. In particular, in the case of strictly adiabatic evolution of the neutron spin phase during the time, while the neutron transits the distance L in a noncollinear field making (for definiteness sake) a turn, the dynamic phase is defined by

$$\Delta\beta = \beta_- - \beta_+ = \omega_L t\,,$$

where ω_L is the Larmore frequency of the magnetic neutron moment in the field $\mathbf{H}$, t is the time-of-flight of distance L in the field induced. In this experiment, the time $t = L/v$ is naturally equal to the field cycle T for any neutron velocity. Since the neutron velocity and wavelength λ are connected by the simple relation $v = \mathrm{const}/\lambda$, the dynamic phase is directly proportional to the neutron wavelength:

$$\Delta\beta = \mathrm{const}\times\omega_L\lambda = A\lambda\,.$$

In so doing the measured components of the polarisation vector as well as (31)–(33) are the sines or cosines of the sum $\Delta\alpha$ of the dynamic and geometric phases. Based on polarisation analysis in the framework of time-of-flight technique with the help of a *polychromatic* beam, we can estimate the geometric phase contribution to the total phase

$$\Delta\gamma = \mathrm{const}$$

by the phase shift in oscillating dependence $P_i(\lambda)$:

$$P_i \propto \cos(\Delta\beta + \Delta\gamma) = \cos(A\lambda + \Delta\gamma).$$

If the polarisation is examined on a *monochromatic* neutron beam (($\lambda = \mathrm{const}$)), we can separate the adiabatic Berry phase by calculating the dependence of the total phase on ω_L, i.e. on the value of the magnetic field strength averaged over the contour, keeping the contour geometry constant. It was this technique that Bitter and Dubbers used in 1987 in the first experiment on measuring the Berry phase with monochromatic polarised neutrons, performed at the reactor of the Laue-Langevin Institute [13]. Fig. 5 shows the installation scheme constructed by Bitter and Dubbers as well as the dependence of the polarisation and the total phase on the current I passing through a helicoidal coil. They used two solenoids: a helicoidal one to rotate the magnetic field vector along the neutron trajectory, and a coaxial to it one intended for the field component along the neutron trajectory. It is particularly remarkable that this method enables one to observe phases divisible by 2π, when measuring the total phase in the nonadiabatic region (see Fig. 5b).

Richardson's experiments [14] provided support, in particular, for the additivity of the arising geometric phase. Rotations of the magnetic field intensity vector through 4π, 6π were accompanied by the arising two-fold, three-fold geometric phase, respectively. Weinfurter and Badurek observed in their experiments [15] some specific effects caused by the geometric phase appearance, such as the suppression of modulations in measuring the polarisation vector components and a linear shift of the measured spectra along the parameter characterising the evolution noncyclicity.

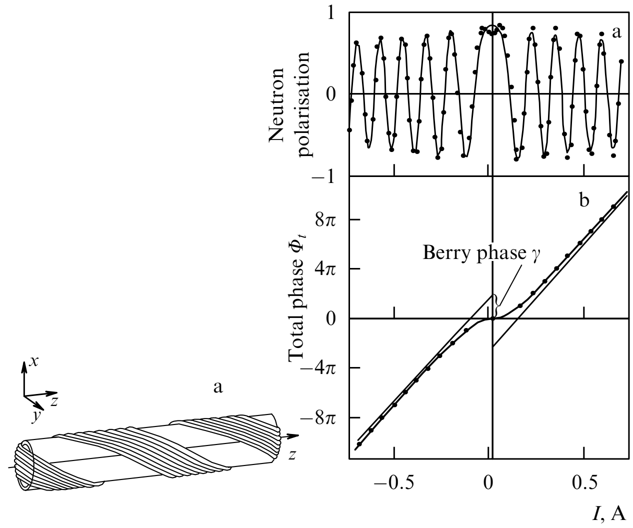

Figure 5. The experiment by Bitter and Dubbers [13] on measuring the geometric phase in a precessing field: (a) the helicoidal coil for a dextrorotating magnetic field. In the reference frame pertaining to a neutron, the magnetic field rotates in the plane perpendicular to the neutron velocity. The neutron beam is directed along the z axis; (b) the dependence of the polarization vector component perpendicular to the magnetic field (as it leaves the helicoidal coil) on the adiabaticity parameter proportional to the current I across the helicoidal coil; (c) observed and calculated phase shifts Φ_t.

The details of these experiments are left beyond the scope of this paper and can be found in the original publications. On the whole, within the measurement accuracy they demonstrated that the Berry phase is equal to a solid angle traced by the vector of a helicoidal magnetic field. At the same time, our measurements [23] (Fig. 6) of the geometric phase performed on a more complex field than a constant evenly precessing magnetic field, demonstrated that the range of parameter ω_L/ω can hardly be divided into the adiabatic and nonadiabatic parts. The reason is that on some local segment of the contour the adiabaticity can fail even for very slow neutrons. Processing the results of this time-of-flight experiment on a polychromatic neutron beam by the precessing field model made it clear that the model is inadequate to study complex contours and there is need in a more sophisticated algorithm (see the Appendix) to calculate the geometric phase without dividing it into the adiabatic and nonadiabatic components.

7. Conclusions

Thus, the Berry phase is not a purely theoretical concept. The dependence of the wave function phase on the geometry of the space of parameters determining the Hamiltonian of the quantum system is an experimentally seen phenomenon. In the context of the neutron optics, the Berry prediction for the adiabatic evolution of the neutron spin (1/2) in simple helicoidal fields is supported by experiments revealing an additional rotation of the polarisation vector component perpendicular to the field by an angle approximately equal to the solid angle traced by the magnetic field strength vector :

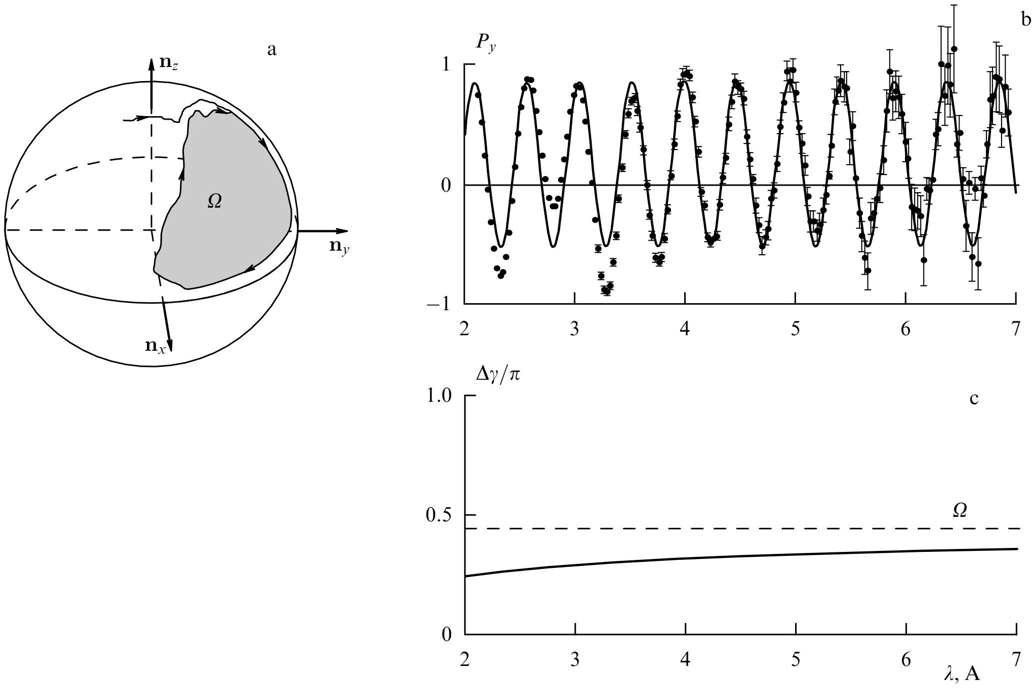

Figure 6. The time-of-flight experiment performed in LNP JINR [23]: (a) contour on the Poincare sphere corresponding to the stationary configuration of the magnetic field realized in the experiment; (b) the solid line corresponds to fitting of the experimental dependence $P_y(\lambda)$ in accordance with the method of least squares and is based on the precessing field model; (c) the calculated dependence of the geometric phase on the neutron wavelength. The deviation from the solid angle Ω, which was independently measured by another method, stands at 10 – 15%.

$$\Omega = 2\pi(1-\cos\theta)\,n,$$

where θ is the angle, and n is the helicoid revolution number. It is clear now that the neglect of the geometric phase can lead in some cases to a systematical error in interpreting the results of experiments with polarised neutrons based on the spin precession method (the spin echo method, the three-dimensional polarisation analysis). So far the accuracy of the measurements does not enable us to conclude that the measured angles of the vector rotation P are exactly equal to the solid angle traced by the magnetic field strength vector. These first experiments only estimate the phase behaviour. But the neutron experiments raise also new questions to the measurement of geometric phases in the case of magnetic fields with complicated geometry relative to the nonadiabatic effects.

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8. Appendix

We extend the Berry approach to the case of nonadiabatic evolution for arbitrary contours.

According to Berry's paper [2] the most general form for a wave function of an evolving quantum system with due regard to possible geometric phases is

$$\begin{aligned}\Psi\left(q,\mathbf{R}(t),t\right) &= \sum_n C_n\big(\mathbf{R}(t),t\big)\,\psi_n\left(q,\mathbf{R}(t)\right)\\ &= \sum_n \exp(\mathrm{i}\alpha_n)\,C_n\big(\mathbf{R}(0),0\big)\,\psi_n\left(q,\mathbf{R}(t)\right), \qquad (\mathrm{A}.1)\end{aligned}$$

where $\psi_n\left(q,\mathbf{R}(t)\right)$ are the eigen states of the Hamiltonian $H\big(\mathbf{R}(t)\big)$ and

$$\alpha_n = \int_0^t \frac{\partial\alpha_n}{\partial\tau}\,\mathrm{d}\tau + \int_{\mathbf{R}(0)}^{\mathbf{R}(t)} \vec{\nabla}_{\mathbf{R}}\alpha_n\,\mathrm{d}\mathbf{R}\,.$$

In essence, these quantities, i.e. the phase differences of expansion coefficients at the initial and arbitrary instances of time, the so-called total phases, determine the evolution of the system.

The first integral in the exponent presents the known dynamic phase of the wave function:

$$\beta_n = \int_0^t \frac{\partial\alpha_n}{\partial\tau}\,\mathrm{d}\tau = -\frac{1}{\hbar}\int_0^t E_n(\tau)\,\mathrm{d}\tau\,. \qquad (\mathrm{A}.2)$$

However, below we discuss the second (curvilinear) integral:

$$\gamma_n = \int_{\mathbf{R}(0)}^{\mathbf{R}(t)} \vec{\nabla}_{\mathbf{R}}\alpha_n\,\mathrm{d}\mathbf{R}\,. \qquad (\mathrm{A}.3)$$

In fact, it is this quantity that was introduced in [2].

Let us substitute expansion (A.1) into the Schrödinger equation (the spectrum $E_n(t)$ is assumed to be nondegenerate):

$$\begin{aligned}&\mathrm{i}\hbar\sum_m C_m\exp\left[\mathrm{i}\gamma_m - \frac{\mathrm{i}}{\hbar}\int_0^t E_m(\tau)\,\mathrm{d}\tau\right]\\ &\qquad\times\left\{\dot{\psi}_m + \mathrm{i}\psi_m\dot{\gamma}_m - \frac{\mathrm{i}}{\hbar}\,\psi_m E_m\right\}\\ &\qquad= \hat{H}\big(\mathbf{R}(t)\big)\sum_m C_m\psi_m\exp\left[\mathrm{i}\gamma_m - \frac{\mathrm{i}}{\hbar}\int_0^t E_m(\tau)\,\mathrm{d}\tau\right],\end{aligned}$$

or

$$\mathrm{i}\hbar \sum_m C_m \exp\left[\mathrm{i}\gamma_m - \frac{\mathrm{i}}{\hbar}\int_0^t E_m(\tau)\,\mathrm{d}\tau\right]\{\dot{\psi}_m + \mathrm{i}\psi_m\dot{\gamma}_m\} = 0.$$

Hence, taking advantage of the orthogonality of the states ψ_m, we express the geometric phases as

$$\dot{\gamma}_n = \frac{\mathrm{i}}{C_n}\sum_m C_m \exp\left[\mathrm{i}\,(\gamma_m - \gamma_n) - \frac{\mathrm{i}}{\hbar}\int_0^t \big(E_m(\tau) - E_n(\tau)\big)\,\mathrm{d}\tau\right](\psi_n, \dot{\psi}_m), \tag{A.4}$$

where $C_n = C_n\big(\mathbf{R}(0), 0\big) \neq 0$.

Since the time dependence is implicit, this set of equations can be transformed as follows

$$\vec{\nabla}_{\mathbf{R}}\gamma_n = \frac{\mathrm{i}}{C_n}\sum_m C_m \exp\left[\mathrm{i}\,(\gamma_m - \gamma_n) - \frac{\mathrm{i}}{\hbar}\int_0^t \big(E_m(\tau) - E_n(\tau)\big)\,\mathrm{d}\tau\right](\psi_n, \vec{\nabla}_{\mathbf{R}}\psi_m). \tag{A.5}$$

The vector fields arising on the right-hand side of the equation

$$\mathbf{A}_{nm} = \mathrm{i}\,(\psi_n, \vec{\nabla}_{\mathbf{R}}\psi_m)$$

are conventionally termed 'the fields induced by $\psi_n\,\big(q, \mathbf{R}(t)\big)$'. In the case studied of evolutions of a neutron spin, the indices m and n take on two values $+$ and $-$.

At first sight, it would seem that by presenting the expansion coefficients in (A.1) in the form

$$\exp(\mathrm{i}\alpha_n)\, C_n\,\big(\mathbf{R}(0), 0\big),$$

we add complexity to the standard linear set of equations used in the Dirac method for the evolution coefficients. However, in the analysis of a particular problem, both the standard representation and the representation via the phase components can be of use.

The adiabatic evolution corresponds to the condition

$$(\psi_n, \dot{\psi}_m) = 0, \qquad n \neq m. \tag{A.6}$$

Consequently, in the adiabatic limit equations (A.3) become much more simple and take the form (1)

$$\dot{\gamma}_n = \mathrm{i}\,(\psi_n, \dot{\psi}_n),$$

or

$$\vec{\nabla}_{\mathbf{R}}\gamma_n = \mathrm{i}\,(\psi_n, \vec{\nabla}_{\mathbf{R}}\psi_n) = \mathbf{A}_{nn}.$$

To calculate directly the total geometric phase in a nonadiabatic approximation, one should in addition to (14), (15) determine the nondiagonal vector fields

$$\mathbf{A}_{+-} = \mathrm{i}\,(\psi_+, \vec{\nabla}\psi_-) = \left(0, 0, -\frac{1}{2H\sin\theta}\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}\right), \tag{A.7}$$

$$\mathbf{A}_{-+} = \mathbf{A}_{+-}, \tag{A.8}$$

substitute the matrix elements found into (A.3) and solve the corresponding set of two differential first-order equations. This system can be easily solved for a precessing field and we will not concentrate our attention on the details. As a result, we obtain expressions (27), (28) for the Aharonov–Anandan phases. Thus, in the case of the precessing field we strictly proved the equality between the nonadiabatic phases calculated by the extended Berry approach and those calculated by the Aharonov–Anandan method. It should be mentioned that in contrast to the Aharonov–Anandan approach the extended Berry consideration is not limited by the cyclicity of wave functions.

References

1. Halpern O, Holstein T *Phys. Rev.* **59** 960 (1941)
2. Berry M V *Proc. Roy. Soc. London A* **392** (1802) 45 (1984)
3. Hannay J H *J. Phys. A* **18** 221 (1985)
4. Berry M V *J. Phys. A* **18** 15 (1985)
5. Aharonov Y, Anandan J *Phys. Rev. Lett.* **58** 1593 (1987); Moore D J *Phys. Rep.* **210** (1) 1 (1991)
6. Mukunda N, Simon R *Ann. of Phys.* **228** 205 (1993) (and refs. therein)
7. Berry M V *Phys. Today* **34** (Dec 26 1990)
8. Anandan J *Nature* **360** 307 (1990)
9. Vinitskiĭ S I et al. *Usp. Fiz. Nauk* **160** (6) 1 (1990) [*Sov. Phys. Usp.* **33** (6) 403 (1990)]
10. Klyshko D N *Usp. Fiz. Nauk* **163** (11) 1 (1993) [*Phys. Usp.* **36** 1005 (1993)]
11. Rytov S M *Dokl. Akad. Nauk SSSR* **18** 263 (1938); Vladimirskiĭ V V *Dokl. Akad. Nauk SSSR* **31** 222 (1941); Pancharatnam S *Proc. Indian Acad. Sci. A* **44** (5) 247 (1956); *A* **46** (4) 280 (1957)
12. Tomita A, Chiao R Y *Phys. Rev. Lett.* **57** 937 (1986)
13. Bitter T, Dubbers D *Phys. Rev. Lett.* **59** 251 (1987)
14. Richardson D J et al. *Phys. Rev. Lett.* **61** 2030 (1988)
15. Weinfurter H, Badurek G *Phys. Rev. Lett.* **64** 1318 (1990)
16. Landau L D, Lifshitz E M *Quantum Mechanics* (Oxford: Pergamon Press, 1977)
17. Wang Shun-Jin *Phys. Rev. A* **42** 5103 (1990)
18. Barut A O *Phys. Rev. A* **47** 2581 (1993)
19. Datta N, Ghosh G, Engineer M H *Phys. Rev. A* **40** 526 (1989)
20. O'Brien M C M, Chancey C C *Am. J. Phys.* **61** 688 (1993); Auerbach A, Manini N, Tosatti E *Phys. Rev. B* **49** 12998 (1994)
21. Thouless D J et al. *Phys. Rev. Lett.* **49** 405 (1982)
22. Tycko R *Phys. Rev. Lett.* **58** 2281 (1987)
23. Korneev D A, Bodnarchuk V I, Davtyan L S *Physica B* **213/214** 993 (1995)
24. Bakai A S, Stepanovskiĭ Yu P *Adiabaticheskie invarianty* (Adiabatic Invariants) (Kiev: Naukova Dumka, 1981)