

REVIEWS OF TOPICAL PROBLEMS

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Singularities of continuation of wave fields

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Abstract. **The state of the art of the problem of the analytical continuation of wave fields is reviewed. The problem is a multidisciplinary subject which involves radiophysics, acoustics, and optics on the one hand, and mathematical physics and the theory of differential equation, on the other. The qualitative aspects of the problem are examined. A computational algorithm for field singularities is given. The relation between the singularities and the 'computational catastrophes' of scattering models is discussed. To illustrate the theoretical material, numerous examples are discussed in great detail.**

1. Introduction

The paper is devoted to the problem of continuation of wave (electromagnetic) fields to the so-called 'nonphysical' domain of the space, and to the main statements of the theory of such continuations.

At present this topic is seriously investigated in the worldwide (mainly mathematical) literature. Here one can find original papers as well as reviews. On the opposite, there is a serious gap in this theme in physical literature.

First, we shall give a brief description of the problem dealt with in this paper.

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Let us try to explain what is the continuation of the wave field and where it can be used. To answer this question, we imagine the following situation. Suppose that we are looking at the image of some object in an ideal plane mirror. What happens? Everybody knows that we do not see the mirror itself. More precisely, we can see the defects on the surface of the mirror, dust, etc. (all this is absent for the ideal mirror). We see the images of all objects positioned before the mirror (in the 'physical' region), and these images appear to be placed behind this mirror (in the 'nonphysical' region of the space). The reason is that our brain is analyzing information about the electromagnetic field (light rays) obtained through our eyes, and reconstructs objects which could originate this electromagnetic field being placed behind the mirror. In other words, *if we place the real objects identical to the images we see in the behind-the-mirror space, and then remove the mirror, the field in the physical region remains unchanged.* In fact, our brain unconsciously carries out the procedure of continuation of the wave field to the nonphysical domain and reconstructs there 'models' of objects originating the field obtained by our eyes in the physical region.

This observation was a starting point for the series of fruitful ideas, in particular, for the method of mirror images. This method, previously used for plane reflecting surfaces, turned out to be so simple and effective that there arise a natural desire to generalize it to the case of non-plane reflecting surfaces. To do this, let us continue our experiment curving steadily the reflecting surface. It is evident that the images behind the mirror will be changed; the impression arises that the behind-the-mirror space begins to be deformed as a continuous media. In this process the part of the image (or even the image as a whole) can disappear (this effect is well-known for any person who looked into the curved mirror — let us remember the laugh room). This effect can be associated with appearance of 'folds' in the hypothetical behind-the-mirror media, that is, such regions where two (or more) images exist at the same time. An image can disappear being 'hidden to such a fold'.

Let us consider the appearance of such a fold on a simple example (for simplicity the deformation in this example is *not smooth*; however, all main effects are the same for smooth deformations as well). Let us deform the mirror by cutting it into two parts at a point O and turning these parts into the 'behind-the-mirror' space (see Fig. 1).

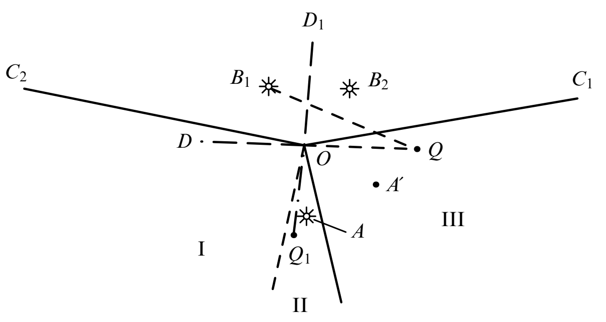

Figure 1.

Suppose that a source of light (a lamp) is placed at some point A in the physical region on the bisectrix of the angle between continuations of the mirror lines OC_1 and OC_2. In accordance with the geometrical optics laws one can construct two images B_1 and B_2 of this source with respect to the mirrors OC_1 and OC_2, respectively. From the point Q placed in region III, the observer will see the image B_1 only (at the point B_2 he will see the object placed at the point A' of the physical space). However, being replaced to region I, the observer will see the image of our source of light at the point B_2. So, we observe that the continuation of the wave field to the nonphysical region through the left half of the mirror is different from that through the right part. There exist at least *two* continuations of the wave field in the behind-the-mirror space and the observer sees one of them depending on the position in the physical space. Therefore, it can be stated that, say, at the point B_2 there exist two images (at one and the same time!). We shall assume that these two images are placed on different 'sheets' of a kind of a Riemannian surface which is mapped onto the plane with folds†. The observer can see the objects placed on one of the sheets, according to his position, which are reachable from the observation point along a direct ray. The point O appears to be a ramification point of this Riemannian surface; the observer cuts (unconsciously) this surface along the ray OD originated from the point O along the direction of the segment QO.

When the observer is moving from the point Q to the region II across the boundary between regions II and III, he fails to see the image of the light source at the moment he intersects this boundary; the image will be 'hidden in the fold'. However, up to the moment of intersection of the boundary between regions II and I, he will not see the image B_2 as well. Can one conclude that the field reflected from the source will be absent in the region II at all? Clearly, such a conclusion (being valid from the viewpoint of the 'pure' geometrical optics) is not true. So, what the observer will see from, say, point Q_1 of the region II? Making the cut of our Riemannian surface along the ray OD_1, the observer reconstructs the field having a singularity on the cut (he sees the field existing on different sheets of the Riemannian surface on each side of the cut). Hence, each point of the cut begins to 'emanate'; all this radiation will come to the observation point Q_1 along the ray OQ_1, and the observer will see the cut OD_1 as a bright point O. Clearly, the same situation takes place in the regions I and III (the cut exists for observation point in these regions as well).

So, what are the conclusions we can make on the basis of the example considered?

First, using the method of mirror images one obtains *ramifying* continuation of the wave field (the careful reader could notice that for smaller values of the angle between OC_1 and OC_2 the images B_1 and B_2 can be placed even over physical region of the space but on the *nonphysical* sheet of the Riemannian surface of the field continued).

Second, when using this method one should 'correct' its initial version adding some *integrals over cuts* to the field induced by the images seen from the observation point.

In the case of *smooth* deformation of the mirror the situation is similar, but the ramification points (folds) are located not on the mirror surface but somewhere in the behind-the-mirror space. For example, if the deformed mirror has a parabolic form (see Fig. 2), the ramification point O coincides with the focal point of the parabola.

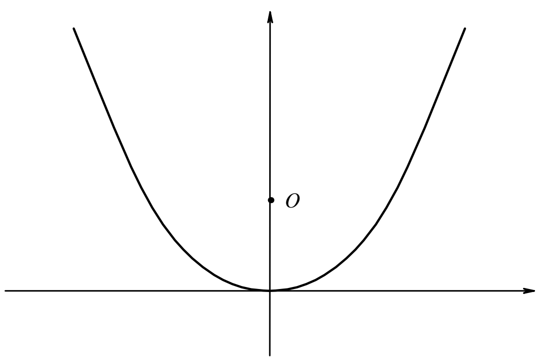

Figure 2.

The examples above show that whereas the field continuation and its ramification points are of *objective* nature, the cuts are *subjective* (for example, they depend on the position of the observation point). These cuts are chosen (unconsciously or not) by the observer, and serve him to choose a *single-valued* branch from *multiple-valued* continuation. The choice of a system of cuts is determined by the intention (which again can be unconscious) to reconstruct the real field in some subdomain of the physical region with the help of 'images' of objects placed on the nonphysical part of the Riemannian surface of continuation.

So, when modeling the field by the mirror image method for non-plain reflecting surfaces, the system of cuts can be chosen more or less arbitrarily. Clearly, if our aim is to model the field in the *whole* physical region, the cuts are to be done *only* in the non-physical regions (we recall that 'non-physical region' can include 'non-physical sheets' of the continuation laying over the physical region of the space). There are no other fixation of the set of cuts, and its choice can be governed by considerations of convenience, symmetry, etc.

† This situation is quite similar to that for the analytic continuation of functions of a complex variable which leads, as is known, to the notion of ramifying analytic function. Therefore, we use here the terminology of the theory of analytic functions. The reader will see below that the connection between the theory of wave fields continuation and the theory of complex-analytic functions is not only terminological.

From the mathematical viewpoint the idea of continuation of the wave field is also quite natural. The matter is that the wave field, being a real-analytic function of the spatial variables, has not more that one continuation to the 'non-physical' region. This continuation is, as a rule, a ramifying real-analytic function. Under some regularity conditions (the analyticity of the boundary data and the boundary itself), such continuation exists everywhere except for the set of singularities lying in the non-physical region and having in generic position the dimension by two units less than that of the space. The continuation problem is, hence, reduced to:

— formulation of the regularity conditions for existence of the continuation of the initial wave field everywhere except for a (real-analytic) set of singularities, which is the theoretical aspect of the problem;

— localization of singularities of the continued field, which is the computational aspect of the problem.

Why is the above problem important and why is so much attention now paid to it? The matter is that the above-described generalization of the method of mirror images to non-plain reflecting surfaces led to a series of methods for solving boundary value problems for harmonic oscillations in electrodynamics, acoustics, optics, quantum scattering theory, elasticity theory and others. *Computational algorithms* associated with these methods are essentially based on information about analytic properties of solutions, in particular, about location of singularities of the continuation of the field to the non-physical domain. Such information can serve as the initial point for constructing *qualitative models* of different kind in problems of wave scattering or potential theory. Finally, information about analytic properties of the wave field is crucial for investigation of *inverse scattering problems* and antennas. For example, it is well-known that only entire functions of finite power can serve as diagrams for antennas with plain radiating opening. The function class of functions which can serve as scattering diagrams or radiation diagrams for antennas with non-plain radiating openings can be strictly described only with the help of information about the continuation of the wave field to the non-physical region of the space.

We outline the contents of the paper.

Section 2 written by B Sternin and V Shatalov contains the consideration of the mathematical theory of continuation of electromagnetic wave fields. Here two types of continuation problems are discussed: the sweeping of current and boundary value problems of electrodynamics.

The problems of the first type (the statement of such problems for the static case goes back to classical works of Poincare, Gerglotz, and Schwarz) involve the continuation of a wave field originated by a known system of currents inside the domain initially occupied with these currents. Such problems arise, for example, in construction of antennas of minimal size emanating given fields.

The problems of the second type are, in essence, the problems of mathematical diffraction theory. Here we investigate the continuation of an electromagnetic field obtained by scattering of a wave on obstacles into the domain initially occupied by these obstacles. As mentioned above, such problems arise in consideration of computational algorithms of diffraction theory, in investigations of the radiation diagrams of antennas, etc.

The investigation of the continuation of wave fields in Section 2 is based on the theory of differential equations in complex domains worked out by B Sternin and V Shatalov [1]. With the help of this theory, one can obtain explicit formulas allowing to *localize singularities* of the continuation of wave fields of the types described above.

One of the most remarkable facts here is the *reflection formula* for the Helmholz equation. This formula, being a generalization of the Schwarz symmetry principle for harmonic functions, differs essentially from the latter by its nonlocal character. It allows one not only to determine the location of singularities but also to *synthesize* in the explicit form the currents originating a given wave field.

We also consider the problems of both above types for domains bounded by *piecewise-analytic boundaries*. We investigate singularities of wave fields arising at angle points of such domains, as well as singularities obtained by 'multiple reflections' of a wave field from different parts of the boundary of such a domain.

Section 3 written by A Kyurkchan expounds on analytic representations of wave fields used for solving boundary value problems of refraction theory. They include representations in the form of wave potentials, expansions in series and integrals in plane waves (the Rayleigh series and Sommerfeld – Weil integrals), expansions in series in cylindrical and spherical harmonics, and the Wilcox series (in inverse powers of distance). It is shown that the boundaries of applicability of such representations are determined by the *geometry of the set of singularities* associated with the continuation of the wave field. The exact bounds of existence domains are established for all listed representations. The connection between the asymptotics of the diagram of the wave field near infinite point on the complex plane of the observation angle and the singularities of continuation of the wave field are established.

Here we also consider some *computational methods* for boundary value problems of scattering theory, known in the literature as the Waterman method (or zero-field method), the MMM (Meisel – Merril – Mille) method, the method of T-matrices, method of auxiliary currents and discrete sources, and the method of diagram equations which reduces the boundary value problem to an equation for scattering diagram, or to a system of equations for coefficients of expansion of the diagram in some basis. The reasons underlying instabilities of these methods and their connection with analytic properties of solutions are discussed. The conditions of existence of solutions to inverse problems of scattering theory and antennas are established. Examples are presented.

The list of references to this paper contains both classical papers and the most recent achievements in this field of knowledge.

2. Mathematical theory of continuation of wave fields

Here we present a method of continuation of wave fields outside their initial domain of definition and discuss the localization of singularities of the resultant continuation. This method is based on the complex theory of differential equations worked out in the series of works by B Sternin and V Shatalov (see Ref. [1] and the bibliography therein) in recent years. The exposition will be carried out for solutions of the Helmholz equation

$$(\Delta + k^2)\, u(x) = 0 \tag{1}$$

in $\mathbf{R}^m$ ($m = 2$ or $m = 3$), though the methods to be presented are applicable in a much more general situation.

2.1 Statement of problem and review of main methods

Let us first formulate the mathematical statement of the continuation problem. Let $D \subset \mathbf{R}^m$ be some domain. The domain D can be viewed as the domain occupied by a scatterer, antenna, etc. Let a solution $u(x)$ to Eqn (1) be given in the complement $\mathbf{R}^m \backslash D$ of the domain D. Suppose that the boundary Γ of D is a closed analytic surface (curve) in $\mathbf{R}^m$; later on we restrict ourselves to the case when the equation of the boundary Γ is given by a polynomial. The statement of the continuation problem is as follows:

Investigate the possibility of continuation of the function $u(x)$ as a solution to the Helmholz equation (1) *to inner points of the domain D and localize the singularities of such continuation* (*that is, determine their position*).

Before solving this problem, let us make some remarks.

First, we note that this solution is *unique* if the continuation problem is solvable. This follows from the real analyticity of solutions to the Helmholz equation (see, e.g. Refs [4, 5]) and from the uniqueness theorem for analytic functions.

Second, the problem in hand can have no solution at all. For example, the solution of the exterior Dirichlet problem for Eqn (1)

$$(\Delta + k^2)\, u(x) = 0\,,$$
$$u(x)\Big|_{\Gamma} = \varphi(x)$$

with non-analytic data $\varphi(x)$ cannot be continued through the boundary Γ at any its point (in the opposite case the function $\varphi(x)$ is analytic as a trace of analytic function $u(x)$ on the analytic surface Γ). Hence, it is necessary to add some constraints in the statement of the problem in order to guarantee the existence of the continuation of the solution to inner points of the domain D.

Third, the analysis shows that the continuation of $u(x)$ inside the domain D is, in general, a *ramifying* function having singularities at some points of D. Hence, for separating a single-valued solution one has to introduce a system of cuts on which the obtained continuation will have jumps (discontinuities of first kind).

There exist at least two statements of the continuation problem for which the solvability can be guaranteed. These are, first, the continuation of solutions to exterior boundary value problems with analytic input data and, second, the continuation of wave fields induced by a given (analytic) current distribution in D inside the domain initially occupied by currents as a solution to a homogeneous equation. The first problem arises, for example, when considering the diffraction of an electromagnetic wave on a scatterer, the second, called 'sweeping of charge' in the case of static fields, when considering the optimization of the size of the antenna originating a given electromagnetic field.

Let us briefly review methods used for solving continuation problems. It seems that the first method used for solving problems of this kind was that based on the *Schwarz symmetry principle* (see Refs [6–9]). This method (in its initial formulation) was applicable for the continuation of *static* fields. It is based on the fact that any harmonic function vanishing on an analytic curve Γ satisfies the relation

$$u(x, y) + u\big(R(x, y)\big) = 0\,, \tag{2}$$

where $(x, y) \to R(x, y)$ is the anticonformal mapping defined in a neighborhood of the curve Γ, which interchanges the parts U_1 and U_2 of this neighborhood with mutual boundary Γ. We remark that the mapping R depends only on the curve Γ. The modification of Eqn (2) to the Neumann conditions [when the normal derivative $\partial u / \partial n$ of the considered harmonic function $u(x)$ vanishes on Γ] has the form

$$u(x, y) - u\big(R(x, y)\big) = 0\,. \tag{3}$$

However, later it was found out that the symmetry principle in the form (2), (3) *is not valid* (even in two dimensions) for the Helmholz equation (1); in three dimensions, even for the Laplace equation this principle is valid only if the surface Γ is a plane or a sphere (see Ref. [10]). The reason is that the connection between values of function $u(x)$ [which satisfies Eqn (1)] in domains U_1 and U_2 has *not a pointwise* [as in (2), (3)] but *integral character*. The corresponding *reflection formula* will be discussed later.

The further progress in the investigation of continuation problems was due to works by I Vekua [11] and H Lewy [12]. In these papers the continuation problem is investigated in the two-dimensional case, and the investigation method is based on the Riemann method of solving the Cauchy problem. These authors clarify the fact that to investigate the continuation problem in the *real* space it is natural to extend the equation considered to the *complex* domain. The reason for this extension is that the Helmholz equation (1) (as well as the simpler Laplace equation) has *complex* characteristics, along which the singularities of solutions to these equations propagate. So, to solve the continuation problem it is natural to use the following scheme (see Refs [13–15]):

(a) reduce the problem to some problem in the *complex* domain;

(b) investigate the singularities of solutions to the complex problem obtained;

(c) obtain the singularities of the initial *real* problem as the 'trace' of the complex singularities for real values of the variables.

We also mention here works by P Garabedian [16, 17], who investigated the character of reflection formulas in the multidimensional case. However, in these works the reader will find no explicit reflection formulas applicable to the solution of the problem of localization of singularities.

In recent years the methods based on the integral equation technique have spread widely. This technique was used by many authors; we mention here papers by R Millar [18, 19] (for a more detailed presentation see the review [14] and the bibliography therein, as well as recent papers [20, 21] by the authors). We should also mention a series of works by D Khavinson and H Shapiro (see also the review [14]), who used the method of unitary solution to the equation in question.

Finally, the method of integral transforms of complex-analytic functions worked out by B Sternin and V Shatalov (see Refs [1, 22–24]) gives an opportunity to investigate the continuation problems in a general situation. The discussion of this method is presented below.

2.2 Reduction to the complex Cauchy problem

Here we illustrate the application of the methods of complex theory of differential equations on the example of the 'sweeping-of-charge' problem (see Refs [1, 13–15, 25]) (possibly, for the Helmholz equation it would be better to call this problem 'sweeping of sources' or 'sweeping of currents', but this terminology is not in custom). The consideration of methods of the continuation of solutions to boundary value problems will be carried out in Section 2.6 for the case of a piecewise-analytic boundary.

For any (generalized) function $f(x)$ we denote by U^f the solution to the following problem:

$$(\Delta + k^2)U^f(x) = f(x)\,, \tag{4}$$

which satisfies the radiation conditions at infinity. To begin with, let the function $f(x)$ coincide in D with some function possessing an analytic continuation to all complex values of x [this function will also be denoted by $f(x)$], and vanishing outside D. The solution $u(x)$ to the problem (4) can be viewed as an electromagnetic field induced by the current distribution $f(x)$ in the domain D (the antenna). The sweeping charge problem has in this case the following formulation: *find a* (*generalized*) *function* $w(x)$ *with as small support* (*lying inside the domain* D) *as possible, such that the solution* U^w *to the equation*

$$(\Delta + k^2)U^w(x) = w(x)$$

coincides with U^f *outside the domain* D.

So, one should find a current distribution $w(x)$ which occupies less volume compared with the initial one but originates the same electromagnetic field outside D.

To construct the function $w(x)$ let us consider the difference $u(x) = U^f(x) - U^w(x)$. Since the support supp w of the function $w(x)$ is lying inside D (Fig. 3), the function $u(x)$ satisfies the equation

$$(\Delta + k^2)\,u(x) = f(x)\,. \tag{5}$$

in some neighborhood of the boundary Γ. Besides, since $U^f(x) = U^w(x)$ outside D, the function $u(x)$ vanishes identically outside D.

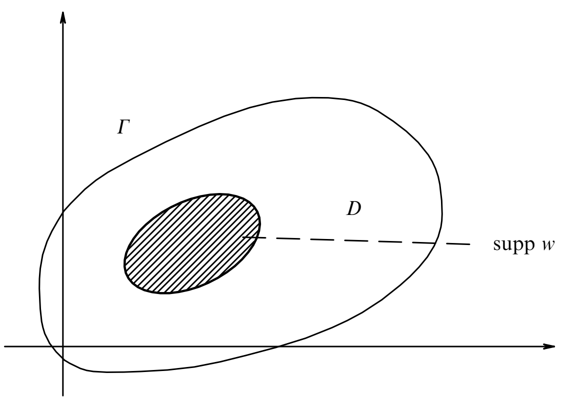

Figure 3.

The inclusion $f(x) \in L_2(\mathbf{R}^m)$ ($L_2(\mathbf{R}^m)$ is the space of functions which are square-integrable in $\mathbf{R}^m$) together with Eqn (5) shows that $u(x)$ is continuous on the boundary Γ with its first derivatives. Therefore, we have

$$u\Big|_\Gamma = \frac{\partial u}{\partial n}\Big|_\Gamma = 0\,.$$

So, the function $u(x)$ satisfies the following Cauchy problem

$$(\Delta + k^2)\,u(x) = f(x)\,,$$

$$u\Big|_\Gamma = \frac{\partial u}{\partial n}\Big|_\Gamma = 0\,. \tag{6}$$

in the domain $D\backslash\operatorname{supp} w$. Conversely, if $u(x)$ is a solution to the problem (6), we can put

$$U^w(x) = U^f(x) - u(x) \quad \text{in} \quad D\backslash\operatorname{supp} w\,,$$
$$U^w(x) = U^f(x) \quad \text{outside} \quad D$$

and continue this function inside the domain supp w in an arbitrary way (the values of U^w will possibly be changed in an arbitrarily small neighborhood of supp w). So, the solution of continuation problem is reduced to the solution of the Cauchy problem (6) inside the domain D. The support of the future 'swept charge' $w(x)$ is, clearly, the set of singularities of the solution $u(x)$ to the problem (6) (more precisely, an arbitrarily small neighborhood of this set).

As was already mentioned, the singularities of solutions to problems of the type (6) are naturally investigated in the complex space. So, to construct a solution to the problem (6) we consider the *complexification* of this problem:

$$(\Delta + k^2)u_c(x) = f_c(x)\,,$$
$$u_c(x) \text{ has zero of second order on } \Gamma_c\,, \tag{7}$$

where $x = (x^1, \ldots, x^m)$ are now *complex* variables in the complex Cartesian space $\mathbf{C}^m$, the right-hand part $f_c(x)$ is an *analytic continuation* of the function $f(x)$ to the complex values of its variables, and Γ_c is a *complexification* of the surface Γ, that is the set in $\mathbf{C}^m$, determined by the same equation as Γ (we recall that, by supposition, the equation of the surface Γ is a polynomial one). Clearly, if $u_c(x)$ is a solution to the problem (7), then the restriction of this function to real values of the variable x lying inside D gives a solution to the problem (6). And *real singularities of the function* $u(x)$ *are intersections of complex singularities of* $u_c(x)$ *with the real domain* D. This intersection gives us exactly the support of the 'swept charge'.

The above reduction of the continuation problem to the Cauchy problem (7) shows that the singularities of the continued field are 'brought' to points of the domain D from the complex space along (complex) characteristics of the Helmholz equation (1).

2.3 Integral transform and solutions to the Cauchy problem

The solution to the Cauchy problem (7) can be obtained with the help of the above-mentioned integral transform of complex-analytic functions. In this section, we shall discuss main definitions and theorems connected with this integral transform. Necessarily, the exposition will be short; the reader can find all details in the book [1].

Let $f(x)$ be an analytic (possibly, many-valued) function of complex variables $x = (x^1, \ldots, x^m)$.

The function

$$\overset{\vee}{f}(p) = \overset{\vee}{f}(p_0, p_1, \ldots, p_m) = \int_{h(p)} \operatorname{Res} \frac{f(x)\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m}{p_0 + p_1 x^1 + \ldots + p_m x^m}\,, \tag{8}$$

will be called the *Laplace – Radon transform* of $f(x)$. Here $\wedge$ is the sign of interior product of forms (see, e.g., Ref. [26]), Res is the residue at the plane

$$L_p = \{x | p_0 + p_1 x^1 + \ldots + p_m x^m = 0\} \tag{9}$$

(see Refs [1, 27 – 29]), and $h(p)$ is an $(m-1)$-dimensional surface (homology class), lying in L_p with the boundary in Γ_c. Let us illustrate the definition (8) in the two-dimensional case ($m = 2$).

Since the function $\check{f}(p)$ given by Eqn (8) is a homogeneous function of degree -1 in the variables p, it is sufficient to compute the integral (8) at $p_1 = 1$. For such a case the integrand equals

$$\operatorname{Res} \frac{f(x^1, x^2)\, dx^1 \wedge dx^2}{x^1 + (p_0 + p_2 x^2)} . \tag{10}$$

The residue (10) can be computed, for example, in variable x^1 as usual one-dimensional residue; the differential dx^2 in this computation is ignored and is transferred to the result without changes. Since the form (10) has a polar singularity of the first order at $x^1 = -(p_0 + p_2 x^2)$, we obtain

$$\operatorname{Res} \frac{f(x^1, x^2)\, dx^1 \wedge dx^2}{x^1 + (p_0 + p_2 x^2)} = f(-p_0 - p_2 x^2, x^2)\, dx^2 . \tag{11}$$

The expression (11) is a differential form on the complex plane $\mathbf{C}$ of the variable x^2. We remind the reader (see, for example, Ref. [26]) that the residue of 2-form with a singularity on hyperplane is a 1-form on the same plane. This form can be integrated over any curve $h(p)$ lying in this plane with boundary in Γ_c (Fig. 4). It is clear that the boundary points of the contour $h(p)$ being points of intersection of the plane (9) will be changed when p changes. This leads to the dependence of the contour $h(p)$ on p. So, in the two-dimensional case we have

$$\check{f}(p_0, 1, p_2) = \int_{h(p)} f(-p_0 - p_2 x^2, x^2)\, dx^2 .$$

Clearly, one could compute the function $\check{f}$ at $p_2 = 1$, using the variable x^2 for computation of the residue. It is easy to see that the result of such computations is the same; in doing so it is necessary to take into account the anticommutativity of the exterior product $dx^1 \wedge dx^2 = -dx^2 \wedge dx^1$.

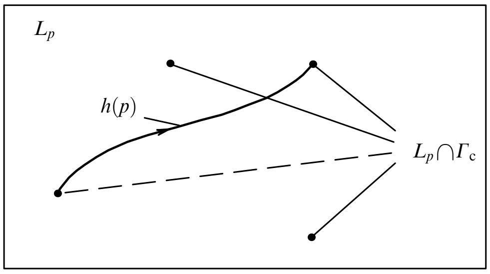

Figure 4.

The examination of singularities of integrals of the type (8) via the known singularities of the function $f(x)$ plays an important role in what follows. The reasons of appearance of singularities of the integrals are as follows:

A. The coincidence of boundary points of the contour $h(p)$ with each other; at such ps the contour $h(p)$ 'vanishes' (this fact gave the name 'vanishing homology classes' for such contours; see Refs [1, 27, 28]).

B. The coincidence of one of boundary points of the contour $h(p)$ with one of singular points of the function $f(x)$.

C. 'Pinching the contour' $h(p)$ between some two singular points of the function $f(x)$ (such phenomenon is known as 'pinch' in the Feynmann integrals theory; see Ref. [28]).

D. The motion of one of boundary points of the contour $h(p)$ to infinity.

The situations A – C can be easily described in terms of relative positions of the plane L_p, the surface Γ_c and the singularity set of the function $f(x)$. Let us present this description.

The situation A takes place only when the plane L_p is either tangent to Γ_c or passes through one of singular points of this surface.

The situation B takes place only when the plane L_p passes through one of the intersection point between Γ_c and the set of singular points of $f(x)$.

The situation C takes place if the plane L_p is tangent to the set of singularities of the function $f(x)$ or goes through one of the singular points of this set.

The situation D seems a little bit different from the three above situations. However, this situation can be described in similar terms if we include into consideration the 'infinite' (improper) points of the complex space $\mathbf{C}^2$. Below we shall illustrate this technique on examples. Here we only remark that infinite points of the complex space can originate real singularities in continuation problems. Such a situation takes place, for example, while considering the two-dimensional scattering problem on the disk or the three-dimensional scattering problem on the sphere.

In the three-dimensional case the investigation of singularities of integral (8) is a little bit more complicated. To investigate this problem one needs to perform the *stratification* of the union of Γ_c with the set of singularities of the function $f(x)$, that is, to decompose this union in smooth manifolds (strata) of different dimension and to consider the tangency between each of these strata and the plane L_p (see, e.g. Ref. [1]).

Let us now describe the main properties of the Laplace – Radon transform†.

First, this transform is *invertible*, and the inverse transform has a similar form. The role of the plane L_p in the inverse transform plays the plane

$$\widetilde{L}_x = \{p | p_0 + p_1 x^1 + \ldots + p_m x^m = 0\} .$$

Second, the following *commutation formula* with differentiation

$$\left(\frac{\partial f}{\partial x^i}\right)^{\vee}(p) = -p_i \frac{d}{dp_0} \check{f}(p) .$$

takes place for this transform.

We do not concentrate on the description of the function classes used for the Laplace – Radon transform since this information is not directly used in the investigation of singularities of continuation of solutions to the Helmholz equation.

† The reader can find the details in Ref. [1].

Now we are able to describe the procedure of solving the problem (7). Applying the Laplace–Radon transform to the equation involved into (7), we obtain the equation

$$(p_1^2 + \ldots + p_m^2)\,\frac{d^2\check{u}(p)}{dp_0^2} + k^2\check{u}(p) = \check{f}(p)\,. \tag{12}$$

One can show that if the function $u(x)$ satisfies zero Cauchy data on the surface Γ_c, then the function $\check{u}(p)$ satisfies zero Cauchy data on the *Legendre transform* $\mathcal{L}\Gamma_c$ of the surface Γ_c. This transform is defined as

$$\mathcal{L}\Gamma_c = \overline{\{p|L_p \text{ is tangent to } \Gamma_c \text{ at some regular point}\}}\,, \tag{13}$$

and the bar denotes the closure. Supplying the equation obtained with zero Cauchy data on $\mathcal{L}\Gamma_c$, for the function $\check{u}(p)$ we obtain the Cauchy problem for an ordinary differential equation (12) (more precisely, for the family of such equations) with constant coefficients, which can be solved explicitly. We remark that the singularities of solution $\check{u}(p)$ to this Cauchy problem are:

(1) singularities of the right-hand part $\check{f}(p)$;

(2) singularities originated by singular points of initial surface (13);

(3) singularities originated by the degeneracy of Eqn (12); these singularities arise at

$$p_1^2 + \ldots + p_m^2 = 0\,. \tag{14}$$

All these singularities can be *computed explicitly* with the help of only algebraic operations even without solving the Cauchy problem for Eqn (12). This gives a possibility of investigating the continued field by *purely algebraic methods*. In the next section we illustrate the computation of singularities on examples.

2.4 Examples

Let us first consider the 'sweeping-of-the charge' problem for a unit circle in the plane:

$$\Delta u(x^1, x^2) + k^2 u(x^1, x^2) = f(x)\,,$$

where

$$f(x) = \begin{cases} 1 & \text{for } (x^1)^2 + (x^2)^2 \leqslant 1\,, \\ 0 & \text{for } (x^1)^2 + (x^2)^2 > 1\,. \end{cases}$$

The corresponding Cauchy problem is

$$\begin{gathered} \Delta u(x^1, x^2) + k^2 u(x^1, x^2) = 1\,, \\ u(x^1, x^2) \text{ has zero of the second order} \\ \text{on } \Gamma_c = \{(x^1)^2 + (x^2)^2 = 1\}. \end{gathered} \tag{15}$$

Let us compute the Laplace–Radon transform $\check{f}(p)$ for the right-hand side $f(x)$ of problem (15) equal to 1. Due to Eqn (8) we have

$$\check{f}(p_0, 1, p_2) = \int_{h(p)} dx^2\,, \tag{16}$$

where $h(p)$ is the curve connecting points of intersection of the plane L_p with the curve Γ_c. The equations of this intersection are

$$p_0 + x^1 + p_2 x^2 = 0\,,$$
$$(x^1)^2 + (x^2)^2 = 1\,,$$

hence

$$x^2 = \frac{1}{p_2^2 + 1}\left(-p_0 p_2 \pm \sqrt{p_2^2 + 1 - p_0^2}\right).$$

So, integral (16) equals

$$\check{f}(p_0, 1, p_2) = \frac{2\sqrt{p_2^2 + 1 - p_0^2}}{p_2^2 + 1}\,.$$

Taking into account the degree of homogeneity of the function f, we finally obtain

$$\check{f}(p) = \frac{2\sqrt{p_1^2 + p_2^2 - p_0^2}}{p_1^2 + p_2^2}\,. \tag{17}$$

Function (17) has singularities on sets $p_0^2 = p_1^2 + p_2^2$ and $p_1^2 + p_2^2 = 0$. Let us present a geometrical treatment of these singularities. First, it is evident that singularities at $p_0^2 = p_1^2 + p_2^2$ arise due to the tangency of the plane L_p with the surface Γ_c (see case A above). Therefore,

$$p_0^2 = p_1^2 + p_2^2$$

is the equation of the Legendre transform of the set $\Gamma_c = \{(x^1)^2 + (x^2)^2 = 1\}$.

To describe geometrically the singularities of function (17) which lie on the set $p_1^2 + p_2^2 = 0$ one needs to introduce the 'infinite' (improper) points of the complex space $\mathbf{C}^2$. To do this, we represent each point $(x^1, x^2) \in \mathbf{C}^2$ as a ray in the complex space $\mathbf{C}^3$ with coordinates (x^0, x^1, x^2), containing the origin and the point $(1, x^1, x^2)$ (see Fig. 5, where the real analogue of the considered situation is drawn). So, each point $(x^0, x^1, x^2) \in \mathbf{C}^3$ with $x^0 \neq 0$ represents the point $(x^1/x^0, x^2/x^0)$ of the space $\mathbf{C}^2$.

It is clear that this point is not changed under the multiplication of all the coordinates (x^0, x^1, x^2) by one and the same nonvanishing number λ, and, hence, depends only on the proportion $x^0 : x^1 : x^2$. The proportion $x^0 : x^1 : x^2$, corresponding to $x^0 = 0$ does not determine any point from $\mathbf{C}^2$; these are exactly improper points of $\mathbf{C}^2$. The set of all proportions $x^0 : x^1 : x^2$, with at least one nonvanishing x^i is

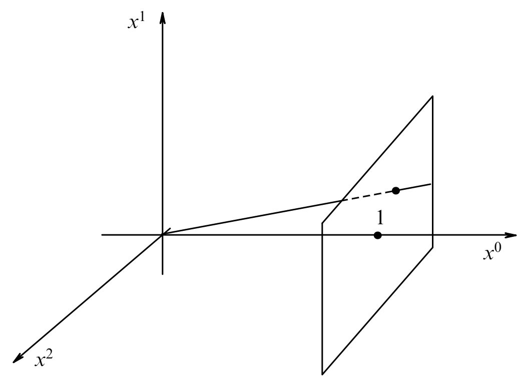

Figure 5.

called the *complex projective space* and is denoted by $\mathbf{CP}^2$. In the neighborhood $x^0 = 0$, $x^1 \neq 0$ coordinates on $\mathbf{CP}^2$ are $x^0/x^1, x^2/x^1$, and in the neighborhood $x^0 = 0, x^2 \neq 0$ they are $x^0/x^2, x^1/x^2$. Clearly, the equation of Γ_c in the space $\mathbf{CP}^2$ has the form

$$\left(\frac{x^1}{x^0}\right)^2 + \left(\frac{x^2}{x^0}\right)^2 = 1 \quad \text{or} \quad (x^1)^2 + (x^2)^2 = (x^0)^2\,. \tag{18}$$

The intersection of the set (18) with the set $\{x^0 = 0\}$ of improper points of $\mathbf{CP}^2$ is a pair of points

$$(0, 1, \mathrm{i})\,, \qquad (0, 1, -\mathrm{i})\,. \tag{19}$$

Clearly, if the plane L_p (in $\mathbf{CP}^2$) is approaching to the plane containing one of the points (19), then one of the points of intersection $L_p \cap \Gamma_c$ goes to infinity. Since the equation of L_p in coordinates (x^0, x^1, x^2) of the space $\mathbf{CP}^2$ is

$$x^0 p_0 + x^1 p_1 + x^2 p_2 = 0\,,$$

this situation takes place for $p_1 = \pm \mathrm{i} p_2$, that is, for $p_1^2 + p_2^2 = 0$. This describes the corresponding singularities of function (17).

Now we write down the problem (12) for the case considered:

$$(p_1^2 + p_2^2)\,\frac{\mathrm{d}^2 \overset{\vee}{u}}{\mathrm{d}p_0^2} + k^2 \overset{\vee}{u} = \frac{2\sqrt{p_1^2 + p_2^2 - p_0^2}}{p_1^2 + p_2^2}\,,$$

$\overset{\vee}{u}$ has zero of the second order for $p_0^2 = p_1^2 + p_2^2$.

It is easy to see that the singularities of the solution $\overset{\vee}{u}$ are lying in one of the following three sets:

$$\begin{aligned} &p_1 = \mathrm{i} p_2\,,\\ &p_1 = -\mathrm{i} p_2\,,\\ &p_1^2 + p_2^2 = p_0^2\,. \end{aligned} \tag{20}$$

Therefore, the singularities of the solution $u(x)$ to the problem (15) are lying at such (x^1, x^2) that the plane $\widetilde{L}_x$ passes through one of the points of intersection of surfaces (20) or is tangent to one of these surfaces. The main interest for us makes the case when $\widetilde{L}_x$ passes through the points $(0, 1, \mathrm{i})$ and $(0, 1, -\mathrm{i})$ of intersection of the first two lines (20) with the third. This condition gives $x^1 = \pm \mathrm{i} x^2$ or

$$(x^1)^2 + (x^2)^2 = 0\,. \tag{21}$$

It can be shown that the cases when $\widetilde{L}_x$ passes through the point $(1, 0, 0)$, as well as when $\widetilde{L}_x$ is tangent to one of surfaces (20) do not lead to the appearance of new singularities of the function $u(x)$. So, the set of *complex* singularities of solutions to the problem (15) is given by formula (21).

As a consequence of Section 2.2 one can conclude that the support of the 'swept charge' in the problem under consideration can be found as an intersection of the set (21) with the real space $\mathbf{R}^2$, that is, the center of the circle $x^1 = x^2 = 0$. We remark that the singularities of continuation has 'come' to the center of the circle along the complex lines $x^1 = \pm \mathrm{i} x^2$ from *infinite* points of the surface Γ_c.

Let us present here (without computations) two three-dimensional examples of investigation of the singularities of continuation. In these examples $f(x^1, x^2, x^3)$ is a function possessing continuation up to an *entire* function of complex arguments.

Consider the problem

$$\Delta u(x) + k^2 u(x) = \begin{cases} f(x) & \text{for} \quad x^3 \geqslant (x^1)^2 + (x^2)^2\,,\\ 0 & \text{for} \quad x^3 < (x^1)^2 + (x^2)^2\,. \end{cases} \tag{22}$$

For this problem the singularities of analytic continuation $u_c(x)$ have the form

$$(x^1)^2 + (x^2)^2 = 0\,, \qquad (x^1)^2 + (x^2)^2 + \left(x^3 - \frac{1}{4}\right)^2 = 0\,.$$

So, the ray coming from the point $(0, 0, 1/4)$ to $+\infty$ along the axis x^3 is exactly the support of the 'swept charge' for the problem (22).

For the problem

$$\Delta u(x) + k^2 u(x) = \begin{cases} f(x) & \text{for} \quad (x^1)^2 + (x^2)^2 + \dfrac{(x^3)^2}{a^2} \leqslant 1\,,\\ 0 & \text{for} \quad (x^1)^2 + (x^2)^2 + \dfrac{(x^3)^2}{a^2} > 1 \end{cases}$$

the singularities of the function $u_c(x)$ are given by

$$(x^1)^2 + (x^2)^2 + \left(x^3 \pm \sqrt{a^2 - 1}\,\right)^2 = 0 \ \text{ и } \ (x^1)^2 + (x^2)^2 = 0\,.$$

Hence, the singularities of continuation of the function $u(x)$ inside the ellipsoid

$$(x^1)^2 + (x^2)^2 + \frac{(x^3)^2}{a^2} \leqslant 1$$

are lying on the segment $\left[-\sqrt{a^2 - 1},\ \sqrt{a^2 - 1}\,\right]$ of the x^3-axis for $a > 1$, or on the circle $x^3 = 0$, $(x^1)^2 + (x^2)^2 = 1 - a^2$ in the case $a < 1$. The support of the swept charge for $a > 1$ coincides with the mentioned segment. If $a < 1$, then one can choose the disk $x^3 = 0$, $(x^1)^2 + (x^2)^2 \leqslant 1 - a^2$ as the support of the swept charge; the necessity of adding such a disk to the set of singularities is due to the necessity of introducing a cut inside the ellipsoid for separating a single-valued branch of the continuation.

2.5 The reflection formula for the Helmholz equation

Here we consider the formula generalizing the Schwarz symmetry principle (2), (3) to the Helmholz equation. This formula will be used in the following section for investigating singularities of continuation for boundary value problems. We present here the formulations only; the reader can find the details in Refs [30, 31].

So, let Γ be an algebraic curve in the space $\mathbf{R}^2$, dividing this space into two domains U_1 and U_2, and let $u(x^1, x^2)$ be a solution to Eqn (1) vanishing on Γ. *Reflection formula* is a formula expressing the value of function $u(x)$ at an arbitrary point x_0 of the domain U_1 via values of the same function in the domain U_2.

To write down the reflection formula for the Helmholz equation, we need the notion of the *Schwarz function* of the curve Γ. Let

$$\varphi(x^1, x^2) = 0\,, \tag{23}$$

be the equation of Γ, where $\varphi(x)$ is a polynomial in x^1, x^2 with real coefficients. Equation (23) can also be viewed as the equation of the complexification Γ_c of the curve Γ if x^1 and x^2 are complex variables. After the change of variables

$$z = x^1 + ix^2, \qquad \zeta = x^1 - ix^2$$

the equation of the curve Γ_c is rewritten as

$$\Phi(z,\zeta) = \varphi\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right) = 0. \tag{24}$$

Denote by $\zeta = S(z)$ the solution of Eqn (24) with respect to ζ, and by $z = \widetilde{S}(\zeta)$, the solution of this equation with respect to z. The function $S(z)$ is called the *Schwarz function* of the curve Γ (see Refs [6–10, 14] and others). Clearly, the functions $S(z)$ and $\widetilde{S}(\zeta)$ are algebraic functions of their arguments. Then, if $\operatorname{grad}\varphi \neq 0$ on Γ, these functions are regular in a neighborhood of the real part Γ of the curve Γ_c.

Identifying the space $\mathbf{R}^2$ with the complex plane $\mathbf{C}$ by the relation $z = x^1 + ix^2$, we define the mapping R as

$$R(x^1, x^2) = R(z) = \overline{S(z)}, \tag{25}$$

where the bar stands for the complex conjugation. The mapping (25) possesses the following properties:

1. The mapping R is defined and regular in a neighborhood of the curve Γ.
2. The mapping R interchanges the domains U_1 and U_2 inside this neighborhood.
3. The mapping R is an idempotent, that is, $R\big(R(x,y)\big) = (x,y)$.
4. The mapping R is identical on Γ.

We remark that the above-defined mapping R is exactly the mapping involved in formulas (2) and (3), describing the Schwarz symmetry principle.

Let $V_0(x^1,x^2,y^1,y^2)$ be the Riemann function for the Helmholz operator. It is known that

$$V_0(x^1,x^2,y^1,y^2) = J_0\left(k\sqrt{(x^1-y^1)^2+(x^2-y^2)^2}\right),$$

where J_0 is the Bessel function of the zeroth order. In particular, the function V_0 is an entire function of its arguments. We define the functions V_1 and V_2 as solutions to the following two Cauchy–Goursat problems in the space $\mathbf{C}^2$:

$$\begin{aligned}&(\Delta_y + k^2)\, V_1(x,y) = 0,\\ &V_1(x,y) = V_0(x,y) \quad \text{for} \quad y \in \Gamma_c,\\ &V_1 = 1 \quad \text{for} \quad y^1 - iy^2 = S(x^1 + ix^2)\end{aligned} \tag{26}$$

and

$$\begin{aligned}&(\Delta_y + k^2)\, V_2(x,y) = 0,\\ &V_2(x,y) = V_0(x,y) \quad \text{for} \quad y \in \Gamma_c,\\ &V_2 = 1 \quad \text{for} \quad y^1 + iy^2 = \widetilde{S}(x^1 - ix^2).\end{aligned} \tag{27}$$

The solutions to problems (26) and (27) exist in $\mathbf{C}^2$ as ramifying analytic functions in (x,y), and the singularities of these functions (both in x and in y) coincide with the singularities of the functions $S(z)$ and $\widetilde{S}(\zeta)$. Let us introduce the function

$$V(x,y) = V_1(x,y) - V_2(x,y),$$

which will be called the *reflected Riemann function* corresponding to the curve Γ.

Now the reflection formula is

$$\begin{aligned}u(x) = -u\big(R(x)\big)& \\ + \frac{1}{2i}\int_\Gamma^{R(x)} \Bigg\{ &\left[u(y)\,\frac{\partial V}{\partial y^1}(x,y) - V(x,y)\,\frac{\partial u}{\partial y^1}(y)\right] dy^2 \\ &- \left[u(y)\,\frac{\partial V}{\partial y^2}(x,y) - V(x,y)\,\frac{\partial u}{\partial y^2}(y)\right] dy^1 \Bigg\},\end{aligned} \tag{28}$$

where the integral is taken over any curve connecting some point of Γ with $R(x)$. We remark that the integral on the right of Eqn (28) does not depend on the choice of such a curve.

The classical symmetry formulas of the form (2) for the Laplace equation and for the Helmholz equation with the straight line as Γ are particular cases of formula (28). For these cases the straightforward computation shows that $V = 0$ and, hence, the integral term on the right of Eqn (28) vanishes. However, even in the case when Γ is a circle ($k \neq 0$), it is easy to see that the integral term in this formula does not vanish; this is exactly the reason why the 'pure' symmetry principle of the form (2) is not valid for the Helmholz equation.

2.6 Continuation of solutions for domains with piecewise-analytic boundary

Consider first the sweeping-of-charge problem. Let $D \subset \mathbf{R}^m$, $m = 2$ or $m = 3$, be a domain whose boundary Γ consists of r pieces Γ_j, $j = 1, \ldots, r$, with equations

$$P_j(x) = 0, \tag{29}$$

where $P_j(x)$ are irreducible polynomials, and let $U(x)$ be a solution to the equation

$$(\Delta + k^2)U = f. \tag{30}$$

Here, as above, the function f vanishes outside D and coincides with some entire function (also denoted by f) inside this domain.

To simplify our considerations, we suppose that the intersections of all characteristic conoids† of complexifications of surfaces Γ_j and Γ_l, $j \neq l$ with the real space $\mathbf{R}^m$ do not intersect one another and have no common points with the boundary of the domain D.

Let us consider the problem of continuation of the function U as a solution to the homogeneous Helmholz equation into the domain D with boundary consisting of two pieces Γ_1 and Γ_2 (Fig. 6). So, let the function $U(x)$ be a

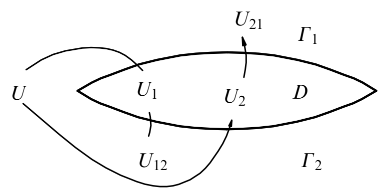

Figure 6.

† By a *characteristic conoid* of a surface one calls the union of all characteristics emanated from characteristic points of this surface; the exact definitions see in Ref. [1].

solution to the homogeneous equation

$$(\Delta + k^2)U = 0\,.$$

given in $\mathbf{R}^m \setminus D$. Denote by $\widetilde{U}_1$ the difference

$$\widetilde{U}_1 = U - u\,.$$

For the function $\widetilde{U}_1$ to be a continuation of a solution of homogeneous equation into the domain D across the boundary part Γ_1, the function u must satisfy the equation

$$(\Delta + k^2)u = f$$

and have zero of the second order on the surface Γ_1. However, the function $\widetilde{U}_1$, constructed in such a way, will not be a solution to the homogeneous equation outside the domain D (near the surface Γ_2), since

$$L\widetilde{U}_1 = LU - Lu = -f\,. \tag{31}$$

However, if we introduce the cut along Γ_2, then the function $\widetilde{U}_1$ in $\mathbf{R}^m$ will be a solution to the homogeneous equation, having inside D singularities corresponding to the intersection of the characteristic conoid of the complexification of the surface Γ_1 with the real space $\mathbf{R}^m$.

Continuing the solution to the homogeneous equation across the boundary part Γ_2, one can construct (in similar fashion) the function $\widetilde{U}_2$, which is a solution to the homogeneous equation in $\mathbf{R}^m$ with the cut along the surface Γ_1.

So, the solution is given on the Riemannian surface which covers the domain D at least twice.

Now the continuation of the function $\widetilde{U}_1$ across the surface Γ_2 leads to the function

$$\widetilde{U}_{12} = \widetilde{U}_1 - v\,,$$

where the function v satisfies Eqn (31) and has zero of order two on Γ_2. Similarly, the continuation of solution $\widetilde{U}_2$ across Γ_1 gives the function $\widetilde{U}_{21}$. Continuing the functions $\widetilde{U}_{12}$ and $\widetilde{U}_{21}$ across Γ_1 and Γ_2, respectively, we obtain the functions $\widetilde{U}_{121}$ and $\widetilde{U}_{212}, \ldots$

The above-described procedure defines a Riemannian surface on $\mathbf{R}^m$ which has ramifications at points of intersection of boundary parts Γ_1 and Γ_2, as well as at points of intersection of characteristic conoids of complexifications of the surfaces Γ_1 and Γ_2 with the real space $\mathbf{R}^m$.

So, the following statement is valid:

Theorem 1. *Under the above assumptions, the solution to Eqn* (30) *can be continued as a solution to the homogeneous Helmholz equation up to a ramifying analytic function given on the above-constructed Riemannian surface.*

Let us now turn to the consideration of boundary value problems. These problems will be investigated in the two-dimensional case with the help of the reflection formula. Consider the domain $D \subset \mathbf{R}^2$, with a piecewise-analytic boundary $\Gamma = \cup_j \Gamma_j$, $j = 1, \ldots, r$, such that the components Γ_j of this boundary are described , as above, by Eqns (29). Let the function $u(x, y)$ be a solution to the homogeneous equation

$$(\Delta + k^2)u = 0 \tag{32}$$

in D which satisfies the conditions

$$u(x, y)\Big|_{\Gamma_j} = f_j(x) \tag{33}$$

on the boundary of this domain. It is supposed that the functions f_j satisfy compatibility conditions on the neighboring parts of the boundary.

Suppose, for simplicity, that the boundary of D consists of three algebraic parts Γ_1, Γ_2, and Γ_3 (Fig. 7). It is also supposed that the singularities of the Schwarz function $S_j(x + \mathrm{i}y)$ of each of the curves Γ_j, $j = 1, 2, 3$ do not lie on the boundary of the domain D.

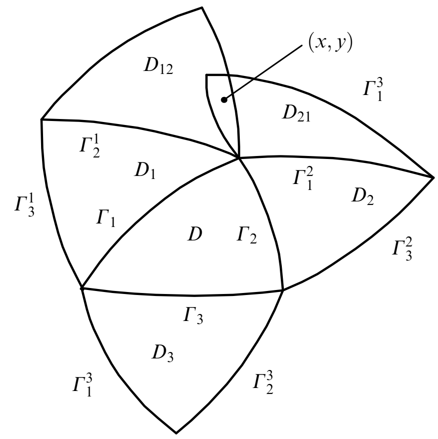

Figure 7.

The mapping R_j: $D \to D_j$, determined by the formula

$$R_j(x, y) = \overline{S_j(x + \mathrm{i}y)} \tag{34}$$

corresponds to each part Γ_j of the boundary. Under these mappings the curves Γ_l are taken into the curves Γ_l^j, and the curve Γ_j is taken into itself (see Fig. 7), since $R_j(x, y)\big|_{\Gamma_j} = \mathrm{id}$. By D_j we denote the image of the domain D under the mapping R_j.

The function $u(x, y)$ can be continued into the domain D_j, and the continuation is performed as follows. We use the reflection formula

$$u(x_0, y_0) = -u\big(R_j(x_0, y_0)\big) + \frac{1}{2\mathrm{i}} \int_{\Gamma_j}^{R_j(x_0, y_0)} \omega_j + F\left[f_j(x, y)\right] \tag{35}$$

near the boundary. Then, moving the point $(x_0, y_0) \in D_j$ along some curve in such a way that its image remains in the domain D, we perform the analytic continuation of the integral on the right of Eqn (35). Here $(x_0, y_0) \in D_j$, the mappings R_j are defined by formulas (34), ω_j is a known differential form depending in a linear way on function $u(x, y)$ and its derivatives of the first order with coefficients determined by the curve Γ_j [see formula (28) above], F is a known functional, and the integral is taken over any curve connecting Γ_j with the point† $R_j(x_0, y_0) \in D$.

The domains D_j can be 'glued' to the domain D along the corresponding curves Γ_j. By the same procedure each domain D_j can be reflected from the curve Γ_l^j, $l \neq j$ to the domain D_{jl}

† The above discussion on the reflection formula refers to the case when the function under consideration satisfies the *homogeneous* boundary condition. The generalization of this formula to the case of nonhomogeneous boundary conditions (33) is also possible. The reader can find the details in Ref. [1].

(see, for example, the domain D_{12} on Fig. 7), and the function $u(x, y)$ can be continued to this domain as well.

We remark that all reflected domains have to be considered on the corresponding Riemannian surface. The reflected domains can contain the singularities of the corresponding Schwarz functions, leading to singularities of the mappings R_j. So, we obtain a Riemannian surface to which the function $u(x, y)$ can be continued.

It is clear that the values of the function $u(x, y)$ continued along a curve coming to a point (x, y) through domains D_1 and D_{12} will not coincide with values of the same function obtained by the continuation through the domains D_2 and D_{21}. This shows that the points of intersection of different parts of the boundary are the ramification points of the continuation in question.

Hence, the constructed Riemannian surface ramifies at points of intersection between different analytic parts of the boundary (of the domain D or the corresponding reflected domains), as well as at points of singularity of the Schwarz functions of curves, corresponding to these parts.

So, let $u(x, y)$ be an arbitrary solution to the Helmholz equation (32) satisfying the boundary conditions (33). The following statement is valid:

Theorem 2. *Under the above-formulated conditions the function $u(x, y)$ can be continued up to a ramifying analytic function defined on the Riemannian surface constructed above. The continuation to each reflected domain can be obtained by step-by-step application of the reflection formula* (35).

3. Applied aspects of the continuation theory of wave fields

To be definite, in this section we shall consider exterior diffraction and scattering problems. This is connected with the fact that up to now the most of results on the discussed topic are obtained exactly for these problems. The understanding that there exists the problem of continuation of a solution to a boundary-value problem out of its initial domain of definition had come after a series of failures of attempts to numerically realize classical and other computational schemes. The matter is that working out some computational algorithm one often has in mind some representation of the searched solution. The choice of specific representation in strong extent determines certain analytic properties of the solution in question. For example, the solution of a boundary value problem can be represented in the form of expansion in some full system of functions each satisfying the equation considered (say, the Helmholz equation), and some additional conditions (say, the radiation condition). The coefficients of this expansion are found from the boundary condition. This is, clearly, one of the simplest schemes of constructing the computational algorithm of solving a boundary value problem. There exist a lot of different schemes which will be considered below. Here we mention only one important observation. The described scheme (and its analogues) can be successfully realized only in the case when the existence domain of the chosen analytic representation includes the domain (together with its boundary) in which the solution is searched. Unfortunately, this condition is valid very rarely. The latter fact was clarified quite recently, though the reason for this phenomenon is clear enough in most cases. The matter is that a solution to the Helmholz equation (or to Maxwell's equations in a stationary case) is a real-analytic function which vanishes at infinity (according to the radiation condition) and, hence, this solution has singularities near the origin. These singularities naturally lie outside the domain where the solution is searched, or on its boundary. However, the geometry of singularities determines boundaries of existence for analytic representations of the solution, and these boundaries coincide with the boundary of the main domain only in exceptional cases (for example, for a sphere).

Let us discuss this question in more detail. For simplification of the presentation we shall consider only scalar wave fields. We remark that all the results obtained for this case remain valid in the vector case as well.

3.1 Analytic representation of wave fields. Diagram of wave field

So, the wave field is understood as a solution u to the homogeneous Helmholz equation

$$(\Delta + k^2)u = 0\,, \tag{36}$$

satisfying at infinity the condition of, say, the following form

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} + \mathrm{i}ku\right) = 0 \tag{37}$$

($k = \mathrm{const}$ is the wave number), and the boundary conditions given on some surface Γ. Here we restrict ourselves to the consideration of only two types of this surface: the closed surface bounding a compact body (a domain D), and infinite periodic surface. (In the latter case the condition at infinity differs from that formulated above). If Γ is the boundary of a compact body, then the wave field u is often represented in the form of the following two expansions [32–35]:

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{mn}(-\mathrm{i})^{n+1} h_n^{(2)}(kr) P_n^m(\cos\theta) \exp(\mathrm{i}m\varphi)\,, \tag{38}$$

$$u(r, \theta, \varphi) = \frac{\exp(-\mathrm{i}kr)}{kr} \sum_{p=0}^{\infty} \frac{a_p(\theta, \varphi)}{(kr)^p}\,. \tag{39}$$

Expansion (38) is often referred as the Rayleigh representation or as series in wave harmonics (metaharmonic functions [36]). Relation (39) is called the Atkinson–Wilcox series. In these formulas r, θ, φ are spherical coordinates of the observation point, $h_n^{(2)}(kr)$ is the spherical Hankel function of the second kind (nth order), $P_n^m(\cos\theta)$ is the adjoint Legendre polynomial. For vector fields, analogous expansion formulas are presented, for example, in Refs [34, 35, 37].

From (39) one can see that the following asymptotic formula

$$u(r, \theta, \varphi) = \frac{\exp(-\mathrm{i}kr)}{kr} f(\theta, \varphi) + O\left(\frac{1}{(kr)^2}\right) \tag{40}$$

is valid as $kr \to \infty$. Here $f(\theta, \varphi) = a_0(\theta, \varphi)$ is the wave field diagram. All other coefficients in Eqn (39) can be expressed in terms of the diagram $f(\theta, \varphi)$ with the help of the following recurrent relations

$$a_p = \frac{\mathrm{i}}{2p}\left[p(p-1) + D\right] a_{p-1}\,, \qquad a_0 \equiv f\,, \tag{41}$$

where

$$D = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

is the Beltrami operator.

It is easy to show (see, e.g. Ref. [38]), that the Fourier series for the diagram has the form

$$f(\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_{mn}P_n^m(\cos\theta)\exp(\mathrm{i}m\varphi)\,, \tag{42}$$

and the coefficients a_{mn} in the expansions (42) and (38) are the same. So, from the relation (40) one can see that in the so-called far zone, that is, for $kr \geqslant 1$, the wave field is completely determined by the function $f(\theta,\varphi)$ — the diagram of the wave field [up to the values of the order of $1/(kr)^2$]. On the other hand, the relations (38) and (39) together with equalities (41) and (42) allow one to reconstruct the wave field everywhere except for some neighborhood of the origin provided that the diagram $f(\theta,\varphi)$ is known. The boundary of this neighborhood will be determined in the next section. We remark that the series (38) and (39) become identical in the domain of their existence after some rearranging of their terms [35].

In the two-dimensional case the Rayleigh expansion has the following form

$$u(r,\varphi) = \sum_{n=-\infty}^{\infty} a_n(-\mathrm{i})^n H_n^{(2)}(kr)\exp(\mathrm{i}n\varphi)\,, \tag{43}$$

where r,φ are polar coordinates of the observation point, and a_n are the Fourier coefficients of the diagram $f(\varphi)$ of the wave field determined by the relation

$$u(r,\varphi) = \sqrt{\frac{2}{\pi kr}}\exp\left(-\mathrm{i}kr + \frac{\mathrm{i}\pi}{4}\right)\left[f(\varphi) + O\left(\frac{1}{kr}\right)\right]. \tag{44}$$

Here, similar to the three-dimensional case, the relation (43) allows one to reconstruct the field everywhere except for some neighborhood of the origin provided that the diagram (that is, the asymptotics of the field in the far zone) is known. In contrast to the three-dimensional case, the series (43) cannot be rearranged into the expansion in powers of $1/kr$. The corresponding expansion has the form [35]:

$$u(r,\varphi) \sim \sqrt{\frac{2}{\pi kr}}\exp\left(-\mathrm{i}kr + \frac{\mathrm{i}\pi}{4}\right)\sum_{n=0}^{\infty}\frac{\widehat{P}_n}{(kr)^n}f(\varphi)\,,$$

where

$$\widehat{P}_n = \left(\frac{\mathrm{i}}{2}\right)^n\frac{1}{n!}\prod_{s=1}^{n}\left[\left(s-\frac{1}{2}\right)^2 + \frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\right],$$

and has the character of asymptotic series.

There exist one more method of analytic representation of wave field — integrals over plane waves or the Sommerfeld–Weil representation [38, 39]. In the three-dimensional case this representation has the form

$$u(r,\theta,\varphi) = \frac{1}{2\pi\mathrm{i}}\int_0^{2\pi}\int_0^{\pi/2+\mathrm{i}\infty} f(\alpha,\beta)$$
$$\times\exp\Big\{-\mathrm{i}kr\big[\sin\theta\sin\alpha\cos(\varphi-\beta)+\cos\theta\cos\alpha\big]\Big\}\sin\alpha\,\mathrm{d}\alpha\,\mathrm{d}\beta\,. \tag{45}$$

In (45), the integration over the parameter α is fulfilled along the path lying in the complex plane $\alpha = \alpha_1 + \mathrm{i}\alpha_2$. Therefore, the diagram $f(\alpha,\beta)$ of the wave field must be analytically continuable to the whole complex plane $\alpha = \alpha_1 + \mathrm{i}\alpha_2$. We remark that Eqn (45) remains unchanged in the vector case as well.

For two-dimensional wave fields we have

$$u(r,\varphi) = \frac{1}{\pi}\int_{-\mathrm{i}\infty}^{\pi+\mathrm{i}\infty} f(\psi)\exp\big[-\mathrm{i}kr\cos(\psi-\varphi)\big]\,\mathrm{d}\psi\,. \tag{46}$$

Here the integration is fulfilled over a contour in the complex plane $\psi = \alpha + \mathrm{i}\beta$. The integrals (45) and (46) converge, as it will be shown, in a half-space bounded by some plane $z = z_0$ (by a straight line $y = y_0$ in two dimensions). Rotating the coordinate system, one can perform the analytic continuation of the wave field to a neighborhood of the origin with the help of Eqns (45) and (46). The modifications of the above plane-wave representations which converge outside a convex neighborhood of the origin are presented in Refs [40–42].

In considering problems of scattering of wave fields by periodic boundaries between two media, the representation of wave field by series in plane waves (the Rayleigh series) is widely used. In the two-dimensional case such a representation has the following form [43]:

$$u(x,y) = \frac{2}{b}\sum_{n=-\infty}^{\infty} g_0(w_n)\frac{\exp(-\mathrm{i}w_n x)\exp(-\mathrm{i}v_n y)}{v_n}\,. \tag{47}$$

Here b is the period of the boundary described by the equation $y = f(x) = f(x+b)$,

$$w_n = \frac{2\pi}{b}\,n + k\sin\theta\,, \qquad v_n = \sqrt{k^2 - w_n^2}\,, \tag{48}$$

and one chooses a branch of the square root such that $\operatorname{Re} v_n \geqslant 0$, $\operatorname{Im} v_n \leqslant 0$. Here θ is the incidence angle of the plane wave, and $g_0(w_n)$ is the diagram of the central period (see Section 3.2).

The series (47) converges, as well as integral (46) does, in some half-plane $y > y_0$. The scattering coefficient of spectral order m is connected with the quantity $g_0(w_m)$ by the following relation

$$R_m = \frac{2}{b}\frac{g_0(w_m)}{v_m}\,. \tag{49}$$

So, one can reconstruct the wave field everywhere except for a neighborhood $y \leqslant y_0$ of the Ox axis by the scattering coefficients of all spectral orders, or, what is the same, by the diagram of one period.

In three dimensions, the corresponding representation has quite similar character (see, e.g. Refs [15, 44, 45]).

All these representations of wave fields can be obtained from the wave potential representation

$$u(\mathbf{r}) = \int_\Gamma\left[\mu(\mathbf{r}')\frac{\partial G(\mathbf{r};\mathbf{r}')}{\partial n'} - \nu(\mathbf{r}')G(\mathbf{r};\mathbf{r}')\right]\mathrm{d}s'\,. \tag{50}$$

Here $\mathbf{r}$ is the radius of the observation point, $\mathbf{r}'$ is the radius of the integration point on Γ which is some smooth enough closed surface (a contour in two dimensions), $\partial/\partial n'$ denotes differentiation along the exterior normal of Γ, μ and ν are densities of potentials of double and single layers, respectively, and $G(\mathbf{r};\mathbf{r}')$ is the Green function. It has the form

$$G(\mathbf{r};\mathbf{r}') = \frac{\exp\left(-\mathrm{i}k|\mathbf{r}-\mathbf{r}'|\right)}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{51}$$

in the case of three dimensions and

$$G(\mathbf{r};\mathbf{r}') = \frac{1}{4\mathrm{i}}\, H_0^{(2)}\left(k|\mathbf{r}-\mathbf{r}'|\right) \tag{52}$$

in two dimensions, where $H_0^{(2)}$ is the second-kind Hankel function of the zeroth order. Finally, for two-dimensional problems with periodic boundary

$$G(\mathbf{r};\mathbf{r}') = \frac{1}{2b\mathrm{i}} \sum_{n=-\infty}^{\infty} \frac{\exp\left[-\mathrm{i}w_n(x-x') - \mathrm{i}v_n|y-y'|\right]}{v_n}\,. \tag{53}$$

The following expressions

$$\mu = u(\mathbf{r}') \qquad \text{or} \qquad \mu = \hat{u}(\mathbf{r}')\,,$$

$$v = \left.\frac{\partial u}{\partial n'}\right|_\Gamma \qquad \text{or} \qquad v = \left.\frac{\partial \hat{u}}{\partial n'}\right|_\Gamma\,,$$

are commonly used for the densities μ and v of potentials. Here $u(\mathbf{r}')$, $\partial u/\partial n'|_\Gamma$ are the values of the wave field and its normal derivative on Γ. The quantity $\hat{u}$ is the total field, that is, $\hat{u} = u + u^0$, where u^0 is some given (initial) wave field satisfying a nonhomogeneous Helmholz equation.

Representations similar in their sense take place in the vector case as well (see, e.g. Refs [33, 34]).

From the preamble to this section it is clear that all the singularities of the wave field $u(\mathbf{r})$ have to be contained inside the surface (contour) Γ. This circumstance, being very important, is crucial for some numerical methods (see Section 3.4).

3.2 Analytic properties of diagram and boundaries of existence domains of analytic representations

Using asymptotic expansions (51) and (52) of the Green function for $r \gg r'$, $kr \gg 1$, we obtain the following expressions for the wave field diagram:

$$f = \kappa \int_\Gamma \left(\mu\,\frac{\partial}{\partial n'} - v\right) \exp(\mathrm{i}kr'\cos\gamma)\,\mathrm{d}s'\,, \tag{54}$$

where κ is a numerical coefficient; $\kappa = k/4\pi$, $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$ in three dimensions, and $\kappa = 1/4\mathrm{i}$, $\cos\gamma = \cos(\varphi-\varphi')$ in two dimensions.

In scattering problems with periodic boundary the diagram $g_0(w)$ is given by the following integral [43]:

$$g_0(w) = \frac{1}{4\mathrm{i}} \int_{-b/2}^{b/2} \left(\mu\,\frac{\partial}{\partial n'} - v\right) \times \exp\left[\mathrm{i}wx + \mathrm{i}vf(x)\right] \sqrt{1+f'^2(x)}\,\mathrm{d}x\,, \tag{55}$$

where $v = \sqrt{k^2-w^2}$, $\operatorname{Im} v \leqslant 0$.

To shorten the exposition, we consider below mainly the two-dimensional case formulating only the final results in three dimensions. So, let us consider the integral (54), where for simplicity we put $\mu = 0$. This corresponds to the case

$$\mu = \hat{u}(\mathbf{r}')\Big|_\Gamma\,, \qquad v = \left.\frac{\partial \hat{u}}{\partial n'}\right|_\Gamma\,,$$

when the Dirichlet condition $\hat{u}\big|_\Gamma = 0$ is given on Γ. Integral (54) becomes

$$f(\varphi) = \frac{\mathrm{i}}{4} \int_\Gamma \frac{\partial \hat{u}}{\partial n'} \exp\left[\mathrm{i}kr'\cos(\varphi-\varphi')\right] \mathrm{d}s'\,. \tag{56}$$

Let $r = \rho(\varphi)$ be an equation of Γ in a polar coordinate system. Then the relation (56) takes the form

$$f(\psi) = \int_0^{2\pi} q(\varphi) \exp\left[\mathrm{i}k\rho(\varphi)\cos(\psi-\varphi)\right] \mathrm{d}\varphi\,. \tag{57}$$

Here we have performed the change of variables $\varphi \to \psi$, $\varphi' \to \varphi$ and introduced the notation

$$q(\varphi) = \frac{\mathrm{i}}{4}\left.\left[\rho(\varphi)\,\frac{\partial \hat{u}}{\partial r} - \frac{\rho'(\varphi)}{\rho(\varphi)}\,\frac{\partial \hat{u}}{\partial \varphi}\right]\right|_{r=\rho(\varphi)}\,.$$

It is easy to see that the diagram $f(\psi)$ is continuable to the whole complex plane $\psi = \alpha + \mathrm{i}\beta$ as an entire function [40, 41]. Using the simple relation

$$\cos(\psi-\varphi) = \frac{1}{2}\left\{R\left[\cos(\varphi-\alpha) \pm \mathrm{i}\sin(\varphi-\alpha)\right] + \frac{1}{R}\left[\cos(\varphi-\alpha) \mp \mathrm{i}\sin(\varphi-\alpha)\right]\right\},$$

where $R = \exp|\beta|$, the upper sign is taken for $\beta > 0$, and the lower for $\beta < 0$, it is easy to show that as $R \to \infty$ the following asymptotic relations

$$f\left(\alpha \pm \mathrm{i}|\beta|\right) = f_\mp^{\mathrm{E}}(z_\mp)\left[1 + O\left(\frac{1}{R}\right)\right] \tag{58}$$

take place, where

$$f_\mp^{\mathrm{E}}(z_\mp) = \int_0^{2\pi} q(\varphi) \exp\left[\frac{\mathrm{i}k\rho(\varphi)}{2}\,\exp(\pm\mathrm{i}\varphi)z_\mp\right] \mathrm{d}\varphi \tag{59}$$

are entire functions of finite degree of the variables

$$z_\mp = \exp\left(|\beta| \mp \mathrm{i}\alpha\right)\,.$$

Actually, evaluating integrals (57) by the Schwarz inequality, we obtain

$$\max_{|z|=R}\left|f_\mp^{\mathrm{E}}(z)\right| < \exp\left[R\left(\frac{kr_0}{2} + \varepsilon\right)\right],$$

where $r_0 = \max_\varphi \rho(\varphi)$ is the radius of the minimal circle containing the surface Γ, $\varepsilon > 0$ is an arbitrary number. From this estimate, since ε is arbitrary, we obtain

$$\overline{\lim_{R\to\infty}}\, \frac{\ln \max_{|z|=R}\left|f_\mp^{\mathrm{E}}(z_\mp)\right|}{R} = \sigma_\mp \leqslant \frac{kr_0}{2}\,, \tag{60}$$

that is, $f_\mp^{\mathrm{E}}(z_\mp)$ are entire functions of the first order with finite degree $\sigma_\mp$ which does not exceed $kr_0/2$. This result remains valid in the case when $\mu \not\equiv 0$ on Γ as well. So, if a function $f(\psi)$ is a diagram of a wave field and, hence, is representable as integrals of the type (57) or (54), where μ and v are integrable functions (currents) on Γ, then this diagram is analytically continuable to the whole complex plane $\psi = \alpha + \mathrm{i}\beta$ and near the infinite point of this plane (for $|\beta| \to \infty$) it asymptotically coincides with some entire function of the complex variable $z = \exp\big(|\beta| - \mathrm{i}\alpha \operatorname{sign}\beta\big)$ [see Eqn (58)]. The quantity $\sigma = \max(\sigma_+, \sigma_-)$ is called the degree of the wave field diagram.

As is well-known [32, 46], the diagram of the system with a plane aperture is an entire function of finite degree. In contrast, the diagram of the system of currents distributed on some closed surface Γ which is not a part of the plane, is not an entire function of exponential type, but just coincides [in the sense of asymptotic equality (58)] with some entire function of finite degree in a neighborhood of infinity. Nevertheless, this is sufficient for using numerous properties of entire functions of exponential type.

One of such properties important for what follows is the fact that the coefficients $c_n^{\mp}$ of everywhere convergent series

$$f_{\mp}^{\mathrm{E}}(z_{\mp}) = \sum_{n=0}^{\infty} c_n^{\mp} z_{\mp}^n \tag{61}$$

satisfy the following limit relations [46]:

$$\overline{\lim_{n\to\infty}} \sqrt[n]{n!|c_n^{\mp}|} = \sigma_{\mp}\,. \tag{62}$$

These relations allow us to establish exact boundaries for domains of representations of wave field by series (38) and (43) over metaharmonic functions, as well as by Atkinson – Wilcox series (39).

Let us now consider the series (43). It is easy to see that its coefficients a_m, can be found from the following relations

$$a_m = \mathrm{i}^m \int_0^{2\pi} q(\varphi) J_m\big(k\rho(\varphi)\big) \exp(-\mathrm{i}m\varphi)\,\mathrm{d}\varphi\,. \tag{63}$$

Using the asymptotics of the Bessel function as $|m| \to \infty$, from (63) we obtain

$$a_m = \frac{\mathrm{i}^{|m|}}{|m|!} \int_0^{2\pi} q(\varphi) \left(\frac{k\rho(\varphi)}{2}\right)^{|m|} \exp(-\mathrm{i}m\varphi)\,\mathrm{d}\varphi \left[1 + O\left(\frac{1}{m}\right)\right]. \tag{64}$$

On the other hand, using relation (59) for the coefficients $c_n^{\mp}$ of series (61), we have

$$c_n^{\mp} = \frac{\mathrm{i}^n}{n!} \int_0^{2\pi} q(\varphi) \left(\frac{k\rho(\varphi)}{2}\right)^n \exp(\pm \mathrm{i}n\varphi)\,\mathrm{d}\varphi\,, \qquad n \geqslant 0\,. \tag{65}$$

Comparing formulas (64) and (65) we find that the following relations

$$a_{|m|} = c_m^+ \left[1 + O\left(\frac{1}{m}\right)\right], \qquad a_{-|m|} = c_m^- \left[1 + O\left(\frac{1}{m}\right)\right] \tag{66}$$

are valid as $|m| \to \infty$. Let us introduce the functions $g_+(\xi_+)$ and $g_-(\xi_-)$, associated via the Borel transform to the functions $f_+^{\mathrm{E}}(z_+)$ and $f_-^{\mathrm{E}}(z_-)$, respectively [47]

$$g_{\pm}(\xi_{\pm}) = \sum_{n=0}^{\infty} n!\, \frac{c_n^{\pm}}{\xi_{\pm}^{n+1}}\,. \tag{67}$$

The functions $g_+(\xi_+)$ and $g_-(\xi_-)$ are regular in domains $|\xi_+| > \sigma_+$ and $|\xi_-| > \sigma_-$, respectively [46]. Expansion (43) can be written down in the form

$$\begin{aligned} u(r,\varphi) &= \sum_{m=0}^{\infty} a_m (-\mathrm{i})^m H_m^{(2)}(kr) \exp(\mathrm{i}m\varphi) \\ &\quad + \sum_{m=1}^{\infty} a_{-m} (-\mathrm{i})^m H_m^{(2)}(kr) \exp(-\mathrm{i}m\varphi) \\ &\equiv F_1(r,\varphi) + F_2(r,\varphi)\,. \end{aligned}$$

Using the asymptotics for the functions $H_m^{(2)}(kr)$ as $m \to \infty$ one can show that the convergence of the series for the functions F_1 and F_2 is equivalent to the convergence of the following two series

$$\begin{aligned} \widehat{F}_1(w_1) &= \frac{\mathrm{i}}{\pi} \sum_{m=1}^{\infty} a_m (m-1)!\, w_1^{-m}\,, \\ \widehat{F}_2(w_2) &= \frac{\mathrm{i}}{\pi} \sum_{m=1}^{\infty} a_m (m-1)!\, w_2^{-m}\,, \end{aligned} \tag{68}$$

respectively, where

$$w_1 = \frac{\mathrm{i}kr}{2} \exp(-\mathrm{i}\varphi)\,, \qquad w_2 = \frac{\mathrm{i}kr}{2} \exp(\mathrm{i}\varphi)\,. \tag{69}$$

Comparing now the power series (68) and (67), we see that the regularity domains for the functions F_1 and g_+, as well as for F_2 and g_- coincide. In other words, the function F_1 is regular in the domain

$$|w_1| \equiv \frac{kr}{2} > \sigma_+\,,$$

and F_2, in the domain

$$|w_2| \equiv \frac{kr}{2} > \sigma_-\,.$$

So, we have found out that the expansion (43) is valid in the domain

$$\frac{kr}{2} > \sigma \equiv \max(\sigma_+, \sigma_-)\,. \tag{70}$$

Since the expansion (43) is a power series in its principal part [see Eqns (68)], then there exists at least one singular point of the function $u(r,\varphi)$ on the circle $r = 2\sigma/k$. Clearly, this point can be located only inside the domain D or on its boundary Γ. Hence, the relation (43) allows one to continue the wave field $u(r,\varphi)$ inside the domain D up to the circle $r = 2\sigma/k$.

Consider now the representation (46). The integral (46) converges or not depending on the behavior of the integrand near points $-\mathrm{i}\infty$, $\pi + \mathrm{i}\infty$. The expression in the exponent of the integrand in Eqn (46) has the form

$$-\frac{\mathrm{i}kr}{2}\, z \exp(\mathrm{i}s\varphi) \left[1 + O\left(\frac{1}{z^2}\right)\right], \qquad s = \pm 1\,,$$

and for function $f(\psi)$ the asymptotics (58) takes place. So, the convergence of integral (46) is equivalent to the existence of the following two integrals

$$g_+(w_1; 0) = \int_0^{\infty \exp(\mathrm{i}0)} f_+^{\mathrm{E}}(\xi_+) \exp(-w_1 \xi_+)\,\mathrm{d}\xi_+\,,$$

$$g_-(w_2; -\pi) = \int_0^{\infty \exp(-\mathrm{i}\pi)} f_-^{\mathrm{E}}(\xi_-) \exp(-w_2 \xi_-)\,\mathrm{d}\xi_-\,,$$

which are Borel transforms of the functions f_{+}^{E} and f_{-}^{E} for half-planes

$$\mathrm{Re}\{w_1 \exp(\mathrm{i}0)\} > h_{+}(0)\,, \qquad \mathrm{Re}\{w_2 \exp(-\mathrm{i}\pi)\} > h_{-}(-\pi)\,,$$

where the functions g_{+} and g_{-} are regular. Here $h_{\pm}(\gamma)$ are indicatrices of growth [46] of functions $f_{\pm}^{\mathrm{E}}(\xi_{\pm})$. In view of Eqn (69) we deduce that integral (46) converges in the domain

$$r\sin\varphi \equiv y > \frac{2}{k}\,\sigma_{\mathrm{S}} \equiv \frac{2}{k}\max\left[h_{+}(0), h_{-}(-\pi)\right]\,, \qquad (71)$$

and the function $u(r,\varphi)$ has at least one singular point on the straight line $y = 2\sigma_{\mathrm{S}}/k$, and this point is located inside D or on its boundary.

In papers [40, 41], the following modification of representation (46)

$$u(r,\varphi) = \frac{1}{\pi}\int_{-\pi/2-\mathrm{i}\infty}^{\pi/2+\mathrm{i}\infty} f(\varphi+\psi)\exp(-\mathrm{i}kr\cos\psi)\,\mathrm{d}\psi\,. \qquad (72)$$

was suggested.

It can be shown that the integral (72) is regular in the domain $\mathbf{R}^2 \backslash \overline{B}_0$, where $\overline{B}_0$ is a convex hull of singularities of continuation of the function $u(r,\varphi)$ inside D. From the above considerations it is clear that $\overline{B}_0$ is a smallest closed convex set containing sets $\overline{B}_{+}$ and $\overline{B}_{-}$ (conjugated diagrams of functions f_{+}^{E} and f_{-}^{E}, respectively. So, the integral (72) allows one to perform analytic continuation of the function $u(r,\varphi)$ into the domain D inside Γ up to the boundary of the set $\overline{B}_0$.

Representation (47) of the wave field scattered by the periodical boundary in the form of series over plane waves regularly (absolutely and uniformly) converges in any closed domain lying inside the half-plane (71) [43].

In three dimensions, for the diagram $f(\theta,\varphi)$ of the wave field there is an asymptotic equality similar to (58)

$$f(\theta,\varphi) = f^{\mathrm{E}}(z,\varphi)\left[1 + O\left(\frac{1}{R}\right)\right]\,, \qquad (73)$$

on the complex plane $\theta = \theta_1 + \mathrm{i}\theta_2$ [38], where

$$z = R\exp(\pm\mathrm{i}\theta_1)\,, \qquad R = \exp|\theta_2|\,,$$

$f^{\mathrm{E}}(z,\varphi)$ is an entire function in z of finite degree σ not exceeding $kr_0/2$, where r_0 is the radius of the minimal sphere containing Γ. As above, the quantity σ is called *the degree of the diagram*. Series (38) and (39) converge in the domain $r > 2\sigma/k$, and the corresponding plane-wave representations do this in the half-space $z \equiv r\cos\theta > 2\sigma_{\mathrm{S}}/k$, where $\sigma_{\mathrm{S}} = h(-\pi/2)$, $h(\gamma)$ is the indicatrix of the growth of the entire function $f^{\mathrm{E}}(z,\varphi)$, and quantities σ, σ_{S} are characteristics of the growth of the function f^{E} independent from φ.

In many situations (for example, for scattering of a plane wave by an obstacle bounded by a smooth surface Γ, as shown below in Section 3.3; see also Refs [44, 15, 48]) the quantities σ and σ_{S} can be found with the help of rather simple relations. In two dimensions these relations have the following form [40, 41]

$$\sigma = \max_{\varphi_0,\,s}\left|\frac{k\rho(\varphi_0)}{2}\exp(\mathrm{i}s\varphi_0)\right|\,,$$

$$\sigma_{\mathrm{S}} = \max_{\varphi_0,\,s}\,\mathrm{Re}\left\{\frac{k\rho(\varphi_0)}{2}\exp\left[\mathrm{i}s\left(\varphi_0 - \frac{\pi}{2}\right)\right]\right\}\,, \qquad (74)$$

where φ_0 are the roots of the equations

$$\left.\frac{\rho'(\varphi)}{\rho(\varphi)}\right|_{\varphi=\varphi_0} = -\mathrm{i}s\,, \qquad \exp(\mathrm{i}s\varphi_0) = 0\,, \qquad s = \frac{\beta}{|\beta|}\,, \qquad (75)$$

and the maximum in Eqn (74) have to be searched among the roots (75) which are taken inside a contour C by a mapping, say, of the form $\xi = \rho(\varphi)\exp(\mathrm{i}\varphi)$ that takes points of the contour Γ to the contour C on the plane $z = r\exp(\mathrm{i}\varphi)$.

In three dimensions [38]

$$\sigma = \max_{\theta_0^s\varphi_0,\,s}\left|\frac{k\rho(\theta_0^s,\varphi_0)}{2}\exp(\mathrm{i}s\theta_0^s)\right|\,,$$

$$\sigma_{\mathrm{S}} = \max_{\theta_0^s\varphi_0,\,s}\,\mathrm{Re}\left\{\frac{k\rho(\theta_0,\varphi_0)}{2}\exp(\mathrm{i}s\theta_0^s)\right\}\,, \qquad (76)$$

where θ_0^s, φ_0 are found from the conditions

$$\left.\frac{\rho'_\theta(\theta,\varphi)}{\rho(\theta,\varphi)}\right|_{\theta_0^s,\,\varphi_0} = -\mathrm{i}s\,, \qquad \left.\rho'_\varphi(\theta,\varphi)\right|_{\theta_0^s,\,\varphi_0} = 0\,,$$

$$\exp(\mathrm{i}s\theta_0^s) = 0\,, \qquad s = \pm 1\,, \qquad (77)$$

$r = \rho(\theta,\varphi)$ is the equation of the surface Γ in spherical coordinates. Here again the maximum in Eqn (76) is searched among those roots of equations (77) which are taken inside contours C_φ which are images of sections of surface Γ by the plane $(\varphi, \varphi+\pi)$ on the plane $z = r\exp(\mathrm{i}\alpha)$ under the mapping $\xi = \rho(\theta,\varphi)\exp(\mathrm{i}\theta)$.

Finally, for scattering of the plane wave by the periodic boundary $y = f(x)$ we have [43]

$$\sigma_{\mathrm{S}} = \max_{x_0,\,z=\pm R}\mathrm{Re}\{F(x_0)\}\,. \qquad (78)$$

Here

$$F(x) = -s\,\frac{k}{2}\left[x - \mathrm{i}sf(x)\right]\exp(-\mathrm{i}s\beta)\,,$$

and x_0 are the roots of the equation

$$\left.f'(x)\right|_{x=x_0} = -\mathrm{i}s\,.$$

The quantities s and β are here of the same sense as in relations (75). If the boundary Γ has non-analytic points, the maximum in relations (74), (76), and (78) must be taken over these points as well.

A lot of examples of usage of the above relations are given in Refs [15, 38, 42–44] and others.

3.3 Inverse problems in theory of scattering and antennas

The results of the previous section allow one to establish the existence conditions for solutions of inverse problems of theory of scattering and antennas [40, 41, 48]. These conditions are worth formulating in the form of the following theorems.

Theorem 3. *The function $f(\alpha)$ given on the interval $[0, 2\pi]$ and analytically continuable to the whole complex plane $\psi = \alpha + \mathrm{i}\beta$, is a diagram of a wave field regular outside the circle of the radius a iff asymptotic equalities* (58) *take place as $|\beta| \to \infty$. In* (58) *$f_{\mp}^{\mathrm{E}}(z_{\mp})$ are entire functions in $z_{\mp} = \exp(|\beta| \mp \mathrm{i}\alpha)$ of finite degree $\sigma \leqslant ka/2$.*

In three dimensions a similar theorem is formulated as follows:

Theorem 4. *A regular function $f(\theta_1, \varphi)$ defined on the unit sphere $(0 \leqslant \theta_1 \leqslant \pi, 0 \leqslant \varphi \leqslant 2\pi)$ and analytically continuable to the whole complex plane $\theta = \theta_1 + \mathrm{i}\theta_2$, is a diagram of a wave field regular outside the sphere of the radius a iff asymptotic equality* (73) *takes place as* $\exp|\theta_2| \to \infty$. *In this equality $f^{\mathrm{E}}(z, \varphi)$ is some entire function in*

$$z = \exp\left(|\theta_2| - \mathrm{i}\,\frac{\theta_2}{|\theta_2|}\,\theta_1\right)$$

of finite degree not exceeding $ka/2$.

Similar in sense theorems, though in much more complicated formulation, are presented in Ref. [32]. These theorems allow one, in particular, to prove the uniqueness of inverse scattering problems, as well as to show approaches to the problem of determinating the form of a scatterer via a given diagram. We do not discuss these questions here; the reader can find such a discussion in the monograph [32] and in works cited there.

Let us consider here a simpler, but at the same time important, question about the sense of restrictions contained in the formulations of the above theorems.

It is well-known (see, e.g. Ref. [46]) that the antenna with plane opening possesses the diagram whose width is strongly connected with the size of the opening and cannot be done arbitrarily small without a strong decrease in the radiating power which is comparable with the input power of the antenna. Antennas with a significant fraction of non-radiating power are called *overdirected*. The distribution of currents (fields) at the opening of such antennas is of rapidly oscillating character and cannot be easily realized [49]. The mathematical problem of finding such distributions via a given diagram appears to be unstable and a small change in the input data essentially changes the distribution of field on the aperture.

Let us consider an example illustrating the importance of the restrictions involved into the above-formulated theorems. As such an example, we consider the problem of synthesis of antenna lattice formed by open ends of waveguides located on the arc of the circle of radius a [50, 51] (a two-dimensional problem). The boundaries of the waveguides are orthogonal to the side surface of ideally conductive cylinder.

The diagram $f(\varphi)$ can be realized by such a lattice in accordance to theorem 3 only in the case when the set of singularities of the field corresponding to this diagram is contained as a whole inside the mentioned circle of radius a.

Consider the realization of the so-called Gaussian diagram

$$f(\varphi) = \exp\left(-p\sin^2\frac{\varphi}{2}\right). \tag{79}$$

It is easy to see that for this diagram

$$f^{\mathrm{E}}(z) = \exp\left(-\frac{p}{2} + \frac{pz}{4}\right). \tag{80}$$

Generalizing (71), we introduce the quantity

$$\sigma_{\mathrm{S}}(\gamma) = \max\{h_+(\gamma),\, h_-(\pi - \gamma)\}. \tag{81}$$

Then in the coordinate system turned through the angle γ the field corresponding to the diagram (79) will be regular in the domain $y_\gamma > (2/k)\sigma_{\mathrm{S}}(\gamma)$ [see Eqn (71)], where y_γ is the ordinate in the coordinate system turned by the angle γ. In other words, the field having the diagram (79) is regular outside the half-plane

$$y_\gamma \leqslant \frac{2}{k}\,\sigma_{\mathrm{S}}(\gamma), \tag{82}$$

and has at least one singular point on the line $y_\gamma = (2/k)\sigma_{\mathrm{S}}(\gamma)$. Hence, the domain $\overline{B}_0$ obtained as intersection of planes (82) for all angles γ is exactly the convex hull of the singularities of the field with diagram (79). Since [46]

$$h(\gamma) = \overline{\lim_{R\to\infty}}\,\frac{\ln\left|f^{\mathrm{E}}\big(R\exp(\mathrm{i}\gamma)\big)\right|}{R},$$

then, taking into account (81), we obtain that for the example considered

$$\sigma_{\mathrm{S}}(\gamma) = \frac{p}{4}\,|\cos\gamma|. \tag{83}$$

Hence, in this case the set $\overline{B}_0$ is a segment $-p/2k \leqslant y \leqslant p/2k$ of the Oy axis.

As follows from (71), (81), and (83), the degree σ of function (80) equals $\sigma = p/4$. Hence, in accordance with theorem 3, the diagram (79) can be realized by the considered lattice only in the case when the radius a of the cylinder is greater that $p/4$.

Figure 8 shows continuous amplitude, and Fig. 9, phase distributions of the component E_φ of the field realizing diagram (79) with the degree $p = 15$. Here the numbers of curves correspond to the following values of the radius ka of the cylinder: *1* — $ka = 40$, *2* — $ka = 20$, *3* — $ka = 16$, *4* — $ka = 8$. It is seen that the last distribution (curve *4*) is close to 'overdirected', though the circle with $ka = 8$ encircles the segment $\overline{B}_0$ (the critical radius of the given diagram is $ka = 7.5$).

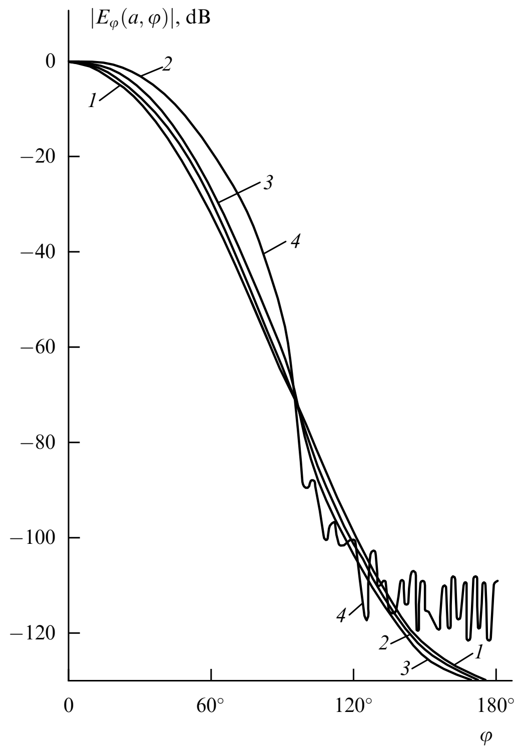

Figure 8.

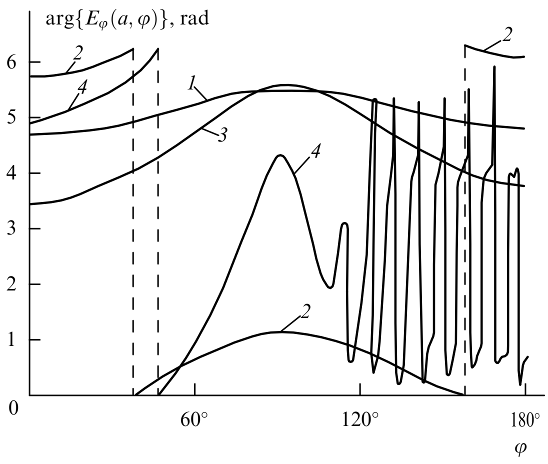

Figure 9.

As can be seen from the example considered, the criteria of realizability of the diagram involved into theorem 3 can be generalized, for instance, in the form of the following affirmation [50, 51]:

Let $\overline{B}$ *be a convex closed domain. Then a function* $f(\alpha)$ *given on the segment* $0 \leqslant \alpha \leqslant 2\pi$ *and analytically continuable to the whole complex plane* $\psi = \alpha + \mathrm{i}\beta$ *can be realized as a diagram of integrable currents distributed on an arbitrary closed Lyapunov contour encircling* $\overline{B}$*, iff asymptotic expansions* (58) *take place as* $|\beta| \to \infty$*, where* $f^{\mathrm{E}}_{\mp}(z_{\mp})$ *are entire functions of finite degree in* $z_{\mp} = \exp\big(|\beta| \mp \mathrm{i}\alpha\big)$*, and the corresponding set* $\overline{B}_0$ [*the convex hull of singularities of the field with the diagram* $f(\alpha)$] *satisfies the relation* $\overline{B}_0 \subseteq \overline{B}$.

Further generalizations of this criterion are given in Ref. [50]. In short words, all the versions of realizability criteria contain one and the same requirement: *the singularities of the wave field corresponding to a given diagram must be contained in the support of currents realizing this diagram.*

In different interpretations, this requirement is also crucial in a lot of widely used methods of solving direct problems of diffraction theory.

Information on the localization of singularities of the wave field is also very important for solving inverse scattering problems (reconstruction of the form of a scatterer by its diagram) [32]. For example, knowledge of the geometry of the set $\overline{B}_0$ (the convex hull of singularities of the wave field) allows one to evaluate the dimensions of the scatterer [32]. The above relations (70), (71), (74), and (76) make it possible to find the set $\overline{B}_0$ in the investigations of scattering of the plane wave. In papers [15, 40–44] one can find a lot of examples of finding the set $\overline{B}_0$ for scatterers of different geometry. So, for example, for the diffraction of a plane wave on a three-axis ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

for $a \geqslant b \geqslant c$ the set $\overline{B}_0$ is an ellipse in the plane xy with semiaxes $\sqrt{a^2 - c^2}$, $\sqrt{b^2 - c^2}$. For $a = b = c$ the ellipsoid reduces to a sphere, and the singularity set becomes a point in its center. The question why we see the sphere instead of its center is quite natural. It seems that the complete answer to this question cannot be obtained in the framework of the pure diffraction theory. However, it is clear that under one and the same illumination conditions an uneven sphere will be better recognized than the polished one. This fact is quite understandable form the viewpoint of the singularity theory. The matter is that the uneven sphere is in fact some body with the form close to the spherical one whose surface can be represented (in the first approximation) by a function with a piecewise-continuous derivative. So, singular points of the field scattered by such a body will be located, in particular, on the surface of this body. Moreover, the closer the form of the body is to the spherical one, the more such singularities will occur. In the limit, these singularities will fill the spherical surface densely, but their 'amplitude' will become very small comparable with the amplitude of the singularity in the center of the sphere, where, by the way, the field has an essential singularity [44, 15].

From the example considered it is clear that for the scatterer to be 'unseen', that is, hardly recognized, its form must be as close to the ideal analytic surface as possible. It must have no angles or other irregularities.

These considerations are, surely, of intuitive character. The clear qualitative and quantitative recommendations can be given only on the basis of a detailed analysis of the process of wave scattering by bodies of different geometry and with different media parameters, which can be carried out on the basis of corresponding mathematical models.

3.4 Analytical properties of wave fields and computational methods of solving diffraction theory problems

Mathematical models of scattering and diffraction of waves are constructed using different numerical methods. The most rigorous methods are based on some analytical representations of the wave (or diffraction) field. The mostly used are representations in wave harmonics (metaharmonic functions) of the form (38) or (39), in the form of series in plane waves (47), and by wave potentials (50). Then, with the help of the boundary condition the problem is reduced to a system of algebraic equations, or to an integral or integrodifferential equation with respect to the potential density (surface or auxiliary current [52, 53]). On the basis of the above discussion, we can suppose that the methods of such kind will lead to the correct computational algorithms only with taking into account the analytic properties of the solution. The existing literature on numerical methods of solving boundary value problems of electrodynamics and acoustics confirms this supposition. So, the algorithms using expansions of the form (38) or (43) [3] turn out to be stable only in the case when the minimal sphere (circle) encircling all singularities of the wave field is contained as a whole inside the scatterer, that is, when the inequality

$$\rho_{\min} > \frac{2\sigma}{k} \tag{84}$$

holds. Here $\rho_{\min}$ is the radius of the minimal sphere inside the scatterer surface Γ, and σ is the degree of the diagram of the wave field (see Section 3.2). This fact, at first glance, contradicts the known Vekua theorem on the completeness of metaharmonic functions [36]

$$\psi_{nm}(r, \theta, \varphi) = h_n^{(2)}(kr) P_n^m(\cos\theta) \exp(\mathrm{i}m\varphi)\,,$$
$$\psi_n(r, \varphi) = H_n^{(2)}(kr) \exp(\mathrm{i}n\varphi)$$

on any Lyapunov surface (contour), as well as of any linear combination of these functions in any closed subdomain outside the scatterer. The explanation to this contradiction is that (see Ref. [54]) the series in such functions giving the best approximation of the wave field everywhere outside the scatterer is not, in general, absolutely convergent. Being rewritten in the form of the Rayleigh series [Eqn (38) or Eqn (43)] it may be rearranged (as already noted) in its infinite remainders to a series in powers of $1/r$ and, hence, converges only outside a sphere (circle) passing through the most distant singular point of the wave field.

Condition (84) for a scatterer, say, of an ellipsoidal form means that the eccentricity ε of the ellipsoid must satisfy the condition $\varepsilon < 1/\sqrt{2}$ [15, 41].

A similar situation also arises while using expansions in plane waves of the form (47). Namely, algorithms based on the usage of expansions of the form (47) [2] are stable only under the condition

$$y_{\min} > \frac{2\sigma_{\mathrm{S}}}{k}\,, \tag{85}$$

where $y_{\min}$ is a minimal ordinate of the surface $y = f(x)$, and the quantity σ_{S} is defined by (71). For example, in scattering of a plane wave by the surface

$$y = a\cos\left(\frac{2\pi}{b}\,x\right),$$

$a > 0$, b is the period, condition (85) is reduced to the following restriction [43]:

$$\frac{2\pi a}{b} < 0.447743\ldots$$

For a trokhoid given by

$$x = a(t + \tau\cos t)\,, \qquad y = a\tau\sin t\,,$$

$$a > 0\,, \qquad 0 < \tau \leqslant 1\,, \qquad 0 \leqslant t < 2\pi\,,$$

the condition (85) restricts the entire mentioned range of the parameter τ by values $\tau < 0.2784613\ldots$ [55].

In practical numerical investigations the restriction of the form (84), (85) appear in the form of the so-called 'computational catastrophes', that is, complete loss of the stability of the algorithm during the attempts of increasing the computational accuracy by enlarging dimensions of the corresponding algebraic systems.

Below, we shall consider an example of occurrence of a catastrophe of such kind in connection with another class of numerical methods, that is, the methods based on representations of the form (50). Using such representations, it is possible to reduce the boundary value problem to integral equations of the first or second kind with respect to the surface currents [32], that is, the currents distributed directly on the surface Γ of the scatterer. In recent years, the methods based on the representations of the form (50) (such as method of auxiliary currents [52, 53] and the method of discrete sources [56, 57]) have gained widespread acceptance. In these methods the support of currents is not the scatterer surface, but some other surface Σ lying inside the scatterer. These methods are preferable due to their algorythmical simplicity, universality and high speed of action. However, the efficiency of these methods depends strongly on the choice of the auxiliary surface Σ — the support of currents. The first version of the methods in question, the nonorthogonal series method, was presented in Ref. [58], where, in particular the completeness of the system of functions $G(\mathrm{r}, \mathbf{r}_n)$ was proved in the case of arbitrary location of points $\mathbf{r}_n$ inside Γ when the points defined by the radius-vector $\mathbf{r}$ lie on an arbitrary Lyapunov surface Γ [see Eqns (51) and (52)]. The reason of this phenomenon is quite clear in view of the results of the preceding section. Solving a boundary value problem by the auxiliary current method or by its discrete analogue, one searches densities of wave potentials distributed on an auxiliary surface Σ which realize the required wave (diffraction) field. The distribution of these densities is found from the boundary value condition. The integral equation of the first kind with a smooth kernel arising in this process has, for instance, the following form [52, 53]:

$$\int_{\Sigma} v(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\,\mathrm{d}\sigma' = u^0(\mathbf{r})\,, \qquad \mathbf{r}' \in \Sigma\,, \qquad \mathbf{r} \in \Gamma\,. \tag{86}$$

We see that this approach has a lot in common with solving an inverse scattering problem: in both cases the problem is reduced to an integral equation of the first kind. It is clear that for the given right-hand side $u^0(\mathbf{r})$ Eqn (86) may have no solutions. On the other hand, it is easy to show [52, 53] that if the surface Σ encircles all singularities of the wave field $u(\mathbf{r})$, Eqn (86) has a solution. More precisely, the following theorem takes place [52, 53, 56]:

Let Σ be an arbitrary closed nonresonance Lyapunov surface in D (D is a domain inside Γ). Then necessary and sufficient condition of solvability of equation of the type (86) *in $L_p(\Sigma)$, $p \geqslant 1$ is that the surface Σ encircles the set of singularities of the scattered field continued inside D.*

The requirement of the absence of resonances, that is, of the absence of free oscillations in the domain inside Σ on the given frequency $f = kc/2\pi$, is connected with the form of Eqn (86). If one reduces the boundary value problem to equations containing both densities μ and v of potentials, then this requirement may be omitted. The condition that all singularities of the wave field are encircled by the surface Σ is needed for the discrete analogue of the method as well [52, 53].

Let us consider an example illustrating the above considerations. Figures 10 and 11 show the results of solving the problem of scattering of the plane wave by the elliptic cylinder with semiaxes $ka = 3$; $kb = 1.2$; $kf = 2.7495\ldots$ ($2f$ is an interfocal distance) by the method of discrete sources [59]. Integral in Eqn (86) was replaced by a sum with the help of the rectangle formula, and right- and left-hand parts were equated at the so-called collocation points the number of which can exceed the number of terms in the sum approximating the integral, that is, the number of sources. In the latter case we obtain an overdetermined algebraic system.

The singularities of the wave field in this problem are located at focal points [44, 15]. Figure 10 shows the dependence of the remainder δ in the boundary value condition on the parameter kd characterizing the distance between the auxiliary contour Σ and the contour Γ of the cross-section of the cylinder. As Σ we have chosen the cofocal ellipse with the small axis $b_1 = b - d$. Numbers *1, 2, 3* denote graphs obtained when 30, 80, and 120 sources, respectively, were used. It is seen that for the correct choice of the auxiliary contour (so that it encircles the interfocal segment), one can obtain very high accuracy of computations, and no destruction of the algorithm occurs even for large dimensions of algebraic systems. The value of the error in the boundary value condition is the most pertinent and very sensible indicator of the correctness of the solution obtained since

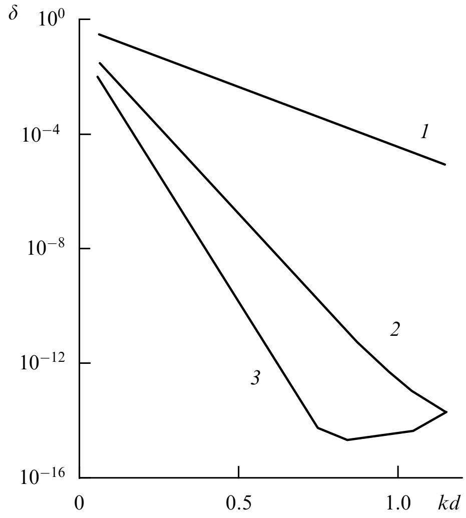

Figure 10.

the integral characteristics (such as the scattering diagram) are computed with much more high accuracy then boundary conditions. However, the norm of current on the auxiliary contour can also serve as an indicator of the algorithm stability.

In Figure 11 we present the graphs of the dependence of the norm of current $\|v(\mathbf{r}')\|$ on the parameter kg defined by the relation $g = a - a_1$, where a_1 is the length of the large semiaxis of the auxiliary elliptic contour with small semiaxis equal to $kb_1 = 0.7$. Curves *2* and *3* correspond to dimensions of algebraic system 60×60 and 120×120. Curve *1* corresponds to the case when the number of collocation points was five times more that the number of discrete (auxiliary) sources (the dimension of the algebraic system is 60×300). It can be seen that as soon as the auxiliary contour fails to encircle the interfocal segment (for $kg > 0.25$), the computational catastrophe occurs: the norm of current increases the faster, the larger is the dimension of the algebraic system, that is, the higher is the potential accuracy of computations (see Fig. 10).

So, the considerations of this subsection show the importance of taking into account the analytical properties of solutions for constructing correct computational algorithms of solving boundary value problems of diffraction theory. In the first part of the paper it was shown that this information can be obtained *a priori*, that is, before solving the corresponding boundary value problem. In the next section we shall discuss a method of solving boundary value problems of diffraction theory where this *a priori* information about analytical properties of solutions is used in working out and verification of the computational algorithm.

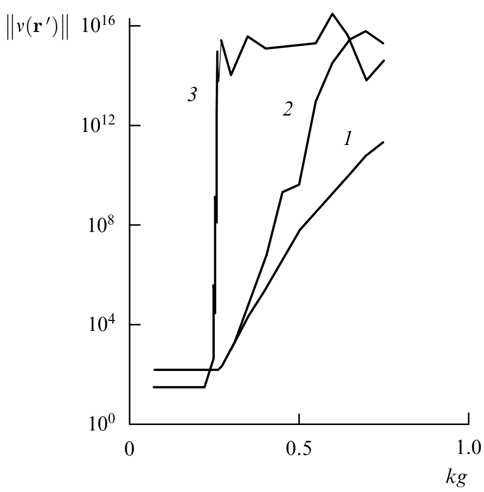

Figure 11.

3.5 Method of diagram equations

This method was applied to solving the problem of scattering waves by a compact obstacle [42, 60], a periodic boundary between two media [55, 61], the group of scatterers [62], and to some other problems. The algorithms constructed on the basis of this method proved to be highly effective and stable. In this method, the boundary value problem is reduced to an integrooperator equation with respect to the diagram of the wave field or to an algebraic system with respect to coefficients of expansion of this diagram in some basis. The resultant equations allow one to find the diagram directly without the auxiliary stage of computing the densities of wave potentials (currents) on the scatterer surface (or on the boundary between media). Besides, since the diagram is almost invariant (see Section 3.3) with respect to the choice of the support of 'currents', it is possible to obtain the correct algorithms of solving the problem of scattering of waves by the group of scatterers up to their mutual contact [62]. Other merits of this method will be discussed below.

To be definite, let us consider a concrete realization of the method on the example of the problem of scattering of a sound wave on a three-dimensional compact scatterer with Dirichlet conditions on the boundary [42]. In this case the representation (54) takes place for the scattering diagram, where we have to put $\mu = 0$ in accordance to the boundary condition. Then, using a modification of integral (45) allowing the reconstruction of the wave field $u(r, \theta, \varphi)$ everywhere outside the convex hull $\overline{B}_0$ of singularities of the function u via the diagram $f(\theta, \varphi)$, one can (if the surface $\varGamma$ encircles $\overline{B}_0$) find the density of the potential $v(\varGamma)$ [see the text after Eqn (53)].

Finally, substituting the expression found for $v(\varGamma)$ into the representation (54), we obtain the required integrooperator equation for the diagram $f(\theta, \varphi)$. If one substitutes the expansion of the diagram into the series in spherical harmonics

$$f(\theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{mn} P_n^m(\cos\theta) \exp(\mathrm{i}m\varphi) \tag{87}$$

into the equation obtained and performs the integration over the surface of a unit sphere, the following algebraic system can be found

$$a_{nm} = a_{nm}^0 + \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} G_{nm,\,\nu\mu}\, a_{\nu\mu}\,, \tag{88}$$

for coefficients a_{nm} of expansion (87). In deriving it, one should take into account the orthogonality of functions $P_n^m(\cos\theta)\exp(\mathrm{i}m\varphi)$. In the latter formula

$$a_{nm}^0 = \mathrm{i}^n (2n+1) \frac{(n-m)!}{(n+m)!} \int_0^{2\pi} \int_0^{\pi} q^0(\theta, \varphi)\, j_n\big(k\rho(\theta, \varphi)\big) \times P_n^m(\cos\theta) \exp(-\mathrm{i}m\varphi)\, \mathrm{d}\theta\, \mathrm{d}\varphi\,, \tag{89}$$

$$G_{nm,\nu\mu} = \mathrm{i}^{n-\nu}(2n+1)\frac{(n-m)!}{(n+m)!}$$
$$\times \frac{\mathrm{i}}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left\{k^2\rho^2(\theta,\varphi)h_\nu^{(2)\prime}(k\rho)P_\nu^\mu(\cos\theta)\sin\theta\right.$$
$$- k\rho_\theta' h_\nu^{(2)}(k\rho)\frac{\mathrm{d}}{\mathrm{d}\theta}\left[P_\nu^\mu(\cos\theta)\right]\sin\theta$$
$$\left.- \mathrm{i}\mu\frac{k\rho_\varphi'}{\sin\theta}h_\nu^{(2)}(k\rho)P_\nu^\mu(\cos\theta)\right\}j_n(k\rho)$$
$$\times P_n^m(\cos\theta)\exp\left[\mathrm{i}(\mu-m)\varphi\right]\mathrm{d}\theta\,\mathrm{d}\varphi\,, \qquad (90)$$

and

$$q^0(\theta,\varphi) = -\frac{k}{4\pi}\left[\rho^2(\theta,\varphi)\sin\theta\,\frac{\partial u^0}{\partial r} - \rho_\theta'(\theta,\varphi)\sin\theta\,\frac{\partial u^0}{\partial\theta}\right.$$
$$\left.\left. - \frac{\rho_\varphi'(\theta,\varphi)}{\sin\theta}\frac{\partial u^0}{\partial\varphi}\right]\right|_{r=\rho(\theta,\varphi)}\,,$$

$j_n(x)$ is the spherical Bessel functions of order n.

We see that, in contrast to traditionally used methods, the matrix elements of algebraic system (88) are determined by two-dimensional integrals (and not four-dimensional as with traditional methods such as, for example, the method of integral equations with respect to surface currents).

The matter is that in the current integral equations method the reduction to an algebraic system is performed with the help of some projection relations. To do this it is necessary to integrate twice over the surface Γ: first in the process of substitution of the expansion, say, in metaharmonic functions, of the searched current into the integral equation, and, second, when projecting the equality obtained to some basis. In the method under consideration the second integration is fulfilled over a unit sphere, where spherical functions are orthogonal. In the case of a body of rotation the system (88) and formulas (89) and (90) are essentially simplified [42].

The system (88) was obtained from the integral equation which was valid under the condition that the set $\overline{B}_0$ is contained as a whole inside Γ. However, deducing the system of equations we had to perform a series of mathematical operations which could change the conditions of existence of solutions. If one performs the estimate of the asymptotics (as $n\to\infty$, $\nu\to\infty$) of the matrix elements and right-hand sides of the system, then, using some sufficient criterion of solvability of infinite algebraic system by the reduction method, the conditions required can be obtained.

Performing corresponding estimates of integrals (90), we obtain [42], that, for example, for $n \gg \nu$

$$|G_{nm,\nu\mu}| \leqslant \text{const}\times\frac{\sigma_1}{nn!}\,,$$

where σ_1 coincides with the above defined parameter σ [see the relations (76) and (77)]. Similar, for $\nu \gg n$ one can obtain that

$$|G_{nm,\nu\mu}| \leqslant \text{const}\times\frac{\nu!}{\sigma_2^n}\,,$$

where the parameter σ_2 is defined by the relation

$$\sigma_2 = \min_{\theta_0^s,\varphi_0,s}\left|\frac{k\rho(\theta_0^s,\varphi_0)}{2}\exp(\mathrm{i}s\theta_0^s)\right|\,,$$

and the minimum is searched on the set of roots of Eqn (77) which are taken by the change of variable $\xi = \rho(\theta,\varphi)\exp(\mathrm{i}\theta)$ outside contours C_φ on the plane $z = r\exp(\mathrm{i}\alpha)$ [see the text after Eqns (77)].

For right-hand sides a_{nm}^0 the asymptotic estimate of integrals (89) gives [42]:

$$|a_{nm}^0| \leqslant \text{const}\times\frac{\sigma^n}{nn!}\,, \qquad \sigma = \max(\sigma_1,\sigma_0)\,,$$

where $\sigma_0 = kr_0/2$ and r_0 is the distance to the furthest from the origin point inside Γ corresponding to a singularity of the continuation of the function $q^0(\theta,\varphi)$ to the domain of complex angles θ. For the case when u^0 is a field of a plane wave, we have $\sigma_0 = 0$, that is $\sigma = \sigma_1$.

As we have already mentioned, the quantity $2\sigma/k$ equals the distance from the origin to the furthest singularity of the continuation of diffraction field inside Γ. Conversely, the quantity $2\sigma_2/k$ equals the distance from the origin to the nearest singular point occurring during the continuation of the field from the boundary of Γ into the domain interior with respect to Γ.

If we replace the unknown coefficients a_{nm} in system (88) by new ones putting

$$a_{nm} = \frac{\sigma^n}{n!}x_{nm}\,,$$

then for coefficients x_{nm} we obtain a new algebraic system which is solvable under the condition

$$\sigma_2 > \sigma\,, \qquad (91)$$

that is, in the case when the domain to which the exterior field is extendible has a nonempty intersection with the domain to which the interior field might be extendible (if, for example, we solve the interior boundary value problem). Condition (91) preserves its form, and the quantities σ and σ_2 preserve their meaning independently of the type of boundary conditions on Γ.

Let us consider two examples. Figure 12 shows the scattering diagram of the spheroid with semiaxes $ka = kb = 5$, $kc = 0.125$ for the case of a plane wave incident along the rotation axis (solid line) [59]. The dashed line shows the dependence of azimuthal component of the vector scattering diagram of a plane electromagnetic wave falling

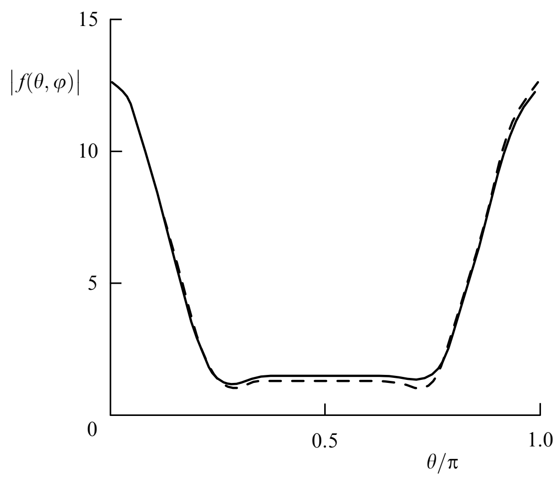

Figure 12.

normally on a thin disk of the radius $ka = 5$. This dependence is taken from Ref. [63].

The Table contains data illustrating the convergence in computation of the diffraction on the spheroid with semiaxes $ka = kb = 5$, $kc = 0.5$. Here cond is a conditional number of the system of equations, S_i is the integral cross-section of scattering, S_0 is the value of this parameter obtained from optical theorem, N is an upper limit of summation over ν in the system of the form (88).

Table

	$N = 7$	$N = 8$	$N = 9$
cond	~ 1	~ 1	~ 1
$S_i/2\pi a^2$	1.0330	1.0330	1.0330
$\|1 - S_0/S_i\|$	8×10^{-3}	6×10^{-4}	4×10^{-4}

It can be seen that the results become stable in the second or third sign for $N \geqslant kd/2 + 2$, where d is a maximal size of the scatterer. This conclusion also remains valid when one solves scattering problems for obstacles with another geometry.

This remarkable fact of high convergence of algorithms of the diagram equations method has a simple explanation. The matter is that in this method the problem is formulated directly for the scattering diagram (or its Fourier coefficients). On the other hand, the structure of the diagram, that is, its width, the width and the level of its main petal etc. is determined mainly by the ratio of the size of the scatterer to the wavelength. Therefore, almost independently of the form of the scatterer, the convergence is the same as with the solution of the problem of scattering on a sphere surrounding the scatterer.

The second example shows that the condition (91) is not only sufficient, but also necessary for the system of algebraic equations with respect to coefficients x_{nm} to be solvable by the reduction method. Figure 13 illustrates the dependence of the quantity $|1 - S_0/S_i|$ on the ratio a/c when $kc = 1$ for the body formed by the rotation of the curve $\rho(\theta) = c\cos^2\theta + a\sin^2\theta$ around the polar axis. Condition (91) for this case has the form $a/c > 0.5$. From Fig. 13 one can see that for $a/c \approx 0.5$ the qualitative jump occurs: for $a/c > 0.5$ with the growth of N the accuracy is increased, and for $a/c < 0.5$ it decreases, that is, the algorithm is destroyed.

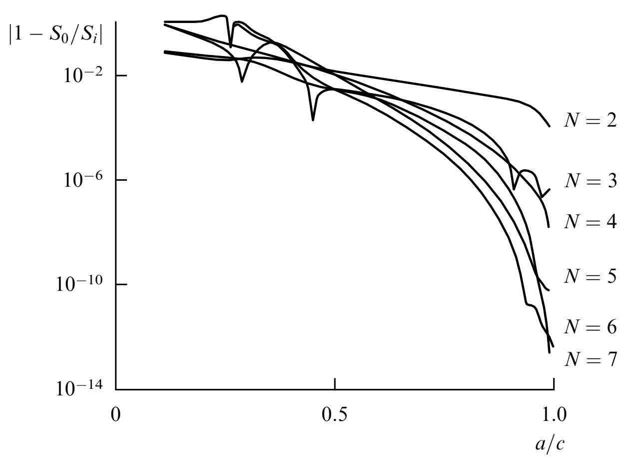

Figure 13.

The scatterers for which the boundary Γ satisfies the inequality $\sigma_2 > \sigma_1$, are called *weakly non-convex* in Refs [42, 60]. Such are, in particular, all convex scatterers. The class of weekly non-convex scatterers is much wider than the set of bodies for which the methods based on the representation of the wave field in the form of expansion (38) are applicable (that is, those for which the so-called Rayleigh hypothesis is valid). At the same time, as shown by computations, the diagram equations method allows one to construct computational algorithms with very high speed of convergence close to that achieved for the method of separation of variables (when the latter is applicable).

4. Conclusions

Summing up the above material, we can conclude that the mathematical part of the problem of continuation of solutions to Maxwell and Helmholz equations is solved in principle. In particular, there is full clarity with the localization of singularities of continuation. Concerning the physical or applied part of the problem, here the work, in the author's opinion, is just begun. We come to the analogue of the situation that existed in the sixties — seventies after it was found out that the diagrams of the so-called aperture antennas are related to some class of entire functions. This fact then allowed joining different questions of antenna constructing into the framework of the unified synthesis theory.

In the present review, we tried, in particular, to demonstrate that the wave fields continuation theory not only allows one to obtain a deeper understanding of such a complicated phenomenon as the diffraction of electromagnetic and sound waves, but also gives concrete recommendations on creation of highly effective and stable algorithms of mathematical modelling of diffraction and scattering problems.

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