# Waves in weakly anisotropic 3D inhomogeneous media: quasi-isotropic approximation of geometrical optics 

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#### Abstract

The quasi-isotropic approximation (QIA) of geometrical optics is outlined. The main idea of the method is that electromagnetic waves in weakly anisotropic media preserve their transverse structure as they do in isotropic media. Advantages of the QIA are illustrated by considering electromagnetic wave propagation in plasma, a number of optical problems (liquid crystals, hiral media, single mode optical fibres), acoustical problems of weakly anisotropic elastic media,


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and quantum mechanical polarisation effects of the Stern-Gerlach type. New modifications of the QIA are presented, namely the method of split rays and the synthetic approach, the latter being applicable even for strongly anisotropic media.

## 1. Introduction

### 1.1 Transformation of polarisation of a vector field in weakly anisotropic media

Problems of propagation of electromagnetic, elastic or other waves in anisotropic media occupy a significant place in wave physics. Among them, exploring waves in weakly inhomogeneous media is one of the most important. One faces with this problem in many fields of physics. The propagation of electromagnetic waves of different bands in a weakly magnetised plasma (laboratory, ionospheric, near-solar, interstellar), passing of electromagnetic waves through a condensed matter, pieso- and ferromagnetics, light waves in hiral media, liquid crystals, polarisation phenomena in deformed light guides, propagation of acoustic waves in weakly anisotropic and/or weakly deformed elastic media, splitting of beams of particles with spin in magnetic fields -
that is the list, by no means complete, of questions connected with the theory of wave processes in weakly anisotropic media.

The main feature of inhomogeneous weakly anisotropic media is their capability of changing substantially the polarisation of a vector field of a specific physical nature. Polarisation changes in weakly anisotropic regions of inhomogeneous medium owe their existence to a strong interconversion of normal modes under conditions of slowly arising or disappearing polarisation degeneracy. In an isotropic medium, the electromagnetic field is represented by a superposition of two normal waves, each characterised by a definite, intrinsic polarisation, while in the isotropic medium transversal electromagnetic waves are polarisation-degenerate, and the state of their polarisation is not fully defined.

Prior to entering a strongly anisotropic medium, the wave with a degenerate polarisation undergoes a metamorphosis in a weakly anisotropic layer that matches the isotropic and anisotropic media (see Fig. 1). Within the anisotropic media, such a wave converts into a superposition of independent normal waves.


Figure 1. Transformation of a transversal wave with a degenerate polarisation into a superposition of normal waves occurs in the region of strong interaction of normal waves $\delta<1$, in a weakly anisotropic layer separating isotropic and anisotropic media. QIA provides matching between polarisation degenerate transversal waves in isotropic media (the Rytov method) and independent normal waves in anisotropic media (the Courant-Lax method).

Linear coupling of waves in an inhomogeneous isotropic medium which is associated with lifting of polarisation degeneracy is a physical problem of general character, important both theoretically and practically. This review is dedicated to the analysis of this problem.

### 1.2 Basic methods of description of wave field in weakly anisotropic media

Let us assume that an electromagnetic wave of given polarisation (say, of a linear one) propagates from an isotropic medium into an anisotropic one. In a weakly
anisotropic medium, normal waves obey coupled equations thus allowing one to characterise them as linearly interacting normal waves. The finite state of the polarisation ('the limiting polarisation') of a vector wave field depends on the nature of inhomogeneity encountered in a weakly anisotropic medium. Describing the process of wave transformation in such a medium presents a rather complex mathematical problem not fully solved up to now.

A satisfactory theory of linear interaction of normal modes was elaborated by Budden [1, 2] for a plane-layered plasma (the Budden method). In a layered medium, Maxwell's equations transform to the system of four coupled ordinary differential equations, and by applying the Budden method one may reduce substantially the order of that system (it becomes a second-order). The Budden method is presented in many text-books [3-5], and is widely used in applied investigations.

Initially, the Budden method dealt with electromagnetic waves in plasma, however, later, Budden's results were extended to arbitrary dielectric layered media described by a tensor $\varepsilon_{i k}(z)[6,7]$. A comprehensive analysis of results that follow from the Budden method has been carried out by Zheleznyakov, V Kocharovsky, and Vl Kocharovsky [8].

Another approach to the description of waves in inhomogeneous media has been suggested by Kravtsov [9]. That approach - the quasi-isotropic approximation (QIA) of geometrical optics - is not limited to plane-layered media, and applies to arbitrary 3D inhomogeneous media. Similar to the Budden method, in the framework of the QIA, a medium is assumed to be smoothly inhomogeneous, i. e. the geo-metric-optical parameter is supposed to be small:

$$
\begin{equation*}
\mu \sim \frac{1}{k_{0} l} \ll 1, \tag{1.1}
\end{equation*}
$$

where $l$ is the characteristic parameter of medium inhomogeneity, $k_{0}=\omega / c$.

The QIA is based on the idea that to zero approximation an electromagnetic field has the same transversal structure as it would have in an isotropic medium. In contrast to the case of an isotropic medium, where the polarisation of the field vector in a plane perpendicular to a ray is arbitrary, in the anisotropic medium it is uniquely defined by the anisotropy tensor,

$$
\begin{equation*}
v_{i k}=\varepsilon_{i k}-\varepsilon_{0} \delta_{i k} \tag{1.2}
\end{equation*}
$$

Here $\varepsilon_{i k}$ is the tensor of dielectric permittivity, $\varepsilon_{0}$ is its main isotropic part, for example, $\varepsilon_{0}=(1 / 3) \mathrm{Sp} \hat{\varepsilon}$. The smallness of anisotropy is characterised by the parameter

$$
\begin{equation*}
\mu_{1}=\max _{i, k}\left|v_{i k}\right| \ll 1 \tag{1.3}
\end{equation*}
$$

which, in the framework of QIA, serves as a parameter of asymptotic expansion of the wave field, as does the small geometric-optical parameter $\mu$.

The ratio

$$
\begin{equation*}
\delta=\frac{\mu_{1}}{\mu} \tag{1.4}
\end{equation*}
$$

serves as the measure of anisotropy strength. For $\delta \gg 1$, the QIA equations reduce to those for independent normal waves, i.e. to the Courant - Lax equations [10-12], whereas
for $\delta \rightarrow 0$ they reduce to the equations of geometrical optics for an isotropic medium (the Rytov method [13]). Thus, the QIA allows one to trace a continuous transition from transversal waves in isotropic media $(\delta \rightarrow 0)$ to independent normal waves in strongly anisotropic media ( $\delta \gg 1$ ), see Fig. 1. Quite recently this was thought of as being difficult to attain [12].

In time that followed the QIA equations were subjected to analysis and generalisations. To solve the QIA equations, Naĭda applied the methods of the perturbation theory [1417], suggested the method of split rays and formulated QIA equations for the electric induction vector [18, 19], derived QIA equations for electromagnetic waves in moving media [20] and for acoustic waves in weakly anisotropic elastic media [21, 22]. Together with Prudkovskǐ̆, he formulated QIA equations for the quantum mechanical problem concerned with splitting of beams of particles with different spin states in a magnetic field [23].

As it turns out the QIA equations are simpler in form than the Budden equations and are frequently more convenient in specific calculations, although, by their universality with respect to the degree of anisotropy $\mu_{1}$, they compare unfavourably with the Budden equations: while the QIA requires the smallness of two parameters, $\mu \ll 1$ and $\mu_{1} \ll 1$, the Budden method applies if small is only the geometricoptical parameter $\mu$ whereas the parameter of anisotropy, $\mu_{1}$, can be on the order of unity. Later, however, investigators succeeded in modifying the QIA to incorporate the wave transformation in strongly anisotropic media with $\mu_{1} \sim 1$ (see Section 3.4).

The QIA makes it possible to calculate effects of quasitransversal (in respect to an external magnetic field) electromagnetic field propagation in 3D inhomogeneous plasma [19, $24,25]$, to analyse a number of ionospheric propagation problems [26], to clarify polarisation peculiarities of scattering of radio-waves in the polar ionosphere [25, 27, 28], to calculate the depolarisation of electromagnetic waves in a randomly inhomogeneous plasma [29, 30], to analyse systematically the effects of linear transformation of waves which enter 3D inhomogeneous plasma [8].

This has to be supplemented by a number of relatively new phenomena which can be described with the help of the QIA: the linear wave interaction in the region of neutral magnetoactive field in plasma [5], effect of 'tangent conical refraction' [31], interaction of helical waves in liquid crystals [32], transformation of light polarisation in single-mode optical light guides (see Section 5.3), and others.

This review is aimed at a systematic description of different modifications of the QIA as applied to various polarisation effects in inhomogeneous weakly anisotropic media.

Modifications and generalisations of the QIA are outlined in Section 3. Section 3.3 deals with a conceptually new approach (the method of split rays), which accounts for ray splitting that accompanies the decomposition of the total field into independent normal waves. The method of split rays admits an effective generalisation even on strongly anisotropic inhomogeneous media. Similarly to the Budden method, such a generalisation is not restricted to small $\mu_{1}$, however unlike that method, applicable only to plane-layered inhomogeneous media, it enables description of wave transformation in arbitrary 3D inhomogeneous media.

Selected sections are devoted to electromagnetic waves in magnetoactive plasma (Section 4), light waves in deformed
media and fibres (Section 5), and to acoustic waves in deformed elastic media (Section 6).

The idea itself of considering the anisotropy as a small perturbation has already been formulated by Pauli [33] in respect to the Dirac equation, and was concerned with the experiment of the Stern-Gerlach type. The anisotropy in this case occurs due to a weak magnetic field. In time that followed Pauli's results were improved first by Galanin [34], and then by Rubinow and Keller [35], although without account for splitting of beams of polarised particles in a magnetic field. Most clearly the spin motion along the split trajectories is presented in a book by Akhiezer and Berestetskiĭ [36]. We decided that for methodological reasons it would be instructive to elucidate the behaviour of particle spinor wave functions in a magnetic field, treating the problem from the QIA positions. This is done in Section 7.

## 2. Quasi-isotropic approximation of geometrical optics of 3D anisotropic media

### 2.1 General scheme of the geometrical optics method

The eikonal substitution and the eikonal equation. The basic ideas of the geometrical optics method will be recalled by the example of Maxwell's equations [37]. For simplicity, we shall confine ourselves to the case of a monochromatic wave (with dependence on time as $\exp (-\mathrm{i} \omega t)$ ) in an inhomogeneous gyrotropic stationary medium. We disregard absorption and set the magnetic permittivity at zero. Under these conditions, the electric field vector $\overrightarrow{\mathcal{E}}$ obeys the equation

$$
\begin{equation*}
k_{0}^{2} \stackrel{\rightharpoonup}{\mathcal{E}}-\operatorname{rot} \operatorname{rot} \overrightarrow{\mathcal{E}}=0 \quad\left(k_{0}=\frac{\omega}{c}, \quad \varepsilon_{m n}^{*}=\varepsilon_{m n}\right) \tag{2.1}
\end{equation*}
$$

If electromagnetic waves propagate in smoothly inhomogeneous media, it is natural to solve Maxwell's equations using the geometric optics method. Assume, for definiteness, that an electromagnetic wave with a 'wide' phase front is specified upon entering an inhomogeneous medium. The task is to find the field at all points of the inhomogeneous medium, based on the assumption that the medium varies smoothly at the wavelength scale.

The main geometric-optical technique is the eikonal substitution,

$$
\begin{equation*}
\overrightarrow{\mathcal{E}}(\mathbf{r})=\mathbf{E}(\mathbf{r}) \exp [\mathrm{i} \varphi(\mathbf{r})] \tag{2.2}
\end{equation*}
$$

which enables one to separate fast oscillations of the wave field and relatively slow (by a factor of $k_{0} l$ slower) variations in parameters of medium and in wave parameters associated with them.

Eikonal substitution (2.2) reduces vector equation (2.1) to the form

$$
\begin{equation*}
k_{0}^{2} \hat{\varepsilon} \mathbf{E}+[\mathbf{k}[\mathbf{k} \mathbf{E}]]-\mathrm{i}([\mathbf{k}, \operatorname{rot} \mathbf{E}]+\operatorname{rot}[\mathbf{k} \mathbf{E}])-\operatorname{rot} \operatorname{rot} \mathbf{E}=0 \tag{2.3}
\end{equation*}
$$

where $\mathbf{k}=\nabla \varphi$ is the local wave vector.
To zero approximation, we retain only terms quadratic in $k_{0}$ and $\mathbf{k}$ in $\operatorname{Eqn}$ (2.3). That yields a vector equation for $\mathbf{E}^{(0)}$ :

$$
\begin{equation*}
k_{0}^{2} \hat{\varepsilon} \mathbf{E}^{(0)}+\left[\mathbf{k}\left[\mathbf{k} \mathbf{E}^{(0)}\right]\right]=0 \tag{2.4}
\end{equation*}
$$

which is equivalent to the system of three equations for components $E_{x}^{(0)}, E_{y}^{(0)}, E_{z}^{(0)}$. As is well-known, the solvability
condition for that system leads to the eikonal equation:

$$
\begin{equation*}
\operatorname{det}\left(k_{0}^{2} \varepsilon_{m n}+k_{m} k_{n}-\delta_{m n} \mathbf{k}^{2}\right)=0 \quad\left(k_{m}=\frac{\partial \varphi}{\partial x_{m}}\right) . \tag{2.5}
\end{equation*}
$$

The ratio $n(\mathbf{r}, \mathbf{k} /|\mathbf{k}|)=|\mathbf{k}| / k_{0}$ is the refraction index which, in turn, satisfies the algebraic equation

$$
\begin{equation*}
\operatorname{det}\left(n^{-2} \varepsilon_{m n}+t_{m} t_{n}-\delta_{m n}\right)=0, \tag{2.6}
\end{equation*}
$$

where $\mathbf{t}=\mathbf{k} /|\mathbf{k}|$ is a unity vector in the direction $\mathbf{k}$. For $t_{x}=t_{y}=0, t_{z}=1$ we may obtain the following well-known formula:

$$
\begin{equation*}
n_{1,2}^{-2}=\frac{1}{2}\left(\chi_{x x}+\chi_{y y}\right) \pm\left[\frac{1}{4}\left(\chi_{x x}-\chi_{y y}\right)^{2}+\left|\chi_{x y}\right|^{2}\right]^{1 / 2} \tag{2.7}
\end{equation*}
$$

where $\chi_{x x}, \ldots$ stand for the components of the inverse tensor of dielectric permittivity $\hat{\chi}=\hat{\varepsilon}^{-1}$ [38].

Hamiltonian equations for rays. Viewed as an algebraic equation in respect to the frequency $\omega=k_{0} c$, local dispersion relation (2.5), as a rule, has among its roots two positive ones:

$$
\begin{equation*}
\omega=\Omega_{1,2}(\mathbf{r} ; \mathbf{k}) \tag{2.8}
\end{equation*}
$$

In an anisotropic medium, $\Omega_{1} \neq \Omega_{2}$ except for special directions. We shall assume that $\Omega_{1}>\Omega_{2}$, and that $\Omega_{1}$ corresponds to the extraordinary mode while $\Omega_{2}$ refers to the ordinary one. In isotropic media $\Omega_{1}(\mathbf{r} ; \mathbf{k})=\Omega_{2}(\mathbf{r} ; \mathbf{k})=$ $\varepsilon^{-1 / 2}|\mathbf{k}|$.

For both isotropic and anisotropic media, rays corresponding to zero approximation in the form of Eqns (2.4) and (2.5) obey the Hamiltonian equations

$$
\begin{equation*}
\frac{\mathrm{d} \mathbf{r}}{\mathrm{~d} t}=\frac{\partial \Omega_{a}}{\partial \mathbf{k}}, \quad \frac{\mathrm{~d} \mathbf{k}}{\mathrm{~d} t}=-\frac{\partial \Omega_{a}}{\partial \mathbf{r}} \tag{2.9}
\end{equation*}
$$

where $t$ is the time and $\partial \Omega_{a} / \partial \mathbf{k}$ is the group velocity. Here $a$ is the wave polarisation index: $a=1$ for extraordinary and $a=2$ for ordinary waves.

In an isotropic media, Eqns (2.9) related to different values of $a$ do coincide. They admit a simplification in accordance with Eqns (2.9):

$$
\begin{equation*}
\frac{\mathrm{d} \mathbf{r}}{\mathrm{~d} s}=\frac{\mathbf{k}}{|\mathbf{k}|}, \quad \frac{\mathrm{d} \mathbf{k}}{\mathrm{~d} s}=k_{0} \nabla n, \quad \mathrm{~d} s=\mathrm{d} t \frac{\mathrm{~d} \Omega}{\mathrm{~d} \mathbf{k}} \tag{2.10}
\end{equation*}
$$

Eikonals $\varphi_{1}$ and $\varphi_{2}$ of normal waves are given by formulae

$$
\begin{equation*}
\varphi_{a}=\int \mathbf{k}_{a} \mathrm{~d} \mathbf{r}_{a}=\left.\int \mathbf{k}_{a} \frac{\partial \Omega_{a}}{\partial \mathbf{k}}\right|_{\mathbf{k}=\mathbf{k}_{a}} \mathrm{~d} t, \quad a=1,2 \tag{2.11}
\end{equation*}
$$

Eqn (2.11) simplifies in the case of isotropic medium:

$$
\begin{equation*}
\varphi=k_{0} \int \varepsilon^{1 / 2} \mathrm{~d} s=k_{0} \int n \mathrm{~d} s \tag{2.12}
\end{equation*}
$$

All formulae for rays, eikonals, and refraction indices related to ordinary and extraordinary rays in an anisotropic media join continuously their respective 'isotropic' counterparts at a continuous transition $\left(v_{\alpha \beta} \rightarrow 0\right)$ from the anisotropic to isotropic media. Such a continuity is not exhibited by the equations for wave amplitudes taken from common theories of geometrical optics, i.e., on the one hand, by the Courant equations for anisotropic media, and on the other by the Rytov equations.

Independent normal waves. In the case of anisotropic medium, to zero approximation, Eqn (2.4) defines uniquely the polarisation of each of the solutions $\varepsilon_{1}^{(0)}$ and $\varepsilon_{2}^{(0)}$ of Eqn (2.1) (or each of the solutions $\mathbf{E}_{1}^{(0)}$ and $\mathbf{E}_{2}^{(0)}$ of Eqn (2.3)) that correspond, respectively, to extraordinary and ordinary waves:

$$
\begin{equation*}
\mathbf{E}_{a}^{(0)}=C_{a} \mathbf{e}_{a}, \quad \overrightarrow{\mathcal{E}}_{a}^{(0)}=C_{a} \mathbf{e}_{a} \exp \left(\mathrm{i} \varphi_{a}\right) \quad(a=1,2) \tag{2.13}
\end{equation*}
$$

Polarisation vectors $\mathbf{e}_{1}$ and $\mathbf{e}_{2}$ satisfy vector equation (2.4). According to the rules defined by Eqn (2.4), the polarisation of each from normal waves follows the turns of the medium anisotropy axes given by the tensor $\hat{\varepsilon}(\mathbf{r})$.

Expressions for $\mathbf{e}_{1}$ and $\mathbf{e}_{2}$ are the simplest in a right orthogonal system with real orths $\mathbf{q}_{1}, \mathbf{q}_{2}$, and $\mathbf{t}$. As is known [38], in this case

$$
\begin{align*}
& \mathbf{d}_{1}=\left(e_{11} \mathbf{q}_{1}+e_{12} \mathbf{q}_{2}\right) n_{1}^{2}, \quad \mathbf{d}_{2}=\left(e_{21} \mathbf{q}_{1}+e_{22} \mathbf{q}_{2}\right) n_{2}^{2}, \quad \mathbf{d}_{3}=0, \\
& \mathbf{e}_{1}=\hat{\chi} \mathbf{d}_{1}=e_{11} \mathbf{q}_{1}+e_{12} \mathbf{q}_{2}+e_{13} \mathbf{t} \\
& \mathbf{e}_{2}=\hat{\chi} \mathbf{d}_{2}=e_{21} \mathbf{q}_{1}+e_{22} \mathbf{q}_{2}+e_{23} \mathbf{t} \tag{2.14}
\end{align*}
$$

where refraction indices $n_{1,2}$ are given by formulae (2.7). $e_{i j}$ stands here for the values

$$
\begin{array}{ll}
e_{11}=e_{22}=\left(1+K_{1}^{2}\right)^{-1 / 2}, & e_{12}=e_{21}=-\mathrm{i} K_{1}\left(1+K_{1}^{2}\right)^{-1 / 2}, \\
e_{13}=n_{1}^{2}\left(\chi_{31} e_{11}+\chi_{32} e_{12}\right), & e_{23}=n_{2}^{2}\left(\chi_{31} e_{21}+\chi_{32} e_{22}\right), \tag{2.15}
\end{array}
$$

that correspond to the transversal (in respect to $\mathbf{t}$ ) component of $\mathbf{e}_{a \perp}$ normed to unity: $\left|\mathbf{e}_{a \perp}\right|^{2}=1$. Additionally, in (2.15) the following notations are introduced:

$$
\begin{align*}
& K_{1}=J K^{J}, \quad J=-\operatorname{sgn} \operatorname{Im} \chi_{12}, \quad K=Q-\left(1+Q^{2}\right)^{1 / 2} \\
& Q=\frac{\mathrm{i}\left(\chi_{22}-\chi_{11}\right)}{\chi_{12}} . \tag{2.16}
\end{align*}
$$

Courant and Lax [10-12] derived an ordinary differential equation for each of amplitudes $C_{a}$ in formulae (2.13). Given these amplitudes, expressions (2.13) approximate reasonably the exact solutions, although only for a strong anisotropy, when the parameter $\delta=\mu_{1} / \mu$ is large, $\delta \gg 1$. They do not hold for a weak anisotropy. There are cases for which in the framework of the normal wave method the wave transformation can be calculated only by the perturbation theory [3, 38a].

The polarisation structure of a field in an isotropic medium. In the case of isotropic medium, Eqn (2.4) no longer defines uniquely the orientation of the vector amplitude $\mathbf{E}^{(0)}$. This corresponds to the polarisation degeneration. From Eqn (2.4) in this case it only follows that

$$
\begin{align*}
& \mathbf{E}^{(0)}=C_{1} \mathbf{q}_{1}+C_{2} \mathbf{q}_{2}, \\
& \overrightarrow{\mathcal{E}}^{(0)}=\left(C_{1} \mathbf{q}_{1}+C_{2} \mathbf{q}_{2}\right) \exp \left(\mathrm{i} k_{0} \int n \mathrm{~d} s\right), \tag{2.17}
\end{align*}
$$

where $\mathbf{q}_{1}$ and $\mathbf{q}_{2}$ are arbitrary linearly independent vectors, perpendicular to the tangent $t$. In particular, one may take the normal $\mathbf{n}$ and binormal $\mathbf{b}$ to the ray, respectively, as $\mathbf{q}_{1}$ and $\mathbf{q}_{2}$ [13, 37], so

$$
\begin{equation*}
\mathbf{E}^{(0)}=E_{n} \mathbf{n}+E_{b} \mathbf{b}, \quad \overrightarrow{\mathcal{E}}^{(0)}=\left(E_{n} \mathbf{n}+E_{b} \mathbf{b}\right) \exp \left(\mathrm{i} k_{0} \int n \mathrm{~d} s\right) \tag{2.18}
\end{equation*}
$$

Thus, in the case of isotropic medium the general expression for zero approximation (2.18) differs drastically from the respective form (2.13) for independent normal waves. Accordingly, for amplitudes $C_{1}$ and $C_{2}$ one obtains not a single equation, but the system of two ordinary differential equations. A formal difference between Eqns (2.13) and (2.18) greatly complicates a smooth conjunction of polarisation-degenerate transversal waves in an isotropic medium with independent normal waves in an anisotropic medium that long it was not even clear how to tackle that problem.

However, smooth matching of waves propagating from an isotropic medium into an anisotropic one could be obtained in the framework of the quasi-isotropic approximation (QIA) which, on the one hand, preserves the transversal structure of the field in the isotropic medium, and, on the other, admits a transition to normal waves in essentially anisotropic medium.

Universal geometric-optical procedure. In order to correlate different, at first glance even incompatible, variants of the ray theory, it is convenient to write Maxwell's equations in a form from which all known variants of the ray method would be readily apparent. Daring to overestimate to some extent the significance of the suggested approach and yet not willing to refuse a relevant term, we term the procedure to be stated below a universal one, the more so as the procedure also leads to some new results, for instance, to ray splitting.

In a 3D inhomogeneous medium the construction of ray solutions simplifies since the vector of electric induction $\mathbf{D}=\hat{\varepsilon} \overrightarrow{\mathcal{E}}$ is almost transversal (it is strictly transversal to zero approximation) with respect to the wave vector $\mathbf{k}$. That transversality takes place both in isotropic and anisotropic media. Therefore it is natural to use the right triple of orths $\mathbf{q}_{1}$, $\mathbf{q}_{2}$, and $\mathbf{t}$, such that $\left(\mathbf{q}_{m}^{*}, \mathbf{q}_{n}\right)=\delta_{m n}(m, n=1,2)$, in Eqn (2.3).

We make the substitution

$$
\begin{equation*}
\mathbf{E}=E_{1} \mathbf{q}_{1}+E_{3} \mathbf{q}_{2}+E_{3} \mathbf{t} \tag{2.19}
\end{equation*}
$$

into Eqn (2.3) and introduce the designation

$$
\begin{equation*}
\mathbf{D}=\hat{\varepsilon} \mathbf{E}=D_{1} \mathbf{q}_{1}+D_{2} \mathbf{q}_{2}+D_{3} \mathbf{t} \tag{2.20}
\end{equation*}
$$

On inserting (2.19) into Maxwell's equations (2.3) we multiply them scalarly by some as yet arbitrary vectors $\mathbf{Q}_{1}$ and $\mathbf{Q}_{2}$ (not necessarily perpendicular to the ray, but not aligned with it), and also by orth $t$. Then we find

$$
\begin{align*}
& \mathbf{Q}_{1}\left(\mathbf{D}-n^{2}[\mathbf{t}[\mathbf{t} \mathbf{E}]]\right)-\mathrm{i} k_{0}^{-1} \mathbf{Q}_{1}(n[\mathbf{t}, \operatorname{rot} \mathbf{E}]+\operatorname{rot}[n \mathbf{t}, \mathbf{E}]) \\
& +k_{0}^{-2} \mathbf{Q}_{1} \operatorname{rot} \operatorname{rot} \mathbf{E}=0, \\
& \mathbf{Q}_{2}\left(\mathbf{D}-n^{2}[\mathbf{t}[\mathbf{t} \mathbf{E}]]\right)-\mathrm{i} k_{0}^{-1} \mathbf{Q}_{2}(n[\mathbf{t}, \operatorname{rot} \mathbf{E}]+\operatorname{rot}[n \mathbf{t}, \mathbf{E}]) \\
& +k_{0}^{-2} \mathbf{Q}_{2} \operatorname{rot} \operatorname{rot} \mathbf{E}=0 ;  \tag{2.21}\\
& D_{3}-\mathrm{i} k_{0}^{-1} \mathbf{t}(n[\mathbf{t}, \operatorname{rot} \mathbf{E}]+\operatorname{rot}[n \mathbf{t}, \mathbf{E}])+k_{0}^{-2} \mathbf{t} \operatorname{rot} \operatorname{rot} \mathbf{E}=0 \text {. } \tag{2.22}
\end{align*}
$$

The choice of $\mathbf{q}_{1}$ and $\mathbf{q}_{2}$, as well as $\mathbf{Q}_{1}$ and $\mathbf{Q}_{2}$ vectors is dictated by reasons of convenience.

The advantages of the equations in the form (2.21) and (2.22) as confronted with (2.3) are that Eqn (2.22) makes explicit the smallness of the transversal component $D_{3}$. As a result we have every reasons to exclude completely the transversal component $D_{3}^{(0)}$ from the zero approximation and set $D_{3}^{(0)}=0$.

Universal relationships (2.21) and (2.22) enable the derivation of ordinary differential equations for amplitude coefficients under all modifications of the geometrical optics method: the Rytov method (Section 2.2), the method of independent normal waves (the Courant-Lax method, Section 2.3), and the quasi-isotropic approximation (Section 2.4). These relationships apply as well to the method of split rays (Sections 3.3 and 3.4).

### 2.2 Geometrical optics of isotropic media (the Rytov method)

In an isotropic medium the polarisation degeneracy occurs and the zero-order field admits form (2.18), i. e. in this case $\mathbf{q}_{1}=\mathbf{n}$ and $\mathbf{q}_{2}=\mathbf{b}$. In universal relationships (2.21) one may conveniently choose $\mathbf{Q}_{1}=\mathbf{n}$ and $\mathbf{Q}_{2}=\mathbf{b}$.

Then from (2.21) the system of two equations for the amplitudes of zero approximation, $E_{n}$ and $E_{b}$, follows [13, 37]:

$$
\begin{align*}
& \frac{\mathrm{d} E_{n}}{\mathrm{~d} s}-T^{-1} E_{b}+E_{n}\left(\mathrm{~d} \ln \frac{\varepsilon^{1 / 4}}{\mathrm{~d} s}+\frac{1}{2} \operatorname{div} \mathrm{t}\right)=0 \\
& \frac{\mathrm{~d} E_{b}}{\mathrm{~d} s}+T^{-1} E_{n}+E_{b}\left(\mathrm{~d} \ln \frac{\varepsilon^{1 / 4}}{\mathrm{~d} s}+\frac{1}{2} \operatorname{div} \mathrm{t}\right)=0 \tag{2.23}
\end{align*}
$$

where $T=\mathbf{b} \mathrm{d} \mathbf{n} / \mathrm{d} s$ is the radius of the ray twisting. The last terms in each of the equations of system (2.23) can be easily eliminated by introducing the normed amplitudes $\Gamma_{n}$ and $\Gamma_{b}$ :

$$
\begin{equation*}
\overrightarrow{\mathcal{E}}=\Phi_{0} \varepsilon^{-1 / 4}\left(\Gamma_{n} \mathbf{n}+\Gamma_{b} \mathbf{b}\right) \exp \left(\mathrm{i} k_{0} \int n \mathrm{~d} s\right) \tag{2.24}
\end{equation*}
$$

where $\Phi_{0}$ is a real-valued function that satisfies the law of energy conservation along the ray tube $\operatorname{div}\left(\Phi_{0}^{2} \mathbf{t}\right)=0$. The intensity of normed amplitudes equals unity: $\left|\Gamma_{n}\right|^{2}+\left|\Gamma_{b}\right|^{2}=1$.

In conjunction with Eqn (2.24), Eqns (2.23) can be rewritten as

$$
\begin{equation*}
\frac{\mathrm{d} \Gamma_{n}}{\mathrm{~d} s}-T^{-1} \Gamma_{b}=0, \quad \frac{\mathrm{~d} \Gamma_{b}}{\mathrm{~d} s}+T^{-1} \Gamma_{n}=0 \tag{2.25}
\end{equation*}
$$

From Eqns (2.25), the well-known Rytov equation [13] follows:

$$
\begin{equation*}
\frac{\mathrm{d} \vartheta}{\mathrm{~d} s}=-T^{-1} \tag{2.26}
\end{equation*}
$$

for the angle

$$
\vartheta=\arctan \frac{E_{b}}{E_{n}}=\arctan \frac{\Gamma_{b}}{\Gamma_{n}}
$$

between the vector $\mathbf{E}$ and the orth $\mathbf{n}$. These equations define fully the zero-order field in an isotropic medium. The Rytov law of polarisation rotation (2.26) forms a particular case of a more general law which is referred to as the Berry effects [ 80 , 80a].

### 2.3 Noninteracting normal waves in an anisotropic medium (the Courant-Lax method)

In the theory by Courant and Lax an ordinary differential equation (the transfer equation), localised along rays (2.9), is derived for each of scalar amplitudes $C_{1}$ and $C_{2}$ in Eqn (2.13). As related, for instance, to the extraordinary ray (i. e. in respect to $C_{1}$ ) that equation is

$$
\begin{equation*}
\frac{\mathrm{d} C_{1}}{\mathrm{~d} t}+P^{1} C_{1}=0 \tag{2.27}
\end{equation*}
$$

where $P^{1}=(1 / 2) c\left(\mathbf{h}_{1}^{*} \operatorname{rot} \mathbf{e}_{1}-\mathbf{e}_{1}^{*} \operatorname{rot} \mathbf{h}_{1}\right)$.
In the first order of the perturbation theory, the amplitude $C_{2}^{(1)}$ is given by the expression

$$
\begin{equation*}
C_{2}^{(1)}=\mathrm{i} k^{-2}\left(n_{1}^{2}-n_{2}^{2}\right)^{-1} \mathbf{e}_{2}^{*}\left\{\left[\mathbf{k}_{1}, \operatorname{rot}\left(C_{1} \mathbf{e}_{1}\right)\right]+\operatorname{rot}\left[\mathbf{k}_{1}, C_{1} \mathbf{e}_{1}\right]\right\} . \tag{2.28}
\end{equation*}
$$

It is readily seen that correction (2.28) diverges as one approaches an isotropic medium where $n_{1}=n_{2}$. That implies that the Courant-Lax method no longer applies (it diverges) as $\delta \ll 1$ and is not capable of solving the problem of limiting polarisation.

### 2.4 Waves in weakly anisotropic medium.

## The quasi-isotropic approximation (QIA)

Basic equations of the QIA. The quasi-isotropic approximation is founded on the choice of the zero-approximation solution $\mathcal{E}^{(0)}$ in the form (2.18) as if there were no anisotropy $[9,37]$, i.e. if the anisotropy tensor $v_{m n}=\varepsilon_{m n}-\varepsilon_{0} \delta_{m n}$ were equal to zero. In the framework of that approach the zero approximation looks like a transversal wave (2.24) where the replacement $\varepsilon \rightarrow \varepsilon_{0}$ is made.

Hence, we apply the isotropic eikonal formula (2.24) to describe the waves in a weakly anisotropic medium. The orths $\mathbf{t}, \mathbf{n}$, and $\mathbf{b}$ are the tangent, normal, and binormal to the 'isotropic' ray. The ray obeys the Hamiltonian equations (2.10) in which a substitution of $n=n_{0}=\varepsilon_{0}^{1 / 2}$ has to be made. Fig. 1 shows the 'isotropic' ray by a dashed line. In what follows we consider a modified variant of the method in which an 'isotropic' ray is replaced by 'split' rays that correspond to normal waves (method of split rays, Sections 3.3 and 3.4).

Solvability conditions (2.21) of the first approximation equations attain (if orth $\mathbf{n}$ and $\mathbf{b}$ are used) the form

$$
\begin{align*}
& \frac{\mathrm{d} \Gamma_{n}}{\mathrm{~d} s}-\frac{1}{2} \mathrm{i} k_{0} n_{0}^{-1}\left(v_{n n} \Gamma_{n}+v_{n b} \Gamma_{b}\right)-T^{-1} \Gamma_{b}=0, \\
& \frac{\mathrm{~d} \Gamma_{b}}{\mathrm{~d} s}-\frac{1}{2} \mathrm{i} k_{0} n_{0}^{-1}\left(v_{b n} \Gamma_{n}+v_{b b} \Gamma_{b}\right)+T^{-1} \Gamma_{n}=0 . \tag{2.29}
\end{align*}
$$

Here indices $n$ and $b$ refer, respectively, to the normal $\mathbf{n}$ and binormal $\mathbf{b}$ to the ray.

By introducing the parameter of polarisation, $\vartheta=\arctan E_{b} / E_{n}$ (which, generally speaking, is complex), we may rewrite Eqns (2.29) as a variant of the Riccati equation [9, 37]:

$$
\begin{align*}
\frac{\mathrm{d} \vartheta}{\mathrm{~d} s} & +\left[T^{-1}+\frac{1}{2} \mathrm{i} k_{0} n_{0}^{-1}\left(v_{n b}-v_{b n}\right)\right] \\
& +\frac{1}{4} \mathrm{i} k_{0} n_{0}^{-1}\left[\left(v_{n n}-v_{b b}\right) \sin 2 \vartheta-\left(v_{n b}+v_{b n}\right) \cos 2 \vartheta\right]=0 . \tag{2.30}
\end{align*}
$$

The complex angle $\vartheta$ characterises all parameters of the polarisation ellipse. Its real part $\vartheta^{\prime}=\operatorname{Re} \vartheta$ gives the inclination of the larger ellipse axis with respect to the normal $\mathbf{n}$ to the ray. Hyperbolic tangent of the imaginary part $\vartheta^{\prime \prime}=\operatorname{Im} \vartheta$ equals to the ratio of small axis $b$ to the large one, $a$ : $\left|\tanh \vartheta^{\prime \prime}\right|=b / a$, while the sign of $\vartheta^{\prime \prime}$ characterises the sense of rotation of field vectors: if $\vartheta^{\prime \prime}>0$ the vector rotates clockwise, and if $\vartheta^{\prime \prime}<0$ it rotates counterclockwise viewed along the ray.

For an isotropic medium ( $\hat{v}=0$ ), Eqns (2.29) reduce to Rytov equations (2.25) while Eqn (2.30) of the Riccati type becomes the Rytov law (2.26) of polarisation plane rotation.

The QIA equations in moving axis. In many particular cases, eikonal formula (2.24) and Eqns (2.29) can be conveniently written in moving axes $\mathbf{q}_{1}(s)$ and $\mathbf{q}_{2}(s)$ satisfying the condition of orthogonality $\mathbf{q}_{1} \perp \mathbf{q}_{2} \perp \mathbf{t}$. The eigenaxes of two-dimensional tensor with components $\operatorname{Re}\left(\varepsilon_{n n}, \varepsilon_{n b}, \varepsilon_{b n}, \varepsilon_{b b}\right)$ can be used for $\mathbf{q}_{1}$ and $\mathbf{q}_{2}$ axes.

Setting $\mathbf{q}_{1}=\mathbf{n} \cos \psi+\mathbf{b} \sin \psi$ and $\mathbf{q}_{2}=-\mathbf{n} \sin \psi+\mathbf{b} \cos \psi$, where $\psi$ is the variable angle measured from the normal $\mathbf{n}$ to the orth $\mathbf{q}_{1}, \psi=\operatorname{arccotan}\left(\mathbf{b q}_{1} / \mathbf{n} \mathbf{q}_{1}\right)$, and making use of the substitution

$$
\begin{equation*}
\overrightarrow{\mathcal{E}}=\mathbf{E} \exp \left(\mathrm{i} k_{0} \int n_{0} \mathrm{~d} s\right), \quad \mathbf{E}=\Phi_{0} \varepsilon_{0}^{-1 / 4}\left(\Gamma_{1} \mathbf{q}_{1}+\Gamma_{2} \mathbf{q}_{2}\right) \tag{2.31}
\end{equation*}
$$

we derive the system of equation for $\Gamma_{1,2}$ :

$$
\begin{align*}
& \frac{\mathrm{d} \Gamma_{1}}{\mathrm{~d} s}-\frac{1}{2} \mathrm{i} k_{0} \varepsilon_{0}^{-1 / 2}\left(v_{11} \Gamma_{1}+v_{12} \Gamma_{2}\right)-T_{\mathrm{ef}}^{-1} \Gamma_{2}=0 \\
& \frac{\mathrm{~d} \Gamma_{2}}{\mathrm{~d} s}-\frac{1}{2} \mathrm{i} k_{0} \varepsilon_{0}^{-1 / 2}\left(v_{21} \Gamma_{1}+v_{22} \Gamma_{2}\right)+T_{\mathrm{ef}}^{-1} \Gamma_{1}=0 \tag{2.32}
\end{align*}
$$

Indices 1, 2 here correspond to orths $\mathbf{q}_{1}, \mathbf{q}_{2}: v_{\alpha \beta}=\left(\mathbf{q}_{\alpha}, \hat{v} \mathbf{q}_{\beta}\right)$ while the value

$$
\begin{equation*}
T_{\mathrm{ef}}^{-1}=T^{-1}+\frac{\mathrm{d} \psi}{\mathrm{~d} s} \tag{2.33}
\end{equation*}
$$

represents the effective ray twisting in moving axes $\mathbf{q}_{1}, \mathbf{q}_{2}$. It can be shown that the quasi-isotropic approximation is invariant to the first order in $\mu_{1}$ with respect to the mentioned variations of function $\varepsilon_{0}(\mathbf{r})$.

Passing from the QIA to 'simplified' normal waves. In a homogeneous medium, for which $T^{-1}=0, \Phi_{0}=$ const, and $\varepsilon_{\alpha \beta}=$ const, equations (2.29) lead to expressions that resemble independent normal waves:

$$
\begin{equation*}
\overrightarrow{\mathcal{E}}_{1,2}=C_{1,2} \tilde{\mathbf{e}}_{1,2} \exp \left(\mathrm{i} k_{0} \tilde{\mathbf{n}}_{1,2} s\right) \tag{2.34}
\end{equation*}
$$

Their difference from exact normal waves is that instead of exact polarisation vectors $\mathbf{e}_{1,2}$ and refraction indices $n_{1,2}$ they involve simplified expressions $\tilde{\mathbf{e}}_{1,2}$ and $\tilde{n}_{1,2}$ which differ from the exact values (2.7) and (2.15) only in the secondorder terms in respect to anisotropy $\mu_{1}^{2} \sim\left|v_{\alpha \beta}\right|^{2}$. Therefore the expressions (2.34) could be termed as 'simplified' normal waves.

Methods of solution of QIA equations. QIA equations (2.29) form the system of two linked ordinary differential equations with variable coefficients. Such equations are encountered in many fields of physics being applied to the description of a number of similar phenomena: adiabatic perturbations in nonstationary problems of quantum mechanics and the phenomenon of change of spin state in beams of polarised particles [39], linear interaction of normal oscillations in nonstationary systems [40], wave transformation in nonstationary waveguides [41]. For solution of linked equations, special methods are developed. They are described, for example, in Ref. [42].

The methods employed to solve mentioned problems could be useful in solving the QIA equations. In particular, of use might be various variants of perturbation theory and
asymptotic methods, the method of coefficient linearisation, and a few other approaches. We shall present some examples below.

## 3. Modifications and generalisations of QIA equations

### 3.1 QIA equations for the vector of electric induction

The assumption that the field $\overrightarrow{\mathcal{E}}$ is transversal adopted in the primary version of the QIA holds true only in the framework of zero approximation. In reality, the transversal field component $E_{3}$ differs from zero and, in principle, could be found in the first approximation by parameter $\mu_{1}$. Nevertheless, if one formulates QIA equations for the induction vector $\overrightarrow{\mathcal{D}}$, which always has the property of being perpendicular to the wave vector for plane waves in a homogeneous medium, then the transversal component $E_{3}$ can be found in zero approximation. The modification of the QIA for the vector $\overrightarrow{\mathcal{D}}$ was implemented by Naĭda $[18,19,31]$.

In the framework of the standard QIA the relation between vectors $\mathbf{D}$ and $\mathbf{E}$ in zero approximation looks like

$$
\begin{equation*}
D_{1}=\varepsilon_{11} E_{1}+\varepsilon_{12} E_{2}, \quad D_{2}=\varepsilon_{21} E_{1}+\varepsilon_{22} E_{2} \tag{3.1}
\end{equation*}
$$

whereas in reality they are tied by the relationships

$$
\begin{align*}
& D_{1}=\varepsilon_{11} E_{1}+\varepsilon_{12} E_{2}+\varepsilon_{23} E_{3}=\operatorname{det}(\hat{\varepsilon}) \frac{\chi_{22} E_{1}-\chi_{12} E_{2}}{\varepsilon_{33}}, \\
& D_{2}=\varepsilon_{21} E_{1}+\varepsilon_{22} E_{2}+\varepsilon_{23} E_{3}=\operatorname{det}(\hat{\varepsilon}) \frac{-\chi_{21} E_{1}+\chi_{11} E_{2}}{\varepsilon_{33}}, \tag{3.2}
\end{align*}
$$

which follow from the condition of transversality of the vector $\mathbf{D}$ in respect to the wave vector $\mathbf{k}$, i.e. from the condition $D_{3}=0$.

Without running the risk of impinging upon the accuracy of the QIA, i. e. with accuracy up to second-order terms in respect to $\mu_{1}$, we replace the multiplier $\operatorname{det}(\hat{\varepsilon}) / \varepsilon_{33}$ in (3.1) by $\varepsilon_{0}^{2}$ and take this into account when passing from Eqn (2.21) to the QIA equations for the vector $\overrightarrow{\mathcal{D}}$. As a result, the form of formula (2.31) and Eqn (2.32) remain unchanged, but components of tensor $v_{i k}$ entering them have to be substituted by values $v_{i k}^{\prime}$ given by formulae

$$
\begin{array}{ll}
v_{11}^{\prime}=\varepsilon_{0}^{2}\left(\varepsilon_{0}^{-1}-\chi_{11}\right), & v_{12}^{\prime}=-\varepsilon_{0}^{2} \chi_{12} \\
v_{21}^{\prime}=-\varepsilon_{0}^{2} \chi_{21}, & v_{22}^{\prime}=\varepsilon_{0}^{2}\left(\varepsilon_{0}^{-1}-\chi_{22}\right) \tag{3.3}
\end{array}
$$

The components of the anisotropy tensor $v_{i k}^{\prime}=$ $\varepsilon_{0}^{2}\left(\varepsilon_{0}^{-1} \delta_{i k}-\chi_{i k}\right)$ used here depart from the original tensor $v_{i k}=\varepsilon_{i k}-\varepsilon_{0} \delta_{i k}$ only in the second order in small parameter $\mu_{1}$, since

$$
\chi_{i k}=\left(\hat{\varepsilon}^{-1}\right)_{i k}=\varepsilon_{0}^{-1} \delta_{i k}-\varepsilon_{0}^{-2}\left(\varepsilon_{i k}-\varepsilon_{0} \delta_{i k}\right)+O\left(\mu_{1}^{2}\right)
$$

After the substitution of $v_{i k}$ for $v_{i k}^{\prime}$, all QIA equations written above (see Section 2.4) for components of $\mathbf{E}$ vector become valid for analogous components of vector $\mathbf{D}$.

Although, formally, the distinctions due to the replacement of $v_{i k}$ by $v_{i k}^{\prime}$ are at first glance insignificant, 'new' QIA equations offer a few advantages over the previous ones. First, the distinctions in quadratic terms in $\mu_{1}$ may become significant if one approaches the applicability boundaries. In
that respect equations for vector $\mathbf{D}$ are more favourable than those for $\mathbf{E}$. Second, the transversality of $\mathbf{D}$ facilitates substantially the description of splitting of 'isotropic' rays into ordinary and extraordinary rays within the domain of their interaction. This aspect of the problem we shall consider in Sections 3.3 and 3.4.

### 3.2 QIA equations in the form of equations for interacting modes

Deformed normal waves. The QIA equations can be given a form describing the interaction of normal waves. Let us take real eigenorths $\mathbf{e}_{1}$ and $\mathbf{e}_{2}$ of two-dimensional tensor $\chi_{n n}, \chi_{n b}$, $\chi_{b n}, \chi_{b b}$ which play the role of vectors of polarisation, for real orths $\mathbf{q}_{1}$ and $\mathbf{q}_{2}$. These orths are orthogonal to the ray.

Passing to the normal waves can be accomplished in two different, yet analogous ways, depending on which of waves, ordinary or extraordinary, we want to approach.

The field complied with the extraordinary wave is written as

$$
\begin{align*}
\overrightarrow{\mathcal{E}}_{\mathrm{e}}= & \Phi_{0} n_{0}^{3 / 2}\left(n_{1}^{-2} C_{1} \mathbf{e}_{1}+n_{2}^{-2} C_{2} \mathbf{e}_{2}\right) \\
& \times \exp \left[\mathrm{i} k_{0} \int\left(\frac{3}{2} n_{0}-\frac{1}{2} n_{0}^{3} n_{1}^{-2}\right) \mathrm{d} s\right] \tag{3.4}
\end{align*}
$$

Projecting Eqn (2.21) on vectors $\mathbf{Q}_{1}=\mathbf{e}_{1}^{*}$ and $\mathbf{Q}_{2}=\mathbf{e}_{2}^{*}$ we obtain equations for determination of amplitude coefficients $C_{1}$ and $C_{2}$ [18]:

$$
\begin{align*}
& \frac{\mathrm{d} C_{1}}{\mathrm{~d} s}+p_{11} C_{1}+p_{12} C_{2}=0 \\
& \frac{\mathrm{~d} C_{2}}{\mathrm{~d} s}+\mathrm{i} k_{0}\left(n_{1}-n_{2}\right) C_{2}+p_{21} C_{1}+p_{22} C_{2}=0 \tag{3.5}
\end{align*}
$$

Here we denoted

$$
\begin{align*}
& p_{11}=p_{22}^{*}=\frac{2 \mathrm{i} J K}{1+K^{2}} T_{\mathrm{ef}}^{-1}=-\frac{\mathrm{i} J}{\left(1+Q^{2}\right)^{1 / 2}} T_{\mathrm{ef}}^{-1} \\
& p_{12}=-p_{21}^{*}=\mathrm{i} \Psi+\frac{K^{2}-1}{1+K^{2}} T_{\mathrm{ef}}^{-1}=\mathrm{i} \Psi-\frac{Q}{\left(1+Q^{2}\right)^{1 / 2}} T_{\mathrm{ef}}^{-1} \tag{3.6}
\end{align*}
$$

Values $J, K$, and $q$ are given by formulae (2.16); the role of vector $\mathbf{t}$ in these formulae is played by the tangent orth to the 'isotropic' ray. Additionally, here $\Psi=(-1 / 2) \mathrm{d} \arctan Q / \mathrm{d} s$.

Linked equations (3.5) for amplitudes $C_{1}$ and $C_{2}$ describe the interaction of normal modes 1 and 2 . In a homogeneous medium where all $p_{\alpha \beta}=0$, Eqn (3.5) admits the solution $C_{1}=\mathrm{const} \neq 0, C_{2}=0$ that conforms to the extraordinary wave $\overrightarrow{\mathcal{E}}_{1}$ described by formula (2.34).

Analogously, when approaching the ordinary wave, we assume

$$
\begin{align*}
\overrightarrow{\mathcal{E}}_{\mathrm{o}}= & \Phi_{0} n_{0}^{3 / 2}\left(n_{1}^{-2} C_{1}^{\prime} \mathbf{e}_{1}+n_{2}^{-2} C_{2}^{\prime} \mathbf{e}_{2}\right) \\
& \times \exp \left[\mathrm{i} k_{0} \int\left(\frac{3}{2} n_{0}-\frac{1}{2} n_{0}^{3} n_{2}^{-2}\right) \mathrm{d} s\right] \tag{3.7}
\end{align*}
$$

where amplitude coefficients $C_{1}^{\prime}$ and $C_{2}^{\prime}$ meet the equations

$$
\begin{align*}
& \frac{\mathrm{d} C_{1}^{\prime}}{\mathrm{d} s}+\mathrm{i} k_{0}\left(n_{2}-n_{1}\right) C_{1}^{\prime}+p_{11} C_{1}^{\prime}+p_{12} C_{2}^{\prime}=0 \\
& \frac{\mathrm{~d} C_{2}^{\prime}}{\mathrm{d} s}+p_{21} C_{1}^{\prime}+p_{22} C_{2}^{\prime}=0 \tag{3.8}
\end{align*}
$$

with coefficients $p_{i k}$ being given by formula (3.6) as previously.

With the accuracy up to terms of $\mu_{1}^{2}$ order, arguments of exponents in Eqns (3.4) and (3.7) are equal, respectively, to $\mathrm{i} k_{0} \int n_{1} \mathrm{~d} s$ and $\mathrm{i} k_{0} \int n_{2} \mathrm{~d} s$. Note that these relationships become exact at choosing $n=n_{1}$ in Eqn (3.4) and at $n=n_{2}$ in Eqn (3.7).

Deformed normal waves. In the region of relatively strong birefringence $\delta \gg 1$, where normal waves interact weakly with each other, solutions to (3.5) and (3.8) can be constructed in a form of deformed normal waves corresponding very closely to the normal waves (2.13). For that purpose we invoke the iterative approach proposed by Naĭda [18]. In zero approximation we assume that there exists a single normal wave in nondeformed form (2.13). That wave we shall take as a 'seed' for the iterative procedure. The extraordinary wave (with index 1) will serve as a seed for Eqn (3.5) at $C_{1}^{(0)} \neq 0$ and $C_{2}^{(0)}=0$, whereas the ordinary wave will serve as a seed for Eqn (3.8) at $C_{1}^{(0)}=0$ and $C_{2}^{(0)} \neq 0$. In the region of interest where the wave interaction is small, the coefficient $k_{0}\left(n_{1}-n_{2}\right)$ is large compared to $1 / l$. Hence, a formal asymptotic expansion in inverse powers of the large parameter $\delta=k_{0}\left(n_{1}-n_{2}\right) l \gg 1$ can be constructed for $C_{1}$ and $C_{2}$.

In the case of the 'extraordinary' seed this implies that initial terms in the expansion of $C_{\alpha}=C_{\alpha}^{(0)}+C_{\alpha}^{(1)}+C_{\alpha}^{(2)}+\ldots$ $(\alpha=1,2)$ in series by inverse powers of large parameter $\delta \gg 1$ should be subjected to the condition

$$
\begin{equation*}
C_{1}^{(0)}\left(s_{\text {in }}\right) \neq 0, \quad C_{2}^{(0)} \equiv 0 \tag{3.9}
\end{equation*}
$$

Since the coefficient $C_{2}^{(0)}$ is equal to zero, the system of equations (3.5) simplifies and defines only the dynamics of behaviour of the coefficient $C_{1}^{(0)}$ :

$$
\frac{\mathrm{d} C_{1}^{(0)}}{\mathrm{d} s}+p_{11} C_{1}^{(0)}=0 .
$$

Subsequent terms of asymptotic expansion in powers of $1 / \delta$ are determined from the recurrent formulae

$$
\begin{align*}
& C_{2}^{(m)}=\mathrm{i} k_{0}^{-1}\left(n_{1}-n_{2}\right)^{-1}\left[\frac{\mathrm{~d} C_{2}^{(m-1)}}{\mathrm{d} s}+p_{21} C_{1}^{(m-1)}+p_{22} C_{2}^{(m-1)}\right], \\
& \frac{\mathrm{d} C_{1}^{(m)}}{\mathrm{d} s}+p_{11} C_{1}^{(m)}+p_{12} C_{2}^{(m)}=0, \quad C_{1}^{(m)}\left(s_{\text {in }}\right) \equiv 0 \quad(m \geqslant 1) . \tag{3.10}
\end{align*}
$$

Similar procedure can be applied to Eqn (3.8) with the 'ordinary' seed $C_{2}^{(0)} \neq 0$ and $C_{1}^{()} \equiv 0$.

The coefficients that one may find for the 'extraordinary' seed (3.9) in the region of weak interaction $\delta=k_{0} l\left|n_{1}-n_{2}\right| \gg 1$ satisfy the estimate

$$
\begin{equation*}
\left|C_{2}\right| \ll\left|C_{1}\right|, \tag{3.11}
\end{equation*}
$$

which corresponds to a weak deformation of normal waves. Under these conditions, the replacements of $n_{2}^{-2}$ by $n_{1}^{-2}$ in the term with small amplitude $C_{2}$ in (3.4), and $n_{0}$ by $n_{1}$ in the argument of exponent in that formula, would not involve large errors. As a result, QIA formula (3.4) for a deformed extraordinary component simplifies and takes the form

$$
\begin{equation*}
\overrightarrow{\mathcal{E}}_{\mathrm{e}}=\Phi_{0} n_{1}^{-1 / 2}\left(C_{1} \mathbf{e}_{1}+C_{2} \mathbf{e}_{2}\right) \exp \left(\mathrm{i} k_{0} \int n_{1} \mathrm{~d} s\right) \tag{3.12}
\end{equation*}
$$

Similar operations with the ordinary seed $C_{1}^{(0)}=0, C_{2}^{(0)} \neq 0$ lead to a deformed ordinary wave

$$
\begin{equation*}
\overrightarrow{\mathcal{E}}_{\mathrm{o}}=\Phi_{0} n_{2}^{-1 / 2}\left(C_{1}^{\prime} \mathbf{e}_{1}+C_{2}^{\prime} \mathbf{e}_{2}\right) \exp \left(\mathrm{i} k_{0} \int n_{2} \mathrm{~d} s\right) \tag{3.13}
\end{equation*}
$$

Of importance is that both solutions, (3.12) and (3.13) do not contain oscillating terms for $\delta=\mu_{1} / \mu \gg 1$. Oscillations in amplitudes of waves (3.12) and (3.13) also do not occur in the region of relatively weak birefringence, $\delta=\mu_{1} / \mu \lesssim 1$, where the iterations cease to converge, but now for another reason, namely, due to the smallness of parameter $\delta \lesssim 1$. Hence wave solutions (3.12) and (3.13), by their phase structure, are close to normal waves (2.13) either for $\delta \gg 1$ or $\delta \lesssim 1$, however they posses a deformed polarisation structure with respect to that of (2.13). The deformation of polarisation is relatively small for $\delta \gg 1$, but it grows (up to $100 \%$ ) for $\delta \leqq 1$. We term the solutions (3.12) and (3.13) by deformed normal waves. With their help, a connection between the QIA and the Budden equations can be readily established, as well as the process of splitting of ordinary and extraordinary rays can be described.

Equations for interacting modes. Their relation to the Budden equations. If small corrections on the order of $\mu$ are ignored, formula (3.4) for the extraordinary seed and a respective formula for the ordinary seed can be presented in a unified way

$$
\overrightarrow{\mathcal{E}}=\Phi_{0}\left(n_{1}^{-1 / 2} F_{1} \mathbf{e}_{1}+n_{2}^{-1 / 2} F_{2} \mathbf{e}_{2}\right) .
$$

Here amplitudes

$$
\begin{aligned}
& F_{1}=C_{1} \exp \mathrm{i} \varphi_{1}+C_{1}^{\prime} \exp \mathrm{i} \varphi_{2} \\
& F_{2}=C_{2} \exp \mathrm{i} \varphi_{1}+C_{2}^{\prime} \exp \mathrm{\varphi} \varphi_{2}
\end{aligned}
$$

satisfy the QIA equations in a form of interacting normal waves

$$
\begin{align*}
& \frac{\mathrm{d} F_{1}}{\mathrm{~d} s}-\mathrm{i} k_{0} n_{1} F_{1}+p_{11} F_{1}-p_{12} F_{2}=0 \\
& \frac{\mathrm{~d} F_{2}}{\mathrm{~d} s}-\mathrm{i} k_{0} n_{2} F_{2}+p_{21} F_{1}+p_{22} F_{2}=0 \tag{3.14}
\end{align*}
$$

with the same coefficients as in Eqns (3.5) and (3.8). In a plane-layered medium these equations reduce to the Budden equations for interacting waves $[1-5]$.

Thus, the QIA matches quite naturally the classical Budden and Courant - Lax methods. The virtues of the QIA are that it applies not only to plane-layered media, as the Budden method does, but also to 3D inhomogeneous media. The drawback is that the QIA equations are limited by the condition of weak anisotropy, $\mu_{1} \ll 1$, while the Budden and Courant-Lax methods are free of these limitations. In Section 3.4 we show how the QIA equations are to be modified in order to fit also the case of strong anisotropy ( $\mu_{1} \sim 1$ ).

### 3.3 The method of split rays in the case of weak anisotropy

QIA equations based on split rays. With all superficial similarity between simplified normal waves (3.12) and (3.13), derived from the QIA, and independent normal waves of the Courant-Lax method, a marked distinction preserves: the basic form of the QIA ignores splitting of the rays into ordinary and extraordinary ones, whereas normal
waves do propagate along different rays. Accordingly, the phases of the simplified waves depart from those given by exact expressions (2.11). However both these shortages can be eliminated relatively easily. The modification of the QIA, suggested by Naĭda $[18,19,31]$ serves for that. We term it the method of split rays.

The basic idea of that approach is in abandoning the isotropic rays and replacing them by the split rays of simplified normal waves (Fig. 1). On the basis of these rays deformed normal waves should be constructed further with the help of equations of (3.5) and (3.8) type. The rays corresponding to the simplified normal waves will satisfy Eqn (2.10) in which standard simplifications of the QIA are to be made, that is, the terms on the $\mu_{1}^{2}$ order are to be omitted.

In order to use conveniently the smallness of anisotropy, $v_{\alpha \beta} \sim \mu_{1} \ll 1$, it is expedient to express orths $\mathbf{q}_{1}$ and $\mathbf{q}_{2}$, perpendicular to the ray, through orths $\tilde{\mathbf{n}}_{1}$ and $\tilde{\mathbf{b}}_{1}$ that would be close, respectively, to the normal $\mathbf{n}$ and binormal $\mathbf{b}$ of the 'isotropic' ray. For example, a vector that is simultaneously perpendicular to the tangent $\mathbf{t}_{1}$ to the ray and to the normal $\mathbf{n}$ to the isotropic ray can be taken as $\tilde{\mathbf{b}}$. Then

$$
\mathbf{q}_{1}=\tilde{\mathbf{n}} \cos \psi+\tilde{\mathbf{b}} \sin \psi, \quad \mathbf{q}_{2}=-\tilde{\mathbf{n}} \sin \psi+\tilde{\mathbf{b}} \cos \psi .
$$

By analogy with Section 3.1 we shall seek the solutions of Eqn (2.1) along extraordinary rays (with index 1) in the form

$$
\begin{align*}
\overrightarrow{\mathcal{E}}_{\mathrm{e}}= & \Phi_{1} n_{1}^{3 / 2} \hat{\chi}\left(\tilde{\Gamma}_{1} \mathbf{q}_{1}+\tilde{\Gamma}_{2} \mathbf{q}_{2}\right) \\
& \times \exp \left[\mathrm{i} \varphi_{1}+\frac{1}{4} \mathrm{i} k_{0} \int n_{1}^{3}\left(n_{1}^{-2}-n_{2}^{-2}\right) \mathrm{d} s_{1}\right] . \tag{3.15}
\end{align*}
$$

Here $\mathrm{d} s_{1}$ is the element of length of the extraordinary ray, $\varphi_{1}$ is the corresponding eikonal calculated by formula (2.11). Values of refraction indices $n_{1}$ and $n_{2}$ in formula (3.15) are those on the extraordinary ray. They are functions of two vector arguments $\mathbf{r}$ and $\mathbf{t}=\mathbf{k}_{1} /\left|\mathbf{k}_{1}\right|$. The amplitude factor $\Phi_{1}$ satisfies the conservation law $\operatorname{div}\left(\Phi_{1}^{2} \mathbf{t}\right)=0$.

Now substitute (3.15) in (2.1) and project the resultant equation on $\tilde{\mathbf{n}}$ and $\mathbf{b}$, i. e., in fact, pass to equations (2.21). After that, neglecting the contributions of $\mu_{1}^{2}$ order, we shall find the system of QIA equations for coefficients $\tilde{\Gamma}_{1}$ and $\tilde{\Gamma}_{2}$ [18, 19]

$$
\begin{align*}
& \frac{\mathrm{d} \tilde{\Gamma}_{1}}{\mathrm{~d} s_{1}}+\frac{\mathrm{i}}{2} k_{0} n_{1}^{3}\left(\Delta \chi \tilde{\Gamma}_{1}+\chi_{12} \tilde{\Gamma}_{2}\right)-T_{\mathrm{ef}}^{-1} \tilde{\Gamma}_{2}=0 \\
& \frac{\mathrm{~d} \tilde{\Gamma}_{2}}{\mathrm{~d} s_{1}}+\frac{\mathrm{i}}{2} k_{0} n_{1}^{3}\left(\chi_{21} \tilde{\Gamma}_{1}-\Delta \chi \tilde{\Gamma}_{2}\right)+T_{\mathrm{ef}}^{-1} \tilde{\Gamma}_{1}=0 \tag{3.16}
\end{align*}
$$

where $\Delta \chi=\left(\chi_{11}-\chi_{22}\right) / 2$. The respective equations for the ordinary ray follow from (3.16) after the replacement of indices coming with $\mathrm{d} s$ and $n$.

Therefore, the QIA equations are split here in two branches that are related to two normal waves (ordinary and extraordinary) and are based on respective rays. In the region of relatively strong birefringence $\delta \gg 1$, the method of split rays provides an asymptotic transition, with a correct phase, to noninteracting normal waves (2.13).

Accuracy estimations. That the QIA equations conform to the Budden equations for a plane layer provides a possibility of firm estimation of the method accuracy, without the analysis of all residual terms omitted in equations (3.16). Taking into account that the error of the Budden method scales as $\mu \sim k_{0}^{-1} l^{-1}$, we conclude that for the region of relatively strong birefringence $\delta=\mu_{1} / \mu \gg 1$ the estimate
holds

$$
\begin{equation*}
\left|\frac{\delta E}{E}\right| \lesssim \max \left(\mu, \mu_{1}\right), \quad \mu_{1}, \mu_{1} \ll 1 \tag{3.17}
\end{equation*}
$$

It can be rewritten in a uniform way

$$
\begin{equation*}
\max _{(s)}\left|\frac{\delta E}{E}\right| \lesssim \max \left[k_{0}^{-1} l_{\mathrm{b}}^{-1}, \max _{(s)} v(s)\right], \tag{3.18}
\end{equation*}
$$

where the symbol $l_{\mathrm{b}}$ implies that the value of scale $l$ is taken at the boundary of the region of interaction where $\delta \approx 1$.

The region of localisation of split rays. Effective mutual transformation of extraordinary and ordinary waves and splitting of a ray into extraordinary and ordinary rays are localised in the region $\delta \lesssim 1$.

Why the linear wave transformation occurs precisely in the region $\delta \lesssim 1$ could be explained in the following way. Within the region of weak birefringence, $\delta \ll 1$, the spatial scale of beating between polarisation components $\Lambda=k^{-1}\left|n_{1}-n_{2}\right|^{-1}$ exceeds substantially the scale $l$ of medium inhomogeneities, so slow beating of scale $\Lambda$ is simply not seen against the background of variable medium parameters. A noticeable ray splitting does not occurs under these conditions. On the other hand, in the region of relatively strong double refraction $(\delta \gg 1)$ waves $\overrightarrow{\mathcal{E}}_{1}$ and $\overrightarrow{\mathcal{E}}_{2}$ are almost independent and thus do not suffer mutual transformation. Hence, the mutual transformation of waves vanishes either for $\delta \ll 1$ or for $\delta \gg 1$ and is localised in the region $\delta \lesssim 1$. These consideration have already been well known since the works by Budden [1-5, 8].

Matching of the QIA solution with normal waves. The matching could be appropriately done at a point within the region of strong birefringence $\delta \gg 1$ which is still close to the interaction region $\delta \sim 1$ [9]. The details of that procedure were ascertained by Naĭda [16, 17].

The matching procedure simplifies considerably if one uses the equations for the electric induction vector in a form written for split rays. In that case, matching is carried out only in the regions of localisation $\delta \sim 1$. Every time three waves with the same value of wave vector $\mathbf{k}$ participate in matching: a wave arriving to the matching point and two normal waves emerging from it.

Linear transformation occurs at all local extreme points of parameter $\delta$. If $\delta \gg 1$ in the region of the extremum, the linear transformation is rather weak there, $\sim \exp (-\delta)$. Therefore local extreme points of $\delta$ in the region $\delta \gg 1$ should be ignored.

On the contrary, the wave interaction in the vicinities of extreme points located within the region of weak birefringence must be accounted for.

The procedure of matching the QIA solution with normal waves is elucidated by typical examples given below.

Example 1. Incidence of a wave on a doubly refractive layer with a single maximum of parameter $\delta$ inside the layer (Fig. 2). Let us assume that an initial point $A$ (see Fig. 2) is characterised by the same direction $\mathbf{t}^{A}$ of wave vectors of ordinary and extraordinary waves. Construction of the field in the vicinity of that point consists of the following steps of matching the QIA solutions with normal waves.
(a) Through point A we draw extraordinary (1) and ordinary (2) rays that correspond to the initial direction $\mathbf{t}^{A}$ of wave vectors. Then we find phases $\varphi_{1}$ and $\varphi_{2}$ along these rays, construct the polarisation orths, $\mathbf{e}_{1}$ and $\mathbf{e}_{2}$, and $\mathbf{e}_{1}^{\prime}$ and


Figure 2. Explanations to the iterative procedure used in the method of split rays in presence of a local maximum of value $\left|n_{1}-n_{2}\right|$. The solid line shows the extraordinary ray, $A$ is the initial point where both rays correspond to the same direction of the wave vector, $B_{1}$ and $B_{2}$ are the points where the value $\left|n_{1}-n_{2}\right|$ reaches extremum; in their vicinity the iterative procedure is constructed. The arrows on each ray show the direction of integration of Eqns (2.15) (for extraordinary seed), or their analogue for an ordinary wave; the arrows near the rays denote the directions of wave propagation.
$\mathbf{e}_{2}^{\prime}$, respectively, together with amplitude functions $\Phi_{1}$ and $\Phi_{2}$ which satisfy the conservation law.
(b) At each of rays we shall find the points $B_{1}$ and $B_{2}$ where parameter $\delta=k l\left|n_{1}-n_{2}\right|$ reaches maximum.
(c) In the region of strong birefringence $\delta \gg 1$, we construct iterative procedures for each ray in both directions from respective points $B_{1}$ and $B_{2}$, using extraordinary or ordinary seeds, for instance,

$$
C_{1}^{(0)}\left(s_{\text {lin }}\right)=1, \quad C_{2}^{(0)}\left(s_{\text {lin }}\right)=0
$$

(d) We employ the amplitudes $C_{1}\left(s_{1}\right), C_{2}\left(s_{1}\right), C_{1}^{\prime}\left(s_{2}\right)$, and $C_{2}^{\prime}\left(s_{2}\right)$ obtained by the iterative procedure as an initial condition for the system of equations (3.16) and for an analogous system for the ordinary wave. With allowance for those initial conditions 'deformed' extraordinary and ordinary waves are constructed on both sides of points $B_{1}$ and $B_{2}$ : (1) the solutions $\tilde{\Gamma}_{1}\left(s_{1}\right)$ and $\tilde{\Gamma}_{2}\left(s_{1}\right)$ of system (3.16) and an expression for the normed field $\overrightarrow{\mathcal{E}}_{\mathrm{e}}^{\text {norm }}\left(s_{1}\right)$ are determined from Eqn (3.15); (2) the solutions $\tilde{\Gamma}_{1}\left(s_{2}\right)$ and $\tilde{\Gamma}_{2}\left(s_{2}\right)$ of system (3.16) (rewritten for ordinary waves) and the expression for $\overrightarrow{\mathcal{E}}_{\mathrm{o}}^{\text {norm }}\left(s_{2}\right)$ are determined from (3.15) rewritten for an ordinary wave.
(e) We determine decomposition coefficients $K_{1,2}$ of the initial field into rays $l$ and 2 :

$$
\begin{equation*}
\overrightarrow{\mathcal{E}}^{A}=K_{1} \overrightarrow{\mathcal{E}}_{\mathrm{e}}^{\mathrm{norm}}(A)+K_{2} \overrightarrow{\mathcal{E}}_{\mathrm{o}}^{\mathrm{norm}}(A) \tag{3.19}
\end{equation*}
$$

Superposition (3.19) of two deformed normal waves,

$$
\overrightarrow{\mathcal{E}}_{\mathrm{e}}\left(s_{1}\right)=K_{1} \overrightarrow{\mathcal{E}}_{\mathrm{e}}^{\text {norm }}\left(s_{1}\right) \quad \text { and } \quad \overrightarrow{\mathcal{E}}_{\mathrm{o}}\left(s_{2}\right)=K_{2} \overrightarrow{\mathcal{E}}_{\mathrm{o}}^{\text {norm }}\left(s_{2}\right),
$$

each being localised on the respective ray, present the desired solution of geometric-optical problem.

Example 2. Observation of a point source through a doubly refractive layer with a single maximum in parameter $\delta=k_{0} l\left|n_{1}-n_{2}\right|$ (Fig. 3). Assume that a point source is located at the point $A$. This implies that on some small sphere $S^{A}$, surrounding point $A$, directions $\mathbf{t}^{A}$ of wave vectors and initial values $\overrightarrow{\mathcal{E}}^{A}$ of fields are specified. We are to find the field $\overrightarrow{\mathcal{E}}$ at a given point $B$.

In this case, in addition to five steps listed above in example 1, the solution also includes one preliminary step: we should find two rays $A^{\wedge} B$ of 1 and 2 types which would join the source $A$ with the observation point $B$. For each of rays the points of intersection with initial sphere $S^{A}$ are to be


Figure 3. Radiation due to a point source $A$ (it is surrounded by a small sphere $S^{A}$ ) in a doubly refractive layer with a local maximum of $\left|n_{1}-n_{2}\right| ; 1$ and 2 are the main rays reaching the point of observation $d, 1^{\prime}$ and $2^{\prime}$ are auxiliary rays, $B_{1,2}$ and $B_{1,2}^{\prime}$ are the points where $\left|n_{1}-n_{2}\right|$ reaches maximum. The arrows on the rays show the directions of iterations; the arrows near the rays denote the directions of wave propagation.
determined. Then the problems breaks into two parts corresponding to the rays $A_{1} B$ and $A_{2} B$. For each of them the problem statement is essentially that of example 1. Indeed, for the ray $A_{\hat{\imath}} B$, initial direction of wave vectors, $\mathbf{t}^{A 1}$, and initial field, $\overrightarrow{\mathcal{E}}_{\mathrm{e}}{ }^{A}$, are given at point $A_{1}$. Now, in accordance with step (a) of example 1 , a ray with the same direction of the wave vector, but corresponding to type 2 of normal wave, should be drawn through the point $A_{1}$. In the same manner, an additional ray is to be drawn through the point $A_{2}$, now corresponding to a normal wave of type 1 . Clearly, these additional rays do not reach point $B$, however they are necessary to determine components of initial fields $\overrightarrow{\mathcal{E}}_{\mathrm{e}}{ }^{A}$ and $\overrightarrow{\mathcal{E}}_{\mathrm{o}}^{A}$ that correspond to rays $A_{\hat{1}} B$ and $A_{2} B$, in accordance with step e).

What follows is clear from Fig. 3. At the point $B$ we obtain the sum of two waves arriving by different rays $A_{1} B$ and $A_{\hat{2}} B$. From point $B$ two sources will be seen, one in polarisation 1 , and the other in polarisation 2. Equations of (3.16) type for ordinary and extraordinary waves are to be solved at all four rays that appear in the problem treatment. For two basic rays $A_{\hat{1}} B$ and $A_{\hat{2}} B$, seed iterations are constructed on both sides of the point with maximum $\delta$, whereas for two additional rays they are constructed only in the direction to the source $A$.

Example 3 . The incidence of a wave on a doubly refractive layer with two maxima and one minimum of parameter $\delta=k_{0} l\left|n_{1}-n_{2}\right|$ (Fig. 4). We imply that in the region of minimum (points $C$ in Fig. 4) the parameter $\delta$ is less than unity: $\delta_{\min } \lesssim 1$, otherwise the coefficients of transformation will be exponentially small.


Figure 4. The iterative procedure in the presence of local minima in $\left|n_{1}-n_{2}\right| ; C_{1}$ and $C_{2}$ are the points of minima locations in which splitting of rays and matching of solutions take place; $B_{1}, \ldots, B_{6}$ are the points of local maxima. The remaining designations are the same as in Fig. 2 and Fig. 3.

The sequence of operations in this case is clear from Fig. 4. Finally, an original ray splits into four rays, two of type 1 and two of type 2.

Criteria of distinguishability of split rays. As is known, the rays have a dual nature. On the one hand, a ray is though of as a mathematical object, i. e. an infinitely thin line in the space, on which in the approximation of geometrical optics the wave field is strung (dressed, sewed). On the other hand, it is a physical object whose parameters, say, the thickness, could be measured. Physical aspects of the concept of a ray have been discussed by Kravtsov and Orlov in article [43] and book [37], and also by Kravtsov in review [44]. In line with that concept, a physical ray is associated with the Fresnel volume surrounding the ray whereas physical distinguishability of rays implies the possibility of separating the Fresnel volumes. (Fresnel volume is the union of all first Fresnel zones strung on the ray).

Similar distinguishability criteria should be found in an anisotropic medium: as soon as Fresnel volumes of rays cease to intersect, the rays can be accepted as existing on their own, i.e. as admitting a distinguishability by means of physical devices. These include orifices, slits, lenses, reflectors, antennae, phased grids, etc.

In an anisotropic medium, the list of devices can be complemented by polarisers performing the polarisation selection, and by travelling wave antennae discriminating between ordinary and extraordinary waves by their phase velocity.

Needless to say, in the interaction region, $\delta \leqq 1$, the rays can not be distinguished by any physical device. It seems to us that the question of distinguishability of rays leaving the interaction region in a weakly anisotropic medium calls for further detailed investigation.

### 3.4 Generalisation of the method of split rays (synthetic approach)

The above-described method of split ray requires anisotropy to be weak, $\mu_{1} \ll 1$. Meanwhile one may abandon even that rather restrictive requirement by using original, not simplified, normal waves without resorting to the expansions of refractive indices and polarisation vectors in small anisotropy parameter $\mu_{1}$.

Such a modification of the method of split ray was first proposed by Naĭda [18]. It was substantiated mathematically by Naĭda and Prudkovskiĭ [23] and came to be known as the 'synthetic approach'. This approach combines the advantages of the QIA in the form of split rays, which is capable of describing the interaction of waves in 3D inhomogeneous media but fails in the case of strongly anisotropic media with $\mu_{1} \sim 1$, with advantages of the Budden method which allows for strong anisotropy, but applies only to layered, i.e. 1D inhomogeneous, media. Thereby the synthetic approach provides a synthesis of the QIA with the Courant-Lax method, which, although it applies to 3D inhomogeneous media and allows for a strong anisotropy, lacks the third component intrinsic to the QIA and the Budden method: the Courant - Lax method fails to describe the transformation of normal waves.

With this synthesis of the QIA, the Budden and Courant Lax methods, it becomes possible to describe the interaction of normal waves even in a strongly anisotropic medium, precisely in the vicinities of peculiar, 'degenerate' directions of wave vector $\mathbf{k}$. In such directions the refractive indices of two normal waves coincide: $n_{1}(\mathbf{r}, \mathbf{k})=n_{2}(\mathbf{r}, \mathbf{k})$. This favours
a strong interaction between waves with near-parallel phase fronts. The first, relatively straightforward attempt [45] to construct the theory of interaction in the vicinities of degenerate directions ('quasi-degenerate' approximation of geometrical optics) has not been fully successful. Here we present more rigorous approach which allows for splitting rays within the interaction region.

Equations for interacting waves in a strongly anisotropic medium could be written by analogy with Section 3.3, with the replacement of simplified normal waves by their original counterparts. We confine ourselves only to the statement of this analogy leaving aside important details related to the substantiation of the method. The synthetic method was first suggested by Naĭda [8] as a natural modification of the Courant-Lax method. The wave splitting method within the QIA was formulated in Ref. [18] as a subasymptotic of the synthetic method under weak anisotropy.

The transition from simplified waves (those that are concerned only with linear in $\mu_{1}$ terms) to original normal waves is possible only under conditions of localised interaction, with the interaction occurring in a finite interval $l$, where $\delta \lesssim 1$. In this interval, simplified normal waves differ but slightly from their full versions, whereas out of the interval $l$ the interaction weakens abruptly - which allows one to abandon the simplified normal waves in favour of their full versions. The advantage is that there is no need to match the solutions of the synthetic modification of the QIA with independent normal waves out of the interaction region since the synthetic solutions transform asymptotically into independent normal waves there [18, 23].

## 4. Electromagnetic waves in an inhomogeneous plasma in a weak magnetic field

### 4.1 Quasi-longitudinal and quasi-transversal propagation

 QIA equations for magnetoactive plasma. We write the QIA equations for normed vector of electromagnetic induction $\overrightarrow{\mathcal{E}}=\varepsilon_{0}^{-1} \mathbf{D}$, where $\varepsilon_{0}=1-v$ is the dielectric permittivity of plasma in absence of magnetic field, $v=4 \pi e^{2} N_{\mathrm{e}} / m \omega^{2}, e$ and $m$ are, respectively, the electron charge and mass, $N_{\mathrm{e}}$ is the electron concentration. By making use of known expressions for components of the tensor of magnetoactive electron plasma [3-5] one may readily obtain the expressions for the components of anisotropy tensor $v_{i k}^{\prime}=\hat{\chi}-\hat{\chi}_{0}, \hat{\chi}=\stackrel{\hat{\varepsilon}^{-1}}{ }$ which enter the QIA equations for components of vector $\overrightarrow{\mathcal{D}}$ :$$
\begin{align*}
& v_{11}^{\prime}=-v_{22}^{\prime}=-\frac{1}{2}(1-v)^{-1} u v \sin ^{2} \alpha, \\
& v_{12}^{\prime}=-v_{21}^{\prime}=\mathrm{i} v u^{1 / 2} \cos \alpha . \tag{4.1}
\end{align*}
$$

Here $u=e H^{0} /(m c \omega)^{2}, H^{0}$ is the magnetic field strength, $\alpha$ stands for the angle between the orth $\mathbf{t}$ tangent to the ray and the vector $\mathbf{H}^{0}$.

Take orths $\mathbf{q}_{1}$ and $\mathbf{q}_{2}$ in such a manner that they form a right coordinate triple with the tangent orth t , and that orth $\mathbf{q}_{2}$ lies in the plane $\mathbf{t}, \mathbf{H}^{0}$ (Fig. 5).

Substituting (4.1) in Eqn (2.32) we find

$$
\begin{align*}
\overrightarrow{\mathcal{E}}= & \Phi_{0} \varepsilon_{0}^{-1 / 4}\left(\Gamma_{1} \mathbf{q}_{1}+\Gamma_{2} \mathbf{q}_{2}\right) \\
& \times \exp \left\{\mathrm{i} k_{0} \int\left[(1-v)^{1 / 2}-\frac{u v\left(1+\cos ^{2} \alpha\right)}{4(1-v)^{3 / 2}}\right] \mathrm{d} s\right\}, \tag{4.2}
\end{align*}
$$



Figure 5. Relative positions of the external magnetic field vector $\mathbf{H}^{0}$, the tangent to a ray $\mathbf{t}$, the normal $\mathbf{n}$, binormal $\mathbf{b}$, and auxiliary orths $\mathbf{q}_{1}$ and $\mathbf{q}_{2}$.
where $\Gamma_{1}$ and $\Gamma_{2}$ obey the equations

$$
\begin{align*}
& \frac{\mathrm{d} \Gamma_{1}}{\mathrm{~d} s}=-\frac{1}{2} \mathrm{i} G(1-v)^{-1} u^{1 / 2} \sin ^{2} \alpha \Gamma_{1}-\left(G \cos \alpha-T_{\text {ef }}^{-1}\right) \Gamma_{2}, \\
& \frac{\mathrm{~d} \Gamma_{2}}{\mathrm{~d} s}=\left(G \cos \alpha-T_{\text {ef }}^{-1}\right) \Gamma_{1}+\frac{1}{2} \mathrm{i} G(1-v)^{-1} u^{1 / 2} \sin ^{2} \alpha \Gamma_{2} \tag{4.3}
\end{align*}
$$

Here

$$
\begin{equation*}
G=\frac{1}{2} k_{0}(1-v)^{-1 / 2} v u^{1 / 2} \tag{4.4}
\end{equation*}
$$

$\psi$ is the angle between the binormal to the ray, $\mathbf{b}$, and the plane $\mathbf{t}, \mathbf{H}^{0}$ (see Fig. 5); $T_{\text {ef }}^{-1}=T^{-1}+\mathrm{d} \psi / \mathrm{d} s$ is the effective twisting of the ray with account for the rotation of magnetic field strength lines around the ray.

Qualitative picture of the interaction of circular polarised waves in an inhomogeneous magnetoactive plasma. The propagation of an electromagnetic wave through a homogeneous magnetoactive plasma is described in the simplest way in two limiting cases: (a) of the longitudinal propagation (the wave vector $\mathbf{k}$ is parallel to the external magnetic field $\mathbf{H}^{0}$, and $\alpha=0$ ); (b) of transversal propagation (wave vector $\mathbf{k}$ is perpendicular to $\mathbf{H}^{0}$, and $\alpha=\pi / 2$ ).

For longitudinal propagation, eigenwaves are polarised circularly which corresponds to the Faraday effect. The transversal propagation is characterised by a linear polarisation of eigenwaves, thus leading to the Cotton-Mouton effect. In intermediate cases the waves are elliptically polarised. Eventually, on both sides of the orthogonality point $Q$ at which $\mathbf{H}^{0} \perp \mathbf{k}$, a transformation of polarisation of normal wave occurs: a wave initially polarised by right circle (far from the point $Q$ ), changes to that polarised elliptically, then (at the point $Q$ ) it becomes linearly polarised, then again it changes to an elliptically polarised wave (with left direction of rotation), and, ultimately, to that polarised by left circle. The evolution of the wave initially polarised by left circle occurs in the reverse order.

Denote by $l_{\perp}$ the characteristic length of the interval at which the transformation of polarisation occurs. From Eqn (2.7) it follows that the difference of refraction indices, $\left|n_{1}-n_{2}\right|$, reaches a local minimum at a point of transversal propagation where $\cos \alpha=0$. The parameter
$\delta=l_{\perp}^{-1} k_{0}^{-1}\left|n_{1}-n_{2}\right|$ also has a local minimum there. Just in the region pointed out the most effective mutual transformation of ordinary and extraordinary waves occurs.

One can easily find a parameter that defines the intensity of mutual transformation of right- and left-polarised waves in the interval of polarisation transformation. Its role plays the phase difference $p=k_{0} l_{\perp}\left|n_{1}-n_{2}\right|$ over the interval of length $l_{\perp}$. If the phase difference $p$ is large,

$$
p=\left.k_{0} l_{\perp}\left|n_{1}-n_{2}\right|\right|_{Q}=\left.k l_{\perp} u v(1-v)^{-3 / 2}\right|_{Q} \gg 1
$$

everywhere in the interval of polarisation transformation the condition of normal wave independence is fulfilled. Hence, at the orthogonality point $Q$ normal waves almost do not transform: the ordinary (for instance, right-polarised) wave remains ordinary (left-polarised) while the extraordinary one (left-polarised) remains extraordinary (right-polarised); accordingly, the coefficient $\eta$ of mutual transformation of right- and left-polarised waves is close to unity.

If the phase difference between polarisation components at the interval $l_{\perp}$ considered is close to zero, waves polarised circularly preserve their polarisation almost unchanged, but change their 'name'. The coefficient of mutual transformation between ordinary and extraordinary waves is, accordingly, close to unity.

The picture of interaction between polarisation modes simplifies markedly in the case of weak external magnetic field. This case is commonly encountered in the solar atmosphere and the Earth's ionosphere. In a weak magnetic field the polarisation of eigenwaves is close to a circular one over a wide range of angles $\alpha$ between the wave vector $\mathbf{k}$ and the external field $\mathbf{H}^{0}$. The distinction of the polarisation from a circular one becomes noticeable only within the cone:

$$
\begin{equation*}
2|\cos \alpha| \lesssim u^{1 / 2}(1-v)^{-1} \sin ^{2} \alpha . \tag{4.5}
\end{equation*}
$$

The interval where condition (4.5) is implemented is called the region of quasi-transversal wave propagation. In a weak magnetic field the dimension $l_{\perp}$ of this region is considerably less then curvature radii of the ray or magnetic strength lines.

Indeed, in Eqns (4.3), terms $G \Gamma_{1,2} \cos \alpha$ correspond to the Faraday effect, while terms $(1 / 2) G(1-v)^{-1} u^{1 / 2} \Gamma_{1,2} \sin ^{2} \alpha$ are related to the Cotton-Mouton effect. 'Faraday' terms prevail over their 'Cotton-Mouton' counterparts for sotermed quasi-longitudinal propagation, when $2|\cos \alpha| \gg$ $(1-v)^{-1} u^{1 / 2} \sin ^{2} \alpha$. In this case cross-terms in Eqns (4.3) are small and the equations describe an independent propagation of two waves polarised circularly. In the range of quasitransversal propagation, where inverse inequality (4.5) holds true, Cotton-Mouton cross-terms, responsible for the wave transformation, are important in (4.3). On the other hand, by virtue of assumption that the parameter $u^{1 / 2}(1-v)^{-1}$ is small, from inequality (4.5) it follows that value of $\cos \alpha$ is also small everywhere in the quasi-transversal region. Based on this we may write

$$
\begin{equation*}
\cos \alpha \cong \frac{s}{\rho}+O\left(\frac{s^{3}}{\rho^{3}}\right), \quad \sin ^{2} \alpha \cong 1-O\left(\frac{s^{2}}{\rho^{2}}\right) \tag{4.6}
\end{equation*}
$$

where $s$ is the ray length measured from the orthogonality point $\cos \alpha=0$, and $\rho=(\mathrm{d} \cos \alpha / \mathrm{d} s)^{-1}$ is the characteristic scale at which angle $\alpha$ varies along the ray. It depends both on the ray curvature and magnetic field configuration. In particular, if the ray is completely in the plane of magnetic
meridian, $\left|\rho^{-1}\right|=\left|\rho_{\mathrm{r}}^{-1} \pm \rho_{H}^{-1}\right|$, where $\rho_{\mathrm{r}}$ is the ray curvature radius, and $\rho_{H}$ is the distance from the orthogonality point to a virtual centre of the magnetic field lines where tangents to lines of $\mathbf{H}^{0}$ intersect.

From expressions (4.6) and estimate (4.5) it follows that in the interval of quasi-transversal propagation $2|s / \rho| \lesssim$ $u^{1 / 2}(1-v)^{-1}$. Therefore the length of the interval of interaction satisfies the estimate

$$
\begin{equation*}
l_{\perp} \sim 2|s| \cong|\rho| u^{1 / 2}(1-v)^{-1} \ll|\rho|, \tag{4.7}
\end{equation*}
$$

i.e. $l_{\perp}$ is small compared to the curvature radius $|\rho|$.

We shall also assume that the inequality

$$
\begin{equation*}
l_{\perp}<l \tag{4.8}
\end{equation*}
$$

is fulfilled, where $l$ is the scale of medium inhomogeneities. If inequality (4.8) holds true one may speak about spatial localisation of the effect.

From estimate (4.7), the estimate follows for the parameter $p$ that defines the intensity of transformation

$$
\begin{equation*}
p=\left.k l_{\perp}\left|n_{1}-n_{2}\right|\right|_{Q} \sim k \rho u^{3 / 2} v(1-v)^{-5 / 2} . \tag{4.9}
\end{equation*}
$$

For the case of $v \ll 1$ this estimate was first found by Cohen [46]. For a general case it was found by Melrose [47] based on the Budden equations and in Ref. [48] based on the method of phase integrals. A detailed analysis of the problem is performed in Ref. [8].

### 4.2 Coefficients of transformation at the interval of quasi-transversal propagation

The QIA equations for waves with a circular polarisation. The effect of interaction of circularly polarised waves in magnetoactive plasma in the region of quasi-transversal magnetic field (for brevity the effect will be referred to as 'quasitransversal' interaction) was invoked for the explanation of peculiarities of solar radiation in radio band [5,46-48] and some anomalies of the Faraday effect in the Earth's ionosphere [49, 50]. All calculations of this effect were conducted in the framework of a simplified problem statement which involved plane waves in a homogeneous plasma placed in an inhomogeneous magnetic field. The only significant analytical result for transformation coefficients was found by Zheleznyakov and Zlotnik [48] who used the phase integrals method.

More radical solution of the problem was suggested by Kravtsov and Naĭda [24] and also by Naǐda [19]. Being interested in the transformation of circularly polarised waves, in Eqn (4.2) we change to variables

$$
\begin{equation*}
\gamma_{1,2}=2^{-1 / 2}\left(\Gamma_{2} \mp \mathrm{i} \Gamma_{1}\right), \tag{4.10}
\end{equation*}
$$

that obey the equations

$$
\begin{align*}
& \frac{\mathrm{d} \gamma_{1}}{\mathrm{~d} s}=\mathrm{i}\left(G \cos \alpha-T_{\mathrm{ef}}^{-1}\right) \gamma_{1}+\frac{1}{2} \mathrm{i} G u^{1 / 2}(1-v)^{-1} \gamma_{2} \sin ^{2} \alpha, \\
& \frac{\mathrm{~d} \gamma_{2}}{\mathrm{~d} s}=\frac{1}{2} \mathrm{i} G u^{1 / 2}(1-v)^{-1} \gamma_{1} \sin ^{2} \alpha-\mathrm{i}\left(G \cos \alpha-T_{\mathrm{ef}}^{-1}\right) \gamma_{2} \tag{4.11}
\end{align*}
$$

and the norming condition $\left|\gamma_{1}\right|^{2}+\left|\gamma_{2}\right|^{2}=1$. If a wave incident from the side of negative $s$ is polarised by a right circle, the system of equations (4.10) should be supplemented by the
initial conditions

$$
\begin{equation*}
\left|\gamma_{1}(-\infty)\right|=\left|\gamma_{1}^{\mathrm{r}}(-\infty)\right|=1, \quad \gamma_{2}(-\infty)=\gamma_{2}^{\mathrm{r}}(-\infty)=0 \tag{4.12}
\end{equation*}
$$

while for polarisation by a left circle it should be supplemented by the initial conditions

$$
\begin{equation*}
\gamma_{1}(-\infty)=\gamma_{1}^{1}(-\infty)=0, \quad\left|\gamma_{2}(-\infty)\right|=\left|\gamma_{2}^{1}(-\infty)\right|=1 \tag{4.13}
\end{equation*}
$$

Analytical solution of the QIA equations for localised interaction. If the region of interaction is localised, i.e. the length of the interaction region, $l_{\perp}$, is small compared not only to $\rho$, but also to a characteristic scale of plasma inhomogeneity, $l$, system (4.11) admits an approximate solution, possessing, nevertheless, a reasonable universality. In fact, for $l_{\perp} \ll l$ the plasma parameters $u$ and $v$, as well as effective twisting $T_{\text {ef }}^{-1}$ within the region of interaction can be replaced by their local values at the point of orthogonality. Then, in conjunction with (4.6), we find the system of equations

$$
\begin{align*}
& \frac{\mathrm{d} \gamma_{1}}{\mathrm{~d} s}=\mathrm{i}\left(G \rho^{-1} s-T_{\text {ef }}^{-1}\right) \gamma_{1}+\frac{1}{2} \mathrm{i} G(1-v)^{-1} u^{1 / 2} \gamma_{2}, \\
& \frac{\mathrm{~d} \gamma_{2}}{\mathrm{~d} s}=\frac{1}{2} \mathrm{i} G(1-v)^{-1} u^{1 / 2} \gamma_{1}-\mathrm{i}\left(G \rho^{-1} s-T_{\text {ef }}^{-1}\right) \gamma_{2} . \tag{4.14}
\end{align*}
$$

> If a dimensionless variable

$$
\begin{equation*}
\xi=\left(\frac{G}{|\rho|}\right)^{1 / 2}\left(s-\rho G^{-1} T_{\text {ef }}^{-1}\right) \tag{4.15}
\end{equation*}
$$

is introduced, Eqns (4.14) take the form

$$
\begin{align*}
& \frac{\mathrm{d} \gamma_{1}}{\mathrm{~d} \xi}=-\mathrm{i} \xi \gamma_{1} \operatorname{sgn} \rho+\frac{1}{2} \mathrm{i} p^{1 / 2} \gamma_{2}, \\
& \frac{\mathrm{~d} \gamma_{2}}{\mathrm{~d} \xi}=\frac{1}{2} \mathrm{i} p^{1 / 2} \gamma_{1}+\mathrm{i} \xi \gamma_{2} \operatorname{sgn} \rho . \tag{4.16}
\end{align*}
$$

Eqns (4.16) contain a single parameter,

$$
\begin{equation*}
p=G u|\rho|=\frac{1}{2} k v u^{3 / 2}|\rho|(1-v)^{-5 / 2}, \tag{4.17}
\end{equation*}
$$

with which we have encountered earlier analyzing the problem qualitatively: it is the phase difference over the interaction interval $l_{\perp}$ (see (4.9)).

One may verify that the component $\gamma_{2}$ satisfies the Weber-Hermite equation and that solution of the system of equations (4.16) could be expressed in terms of functions of parabolic cylinder $D_{n}(z)$. With initial condition (4.12) (initial right circle), the intensities of circularly polarised waves for $\xi \rightarrow+\infty$ are equal to

$$
\begin{align*}
& \left|\gamma_{1}^{\mathrm{r}}(+\infty)\right|^{2}=\exp \left(-\frac{\pi p}{4}\right) \\
& \left|\gamma_{2}^{1}(+\infty)\right|^{2}=1-\exp \left(-\frac{\pi p}{4}\right) \tag{4.18}
\end{align*}
$$

Hence, the value

$$
\begin{equation*}
\eta=1-\exp \left(-\frac{\pi p}{4}\right) \tag{4.19}
\end{equation*}
$$

represent the coefficient of mutual transformation of rightpolarised and left-polarised waves.

It is self-evident that formula (4.19) coincides with an expression for the coefficient of transformation obtained in the framework of 1D problem statement by the method of phase integrals $[48,51]$, and yet the results of the QIA are in many respects more comprehensive. First, the QIA applies for 3D curved rays. Second, the QIA gives not only the value of the transformation coefficient $\eta$, but also the field magnitude at all points of the ray. That makes it possible to calculate $\eta$ even in the cases of a source located inside the interaction region. This is important for ionospheric investigations. Third, it turns out that twisting of the ray and the rotation of the magnetic field strength lines do not influence the value of $\eta$, since effective twisting, $T_{\text {ef }}^{-1}$, has at all dropped out from Eqn (4.16). Finally, fourth, from basic equations (4.11) we may derive that the wave transformation is small in an extended region of quasi-longitudinal propagation and find corrections to transformation coefficient (4.19) due to 'partial localisation' of the interaction effect. We return to this question further.

Possibilities of 'Cotton-Mouton' plasma diagnostics. Unlike the Faraday effect which provides information about integrated parameters of the plasma, the quasi-transversal interaction due to the Cotton-Mouton effect may serve as a source of information about local characteristics of plasma at the orthogonality point. In particular, one may speak about determination of local electron concentration $N_{e}$ with the other parameters being known.

If the effect of quasi-transversal interaction is spatially localised (condition (4.8)), the coefficient of transformation is given by (4.19), and by measured values of $\eta$ one may determine the parameter $p$ which characterises local plasma properties at the point of orthogonality:

$$
\begin{equation*}
p \equiv G_{\mathrm{o}} u_{\mathrm{o}}|\rho|=\frac{4}{\pi} \ln \frac{1}{1-\eta} . \tag{4.20}
\end{equation*}
$$

The capability of that kind, substantiated by the results of Cohen [46], and Zheleznyakov and Zlotnik [48], is widely and successfully used to treat the data on the polarisation of solar irradiance, in particular, to estimate the magnetic field $H_{0}$ of the solar corona based on preliminary estimates of other plasma parameters. However, as applied to ionospheric and laboratory plasma, the possibilities of Cotton-Mouton diagnostics were not discussed in detail, in particular, due to lack of an effective theory of transformation of the normal waves in the case of curved rays.

Under the conditions of the Earth's ionosphere the characteristic scale $\rho$ is $3000-6000 \mathrm{~km}$ (given a dipole model of the magnetic field), whereas the vertical scale for plasma inhomogeneities $l_{\text {vert }} \sim 100 \mathrm{~km}$. By strength of (4.7) the effect of quasi-transversal interaction will be localised for $u^{1 / 2} \ll l_{\perp} /|\rho| \sim 1 / 30-1 / 60$, i.e. for $\lambda \geqslant 4-7 \mathrm{~m}$. Thus, in the ultrahigh frequency range, one can rely on measurements of local electron concentration of ionospheric plasma at the points of orthogonality. For an oblique propagation of radiowaves the effective scale $l$ of inhomogeneities increases several folds (since $l_{\text {horiz }} \sim 1000 \mathrm{~km}$ ), and the threshold wavelength also increases in the same manner (up to $20-30 \mathrm{~m}$ ).

For laboratory plasma, values $N_{\mathrm{e}}, H_{0}$, and $\rho$ may vary within extremely wide limits, hence, it seems quite reasonable that by appropriate choice of frequency and propagation direction for microwave radiation, large volumes of plasma
can be sounded with a view to determine the local electron concentration. That could be favoured by intentionally controlling the magnitude and configuration of the magnetic field provided it is admissible under the conditions of experiment.

On solving plasma diagnosis problems, in addition to the localisation of the effect (condition (4.8)) one should also strive for its better 'visibility' which could be, for instance, characterised by the degree of linear polarisation $\beta_{\text {lin }}=2[\eta(1-\eta)]^{1 / 2}$. This value reaches a maximum at $\eta=1 / 2$ which is attained for $p_{1,2}=4 \pi^{-1} \ln 2=0.88$ and for frequency of radiation

$$
\frac{\omega_{1}}{\omega_{0}}=\left(0.57 \omega_{0}^{-2} \omega_{\text {b.v. }}^{3}|\rho| c^{-1}\right)^{1 / 4}
$$

Generally speaking, at this frequency of 'best visibility' of the effect inequalities (4.4) or (4.8) can be violated. In order to avoid such a violation it is necessary, as follows from simple calculations, for the plasma frequency $\omega_{0}$ to be confined in limits

$$
\omega_{\text {b.v. }}^{3 / 2}\left(\frac{|\rho|}{c}\right)^{1 / 2} \gg \omega_{0} \gg \omega_{\text {b.v. }}^{1 / 2}\left(\frac{|\rho|}{c}\right)^{-1 / 2}
$$

which, in turn, is possible if $\omega_{\text {b.v. }}|\rho| / c \gg 1$. These inequalities specify the ranges of values of $\omega_{\text {b.v. }}$ and $\omega_{0}$, for which both the spatial localisation and good visibility of the effect are possible.

Account for partial localisation of the Cotton-Mouton effect. Estimates of errors due to thr replacement of the exact QIA equations (4.11) by Eqns (4.14) with linearised coefficients can be carried out in various ways.

The simplest way is to solve Eqns (4.11) in the region of quasi-transversal propagation by the perturbation method in the parameter $q=u^{1 / 2} \sin ^{2} \alpha /(2 \cos \alpha)$, which is small precisely in that region. To account for a partial localisation in the region of interaction, $|s| \lesssim l_{\perp}=u^{1 / 2}|\rho|$, we may construct a perturbation theory series based on Eqns (4.14), by considering there terms quadratic in $s$ in expansions of $\cos \alpha$ and $\sin ^{2} \alpha$, which were ignored in Eqn (4.6), and linear in $s$ terms in expansions of plasma parameters $u, v$, and $T_{\text {ef }}^{-1}$. Matching both perturbation series at the boundary of the region of interaction $\left(|s| \sim l_{\perp} \sim|\rho| u^{1 / 2}\right)$ so as to provide the least influence of the matching position, one may find the correction to computed transformation coefficient (4.19).

Ionospheric manifestations of the Cotton-Mouton effect: weak depolarisation of radio-waves. Already first works devoted to the QIA were aimed at the description of radiowave polarisation in the ionosphere. A simple way to describe a weak depolarisation caused by a distributed, nonlocalised Cotton - Mouton effect was suggested in Ref. [9].

Let us write the Riccati equation (2.30) for plasma placed in a weak magnetic field. In compliance with (4.1), we find the equation for $\vartheta$ :

$$
\begin{align*}
\frac{\mathrm{d} \vartheta}{\mathrm{~d} s}= & T^{-1}+\frac{1}{2} k_{0} v(1-v)^{-1} u^{1 / 2} \cos \alpha \\
& +\frac{\mathrm{i}}{4} k_{0} v u \sin ^{2} \alpha \sin 2(\vartheta+\varphi) \tag{4.21}
\end{align*}
$$

The first term in that equation describes the Rytov twisting, the second one describes the Faraday rotation of the polarisation plane, whereas the third imaginary term corresponds to the Cotton-Mouton effect.

For the most part of the rays the third term is small compared to the second. Hence Eqn (4.21) can be solved by an iterative procedure. For that purpose we write Eqn (4.21) in the form

$$
\begin{equation*}
\frac{\mathrm{d} \vartheta}{\mathrm{~d} s}=T^{-1}+\frac{1}{2} k_{0} v(1-v)^{-1} u^{1 / 2} \cos \alpha+\mathrm{i} M(s, \vartheta), \tag{4.22}
\end{equation*}
$$

where $\quad M(s, \vartheta)=-(1 / 2) k_{0} v u \sin ^{2} \alpha \sin 2(\vartheta+\varphi)$. Setting $M=0$, we obtain the zero approximation

$$
\begin{equation*}
\vartheta_{0}(s)=\vartheta(0)+\int_{0}^{s} \frac{\mathrm{~d} s}{T}+\vartheta_{\mathrm{F}}(s) \tag{4.23}
\end{equation*}
$$

where $\vartheta_{\mathrm{F}}$ is the Faraday rotation angle,

$$
\begin{equation*}
\vartheta_{\mathrm{F}}(s)=\frac{1}{2} k_{0} \int_{0}^{s} v(1-v)^{-1} u^{1 / 2} \cos \alpha \mathrm{~d} s . \tag{4.24}
\end{equation*}
$$

The first iteration results in a small imaginary correction to $\vartheta_{0}(s)$ :

$$
\begin{align*}
\vartheta_{1}(s) & =\vartheta_{0}(s)+\mathrm{i} \int_{0}^{s} M\left(s, \vartheta_{0}\right) \mathrm{d} s \\
& =\vartheta_{0}(s)-\frac{1}{4} k_{0} \int_{0}^{s} v u \sin \alpha \sin 2\left(\vartheta_{0}+\varphi\right) \mathrm{d} s . \tag{4.25}
\end{align*}
$$

The imaginary correction $\vartheta_{1}(s)=\mathrm{i} \vartheta_{1}^{\prime \prime}$ is responsible for the transformation of a linearly polarised wave into that polarised elliptically, with the small axis

$$
\begin{equation*}
\tan \vartheta_{1}^{\prime \prime} \approx \operatorname{Im} \vartheta_{1}=\frac{1}{4} k_{0} \int_{0}^{s} v u \sin ^{2} \alpha \sin 2\left(\vartheta_{0}+\varphi\right) \mathrm{d} s \tag{4.26}
\end{equation*}
$$

Thus, for the ionospheric propagation of radio-waves the state of polarisation is characterised by the angle (4.23) at which the large axis of polarisation ellipse turns and by the small axis (4.26) that defines the depolarisation of the wave.

Fuki [25] has constructed another variant of the iterative procedure which possesses an improved convergence. We rewrite Eqn (4.22) in the form of two coupled equations for $\vartheta^{\prime}=\operatorname{Re} \vartheta$ and $\vartheta^{\prime \prime}=\operatorname{Im} \vartheta$ :

$$
\begin{align*}
& \frac{\mathrm{d} \vartheta^{\prime}}{\mathrm{d} s}=T^{-1}+\frac{\mathrm{d} \vartheta_{\mathrm{F}}}{\mathrm{~d} s}+\frac{1}{4} k_{0} v u \sin ^{2} \alpha \cos 2\left(\vartheta^{\prime}+\varphi\right) \sinh 2 \vartheta^{\prime \prime}, \\
& \frac{\mathrm{d} \vartheta^{\prime \prime}}{\mathrm{d} s}=-\frac{1}{4} v u \sin ^{2} \alpha \sin 2\left(\vartheta^{\prime}+\varphi\right) \cosh \vartheta^{\prime \prime}, \tag{4.27}
\end{align*}
$$

where $\vartheta_{\mathrm{F}}$ is the Faraday rotation angle (4.24). From the second equation of Eqns (4.27) it follows that

$$
\begin{equation*}
\sinh 2 \vartheta^{\prime \prime}=\tan \left[\arctan \sinh 2 \vartheta^{\prime \prime}(0)-L\left(s, \vartheta^{\prime}\right)\right] \tag{4.28}
\end{equation*}
$$

where

$$
L\left(s, \vartheta^{\prime}\right)=\frac{1}{2} k_{0} \int_{0}^{s} v u \sin ^{2} \alpha \sin 2\left(\vartheta^{\prime}+\varphi\right) \mathrm{d} s
$$

and $\vartheta^{\prime \prime}(0)$ is the initial value of $\vartheta^{\prime \prime}$.
On substituting (4.28) in the first equation of Eqns (4.27) we find a closed equation for $\vartheta^{\prime}$ which could be rewritten in the integral form:

$$
\begin{equation*}
\vartheta^{\prime}(s)=\vartheta^{\prime}(0)+\int_{0}^{s} \frac{\mathrm{~d} s}{T}+\vartheta_{\mathrm{F}}(s)+R\left(s, \vartheta^{\prime}\right) \tag{4.29}
\end{equation*}
$$

where

$$
\begin{aligned}
R\left(s, \vartheta^{\prime}\right)= & -\frac{1}{4} k_{0} \int_{0}^{s} v u \sin ^{2} \alpha \cos 2\left(\vartheta^{\prime}+\varphi\right) \\
& \times \tan \left[\arctan \sinh 2 \vartheta^{\prime \prime}(0)-\mathrm{L}\left(\mathrm{~s}, \vartheta^{\prime}\right)\right] \mathrm{ds}
\end{aligned}
$$

Since $R$ is small, it is natural to solve Eqn (4.29) by an iterative method. If we assume in zero approximation

$$
\vartheta_{0}^{\prime}(s)=\vartheta^{\prime}(0)+\int_{0}^{s} \frac{\mathrm{~d} s}{T}+\vartheta_{\mathrm{F}}(s)
$$

then, in the $m$-th approximation

$$
\begin{align*}
& \vartheta_{m}^{\prime}(s)=\vartheta_{0}^{\prime}(s)+R\left(s, \vartheta_{m-1}^{\prime}\right) \\
& \sinh 2 \vartheta_{m}^{\prime \prime}(s)=\tan \left[\arctan \sinh 2 \vartheta^{\prime \prime}(0)-L\left(s, \vartheta_{m}^{\prime}\right)\right] \tag{4.30}
\end{align*}
$$

In the simplest case, when an incident wave is linearly polarised and $\vartheta^{\prime \prime}(0)=0$, expression (4.26) follows for the small correction $\vartheta_{1}^{\prime \prime}$ in the first approximation.

For most problems of the ionospheric propagation of high frequency and ultrahigh frequency radio-waves the approximation (4.26) is sufficient to evaluate the depolarisation. Otherwise the iterative scheme (4.30) should be used.

In Ref. [26] detailed calculations of radio-wave polarisation have been carried out for given models of the ionosphere. The QIA equations were solved numerically by the RungeKutta method. According to that reference, the Rytov rotation of the polarisation plane can be disregarded in most cases compared to the Faraday rotation, since the radius of ray twisting, $T$, is commonly extremely large even under the conditions of transition from the diurnal region of the ionosphere to the nocturnal region. It is there the contribution from twisting is maximum, for the structure of diurnal and nocturnal stratosphere is plane-layered, and thus twisting is practically absent.

Polarisation effects at scattering of radio-waves in the polar and equatorial ionosphere. The depolarisation of radio-waves is to be accounted for if there is scattering on irregularities of the polar and equatorial ionosphere. Depolarisation owes its existence to two reasons. They are: (1) the transformation of waves as they propagate from the source to a point of scattering and return back, and (2) the anisotropy of irregularities directly at the point of scattering. As shown in Ref. [27], the local depolarisation at the point of scattering is usually extremely small and the resulting depolarisation is due to summing of polarisation changes along the entire ray.

Fuki [25, 28] subsequently ascertained these results and constructed the correlation matrix of the scattered field with account for the Faraday and Cotton-Mouton effects. This made possible the explanation of the experimental results presented in Ref. [52], which involved measurements of polarisation of radio-waves scattered in the polar ionosphere on irregularities elongated in the direction of the Earth's magnetic field. The results discussed can be used to analyse radio echoes from strongly elongated irregularities of the equatorial ionosphere, as well as incoherent (Thomson) scattering on thermal fluctuations of electron concentration in the ionosphere.

In many cases scattered signals present a hindrance to radio engineering systems of different type (used in wireless communication, radio-location, radio navigation, etc.). In Ref. [53], methods of polarisation damping of influence of
auroral and equatorial radio echoes have been suggested. Given a radar with the main lobe angular width of approximately $1^{\circ}$, the degree of auroral echo suppression due to the polarisation selection may reach $25-30 \mathrm{~dB}$.

### 4.3 Other polarisation effects in plasma

The transformation of normal waves in the region of zero magnetic field in the solar corona. Above, we have confined ourselves to considering only one, though important, question about changes in polarisation in the region of quasitransversal propagation of electromagnetic waves in magnetoactive plasma. When analyzing the effects that could influence the polarisation of electromagnetic waves radiated by the solar corona, Zheleznyakov [54] has discovered that a marked transformation of normal waves occurs also in regions of neutral magnetic field. In magnetic fields of complex structure, the presence of points where the field $\mathbf{H}^{0}$ equals zero is the rule rather than the exception.

In the vicinity of the neutral field point everything looks as if we have encountered the problem of limit polarisation twice: first the normal waves propagating in the magnetoactive plasma become transversal as $\mathbf{H}^{0} \rightarrow 0$, then a transversal wave splits into the superposition of two normal waves past the neutral point. The results of rigorous analysis of this problem are presented in a review article [8].

Transformation of waves in an inhomogeneous plasma in the presence of a shear of magnetic field strength lines. This problem was studied in Ref. [55]. The main result is in establishing the fact that even a small shear of strength lines can lead to a noticeable field transformation. This effect can be used for plasma diagnosis in new ranges of parameters.

Polarisation effects in a moving plasma. Inhomogeneous motion either of laboratory or any other plasma results in a weak anisotropy. In essence, this anisotropy is similar to the optic Maxwell effect (appearance of anisotropy in a shear flow of fluid). Mechanisms which give rise to anisotropy in a plasma, discussed in Refs [56-58], are due to spatial dispersion induced by an inhomogeneous flow of plasma. Clearly, polarisation effects in an inhomogeneously moving plasma can be described in the framework of the QIA.

Weak anisotropy due to the inhomogeneity of medium. The inhomogeneity of a medium which possesses spatial dispersion must inevitably lead to anisotropy, since there exists a distinguished direction $\nabla \varepsilon_{0}$, where $\varepsilon_{0}$ is the permittivity of the isotropic medium in the absence of inhomogeneities. Calculation of the plasma inhomogeneity tensor carried out in Ref. [59] has confirmed this expectation. The anisotropy induced by inhomogeneity turned out to be negligible, however at large distances such anisotropy could be discovered by polarisation methods. In essence, even the Rytov rotation of the polarisation plane might be interpreted as being caused by the appearance of a weak anisotropy due to medium inhomogeneities.

Polarisation effects in a plasma with random inhomogeneities. In a work by Apresyan [29] (see also Ref. [30]) the problem of the influence of weak fluctuations in an anisotropic medium on the polarisation of a wave passing through a thick randomly inhomogeneous layer has been considered. This problem is encountered, in particular, in describing the polarisation of high-frequency waves in magnetoactive cosmic plasma. In the work mentioned the QIA equations with a fluctuating tensor of dielectric permittivity were used. On their basis, equations for the mean value of the Stokes vector, which describes the wave polarisation, were derived. As it
turns out, within the layer the mean Stokes vector tends asymptotically to a specific direction, related to the mean value of the tensor of dielectric permittivity of the medium. Thus the measurement of the mean polarisation provides additional information about mean properties of the medium and in that manner facilitates the solution of inverse problems.

## 5. Optical effects in weakly-anisotropic media

## 5.1 'Tangent' conical refraction

General picture of the effect. As is known, the conical refraction (the internal one) takes place on incidence of a plane wave on a homogeneous crystal, if refraction indices for two types of normal waves with the same direction of $\mathbf{k}$ coincide [51]:

$$
\begin{equation*}
n_{1}(\mathbf{k}, \hat{\varepsilon})=n_{2}(\mathbf{k}, \hat{\varepsilon}) . \tag{5.1}
\end{equation*}
$$

In an anisotropic smoothly inhomogeneous medium condition (5.1) will not be met everywhere in the volume occupied by a wave, as it is in the classical conical refraction effect, but only along a special line (Fig. 6) where the wave vector $\mathbf{k}(\mathbf{r})$ is oriented in a proper way with respect to the main axes of tensor $\hat{\varepsilon}(\mathbf{r})$. That line appears because condition (5.1) corresponds to the intersection of two surfaces in the coordinate space. As a result, instead of the pattern of ray scattering in cone (5.1), associated with the classical effect of the conical refraction, in a smoothly inhomogeneous medium a specific wave transformation is observed, which can be reasonably treated as tangent conical refraction (Naĭda [31]).


Figure 6. The phenomenon of tangent conical refraction is observed in the vicinity of the point $Q$ where the ray intersect a 'critical' line given by condition (5.1). At this line the refractive indices of two normal waves in a weakly anisotropic media coincide. The region of strong transformation of the normal waves is dotted ( $l_{\text {int }}$ and $\rho_{\perp}$ are respectively the transversal and longitudinal scales of the interaction region).
'Tangent' conical refraction can occur in an inhomogeneously deformed crystal, in an inhomogeneously deformed glass, whose optical anisotropy is caused by the elastic-optical effect, and, finally, the effect can be observed in a moving fluid with inhomogeneous velocity field, where the anisotropy is due to the Maxwell effect, and in neodimum glasses subjected to inhomogeneous heat loads in intense lasers [60, 60a].

Let us place between crossed polarisers a doubly refractive medium in which the wave transformation occurs in the vicinity of a point that satisfies (5.1). Because of the wave transformation the interference pattern on the screen will be modified. Fig. 7 shows the character of the interference pattern which we shall discuss below.

At critical points (5.1) the relationships

$$
\chi_{11}=\chi_{22}, \quad \operatorname{Re} \chi_{12}=0 .
$$

hold.
The simplest scheme of observation of the effect. Fig. 7 shows the simplest experimental setup allowing the observation of the effect. It consists of a rectangular glass parallelepiped with length $2 L$ and the side $a$ of its square base ( $L \gtrdot a)$, two crossed polarisers $P_{1}$ and $P_{2}$, the source of monochromatic light, and a screen. The pairs of opposite side faces ( $A B, C D$ and $B^{\prime} C^{\prime}, A^{\prime} D^{\prime}$ ) can be subjected to distributed loads (of compression) which uniformly increase from the central cross section to the end-walls. We shall denote the current value of the external pressure by $P(z) ; P>0$ on compression in the direction of orientation of the output polaroid (axis $x$ ) and $P<0$ on compression in the perpendicular direction; the maximum value of $|P(z)|$ near the endwalls we denote by $P_{\mathrm{m}}$.


Figure 7. The scheme of observation of the tangent conical refraction; $P_{1}$ and $P_{2}$ are the crossed polaroids, $S$ is the screen. A transparent rectangular parallelepiped of length $L_{\max }$ with a narrow cylindrical channel aligned longitudinally is subjected to external mechanical loads applied to the corners of the parallelepiped (they are shown by arrows). Radial stresses due to a compressed fluid in the channel are not displayed. A cross-like interference pattern of interacting polarisation modes is observed on the screen [31].

Along the axis of the parallelepiped a cylindrical channel is drilled. Its radius $R$ is considerably smaller than the width of end-wall face $a$. A fluid at the pressure $P_{1}$ is pumped into the channel. In the vicinity of that channel the glass might be subjected to axisymmetric deformations.

Clearly, if there are no external and internal stresses, the screen will be dark. The same will be observed if there are only external stresses, i. e. $P_{1}=0$. Lightening due to marginal effects can only be seen in the regions adjacent to the external and internal surfaces.

In absence of external stresses there are no 'critical' points (5.1) at the rays, so the linear wave transformation does not occur. If external loads $\left(P_{\mathrm{m}} \neq 0\right)$ are applied simultaneously with internal pressure $P_{1}$, 'critical' lines form in the midplanes $x=0$ and $y=0$, and close to them mutual transformation of normal waves occurs. At a given distance $r$ from the centre the 'critical' points arise when $P_{\mathrm{m}} \geqslant\left|\sigma_{r r}^{0}(r)-\sigma_{\varphi \varphi}^{0}(r)\right|$, where $\sigma_{r r}^{0}$ and $\sigma_{\varphi \varphi}^{0}$ are the radial and azimuthal eigenvalues of the stress
tensor $\hat{\sigma}^{0}$ corresponding to the internal (radial) stresses. For a given $r$, the 'critical' point is created at one of the end-walls for $P_{\mathrm{m}}=\left|\sigma_{r r}^{0}(r)-\sigma_{\varphi \varphi}^{0}(r)\right|$, and as $P_{\mathrm{m}}$ increases it displaces to the central cross-section of the parallelepiped. Correspondingly, there is a rise in the modifications introduced by the mutual transformation of waves into the interference pattern on the screen.

Under the conditions formulated above, i.e. for $\left|\sigma_{r r}^{0}(r)-\sigma_{\varphi \varphi}^{0}(r)\right| \ll P_{\mathrm{m}}$ 'critical' points form on the ray. As a result, on the screen a dark cross will appear against a uniformly lighted background (see Fig. 7), a cross which is wider, the less the ratio $\left|\sigma_{r r}^{0}(r)-\sigma_{\varphi \varphi}^{0}(r)\right| / P_{\mathrm{m}}$. The QIA formulae from Section 2 enable one to calculate the particular parameters of that cross.

Coefficients of transformation of normal waves for a localised conical refraction. A detailed procedure involving calculations of the coefficients and fields in the framework of QIA is presented in Ref. [31]. Practically, it does not differ from the procedure of calculation described in Section 4, as applied to quasi-transversal propagation of waves in a plasma. In the case of longitudinal localisation of the interaction (the length of interaction $l_{\mathrm{int}}$ is less than the scale $l$ of medium inhomogeneities) the coefficients in the QIA equations can be linearised by the distance from the critical point.

In the case of longitudinal localisation, the coefficient $\eta$ of transformation (by intensity) of incident normal linearly polarised wave into a normal wave of the same type with perpendicular polarisation direction is expressed as [31]:

$$
\begin{equation*}
\eta=1-\exp \left(-\frac{\pi p}{4}\right) \tag{5.2}
\end{equation*}
$$

where

$$
\begin{equation*}
p=2 n_{0}^{7} k v^{-1}\left|\chi_{12}(Q)\right|^{2} \tag{5.3}
\end{equation*}
$$

Here $k=\omega / c, n_{0}$ is the refraction index at the 'critical' point $Q$ and

$$
\begin{equation*}
v=\frac{1}{2} n_{0}^{4}\left|\frac{\partial\left(\chi_{11}-\chi_{22}\right)}{\partial s}\right|_{Q} . \tag{5.4}
\end{equation*}
$$

In particular, for a wave passing through the 'critical' point $Q$, the transformation coefficient $\eta$ is equal to zero for a realvalued tensor $\hat{\varepsilon}$ since in that case $p=0$.

For the longitudinal size $h$ of the transformation region Naĭda [31] found the estimate

$$
\begin{equation*}
l_{\mathrm{int}} \sim n_{0}^{1 / 2} k^{-1 / 2} v^{-1 / 2} \tag{5.5}
\end{equation*}
$$

The smallness of this value compared to $l$ defines the applicability condition for formula (5.3).

If, additionally, the transversal size $\rho_{\perp}$ of the transformation region is small, formula (5.3) can be rewritten as

$$
\begin{equation*}
p=\left.2 n_{0}^{7} k v^{-1}\left[\left|\operatorname{Im} \chi_{12}\right|^{2}+X^{2}\left|\frac{\partial \operatorname{Re} \chi_{12}}{\partial X}\right|^{2}\right]\right|_{Q} \tag{5.6}
\end{equation*}
$$

The value of the impact parameter $|X|$ at which in absence of gyration a half-level of brightness is achieved ( $\eta=1 / 2$ ), can be conveniently taken as the parameter $\rho_{\perp}$ :

$$
\begin{equation*}
\rho_{\perp}=\left(2 \pi^{-1} \ln 2\right)^{1 / 2} n_{0}^{-1 / 4} k^{-1 / 2} v^{1 / 2}\left|\frac{\partial \operatorname{Re} \chi_{12}(Q)}{\partial X}\right|^{-1} \tag{5.7}
\end{equation*}
$$

That this value is small compared with $l$ serves as the condition of applicability of formula (5.6), based on linearised coefficients.

### 5.2 Light propagation in hiral media and inhomogeneous liquid crystals

The photoelastic effect usually leads to weak optical anisotropy in elastic bodies. Liquid crystals are also characterised by a relatively weak anisotropy. Correspondingly, it seems reasonable to use the quasi-isotropic approximation of geometrical optics to describe light waves in liquid crystals and dielectrics subjected to inhomogeneous stresses.

One of the interesting objects for analysis is the light propagation in a hiral medium whose optical axis rotates relatively a ray with a definite spatial period. This object is interesting in two respects.

First, historically it was the first example which demonstrated the inapplicability of the geometrical optics in the form of independent normal waves in a limit of weak anisotropy (Ginzburg, [61]). Second, a dielectric with a uniformly rotating axis admits exact solution either of Maxwell's equations [61-63] or any approximate equations that could be imagined, including the QIA equations. So on this example, the applicability conditions of approximate methods could be verified.

The problems of helical wave propagation in inhomogeneous media are of practical importance for the optics of liquid crystals. As shown in Ref. [64], in inhomogeneous (i. e., with inhomogeneous rotation of optical axes) liquid crystals of cholesteric type a linear interaction of helical waves is possible. That reference also contains the analysis of many other aspects of the question as well as an extensive bibliography.

### 5.3 Light polarisation in deformed single-mode light guides $\dagger$

Factors influencing the polarisation state in light guides. Electromagnetic waves in axisymmetric light guides are characterised by a two-fold polarisation degeneration, similar to transversal waves in an isotropic medium. In real light guides, the polarisation degeneracy is lifted because of many factors, such as technological defects in axial symmetry, anisotropy of the fibre material, artificial defects caused by mechanical stresses (photoelasticity), and also bending and twisting of the light guide.

We shall denote by $h_{1}$ and $h_{2}$ the propagation constants of two polarisation modes appearing after the lifting of the degeneration. If $h_{1}$ and $h_{2}$ differ from each other quite strongly ( $\Delta h=\left|h_{1}-h_{2}\right| \gg 1 / l$, where $l$ is the characteristic scale of the longitudinal inhomogeneity of the waveguide), the normal waves can be regarded as independent. In this case, exciting a wave of a definite type, one may safely suggest that the field polarisation in the light guide will be preserved, which is important for practical applications. In the opposite case ( $\Delta h \lesssim 1 / l$ ), an intense transformation of normal waves occurs. Such a transformation is accompanied by an unstable state of the field in the light guide, which is undesirable in communication systems.

To describe the normal wave transformation in light guides it seems natural to resort to equations of the QIA type which would provide a smooth transition both to the
$\dagger$ The material of this section is prepared by Yu A Kravtsov in a collaboration with A N Pilipetskiĭ.
polarisation degeneration $(\Delta h \rightarrow 0)$ and to independent normal waves $(\Delta h \gg 1 / l)$. The role of the anisotropy parameter $\mu_{1}$ here plays the relative difference in the constants of propagation of two polarisation modes, $\mu_{1} \approx|\Delta h| / h$. Taking into account that the parameter $\mu \sim\left(k_{0} l\right)^{-1} \sim(h l)^{-1}$ preserves its common geometrical sense one can easily verify that the product $\Delta h l$ now plays the role of parameter $\delta=\mu_{1} / \mu$.

The quasi-isotropic approximation takes into account almost all factors responsible for the polarisation state in light guides: bending and twisting of a fibre, weak anisotropy of the material, small deviation from the axial symmetry. Thus, the QIA pretends to a unified consideration of all conceivable polarisation effects in light guides, except for, say, an extremely weak scattering on small inhomogeneities.

Local curvilinear coordinates. Small parameters of the problem. On describing electromagnetic waves in a light guide it is expedient to introduce curvilinear coordinates, with the axis $\zeta$ measured along the axis of the light guide (the axis is introduced as loci of centroids of the permittivity distribution $\varepsilon_{0}=(1 / 3) \operatorname{Sp} \varepsilon_{i k}$ in cross-sections), and variables $\xi$ and $\eta$ measured in the transversal plane, with the axis $\xi$ aligned with the normal and $\eta$ with the binormal to the axial line $\mathbf{r}=\mathbf{r}(\zeta)$. As a result, the radius vector for an arbitrary point $\mathbf{r}$ can be represented as the sum

$$
\begin{equation*}
\mathbf{r}(\xi, \eta, \zeta)=\mathbf{n}(\zeta) \xi+\mathbf{b}(\zeta) \eta+\mathbf{r}(\zeta) \tag{5.8}
\end{equation*}
$$

In each cross-section of the waveguide we select the axisymmetric part, $\varepsilon_{a x}(\rho, \zeta)$, dependent on the distance to the axis of the waveguide. The difference

$$
\begin{equation*}
\gamma=\varepsilon(\xi, \eta, \zeta)-\varepsilon_{a x}(\rho, \zeta), \quad \rho=\left(\xi^{2}+\eta^{2}\right)^{1 / 2} \tag{5.9}
\end{equation*}
$$

will then characterise the departure from axial symmetry. We associate with it the small parameter $\mu_{2}=\max \left(|\gamma| / \varepsilon_{0}\right) \ll 1$, preserving the designation $\mu_{1}$ for the small anisotropy $v_{i k}=\varepsilon_{i k}-\varepsilon_{0} \delta_{i k}$.

In addition to small parameters $\mu, \mu_{1}$, and $\mu_{2}$, in this problem a new parameter, $\mu_{3} \sim a / l \ll 1$, arises that characterises the smallness of the radius $a$ of the light guide core as compared to the scale $l$ of inhomogeneities. As the scale $l$, one may take the curvature radius of the axial line $R=1 / K_{1}$ or its twisting radius $T=1 / K_{2}$, so, in fact, $\mu_{3}^{\prime} \sim a K_{1}$ or $\mu_{3}^{\prime \prime}=a K_{2}$. The smallness of the parameter $\mu_{3}$ insures the absence of abrupt bending and twisting of the light guide. We shall assume that all parameters $\mu, \mu_{1}, \mu_{2}$, and $\mu_{3}$ are of the same order of smallness and departing from this we expand the fields in the light guide.

Maxwell's equations in curvilinear coordinates. In a symmetrical single-mode light guide the electric field is a superposition of two polarised states

$$
\begin{equation*}
\mathbf{E}=\left(\Phi_{1} \mathbf{e}_{1}+\Phi_{2} \mathbf{e}_{2}\right) \exp \left(\mathrm{i} h_{0} \zeta\right), \tag{5.10}
\end{equation*}
$$

where $h_{0}$ is the constant of propagation whereas vector functions $\mathbf{e}_{1}$ and $\mathbf{e}_{2}$ are described, for example, in Refs [65, 66]. These functions have both latitudinal and transversal components with the latter orthogonal to each other: $\mathbf{e}_{1 \perp} \mathbf{e}_{2 \perp}=0$.

In a weakly inhomogeneous and weakly anisotropic light guide a field in zero approximation preserves the unperturbed structure (5.10) but to the first order in small parameter $\mu$ scalar amplitudes $\Phi_{1}$ and $\Phi_{2}$ become variable, $\Phi_{1,2}=\Phi_{1,2}(\zeta)$, and interdependent. The link between them could be found by
writing Maxwell's equations in the curvilinear coordinates $\zeta, \eta$, and $\xi$. To zero order in respect to $\mu$ Maxwell's equations are satisfied by strength of (5.10), and the condition of solvability of the first-order equations yields linked equations for $\Phi_{1}$ and $\Phi_{2}$, analogous to the QIA equations. To simplify the analysis we consider separately the contributions coming from bending, twisting, the anisotropy and asymmetry, although in reality they act simultaneously.

The influence of bending and twisting. With account only for bending and twisting we obtain the equations

$$
\begin{align*}
& \frac{\partial \Phi_{1}}{\partial \xi}=-\mathrm{i} h_{0} K_{1}^{2} a_{\mathrm{ef}}^{2} \Phi_{1}+K_{2} \Phi_{2}, \\
& \frac{\partial \Phi_{2}}{\partial \xi}=-K_{2} \Phi_{1}+\mathrm{i} h_{0} K_{1}^{2} a_{\mathrm{ef}}^{2} \Phi_{2}, \tag{5.11}
\end{align*}
$$

where the value $a_{\mathrm{ef}}$ characterises an effective radius of the light guide.

Terms with the twisting $K_{2}$ in Eqns (5.11) describe Rytov rotations of field vectors in respect to the trihedron $\mathbf{n}, \mathbf{b}$, and $\mathbf{t}$ tied to the axis of the light guide. The terms with $K_{1}$ describe birefringence in a bend waveguide. For $K_{1}=$ const and $K_{2}=0$ (light guide circle) from (5.11) the results follow that were derived in Ref. [67] on the basis of another approach. Corrections to the unperturbed propagation constant $h_{0}$ turn out to be different for two polarisations. They are secondorder in parameter $\mu_{3} \sim K_{1} a$. We have artificially attributed them to the first-order terms in Maxwell's equations, but thus we obtained the birefringence due to the light guide bending. Results that follow from Eqns (5.11) compare reasonably with the results of other works [68-70] in which the role of bending and twisting was analysed but on another methodical basis.

Noteworthy is that the Rytov rotation of the polarisation plane, though studied more than 50 years ago [13], was experimentally verified quite recently with the help of light guides [71-73].

The influence of ellipticity (asymmetry) of the core. Expand the asymmetrical part $\gamma$ of the permittivity (5.9) in a Fourier series in an angular variable defined in the plane of the light guide cross section and insert in the solvability conditions of Maxwell's equations. As a result, the differential equations for $\Phi_{1}$ and $\Phi_{2}$ take the form

$$
\begin{equation*}
\frac{\partial \Phi_{1}}{\partial \zeta}=\frac{\mathrm{i} k_{0}}{2}\left(\delta_{c} \Phi_{1}+\delta_{s} \Phi_{2}\right), \quad \frac{\partial \Phi_{2}}{\partial \zeta}=\frac{\mathrm{i} k_{0}}{2}\left(\delta_{s} \Phi_{1}-\delta_{c} \Phi_{2}\right), \tag{5.12}
\end{equation*}
$$

where values $\delta_{c}$ and $\delta_{s}$ characterise the interaction between two polarisation modes due to axial asymmetry of the light guide. From Eqns (5.12) corrections to the propagation constants of polarisation modes are computed, $\Delta h_{1,2}=$ $\pm\left(\delta_{c}^{2}+\delta_{s}^{2}\right)^{1 / 2}$. They compare with results of calculations which could be found in Refs [74-76]. Of note is that simple formulae for $\delta_{c}$ and $\delta_{s}$ enable calculating easily the corrections to propagation constants for an arbitrary asymmetry, say for a fibre with an elliptical core, or with two cores. It seems to us that such an approach is simpler and clearer than the existing ones.

The influence of anisotropy. One distinguishes between two kinds of anisotropy in fibres: 'frozen' anisotropy formed in the process of fibre production, and 'deformational' anisotropy which occurs due to the action of mechanical stresses, in particular, at bending [77-79].

Accounting for the anisotropy of a light guide material results in equations of (5.12) type. The coefficients entering the equations are given by linear combinations of components of the tensor $v_{i k}$ averaged with squared transversal or longitudinal wave functions, or with the product of these functions.

For the sake of brevity, suffice it to say that the transformation of polarised modes is determined not only by the transversal components of the anisotropy tensor, but also by the longitudinal component, which is capable of coupling the polarisation modes. According to these equations, the propagation constants become changed by $\Delta h \sim k_{0}\left|v_{i k}\right|$.

At simultaneous action of many factors the superposition of all factors mentioned above enters the differential equations for $\Phi_{1}$ and $\Phi_{2}$, since the calculations were carried out to the first order in respect to $\mu_{i}$ with the only exception for coefficients characterising the contribution from bending that are proportional to $\mu_{3}^{2} \sim(k a)^{2}$.

The advantage of the approach presented is that we consider all possible factors together, and thus achieve a unified description of the various effects that lift polarisation degeneracy. One more advantage is the ability to compare the action of different factors that lift the degeneracy and to find the conditions of stable polarisation in the light guide. In essence we return to the condition $\delta \sim \Delta h l \gg 1$ discussed at the beginning of the section. This condition ensures the weakness of interaction between polarisation modes. When an opposite condition, $\delta \leqq 1$, is implemented, mutual transformation of modes renders instable the state of polarisation in the light guide. The reader can find additional practical details on the problem in Refs [81-84].

### 5.4 The optical Magnus effect

Noteworthy is one more interesting effect linked with weak optical anisotropy, namely the optical analogue of the Magnus effect. The essence of the optical Magnus effect is that in an inhomogeneous medium a ray is displaced depending on its polarisation [85]. The rays with a right and left circular polarisation displace in different directions.

The optical Magnus effect provides more grounds to note a close connection between the anisotropy and inhomogeneity of a medium. In particular, we have already pointed out in Section 4.3 that an isotropic medium, possessing a spatial dispersion, attains a weak anisotropy if there is an inhomogeneity. In the case of the optical Magnus effect the anisotropy does not occur, but a weaker effect, the polarisa-tion-dependent displacement of rays, is observed.

According to Ref. [85], the optical Magnus effect modifies the ray equation: instead of the first equation in Eqns (2.10) one should write

$$
\frac{\mathrm{d} \mathbf{r}}{\mathrm{~d} s}=\mathbf{t}-\frac{\sigma}{k_{0} n}\left(\mathbf{t} \ln n_{0}\right),
$$

where t is the tangent to a ray, and the value

$$
\sigma=\operatorname{Im}\left(\frac{2 E_{1}^{*} E_{2}}{\left|E_{1}\right|^{2}+\left|E_{2}\right|^{2}}\right)
$$

characterises the degree of circularity of the field (values $E_{1}$ and $E_{2}$ are the projections of the field on two perpendicular orths): for the right circular polarisation $\sigma=+1$, whereas for the left one $\sigma=-1$. In the intermediate cases $-1<\sigma<+1$.

The ray displacement mentioned could be interpreted as an interaction of the photon spin (its polarisation) with a medium inhomogeneity, i.e. as a sort of a spin-orbital photon interaction in an inhomogeneous medium. In a certain sense the effect is opposite to the Rytov rotation of the polarisation plane, in which twisting of a ray influences the field polarisation: under the optical Magnus effect the polarisation itself influences the ray trajectory.

The optical Magnus effect was observed experimentally in light guides as the speckle-picture displacement (turning) upon replacing the right polarisation by the left one [86].

### 5.5 Polarisation effects in nonlinear optics

Intense electromagnetic waves in nonlinear isotropic media induce various polarisation effects: the self-rotation of polarisation ellipse of an intense light wave [87], the appearance of nonlinear anisotropy [88, 89], the self-induced rotation of the polarisation plane in cubic crystals due to the anisotropy of nonlinear absorption [90].

A wide spectrum of nonlinear polarisation phenomena was discussed in the monograph [91] devoted to optical bistability. They include polarisation multistability, polarisation instability and chaos, depolarisation instability at two-photon absorption, polarisation instability in double refractive media, etc. In this journal, reviews of nonlinear polarisation phenomena have been given by Arakelyan [92], and Zheludev [93]. We can also mention recent Refs [94, 95, 120].

In mentioning these works we would like to draw attention to the fact that precisely the QIA ideas are applied in analyzing the nonlinear polarisation effects. The point is that nonlinear corrections to the tensor of electric permittivity are always considered to be small and are accounted in equations for field amplitudes as perturbations. If nonlinear corrections are of tensor character, the polarisation degeneracy is lifted at intensity increasing. As a result, equations for polarisation modes 'hook' each other and one arrives at a system of nonlinear equations of the QIA. It is this dependence of the anisotropy on amplitudes that leads to the nonlinear polarisation effects mentioned above.

## 6. Acoustics of weakly anisotropic media

### 6.1 Quasi-isotropic approximation of geometrical acoustics

In this section we apply the QIA method to acoustic problems (Naĭda [21, 22]). The QIA equations enable matching of the geometrical acoustics of 3D inhomogeneous isotropic media (Refs [96, 97]) with the Courant - Lax method of independent normal waves (the latter was applied to acoustics of 3D inhomogeneous anisotropic media in Refs [98-107]). The need for such a matching is at present dictated by tasks arising in seismology and ultrasonic nondestructive control.

Deformation waves in an inhomogeneous anisotropic medium are described by the equation [102]

$$
\begin{equation*}
\rho \frac{\partial^{2} u_{\alpha}}{\partial t^{2}}=\frac{\partial \sigma_{\alpha \beta}}{\partial x_{\beta}} \tag{6.1}
\end{equation*}
$$

where $\rho$ is the medium density, $\mathbf{u}=\left(u_{1}, u_{2}, u_{3}\right)$ is the displacement vector, $a_{\alpha \beta \gamma v}$ is the tensor of elastic moduli $\left(a_{\alpha \beta \gamma v}=a_{\gamma v \alpha \beta}\right), \sigma_{\alpha \beta}$ is the tensor of stresses, connected with
the displacements $x_{v}$ by the formula

$$
\begin{equation*}
\sigma_{\alpha \beta}=a_{\alpha \beta \gamma v} \frac{\partial u_{\gamma}}{\partial x_{v}} \tag{6.2}
\end{equation*}
$$

The summation from 1 to 3 over recurrent indices is implied everywhere in this section. In a limiting case of isotropic medium, the tensor $a_{\alpha \beta \gamma v}$ is

$$
\begin{equation*}
a_{\alpha \beta \gamma v}^{0}=\lambda^{\prime} \delta_{\alpha \beta} \delta_{\gamma v}+\mu^{\prime}\left(\delta_{\alpha \gamma} \delta_{\beta v}+\delta_{\alpha v} \delta_{\beta \gamma}\right) \tag{6.3}
\end{equation*}
$$

We shall term a medium as a weakly anisotropic one if the difference in phase velocities of transversal modes, $\Delta v_{\perp} / v_{\perp}$, is small as compared to unity: $\mu_{1} \sim \Delta v_{\perp} / v_{\perp} \ll 1$. For $\mu_{1}=0$ the geometrical acoustics of isotropic media [96, 97] holds true, whereas for $\delta \gg \mu_{1} / \mu \gg 1$ the geometrical optics in the form of independent normal waves should be invoked [98-102].

The quasi-isotropic approach describes elastic waves in the intermediate case and in this manner provides a continuous transition from an anisotropic medium to an isotropic one. In a weakly anisotropic medium the Lamé coefficients $\lambda^{\prime}$ and $\mu^{\prime}$ could be taken so that the isotropic tensor $a_{\alpha \beta \gamma v}^{0}$ formed by them differs but slightly from the original tensor $a_{\alpha \beta \gamma v}$, i. e. that the condition of small anisotropy is met:

$$
\begin{equation*}
\mu_{1} \sim\left(\mu^{\prime}+\lambda^{\prime}\right) \Delta a_{\alpha \beta \gamma v}=\left(\mu^{\prime}+\lambda^{\prime}\right)^{-1} \max _{\alpha, \beta, \gamma, v}\left|a_{\alpha \beta \gamma v}-a_{\alpha \beta \gamma v}^{0}\right| \ll 1 . \tag{6.4}
\end{equation*}
$$

We make use of the eikonal substitution

$$
\begin{equation*}
\mathbf{u}=\mathbf{U} \exp (-\mathrm{i} \omega t+\varphi) \tag{6.5}
\end{equation*}
$$

in Eqn (6.2). Having obtained the equation

$$
\begin{gather*}
{\left[\left(\omega^{2} \rho-\mu^{\prime} p^{2}\right) U_{\alpha}-\left(\lambda^{\prime}+\mu^{\prime}\right)(\mathbf{p} \mathbf{U}) p_{\alpha}\right]} \\
=\Delta a_{\alpha \beta \gamma v} p_{\beta} p_{v} U_{\gamma}-\mathrm{i} X_{\alpha}+\ldots \tag{6.6}
\end{gather*}
$$

for amplitude $\mathbf{U}$, where

$$
\begin{aligned}
\mathbf{X}= & \left(\lambda^{\prime}+\mu^{\prime}\right)(\nabla(\mathbf{p} \mathbf{U})+\mathbf{p} \operatorname{div} \mathbf{U})+\mu^{\prime}(\mathbf{U} \operatorname{div} \mathbf{p}+2(\mathbf{p} \nabla) \mathbf{U}) \\
& +(\mathbf{p} \mathbf{U}) \nabla \lambda^{\prime}+\left(\nabla \mu^{\prime}, \mathbf{p}\right) \mathbf{U}+\left(\nabla \mu^{\prime}, \mathbf{U}\right) \mathbf{p},
\end{aligned}
$$

we recall that the eikonal of transversal waves obeys the equation $\omega^{2} \rho-\mu^{\prime}(\Delta \varphi)^{2}=0$ and multiply Eqn (6.6) successively by the vectors of the normal and binormal to the ray. As a result, we find two equations for components of the vector amplitude $\mathbf{U}$.

The substitution

$$
\mathbf{U}=U_{0}(\rho \mu)^{-1 / 4}\left(Q_{n} \mathbf{n}+Q_{b} \mathbf{b}\right)
$$

where $U_{0}$ obeys the law of energy conservation in a ray tube yields for $Q_{n}$ and $Q_{b}$ the system of QIA equations [21]

$$
\begin{align*}
& \frac{\mathrm{d} Q_{n}}{\mathrm{~d} s}+\frac{1}{2} \mathrm{i} \omega \rho^{1 / 2} \mu^{-3 / 2}\left(\Delta a_{n t n t} Q_{n}+\Delta a_{n t b t} Q_{b}\right)-T^{-1} Q_{b}=0 \\
& \frac{\mathrm{~d} Q_{b}}{\mathrm{~d} s}+\frac{1}{2} \mathrm{i} \omega \rho^{1 / 2} \mu^{-3 / 2}\left(\Delta a_{b t n t} Q_{n}+\Delta a_{b t b t} Q_{b}\right)+T^{-1} Q_{n}=0 \tag{6.7}
\end{align*}
$$

Thus Eqns (6.7) posses the same structure as the QIA equations for electromagnetic waves. This enables one almost without modification to extend on acoustics not only various
methods of calculations involving polarisation effects but also certain methods of diagnosis of weakly anisotropic media, known in electrodynamics. Moreover, certain effects can be generalised on acoustics. For instance, the method of polarisation diagnosis of plasma discussed in Section 4 can be applied almost without changes to the acoustic polarisation diagnosis of the pre-stressed media. For the same reason one may anticipate the existence of an acoustic analogue of tangent conical refraction (Section 5.1).

### 6.2 Geometrical acoustics of an isotropic homogeneously stressed medium

Consider an acoustic wave propagating in an anisotropic, pre-deformed medium which, as is known, is similar to an anisotropic medium with the tensor

$$
a_{\alpha \beta \gamma v}=\left.\frac{\partial^{2} W}{\partial\left(\partial w_{\alpha} / \partial x_{\beta}\right) \partial\left(\partial w_{\gamma} / \partial x_{v}\right)}\right|_{w=u^{0}},
$$

where $W$ is the deformation energy per unit of unperturbed volume, and $\mathbf{u}^{0}(\mathbf{r})$ is the displacement vector describing the preliminary deformation. Provided the initial disturbances are small,

$$
a_{\alpha \beta \gamma v}=a_{\alpha \beta \gamma v}^{0}+\Delta a_{\alpha \beta \gamma v},
$$

where

$$
\begin{align*}
& \Delta a_{\alpha \beta \gamma v}=W_{\alpha \beta \gamma v \varepsilon \theta} \frac{\partial u_{\varepsilon}^{0}}{\partial x_{\theta}}, \\
& W_{\alpha \beta \gamma v \varepsilon}=\left.\frac{\partial^{3} W}{\partial\left(\partial w_{\alpha} / \partial x_{\beta}\right) \partial\left(\partial w_{\gamma} / \partial x_{v}\right) \partial\left(\partial w_{\varepsilon} / \partial x_{\theta}\right)}\right|_{\partial w_{\mu} / \partial x_{\theta}=0} . \tag{6.8}
\end{align*}
$$

In a coordinate system where the $x_{1}$ axis is parallel to the wave vector $\mathbf{k}$ one needs to know only $a_{\alpha 1 \gamma 1}$ and $W_{\alpha 1 \gamma 1 \varepsilon \theta}$. As turns out, there are only six different components $W_{\alpha 1 \gamma 1 v e \theta}$ :

$$
\begin{aligned}
W_{111111} & =C_{1}=6 \mu^{\prime}+3 \lambda^{\prime}+2 A+6 B+2 C \\
& =6 \mu^{\prime}+3 \lambda^{\prime}+4 m+2 l, \\
W_{111122} & =W_{111133}=C_{2}=\lambda^{\prime}+2 B+2 C=\lambda^{\prime}+2 l, \\
W_{212111} & =W_{212122}=W_{112121}=W_{211121}=W_{313111} \\
& =W_{313133}=W_{113131}=W_{311131}=W_{3} \\
& =2 \mu^{\prime}+\lambda^{\prime}+\frac{1}{2} A+B=2 \mu^{\prime}+\lambda^{\prime}+m, \\
W_{212133} & =W_{313122}=W_{4}=\lambda^{\prime}+B=\lambda^{\prime}+m-\frac{1}{2} n, \\
W_{112112} & =W_{211112}=W_{113113}=W_{311113}=W_{5} \\
& =\mu^{\prime}+\frac{1}{2} A+B=\mu^{\prime}+m, \\
W_{213123} & =W_{213132}=W_{312132}=W_{312123}=W_{6} \\
& =\mu^{\prime}+\frac{1}{4} A=\mu^{\prime}+\frac{1}{4} n,
\end{aligned}
$$

where $l, m$, and $n$ are the Murnaghan moduli [104]:

$$
n=A, \quad m=\frac{1}{2} A+B, \quad l=B+C .
$$

Calculating components $\Delta a_{\alpha 1 \gamma 1}$ by formulae (6.8) and substituting them in Eqns (6.7) we obtain the QIA equation for a pre-stressed isotropic medium:

$$
\begin{align*}
& \mathbf{u}=U_{0}\left(Q_{n} \mathbf{n}+Q_{b} \mathbf{b}\right)\left(\rho \mu^{\prime}\right)^{-1 / 4} \exp \{-\mathrm{i} \omega t+\mathrm{i} \varphi \\
& \left.\quad-\frac{1}{2} \omega \int \rho^{1 / 2}\left(\mu^{\prime}\right)^{-3 / 2}\left[C_{3} w_{t t}+\frac{1}{2}\left(C_{3}+C_{4}\right)\left(w_{n n}+w_{b b}\right)\right] \mathrm{d} s\right\} \\
& \frac{\mathrm{d} Q_{n}}{\mathrm{~d} s_{0}}=-\mathrm{i} G Q_{n}+\left(T^{-1}-\mathrm{i} H\right) Q_{b} \\
& \frac{\mathrm{~d} Q_{b}}{\mathrm{~d} s_{0}}=\left(-T^{-1}+\mathrm{i} H\right) Q_{n}+\mathrm{i} G Q_{b} \tag{6.9}
\end{align*}
$$

where

$$
\begin{aligned}
& G=\frac{\omega \rho^{1 / 2}}{2 \mu^{3 / 2}} C_{6}\left(w_{n n}-w_{b b}\right), \quad H=\frac{\omega \rho^{1 / 2}}{2 \mu^{3 / 2}} C_{6}\left(w_{n b}+w_{b n}\right), \\
& w_{n n}=n_{\alpha} n_{\beta} \frac{\partial u_{\alpha}^{0}}{\partial x_{\beta}}, \quad w_{n b}=n_{\alpha} b_{\beta} \frac{\partial u_{\alpha}^{0}}{\partial x_{\beta}}, \\
& w_{b b}=b_{\alpha} b_{\beta} \frac{\partial u_{\alpha}^{0}}{\partial x_{\beta}}, \quad w_{t t}=t_{\alpha} t_{\beta} \frac{\partial u_{\alpha}^{0}}{\partial x_{\beta}} .
\end{aligned}
$$

Special calculations of the elastic wave transformation can be carried out for a homogeneous medium, for an axisymmetric torsion of a homogeneous cylinder, and for several other systems [21, 22]. The theory presented can also be applied in seismic sounding tasks (weak anisotropy of the Earth's crust was discovered quite recently [105, 106]), and in problems of acoustics of liquid crystals [107].

## 7. Quantum mechanical analogues of waves in weakly anisotropic media

### 7.1 The Stern-Gerlach effect as birefringence of spinor wave functions in a magnetic field

In this section we compare the QIA with a semiclassical asymptotics of the Pauli equations for spin $1 / 2$ particles in a magnetic field. The case of $1 / 2$ spin and the Pauli equation (as opposed, for instance, to the Dirac equation) are taken here to avoid complex manipulations and demonstrate the essence of novel features brought by the QIA to this well-investigated problem.

There are two approaches to construct the semiclassical approximation for particles with spin: the approach by Pauli and that by de Broglie. The Pauli approach is based on the assumption that the semiclassical trajectory of a particle is not linked with its magnetic momentum. That is quite similar to the early version of the QIA in electrodynamics: in zero order the QIA gives the trajectory (the ray) that does not depend on the spin (the polarisation). In his time Pauli did not succeed in finding a complete (i.e. with account for polarisation) semiclassical asymptotics. This was first done by Galanin [34]. Later the result was rederived more rigorously by Rubinow and Keller [35].

Another approach was outlined by de Broglie [108] who suggested that trajectories of spin $1 / 2$ particles are associated with the magnetic moment already in zero order of semiclassics. De Broglie intended to supplement semiclassical formulae by Pauli by a term that would depend on the magnetic momentum and external magnetic field. Corresponding additional terms would appear in eikonal equations
and Hamiltonian equations. The latter would imply that particle world lines are dependent on particle spin states. This is exactly what occurs in the Stern-Gerlach effect. However de Broglie did not manage to find the expression for that additional term.

The program outlined by de Broglie was realised ten years later by Schiller [109, 110], yet in a very cumbersome manner. Meanwhile the picture of the effect is immediately clarified if one notes that the Stern-Gerlach effect represents the birefringence of spin $\psi$-function by an external magnetic field. Departing from that viewpoint we may find solutions for the Pauli and Dirac equations that are similar to those suggested by electrodynamics of anisotropic inhomogeneous media.

This approach was realised in a paper by Naĭda and Prudkovskiĭ [23] which we shall follow further. By analogy with the procedure of ray splitting in electrodynamics of anisotropic media we obtain geometric-optical solutions of the Pauli equation that correspond to the split rays in the Stern-Gerlach effect, the goal de Broglie was working towards. The solutions admit as particular cases both the Stern-Gerlach type particle trajectories and the solutions of the Pauli-Galanin-Rubinow - Keller type.

### 7.2 QIA equations for a spinor wave function

Let a beam of nonrelativistic polarised spin $1 / 2$ particles (for example, neutrons) enters an inhomogeneous magnetic field (Fig. 8). Such a beam, generally speaking, splits into two beams of different polarisation. The task is to calculate the polarisations, intensities and phases of these beams.


Figure 8. Toward the calculation of the interference pattern for the beam of neutrons at the point $d$ in the Stern-Gerlach experiment; $F F$ is the initial surface of constant phase. Solid (dashed) lines correspond to trajectories of particles with a spin being parallel (antiparallel) to the magnetic field. $B_{1,2,3,4}$ are the points of local maximum of the magnetic field $|\mathbf{H}|$ at the trajectories of particles; in their vicinities the iterative procedure is performed with an ordinary or extraordinary initial conditions. The arrows near the lines show the directions of particle trajectories while the arrows on the trajectories show the direction of integration associated with the iterative procedure [23].

Noteworthy, neutrons as a subject for illustration are more convenient than Ag or K atoms in ${ }^{2} S_{1 / 2}$ states used in the Stern-Gerlach or Frish-Segre experiments on the change of spin state near the neutral point of the magnetic field, the reason being that in atoms we need to account for the magnetic momentum of the nucleus, which complicates the picture considerably (Rabi [117], Motz and Roze [118]).

The solution of the formulated problem reduces to a construction of the appropriate semiclassical solution of the Pauli equation

$$
\begin{equation*}
\mathrm{i} h \frac{\partial \psi}{\partial t}=\left[-\frac{h^{2}}{2 m} \nabla^{2}-\mu_{\mathrm{n}}(\mathbf{H} \boldsymbol{\sigma})\right] \psi, \tag{7.1}
\end{equation*}
$$

where $m$ is the neutron mass, $\mu_{\mathrm{n}}$ is its magnetic momentum, $\mathbf{H}$ stands for the magnetic field, and $\boldsymbol{\sigma}$ is a vector composed from Pauli matrices.

In order to obtain a semiclassical solution to the Pauli equation (7.1) we first make the eikonal substitution

$$
\begin{equation*}
\psi(x)=A(x) \exp [\mathrm{i} \varphi(x)] \tag{7.2}
\end{equation*}
$$

for the spinor two-component wave function $\psi$ and write the eikonal equation with account for the magnetic moment:

$$
\begin{aligned}
\operatorname{det}\left(-\hbar \omega+\frac{\hbar^{2}}{2 m} \mathbf{k}^{2}\right. & \left.-\mu_{\mathrm{n}} \mathbf{H} \boldsymbol{\sigma}\right) \\
& =\left(-\hbar \omega+\frac{\hbar^{2}}{2 m} \mathbf{k}^{2}\right)^{2}-\mu_{\mathrm{n}}^{2} \mathbf{H}^{2}=0
\end{aligned}
$$

where $\mathbf{k}=\nabla \varphi, \omega=-\partial \varphi / \partial t$. Solving this equation we find

$$
\begin{equation*}
\omega=\frac{\hbar}{2 m} \mathbf{k}^{2} \mp \frac{\mu_{\mathrm{n}}}{\hbar^{2}}|\mathbf{H}| . \tag{7.3}
\end{equation*}
$$

The corresponding equation for particle trajectories has the form

$$
\begin{equation*}
m \frac{\mathrm{~d}^{2} \mathbf{r}}{\mathrm{~d} t^{2}}= \pm \mu_{\mathrm{n}}|\mathbf{H}(\mathbf{r})| \tag{7.4}
\end{equation*}
$$

whence follows the expression for the eikonal $\varphi$ :

$$
\varphi=\varphi\left(x_{\mathrm{in}}\right)+\int_{x_{\mathrm{in}}}^{x}(-\omega \mathrm{d} t+\mathbf{k} \mathrm{d} \mathbf{r}) .
$$

Integration here is carried out along the world line (7.4) and the vector $\mathbf{k}$ is replaced by $\mathbf{k}=m \mathbf{v} / \hbar$. The choice of upper (lower) indices in Eqns (7.3) and (7.4) and in subsequent formulae corresponds to spin oriented by (against) the magnetic field. Hence, there are two different eikonals and two different families of the world lines for particles, as de Broglie suggested [108].

Substituting (7.2) in Eqn (7.1) we find a semiclassical equation for the amplitude $A(x)$

$$
\begin{align*}
(-\hbar \omega & \left.+\frac{\hbar^{2}}{2 m} \mathbf{k}^{2}\right) A+\mathrm{i} \hbar\left(\frac{\partial A}{\partial t}+\frac{\hbar}{m} \mathbf{k} \nabla A+A \frac{\hbar}{2 m} \operatorname{div} \mathbf{k}\right) \\
+\mu_{\mathrm{n}}(\mathbf{H} \boldsymbol{\sigma}) A & =\frac{\hbar^{2}}{2 m} \nabla^{2} A \tag{7.5}
\end{align*}
$$

Omitting the small right-hand-side in this formula, we represent the amplitude $A$ as $A=\gamma(x) a(x)$, where $a$ obeys the semiclassical equation

$$
\begin{equation*}
\frac{\mathrm{d} a}{\mathrm{~d} t}-\mathrm{i} \frac{\mu_{\mathrm{n}}}{\hbar}[\mp|\mathbf{H}|+(\mathbf{H} \boldsymbol{\sigma})] a=0 \tag{7.6}
\end{equation*}
$$

and $\gamma(x)$ satisfies the conservation law

$$
\begin{equation*}
\frac{\partial \gamma^{2}}{\partial t}+\operatorname{div}\left(\mathbf{v} \gamma^{2}\right)=0 \tag{7.7}
\end{equation*}
$$

The eikonal substitution (7.2) is quite analogous to the quasi-isotropic substitution (as applied to a neutral spin $1 / 2$ particle) that was used by Pauli, Galanin, and also by Rubinow and Keller. The conservation law (7.7) was derived by Pauli [33]. Eqn (7.6) is equivalent to the precession equation that was found in the nonrelativistic limit by Galanin [34] and then by Rubinow and Keller [35].

One may check this by introducing a new amplitude

$$
\begin{equation*}
\tilde{a}=a \exp \left( \pm \mathrm{i} \int_{0}^{t} \mu_{\mathrm{n}} \frac{|\mathbf{H}|}{\hbar} \mathrm{d} t\right) \tag{7.8}
\end{equation*}
$$

for which from Eqn (7.6) follows the Pauli equation in a reference frame where the particle moving along the world line is at rest:

$$
\begin{equation*}
\mathrm{i} \hbar \frac{\mathrm{~d} \tilde{a}}{\mathrm{~d} t}=-\mu_{\mathrm{n}}(\mathbf{H} \boldsymbol{\sigma}) \tilde{a} . \tag{7.9}
\end{equation*}
$$

Accordingly, along each of the world lines (7.4) the magnetic momentum $\mathbf{M}=\mu_{\mathrm{n}}\left(a^{*}, \boldsymbol{\sigma} a\right)=\mu_{\mathrm{n}}\left(\tilde{a}^{*}, \boldsymbol{\sigma} \tilde{a}\right)$ vector obeys the ordinary precession equation

$$
\begin{equation*}
\frac{\mathrm{d} \mathbf{M}}{\mathrm{~d} t}=2 \mu_{\mathrm{n}} \hbar^{-1}[\mathbf{M H}] . \tag{7.10}
\end{equation*}
$$

Combining formulae (7.2) and (7.8) gives the final eikonal substitution

$$
\begin{equation*}
\psi(x)=\gamma(x) \tilde{a}(x) \exp \left\{\mathrm{i} \int\left[\left(\omega \mp \hbar^{-1} \mu_{\mathrm{n}}|\mathbf{H}|\right) \mathrm{d} t+\mathbf{k} \mathrm{d} \mathbf{r}\right]\right\}, \tag{7.11}
\end{equation*}
$$

which would solve the problem in the de Broglie statement. As one may readily see, formula (7.11) refers to the spin direction being parallel or antiparallel to the current vector $\mathbf{H}$, as it is realised in the Stern-Gerlach effect.

Thus, the QIA asymptotics presented above involves Eqns (7.4) for the world lines, the eikonal substitution (7.11) and Eqns (7.9) for spin, and differs from the Pauli-Galanin method in following aspects:
(a) particles belong to different types of trajectories which correspond to different particle spin orientations with respect to an external magnetic field;
(b) phases of $\psi$-functions contain corrections linked to the magnetic momenta.

### 7.3 Approximation of deformed normal waves

The second derivative of amplitude $A$ entering the right-hand side of Eqn (7.5) is on the order

$$
\begin{equation*}
\left|\nabla^{2} A\right| \sim|A|\left(\frac{\mu_{\mathrm{n}}|\mathbf{H}|}{h v}\right)^{2}+\ldots \tag{7.12}
\end{equation*}
$$

and is related to oscillations in $a$ and $A$ with the Larmour frequency. Corresponding error $\delta a$ associated with solutions of Eqn (7.6) (in respect to that of Eqn (7.5)) will grow along the ray:

$$
\begin{equation*}
\frac{|\delta a|}{|a|} \sim \frac{h}{m} \int\left(\frac{\mu_{\mathrm{n}}|\mathbf{H}|}{\hbar v}\right)^{2} \mathrm{~d} t . \tag{7.13}
\end{equation*}
$$

In those cases when error (7.13) becomes inappropriate, calculations of the beam can be carried out directly by Eqn (7.6), with arbitrary upper and lower signs in Eqns (7.6), (7.3), and (7.4), and by applying a given initial condition for the wave function $\psi_{\text {in }}$ directly to Eqn (7.6). The trajectory splitting in this case, in fact, does not occur since it is not discernible against the background of errors evaluated by Eqn (7.13).

Growing error (7.13) can be eliminated if we formulate the initial condition for Eqn (7.6) in the region of strong magnetic
field which requires the equality

$$
\begin{equation*}
[(\mathbf{H} \boldsymbol{\sigma}) \mp|\mathbf{H}|] a=0 \quad \text { for } \quad|\mathbf{H}|=\max \tag{7.14}
\end{equation*}
$$

to be implemented. In this case the amplitude $a(x)$ in the region of a strong magnetic field will vary smoothly (without oscillations) and instead of estimate (7.12) we find

$$
\left|\nabla^{2} A\right| \sim l^{-2}|A| .
$$

Then errors accumulated in the region of strong magnetic field will not exceed $\mu$. On the other hand, in the region of weak magnetic field the right-hand side of Eqn.(7.5) also is small according to estimate (7.12). Therefore, taking the initial condition to Eqn (7.6) by formula (7.12) provides favourable double asymptotes both on the side of strong magnetic field and weak magnetic field. Naturally, this also promises a high total accuracy satisfying the estimate

$$
\begin{equation*}
\frac{|\delta a|}{|a|} \sim \lambda l_{\mathrm{b}}^{-1}, \quad \lambda=\frac{\hbar}{m v} . \tag{7.15}
\end{equation*}
$$

In the given case, $l_{\mathrm{b}}$ coincides by an order of value with the particle run length for a Larmour period $l \sim v h / \mu_{\mathrm{n}}|\mathbf{H}|=v T_{\mathrm{L}}$.

As in the case of electromagnetic waves, to get the highest accuracy (7.15) we should ascertain the initial condition (7.14) with the help of iterative procedure

$$
\begin{equation*}
a=\left(C_{(0)}^{1}+C_{(1)}^{1}+C_{(2)}^{1}+\ldots\right) a_{\mp}+\left(C_{(1)}^{2}+C_{(2)}^{2}+\ldots\right) a_{ \pm} \tag{7.16}
\end{equation*}
$$

where amplitudes of polarisation modes $a_{+}$and $a_{-}$satisfy the conditions

$$
[\mathbf{H} \boldsymbol{\sigma} \mp|\mathbf{H}|] a_{\mp}=0, \quad a_{+}^{*} a_{+}+a_{-}^{*} a_{-}=1, \quad a_{+}^{*} a_{-}=0 .
$$

Coefficient $C_{(0)}^{1}$ can be found from the equation

$$
\begin{equation*}
\frac{\mathrm{d} C_{(0)}^{1}}{\mathrm{~d} t}+a_{\mp}^{*} \frac{\mathrm{~d} a_{\mp}}{\mathrm{d} t} C_{(0)}^{1}=0,\left.\quad C_{0}^{1}\right|_{t=t_{\mathrm{in}}}=1 \tag{7.17}
\end{equation*}
$$

and the other coefficients $C_{(n)}^{1}, C_{(n)}^{2}$ can be found from the recurrent formulae

$$
\begin{align*}
& C_{(n)}^{2}= \pm \frac{\mathrm{i} h}{2 \mu_{\mathrm{n}}|\mathbf{H}|}\left[\frac{\mathrm{d} C_{(n-1)}^{2}}{\mathrm{~d} t}+\left(a_{ \pm}^{*} \frac{\mathrm{~d} a_{\mp}}{\mathrm{d} t}\right) C_{(n-1)}^{1}\right. \\
&\left.+\left(a_{ \pm}^{*} \frac{\mathrm{~d} a_{ \pm}}{\mathrm{d} t}\right) C_{(n-1)}^{2}\right], \\
& \frac{\mathrm{d} C_{(n)}^{1}}{\mathrm{~d} t}+a_{\mp}^{*} \frac{\mathrm{~d} a_{\mp}}{\mathrm{d} t} C_{(n)}^{1}=-a_{\mp}^{*} \frac{\mathrm{~d} a_{ \pm}}{\mathrm{d} t} C_{(n)}^{2} . \tag{7.18}
\end{align*}
$$

It is convenient to set initial values $C_{(n)}^{1}\left(t_{\text {in }}\right)$ in the last equation to zero.

Series (7.16) represents an asymptotic expansion of a solution to (7.6) in a region of strong field. The convergence condition for series (7.16) is expressed by inequality $l \gg v h / \mu_{\mathrm{n}}|\mathbf{H}|$, so that the value $l_{\mathrm{b}} \sim v h / \mu_{\mathrm{n}}|\mathbf{H}|$ is the marginal one for this inequality. The $\psi$-functions constructed in that manner are analogous to deformed normal waves in electrodynamics (see Section 3).

Striving to achieve the upmost accuracy (7.15) we must apply the procedure of trajectory splitting whose essence is
illustrated by Fig. 8. Through the points of the initial front we draw by two rays corresponding to two types of polarisation. Each of them is constructed by Eqn (7.4) with a respective sign. At the initial points the trajectories have to be perpendicular to the constant phase surface. If each of trajectories possesses a single maximum of $\mathbf{H}$ (as it is assumed in Fig. 8), then, in the vicinity of each trajectory it is necessary to make use of the deformed normal wave approximation and the iterative procedure with a seed (7.16) - (7.18) and with the number of steps

$$
\begin{equation*}
N \approx \frac{\ln \mu}{\ln \left[v \hbar\left(\mu_{\mathrm{n}}|\mathbf{H}| l\right)^{-1}\right]}, \tag{7.19}
\end{equation*}
$$

where $\mu$ is the required accuracy. After that, with the result of iterations as an initial condition one should solve Eqns (7.6), each at its own trajectory and with its own initial condition, departing from the initial point to both sides. We shall denote the solutions that ensue by $b^{ \pm}$. Matching spinors $b^{ \pm}$on the initial front with the initial condition one may find the amplitudes $b^{+}$and $b^{-}$for each of rays. The polarisation modes, constructed in that way,

$$
\begin{equation*}
\psi^{ \pm}(x)=b^{ \pm}(x) \gamma_{ \pm}(x) \exp \left\{\mathrm{i} \int_{x_{\mathrm{in}}}^{x}\left(-\omega_{ \pm} \mathrm{d} t_{ \pm}+\mathbf{k}_{ \pm} \mathrm{d} \mathbf{r}_{ \pm}\right)\right\} \tag{7.20}
\end{equation*}
$$

form the sought-for approximate solution

$$
\begin{equation*}
\psi(x)=\psi^{+}(x)+\psi^{-}(x) . \tag{7.21}
\end{equation*}
$$

Being interested in the interference pattern of the SternGerlach components of the wave, for each type of polarisation one has to draw a single trajectory from the initial front to the point $d$ considered (Fig. 8) and then to implement the procedure described above of construction of the corresponding solutions $\psi^{+}$or $\psi^{-}$by Eqns (7.16) - (7.20) at each of the trajectories. It is this procedure that yields the desired mathematical model of the Stern-Gerlach effect.

If there are more than one local maximum of the value $|\mathbf{H}|$ (as is the case in the Frish-Segre experiment [116]), the procedure described above of splitting the trajectories and matching the solutions needs to be carried out not only at the initial point, but also at each point of local minimum of the value $|\mathbf{H}|$. Accordingly, a seed iteration should be constructed in the vicinity of each local maxima of $|\mathbf{H}|$, and the summation in Eqn (7.21) should be done not over double, but over multiple (quadruple, octuple, etc.) branching of the rays. As one may see, the procedure described is quite similar to the corresponding procedure in electrodynamics of birefringent media (Section 3.3).

For ray splitting we should remember that the intensity of the beams splitting off does not exceed $\exp \left[-v h /\left(\mu_{\mathrm{n}}|\mathbf{H}| l\right)\right]$, and if this factor is small, some splitting could be ignored.

Clearly, the QIA can also be applied to describe polarised beams of atoms and ions in a magnetic field - of course, with account for the magnetic momentum of nucleus. There are no objections for applying the method to the squared Dirac equations.

## 8. Conclusion

When initiating the work on this review the authors did not realise all the applications of the quasi-isotropic approach. It applies to vector field of arbitrary physical nature: electro-
magnetic, elastic, spinor, etc. Most widely the method is used in problems of plasma physics (as applied to a plasma of interplanetary, interstellar, solar, ionospheric, laboratory, or other origin) and in optics of liquid crystals. A novel field for the quasi-isotropic approximation is formed by polarisation phenomena in light guides.

The phenomena of tangent conical refraction in optics and in the theory of elastic waves thus far are of theoretical interest, however a practical interest to arbitrarily deformed optical and acoustical media is well expressed in recent years.

Although we tried to collect together all material related to the problem discussed, a number of interesting questions have still not received our attention.

First, we have confined ourselves to mentioning only a few mechanisms of wave transformation in a plasma, in particular, in the solar corona, however we were to put aside the physics itself of the sources of polarised radiation. Fortunately, we can refer the reader to Ref. [5] and to an excellent review article by Zheleznyakov, V Kocharovskiĭ and V1 Kocharovskiĭ [8] where the problem of sources is studied in detail.

Second, we have restricted ourselves to presenting only fragmentary information on optics of inhomogeneous liquid crystals and have not mentioned the transition of microwaves through weakly anisotropic artificial dielectrics, vegetable cover [119], and other weakly anisotropic objects. Finally, the questions of wave transformation in deformed dielectric and metallic waveguides of square and cylindrical crosssection characterised by the polarisation degeneration were left beyond the scope of this review.

We believe that the concept of quasi-isotropic approximation and particular results obtained with the help of QIA and presented in the review are of interdisciplinary nature and will be useful to a wide audience.

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