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Quantum field renormalization group in the theory of fully developed turbulence

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Contents

Abstract. **Quantum field renormalization group results of the theory of developed turbulence are reviewed. Background information about quantum field renormalization theory, including operator expansion and the renormalization of composite operators is given. As an example problem, the stochastic model of isotropic homogeneous turbulence is considered for which, using the renormalization technique, the existence of infrared scaling with Kolmogorov dimensions is proved. The dimension of composite operators and the infrared asymptotic behavior of various correlation functions are discussed, and numerical amplitude factors of scaling laws are calculated.**

1. Introduction

The renormalization group (RG) method introduced originally in quantum field theory (QFT) to meet the needs of elementary particle physics was successfully applied, in the early 70s, by Wilson and others to the theory of critical phenomena in order to substantiate the critical scale invariance (scaling) and to evaluate universal characteristics of critical behavior (critical exponents and scaling functions) in the form of ϵ-expansions. Later on, it was generalized to other problems exhibiting scaling in the infrared region: critical dynamics, random walks, polymer physics, and finally, the theory of fully developed hydrodynamic turbulence. The present paper is a review of basic results obtained by the RG method in the turbulence theory for more than the past fifteen years.

In contrast to the theory of critical behavior, the RG technique in the theory of a fully developed turbulence is not still generally accepted and it is used in the form of rather different formalisms (the field-theoretical RG, Wilson's recursion relations, iteration averaging over modes of sublattice scales), which greatly complicates mutual understanding of various specialists in this field of activity. Therefore, in this paper, a systematic use is made of the standard field-theoretical RG technique as being reliably based on the quantum field theory of renormalization and well-developed methods of evaluation of RG functions and critical dimensions (analytic regularization, the scheme of minimal subtractions, etc.); and a detailed account is given not only for physical results but also for the RG method as well.

In Section 2 we consider the simplest stochastic model of homogeneous isotropic turbulence of an incompressible fluid and give a proof of the existence of infrared scaling with the Kolmogorov dimensions. In Section 3 we examine more complicated problems of the RG theory of turbulence related to renormalization of composite operators, the Wilson

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operator product expansion, and to the study of infrared asymptotic behavior of various scaling functions.

The RG theory of turbulence raised a lot of new problems which were absent in the theory of critical behavior: 'freezing' of critical dimensions, 'dangerous' composite operators with negative dimensions, and so on. Solution of these problem requires going beyond the scope of ϵ-expansions and simultaneous use of the RG-technique and other methods (infrared perturbation theory, operator product expansion, functional equations of the Schwinger type, etc.). These problems, which still need further investigation, are also addressed in Section 3.

Basic references on the physics of a developed turbulence are [1–4]; on the quantum-field theory of renormalization, [5, 6]; on application of the RG method in the theory of critical behavior, [7–10]; on functional methods of QFT, [11, 12].

2. RG method in the stochastic model of isotropic turbulence

2.1. The Navier–Stokes equation. Phenomenology of a developed turbulence

As a microscopic model of a developed (homogeneous, isotropic) turbulence of an incompressible viscous fluid we consider the stochastic Navier–Stokes equation with an external random force [1, 3, 13, 14]

$$\nabla_t \varphi_i = \nu_0 \Delta \varphi_i - \partial_i p + F_i \,, \qquad \nabla_t \equiv \partial_t + (\varphi\partial) \,. \tag{1}$$

Here φ_i is a transverse (owing to incompressibility) vector field of velocity; p and F_i are, respectively, density and transverse external random force per unit mass of the fluid (all these quantities depend on $x \equiv t, \mathbf{x}$); ν_0 is a kinematic viscosity coefficient; ∇_t is the Galileo-covariant derivative. We consider equation (1) on the whole time axis t and specify its solution by the requirement that φ vanishes for $t \to -\infty$. Distribution of F is supposed to be Gaussian with zero mean and correlator

$$\langle F_i(x) F_j(x') \rangle = \delta(t-t')(2\pi)^{-d} \times \int \mathrm{d}\mathbf{k}\, P_{ij}(\mathbf{k}) d_F(\mathbf{k}) \exp\left[\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{x}')\right] \,, \tag{2}$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ is a transverse projector; $d_F(k)$ is a certain function of $k \equiv |\mathbf{k}|$ and parameters of the model; d is the $\mathbf{x}$-space dimension. The random force serves as a phenomenological model of 'pumping' energy into the system which occurs owing to interaction with large-scale motions. The average power W of pumping is connected with the function d_F in Eqn (2) as follows:

$$W = \frac{d-1}{2(2\pi)^d} \int \mathrm{d}\mathbf{k}\, d_F(\mathbf{k}) \,. \tag{3}$$

In the framework of the stochastic model one may neglect initial and boundary conditions and directly consider homogeneous developed turbulence [1, 13, 14]; in this case the field φ in Eqn (1) corresponds only to the chaotic (pulsation) component of realistic velocity.

Equation (1) is solved by iterations with respect to nonlinearity with a subsequent averaging over distribution of the stochastic force. Quantities to be evaluated are various Green's functions, correlation functions $\langle \varphi(x_1) \ldots \varphi(x_n) \rangle$, and also response functions, i.e., variational derivatives of correlation functions with respect to nonstochastic external forces which may be inserted into the right-hand side of Eqn (1).

A simplified physical picture of turbulence is as follows (Refs [1, 4]): the energy of an external source (our random force) comes to the system from the large-scale motions (vortices) with a characteristic size $l_{\max}$, then it is transferred along the spectrum ('vortex subdivision') owing to nonlinearity of equation (1), and finally at scales $l_{\min}$ ('dissipative length'), where the role of viscosity becomes important, active dissipation comes to the fore. Independent parameters are W, ν_0, and $m \equiv l_{\max}^{-1}$ (the latter will be called 'the mass' in analogy with quantum-field models); all the rest may be expressed through them on the grounds of scaling considerations: at large scales through parameters W, m, at small scales through W, ν_0. For instance, $l_{\min} = 1/\Lambda$, $\Lambda = W^{1/4} \nu_0^{-3/4}$. A developed turbulence is characterized by a large value of the Reynolds number $\mathrm{Re} = (\Lambda/m)^{4/3}$, and as a consequence, by a wide 'inertial range' determined by inequalities $m \ll k \ll \Lambda$ for momenta (wave numbers) and $\omega_{\min} \equiv W^{1/3} m^{2/3} \ll \omega \ll \omega_{\max} \equiv \nu_0 \Lambda^2$ for frequencies [1].

The phenomenological Kolmogorov–Obukhov theory [1] was based on two hypotheses. The first one appeared to be too restrictive and was later modified. We first cite the original version, denoting it with 1′:

Hypothesis 1′ (see Ref. [1], p. 319): distribution of Fourier components $\varphi(\omega, \mathbf{k})$ of the random velocity $\varphi(x) \equiv \varphi(t, \mathbf{x})$ within the range $k \gg m$, $\omega \gg \omega_{\min} = W^{1/3} m^{2/3}$ depends on the total power W of pumping but is independent from 'details of its construction,' in particular it does not depend on the value of m.

Hypothesis 2 (see Ref. [1], p. 321): in the range $k \ll \Lambda$, $\omega \ll \omega_{\max} = \nu_0 \Lambda^2$, the above distribution does not depend on the viscosity coefficient ν_0.

The second hypothesis implies, in particular, that in the range of its applicability the pair correlator of the Fourier components of velocity in the case of a d-dimensional problem can be represented in the form

$$\langle \varphi_i(\omega, \mathbf{k}) \varphi_j(\omega', \mathbf{k}') \rangle = (2\pi)^{d+1} \delta(\omega + \omega') \delta(\mathbf{k} + \mathbf{k}') \times P_{ij}(\mathbf{k}) D(\omega, k) \,,$$

$$D(\omega, k) = W^{1/3} k^{-d-4/3} f\left(\frac{Wk^2}{\omega^3}, \frac{m}{k} \right) , \tag{4}$$

where f is a certain function of two independent dimensionless arguments.

However, according to hypothesis 1′, in the inertial range the dependence on m should vanish, i.e., the function f in Eqn (4) should be finite when its second argument m/k tends to zero. However, it is known for a long time (see Ref. [1]) that this is not the case: owing to the kinematic effect of transfer of vortices by large-scale motions with $k \simeq m$, for dynamic objects of the type of (4) the limit $m/k \to 0$ does not exist. The corrected version of 1' is as follows (see, for instance, Ref. [15]):

Hypothesis 1: within the range $k \gg m$, $\omega \gg (Wm^2)^{1/3}$, simultaneous distribution functions of spatial Fourier components $\varphi(t, \mathbf{k})$ of the random velocity field $\varphi(t, \mathbf{x})$ converge to a finite limit as $m/k \to 0$.

With the aid of hypothesis 2 [or integrating (4) over frequencies ω, ω'] we obtain the following representation for the static pair correlator:

$$\langle \varphi_i(t,\mathbf{k})\varphi_j(t,\mathbf{k}')\rangle = (2\pi)^d\delta(\mathbf{k}+\mathbf{k}')P_{ij}(\mathbf{k})D_{\mathrm{st}}(k)\,,$$
$$D_{\mathrm{st}}(k) = k^{-d}\left(\frac{W}{k}\right)^{2/3} f\left(\frac{m}{k}\right). \qquad (5)$$

In accordance with hypothesis 1, there exists a finite limit $f(0)$ of the function $f(m/k)$ when $m/k \to 0$, and the quantity $f(0)$ appears to be connected with the well-known Kolmogorov constant (see Section 3.9). Representation (5) is valid for all $k \ll \Lambda$, i.e., both in the inertial range $m \ll k \ll \Lambda$, where the function $f(m/k)$ can be replaced by the constant $f(0)$, and in the 'energy-containing region' (pumping range), where it is nontrivial.

Representations of the type of (4) and (5) can be written for more complicated correlation functions with any number of fields φ. All of them are only based on hypothesis 2, and taken as a whole, establish the infrared (since the conditions $k \ll \Lambda$, $\omega \ll \nu_0\Lambda^2$ of hypothesis 2 do not impose restrictions at large scales) scaling invariance (or, simply, scaling) with certain Kolmogorov's dimensions $\Delta_F \equiv \Delta[F]$ for all 'IR-relevant' quantities $F = \{\varphi \equiv \varphi(x),\ m,\ t \sim \omega^{-1},\ r \sim k^{-1}\}$ with W, ν_0 being irrelevant:

$$\Delta_\varphi = -\frac{1}{3}\,, \quad \Delta_t = -\Delta_\omega = -\frac{2}{3}\,, \quad \Delta_k = -\Delta_r = \Delta_m = 1\,. \qquad (6)$$

In conclusion we shall more precisely specify the form of the pumping function d_F which is used in RG theory of turbulence for definition of correlator (2). From a physical standpoint, a realistic pumping should be infrared, i.e., the function d_F should depend on the parameter $m \ll \Lambda$ and the main contribution to integral (3) should come from scales $k \sim m$. On the other hand, to employ the standard field-theoretical RG technique, it is important that the function d_F has a power-like asymptotic form at large k. The function

$$d_F = D_0 k^{4-d}(k^2+m^2)^{-\epsilon} \qquad (7)$$

used in Refs [15, 16] meets this condition. Here $\epsilon > 0$ has the meaning of 'deviation from the logarithmic theory' (for details, see Section 2.4). A logarithmic theory is specified by the value $\epsilon = 0$, whereas the pumping (7) becomes infrared only when $\epsilon > 2$. In the region $0 < \epsilon < 2$ the pumping (7) is ultraviolet (UV) and makes the integral (3) diverge at large k. This implies that it must be cut off at $k \leqslant \Lambda$ and, therefore, the main contribution to this integral comes from scales $k \sim \Lambda$. Thus, in this case $W \sim D_0\Lambda^{4-2\epsilon}$, unlike $W \sim D_0 m^{4-2\epsilon}$ for $\epsilon > 2$.

In most studies on RG theory of turbulence a purely power-like pumping corresponding to $m = 0$ in Eqn (7) is used:

$$d_F = D_0 k^{4-d-2\epsilon}\,. \qquad (8)$$

This can be done if we only deal with the substantiation of IR scaling and calculation of critical dimensions (which do not depend on m for any pumping), and represent other objects of the type of scaling functions only in the framework of perturbation theory in the form of ϵ-expansions. In this case transition to the theory with $m = 0$ is self-consistent since the ϵ-expansion of diagrams always results in finite coefficients when $m \to 0$. However, this does not prove the Kolmogorov hypothesis 1, because for finite ϵ in the realistic range $\epsilon > 2$ of IR pumping the limit $m \to 0$ may not exist (a simple example is the function $m^{2-\epsilon}$: its ϵ-expansion coefficients vanish when $m \to 0$, whereas the function itself diverges if $\epsilon > 2$). Therefore hypothesis 1 can only be considered in the framework of models of the type of (7) with the parameter $m \neq 0$, and beyond the scope of ϵ-expansion. If we do not touch upon these problems, we can use the most simplified model (8). A physical value of ϵ for it is $\epsilon_{\mathrm{p}} = 2$. It corresponds to the boundary of IR-pumping given by Eqn (7), since for $\epsilon > 2$ the integral (3) with the function (8) does not exist owing to IR-divergence, and for $\epsilon < 2$ the pumping is ultraviolet. Note that at $\epsilon = 2$ the parameter D_0 in Eqn (8) acquires the dimension of W. Also note that an idealized pumping caused by infinitely large vortices corresponds to $d_F(\mathbf{k}) \propto \delta(\mathbf{k})$, while the function Ck^{-d}, with a proper choice of the amplitude C, can be regarded as a power-like model of a d-dimensional δ-function (see Section 3.7). A more realistic model of pumping is of course the model (7) or its generalization

$$d_F = D_0 k^{4-d-2\epsilon} h\left(\frac{m}{k}\right), \qquad h(0) = 1\,, \qquad (9)$$

where $h(m/k)$ is an arbitrary 'good enough' function ensuring the convergence of integral (3) at small k and normalized to unity at $k \gg m$, where Eqn (9) turns into (8). Most of the results for the inertial range do not depend on the choice of h, and we will expound them for the model (9) assuming h arbitrary.

2.2 Field-theoretical formulation

It is well-known (Refs [17, 18]) that the initial-value problem for the stochastic equation

$$\partial_t\varphi(x) = V(x;\varphi) + F(x)\,, \qquad \langle F(x)F(x')\rangle = D_F(x,x') \quad (10)$$

in a field or a system of fields $\varphi(x) \equiv \varphi(t,\mathbf{x})$ with an arbitrary functional V which does not contain time derivatives of φ and an arbitrary correlator D_F of the Gaussian random force F, when defined in a standard way as we did it for Eqn (1), is equivalent to a quantum theory of the doubled set of fields $\Phi \equiv \varphi, \varphi'$ with the action

$$S(\Phi) = \frac{\varphi' D_F \varphi'}{2} + \varphi'(-\partial_t\varphi + V)$$
$$\equiv \frac{1}{2}\iint \mathrm{d}x\,\mathrm{d}x'\,\varphi'(x)D_F(x,x')\varphi'(x')$$
$$+ \int \mathrm{d}x\,\varphi'(x)\left[-\partial_t\varphi(x) + V(x;\varphi)\right]. \qquad (11)$$

This means that statistical averages $\langle\ldots\rangle$ of random quantities can be identified with functional averages with the weight $\exp\ S(\Phi)$. Therefore, generating functionals of total $(G(A))$ and connected $(W(A))$ Green's functions for the problem (10) are represented by the functional integral

$$G(A) = \exp W(A) = \int \mathcal{D}\Phi \exp\left[S(\Phi) + A\Phi\right] \qquad (12)$$

with arbitrary sources $A \equiv A_\varphi, A_{\varphi'}$ in the linear form

$$A\Phi \equiv \int \mathrm{d}x\left[A_\varphi(x)\varphi(x) + A_{\varphi'}(x)\varphi'(x)\right]. \qquad (13)$$

The generating functional of 1-irreducible Green's functions is given by the Legendre transform of $W(A)$ with respect to A:

$$\Gamma(\Phi) = W(A) - A(\Phi)\,, \qquad \Phi(x) = \frac{\delta W(A)}{\delta A(x)}\,. \tag{14}$$

Independent argument here is Φ, whereas $A = A(\Phi)$ is implicitly determined by the second equation in (14). The source $A_{\varphi'}$ has the meaning of nonstochastic external force [additional term to V in Eqn (10)], therefore, in particular, the Green function $\langle\varphi\varphi'\rangle$ of model (11) coincides with the simplest response function $\delta\langle\varphi\rangle/\delta A_{\varphi'}\big|_{A=0}$ in the initial-value problem (10).

For brevity in what follows we shall use the compact form of writing used in Eqns (11)–(13), implying all the necessary integrations over the variables $x \equiv t, \mathbf{x}$ and summations over indices of the fields Φ and sources A.

Integral (12) is a standard construction of QFT, therefore, all the Green functions allow standard Feynman diagram representations (see, for instance, Ref. [12]). Diagram lines are represented by elements of a 2×2 matrix $\langle\Phi\Phi\rangle_0$ of bare propagators, which is connected by formula $\langle\Phi\Phi\rangle_0 = K^{-1}$ with the matrix K defining the free (quadratic in Φ) part of action $S_0 = -\Phi K\Phi/2$. If $V = L\varphi +$ nonlinear terms (with a linear operator L, $\operatorname{Re} L \leqslant 0$), bare propagators for action (11) are of the form

$$\langle\varphi\varphi'\rangle_0 = \langle\varphi'\varphi\rangle_0^{\mathrm{T}} = (\partial_t - L)^{-1}\,, \qquad \langle\varphi'\varphi'\rangle_0 = 0\,,$$
$$\langle\varphi\varphi\rangle_0 = \langle\varphi\varphi'\rangle_0 D_F \langle\varphi'\varphi\rangle_0\,, \tag{15}$$

where T denotes operator transpose (permutation of arguments in the coordinate representation and $\partial^{\mathrm{T}} = -\partial$ for derivatives). Propagator $\langle\varphi\varphi'\rangle_0$ is retarded [this is an auxiliary condition for equation (10)], while $\langle\varphi'\varphi\rangle_0$ is advanced. Together with $\langle\varphi'\varphi'\rangle_0 = 0$, this results in vanishing of any 1-irreducible Green's function $\langle\varphi\ldots\varphi\rangle_{\text{1-ir}}$ of fields φ only (i.e., without φ') because any corresponding diagram contains a closed cycle of retarded lines. For the same reason all vacuum loops and all connected functions $\langle\varphi'\ldots\varphi'\rangle_{\mathrm{cn}}$ of fields φ' only (without φ) also vanish [19].

Representation (11), (12) was obtained in Refs [17, 18], but the diagram technique of the type of (15) was formulated earlier in Refs [14, 20–22]. In turbulence theory, it is just the Wyld diagram technique [14]. The functional formulation (11), (12) essentially simplifies derivation of exact functional relations of Schwinger's equations type (see Section 3.1), and what is particularly important, allows one to apply the standard field-theoretical RG technique to the stochastic problem (10).

The authors of Refs [17, 18] added to action (11) a term with a self-contracted line $\langle\varphi\varphi'\rangle_0$ formally generated by determinant of the linear operator $-\partial_t + \delta V/\delta\varphi$. This addition exactly cancels all diagrams with self-contracted lines $\langle\varphi\varphi'\rangle_0$ which appear among others when Feynman's diagram technique is applied to action (11) but which do not arise when diagrams are directly constructed by iterations of the stochastic equation (10). Following Ref. [19], we shall simply set the self-contracted line $\langle\varphi\varphi'\rangle_0$ in diagrams to zero. This definition provides simultaneous vanishing of superfluous diagrams and their compensating contribution to action (11).

If the above general theorem is applied to the Navier–Stokes stochastic equation (1) and (2), we arrive at the theory of two transverse vector fields $\Phi \equiv \varphi, \varphi'$ with the action

$$S(\Phi) = \frac{\varphi' D_F \varphi'}{2} + \varphi'\left[-\partial_t\varphi + \nu_0\Delta\varphi - (\varphi\partial)\varphi\right], \tag{16}$$

where D_F is the correlator (2) of the random force. As the field φ' is transverse, we omitted in Eqn (16) a pure longitudinal contribution $\partial_i p$ coming from Eqn (1).

Bare propagators (15) for the model (16) in the momentum-frequency representation are of the following form:

$$\langle\varphi\varphi'\rangle_0 = \langle\varphi'\varphi\rangle_0^* = (-\mathrm{i}\omega + \nu_0 k^2)^{-1}\,, \qquad \langle\varphi'\varphi'\rangle_0 = 0\,,$$
$$\langle\varphi\varphi\rangle_0 = \frac{d_F(k)}{\omega^2 + \nu_0^2 k^4}\,. \tag{17}$$

Here the function $d_F(k)$ is defined by Eqn (2). All lines are, in vector indices, multiple of the transverse projector $P_{ij}(\mathbf{k})$ which is not written out explicitly in Eqn (17) but will always be meant. Interaction in Eqn (16) is described by a three-leg vertex $-\varphi'(\varphi\partial) = \varphi_i' V_{ijs}\varphi_j\varphi_s/2$ with the vertex factor

$$V_{ijs} = \mathrm{i}(k_j\delta_{is} + k_s\delta_{ij})\,, \tag{18}$$

where $\mathbf{k}$ is the momentum flowing into the vertex through the field φ'. For illustration, Fig. 1 shows diagrams for exact Green's functions $\langle\varphi\varphi\rangle$ and $\langle\varphi\varphi'\rangle$ in the one-loop approximation. Lines of diagrams are associated with bare propagators (17); vertices with factors (18); the crossed end of a line corresponds to the field φ'; the noncrossed one to the field φ. The perturbation expansion parameter (in the QFT language, the coupling constant, or charge) is taken to be $g_0 \equiv D_0/\nu_0^3$ with D_0 from Eqn (9).

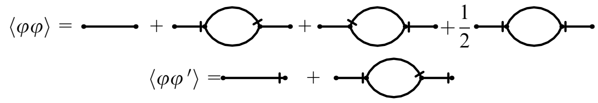

Figure 1.

2.3 IR- and UV-singularities of diagrams in perturbation theory

To illustrate problems arising in perturbation theory for the model (16) with pumping (9) we shall consider the pair correlator of velocity following Ref. [15]. At finite $\epsilon > 0$ all involved diagrams converge in the region of large momenta and frequencies, therefore they can be computed without UV-cut-off Λ. In this method all UV-divergences arise as poles in ϵ, at $\epsilon \to 0$, and the perturbation series for the correlator assumes the form

$$\langle\varphi\varphi\rangle = \langle\varphi\varphi\rangle_0\left[1 + \sum_{n=1}^{\infty}(g_0 k^{-2\epsilon})^n A_n\left(\frac{\omega}{\nu_0 k^2}, \frac{m}{k}, \epsilon\right)\right], \tag{19}$$

where $g_0 = D_0/\nu_0^3$ with D_0 from Eqn (9) and the coefficients A_n have poles in ϵ. From Eqn (19) it is seen that to determine the asymptotic behavior at $k \to 0$ at a fixed charge g_0 and coefficients A_n, it is necessary to sum up the whole series. This is just the first IR-problem to be solved by the RG method. At the level of canonical dimensions the problem is to determine asymptotic behavior at $\lambda \to 0$ of expression (19) with rescaled variables $k \to \lambda k$, $\omega \to \lambda^2\omega$, $m \to \lambda m$, and irrelevant variables g_0, ν_0 kept fixed. To formulate the first IR-problem more accurately, as it follows from the RG-analysis (for details, see Section 2.6), we must replace the canonical values of exponents λ for all IR-relevant variables F by their actual critical dimensions Δ_F, which also must be simultaneously determined (besides $\Delta_k = 1$, which is merely the normalization condition for dimensions).

It is to be noted that the above-formulated IR-problem is nontrivial for any $\epsilon > 0$, including the range $0 < \epsilon < 2$, when pumping (9) is ultraviolet. In this region from Eqn (3) we have $W \simeq D_0 \Lambda^{4-2\epsilon}$ up to a unimportant dimensionless coefficient of order 1, which, together with definitions $\Lambda = (W/\nu_0^3)^{1/4}$ and $g_0 = D_0/\nu_0^3$, gives $g_0 \simeq \Lambda^{2\epsilon}$. Similarly, for the region $\epsilon > 2$ of IR-pumping, from (3) we have $W \simeq D_0 m^{4-2\epsilon}$, hence $g_0 \simeq (\Lambda/m)^4 m^{2\epsilon}$. In both cases the dimensionless expansion parameter $g_0/k^{2\epsilon}$ in the searched IR-asymptotics is not small $[(\Lambda/k)^{2\epsilon}$ in the first case and $\Lambda^4/(k^{2\epsilon} m^{4-2\epsilon})$ in the second], consequently we have to sum series (19), and this is just the first IR-problem. Therefore, the types of pumping and corresponding singularities cannot be identified: even for pumping of the UV-type with $0 < \epsilon < 2$ in Eqn (9) the perturbation series contains IR-singularities which will be summed up by the RG method.

From the above estimates it is clear that these singularities become less strong when $\epsilon > 0$ decreases and would vanish at all at $\epsilon = 0$ if we could pass to this limit in Eqn (19). However, this is impossible since expression (19) contains UV-divergences, i.e., poles in ϵ. Removal of these poles is a classical UV-problem whose general solution is given by the theory of UV-renormalization. This theory gives rise to RG equations which express a simple idea of nonuniqueness of renormalization (for details, see Section 2.4). This explains why genetically related with the problem of UV-divergences, the RG method proves to be a useful tool for solving the problem of IR-divergences, at first sight completely different from that of UV-divergences.

Solution of the first IR-problem will substantiate the Kolmogorov hypothesis 2. Hypothesis 1 is connected with the second IR-problem, i.e., with possible singularities of the coefficients A_n in Eqn (19) when $m/k \to 0$. In our model (9) with a finite ϵ these singularities do exist and the problem is nontrivial. As we said earlier (Section 2.1), the second IR-problem, contrary to the first one, is not solved automatically by resumming series of perturbation theory with the use of the conventional RG technique. Analogous problems also exist in the models of critical behavior and are solved by the Wilson operator product expansion [6, 10]. These problems will be discussed later (Section 3.4), and here we only note that the second IR-problem is always considered within the framework of the general solution to the first one: at first at any fixed ratio m/k only the leading term of the IR-asymptotics at $\lambda \to 0$ is extracted (see above), and then its asymptotic behavior at $m/k \to 0$ is examined.

2.4 UV-renormalization. RG equations

Here we shall give brief necessary information on the quantum-field theory of renormalization and RG technique; a detailed account can be found in monographs [5, 6].

We will consider models whose diagrams may be computed without UV-cut-off Λ (this quantity can be contained only in parameters of the type of g_0) and UV-divergences appear as poles in a certain dimensionless 'parameter of deviation from logarithmic theory' ϵ. These are, in particular, our models of type of (16) and also diverse specific models of the theory of critical behavior with dimensional regularization [8–10].

The procedure of multiplicative renormalization removing UV-divergences (in our case poles in ϵ) is the following: the initial action $S(\Phi)$ is declared to be nonrenormalized; its parameters e_0 (e stands for the whole set of parameters) are called bare parameters and are considered some (to be determined) functions of new renormalized parameters e. The renormalized action is assumed to be the functional $S_{\mathrm{R}}(\Phi) = S(Z_\Phi \Phi)$, where renormalization constants Z_Φ (one per each independent component of the field Φ) are also to be determined. Functional averaging $\langle \ldots \rangle$ defining nonrenormalized total Green's functions $G_n = \langle \Phi \ldots \Phi \rangle$ is performed with the weight $\exp S(\Phi)$; whereas averaging with the weight $\exp S_{\mathrm{R}}(\Phi)$ results in renormalized functions G_n^{R}. Connection between the functionals S and S_{R} leads to interrelation $G_n^{\mathrm{R}} = Z_\Phi^{-n} G_n$ between the corresponding Green functions [here $G_n = G_n(e_0, \epsilon, \ldots)$ by definition, dots denote other arguments like coordinates or momenta], while G_n^{R} and Z_Φ, by convention, are expressed via parameters e. In the framework of perturbation theory there is a one-to-one correspondence $e_0 \Leftrightarrow e$, therefore one may take either e_0 or e for a set of independent variables.

For the following it will be convenient to treat instead of total Green's functions $G_n \equiv \langle \Phi \ldots \Phi \rangle$ their connected $W_n \equiv \langle \Phi \ldots \Phi \rangle_{\mathrm{cn}}$ or 1-irreducible $\Gamma_n \equiv \langle \Phi \ldots \Phi \rangle_{\text{1-ir}}$ parts. The corresponding generating functionals are connected by Eqn (12) and the first equation of (14). With the above rule for renormalization of G_n these relations lead to renormalization rules for W_n and Γ_n:

$$W_n^{\mathrm{R}}(e, \epsilon, \ldots) = Z_\Phi^{-n}(e, \epsilon) W_n\big(e_0(e, \epsilon), \epsilon, \ldots\big)\,,$$
$$\Gamma_n^{\mathrm{R}}(e, \epsilon, \ldots) = Z_\Phi^{n}(e, \epsilon) \Gamma_n\big(e_0(e, \epsilon), \epsilon, \ldots\big)\,. \tag{20}$$

Functions $e_0(e, \epsilon)$, $Z_\Phi(e, \epsilon)$ can be chosen arbitrarily, which corresponds to an arbitrary choice of normalization of the fields and parameters e at a given e_0. The basic claim of the theory of renormalization is that for the so-called multiplicative-renormalizable models there is a choice for these functions such, that $W_n^{\mathrm{R}}(e, \epsilon, \ldots)$ remain finite at $\epsilon \to 0$ and fixed e. With this choice all UV-divergences (poles in ϵ) turn out to be concentrated in the functions $e_0(e, \epsilon)$ and $Z_\Phi(e, \epsilon)$ and disappear from renormalized Green's functions $W_n^{\mathrm{R}}(e, \epsilon, \ldots)$. Note that if any of these three sets of Green's functions (total, connected, 1-irreducible) is UV-finite, other two sets are also UV-finite automatically. For the following we will consider connected functions W_n.

RG equations are written for renormalized functions W_n^{R} which differ from initial nonrenormalized functions W_n only by normalization, and therefore, can equally be used for analysis of critical scaling. We shall demonstrate elementary derivation of RG equations following Ref. [15]. The requirement of elimination of divergences does not determine the functions $e_0(e, \epsilon)$ and $Z_\Phi(e, \epsilon)$ uniquely; there remains a freedom to introduce an extra dimensional parameter, the renormalization mass μ, into these functions (and implicitly into W_n^{R} as well):

$$W_n^{\mathrm{R}}(e, \mu, \epsilon, \ldots) = Z_\Phi^{-n}(e, \mu, \epsilon) W_n\big(e_0(e, \mu, \epsilon), \epsilon, \ldots\big)\,. \tag{21}$$

A change in μ at fixed e_0 produces changes in e, Z_Φ and W_n^{R}, but not in $W_n(e_0, \epsilon, \ldots)$. Denoting differentiation $\mu\partial_\mu$ at fixed e_0 by $\widetilde{\mathcal{D}}_\mu$ and applying it to both sides of equation $Z_\Phi^n W_n^{\mathrm{R}} = W_n$, we get the basic differential RG equation:

$$[\mathcal{D}_{\mathrm{RG}} + n\gamma_\Phi] W_n^{\mathrm{R}}(e, \mu, \epsilon, \ldots) = 0\,, \qquad \gamma_\Phi \equiv \widetilde{\mathcal{D}}_\mu \log Z_\Phi\,. \tag{22}$$

Here summation runs over all renormalized parameters e, and $\mathcal{D}_{\rm RG}$ is the operator $\widetilde{\mathcal{D}}_\mu = \mu\partial_\mu\big|_{e_0}$ expressed through the variables e and μ:

$$\mathcal{D}_{\rm RG} = \widetilde{\mathcal{D}}_\mu = \mu\partial_\mu + \sum_e (\widetilde{\mathcal{D}}_\mu e)\partial_e\,. \tag{23}$$

Coefficients of the differential operator (23) and γ_Φ from Eqn (22) are called RG functions and are derived from various renormalization constants Z. Since the functions $W_n^{\rm R}$ in Eqn (22) are UV-finite, all RG-functions are also UV-finite, i.e., have no poles in ϵ. Note that renormalization formulae (20) for $\Gamma_n^{\rm R}$ result in an equation similar to (22), but with the change $\gamma_\Phi \to -\gamma_\Phi$.

In the general theory of renormalization [6] they distinguish nonrenormalized, S, renormalized, $S_{\rm R}$, and basic, $S_{\rm B}$ actions; the latter is obtained from S when all bare parameters are replaced by their renormalized analogs. UV-divergences are removed after addition to the basic action $S_{\rm B}$ of all necessary counterterms ΔS which are determined by certain rules (see below). If thus obtained renormalized action $S_{\rm R}(\Phi) = S_{\rm B}(\Phi) + \Delta S(\Phi)$ can be reproduced by the above procedure of redefinition of fields and parameters in the initial nonrenormalized action $S(\Phi)$, the model is multiplicative-renormalizable. Therefore, the first step in RG analysis of any model is to explicitly determine all counterterms required for removal of UV-divergences and to verify the multiplicative renormalizability of a theory.

The form of required counterterms is determined from canonical dimensions of 1-irreducible Green functions of the basic theory with the action $S_{\rm B}$. In contrast to static ones, dynamical models of type of (11) are two-scale, i.e., with each quantity F (a field or parameter in the action functional) one may connect [19] two independent canonical dimensions, momentum dimension d_F^k and frequency dimension d_F^ω. They are determined from natural normalization conditions $d_k^k = -d_x^k = 1,\ d_k^\omega = d_x^\omega = 0,\ d_\omega^k = d_t^k = 0,\ d_\omega^\omega = -d_t^\omega = 1$ and from requirement that every term of the action functional be momentum- and frequency-dimensionless. The pair d_F^k and d_F^ω results in a total canonical dimension d_F. This quantity depends on the model; for our model (16) $d_F = d_F^k + 2d_F^\omega$ since the action (16) contains a combination $\partial_t + {\rm const}\times\Delta$, thus $\omega \propto k^2$ with respect to the total dimension (note that there are certain models of critical dynamics with $\omega \propto k^4$, Ref. [23]).

In the theory of renormalization of dynamical models the total dimension d_F plays the same role as the conventional (momentum) dimension in static problems. Canonical dimensions of an arbitrary 1-irreducible Green's function $\Gamma = \langle\Phi\ldots\Phi\rangle_{\text{1-ir}}$ for a d-dimensional problem are given by the relations

$$d_\Gamma^k = d - \sum_\Phi d_\Phi^k\,, \qquad d_\Gamma^\omega = 1 - \sum_\Phi d_\Phi^\omega\,,$$
$$d_\Gamma = d_\Gamma^k + 2d_\Gamma^\omega = d + 2 - \sum_\Phi d_\Phi\,, \tag{24}$$

with summation over all the fields Φ involved into a given function Γ. The total dimension d_Γ in logarithmic theory, i.e., at $\epsilon = 0$, is a formal index $\delta \equiv d_\Gamma(\epsilon = 0)$ of UV-divergence (now we are formulating general rules; in our model (16) dimensions (24) do not depend on ϵ, see Section 2.5). Superficial UV-divergences which require counterterms for removal may appear only in those functions Γ for which δ is a nonnegative integer, see Refs [5, 6].

Analysis of divergences is simplified by the following observations:

(1) For any dynamic model of the type of (11) all 1-irreducible Green's functions of basic fields φ only (without φ') vanish (see Section 2.2) and thus do not generate counterterms.

(2) If for any reason a certain number of external momenta or frequencies is factorized out of all diagrams contributing to a given Green's function, the actual index of divergence δ' turns out to be smaller than $\delta = d_\Gamma(\epsilon = 0)$ by the corresponding number of units (a Green function generates counterterms if δ' is a nonnegative integer).

(3) Sometimes formally dimension-allowed divergences are absent owing to symmetry requirements, say because of Galilean invariance of model (16).

(4) Since all counterterms must be local, nonlocal contributions to the action (if they exist) are never renormalized, i.e., they are the same in $S_{\rm B}(\Phi)$ and $S_{\rm R}(\Phi)$.

These general considerations and formulae (24) allow us to determine for any concrete dynamic model all superficially divergent functions Γ and explicitly obtain the corresponding counterterms $\Delta S(\Phi)$ for the basic action $S_{\rm B}(\Phi)$.

2.5 RG-analysis of stochastic hydrodynamics. IR-scaling

In this section following Ref. [15] we reproduce basic results of the RG-analysis of model (16), which for the first time were obtained in Ref. [16].

We start with a simplified massless model with a power-like pumping (8). The corresponding nonrenormalized and basic actions have the form

$$S(\Phi) = \frac{1}{2}\,g_0 v_0^3 \varphi' k^{4-d-2\epsilon}\varphi' + \varphi'\left[-\partial_t\varphi + v_0\Delta\varphi - (\varphi\partial)\varphi\right], \tag{25}$$

$$S_{\rm B}(\Phi) = \frac{1}{2}\,g\mu^{2\epsilon} v^3 \varphi' k^{4-d-2\epsilon}\varphi' + \varphi'\left[-\partial_t\varphi + v\Delta\varphi - (\varphi\partial)\varphi\right], \tag{26}$$

where the nonlocal contribution is written symbolically, summations over indices of the transverse vector fields $\Phi \equiv \varphi, \varphi'$ and integrations over $x \equiv t, \mathbf{x}$ are implied, μ is the renormalization mass, $e_0 \equiv \{v_0, g_0\}$ are bare parameters, and $e \equiv \{v, g\}$ are the corresponding renormalized parameters.

Canonical dimensions of fields and parameters of the model for an arbitrary space dimension d are presented in Table 1. With these dimensions and Eqn (24) we get $\delta = d_\Gamma = d + 2 - n_\varphi - (d-1)n_{\varphi'}$, where n_φ and $n_{\varphi'}$ are numbers of the corresponding fields in Γ. It is seen from Eqn (18) that each external line φ' in diagrams for Γ produces a factor ∂. Hence, the number of derivatives in a counterterm is not less than $n_{\varphi'}$, and $\delta' = \delta - n_{\varphi'} = d + 2 - n_\varphi - dn_{\varphi'}$ according to the rules of Section 2.4. By the form of δ and δ' we conclude that at $d > 2$ superficial divergences are present only in 1-irreducible functions $\langle\varphi'\varphi\rangle$ ($\delta = 2$, $\delta' = 1$) and $\langle\varphi'\varphi\varphi\rangle$ ($\delta = 1$, $\delta' = 0$), and the corresponding counterterms must contain the symbol

Table 1.

Dimension	F						
	$\varphi(x)$	$\varphi'(x)$	Λ, m, μ	v, v_0	W	g_0	g
d_F^k	-1	$d+1$	1	-2	-2	2ϵ	0
d_F^ω	1	-1	0	1	3	0	0
d_F	1	$d-1$	1	0	4	2ϵ	0

∂. Therefore, the first function generates only the counterterm $\varphi'\Delta\varphi$ but not $\varphi'\,\partial_t\varphi$ although it has the same dimension. The second function generates a counterterm with three fields, $\varphi', \varphi, \varphi$, and one symbol ∂; this term is always reduced to $\varphi'(\varphi\partial)\varphi$ owing to the transversality of all the fields. However, this dimension-allowed counterterm is in fact forbidden by Galilean invariance [16], which requires that operators ∂_t and $(\varphi\partial)$ must enter into counterterms only in the form of a covariant derivative $\nabla_t = \partial_t + (\varphi\partial)$. Therefore, if there is no counterterm $\varphi'\,\partial_t\varphi$, there is no $\varphi'(\varphi\partial)\varphi$ as well. In a particular case $d = 2$ there appears a new superficial divergence in the function $\langle\varphi'\varphi'\rangle_{\text{1-ir}}$ ($\delta = 2$, $\delta' = 0$) which generates the counterterm $\varphi'\Delta\varphi'$. A two-dimensional problem was examined in Refs [24, 25] but with mistakes, a correct analysis can be found in Ref. [26]. For the following we restrict ourselves to the case $d > 2$.

In the case of $d > 2$ only one counterterm $\varphi'\Delta\varphi$ is required. Its addition to Eqn (26) results in the renormalized action

$$S_{\mathrm{R}}(\Phi) = \frac{1}{2}\, g\mu^{2\epsilon}v^3\varphi' k^{4-d-2\epsilon}\varphi' + \varphi'\left[-\partial_t\varphi + Z_v v\Delta\varphi - (\varphi\partial)\varphi\right], \tag{27}$$

where Z_v is the renormalization constant. It is fully dimensionless and thus can depend on the only dimensionless renormalized parameter g (dependence on ϵ and d is always implied).

The explicit form of Z_v depends on a choice of a subtraction scheme. The purpose of a counterterm is to cancel out poles in ϵ in diagrams, therefore the contribution of a counterterm should contain such poles. However, its finite part can be chosen arbitrarily, and its fixation is just the choice of a subtraction scheme. The most convenient for practical computations is the minimal subtraction (MS) scheme [6], in which counterterms contain only poles in ϵ and no finite contributions. Some studies on turbulence theory, for instance, performed in Refs [27, 28], make use of a more traditional scheme in which a freedom of a finite part of renormalization is fixed by prescribing particular values to some functions Γ_n^{R} at an arbitrarily chosen normalization point $k = \mu$, which just introduces the renormalization mass in the given scheme. Physical results do not depend on the choice of a particular subtraction scheme, and this choice is just a question of convenience. We will always utilize the widely accepted MS scheme.

By this scheme only pure poles in ϵ are subtracted from diverging expressions, finite contributions are not changed, and the renormalization constants Z are of the following form:

$$Z = 1 + \sum_{k=1}^{\infty} a_k(g)\epsilon^{-k} = 1 + \sum_{n=1}^{\infty} g^n \sum_{k=1}^{n} a_{nk}\epsilon^{-k}\,. \tag{28}$$

The coefficients a_{nk} in model (16) can depend only on the space dimension d; absence of ϵ in residues $a_k(g)$ is a specific feature of the MS scheme.

The renormalized action (27) is obtained from the nonrenormalized one (25) by the following renormalization of parameters:

$$v_0 = vZ_v\,, \qquad g_0 = g\mu^{2\epsilon}Z_g\,, \qquad Z_g = Z_v^{-3}\,. \tag{29}$$

Fields are not renormalized, i.e., $Z_\Phi = 1$. The model is multiplicative-renormalizable and a standard scheme of derivation of RG equations of type (22) may be used; in this case also $\gamma_\Phi = 0$ because there is no field renormalization. Thus, the model (25) contains anomalously small number of divergences as compared to conventional models of the theory of critical behavior like φ^4: there is only one independent renormalization constant Z_v, whereas Z_g is expressed in terms of Z_v because there is no renormalization of the nonlocal contribution of the random force correlator in equation (27).

All these conclusions remain valid if the power pumping (8) is replaced by a more realistic one, say (7) or (9). In this case the bare parameter m_0 is not renormalized ($m_0 = m$), and relationships (29) are supplemented with the trivial equation

$$m_0 = mZ_m\,, \qquad Z_m = 1\,. \tag{30}$$

The constants Z are calculated directly from diagrams of the basic theory. In a general case, using the renormalization constant Z_F of any quantity F (field or parameter), we may determine the corresponding RG-function $\gamma_F(g)$, anomalous dimension of a given quantity F, and given any charge g (in general any completely dimensionless parameter, which the constants Z depend on, may be considered a charge), we introduce the corresponding β-function $\beta(g)$:

$$\beta \equiv \widetilde{\mathcal{D}}_\mu g\,, \qquad \gamma_F \equiv \widetilde{\mathcal{D}}_\mu \log Z_F = \beta\,\partial_g \log Z_F\,. \tag{31}$$

Here $\widetilde{\mathcal{D}}_\mu \equiv \mu\partial_\mu\big|_{e_0}$ (see Section 2.4). From Eqn (29) we have

$$\gamma_g = -3\gamma_v\,, \qquad \beta = g(-2\epsilon - \gamma_g) = g(-2\epsilon + 3\gamma_v)\,, \tag{32}$$

and the operator $\mathcal{D}_{\mathrm{RG}}$ with $e = \{g, v\}$ takes the form

$$\mathcal{D}_{\mathrm{RG}} = \mathcal{D}_\mu + \beta(g)\partial_g - \gamma_v(g)\mathcal{D}_v\,, \qquad \mathcal{D}_x \equiv x\partial_x\,, \tag{33}$$

where we introduced the notation $\mathcal{D}_x \equiv x\partial_x$ for an argument x of any renormalized Green's function.

A one-loop approximation to Z_v for the model (27) gives [19]

$$Z_v = 1 - \frac{ag}{2\epsilon} + O(g^2)\,, \qquad a = \frac{C_d}{4(d+2)}\,, \tag{34}$$

where

$$C_d \equiv \frac{(d-1)S_d}{(2\pi)^d}\,, \qquad S_d \equiv \frac{2\pi^{d/2}}{\Gamma(d/2)}\,, \tag{35}$$

and S_d in Eqn (35) is the area of a unit sphere in a d-dimensional space, $\Gamma(x)$ is the γ-function, $a = 1/20\pi^2$ for $d = 3$.

Using formulas (32), we obtain RG functions in the one-loop approximation:

$$\gamma_v(g) = ag + O(g^2)\,, \qquad \beta_g = -2\epsilon g + 3ag^2 + O(g^3)\,. \tag{36}$$

The coefficient a in Eqn (34) is positive, which ensures that at small $\epsilon > 0$ in the physical region $g > 0$ there is an IR-stable (IR-attracting) fixed point $g_* = 2\epsilon/3a + O(\epsilon^2)$ of the RG equation $\beta(g_*) = 0$, $\beta'(g_*) > 0$. From the condition $\beta(g_*) = 0$ and the last of equations (32) the value $\gamma_v(g_*)$ may be found exactly without computing diagrams: $\gamma(g_*) = 2\epsilon/3$, and no corrections of order of ϵ^2, ϵ^3, etc. We explicitly evaluated the constant (34) from diagrams only to verify that the coefficient a is positive, i.e., to prove the existence of g_* in the region $g > 0$.

From a solution to the RG equation it follows (see Section 2.6) that if there is an IR-stable fixed point, the leading terms of IR-asymptotic expansion of Green's functions W_n^{R} in any one-charge model obey RG equation (22) with $g \to g_*$. For our model in view of Eqn (33) and $\gamma_\Phi = 0$, from Eqn (22) we obtain

$$(\mathcal{D}_\mu - \gamma_v^* \mathcal{D}_v) W_n^{\mathrm{R}} = 0\,, \qquad \gamma_v^* \equiv \gamma_v(g_*) = \frac{2\epsilon}{3}\,. \tag{37}$$

Canonical scale invariance is expressed by the equations

$$\begin{aligned} &\left(\sum_F d_F^k \mathcal{D}_F - d_{W_n}^k\right) W_n^{\mathrm{R}} = 0\,, \\ &\left(\sum_F d_F^\omega \mathcal{D}_F - d_{W_n}^\omega\right) W_n^{\mathrm{R}} = 0\,, \end{aligned} \tag{38}$$

where F is the set of all arguments of W_n^{R}; $n = \{n_\varphi, n_{\varphi'}\}$ and $d_{\dots}^{k,\omega}$ are canonical dimensions of F and W_n^{R}. Each of equations (37), (38) describes scale invariance under rescaling of those variables, derivatives with respect to which are involved into the differential operator. If we are interested in scaling behavior when some of the variables F are fixed, we should eliminate the corresponding derivatives $\mathcal{D}_F$ by combining the available equations. For instance, in a model of type of (16) with $D_0 = g_0 v_0^3$, functions W_n^{R} in the coordinate representation correspond to $F = \{t, \mathbf{x}, \mu, v, g, m\}$, where t and $\mathbf{x}$ are the sets of all times and coordinates, respectively; and we are interested in scaling behavior under rescaling of t, $\mathbf{x}$, m at fixed μ, v, g. Substituting required dimensions from Table 1 into Eqns (37), (38) and excluding $\mathcal{D}_\mu$ and $\mathcal{D}_v$, we arrive at the searched equation of the critical IR-scaling:

$$(-\mathcal{D}_x + \Delta_t \mathcal{D}_t + \Delta_m \mathcal{D}_m - \Delta_n) W_n^{\mathrm{R}} = 0\,. \tag{39}$$

Here the coefficients

$$\Delta_t = -\Delta_\omega = -2 + \gamma_v^*\,, \qquad \Delta_n = d_{W_n}^k + \Delta_\omega d_{W_n}^\omega\,,\ \Delta_m = d_m^k \tag{40}$$

have the meaning of the corresponding critical dimensions. Substitution of the known [see Eqn (37) and Table 1] values $\gamma_v^* = 2\epsilon/3$, $d_{W_n}^k = -n_\varphi + (d+1) n_{\varphi'}$, $d_{W_n}^\omega = n_\varphi - n_{\varphi'}$ (dimension of a connected function W_n^{R} is equal to the sum of dimensions of the involved fields) into Eqn (40) results in the following expressions for the critical dimensions:

$$\begin{aligned} &\Delta_\varphi = 1 - \frac{2\epsilon}{3}\,, \qquad \Delta_{\varphi'} = d - \Delta_\varphi\,, \\ &\Delta_t = -\Delta_\omega = -2 + \frac{2\epsilon}{3}\,, \qquad \Delta_m = 1\,. \end{aligned} \tag{41}$$

There are no corrections of order of ϵ^2, ϵ^3 or higher, and at the physical value $\epsilon = 2$ these dimensions coincide with the Kolmogorov dimensions (6).

This is just the main result of paper [16] reproduced later by many authors. Recall that the exact RG equation (22) takes the form (37) only for IR-asymptotics of Green's functions. Analysis of the exact equation, conditions for entering in the critical regime, and computation of amplitude factors for scaling laws are presented in Section 2.6.

2.6 Solution of RG equations. Invariant variables. RG-representations of correlation functions

Consider RG equations for a pair correlator $D = \langle \varphi\varphi \rangle$ of the velocity for the model (16) with pumping (9) at $d > 2$ [15]. Since there is no renormalization of fields (Section 2.5), their renormalized Green's functions W_n^{R} coincide with nonrenormalized ones W_n, the only difference is in the choice of variables and in the form of perturbation theory (expansion in g or in g_0). From dimensional considerations we have (the transverse projector is everywhere omitted)

$$D = v k^{-d} R(s, g, z, u)\,, \qquad s = \frac{k}{\mu}\,, \qquad z = \frac{\omega}{v k^2}\,, \qquad u = \frac{m}{k}\,, \tag{42}$$

where R is a function of completely dimensionless arguments. The correlator $D = W_2^{\mathrm{R}}$ obeys the RG equation (22) with $\gamma_\Phi = 0$ and $\mathcal{D}_{\mathrm{RG}}$ defined by Eqn (33):

$$\mathcal{D}_{\mathrm{RG}} D = 0\,, \qquad \mathcal{D}_{\mathrm{RG}} = \mathcal{D}_\mu + \beta(g)\partial_g - \gamma_v(g)\mathcal{D}_v\,. \tag{43}$$

From Eqns (42) and (43) it follows that

$$D = \bar{v} k^{-d} R(1, \bar{g}, \bar{z}, \bar{u})\,, \qquad \bar{z} = \frac{\omega}{\bar{v} k^2}\,, \qquad \bar{u} = u = \frac{m}{k}\,. \tag{44}$$

Invariant variables $\bar{g}$, $\bar{v}$, $\bar{u}$ are first integrals of equation (43) depending on the scale parameter $s \equiv k/\mu$ and normalized to g, v, u at $s = 1$, respectively. Equation $\bar{u} = u = m/k$ holds because $\mathcal{D}_m$ is absent in operator (43), for there is no renormalization of parameter m, see Eqn (30). Equation (44) is valid since both sides of it satisfy RG equations and coincide at $s = 1$ in view of the normalization conditions for invariant variables.

For static (simultaneous) correlator

$$D_{\mathrm{st}} = \frac{1}{2\pi} \int \mathrm{d}\omega\, D = v^2 k^{2-d} R(s, g, u) \tag{45}$$

an analog of Eqn (44) is the representation (provided $\bar{u} = u$)

$$D_{\mathrm{st}} = \bar{v}^2 k^{2-d} R(1, \bar{g}, u)\,. \tag{46}$$

For operator (43) the invariant charge $\bar{g} = \bar{g}(s, g)$ is implicitly determined by relations

$$\log s = \int_g^{\bar{g}} \frac{\mathrm{d}x}{\beta(x)}\,, \qquad \bar{g}(1, g) = g\,, \tag{47}$$

and for the invariant viscosity $\bar{v} = \bar{v}(s, v, g)$ we have

$$\bar{v} = v \exp\left[\int_{\bar{g}}^{g} \gamma_v(x)\, \frac{\mathrm{d}x}{\beta(x)}\right] = \left(\frac{g v^3}{\bar{g} s^{2\epsilon}}\right)^{1/3} = \left(\frac{g_0 v_0^3}{\bar{g} k^{2\epsilon}}\right)^{1/3}. \tag{48}$$

The second equation in (48) follows from the first one and relations (32) between RG-functions; whereas the third follows from the second and renormalization formulae for parameters (29). In the one-loop approximation (36) relations (47) result in the following expression for $\bar{g}$:

$$\bar{g}(s, g) = \frac{g g_*}{g_* s^{2\epsilon} + g(1 - s^{2\epsilon})}\,. \tag{49}$$

Here $g_* = 2\epsilon/3a$ with the constant a given by Eqn (34) is the coordinate of a fixed point in this approximation.

Expressions of type of (44) and (46) with explicit form of the invariant variables of type of (47) and (48) will be called RG-representations of the corresponding Green's functions. Evaluating RG-functions (31) and functions R in Eqns (42), (45) to a finite order of the renormalized perturbation theory in g and substituting them into the RG-representation, we obtain the corresponding approximation of improved perturbation theory with contributions of higher order partially summed up.

Formulae (47) and (48) represent the invariant variables $\bar{e} \equiv \{\bar{g}, \bar{v}\}$ in terms of renormalized parameters $e \equiv \{g, v\}$ and the scale variable $s = k/\mu$: $\bar{e} = \bar{e}(e, s)$, or vice versa, $e = e(\bar{e}, s)$. Further it will be convenient to express $\bar{e}$ in terms of momentum k and bare parameters $e_0 \equiv \{g_0, v_0\}$, which have a direct physical meaning in our model. This can be done since parameters e_0 are renormalization-invariant ($\mathcal{D}_{\mathrm{RG}} e_0 = 0$ by definition of operator $\mathcal{D}_{\mathrm{RG}}$, see Section 2.4), i.e., they are first integrals of the RG equation and, consequently, are some functions of another complete set of first integrals, in this case of variables $\bar{e}$. By dimensional considerations (completely dimensionless parameters are only g and $\bar{g}$) and in view of Eqns (29), (47), (48) and normalization conditions, we obtain

$$g_0 k^{-2\epsilon} = \bar{g} Z_g(\bar{g}), \qquad v_0 = \bar{v} Z_v(\bar{g}). \tag{50}$$

Here the renormalization constants Z can in turn be expressed via simpler objects, RG-functions (32). Formulae for the charge renormalization (29) and definition of the β-function (31) result in the relation $\partial_g \log Z_g = -1/g - 2\epsilon/\beta(g)$. Integrating it under the normalization condition $Z_g(0) = 1$ we obtain

$$Z_g(g) = \exp\left\{ -\int_0^g \mathrm{d}x \left[\frac{2\epsilon}{\beta(x)} + \frac{1}{x} \right] \right\}. \tag{51}$$

Note that for the β-function (32) the term $1/x$ in Eqn (51) provides convergence at $x = 0$. Furthermore, from the above expression for $\partial_g \log Z_g$ it follows that $\partial_g \log(g Z_g) = -2\epsilon/\beta(g) \geqslant 0$, which indicates that the function $g Z_g(g)$ monotonously increases in the interval $0 < g < g_*$, where the β-function is negative. When $g \to g_*$, the function $g Z_g(g)$ diverges.

The behavior (ensured at small ϵ and supposed at large ϵ) of functions $\beta(g)$ and $g Z_g(g)$ are shown in Fig. 2 by solid and dashed lines, respectively, also there is a graphic solution to the renormalization equation $g_0 \mu^{-2\epsilon} = g Z_g(g)$ in g [see Eqn (29), an analogous relation between the quantities $g_0 k^{-2\epsilon}$ and $\bar{g}$ is the first equation in (50)]. From the figure it is clear that:

(1) Values of g and $\bar{g}$ always lie in the interval $[0, g_*]$ irrespective of the magnitude of the bare charge, i.e., g and $\bar{g}$ are small at small $g_* \sim \epsilon$.

(2) $\bar{g}(s, g) \to g_* \sim \epsilon$ when $s \equiv k/\mu \to 0$ and $\bar{g}(s, g) \to 0$ when $s \to \infty$, i.e., a fixed point $g_* \sim \epsilon$ is IR-attracting (a synonym is IR-stable), whereas the trivial fixed point $g_* = 0$ is UV-attracting. Each of these fixed points determines the corresponding asymptotics of Green's functions.

Let us write out general formulae (50) in the one-loop approximation (36). From Eqn (51) we obtain $Z_g(g) = g_*/(g_* - g)$, and hence, $g_0 \mu^{-2\epsilon} = g g_*/(g_* - g)$, in accordance with Eqn (29). Now we may express the invariant variables (47) and (48) in the form

$$\bar{g} = g_* \left(1 + \frac{g_*}{g_0 k^{-2\epsilon}} \right)^{-1}, \qquad \bar{v} = v_0 \left(1 + \frac{g_0 k^{-2\epsilon}}{g_*} \right)^{1/3}, \tag{52}$$

which does not contain the renormalization mass μ.

2.7 IR-scaling at fixed g_0 and v_0

Consider the IR asymptotic behavior of correlators (42) and (45). It follows from the general RG-representations (44) and (46) if we replace all invariant variables $\bar{e}$ by their asymptotic expansions, which further shall be denoted by $\bar{e}_*$. A key point is that in the given approximation $\bar{g}_* = g_* = \text{const}$ is the coordinate of an IR-attracting fixed point, whereas other quantities of the type of $\bar{v}_*$ remain nontrivial functions of their arguments. Thus, for the leading term of IR-asymptotics of the dynamic correlator (44) we get

$$D = \bar{v}_* k^{-d} f(\bar{z}_*, u), \qquad \bar{z}_* \equiv \frac{\omega}{\bar{v}_* k^2}, \qquad f(z, u) \equiv R(1, g_*, z, u), \tag{53}$$

and analogously for the static correlator (46) we get

$$D_{\mathrm{st}} = \bar{v}_*^2 k^{2-d} f(u), \qquad f(u) \equiv R(1, g_*, u), \tag{54}$$

where $\bar{v}_*$ is the IR asymptotic form of the invariant variable (48):

$$\bar{v}_* = \left(\frac{g_0 v_0^3}{g_* k^{2\epsilon}} \right)^{1/3} = \left(\frac{D_0}{g_*} \right)^{1/3} k^{-2\epsilon/3}. \tag{55}$$

Parameters g_0 and v_0 in $\bar{v}_*$ are grouped into the combination $D_0 \equiv g_0 v_0^3$ which enters into the random force correlator (9).

Asymptotic representations similar to Eqns (53) and (54) can be derived for any Green's function; they always obey the simplified RG-equation (43) with the change $g \to g_*$.

For fixed g_0 and v_0 formulae (53)–(55) correspond to the IR-scaling with critical dimensions (41) regardless of the explicit form of scaling functions f in Eqns (53) and (54). When dealing with Eqn (41), one should take into account that in the coordinate representation the critical dimension of any connected Green's function is equal to the sum of dimensions of all the involved fields, whereas in the momentum representation we have

$$\Delta\big[D(k, \omega)\big] = 2\Delta_\varphi - \Delta_\omega - d, \qquad \Delta\big[D_{\mathrm{st}}(k)\big] = 2\Delta_\varphi - d. \tag{56}$$

The scaling functions f are expressed by formulae (53) and (54) through the corresponding functions R in correlators (42)

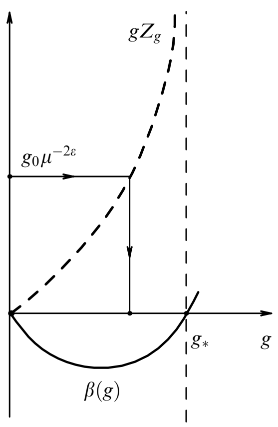

Figure 2.

and (45). The functions R can be computed directly from diagrams in the form of a series

$$R(g,\ldots)=\sum_{n=1}^{\infty} g^n R_n(\ldots) \qquad (57)$$

of the renormalized perturbation theory; dots denote all arguments of R different from g. Making the change $g \to g_*$, expanding then g_* and R_n in ϵ, and grouping contributions of the same order, we obtain from Eqn (57) the ϵ-expansions of the corresponding scaling functions:

$$f(z,u)=\sum_{n=1}^{\infty}\epsilon^n f_n(z,u)\,,\qquad f(u)=\sum_{n=1}^{\infty}\epsilon^n f_n(u)\,. \qquad (58)$$

It is important that evaluation of Eqn (58) to any finite order in ϵ requires only a finite number of diagrams since $g_* \sim \epsilon$ and the coefficients R_n in Eqn (57) for renormalized correlators do not contain poles in ϵ which could compensate the smallness of $g_* \sim \epsilon$. Note also that all the coefficients f_n in Eqn (58) in the model under consideration are finite in the limit $u \equiv m/k \to 0$, in accordance with the Kolmogorov hypothesis $1'$ (though this does not prove it at finite ϵ, see Section 2.1).

In the lowest (first) order of renormalized perturbation theory for model (9) we have

$$D=\frac{g\nu^3\mu^{2\epsilon}k^{4-d-2\epsilon}h(u)}{|\mathrm{i}\omega+\nu k^2|^2}\,,\qquad D_{\mathrm{st}}=\frac{g\nu^2\mu^{2\epsilon}k^{2-d-2\epsilon}h(u)}{2}\,. \qquad (59)$$

The first expression was derived by substitution of the function (9) with $D_0=g_0\nu_0^3$ into the corresponding bare correlator (17) and subsequent change $\nu_0 \to \nu$, $g_0 \to g\mu^{2\epsilon}$; whereas the second expression was obtained by integration of the first one over ω according to Eqn (45). From formulae (59) it is easy to determine the first terms of the corresponding series (57) and (58).

In turbulence theory instead of the static correlator (45) a (one-dimensional) spectrum $E_1(k)$ of pulsation energy is often considered. It is connected with Eqn (45) by the formula

$$E_1(k)=\frac{C_d}{2}\,k^{d-1}D_{\mathrm{st}}(k) \qquad (60)$$

with C_d given by Eqn (35). Its RG-representation automatically follows from Eqn (54). In the lowest order we get from Eqn (59)

$$E_1(k)=\frac{1}{4}\,C_d\,\bar g\,\bar\nu^2 k h(u) \qquad (61)$$

with $\bar g$ and $\bar\nu$ given by (52). In the IR-range $k^{2\epsilon}/g_0 \to 0$ we obtain

$$E_1(k)=(C_d D_0)^{2/3}\left[\frac{\epsilon(d+2)}{24}\right]^{1/3} k^{1-4\epsilon/3}h(u)\,. \qquad (62)$$

When restricted to the inertial range $u \equiv m/k \ll 1$, the function $h(u)$ in these formulae turns into $h(0)=1$ in accordance with Eqn (9).

We will complete this section with a historical reference. As we have already said, the existence of scaling with dimensions (41) was established in Ref. [16]. Explicit one-loop expressions for the spectrum (62) and the invariant viscosity (52) in the framework of model (8), i.e., with $h=1$ in Eqn (62), were first obtained in Ref. [29] for $d=3$ with the help of a special procedure similar to the RG technique in the form of the Wilson recursion relations [30] but without conventional scale transformations of parameters. It was important that the quantity $\bar\nu$ was identified with an effective turbulent viscosity which phenomenologically describes the influence of small-scale components of the velocity field on the large-scale ones [1]. The result of Ref. [29] was then reproduced (for arbitrary $d>2$) and employed in many studies, for instance, in papers [31 – 33, 27]. In Ref. [27] this was obtained in a more self-consistent way with the use of the standard field-theoretical RG technique.

Attempts to go beyond the scope of the simplest approximations (52) and (62) were made in Refs [28, 34]. In particular, in Ref. [34] the coefficient R_2 in the representation (57) for the function (42) was computed for $u=0$ and $s=1$.

However, this is not enough for evaluation of the corresponding coefficient in Eqn (58), since in computation of coordinate of a fixed point g_* the contribution of terms of order ϵ^2 must be taken into account, and this has not been carried out yet (it requires evaluation of two-loop diagrams, whereas for evaluation of R_2 it is enough to consider one-loop diagrams). In Ref. [28] a correction to the expression (62) connected with finite (without poles in ϵ) renormalization of fields was calculated in the one-loop approximation. This is needed for the renormalization scheme used in Refs [27, 28], and corresponds in the MS scheme (where the field renormalization is absent) to the one-loop correction R_2 in representation (57) for the function (45). However, in this paper two-loop corrections to RG-functions also were not computed, and since they give contributions of the same order to RG-representations, the calculation of Ref. [28] also is not complete.

Thus, at present, particular formulae of the type of (62) are known only with one-loop accuracy, which corresponds to the lowest order in ϵ on representation (58) of scaling functions. Calculation of subsequent terms in expansions (58) is a complicated but exclusively technical problem. Moreover, there is no much sense in this work since the problems connected with the Kolmogorov hypothesis 1 cannot be solved within the framework of ϵ-expansion (see Section 2.1), and qualitatively, it is unlikely that inclusion of ϵ-corrections at a large real value $\epsilon_p \geqslant 2$ may give practically more accurate results. In fact, it is only important that RG-representations of the type of (44) and (46) are exact formulae which guarantee the IR-scaling with exact dimensions (41).

2.8 IR-scaling at fixed W and ν_0: independence of ν_0 and 'freezing' of critical exponents at $\epsilon > 2$

In this section we present the proof (Ref. [15]) that Green's functions for the models (16) and (9) do not depend on the viscosity coefficient ν_0 in the whole range $\epsilon > 2$ of IR-pumping and that in this region critical dimensions (41) are frozen at Kolmogorov's values (6).

Expressions (41) obtained in Section 2.5 for critical dimensions correspond to IR-scaling at fixed g_0 and ν_0, or g, ν, and μ in renormalized terms. For models considered in the theory of critical behavior, the problem is always formulated in this way, and formulae of the type of (41) are final answers (although usually series in ϵ are not truncated). In the given case it is not so because the Kolmogorov – Obukhov theory [1] deals with scaling at fixed ν_0 and W. The pumping power W is related to the parameter $D_0=g_0\nu_0^3=g\nu^3\mu^{2\epsilon}$ in Eqn (9) by formula (3), and in order to get final answers it is necessary to express g_0 via W.

Taking for definiteness the model (7) and evaluating integral (3) with the UV-cut-off $\Lambda = (W/\nu^3)^{1/4}$, we obtain

$$W = \frac{D_0 C_d}{4(2-\epsilon)}\left[(\Lambda^2+m^2)^{2-\epsilon} + \frac{m^{4-2\epsilon}}{1-\epsilon} - \frac{(2-\epsilon)m^2(\Lambda^2+m^2)^{1-\epsilon}}{1-\epsilon}\right] \tag{63}$$

with C_d from Eqn (35). The Reynolds number $\mathrm{Re} \simeq (\Lambda/m)^{4/3}$ for a developed turbulence is very large, therefore from Eqn (63) it follows

$$D_0 \equiv g_0 \nu_0^3 = W B(\Lambda, m, \epsilon) \simeq \begin{cases} c_1 W \Lambda^{2\epsilon-4} & \text{when} \quad 2 > \epsilon > 0\,, \\ c_2 W m^{2\epsilon-4} & \text{when} \quad \epsilon > 2\,, \end{cases} \tag{64}$$

where $c_1 = 4(2-\epsilon)/C_d$, $c_2 = c_1(1-\epsilon)$. Definition of the function $B(\Lambda, m, \epsilon)$ in Eqn (64) is clear from comparison with Eqn (63), specifically, $B(\Lambda, m, 2) = 2/\left[C_d \log(\Lambda/m)\right]$ at $\epsilon = 2$. Simple approximations (64) are valid outside of the transitional region around $\epsilon = 2$, its width is not large (of order of $1/\log \mathrm{Re}$), therefore in what follows we shall consider approximations (64) to be valid everywhere up to $\epsilon = 2$. Note that representation of the type of (64) holds for any model of the type of (9). The choice of a particular model affects only nonessential, in what follows, coefficients c_1, c_2 and the shape of functions $B(\Lambda, m, \epsilon)$ in a narrow transitional region near $\epsilon = 2$.

Independence of the IR asymptotic behavior of Green's functions from the viscosity coefficient ν_0 in the region $\epsilon > 2$ is obvious from Eqn (64) and the observation (first mentioned in Ref. [29]) that the parameters g_0 and ν_0 enter into expressions of the type of (53) only as a combination $g_0\nu_0^3 \equiv D_0$ in representation (55) for $\bar{\nu}_*$. In the region $0 < \epsilon < 2$ this will not take place since here remains the dependence on ν_0 through Λ, see Eqn (64).

This statement proves the Kolmogorov hypothesis 2 for $\epsilon > 2$ (see Section 2.1), which automatically results in the IR-scaling with Kolmogorov's dimensions (6). We shall explain the mechanism of 'freezing' of exponents (41) at their Kolmogorov's values for $\epsilon = 2$ throughout the whole range $\epsilon > 2$.

As said above, in the Kolmogorov–Obukhov theory fixed (and in this sense 'critically dimensionless') parameters are ν_0 and W, and, consequently, the function $\Lambda = (W/\nu_0^3)^{1/4}$, whereas m is a dimensional parameter with $\Delta_m = 1$. From Eqn (64) it is seen that when $0 < \epsilon < 2$, parameters D_0 and g_0 are critically dimensionless, and for $\epsilon > 2$ they acquire the critical dimension

$$\Delta[D_0] = \Delta[g_0] = 2\epsilon - 4\,, \qquad \epsilon > 2\,. \tag{65}$$

Expressions (41) follow from the formulae (53) and (55) of critical scaling under the assumption that g_0 is dimensionless, therefore, they are valid only in the range $0 < \epsilon < 2$. For $\epsilon > 2$, new values of dimensions $\Delta'_\varphi = \Delta_\varphi + \Delta[D_0]/3$, $\Delta'_\omega = \Delta_\omega + \Delta[D_0]/3$, $\Delta'_m = \Delta_m = 1$ are obtained if one takes Eqn (65) into account in formulae (53) and (55). These values do not depend on ϵ and coincide with values (41) at $\epsilon = 2$, i.e., with the Kolmogorov dimensions (6). This is just the essence of the statement about freezing of dimensions in the region $\epsilon \geqslant 2$ [15]. It is consistent with the Kolmogorov hypothesis 1 in the sense that dimensions characterize the behavior of Green's functions also in the inertial range, where no dependence should occur on 'details of the pumping mechanism' (Section 2.1); and the choice of a particular value of $\epsilon > 2$ in the range of IR-pumping is just one of those 'details.'

Thus, the standard RG-analysis of the model (16), (9) in terms of the variables W, ν_0, m allows us to prove independence of Green's functions from the viscosity throughout the whole range $\epsilon > 2$ of IR-pumping, which proves the Kolmogorov hypothesis 2 for this region. The main unsolved problem is the dependence of scaling functions on the argument m/k. To prove hypothesis 1, it is necessary to prove that there exists a finite limit, when $m \to 0$, for static correlators for all $\epsilon > 2$. As already said, these questions go beyond the scope of the RG method and will be discussed in the next section.

3. Composite operators, operator product expansion, the first Kolmogorov hypothesis

3.1 Renormalization of composite operators. Use of the Schwinger equations and Galilean invariance

By composite operators F we call any local (unless opposite is specified) monomials or polynomials constructed of fields and their derivatives at one point, for instance, $\varphi^2(x)$, $\varphi'(x)\Delta\varphi(x)$, $\varphi_i(x)\partial_k\varphi_l(x)$. For our model these operators in detail are written as $F = F(x;\Phi)$, $x = t, \mathbf{x}$, and the functional argument Φ will usually be omitted. Since field arguments in Green's functions containing a composite operator $F(x)$ do coincide, there arise extra UV-divergences. They are removed by a specific procedure of renormalization of composite operators (details will be given below); for renormalized Green's functions we can write conventional RG equations which predict IR-scaling with definite critical dimensions $\Delta[F]$ of some 'basic' operators F. Due to renormalization the value of $\Delta[F]$ in a general case does not coincide with a simple sum of critical dimensions of fields and derivatives entering into a given operator.

Study of renormalization of composite operators is important both in itself, since their dimensions and correlation functions can be measured experimentally and for some operators such data are available [35–38], and in view of the problem of proving the Kolmogorov hypothesis 1 (see Section 3.5). The general theory of renormalization of composite operators can be found, for instance, in monographs [6, 9, 10]; below we present only the most necessary information.

A composite operator F is called UV-finite if all Green's functions of the type of $\langle F(x)\Phi(y_1)\ldots\Phi(y_n)\rangle$ with one operator F and any number $n \geqslant 0$ of simple fields Φ are finite (no poles in ϵ). Here $\langle\ldots\rangle$ is understood as averaging with the weight $\exp S_\mathrm{R}(\Phi)$. The generating functional of connected functions of this type is the following:

$$\langle F\rangle_A = \frac{\int \mathcal{D}\Phi F \exp\left[S_\mathrm{R}(\Phi) + A\Phi\right]}{\int \mathcal{D}\Phi \exp\left[S_\mathrm{R}(\Phi) + A\Phi\right]}\,. \tag{66}$$

Therefore, the operator F is UV-finite if and only if the functional $\langle F\rangle_A$ defined by Eqn (66) is UV-finite. Note that the simplest operators $F = 1$ and $F = \Phi(x)$ are UV-finite and that the UV-finiteness of F leads to the UV-finiteness of all operators ∂F, $\partial\partial F$ with any number of external derivatives with respect to $x = t, \mathbf{x}$.

In a general case, composite operators are renormalized with mixing, i.e., an UV-finite renormalized operator is a linear combination of nonrenormalized ones, and vice versa. A complete basis for composite operators is composed of all kinds of local monomials $F_\alpha(x)$ (α is an enumerating index), including the simplest ones $F = 1, \varphi, \varphi'$. Monomials are classified according to the magnitude of dimension $d[F] \equiv d_F$ ('dimension,' if not defined precisely, is always understood as a 'net canonical dimension') which is equal to the sum of dimensions of all fields and derivatives involved in F (see Table 1 in Section 2.5). For a given monomial F_α, an UV-finite renormalized operator $F_\alpha^{\mathrm{R}} = F_\alpha +$ counterterms is constructed uniquely (for a given subtraction scheme). The counterterms are linear combinations of the F_α itself and (possibly) of other nonrenormalized monomials F_β mixed in F_α and meeting the obligatory condition $d[F_\beta] \leqslant d[F_\alpha]$. Coefficients for all operator counterterms in the MS scheme contain only poles in ϵ.

According to the magnitude of d_F monomials are grouped into finite families. If we add to a given family with a given d_F all 'junior' families of lower dimension $d_F - 2, d_F - 4, \ldots$, we obtain a closed system, all monomials of which can mix in renormalization only between themselves.

The renormalization matrix Z_F and the corresponding matrix γ_F of anomalous dimensions (RG-functions) for a given closed system $F \equiv \{F_\alpha\}$ are defined by the relations

$$F_\alpha = \sum_\beta (Z_F)_{\alpha\beta} F_\beta^{\mathrm{R}}, \qquad \gamma_F = Z_F^{-1} \widetilde{\mathcal{D}}_\mu Z_F . \tag{67}$$

The matrix Z_F in Eqn (67) is an analog of renormalization constants of fields Z_Φ (see Section 2.4), and can be calculated directly from diagrams for composite operators as a series in g, and with Z_F at hand we may determine the matrix of anomalous dimensions $\gamma_F(g)$ from Eqn (67) in the same from. Its value $\gamma_F^* \equiv \gamma_F(g_*)$ at a fixed point of the β-function (32) determines the contribution generated by renormalization of operators (67) to the total matrix of critical dimensions

$$\Delta_F = d_F^k + \Delta_\omega d_F^\omega + \gamma_F^* = d_F - \gamma_\nu^* d_F^\omega + \gamma_F^* , \tag{68}$$

which enters into the equation of critical scaling for composite operators:

$$(-\mathcal{D}_{\mathrm{x}} + \Delta_t \mathcal{D}_t + \mathcal{D}_m) F_\alpha^{\mathrm{R}}(x) = \sum_\beta (\Delta_F)_{\alpha\beta} F_\beta^{\mathrm{R}}(x) . \tag{69}$$

This equation is written in the notation (39), d_F^k, d_F^ω, d_F in Eqn (68) are diagonal matrices of the corresponding dimensions (Section 2.4) of the system of operators under consideration. Note that diagonal elements of Δ_F without γ_F^* have the meaning of the sum of critical dimensions of the fields and derivatives involved into a given operator; γ_F^* is an additional term coming from renormalization of operators. If a given F is not renormalized [i.e., $F^{\mathrm{R}}(x) = F(x)$], then its critical dimension is a simple sum of dimensions (41) of all fields and derivatives.

Definite critical dimensions are inherent not in the F_α^{R} themselves, but in those linear combinations of basic operators

$$\bar{F}_\alpha^{\mathrm{R}} = \sum_\beta (U_F)_{\alpha\beta} F_\beta^{\mathrm{R}} , \tag{70}$$

which diagonalize the matrix Δ_F in Eqn (69) under the change $F^{\mathrm{R}} \to \bar{F}^{\mathrm{R}}$: $\Delta_F \to \bar{\Delta}_F = U_F \Delta_F U_F^{-1} = \mathrm{diag}$. This requirement determines the matrix U_F in Eqn (70); diagonal elements $\bar{\Delta}_F$ are critical dimensions of basic operators (70) searched for. In a massless model, when there are no admixtures of junior (with smaller d_F) operators, the matrix U_F is determined from the condition $U_F \Delta_F U_F^{-1} = \mathrm{diag}$, and critical dimensions are eigenvalues of Δ_F. However, in a mass-dependent model of type of (9), since all matrices are block-triangular (see above), the critical dimensions of senior (with maximal d_F) operators can also be determined from eigenvalues of the senior – senior block of operators entering in Δ_F. Complete evaluation of the matrix Δ_F is required only for determining admixtures of junior operators in Eqn (70).

In what follows we will always assume that diagonal elements of U_F equal unity. Under such an auxiliary condition the matrix U_F in Eqn (70) is determined from Δ_F uniquely (if there is no accidental degeneration), therefore there is a definite correspondence between F_α and $\bar{F}_\alpha^{\mathrm{R}}$, and a definite meaning of a critical dimension associated with a given F_α:

$$\Delta_{\mathrm{as}}[F_\alpha] \equiv \Delta[\bar{F}_\alpha^{\mathrm{R}}] . \tag{71}$$

If a certain set of composite operators $\{F_\alpha\}$ closed with respect to renormalization is splitted into two subsets $\{F_\beta\}$ and $\{F_\gamma\}$ so, that any operator F_γ is not admixed to any F_β under renormalization, then the matrix Z_F in Eqn (67) is block-triangular, i.e., $(Z_F)_{\beta\gamma} = 0$, and such are matrices Δ_F in Eqn (68), and U_F in Eqn (70). In this case dimensions $\Delta_{\mathrm{as}}[F_\beta]$ associated with the set F_β and the corresponding basic operators $\bar{F}_\beta^{\mathrm{R}}$ are entirely defined by the block $(Z_F)_{\beta\beta'}$, whereas the block $(Z_F)_{\gamma\gamma'}$ determines the dimensions $\Delta_{\mathrm{as}}[F_\gamma]$, but not the basic elements $\bar{F}_\gamma^{\mathrm{R}}$ [they contain an admixture of operators F_β^{R} determined by nonzero matrix elements $(U_F)_{\gamma\beta}$].

So, the set $\{F_\beta\}$ turns out to be closed also with respect to renormalization. Possible examples are all junior operators of any closed set, all Galilean-invariant operators (see below), all operators of the form ∂F, etc.

At the same time, in the expansion of complete sets $\{F_\alpha\}$ or $\{F_\alpha^{\mathrm{R}}\}$ over basis $\bar{F}_\alpha^{\mathrm{R}}$, operators of the form $\bar{F}_\beta^{\mathrm{R}}$ contribute generally to all F_α, whereas operators $\bar{F}_\gamma^{\mathrm{R}}$ enter only into expansions of operators F_γ. In particular, if an operator F_γ is unique, contribution of the dimension $\Delta_{\mathrm{as}}[F_\gamma]$ associated with it only appears in the expansion of F_γ itself and is absent in all other F_β. Examples are an operator of the form of φ^n in the set of all operators with $d_F \leqslant n$ (see below), and, in a more general formulation, all exceptional operators discussed in Section 3.3.

Information on renormalization of composite operators can sometimes be obtained with the help of various Schwinger's equations and Ward's identities for Galilean transformations without evaluation of diagrams.

In a broad sense of the word, any relations of the type of

$$\int \mathcal{D}\Phi \, \frac{\delta X(\Phi)}{\delta \Phi(x)} = 0$$

which state that any (functional) integral of the total (variational) derivative is equal to zero, are called the Schwinger equations, see Ref. [12]. Specifically, for the model (16) the following relations are valid:

$$\int \mathcal{D}\Phi \, \frac{\delta\{\exp[S_{\rm R}(\Phi) + A\Phi]\}}{\delta\varphi_i'(x)} = 0 \,,$$

$$\int \mathcal{D}\Phi \, \frac{\delta\{\varphi_i(x)\exp[S_{\rm R}(\Phi) + A\Phi]\}}{\delta\varphi_i'(x)} = 0 \,.$$

Using notation (12) and (66), we can rewrite them in the form

$$\left\langle \frac{\delta S_{\rm R}(\Phi)}{\delta\varphi_i'(x)} \right\rangle_A = -A_i^{\varphi'}(x) \,, \tag{72}$$

$$\left\langle \frac{\varphi_i(x)\delta S_{\rm R}(\Phi)}{\delta\varphi_i'(x)} \right\rangle_A = -\left\langle \varphi_i(x) A_i^{\varphi'}(x) \right\rangle_A = -A_i^{\varphi'}(x)\,\frac{\delta W_{\rm R}(A)}{\delta A_i(x)} \,. \tag{73}$$

The right-hand sides of these equations are UV-finite and have definite critical dimensions [the functional $W_{\rm R}(A)$ is dimensionless, and critical dimensions of sources A are expressed through the known dimensions of fields by the requirement that constituents of the linear form (13) are dimensionless]. Therefore, operators in $\langle \ldots \rangle$ on the left-hand sides also exhibit the same properties: these are UV-finite operators of the type of (70) with the known (from the form of right-hand sides) critical dimensions. This also allows us to obtain information on the renormalization matrices Z_F entering into given operators of local monomials (see an example in Section 3.2).

Another method is to use the Ward identities for Galilean transformations $\Phi(x) \to \Phi_v(x)$ with an arbitrary varying velocity $v \equiv \{v_i(t)\}$ sufficiently rapidly decreasing at $|t| \to \infty$:

$$\varphi_v(x) = \varphi(x_v) - v(t) \,, \qquad \varphi_v'(x) = \varphi'(x_v) \,, \qquad x \equiv (t, \mathbf{x}) \,,$$

$$x_v \equiv \big(t, \mathbf{x} + \mathbf{u}(t)\big) \,, \qquad \mathbf{u}(t) \equiv \int_{-\infty}^{t} {\rm d}t'\, v(t') \,. \tag{74}$$

By a strict Galilean invariance we call equations of the type of $H(\Phi) = H(\Phi_v)$ for a functional and $F(x;\Phi_v) = F(x_v;\Phi)$ for a composite operator if they are valid for an arbitrary transformation (74); and by a simple Galilean invariance we mean the same properties but fulfilled only for usual transformations with $\mathbf{v} = {\rm const}$ and $\mathbf{u} = \mathbf{v}t$. For instance, the functional (27) is invariant but not strictly invariant since $S_{\rm R}(\Phi_v) = S(\Phi) + \varphi'\partial_t v$ for it. Strictly invariant are only operators constructed of invariant cofactors φ', $\partial\varphi$ and their covariant derivatives (∂ and ∇_t), for instance, $\partial\varphi\partial\varphi$, $\varphi'\nabla_t\partial\varphi$. The cofactor $\nabla_t\varphi$ is invariant but not strictly, whereas the factors φ and ∂_t are not invariant.

All exact relationships derived by a group change of variables [in our case (74)] in a functional integral may be considered to be the Ward identities in a broad sense of the word. For the first time the Ward identities for transformations (74) were employed in Ref. [16] to prove the absence of renormalization of the vertex in the model (16). Similar identities for time-dependent gauge transformations in critical dynamics were derived earlier in Ref. [39]. The Ward identities including composite operators were examined in Ref. [40]. Below we give the most general and clear formulation of consequences of the Galilean invariance obtained in Ref. [41].

In a general case, in the compact notation $F(x;\Phi) \equiv F(x)$, $F(x;\Phi_v) \equiv F_v(x)$, we have:

$$F_v(x) = F(x_v) + \sum_{k \geqslant 1} v^k F_k(x_v) + \ldots \tag{75}$$

Here dots denote all kinds of contributions with derivatives of v with respect to t. The additional term breaking the Galilean invariance is a polynomial in the velocity and its derivatives; coefficients of the type F_k are local operators of junior dimension; it is implied that they have vector indices contracted with indices of v. For any particular operator F it is not difficult to write the complete expression (75), and it always contains a finite number of contributions.

In Ref. [41] it is shown with the help of the Ward identities that renormalization $F(x) \to F^{\rm R}(x)$ in the framework of the standard MS scheme commutes with transformation (75):

$$(F^{\rm R})_v(x) = (F_v)^{\rm R}(x) = F^{\rm R}(x_v) + \sum_{k \geqslant 1} v^k F_k^{\rm R}(x_v) + \ldots \,, \tag{76}$$

$$\big(\text{counterterms } F(x)\big)_v = \text{counterterms } F_v(x) \,. \tag{77}$$

From relations (76) and (77) a number of useful consequences follows:

(1) For any (strictly) Galilean-invariant operator F the corresponding operator $F^{\rm R}$ and the sum of counterterms are also (strictly) invariant. Thus, critical dimensions associated with Galilean-invariant operators and basis elements are entirely determined by mixing of invariant operators between themselves only.

(2) An operator of the type of φ^n (indices are either free or with any contractions) can be admixed in renormalization neither to itself nor to another operator of the same dimension $d_F = n$. Indeed, if counterterms for F contain φ^n, the left-hand side of equation (77) contains a contribution v^n. However, it cannot be on the r.h.s.: at $d_F = n$ the operator F contains at most n fields φ, and if their number is smaller than n, there is no v^n in F_v; whereas if $F = \varphi^n$, then the contribution to F_v is v^n, but it disappears from the counterterms because it has no UV-divergences. The contradiction obtained proves the statement. It then follows that the critical dimension (71) associated with an operator of the type of φ^n has no corrections coming from γ_F^*, i.e., it is reduced to the sum of critical dimensions of the cofactors:

$$\Delta_{\rm as}[\varphi^n] = n\Delta_\varphi = n\left(1 - \frac{2\epsilon}{3}\right) . \tag{78}$$

Some generalizations of this equation will be discussed in Section 3.3.

In a general case, by substituting Eqn (67) into Eqn (76), one can obtain relations between matrices Z_F of the initial system F and analogous matrices for operators F_k in Eqns (75)–(77) with lower dimensions d_F.

3.2 Composite operators in energy-momentum conservation laws

The equation $\langle F_1 \rangle_A = \langle F_2 \rangle_A$ for functionals of type of (66) is equivalent to the equation $F_1 = F_2$ for the corresponding operators (random quantities). Therefore, relations (72) and (73) for a model like (16) can be rewritten as equations for composite operators [40]:

$$\partial_t\varphi_i + \partial_s\Pi_{is} = D_{is}^F\varphi_s' + A_i^{\varphi'} \,, \tag{79}$$

$$\partial_t E + \partial_i S_i = \dot{E}_{\rm dis} + \varphi_i D_{is}^F \varphi_s' + \varphi_i A_i^{\varphi'} \,. \tag{80}$$

They express the conservation laws of momentum and energy (all quantities per unit mass): φ_i is the momentum density, $E \equiv \varphi^2/2$ is the energy density, Π_{is} is the stress tensor, S_i is the density vector of the energy flux, $\dot{E}_{\rm dis}$ is the rate of energy dissipation; the right-hand sides contain the contribution of nonstochastic external force $A_i^{\varphi'}$ (see Section 2.2) and the contribution of the random force with its correlator (2). The explicit expressions are as follows:

$$\Pi_{is} = p\delta_{is} + \varphi_i\varphi_s - \nu_0(\partial_i\varphi_s + \partial_s\varphi_i)\,, \tag{81}$$

$$S_i = p\varphi_i - \nu_0\varphi_s(\partial_i\varphi_s + \partial_s\varphi_i) + \frac{\varphi^2\varphi_i}{2}\,, \tag{82}$$

$$\dot{E}_{\mathrm{dis}} = -\frac{\nu_0(\partial_i\varphi_s + \partial_s\varphi_i)^2}{2}\,. \tag{83}$$

Here $\nu_0 = \nu Z_\nu$, and p is a nonlocal composite operator

$$p = -\frac{\partial_i\partial_s}{\Delta}(\varphi_i\varphi_s)\,, \tag{84}$$

having the meaning of pressure. The contributions with p to Eqns (79) and (80) arise because the result of formal differentiation of $S_{\mathrm{R}}(\Phi)$ with respect to $\varphi'(x)$ in Eqns (72) and (73) should be contracted with the transverse projector $P_{ij}^{\perp} = \delta_{ij} - \partial_i\partial_j/\Delta$, since the field φ' is transverse.

We stress that equations (79) and (80) do not contain the symbol of averaging $\langle\ldots\rangle$, and hold for operators (random quantities) themselves, not only for their mean values. Formula (3) for the pumping power W is a result of averaging of the corresponding contribution to Eqn (80):

$$W = \langle\varphi D^F\varphi'\rangle \equiv \int \mathrm{d}x'\, D_{is}^F(x,x')\langle\varphi_i(x)\varphi_s'(x)\rangle\,. \tag{85}$$

Since the correlator (2) is of δ-shape in time and symmetric in t, t', we should put the response function $\langle\varphi\varphi'\rangle$ in Eqn (85) at $t = t'$ to be the half-sum of limits from above and below:

$$\langle\varphi_i(x)\varphi_s'(x')\rangle = \begin{cases} P_{is}^{\perp}\delta(\mathbf{x}-\mathbf{x}') & \text{at} \quad t = t'+0\,, \\ 0 & \text{at} \quad t = t'-0\,. \end{cases} \tag{86}$$

If substituted into Eqn (85), this equation leads to expression (3).

Correlation functions of composite operators can, in principle, be measured experimentally. Real observables are quantities of the type of (81)–(83) constructed from non-renormalized monomials and bare parameters. In particular, there are data on the static pair correlator of the dissipation operator (83) in the inertial range that testify to the presence of scaling with the critical dimension $\Delta[\dot{E}_{\mathrm{dis}}] = 0.2$, Refs [35–38]. It rather strongly differs from its canonical dimension $d[\dot{E}_{\mathrm{dis}}] = 4$, which should be a result of the renormalization-generated 'anomaly' of the type of contributions of order ϵ to Eqn (41) from the anomalous dimension γ_ν^*.

The first question that arises in a theoretical study of an operator of the type of (81)–(83) is whether a given operator F has a definite critical dimension $\Delta[F]$ at all. The problem is nontrivial since it is just basic operators (70) that possess definite dimensions, and F in a general case is a linear combination of those operators with different critical dimensions, and (if so) this should be taken into account for analyzing experimental data. Therefore the operator F, with an exception for some rare cases, cannot be considered separately: to expand it over the basis (70) we first have to construct it, and this requires to analyse renormalization of the whole closed system of operators which the operator F belongs to.

This analysis for all operators contained in Eqns (79) and (80) was carried out in Ref. [40] within the framework of the MS scheme for the massless model (25). Note that the absence of a mass m removes admixtures of junior operators to the senior ones and does not distort critical dimensions of the latter, since they are uniquely determined by the senior–senior block because the matrix (68) is block-triangular (see Section 3.1).

The central result of paper [40] is that all operators in Eqns (79) and (80) are UV-finite and possess definite critical dimensions equal to

$$3 - \frac{4\epsilon}{3} = \Delta_\varphi + \Delta_\omega = \Delta[\partial_t\varphi_i] = \Delta[\partial_s\Pi_{is}] = \Delta[A_i^{\varphi'}] \tag{87}$$

for all terms in Eqn (79), and

$$4 - 2\epsilon = 2\Delta_\varphi + \Delta_\omega = \Delta[\partial_t E] = \Delta[\partial_i S_i] = \Delta[\dot{E}_{\mathrm{dis}}] \tag{88}$$

for all terms in (80). The operators $E = \varphi^2/2$ and Π_{is} are UV-finite and have dimensions

$$2 - \frac{4\epsilon}{3} = 2\Delta_\varphi = \Delta[E] = \Delta[\Pi_{is}]\,, \tag{89}$$

and the vector of energy-flux density S_i defined by Eqn (82) is a sum of the UV-finite contribution with dimension $3 - 2\epsilon$, an additional term $c\Delta\varphi_i$ with a different dimension $2 + \Delta_\varphi = 3 - 2\epsilon/3$, and an UV-divergent coefficient c. As this additional term is transverse, it does not contribute to equation (80) and can be neglected, which is reduced to a physically admissible redefinition of the vector S_i.

We shall explain the technique of treating composite operators by this example. We start with the scalar $F(x) = \varphi^2(x) = 2E(x)$ with $d_F = 2$. The massless model does not contain other scalars with $d_F = 2$, and the scalar cannot be admixed to itself [consequence 2 of relations (76) and (77)], therefore the operator is not renormalized, $F^{\mathrm{R}} = F$, and has a definite dimension $\Delta[F] = 2\Delta_\varphi$ in accordance with Eqn (78).

Consider equation (79). All contributions to it, except $\partial_s\Pi_{is}$, are obviously UV-finite and thus are not renormalized, therefore it is also UV-finite owing to equation (79) [recall that it is equivalent to the Schwinger equation (72)]. Renormalization of any operator of the type of ∂F is performed by renormalization of the operator F itself [i.e., $(\partial F)^{\mathrm{R}} = \partial(F^{\mathrm{R}})$], therefore it is sufficient to consider the renormalization 3×3 matrix (67) for three symmetric tensor operators with $d_F = 2$ entering into Eqn (79):

$$F_1 = \partial_i\varphi_k + \partial_k\varphi_i\,, \qquad F_2 = \varphi_i\varphi_k\,, \qquad F_3 = \delta_{ik}\,p\,. \tag{90}$$

The last operator is nonlocal but is needed since it enters into Eqn (79). The general rule that counterterms must be local (Section 2.4) is also valid for composite operators: nonlocal operators cannot be counterterms, and thus, during renormalization, cannot be admixed to other operators, in particular to themselves. (However, in renormalization local operators can be admixed to the nonlocal ones.) In view of its dimension and structure, the operator $\varphi^2\delta_{ik}$ is permissible in Eqn (90); but it is not renormalized (see above) and is not admixed to other operators [corollary 2 of relations (76) and (77)], thus, it is completely split off.

The operator F_1 in Eqn (90) is a multiple of a simple field φ and is not renormalized (i.e., $F_1 = F_1^{\mathrm{R}}$ or $Z_{11} = 1$, $Z_{12} = Z_{13} = 0$); for the second and third operators we have $F_2 = Z_{21}F_1^{\mathrm{R}} + Z_{22}F_2^{\mathrm{R}}$, $F_3 = Z_{31}F_1^{\mathrm{R}} + Z_{32}F_2^{\mathrm{R}} + F_3^{\mathrm{R}}$ (nonlocal F_3 is not admixed). The constants Z are connected with each other, since from Eqns (84) and (90) we get $F_3 = -P^{\|}F_2$ up to the factor δ_{ik}, which entails $F_3^{\mathrm{R}} = -P^{\|}F_2^{\mathrm{R}}$ because external factors like the projector $P_{ij}^{\|} \equiv \partial_i\partial_j/\Delta$ do not affect renormalization. From this result and taking into account that φ is transverse, we obtain

$$F_3 = Z_{31}F_1^{\mathrm{R}} + Z_{32}F_2^{\mathrm{R}} + F_3^{\mathrm{R}} = -P^{\|}F_2$$
$$= -P^{\|}[Z_{21}F_1^{\mathrm{R}} + Z_{22}F_2^{\mathrm{R}}] = -Z_{22}P^{\|}F_2^{\mathrm{R}} = Z_{22}F_3^{\mathrm{R}}\,.$$

Comparing the coefficients of F_i^{R}, we find $Z_{31} = Z_{32} = 0$, $Z_{22} = 1$. The remaining unknown element Z_{21} can be found by requiring that the divergence

$$\partial\Pi = \partial[F_3 + F_2 - \nu Z_\nu F_1] = \partial[F_3^{\mathrm{R}} + Z_{21}F_1^{\mathrm{R}} + F_2^{\mathrm{R}} - \nu Z_\nu F_1^{\mathrm{R}}]$$

be UV-finite (see above). Three operators ∂F^{R} are independent, therefore the coefficient for each of them should be UV-finite, and consequently, it should coincide with its UV-finite part. For the coefficient for $\partial F_1^{\mathrm{R}}$ we obtain $Z_{21} - \nu Z_\nu = -\nu$, and thus, $Z_{21} = \nu(Z_\nu - 1)$. We used the fact that in the MS scheme constants of the type of Z_ν and diagonal elements of any matrix Z_F are of the form of Eqn (28), so that their UV-finite part is unity, and all nondiagonal elements of Z_F consist only of poles in ϵ and thus have no finite part.

Therefore, for the system (90) the matrices Z_F and γ_F in Eqn (67) can be found without computing diagrams:

$$Z_F = \begin{pmatrix} 1 & 0 & 0 \\ \nu(Z_\nu - 1) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \gamma_F = \begin{pmatrix} 0 & 0 & 0 \\ \nu\gamma_\nu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (91)$$

Here the RG-function γ_ν is defined by Eqn (32). Using the known (see Table 1 in Section 2.5) canonical dimensions $d_F = 2, 2, 2$; $d_F^\omega = 1, 2, 2$ of the operators (90), we may construct the matrix of critical dimensions (68) and the matrix U_F from Eqn (70) which diagonalizes Eqn (68):

$$\Delta_F = \begin{pmatrix} 2-\gamma_\nu^* & 0 & 0 \\ \nu\gamma_\nu^* & 2-2\gamma_\nu^* & 0 \\ 0 & 0 & 2-2\gamma_\nu^* \end{pmatrix}, \quad U_F = \begin{pmatrix} 1 & 0 & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (92)$$

Here $\gamma_\nu^* = 2\epsilon/3$ according to Eqn (70). Since the matrix Δ_F is triangular, its diagonal elements $\Delta[F] = 2-\gamma_\nu^*$, $2-2\gamma_\nu^*$, $2-2\gamma_\nu^*$ are critical dimensions searched for, and the corresponding basic operators (70) have just these dimensions, i.e., $\bar F_1^{\mathrm{R}} = F_1^{\mathrm{R}}$, $\bar F_2^{\mathrm{R}} = F_2^{\mathrm{R}} - \nu F_1^{\mathrm{R}}$, $\bar F_3^{\mathrm{R}} = F_3^{\mathrm{R}}$. The tensor (81) expressed initially through nonrenormalized operators (90) can be expressed with the help of Eqn (67) in terms of renormalized operators, and then in terms of the basic operators (70):

$$\Pi = F_3 + F_2 - \nu Z_\nu F_1 = F_3^{\mathrm{R}} + \nu(Z_\nu - 1)F_1^{\mathrm{R}}$$
$$+ F_2^{\mathrm{R}} - \nu Z_\nu F_1^{\mathrm{R}} = \bar F_2^{\mathrm{R}} + \bar F_3^{\mathrm{R}}\,.$$

This result is expressed in terms of two operators $\bar F^{\mathrm{R}}$ with the same (see above) dimension $\Delta[F] = 2 - 2\gamma_\nu^* = 2 - 4\epsilon/3$.

We presented a detailed analysis of the renormalization of a simple system (90) by way of example; more complicated systems may be considered in a similar way [40].

To analyse the vector (82), one should investigate into the closed system of six vector operators with $d_F = 3$:

$$\partial_t\varphi_i\,,\quad \Delta\varphi_i\,,\quad \partial_i\varphi^2,\quad \partial_s(\varphi_i\varphi_s),\quad \varphi_i\varphi^2,\quad \varphi_i p\,. \qquad (93)$$

An analogous system for scalars in Eqn (80) consists of seven operators with $d_F = 4$:

$$\partial_t\varphi^2,\quad \Delta\varphi^2,\quad \partial_i\partial_s(\varphi_i\varphi_s),\quad \partial_i(\varphi_i\varphi^2),\quad \varphi\Delta\varphi,\quad \varphi^4,\quad \partial_i(\varphi_i p)\,. \qquad (94)$$

From dimensional considerations it seems that we need to add the nonlocal operator $\varphi D^F\varphi'$ from Eqn (80), and in a particular case of $d = 3$, also the operator $(\varphi')^2$, which has at $d = 3$ the same dimension $d_F = 4$. But these two operators are not renormalized and are not admixed to operators involved into the system (94), therefore this system can be considered independently [19]. In Ref. [40] it was shown that employing Schwinger's equations and Ward's identities, one can determine the renormalization 6×6 matrix for the system (93) up to two nondiagonal elements remaining unknown; and the 7×7 matrix (94) up to three nondiagonal elements remaining unknown; all other are expressed in terms of $Z\nu$. The remaining unknown elements do not influence critical dimensions and are unimportant for the analysis of contributions to equation (80). The searched critical dimensions Δ_F for system (93) turn out to be equal to $3-\gamma_\nu^*$, $3-2\gamma_\nu^*$ (triple degeneration), and to $3-3\gamma_\nu^*$ (double degeneration), and for family (94) we have $4-2\gamma_\nu^*$ (double), $4-3\gamma_\nu^*$ (quadruple), and $4-4\gamma_\nu^*$, where $\gamma_\nu^* = 2\epsilon/3$ [40].

The dissipation operator (83) is a linear combination of nonrenormalized monomials F_i from system (94) [enumeration follows the sequence in (94)] and upon renormalization is reduced to one of the basic operators (70) for that system:

$$\dot E_{\mathrm{dis}} = \nu_0\left[F_5 - F_3 - \frac{F_2}{2}\right] = \nu\left[F_5^{\mathrm{R}} - F_3^{\mathrm{R}} - \frac{F_2^{\mathrm{R}}}{2}\right] = \nu\bar F_5^{\mathrm{R}}\,. \quad (95)$$

Dimension $\Delta[\dot E_{\mathrm{dis}}] = 4 - 2\epsilon$ is given by Eqn (88). For physical value $\epsilon = 2$ we get $\Delta[\dot E_{\mathrm{dis}}] = 0$, which does not completely coincide with the expected experimental value 0.2 [35–38]. Nevertheless, this is not so important as compared to the main result, i.e., transformation of the canonical dimension $d_F = 4$ into the critical one Δ_F owing to of contributions of anomalous dimensions of order ϵ.

3.3 Critical dimensions of senior operators

When we discuss the problems concerning the Kolmogorov hypothesis 1, the decisive role belongs to the information on critical dimensions of Galilean-invariant operators (70) with $\langle \bar F_\alpha^{\mathrm{R}}\rangle \neq 0$ in a mass-dependent model (see Section 3.4). This condition eliminates, in particular, all operators of the type of ∂F for which $\langle \partial F\rangle = \partial\langle F\rangle = 0$ owing to the translation invariance. These data are scarce, and in this section we present all the information we know. The class of scalars with $d_F = 4$ has only one independent monomial of that type, namely, $F = \partial_i\varphi_k\partial_i\varphi_k$. The corresponding basic operator (70) is the dissipation operator (83) with the critical dimension exactly known from (88) [19].

In the class of the needed type with $d_F = 6$ there are two independent monomials, namely,

$$F_1 = \partial_i\varphi_k\partial_i\varphi_l\,\partial_k\varphi_l\,, \qquad F_2 = \partial^2\varphi_i\partial^2\varphi_i\,, \qquad (96)$$

where $\partial^2 \equiv \Delta$ is the Laplacian. At first sight we can add one more independent monomial $\partial_i\varphi_k\partial_k\varphi_l\partial_l\varphi_i$, but in fact it is reduced to a sum of three nonessential terms of the type of ∂F. At $d = 3$ the system (96) can be enlarged with two other operators, $F_3 = \partial_i\varphi_k'\partial_i\varphi_k'$ and $F_4 = \varphi_i'\varphi_k'\partial_k\varphi_i$, but they are nonessential since both of them have zero mean $\langle F\rangle = 0$

values because of closed cycles of retarded lines in the corresponding diagrams (Section 2.2). This property also survives after renormalization, since the operators (96) with $\langle F\rangle \neq 0$ do not admix to them for the same reason (cycles).

Critical dimensions associated with two essential operators (96) have been calculated in Ref. [42]. One of those dimensions was determined exactly for arbitrary d (it is d-independent) with the aid of the Schwinger equations, but the second dimension was evaluated only for $d = 3$ and in the one-loop approximation:

$$\Delta_1 = 6 - 2\epsilon\,, \qquad \Delta_2 = 6 - \frac{8\epsilon}{7} + O(\epsilon^2)\,. \tag{97}$$

For irrelevant operators $F_{3,4}$ it was found that $\Delta_3 = 6 - 2\epsilon/9 + O(\epsilon^2)$ and $\Delta_4 = 6 + 2\epsilon/3$ (the last value is exact).

In Ref. [43] critical dimensions of all relevant operators of the type of $\partial\varphi\partial\varphi$ with canonical dimensions $d_F = 4$ were evaluated at arbitrary d in the one-loop approximation. They include the scalar $\partial_i\varphi_k\partial_i\varphi_k$ discussed earlier, two independent irreducible tensors F_{ik} of the second rank,

$$\text{I.p. } [\partial_i\varphi_l\partial_k\varphi_l]\,, \qquad \text{I.p. } [\partial_l\varphi_i\partial_l\varphi_k]\,, \tag{98}$$

and three independent irreducible tensors F_{iklm} of the fourth rank:

$$\begin{aligned} &\text{I.p. } [\partial_i\varphi_l\partial_k\varphi_m \pm \partial_l\varphi_i\partial_m\varphi_k]\,,\\ &\text{I.p. } [\partial_i\varphi_k\partial_l\varphi_m + \partial_i\varphi_k\partial_m\varphi_l + \partial_k\varphi_i\partial_l\varphi_m + \partial_k\varphi_i\partial_m\varphi_l]\,. \end{aligned} \tag{99}$$

We denote with 'I.p.' the operation of selection of an irreducible part, i.e., subtraction of appropriate expressions with δ-symbols ensuring that the expressions obtained are traceless, i.e., the contraction over any pair of indices is zero. For instance,

$$\text{I.p. } [\partial_i\varphi_l\partial_k\varphi_l] = \partial_i\varphi_l\partial_k\varphi_l - \frac{\delta_{ik}\partial_m\varphi_l\partial_m\varphi_l}{d}\,.$$

Critical dimensions associated with tensors (98) in the one-loop approximation at arbitrary d are given by

$$\Delta_F = 4 - \frac{4\epsilon}{3} + \frac{2\epsilon}{3d(d-1)(d+4)} \times \left[4 + 8d - d^3 \pm \sqrt{16 - 16d^2 - 4d^3 + 5d^4}\right] + O(\epsilon^2) \tag{100}$$

(for $d = 2, 3, 4$ the square root is an integer), and for operators (99) we get

$$\Delta_F = \begin{cases} 4 - \dfrac{4\epsilon}{3} - \dfrac{8\epsilon(12 - 4d + d^3)}{3d(d-1)(d+4)(d+6)} + O(\epsilon^2)\,,\\[2mm] 4 - \dfrac{4\epsilon}{3} + \dfrac{8\epsilon(d+2)}{3d(d-1)(d+4)} + O(\epsilon^2)\,,\\[2mm] 4 - \dfrac{4\epsilon}{3} + \dfrac{4\epsilon}{3d} + O(\epsilon^2)\,. \end{cases} \tag{101}$$

At $d = 3$ we have $\Delta_F = \{4 - 32\epsilon/21,\ 4 - 10\epsilon/9\}$ for (98) and $\Delta_F = \{4 - 32\epsilon/21,\ 4 - 64\epsilon/63,\ 4 - 8\epsilon/9\}$ for Eqn (99); at $\epsilon = 2$ all these dimensions are strictly positive.

Passing to operators of higher dimensions, we first mention the statement (78) for an arbitrary in the index structure operator of the type of φ^n. For the first time it was given in Ref. [15], a formal proof based on the Ward identities of Ref. [40] was presented in Ref. [44]; within a certain generalization of the Wilson RG it was derived also in Ref. [45] (see below).

The proof of Eqn (78) given in Section 3.1 can be generalized to some other classes of operators. For instance, for a monomial constructed only from symbols φ and ∂_t (we shall denote it with $F = \partial_t^m\varphi^n$) with a particular set (free or with contractions) vector indices, we have [42]:

$$\Delta_{\text{as}}[\partial_t^m\varphi^n] = n\Delta_\varphi + m\Delta_\omega = n\left(1 - \frac{2\epsilon}{3}\right) + m\left(2 - \frac{2\epsilon}{3}\right). \tag{102}$$

For the elements of the nonrenormalized basis instead of the monomials $\partial_t^m\varphi^n$ we can choose polynomials, which are obtained from them by substitution of the covariant derivatives ∇_t for all operations ∂_t. The rule (102) remains valid for all these operators, too.

Let us now formulate the general rule. We will call a given element F (a monomial or a polynomial) of the chosen nonrenormalized basis an 'exceptional element' if its Galilean transformation (75) contains an exceptional UV-finite contribution that cannot be generated by any other nonrenormalized basis element with the same d_F. The initial F cannot be a counterterm, i.e., in renormalization it can admix neither to itself nor to other elements, and the critical dimension (71) associated with it obeys the rule

$$\Delta_{\text{as}}[\text{except. } F] = \sum \Delta[\,\text{factor of } F\,]\,. \tag{103}$$

Equations (78) and (102) are just particular cases of the general rule (103). It may also be applied to other cases, for instance, to the operator $F = \varphi'(\nabla_t\varphi)^n$ which is also exceptional. To avoid misunderstanding, we must note that the concept of exceptionality can depend on the choice of the nonrenormalized basis, but the complete set of critical dimensions of the system under consideration does not depend on this choice.

Almost all the known results about critical dimensions of composite operators in the model (16) were obtained within the standard quantum-field theory of renormalization. The only exception is the work [45], where the authors made an attempt to generalize the Wilson RG technique to the case of some composite operators. In Ref. [45] relation (78) was obtained for operators φ^n, and the following result for the contribution $F \equiv \nu_0\partial_i\varphi_k\partial_i\varphi_k$ to the dissipation operator (83):

$$\Delta[F^n] = n\Delta[F] = n(4 - 2\epsilon)\,. \tag{104}$$

However, equations of motion for composite operators used in Ref. [45] (in the field-theoretical language they are associated with certain senior Schwinger's equations) do not contain the random-force contributions of the type of terms with D_F in Eqns (79) and (80), which leads to the loss of part of diagrams (the necessasity to take the random-force contribution into account was noted in a subsequent Ref. [46]). Also in these works the possibility of mixing of operators in renormalization is not discussed at all. Therefore, it is difficult to consider the proof given in Ref. [45] reliable.

Scalar Galilean-invariant operators with $d_F = 8$ including the operator $\dot{E}^2_{\text{dis}}$ with $\dot{E}_{\text{dis}}$ from Eqn (83) were investigated in Ref. [47]. It turns out that the operator $\dot{E}^2_{\text{dis}}$ unlike $\dot{E}_{\text{dis}}$ itself is not UV-finite (which would have ensured the fulfillment of equation (104) for $n = 2$) or even multiplicatively renormalizable. With the help of the Schwinger equations the authors of

Ref. [47] found exact critical dimensions of some operators with $d_F = 8$, but all of them become negative only when $\epsilon > 3$. On the other hand, there are arguments, Refs [48, 49], in favor of the existence of an infinite number of operators with the same symmetry which become dangerous at $\epsilon > 2$ together with $\dot{E}_{\mathrm{dis}}$, so the problem of validity of Eqn (104) remains open.

Also note that throughout Sections 3.1 – 3.3 by critical dimensions we understood dimensions of the type of (41) describing IR-scaling at fixed g_0, ν_0 (see Section 2.7). Transition to the scaling at fixed W, ν_0 in the Green functions of composite operators is accomplished like that for the Green functions of simple fields (see Section 2.8). The standard formula (68) remains valid at $\epsilon < 2$, and when $\epsilon > 2$, it is replaced by the following expression [49]:

$$\Delta_F = d_F^k + \frac{2d_F^\omega}{3} + \frac{2\gamma_F^*}{\epsilon} = d_F - \frac{4d_F^\omega}{3} + \frac{2\gamma_F^*}{\epsilon} \,. \tag{105}$$

With the use of Eqn (105) it can, in particular, be verified that dimensions (78) associated with φ^n and dimension $\Delta_F = 4 - 2\epsilon$ of the dissipation operator (83) are frozen throughout the whole region $\epsilon > 2$ at their Kolmogorov's values at $\epsilon = 2$. We will not dwell upon this in detail since for the following just dimensions corresponding to the scaling at fixed g_0, ν_0 will play the main role in the analysis of IR asymptotic behavior of scaling functions (see Sections 3.4 and 3.5).

3.4 Investigation of the asymptotics at $m \to 0$ with the use of operator product expansion

The RG-representations (53) and (54) describe the IR-asymptotics at $k \sim m \to 0$, $\omega \sim k^{\Delta_\omega} \to 0$ of correlation functions of the model (16) and (9) at an arbitrary fixed $u \equiv m/k$. To the inertial range there corresponds an additional condition $u \ll 1$. From Eqns (55) and (54) it follows that the first Kolmogorov hypothesis for a simultaneous pair correlator (see Section 2.1) is equivalent to the requirement of cancellation of the m-dependence in the combination $D_0^{2/3} f(u)$ when $u \to 0$, and taking relations (64) into account we obtain [15]:

$$f(u) = \begin{cases} \text{const} = f(0) & \text{for} \quad 0 < \epsilon \leqslant 2 \,, \\ \text{const} \times u^{4(2-\epsilon)/3} & \text{for} \quad \epsilon \geqslant 2 \,. \end{cases} \tag{106}$$

The scaling functions in Eqns (53) and (54) are not determined by the RG equations. A standard method for their computation is the ϵ-expansion (58). From the analysis of diagrams of perturbation theory it is known that the coefficients f_n of the ϵ-expansion in Eqn (58), when $u \to 0$, have only weak singularities of the type of $u \log u$, i.e., they are finite at $u = 0$. Therefore the Kolmogorov hypothesis 1 can be postulated in the framework of the ϵ-expansion, which, however, does not prove it at finite ϵ, since for any arbitrarily small value of ϵ there are diagrams which diverge at $m \to 0$. A formal formulation of the problem is that it is necessary to sum the ϵ-expansion series (58) when ϵ is assumed to be small and under the additional condition $\epsilon \log u \sim 1$.

Like in the theory of critical behavior [6, 10], a problem of this sort is solved with the aid of the well-known Wilson operator product expansion [a synonym is the short-distance expansion (SDE)]: in accordance with SDE, the product $\varphi(x_1)\varphi(x_2)$ of two renormalized field operators at $\mathbf{x} \equiv (\mathbf{x}_1 + \mathbf{x}_2)/2 = \text{const}$, $t \equiv (t_1 + t_2)/2 = \text{const}$, $\mathbf{r} \equiv \mathbf{x}_1 - \mathbf{x}_2 \to 0$, $\tau \equiv t_1 - t_2 \to 0$, may be represented in the following form [15]:

$$\varphi(x_1)\varphi(x_2) = \sum_\alpha C_\alpha(\mathbf{r}, \tau) \bar{F}_\alpha^{\mathrm{R}}(\mathbf{x}, \mathbf{t}) \,. \tag{107}$$

Here the coefficients C_α are regular in m^2; $\bar{F}_\alpha^{\mathrm{R}}$ are all kinds of renormalized basic operators of the type of (70) with definite critical dimensions Δ_α consistent with symmetry of the left-hand side. The renormalized correlator $D = \langle\varphi\varphi\rangle$ is obtained by averaging the expression (107) with the weight $\exp S_{\mathrm{R}}$; as a result the quantities $\langle\bar{F}_\alpha^{\mathrm{R}}\rangle$ appear on the right-hand side. Their asymptotic behavior at $m/\mu \to 0$ is obtained from the corresponding RG-equations (see Section 3.1) and is of the form

$$\langle\bar{F}_\alpha^{\mathrm{R}}\rangle \propto m^{\Delta_\alpha/\Delta_m} \,. \tag{108}$$

[For the model (16) $\Delta_m = 1$]. Thus, the operator product expansion (107) results in the following representation for the scaling function of the correlator $D = \langle\varphi\varphi\rangle$:

$$f(u, \ldots) = \sum_\alpha A_\alpha(u, \ldots)\, u^{\Delta_\alpha} \,. \tag{109}$$

Here the coefficients A_α are regular in u^2, dots in Eqn (109) denote possible extra arguments of f [ϵ for the static function (54) and ϵ, z for the dynamic one, Eqn (53)]. Note that simple regularity of the functions C_α, A_α in $m \propto u$ takes place only for the standard formulation of the asymptotic problem with fixed g_0 (see Section 2.8), therefore, dimensions Δ_α in Eqn (109) are given by the relation (41) for any $\epsilon > 0$.

Representation (109) solves the problem of summation of ϵ-expansions of the scaling function at $\epsilon \log u \sim 1$. When $u \to 0$, the leading contributions to Eqn (109) are those with minimal values of the dimensions Δ_α; in the framework of ϵ-expansions with minimal d_α at $\epsilon = 0$, i.e., with a minimum number of fields and derivatives. The operators with $\Delta_\alpha < 0$, if they exist, will be called 'dangerous' [15]; they are associated with the contributions to Eqn (109) divergent at $u \to 0$.

The problem of dangerous operators does really exist in the framework of the model (16) beyond the scope of the ϵ-expansion. Specifically, from Eqn (78) it is seen that all operators associated with integer powers of the velocity field become dangerous when $\epsilon \geqslant 3/2$, i.e., before entering into the region $\epsilon \geqslant 2$ of real IR-pumping. With increasing ϵ other dangerous operators may appear, for instance, when $\epsilon \geqslant 2$ the dissipation operator (83) becomes dangerous. Its dimension is known exactly [19]: $\Delta_F = 4 - 2\epsilon$, see Section 3.2. On the basis of results of Refs [19, 40], it was assumed in Ref. [15] that in the region $0 < \epsilon < 2$ only operators associated with φ^n may be dangerous. In the region $3/2 < \epsilon < 2$ their contributions to Eqn (109) diverge when $m \to 0$ and must be summed up. This was done in Ref. [15]. But the assumption made in Ref. [15] is invalid since it follows from Eqn (102) that operators associated with $\partial_t^p \varphi^n$ (i.e., operators with an arbitrary arrangement of p symbols ∂_t inbetween n factors φ) may also become dangerous at $\epsilon < 2$ if $p < n/2$: the dimensions (102) vanish at $\epsilon = 3(2p + n)/2(p + n)$. However, it was shown in Ref. [42] that the technique of summation [15] may be readily generalized to operators of this sort as well. It will be expounded in the next section.

3.5 Substantiation of the Kolmogorov hypothesis 1 in the interval $0 < \epsilon < 2$ within infrared perturbation theory

Relevant summation of the contributions of all dangerous operators of the type of φ^n was performed in Ref. [15] with the use of a version of infrared perturbation theory (IRPT), first suggested in quantum electrodynamics [11] and widely used in turbulence theory for studying IR-singularities in self-consistency equations [50 – 54].

The formulation of IRPT employed in Ref. [15] is based on splitting the fields in the functional integral (102) into 'soft' (long wavelength) and 'hard' (short wavelength) components with a subsequent negation of space-time (or only of space, Refs [42, 55]) inhomogeneity of the soft field. For the correlator in the inertial range summation of contributions of all dangerous operators of the type of φ^n in Eqn (107) leads (we expound the results of Ref. [15] in a later formulation [56]) to the expression

$$\langle\varphi(x_1)\varphi(x_2)\rangle = \sum_{n=0}^{\infty}\frac{\tau^n}{n!}\langle(\varphi\partial)^n\rangle C_0(\mathbf{r},\tau)+\ldots \tag{110}$$

Here C_0 is the coefficient for $\bar{F}_\alpha^{\rm R}=1$ in Eqn (107); indices $\varphi\equiv\varphi_i(\mathbf{x},t)$ in the form $\varphi\partial$ are contracted with those of the gradient $\partial\equiv\partial/\partial r_i$; and dots denote contributions of all the remaining operators. Since Eqn (110) includes all nonrenormalized monomials of the type of φ^n, there (and only there) all dangerous operators associated with them with dimensions (78) are contained [see the remark in the text after formula (78)]. Expression (110) may be rewritten in a compact form:

$$\langle\varphi(x_1)\varphi(x_2)\rangle = \langle\langle C_0(\mathbf{r}+\mathbf{v}\tau,\tau)\rangle\rangle+\ldots \tag{111}$$

Here $\langle\langle\ldots\rangle\rangle$ denotes averaging over a random time-independent quantity $\mathbf{v}$, whose distribution is defined by the relations

$$\langle\langle v^n\rangle\rangle = \langle\varphi^n(x)\rangle = \text{const}\times m^{n\Delta_\varphi}+\ldots \tag{112}$$

with free vector indices. Moments (112) are nonzero only for even n; the contribution $m^{n\Delta_\varphi}$ comes from a dangerous operator; dots stand for unessential (less singular) terms coming from possible admixtures in φ^n. Representation (111) and its analog for the response function $\langle\varphi\varphi'\rangle$ were obtained in Ref. [15] under the assumption that self-interaction of hard fields was negligible. This corresponds to the lowest order of perturbation theory in g_0 for expressions inserted into $\langle\langle\ldots\rangle\rangle$. They coincide with the corresponding massless bare Green's functions (17). A generalized formulation of Eqn (111) with the exact function C_0 from Eqn (107) was obtained in Ref. [44]. In Ref. [56] it was shown that relations of the type of (110) – (112) can also be derived without IRPT by using only the short-distance expansion (SDE) for the product $\varphi\varphi$ itself and not only for its average $\langle\varphi\varphi\rangle$. This allows to get rid of the IRPT artificial parameter, the momentum k_*, which separates the regions of soft ($k\leqslant k_*$) and hard ($k>k_*$) momenta; note also that the SDE technique contains a constructive recipe for evaluation of all Wilson's coefficients C_α from Eqn (107), in particular the coefficient C_0 involved into Eqn (111). An important generalization of relations (110) and (111) was found in Ref. [42], where summation was performed not only of contributions of all operators of the type of φ^n, like in Eqn (110), but also of all other operators of the type of $\partial_t^p\varphi^n$, which can also become dangerous at $\epsilon<2$ (see Section 3.3). Instead of $\mathbf{v}\tau\equiv\mathbf{v}(t_1-t_2)$ in Eqn (111) appears an integral over the interval $[t_1,t_2]$ of a time-dependent velocity $\mathbf{v}(t)$ with a known distribution. It is important that this integral vanishes at $t_1=t_2$ like $\mathbf{v}\tau$ in Eqn (111), i.e., the given classes of dangerous operators do not contribute to the static correlator with $t_1=t_2$. But at $t_1-t_2\equiv\tau\neq0$ the singular dependence on m through the distribution (112) survives in Eqn (111), i.e., the Kolmogorov hypothesis 1 is not extended to the dynamic correlator [15]. Note that representations of the type of (111) are well-known in the technique of self-consistency equations (see, for instance, Refs [50 – 54]) and have a simple physical interpretation: they describe the kinematic effect of transport of turbulent vortices as a whole by the large-scale field $\mathbf{v}$ [1]. A new feature of the approach [15] and subsequent investigations into the topic is a combination of representations (111) with RG-representations by means of relations (108), (109), and the use of RG and SDE technique for the calculation of critical dimensions Δ_α and coefficients A_α in Eqn (109).

As was said above, all dangerous at $\epsilon>3/2$ operators of the type of φ^n, $\partial_t^p\varphi^n$ do not contribute to the static correlator, but the Kolmogorov hypothesis 1 can be broken by other dangerous operators which appear as ϵ grows. In particular, when $\epsilon>2$ the dissipation operator (83) with the exactly known dimension $\Delta_F=4-2\epsilon$ and all its powers become dangerous provided the Eqn (104) is correct; with the further increase of ϵ other operators also can become dangerous. Investigation of these problems is simplified to some extent if one takes the results of Ref. [57] (see also Ref. [42]) into account. It is shown there that the product of two fields with coinciding times and contracted vector indices after a certain subtraction (see below) is a strictly Galilean-invariant object (in the sense of Section 3.1) and its SDE only contains operators with the same symmetry:

$$\left[\varphi_i(x_1)\varphi_i(x_2)-\frac{(\varphi^2)^{\rm R}(x_1)+(\varphi^2)^{\rm R}(x_2)}{2}\right]\Bigg|_{t_1=t_2} = \sum_{\rm invar} C_\alpha(\mathbf{r})\bar{F}_\alpha^{\rm R}(x)\,. \tag{113}$$

Here summation runs over all strictly Galilean-invariant operators and $(\varphi^2)^{\rm R}(x)=\varphi^2(x)+\text{const}$ in a mass-dependent model (in a massless model φ^2 is not renormalized). Note that Eqn (113) can contain any invariant operators, in particular tensor ones, the tensor indices are naturally contracted with analogous indices of the coefficients C_α. Without loss of generality it can be assumed that the expansion is carried out over irreducible tensors (for examples, see Section 3.3), then the contribution to the correlator $\langle\varphi\varphi\rangle$ will only come from scalars since the averages (108) for nonscalar irreducible tensors equal zero. For the same reason, contributions to the correlator from all operators of the form ∂F with external derivatives disappear ($\langle\partial F\rangle=\partial\langle F\rangle=0$ owing to translation invariance). Subtracted terms on the left-hand side vanish in the momentum representation after averaging for $k\neq0$.

Thus, the contribution to the SDE-representation (109) of the scaling function of the static correlator comes only from strictly Galilean-invariant scalar operators without external derivatives constructed from the field φ only (Section 3.4). All such operators have an integer even dimension $d_F=2n=0,2,4\ldots$ For small d_F there is a comparatively small number of such operators: $d_F=0$ corresponds to $F=1$; there are no such operators at $d_F=2$; for $d_F=4$ there is the only (up to equivalence with respect to additions of the type of ∂F) operator, namely the dissipation operator (83) with dimension $\Delta=4-2\epsilon$; for $d_F=6$ there are two independent operators (96) with dimensions (97), in the neighborhood of

$\epsilon = 2$ they are not dangerous. The classification with respect to d_F corresponds to a classification for the contributions with respect to powers of $u^{2n+O(\epsilon)}$ in the SDE-representation (109), which is generally accepted in the theory of critical behavior (and is the only possible in the framework of the ϵ-expansion). It follows from the above that the first terms of SDE representation up to the contributions of order of $u^{6+O(\epsilon)}$ and higher in our case are of the following form:

$$f(u) = a_1 + a_2 u^2 + a_3 u^4 + a_4 u^{4-2\epsilon} + \ldots \tag{114}$$

With the known ϵ-expansions of the function $f(u)$ as a whole and exponents Δ_α, we may obtain the coefficients a_i in Eqn (114) in the form of ϵ-expansions; each of them begins with the power ϵ or higher.

Representation (114) can be verified directly by expanding diagrams for the model (16) in powers of m^2. From Eqn (114) it follows that coefficients for 1 and m^2 should be finite, but even the next term of the Taylor series in m^2 does not exist because of the IR-singularity of the type of $m^4 \log m$. After evaluation of the coefficient for $m^4 \log m$ in one-loop diagrams (see Fig. 1) for the static correlator, we can determine the coefficient a_4 in Eqn (114) in the lowest (first) order in ϵ. This calculation for the model (9) with an arbitrary $d > 2$ gives [58]

$$a_1 = \frac{\epsilon}{3a} + O(\epsilon^2)\,, \qquad \frac{a_4}{a_1} = \frac{\alpha_2}{3} + O(\epsilon)\,, \tag{115}$$

where a is the constant defined by Eqn (34); α^2 is the coefficient for u^4 in the expansion of the function $h(u)$ from Eqn (9) in powers of $u \equiv m/k$: $h(u) = 1 + \alpha_1 u^2 + \alpha_2 u^4 + \ldots$ Note that contributions of the type of m^2 coming from certain diagrams shown in Fig. 1 contain IR-divergences, but they cancel out when summed in accordance with the representation (114).

On the grounds of the results obtained for investigated operators of junior dimensions we may suppose that at $\epsilon < 2$ only operators of the type of φ^n and $\partial_t^p \varphi^n$ can be dangerous. They only contribute to the dynamic correlator, and this is consistent with the Kolmogorov hypothesis 1 or, equivalently, with Eqn (106), in the region $0 < \epsilon < 2$. When passing across the boundary $\epsilon = 2$, the dissipation operator and (probably) all its powers become dangerous. There is a hope that if we could sum them up, this would confirm the Kolmogorov hypothesis 1 (106) in some interval $\epsilon \geqslant 2$ near the boundary $\epsilon = 2$.

In Refs [31, 32], the averages of nonrenormalized operators calculated with the use of an iteration procedure similar to the Wilson RG-transformations were discussed. In particular, there was considered the so-called asymmetry (skewness) factor defined by the formula [1, 2]

$$S = -\frac{\left\langle (\partial_1 \varphi_1)^3 \right\rangle}{\left\langle (\partial_1 \varphi_1)^2 \right\rangle^{3/2}}\,. \tag{116}$$

Its value was found to be $S = 0.49$ [31, 32], or, upon correction of a discovered error, $S = 0.59$ [59]. It seems difficult to interpret this procedure in terms of the field-theoretical RG. The author of Ref. [60] found it doubtful that it could be possible to calculate averages of the type of Eqn (116), which are mainly determined by the dissipation range, with the aid of a fixed RG point describing the IR asymptotic behavior. This is really the case: the average $\langle F \rangle$ of a nonrenormalized operator in the framework of a theory with the cut-off Λ generally contains a Λ-divergent regular contribution coming from the first terms of its Taylor expansion in powers of m^2 with positive powers of Λ in the coefficients. It mainly comes from momenta $k \sim \Lambda$, contrary to a singular contribution coming from the region $k \ll \Lambda$. In renormalized theory the regular contribution is removed by the Λ-renormalization procedure (for operators it is usually reduced to a simple subtraction of the regular part $\langle F \rangle$ from F), upon which UV-divergences remain only in the form of poles in ϵ, and they may be removed by a multiplicative ϵ-renormalization (Section 2.4). Note that Eqn (107) contains renormalized operators; and Eqn (108) gives the leading singular contribution to their averages.

3.6 IR asymptotic behavior of the triple velocity correlator

IR asymptotic behavior of the triple correlator of velocity field was investigated in Refs [13, 51, 61–65, 43]. It is of interest in view of the problem of proving cancellation of IR-divergences in skeleton diagrams for self-consistency equations [51], and for analysis of the spectral energy-balance equation [1, 2, 61, 62], see also Section 3.7. This problem was studied within the framework of RG and SDE technique in Refs [64, 65, 43].

Consider a renormalized static triple correlator of the velocity field for the massless model (8) with $0 < \epsilon \leqslant 2$ in the $\mathbf{k}$-representation:

$$\left\langle \varphi_{i_1}(\mathbf{k}_1)\varphi_{i_2}(\mathbf{k}_2)\varphi_{i_3}(\mathbf{k}_3)\right\rangle \equiv (2\pi)^d \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)\mathcal{D}_A(\mathbf{k})\,,$$
$$A \equiv \{i_1, i_2, i_3\}\,, \qquad \mathbf{k} \equiv \{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\}\,. \tag{117}$$

From dimensional considerations we get

$$\mathcal{D}_A(\mathbf{k}) = \nu^3 k^{3-2d} R_A(s, g, \mathbf{n})\,, \qquad s \equiv \frac{k}{\mu}\,,$$
$$k^2 \equiv k_1^2 + k_2^2 + k_3^2\,, \qquad \mathbf{n} \equiv \left\{ \mathbf{n}_i \equiv \frac{\mathbf{k}_i}{k}\,,\ i = 1, 2, 3 \right\}. \tag{118}$$

According to Eqn (37) the function (118) obeys the RG-equation $\mathcal{D}_{\mathrm{RG}}\mathcal{D}_A(\mathbf{k}) = 0$ with $\mathcal{D}_{\mathrm{RG}}$ given by Eqn (33), therefore the RG-representation analogous to Eqn (46),

$$\mathcal{D}_A = \bar{\nu}^3 k^{3-2d} R_A\big(1, \bar{g}(s), \mathbf{n}\big)\,, \tag{119}$$

is valid for it. In the asymptotic region $s \to 0$ we have $\bar{g} \to g_*$ and $\bar{\nu} \to \nu_*$ according to Eqn (55):

$$\mathcal{D}_A \simeq \left(\frac{D_0}{g_*}\right) k^{3-2d-2\epsilon} R_A(1, g_*, \mathbf{n})\,. \tag{120}$$

Representation (120) describes IR asymptotic behavior of the correlator when $k \to 0$ and $\mathbf{n} = \mathrm{const}$, see Section 2.7. The SDE technique (Section 3.4) allows to study the asymptotic behavior of $\mathcal{D}_A(\mathbf{k})$ in three regions of the type of $k_1 \ll k_2 \simeq k_3$ ($\sum_i k_i = 0$), i.e., when $n_i \ll 1$ in the scaling function from Eqn (120), which is necessary for some problems (see Sections 3.7 and 3.8). In view of the symmetry of $\mathcal{D}_A(\mathbf{k})$ in momenta it is sufficient to determine its asymptotic behavior in one of the regions, say at $n_1 \ll 1$.

This was done in Refs [64, 65, 43] with the aid of operator product expansion (107) for fields with coinciding times. Substitution of Eqn (107) into the triple correlator

$$\mathcal{D}_{ijl}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \equiv \left\langle \varphi_i(\mathbf{x}_1)\varphi_j(\mathbf{x}_2)\varphi_l(\mathbf{x}_3)\right\rangle$$

gives

$$\mathcal{D}_{ijl}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \sum_\alpha C_{jl,\alpha}(\mathbf{r}_{23}) \langle \varphi_i(\mathbf{x}_1) \bar{F}^{\mathrm{R}}_\alpha(\mathbf{x}_{23}) \rangle \,, \qquad (121)$$

where $\mathbf{r}_{23} \equiv \mathbf{x}_2 - \mathbf{x}_3$, $\mathbf{x}_{23} \equiv (\mathbf{x}_2 + \mathbf{x}_3)/2$. The asymptotic range $|\mathbf{r}_{23}| \ll |\mathbf{x}_1 - \mathbf{x}_{23}|$ in the coordinate representation corresponds to $k_1 \ll k_2 \simeq k_3$ in Eqn (118).

The product $\varphi_j(\mathbf{x}_2)\varphi_l(\mathbf{x}_3)$ with coinciding times is not a Galilean-invariant object, therefore its SDE must also contain non-invariant operators. However, they will not contribute to the triple correlator (117) with $k_i \neq 0$ because of its invariance. Therefore, in Eqn (121) we can restrict ourselves to strictly Galilean-invariant operators; the leading term of the asymptotic expression searched for is determined by the operator with the minimal value of Δ_F. All scalars can be neglected since the Fourier transform of the correlator $\langle \varphi_i(\mathbf{x}_1) \bar{F}^{\mathrm{R}}_\alpha(\mathbf{x}_{23}) \rangle$ in Eqn (121) corresponding to them is a vector proportional to k_{1i} and obviously cannot contribute to the correlator (117) which is transverse in all the external momenta.

Among all nonscalar strictly invariant operators studied earlier (Sections 3.1–3.3) it is the operator $F = \partial_j\varphi_l + \partial_l\varphi_j$ from system (90) which has the minimum dimension, $\Delta_F = \Delta_\varphi + 1 = 2 - 2\epsilon/3$, at $\epsilon \sim 2$, $d > 2$. The expression on the l.h.s. of SDE for $\varphi_i\varphi_j$ has dimension $2\Delta_\varphi$, therefore the dimension of any of the coefficients C_α is uniquely determined by the dimension of $\bar{F}^{\mathrm{R}}_\alpha$: $\Delta[C_\alpha] = 2\Delta_\varphi - \Delta[\bar{F}^{\mathrm{R}}_\alpha]$. For our operator F we get $\Delta[C] = 2\Delta_\varphi - \Delta[F] = \Delta_\varphi - 1 = -2\epsilon/3$, from which it follows in the massless model that $C(r) \propto |r|^{2\epsilon/3}$. The structure of indices of the corresponding contribution to Eqn (117) is determined by the transversality of fields and by requirements of symmetry. The final result for the asymptotics in Eqn (117) searched for in the range $k_1 \ll k_2 \simeq k_3$ looks as follows [64, 65, 43]:

$$\begin{aligned}\mathcal{D}_{ijl}(\mathbf{k}) \simeq \frac{\mathrm{i}g_* D_0}{8}\, & k_1^{2-d-4\epsilon/3} k_2^{-d-2\epsilon/3} P_{in}(\mathbf{k}_1) \\ & \times \Big\{ a_1 k_{1m} \big[P_{lm}(\mathbf{k}_2) P_{jn}(\mathbf{k}_2) + P_{jm}(\mathbf{k}_2) P_{ln}(\mathbf{k}_2) \big] \\ & + \frac{a_2 P_{lj}(\mathbf{k}_2) k_{2n} (\mathbf{k}_1 \mathbf{k}_2)}{k_2^2} \Big\} \,. \end{aligned} \qquad (122)$$

Here P is the transverse projector; $a_{1,2}$ are numerical coefficients which can be obtained as ϵ-expansions. With the one-loop approximation for the correlator (117) they can only be found in the leading order in ϵ, which gives $a_1 = -1$, $a_2 = 2 - d$.

A representation similar to Eqn (122) was obtained earlier in Ref. [63] on the basis of direct analysis of skeleton diagrams. This representation coincides with Eqn (122) at $\epsilon = 2$ and differs for an arbitrary ϵ. This appears to be in contradiction with the representation (120), i.e., with the total dimension of correlator (117). Therefore we consider the representation (122) to be correct.

Calculating the function R in Eqn (118) as a series of renormalized perturbation theory in g and combining the result with (120) and (122), we obtain the corresponding approximation of improved perturbation theory (Section 2.7). It is not unique to any finite order since it depends, for instance, on a momentum or a combination of momenta which enter into the scale variable s in Eqn (118). Different versions differ from each other by a subsequent order of smallness at small ϵ. In the lowest order one can take the correlator searched for in the form

$$\begin{aligned}\mathcal{D}(\mathbf{k}) = & \frac{\mathrm{i}g_* D_0 (k_1 k_2 k_3)^{2-d-4\epsilon/3}}{4(k_1^{2-2\epsilon/3} + k_2^{2-2\epsilon/3} + k_3^{2-2\epsilon/3})} P_{is}(\mathbf{k}_1) P_{jt}(\mathbf{k}_2) P_{lp}(\mathbf{k}_3) \\ & \times \big[k_1^{d-2+4\epsilon/3} (k_{1t}\delta_{sp} + k_{1p}\delta_{st}) + k_2^{d-2+4\epsilon/3} (k_{2s}\delta_{tp} + k_{2p}\delta_{st}) \\ & + k_3^{d-2+4\epsilon/3} (k_{3s}\delta_{tp} + k_{3t}\delta_{sp}) \big] \,. \end{aligned} \qquad (123)$$

Expression (123) meets the needed properties of symmetry and transversality, is in agreement with the first order of ϵ-expansion, and exhibits the correct asymptotic behavior (122) in all regions $n_i \ll 1$ with coefficients $a_1 = -1$, $a_2 = 2 - d - 4\epsilon/3$ different from the one-loop analogues, $a_1 = -1$, $a_2 = 2 - d$, only by contributions of order ϵ. In fact, it coincides with the known EDQNM-approximation [2, 62] up to a fixed overall amplitude factor. Therefore all the results known for this approximation are automatically valid for representation (123), see Sections 3.7 and 3.8.

3.7 RG-approach and equation of the spectral energy balance

A central role in the phenomenological description of a developed turbulence belongs to the equation of spectral energy balance [1, 2]:

$$\partial_t E(k) = -\dot{E}_{\mathrm{dis}}(k) + T(k) + \dot{E}_{\mathrm{pump}}(k) \,. \qquad (124)$$

Here $E(k) = (d-1)D(k)/2$ [with the function D from the static correlator (5)] is the spectral energy density [all quantities per unit phase volume $\mathrm{d}\mathbf{k}/(2\pi)^d$, i.e., $E = (2\pi)^{-d} \int \mathrm{d}\mathbf{k}\, E(k)$, etc.]; $\dot{E}_{\mathrm{dis}} = (d-1)\nu_0 k^2 D(k)$ is the dissipation rate; $\dot{E}_{\mathrm{pump}}(k)$ is the external source of energy pumping; and

$$T(k) = \int \mathrm{d}\mathbf{x} \exp(-\mathrm{i}\mathbf{k}\mathbf{r}) T(\mathbf{r}) \,, \quad T(\mathbf{r}) \equiv -\langle \varphi_i(\mathbf{x}) \varphi_j(\mathbf{y}) \partial_j \varphi_i(\mathbf{y}) \rangle \,, \qquad (125)$$

(with $\mathbf{r} \equiv \mathbf{x} - \mathbf{y}$) is the so-called transfer integral connected in the momentum representation with the static correlator (117) by the relation

$$T(k) = (2\pi)^{-d} \int \mathrm{d}\mathbf{k_1}\, \mathrm{d}\mathbf{k_2}\, \delta(\mathbf{k} + \mathbf{k_1} + \mathbf{k_2}) \mathrm{i} k_j \mathcal{D}_{ljl}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \,. \qquad (126)$$

At $\mathbf{x} = \mathbf{y}$ the coordinate function T, Eqn (125), vanishes owing to translation invariance (the symbol ∂ may be moved out from $\langle \ldots \rangle$), therefore

$$\int \mathrm{d}\mathbf{k}\, T(k) = 0 \;\Rightarrow\; T(k) = -(2\pi)^d \frac{\partial J_i(\mathbf{k})}{\partial k_i} \,, \qquad (127)$$

where $J_i(\mathbf{k})$ is the vector of density of the energy flux along the spectrum [$(2\pi)^d$ provides a correct normalization]. Due to isotropy we have

$$J_i(\mathbf{k}) = k_i J(k) \,, \qquad I(k) = S_d J(k) k^d \,, \qquad (128)$$

where $I(k)$ is the total energy flux to the outside through the surface of a sphere with radius k in the momentum space; S_d is the area (35) of a unit d-dimensional sphere. The flux I should satisfy the conditions $I(0) = I(\infty) = 0$. Integrating (124) and taking Eqn (127) into account, we can derive the energy balance equation for any spherical layer. Specifically, for the total integral we have

$$\dot{E}_{\rm dis} = (2\pi)^{-d}(d-1)\nu_0 \int \mathrm{d}\mathbf{k}\, k^2 D(k) = (2\pi)^{-d} \int \mathrm{d}\mathbf{k}\, \dot{E}_{\rm pump}(k) \equiv W\,. \tag{129}$$

When applied to the balance equation (129), basic assumptions of the phenomenology are reduced to the following [1, 2]:

(1) Both the viscosity and energy pumping are unessential in Eqn (124) in the inertial range $m \ll k \ll \Lambda$, therefore in a stationary problem we have

$$T(k) = 0 \tag{130}$$

up to corrections in positive powers of m/k and k/Λ.

(2) In the inertial range the function $\mathcal{D}_{ijl}$ in Eqn (126) obeys the relation of generalized homogeneity

$$\mathcal{D}_{ijl}(\lambda\mathbf{k}_1, \lambda\mathbf{k}_2, \lambda\mathbf{k}_3) \simeq \lambda^{3\Delta_\varphi - 2d}\mathcal{D}_{ijl}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \tag{131}$$

up to corrections in higher powers of λ unessential when $\lambda \to 0$.

(3) Integral (126) is IR- and UV-finite, i.e., we may neglect the contributions to it coming from the ranges $k_i \leqslant m$ and $k_i \geqslant \Lambda$, where relation (131) is not valid.

From assumptions 2 and 3 it immediately follows that in the inertial range

$$T(k) = T_0(y)k^{-y} + \ldots\,, \qquad y \equiv d - 1 - 3\Delta_\varphi\,. \tag{132}$$

Here the traditional notation is introduced: y for the exponent and $T_0(y)$ for the amplitude of the leading term. From Eqn (132) and (127) we derive $J_i(\mathbf{k}) = (2\pi)^{-d}k_i k^{-y}T_0(y)/(y-d)$, and hence, for the energy flux in Eqn (128) we obtain $I(k) = (2\pi)^{-d}k^{d-y}S_d T_0(y)/(y-d)$. It is seen that condition (130) [i.e., $T_0(y) = 0$] is equivalent to the requirement that the energy flux be constant, and (for a nonzero flux) it holds only at $y = d$, i.e., only at the Kolmogorov value $\Delta_\varphi = -1/3$ in Eqn (132). Assuming that $T_0(y)$ is analytic in the vicinity of $y = d$, we have $T_0(y) = T_0'(d)(y-d) + \ldots$, and equating the corresponding energy flux $I = (2\pi)^{-d}S_d T_0'(d)$ in the inertial range to the total energy of dissipation (or pumping) (129), we arrive at the relation

$$\dot{E}_{\rm dis} = W = (2\pi)^{-d}S_d T_0'(d)\,, \tag{133}$$

which connects $\dot{E}_{\rm dis}$ through Eqns (126) and (132) with the triple correlator.

Carrying differentiating with respect to y in Eqn (133) explicitly, we can rewrite it in a more convenient form. To do it, we must extract the factor $y - d$ from integral (126). It can be shown (see Ref. [2], p. 317; Ref. [22]) that this integral does not change if the integrand is multiplied by the factor $-(k_2/k)^{y-d}$, and consequently, by the combination $\left[1 - (k_2/k)^{y-d}\right]/2$ from which the factor $y - d$ is extracted explicitly. This transformation of integral (126) allows us to rewrite Eqn (133) in the form

$$\dot{E}_{\rm dis} = W = \frac{S_d k^d}{2(2\pi)^d} \iint \mathrm{d}\mathbf{k}_1\, \mathrm{d}\mathbf{k}_2\, F(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \log \frac{k}{k_2}\bigg|_{y=d}\,, \tag{134}$$

where $F(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)$ is the integrand of (126) with the Kolmogorov value of dimension $y = d$. The factor k^d can be included into the integrand of Eqn (134), this is equivalent to making by the change $\mathbf{k}_i \to \mathbf{k}_i/k$ all momenta dimensionless.

It is important that the major contribution to Eqn (134) comes from the inertial range since the convergence properties of integrals (126) and (134) are the same.

Usually assumptions 1 – 3 are considered on the basis of some approximations, for instance, on the basis of EDQNM-approximation [2, 62] mentioned in Section 3.6. Following Ref. [64] we shall see how these properties are fulfilled, using RG and SDE and considering the model (16) by way of example. The initial relation (124) is obtained by multiplication of equation (79) with $A^{\varphi'} = 0$ by $\varphi_i(x')$ with $t' = t$, averaging $\langle \ldots \rangle$, and by Fourier-transform with respect to $\mathbf{x} - \mathbf{x}'$. The factors $d - 1$ result from the contraction over indices in transverse projectors, the role of $\dot{E}_{\rm pump}(k)$ is played by the quantity $(d-1)d_F(k)/2$ [see the text after formula (85)] with the function d_F from Eqn (2). For model (9) at any $\epsilon > 0$ and $k \ll \Lambda$ the contribution of dissipation to Eqn (124) can always be neglected in comparison to the contribution of pumping (obviously from the estimate based on critical dimensions). Therefore for the stationary problem from Eqn (124) we have in the whole IR-region $k \ll \Lambda$

$$T(k) = -\dot{E}_{\rm pump} = -\frac{(d-1)d_F(k)}{2}\,. \tag{135}$$

From Eqns (135) and (9) for the inertial range we obtain

$$T(k) = T_0 k^{-d+4-2\epsilon}\,, \qquad T_0 = -\frac{(d-1)D_0}{2}\,, \tag{136}$$

where the constant D_0 is expressed in terms of $W = \dot{E}_{\rm dis}$ by relations (64). When comparing Eqns (136) and (132) it is necessary to take into account that Δ_φ in Eqn (132) is the Kolmogorov dimension defined by relations (41) for $\epsilon \leqslant 2$ and by (6) for any $\epsilon \geqslant 2$ (the effect of freezing, see Section 2.8). This means that the role of y in Eqn (132) is played by the quantity

$$y \equiv d - 1 - 3\Delta_\varphi = \begin{cases} d - 4 + 2\epsilon & \text{for} \quad \epsilon \leqslant 2\,, \\ d & \text{for} \quad \epsilon \geqslant 2\,. \end{cases} \tag{137}$$

From Eqns (64) and (137) it follows that when $\epsilon < 2$, expression (136) for $T(k)$ takes the form (132) with the amplitude $T = \text{const} \times \Lambda^{2\epsilon-4} \neq 0$, i.e., the property (130) for $\epsilon < 2$ is not fulfilled. In contradistinction to the case $\epsilon < 2$, for any $\epsilon > 2$ it is fulfilled identically since the leading contribution of the type of Eqn (132) in this region should be of the form const $\times\, k^{-d}$, in accordance with Eqn (137), whereas it is absent in Eqn (136) and this expression as a whole is just a correction with an additional smallness $(m/k)^{2\epsilon-4}$. Just at the boundary $\epsilon = 2$ it becomes leading and the amplitude $T_0 \propto D_0$ acquires the smallness $1/\log(\Lambda/m)$ (Section 1.8), therefore up to this accuracy the statement (130) is also valid for $\epsilon = 2$.

For the massless model (8) which may be considered only in the region $\epsilon < 2$, the smallness in the amplitude $T_0 \propto D_0$ ensuring (130) appears only in the limit $\epsilon \to 2$ since in this case Eqn (64) assumes the following form:

$$D_0 = (2\pi)^d S_d^{-1}\, \frac{4(2-\epsilon)\Lambda^{2\epsilon-4}}{(d-1)}\, W\,. \tag{138}$$

Owing to the factor $2-\epsilon$, the function (8) with D_0 from Eqn (138) vanishes in the limit $W = \text{const}$, $\epsilon \to 2$ for any $k \neq 0$. More precisely,

$$d_F(k) \to \frac{2(2\pi)^d}{d-1}\,\delta(\mathbf{k}) \quad \text{when} \quad \epsilon \to 2\,, \tag{139}$$

if we take into consideration the known power-like representation of δ-function:

$$\delta(\mathbf{k}) = \lim_{\Delta\to+0}(2\pi)^{-d}\int \mathrm{d}\mathbf{k}\,(\Lambda x)^{-\Delta}\exp(\mathrm{i}\mathbf{k}\mathbf{x})$$
$$= S_d^{-1}k^{-d}\lim_{\Delta\to+0}\left[\Delta\left(\frac{k}{\Lambda}\right)^{\Delta}\right]. \tag{140}$$

Thus, the first of basic assumptions (130) is fulfilled in the model (9) for any $\epsilon \geqslant 2$, and in the model (8) only when $\epsilon \to 2-0$.

Let us now discuss assumptions 2 and 3 from the RG standpoint. Equation (131) is an obvious consequence of the IR-scaling and Kolmogorov hypothesis 1 [which we can dispense with if we dilatate the parameter m in Eqn (131)]. Nevertheless, the possibility of using Eqn (131) as the integrand in Eqn (126) to derive Eqn (132) is nontrivial, since the arguments of two fields in the coordinate triple correlator (125) coincide. Therefore, as a matter of fact, we now deal not with a triple, but with a double correlator of the field φ_i and a composite operator $\varphi_i\varphi_j$ (the symbol ∂ can be moved out from the operator $\langle\ldots\rangle$). In a general case its dimension is not equal to the sum of field dimensions owing to a possible contribution of anomalies γ_F^* (Section 3.1), and (if this contribution exists) the critical dimension of an integral of the type of Eqn (126) is not determined by the dimension of the integrand.

In the given case there is no such danger: the composite operator $\varphi_i\varphi_j$ studied in detail in Section 3.2 is represented as a sum of two operators, $F^{(1)} = \varphi_i\varphi_j - \nu_0(\partial_i\varphi_j + \partial_j\varphi_i)$ and $F^{(2)} = \nu_0(\partial_i\varphi_j + \partial_j\varphi_i)$, with definite critical dimensions $\Delta_1 = 2\Delta_\varphi$ and $\Delta_2 = \Delta_\varphi + 1$. The first of them cancels in Eqn (133) the contribution of pumping; and the second of dissipation. At the Kolmogorov value $\Delta_\varphi = -1/3$ the leading operator is the first one, just it enters in Eqns (132) and (135). Its dimension has no anomaly γ_F^* (Section 3.2), therefore in this case equation (131) indeed results in Eqn (132).

Within the phenomenology this was substantiated by the assumption 3, i.e., by the required convergence of integral (124). From the representation (122) derived by the RG- and SDE-technique it follows that integral (126) does converge for $0 < \epsilon < 3$. Here the upper boundary is determined from the IR-convergence; and the lower boundary from the UV-convergence (for details see Ref. [64]). Note that the boundaries of convergence are the same as in the EDQNM-approximation [62], since it has a correct asymptotic behavior (Section 3.6). (The assumption that there is a possible connection between these boundaries and dimensions of composite operators was first put forward in Ref. [66].)

The general conclusion is that the RG- and SDE-technique allow to substantiate the assumptions 1–3, which were postulated phenomenologically for models of the type of (9) at any $\epsilon \geqslant 2$, and in a simplified model (8) at $\epsilon \to 2$.

3.8 On non-Kolmogorov solutions to the energy balance equation

Besides the solution $y = d$ with a nonzero energy flux, other possible solutions $y = y(d)$ to the equation $T_0(y) = 0$ for the amplitude in Eqn (132) meeting the requirement (130) also may be interesting. They can be sought numerically by computing $T_0(y)$ with the help of relations (126) and (132) on the basis of any scale-invariant model for a triple correlator. In Ref. [62] (see also [64]), with the use of a correlator of the type of (123) the authors obtained the solution $y = y(d)$ shown in Fig. 3 as an $ABCDE$ curve. Following Refs [62, 64], we give it in coordinates $\gamma - d$, where $\gamma \equiv (2y - 2d + 5)/3 = 1 - 2\Delta_\varphi$ is the index of the energy spectrum $E(k) \propto k^{d-1}D(k) \propto k^{-\gamma}$, and it exists only near $d = 2$ (revised values $d_\mathrm{c} = 2.066$, $d_\mathrm{c}' = 2.075$ were presented in Refs [64, 67]). The conventional Kolmogorov solution $y = d$ in this coordinates is associated with the line $\gamma = 5/3$ at any value of d. It is interesting to note that though the $ABCDE$ curve was obtained on the basis of particular approximation (123) for the triple correlator, its limit points A, E at $d = 2$ have a simple physical interpretation in the exact theory. Let us explain this remark in more detail.

A specific feature of a two-dimensional problem is the second (in addition to the energy) conserved quantity, the enstrophy $\langle \mathrm{rot}^2\varphi(\mathbf{k})\rangle \propto k^2 D(k)$ [2], which in terms of the transfer integral means that there is the second conservation law $\int \mathrm{d}\mathbf{k}\,T(k)k^2 = 0$, in addition to Eqn (127). This allows us to introduce the notion of the spectral density of enstrophy flux $k^2T(k) = -\partial J_i^e/\partial k_i$. In the inertial range it follows from Eqn (132) that $J_i^e(\mathbf{k}) = J_0^e k_i k^{2-y}$ and $T_0 = J_0^e(y-4)$. Consequently, at $d = 2$ there may exist a regime with a zero energy flux and a nonzero enstrophy flux. This regime corresponds to $y = 4$, i.e., in accordance with Eqn (132), $\Delta_\varphi = -1$ is the point A in Fig. 3. Formally, we can separate the factor $y-4$ from integral (126) at $d = 2$ in the same way as it was done in the general case for the factor $y - d$ [see the derivation of formula (134) in Section 3.7]. However, it is worth to remember that the value $\Delta_\varphi = -1$ in Eqn (123) corresponds to the boundary value $\epsilon = 3$ at which the IR-divergence appears in Eqn (126) (this is due to the dangerous Galilean-invariant operator $\partial\varphi$ appearing at $\epsilon \geqslant 3$, see Section 3.6). Therefore the problem of a rigorous proof of existence of a solution of this type is nontrivial and at present it is intensively discussed in the literature [68]. The second limit point E in Fig. 3 is associated with the value $\Delta_\varphi = 0$ corresponding to the equidistribution of the enstrophy over spectrum, $k^2D(k) = \text{const}$.

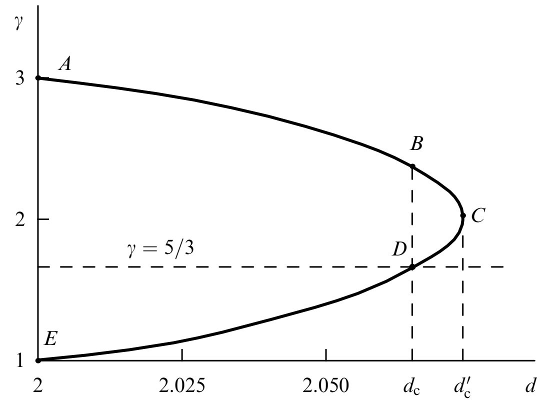

Figure 3.

Of special interest is the point D in Fig. 3 with $d_c \simeq 2.066$ corresponding to intersection of two solutions. At this point, $y = d_c$, and not only the quantity $T_0(y)$ itself vanishes, but also its derivative $T_0'(y)$ and the energy flux (see Section 3.7). Therefore equation (134) cannot be fulfilled at finite values of W and D_0. In accordance with the interpretation given in [62], the region $d < d_c$ corresponds to the inverse energy flux and the Kolmogorov scaling is not realized here. In Ref. [62] it was assumed that with decreasing d the solution $y = d$ in this region 'skips' onto the curve BA.

In Ref. [67] it was stated that equation (134) can be fulfilled only when $d = d_c + O(\epsilon)$, $D_0 \propto \epsilon^{-2}$. Actually equation (134) has been obtained as a limiting case when $\epsilon \to 2$, therefore one cannot speak about its internal consistency when $\epsilon \to 0$. In a more general relation (135), a zero of order of $g_* \propto \epsilon$ at $\epsilon \to 0$ in the amplitude (123) is compensated by the pole $1/\epsilon$, which appears owing to UV-divergence at $\epsilon \to 0$ if we obtain $T(k)$ by evaluation of Eqn (126). This removes the illusory contradiction discovered in Ref. [67].

3.9 Problem of singularities at $\epsilon \to 2$ in a massless model. Calculation of the Kolmogorov constant

The main part of studies on the RG-theory of turbulence has been performed in the framework of a massless model (8) with a pure power pumping, for which the parameters W and D_0 are connected by formula (138). A realistic problem corresponds to the limit $\epsilon \to 2-0$, where the power law turns into a δ-function (139). Correct normalization is ensured by the factor $2-\epsilon$ present in Eqn (138). The parameter $D_0 \propto 2-\epsilon$ enters into the IR asymptotic expressions only through the invariant viscosity (48), therefore in representations of the type of (54), (120) for static correlators it will never be present in expressions for the corresponding scaling functions, but will enter into the overall factor $D_0^{n/3} \propto (2-\epsilon)^{n/3}$, where n is the number of fields in a given correlator [according to the frequency dimension there is one factor per each field, and in the IR asymptotic region it turns into Eqn (55)]. If we believe that there exists a limiting theory at $\epsilon \to 2-0$, we must assume that scaling functions must have singularities of the type of $(2-\epsilon)^{-n/3}$ which compensate zeros $D_0^{n/3} \propto (2-\epsilon)^{n/3}$ in amplitude coefficients. Rigorous proof of the presence of singularities of this kind is a complicated and yet unsolved problem of the theory. It apparently has a certain relationship with the general problem of freezing and substantiation of the Kolmogorov hypothesis 1 for the whole range $\epsilon \geqslant 2$ in a mass-dependent model (Section 2.8). However, the very statement that there exists a limiting massless model with the δ-shaped pumping (139) seems to be rather plausible, in any case, just this hypothetical limit model with $\epsilon \to 2-0$ is in fact a subject of most studies on the RG-theory of turbulence. In this connection we note that the exact relation (86) and convergence of the integral at $\epsilon = 2$ (a consequence of SDE and total dimension) provided by the factor $D_0 \propto (2-\epsilon)$ in the amplitude (123) proves in fact that there is a required pole $1/(2-\epsilon)$ in the scaling function of the triple correlator. This is the only way to make parameter $\dot{E}_{\rm dis} = W$ finite.

Experiment shows that in the inertial range the function $E_1(k)$ (a one-dimensional energy spectrum) connected with the static correlator (5) by formula (60) satisfies the following representation

$$E_1(k) = C_{\rm K} W^{2/3} k^{-5/3} \,. \tag{141}$$

Dimensionless numerical factor $C_{\rm K}$ is called the Kolmogorov constant; its experimental value is $C_{\rm K} = 1.3 \div 2.7$ [38, 69].

The RG-calculation of the Kolmogorov constant in the model (1), (2) was discussed in Refs [31–33, 15, 67, 70–73]. In Refs [31–33], use was made of the Wilson recurrence relations; in Ref. [15], of the field-theoretical formulation. First let us present a brief computation for model (8) at $\epsilon < 2$ given in Ref. [15].

From Eqns (54), (55), (60), and the first equation of (64) we derive

$$E_1(k) = C_{\rm K}(\epsilon) W^{2/3} k^{-5/3} \left(\frac{\Lambda}{k}\right)^{4(2-\epsilon)/3} , \tag{142}$$

where $C_{\rm K}(\epsilon)$ is an analog of the constant $C_{\rm K}$ in Eqn (141) for an arbitrary $\epsilon < 2$:

$$C_{\rm K}(\epsilon) = \left(\frac{2C_d}{g_*^2}\right)^{1/3} (2-\epsilon)^{2/3} f(\epsilon) \,. \tag{143}$$

Here C_d is taken from Eqn (35) and the function $f(\epsilon) \equiv f(u=0,\epsilon)$ comes from Eqn (54). Assuming that the product $(2-\epsilon)^{2/3} f(\epsilon)$ is finite when $\epsilon \to 2$ (see above), and expanding it in a series in ϵ, we arrive at the following expansion of the Kolmogorov constant [15]:

$$C_{\rm K}(\epsilon) = \epsilon^{1/3} \sum_{n=0}^{\infty} p_n \epsilon^n \,. \tag{144}$$

In the lowest approximation $f(\epsilon) = g_*/2$, $g_* = 2\epsilon/3a$ with a from Eqn (34) corresponding to expression (62) for $E_1(k)$. From (144) we obtain

$$C_{\rm K}(\epsilon) = 2\left[\frac{(d+2)\epsilon}{3}\right]^{1/3} , \tag{145}$$

which gives $C_{\rm K} \simeq 3$ at $d = 3$ and a real $\epsilon = 2$ [15].

Expression (145) is the first term of the exact ϵ-expansion given by Eqn (144). Its deviation from experiment is not surprising since there are no obvious reasons for corrections to Eqn (144) to be small. A value of the Kolmogorov constant more close to the experimental one was obtained in Refs [31–33] within the so-called Yakhot–Orszag RG-theory based on a version of the Wilson recurrence relations. We briefly explain their method of calculation. Though equation (3) and its consequence (138) for the power pumping are exact for the stochastic problem (1), the authors of Refs [31–33] do not employ them. The connection between parameters W and D_0 turns out to be a nontrivial problem which in these works is solved with the use of relations (134), (126), (123). The integral (134) is calculated numerically for the particular approximation (123) with the exact value $\epsilon = 2$ in exponents [which corresponds to $y = d$ in the notation (134)], and the first order of the ϵ-expansion of the parameter g_* is substituted into Eqn (123) (this parameter is known only to this order of approximation). For $d = 3$ this leads to the following connection between W and D_0:

$$\left.\frac{D_0}{2\pi W}\right|_{\epsilon=2} \simeq 1.5\ldots \tag{146}$$

Substitution of this expression into Eqn (62) with $h(0) = 1$ results in the value

$$C_{\rm K} \simeq 1.606\ldots \tag{147}$$

of the Kolmogorov constant and is in a good agreement with experiment. (Note that the value $C_K = 1.617$ presented in Refs [31–33] slightly differs from Eqn (147), apparently because of a different accuracy of the numerical integration in Eqn (134). The most accurate value of the constant (147) can be found in Ref. [67].)

From the standpoint of exact theory the above-mentioned method of computation of the constant (147) is not entirely faultless since it does simultaneously negate two qualitative effects: a zero at $\epsilon = 2$ in the ratio D_0/W [a consequence of the exact relation (138)] and a singularity of the scaling function from Eqn (54) (a consequence of the supposed existence of a theory with $\epsilon = 2$, see above) compensating this zero. Therefore, the calculations of Refs [31–33] should be regarded as a special method for computation of the above-mentioned uncertainty $0 \times \infty$, apt for agreement with experiment. However, the authors of Refs [31–33] do not point at this uncertainty, considering the result (146) to be qualitatively correct (instead of exact $D_0/W = 0$ at $\epsilon = 2$). In this case difficulties with the interpretation of the stationary equation of energy balance (124) arise: from Eqn (146) it follows that the pumping contribution in the inertial range at the Kolmogorov value of the field dimension ($\epsilon = 2$) remains finite, and in view of relation (130) (considered in this approach to be valid) it can be canceled only by the dissipation contribution, but this proves impossible when powers of momenta are conventionally estimated from dimensions (Section 3.7). The authors of Refs [31–33] try to find a way out of this contradiction [which, in fact, is illusory, since the pumping contribution is a multiple of $D_0 \propto (2-\epsilon)$ and vanishes at $\epsilon = 2$ in the exact theory] in the following manner. In the Yakhot–Orszag RG-approach it is assumed that the RG-technique in the region $k \ll \Lambda$ leads to an equation analogous to Eqn (124) when the kinematic coefficient of viscosity ν_0 in the dissipation contribution $\dot{E}_{\mathrm{dis}}(k) = (d-1)\nu_0 k^2 D(k)$ is replaced with the k-dependent 'effective eddy viscosity,' whose role is played by the invariant variable $\bar{\nu}_*(k)$ from Eqn (55). Upon this replacement, powers of k in pumping and dissipation contributions appear to be the same at $\epsilon = 2$, and when the explicit form of amplitudes [see Eqns (55) and (61)] is taken into account, these contributions really cancel in the lowest order in ϵ. The condition of cancellation is the relation $f(1, g_*)/g_* = 1/2$ at $\epsilon = 2$ in model (8); it should in essence be considered exact in the Yakhot–Onszag approach though this is not directly noted.

The most detailed exposition of the given approach can be found in Refs [32, 59]; generalization to the problems of a turbulent mixing of a passive impurity, damped turbulence, etc., is given in Refs [32, 33, 45, 46, 74–77], a critical analysis is presented in Refs [60, 70, 78].

From the viewpoint of the standard field-theoretical RG-technique the approach of Refs [31–33] briefly presented above is inconsistent. Moreover, the very problem, the necessity of cancellation of the pumping contribution by something in the inertial range when the condition (130) is satisfied is illusory, since in reality the pumping contribution is a multiple of $D_0 \propto (2-\epsilon)$ and vanishes when $\epsilon \to 2$. In connection with the substitution $\nu_0 \to \bar{\nu}_*(k)$ in equation (124) it should be said that the correct RG-technique allows substantiation of a similar replacement $\nu_0 \to \bar{\nu} \to \bar{\nu}_*(k)$ in renormalization-invariant objects of the type of correlators expressed via renormalized parameters (see Section 2.6), but separate nonrenormalized constants of the type of ν_0 cannot be thus transformed into momentum-dependent quantities: they can be expressed through invariant variables and momentum k, but only in such a combination in which the k-dependence cancels out, see Eqn (50). It also should be noted that the physical idea of mutual cancellation of the contributions of pumping and dissipation in the inertial range contradicts the conventional phenomenology of a developed turbulence, actually denying the very notion of inertial range. A similar 'detailed equilibrium' is realized only in models of the type of critical dynamics [23] describing systems in the state of a thermal equilibrium. They differ qualitatively by the mechanism of pumping ($d_F = \text{const}$ or $\text{const} \times k^2$) from models of a fully developed turbulence.

The last remark concerns the calculation of the Kolmogorov constant (143), see Ref. [70]. The employed relation (138) was obtained by substituting the function (8) into integral (3) with the cut-off $k \ll \Lambda$, where $\Lambda \equiv W/\nu_0^3$ is a parameter determined exactly. In nonphysical conditions of UV-pumping, it is, in principle, equally possible to cut off integral (3) not at Λ but at a certain value of the order of Λ. This will lead to the change $\Lambda \to c\Lambda$ in Eqn (138) with an unknown parameter $c \sim 1$. Thus, an extra factor $c^{2-\epsilon}$ will enter the relation (143), and when expanded in ϵ, this will change (in fact, arbitrarily) the ϵ-expansion coefficients (144) of the Kolmogorov constant. Note that a similar uncertainty is also present if calculations are done with the help of relation (134), since the latter contains only the value of the integral at $y = d$, which corresponds to $\epsilon = 2$, and, therefore, correlators can be determined only up to arbitrary factors of the type of $c^{2-\epsilon}$. Consequently, one should not attach too much importance to the numerical results obtained in the framework of the simplest approximations. Coincidence of values of the Kolmogorov constant obtained by different methods with each other and with the experimental value to an order in magnitude is quite a good result.

3.10 On deviations from Kolmogorov scaling. IR renormalization group

At present there are experimental data which testify to small deviations from predictions of the Kolmogorov theory (Section 2.1) in static correlators of certain Galilean-invariant quantities [35–38]. To be more precise, the point is in deviations of experimentally measurable exponents of the distance $r \equiv |\mathbf{x} - \mathbf{y}|$ from the values predicted by the Kolmogorov theory. At the level of phenomenology they are usually described by extra exponents of $(mr)^{\cdots}$ in correlators (in contradiction with the Kolmogorov hypothesis 1), whereas theoretical models explain them by strongly developed fluctuations of the operator of energy dissipation (83), see, for instance, [79]. Specific discussion in the literature concerns the following two expressions for static ($t = 0$) correlators in the inertial range:

$$\langle F_r^n \rangle = \text{const} \times (Wr)^{n/3} (mr)^{q_n} , \tag{148}$$

$$\big\langle E(\mathbf{x}) E(\mathbf{y}) \big\rangle = \text{const} \times W^2 (mr)^{-\mu} . \tag{149}$$

Here $F_r \equiv \varphi_1(\mathbf{x}) - \varphi_1(\mathbf{y})$, φ_1 is a particular component of the velocity; $E(\mathbf{x}) \equiv \dot{E}_{\mathrm{dis}}(\mathbf{x})$ is operator (83); W is the mean power of the energy pumping (3); $r \equiv |\mathbf{x} - \mathbf{y}|$.

According to the Kolmogorov theory we should have $q_n = \mu = 0$ in Eqns (148), (149); experiment (by exponents of r) gives $0.2 \leqslant \mu \leqslant 0.5$ [35–38]. In Ref. [36], in particular, it was found that $\mu = 0.20 \pm 0.05$. The experimental graph of dependence of q_n on n for even $2 \leqslant n \leqslant 18$ is presented in Ref. [36]. Usually it is supposed that $q_2 \geqslant 0$, though the experimental value of q_2 is indistinguishable from zero [36–38]. With the aid of a coordinate version of equation (124) one may rigorously prove that $q_3 = 0$. For $n \geqslant 4$ the exponents q_n become noticeably different from zero [36].

From the viewpoint of the RG-theory expounded earlier, the quantity (149) possesses a definite critical dimension $2\Delta_E$ with $\Delta_E = 4 - 2\epsilon$ according to Eqn (88); and Eqn (148) is a mixture of terms of different dimensions. This representation reflects only the leading contribution with dimension $n\Delta_\varphi = n(1 - 2\epsilon/3)$. The Kolmogorov values of exponents (148), (149) are obtained at $\epsilon = 2$. The general formulae of IR-scaling (Sections 2.7, 2.8) admit the presence of arbitrary scaling functions $f(mr)$ on the r.h.s. of Eqns (148), (149), and representation (148), (149) is to be understood as the statement concerning the explicit form of the leading term of asymptotic behavior at $mr \to 0$ of these scaling functions.

In Ref. [45] (see also Refs [75, 76]) an attempt was undertaken to interpret Eqns (148), (149) on the basis of the standard RG-theory for a massless model (8) by choosing the real value of ϵ slightly different from $\epsilon = 2$. When applied to Eqns (148), (149), this means that $n/3 + q_n = -n\Delta_\varphi$, $\mu = 2\Delta_E$ with dimensions $\Delta_\varphi = 1 - 2\epsilon/3$, $\Delta_E = 4 - 2\epsilon$ known from Eqns (41), (88), from which it follows that $\mu = 2(4 - 2\epsilon)$, $q_n = -n\mu/6$. With an appropriate choice of $\epsilon < 2$ one can obtain any given $\mu > 0$, but then all q_n will be negative in contradiction with the exact equation $q_3 = 0$ and conventional phenomenological notions, according to which $q_2 \geqslant 0$. Therefore a simple idea of Ref. [45] of the shift of ϵ for explaining representations (148), (149) is not valid, to say nothing of pumping (8) being ultraviolet for $\epsilon < 2$ (see Section 2.1) and being able to lead only to relations of the type of Eqns (148), (149) with the change $m \to \Lambda$, thus coming into contradiction even with the Kolmogorov hypothesis 1.

The authors of Ref. [80] tried to obtain representation (148), (149) by using the RG-technique for a mass-dependent model (9) considering only the contribution of the dissipation operator (83) to SDE-representations of the type of Eqn (109) (it is important that objects in Refs (148), (149) are Galilean-invariant, and Eqn (83) is the only known operator with this symmetry which can be dangerous around $\epsilon = 2$).

This leads to the representations

$$\langle F_r^n \rangle = \text{const} \times (D_0)^{n/3} r^{-n\Delta_\varphi} \left[1 + C_n (mr)^{\Delta_E} + \ldots\right], \qquad (150)$$

$$\left\langle E(\mathbf{x}) E(\mathbf{y}) \right\rangle = \text{const} \times D_0^2 r^{-2\Delta_E} \left[1 + C_E (mr)^{\Delta_E} + \ldots\right], \qquad (151)$$

with a usual $\Delta_\varphi = 1 - 2\epsilon/3$, $\Delta_E = 4 - 2\epsilon$ at any ϵ, with D_0 from Eqn (9) and some universal (depending only on ϵ) coefficients C_n, C_E. For $\epsilon < 2$ the leading contributions at $m \to 0$ to Eqns (150), (151) are the first terms, and this produces the results of Ref. [45]. In Ref. [80] the possibility of a choice of $\epsilon > 2$ was considered. In this case the second terms in Eqns (150), (151) give the leading contribution, and they should be identified with expressions (148), (149), which gives $\mu = \Delta_E = 4 - 2\epsilon < 0$, $q_n = \mu(3 - n)/3$. These answers again do not agree in signs with experimental values and, as noted in Refs [48, 49], contradict the exact equations $q_{2n} \leqslant 2q_n$ which follow from the property that the averaging measure in Eqn (148) is positive. This contradiction can be removed if and only if it is assumed that besides the dissipation operator there exists an infinite number of operators of the same symmetry which become dangerous at $\epsilon = 2$; however, in practice, operators of this type are not known [see the discussion of relation (104) in Section 3.3].

In a series of papers [81–85, 25] attempts were undertaken to investigate the dependence of spectrum (60) on m and thus to determine the exponent q_2 in Eqn (148) with the use of a new 'infrared renormalization group.' The general idea is that IR-divergences (when $m \to 0$) can be handled exactly in the same way as UV-divergences (when $\Lambda \to \infty$). In the formulation of Ref. [85] this reduces to the statement that IR-divergences can be removed by the conventional procedure of multiplicative renormalization of fields and parameters; natural freedom of this procedure leads to the corresponding RG-equations (see Section 2.4). In the earlier formulation of Ref. [81], the RG-equations realizing the same idea were derived by the recurrence procedure with a consecutive elimination of contributions of small rather than large (as usual) momenta of integration. Note that for the first time a similar idea was proposed (but not realized) in report [66], p. 116.

However, the results obtained with the use of the IR renormalization group simply cannot be considered reliable, since general statements necessary for derivation of RG equations (the structure of divergences, multiplicative renormalizability, etc.) have never been proved rigorously here and, in fact, they were assumed to be true in analogy with the corresponding statements of conventional (ultraviolet) renormalization theory. In reality, there is no analogy at all and thus there is no IR-renormalization theory similar to the universal well-developed UV-renormalization theory. The latter is based on two statements: (1) an UV-divergent part of any one-loop 1-irreducible diagram is a simple polynomial in a set of external momenta and frequencies, and its degree is determined from dimension; (2) the same is valid for any multiloop diagram if one removes divergences of all its subdiagrams beforehand. The first of these statements is obvious, and the second is rather nontrivial but it was rigorously proved by many authors for a wide class of quantum-field models (see, for instance,the theorem on R-operation in Ref. [5]). Unlike UV-divergences, there is no simple and universal rule for definition of the general structure in momenta and frequencies for IR-divergent parts even of simplest one-loop diagrams, to say nothing of multiloop ones. The model of turbulence with an effective viscosity used in Refs [81, 82, 85] has been analysed from this standpoint in Ref. [55], where it was shown that the multiple of $\log m$ contributions from one-loop 1-irreducible diagrams do not coincide in the momentum-frequency structure with terms of the action functional, and thus they cannot be removed by multiplicative renormalization. This possibility was realized in the calculation made in Ref. [85] just because the author voluntarily (and illegitimately from the viewpoint of renormalization theory) restricted himself to zero external frequencies.

The general conclusion is that the modern RG-technique ensures the existence of IR-scaling for asymptotic behavior at $m \sim k \to 0$ of static correlators, but does not yet allow to solve exactly the second IR-problem, i.e., to determine dependence on m at $m/k \to 0$. To solve it we apparently need to find all dangerous operators and perform explicit summation of their contributions to SDE-representations for equal-time correlators of the type of Eqns (148), (149), as it has been done with operators of the form of φ^n in representation (110) for non-equal-time Green's functions. It is obviously a complicated problem that requires essential development of the existing technique.

4. Conclusion

In conclusion, we mention the studies on the RG-theory of turbulence which we could not consider in this review because of the lack of volume.

For the first time the RG-technique was successfully applied to hydrodynamics in Ref. [86], where fluid was considered in the state of a thermal equilibrium and substantiation of the equations of hydrodynamics was discussed in connection with the problem of 'long-time tails' of correlation functions.

In a series of works the RG-approach was used to study more realistic models of a fully developed turbulence, which take account of weak [87–90] and strong [91] anisotropy, gyrotropy (the violation of spatial parity) [92, 93], compressibility [74, 94–96], inhomogeneity, damping, and real geometry of the problem [46, 59, 70, 97–100]. The problem of possible IR-relevant corrections to the Navier–Stokes equation was touched upon in Refs [16, 32, 78, 101] and considered in detail in Ref. [102].

The authors of Refs [31–33, 76, 77, 103–105] analysed the turbulent convection of a passive scalar impurity and effects of thermal conductivity. In Ref. [103] substantiation was obtained for the phenomenological Richardson's '4/3 law' for spreading of a cloud of impurity particles and the expression

$$u_* = \frac{1}{2}\sqrt{1 + \frac{8(d+2)}{d}} + O(\epsilon)$$

was found for the inverse effective turbulent Prandtl number reproduced then by many authors. The Batchelor constant was computed in Refs [33, 77] for a spectrum of the passive impurity analogous to Eqn (60); convective turbulence was investigated in Ref. [76].

Magnetohydrodynamic turbulence was studied in Refs [106–109]; with anisotropy, in Ref. [89]; with gyrotropy, in Refs [92, 110]. In Ref. [110] the RG-method was used to examine the phenomenon of a turbulent dynamo, spontaneous generation of a large-scale magnetic field. The Langmuir turbulence of a plasma was studied in Refs [111, 112].

In Refs [65, 93, 97, 98, 100, 113] the model of a developed turbulence based on the principle of maximal randomness of the velocity field [13] and different from Eqn (1) was proposed and thoroughly investigated.

In Ref. [101] (see also references therein and earlier papers [114, 115]) an iteration procedure similar both to the Wilson RG-procedure and to the technique of iterative averaging over modes of lattice scales, which was more traditional for turbulence theory, was developed. Another similar approach was suggested in Ref. [116]. In Refs [117–121] the IR asymptotic behavior of velocity correlators was studied in the framework of the hypothesis about algebra of fluctuating quantities (which is a phenomenological formulation of the operator product expansion, see Section 3.4); in Refs [117, 118] a two-dimensional turbulence was analysed by the technique of conformal-invariant quantum field theories. In Ref. [122] higher orders of the ϵ-expansion were investigated for the exactly solvable model of turbulent convection.

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References

1. Monin A S, Yaglom A M *Statisticheskaya Gidrodinamika* (Statistical Fluid Mechanics) Ch. 2 (Moscow: Nauka, 1967) [Translated into English Vol. 2 (Cambridge, MA: MIT Press, 1975)]
2. Orszag S A, in *Fluid Dynamics, Les Houches, 1973* (Eds R Balian, J-L Peube) (London: Gordon and Breach,1977) p. 235
3. McComb W D *The Physics of Fluid Turbulence* (Oxford: Clarendon, 1990); McComb W D *Rep. Prog. Phys.* **58** 1117 (1995)
4. Landau L D, Lifshits E M *Gidrodinamika* (Hydrodynamics) (Moscow: Nauka,1986) [Translated into English Landau L D, Lifshitz E M *Fluid Mechanics* (London: Pergamon Press, 1959)]
5. Bogoliubov N N, Shirkov D V *Introduction to the Theory of Quantized Fields* (New York: A Willey Interscience, 1959, 1980)
6. Collins J C *Renormalization* (Cambridge: Cambridge Univ. Press, 1984) [Translated into Russian (Moscow: Mir, 1988)]
7. Patashinskiĭ A Z, Pokrovskiĭ V L *Fluktuatsionnaya Teoriya Fazovykh Perekhodov* (Fluctuation Theory of Phase Transitions) (Moscow: Nauka, 1982)
8. Brezin E, Le Guillou J C, Zinn-Justin J, in *Phase Transitions and Critical Phenomena* Vol. 6 (Eds C Domb, M S Green) (New York: Academic, 1976) p.125
9. Amit D J *Field Theory, the Renormalization Group, and Critical Phenomena* (New York: McGraw-Hill, 1978)
10. Zinn-Justin J *Quantum Field Theory and Critical Phenomena* (New York: Clarendon, 1996)
11. Popov V N *Kontinual'nye Integraly v Kvantovoĭ Teorii Polya i Statisticheskoĭ Fizike* (Functional Integral in Quantum Field Theory and Statistical Physics) (Moscow: Atomizdat, 1976)
12. Vasil'ev A N *Funktsional'nye Metody v Kvantovoĭ Teorii Polya i Statistike* (Functional Methods in Quantum Field Theory and Statistical Mechanics) (Leningrad: Leningrad Univ. Press, 1976) [Translated into English (Gordon & Breach, in press)]
13. Kraichnan R H *J. Fluid. Mech.* **5** 407 (1959)
14. Wyld H W *Ann. Phys.* **14** 143 (1961); **9** 1728, 1884 (1966)
15. Adzhemyan L Ts, Antonov N V, Vasiliev A N *Zh. Eksp. Teor. Fiz.* **95** 1272 (1989) [*Sov. Phys. JETP* **68** 733 (1989)]
16. De Dominicis C, Martin P C *Phys. Rev. A* **19** 419 (1979)
17. Janssen H K *Z. Phys. B* **23** 377 (1976); Bausch R, Janssen H K, Wagner H *Z. Phys. B* **24** 113 (1976)
18. De Dominicis C *J. Phys.* (Paris) **37** C1–247 (1976); De Dominicis C, Peliti L *Phys. Rev. B* **18** 353 (1978)
19. Adzhemyan L Ts, Vasiliev A N, Pis'mak Yu M *Teor. Mat. Fiz.* **57** 268 (1983) [*Theor. Math. Phys.* **57** 1131 (1983)]
20. Keldysh L V *Zh. Eksp. Teor. Fiz.* **47** 1515 (1964) [*Sov. Phys. JETP* **20** 1018 (1965)]
21. Martin P C, Siggia E D, Rose H A *Phys. Rev. A* **8** 423 (1973)
22. Zakharov V E, L'vov V S *Izv. Vyssh. Uchebn. Zaved. Radiofiz.* **18** 1470 (1975) [*Radiophys. Quantum Electron.*]
23. Hohenberg P C, Halperin B I *Rev. Mod. Phys.* **49** 435 (1977)
24. Ronis D *Phys. Rev. A* **36** 3322 (1987)
25. Olla P *Phys. Rev. Lett.* **67** 2465 (1991)
26. Honkonen J, Nalimov M Yu *Z. Phys. B* **99** 297 (1996)
27. Teodorovich E V *Izv. Akad. Nauk SSSR Ser. Mekh. Zhidk. Gaza* **22** 29 (1987) [*Fluid Dyn.* **22** 517 (1987)]; *Dokl. Akad. Nauk SSSR* **299** 836 (1988)
28. Teodorovich E V *Zh. Eksp. Teor. Fiz.* **96** 163 (1989) [*Sov. Phys. JETP* **69** 89 (1989)]
29. Fournier J-D, Frish U *Phys. Rev. A* **28** 1000 (1983)
30. Wilson K, Kogut J *The Renormalization Group and the ϵ-Expansion* (New York: Wiley, 1974) [Translated into Russian (Moscow: Mir, 1975)] [Also *Phys. Rep. C* **12** (2) 79 (1974)]
31. Yakhot V, Orszag S A *Phys. Rev. Lett.* **57** 1722 (1986)

32. Yakhot V, Orszag S A *J. Sci. Comput.* **1** 3 (1986)
33. Dannevik W P, Yakhot V, Orszag S A *Phys. Fluids* **30** 2021 (1987)
34. Antonov N V *Zapiski Nauchnykh Seminarov LOMI* **164** 3 (1987)
35. Antonia R A, Satyaprakash B R, Hussian A K M F *J. Fluid. Mech.* **119** 55 (1982)
36. Anselmet F, Gagne Y, Hopfinger E J, Antonia R A *J. Fluid. Mech.* **140** 63 (1984)
37. Kuznetsov V R, Sabel'nikov V A *Turbulentnost' i Gorenie* (Turbulence and Combusion) (Moscow: Nauka, 1986) [Translated into English (New York: Hemisphere Publ. Corp., 1990)]
38. Kuznetsov V R, Praskovskiĭ A A, Sabel'nikov V A *Izv. Akad. Nauk SSSR Ser. Mekh. Zhidk. Gaza* **23** (6) 51 (1988) [*Fluid. Dyn.* **23** 843 (1988)]
39. Halperin B I, Hohenberg P C, Siggia E D *Phys. Rev. B* **13** 1299 (1976)
40. Adzhemyan L Ts, Vasil'ev A N, Gnatich M *Teor. Mat. Fiz.* **74** 180 (1988) [*Theor. Math. Phys.* **74** 180 (1988)]
41. Antonov N V *Vestn. St. Petersburg Univ. Fiz. Khim.* (4) 6 (1992)
42. Adzhemyan L Ts, Antonov N V, Kim T L *Teor. Mat. Fiz.* **100** 382 (1994) [*Theor. Math. Phys.* **100** 1086 (1994)]
43. Adzhemyan L Ts, Borisenok S V, Girina V I *Teor. Mat. Fiz.* **105** 450 (1995)
44. Antonov N V *Zapiski Nauchnykh Seminarov LOMI* **169** 18 (1988)
45. Yakhot V, She Z-S, Orszag S A *Phys. Fluids A* **1** (2) 289 (1989)
46. Yakhot V, Smith L M *J. Sci. Comp.* **7** 35 (1992)
47. Antonov N V, Borisenok S V, Girina V I *Teor. Mat. Fiz.* **106** 92 (1996) [*Theor. Math. Phys.* **106** 75 (1996)]
48. Adzhemyan L Ts et al. *Zapiski Nauchnykh Seminarov LOMI* **224** 43 (1995)
49. Antonov N V, Vasil'ev A N *Teor. Mat. Fiz.* **110** 122 (1997) [*Theor. Math. Phys.*]
50. L'vov V S, Preprint No. 53 IAiE (Novosibirsk: IAiE, SO AN SSSR, 1977)
51. Belinicher V I, L'vov V S *Zh. Eksp. Teor. Fiz.* **93** 533 (1987) [*Sov. Phys. JETP* **66** 303 (1987)]; L'vov V S *Phys. Rep.* **207** 1 (1991)
52. Moiseev S S et al. *Dokl. Akad. Nauk SSSR* **271** 856 (1983) [*Sov. Phys. Dokl.* **28** 634 (1983)]; Moiseev S S, Tur A V, Yanovskiĭ V V *Dokl. Akad. Nauk SSSR* **279** 96 (1984) [*Sov. Phys. Dokl.* **29** 926 (1984)]
53. Sagdeev R Z et al., in *Nonlinear Effects in Plasma Physics and Hydrodynamics* (Ed. R Z Sagdeev) (Moscow: Mir, 1986) p. 138
54. Tur A V, Yanovskiĭ V V, Preprint IKI Pr-1203 (Moscow: IKI AN SSSR, 1986)
55. Antonov N V *Zh. Eksp. Teor. Fiz.* **105** 614 (1994) [*JETP* **78** 327 (1994)]
56. Antonov N V *Zapiski Nauchnykh Seminarov LOMI* **189** 15 (1991)
57. Antonov N V *Vestn. St. Petersburg Univ. Fiz. Khim.* (3) 3 (1992)
58. Adzhemyan L V, Adzhemyan L Ts *Zapiski Nauchnykh Seminarov LOMI* **224** 36 (1995)
59. Smith L M, Reynolds W C *Phys. Fluids A* **4** (2) 364 (1992)
60. Kraichnan R H *Phys. Fluids* **30** 2400 (1987)
61. Kraichnan R H *J. Fluid. Mech.* **9** 1728 (1966)
62. Fournier J-D, Frish U *Phys. Rev. A* **17** 747 (1978)
63. L'vov V, Falkovich G *Phys. Rev. A* **46** 4762 (1992)
64. Adzhemyan L V, Adzhemyan L Ts *Vestn. St. Petersburg Univ. Fiz. Khim.* (3) 3 (1994)
65. Adzhemyan L Ts, Nalimov M Yu *Teor. Mat. Fiz.* **91** 294 (1992)
66. Martin P C, De Dominicis C *Prog. Theor. Phys. Suppl.* (64) 108 (1978)
67. Carati D *Phys. Rev. A* **41** 3129 (1990)
68. Liao W *J. Phys. A* **22** L737 (1989); Ohkitani K *Phys. Fluids A* **2** (9) 1529 (1990); Shivamoggi B *J. Phys. A* **23** 1689 (1990); Novikov E A *Phys. Lett. A* **162** 385 (1992)
69. Yaglom A M *Izv. Akad. Nauk SSSR. Fiz. Atmosfery i Okeana* **17** 1235 (1981)
70. Lam S H *Phys. Fluids A* **4** (5) 1007 (1992)
71. Zhou Y, Vahala G, Hossain M *Phys. Rev. A* **40** 5865 (1989)
72. Carati D *Phys. Fluids A* **2** 1854 (1990)
73. Giles M J *Phys. Fluids* **6** (2) 595 (1994)
74. Staroselsky I et al. *Phys. Rev. Lett.* **65** 171 (1990)
75. Yakhot V, Orszag S A, She Z-S *Phys. Fluids A* **1** (2) 184 (1989)
76. Yakhot V *Phys. Fluids A* **1** (2) 175 (1989)
77. Yakhot V, Orszag S A *Phys. Fluids* **30** (1) 3 (1987)
78. Eyink G L *Phys. Fluids* **6** (9) 3063 (1994)
79. Borgas M S *Phys. Fluids A* **4** (9) 2055 (1992)
80. Adzhemyan L Ts, Antonov N V *Vestn. St. Petersburg Univ. Fiz. Khim.* (2) 53 (1994)
81. Yakhot V *Phys. Rev. A* **23** 1486 (1981)
82. Yakhot V *Phys. Rev. A* **24** 3268 (1981)
83. Levich E *Phys. Lett. A* **79** 171 (1980)
84. Grossmann S, Schnedler E *Z. Phys. B* **26** 307 (1977)
85. Teodorovich E V *Zh. Eksp. Teor. Fiz.* **102** 863 (1992) [*Sov. Phys. JETP* **75** 472 (1992)]
86. Forster D, Nelson D R, Stephen M J *Phys. Rev. Lett.* **36** 867 (1976); *Phys. Rev. A* **16** 732 (1977)
87. Rubinstein R, Barton J M *Phys. Fluids* **30** 2987 (1987)
88. Carati D, Brenig L *Phys. Rev. A* **40** 5193 (1989)
89. Adzhemyan L Ts et al. *Int. J. Mod. Phys B* **9** 3401 (1995)
90. Kim T L, Serdyukov A V *Teor. Mat. Fiz.* **105** 412 (1995)
91. Buša J et al., Preprint JINR E17-95-348 (Dubna: JINR, 1995); Helsinki University Preprint HU-TFT-96-19 (Helsinki: Helsinki University, 1996)
92. Pouquet A, Fournier J-D, Sulem P-L *J. Phys. Lett.* (France) **39** L-199 (1978)
93. Adzhemyan L Ts et al. *J. Phys. II France* **5** 1077 (1995)
94. Adzhemyan L Ts, Nalimov M Yu, Stepanova M M *Teor. Mat. Fiz.* **104** 260 (1995) [*Theor. Math. Phys.* **104** 971 (1995)]
95. Volchenkov D Yu, Nalimov M Yu *Teor. Mat. Fiz.* **106** 375 (1996)
96. Antonov N V, Nalimov M Yu, Udalov A A, Preprint SPbU PI-11-96 (St. Petersburg: SPbGU, 1996) [Submitted to *Teor. Mat. Fiz.*]
97. Adzhemyan L Ts, Nalimov M Yu *Teor. Mat. Fiz.* **96** 150 (1993) [*Theor. Math. Phys.* **96** 872 (1993)]
98. Adzhemyan L Ts, Borisenok S V, Nalimov M Yu *Teor. Mat. Fiz.* **106** 416 (1996) [*Theor. Math. Phys.* **106** 341 (1996)]
99. Adzhemyan L Ts, Gnatich M, Steglik M, Preprint OIYaI P17-94-319 (Dubna: OIYaI, 1994)
100. Nalimov M Yu, Ryzhov A K *Vestn. St. Petersburg Univ. Fiz. Khim.* (1) 72 (1995)
101. McComb W D, Watt A G *Phys. Rev. A* **45** 3507 (1992); **46** 4797 (1992)
102. Antonov N V, Borisenok S V, Girina B I *Teor. Mat. Fiz.* **107** 47 (1996)
103. Adzhemyan L Ts, Vasil'ev A N, Gnatich M *Teor. Mat. Fiz.* **58** 72 (1984) [*Theor. Math. Phys.* **58** 47 (1984)]
104. Teodorovich E V *Prikl. Mat. Mekh.* **52** 218 (1988) [*Appl. Math. Mech.* **52** 170 (1988)]
105. Gnatich M *Teor. Mat. Fiz.* **83** 374 (1990)
106. Fournier J-D, Sulem P-L, Pouquet A *J. Phys. A* **15** 1393 (1982)
107. Adzhemyan L Ts, Vasil'ev A N, Gnatich M *Teor. Mat. Fiz.* **64** 196 (1985) [*Theor. Math. Phys.* **64** 777 (1985)]
108. Camargo S J, Tasso H *Phys. Fluids B* **4** (5) 1199 (1992)
109. Adzhemyan L Ts, Volchenkov D Yu, Nalimov M Yu *Teor. Mat. Fiz.* **107** 142 (1996)
110. Adzhemyan L Ts, Vasil'ev A N, Gnatich M *Teor. Mat. Fiz.* **72** 369 (1987) [*Theor. Math. Phys.* **72** 940 (1987)]
111. Pelletier G *J. Plasma Phys.* **24** 421 (1980)
112. Adzhemyan L Ts et al. *Teor. Mat. Fiz.* **78** 368 (1989) [*Theor. Math. Phys.* **78** 260 (1989)]
113. Adzhemyan L Ts et al. *Czechoslovak J.Phys.* **45** 491 (1995)
114. Rose H A *J. Fluid. Mech.* **81** 719 (1977)
115. Siggia E D *J. Fluid. Mech.* **107** 375 (1981)
116. Eyink G L *Phys. Rev. E* **48** 1823 (1993)
117. Polyakov A M *Nucl. Phys. B* **396** 367 (1993)
118. Falkovich G, Hannany A *Phys. Rev. Lett.* **71** 3454 (1993)
119. Eyink G L *Phys. Lett. A* **172** 355 (1993)
120. Lebedev V V, L'vov V S *Pis'ma Zh. Eksp. Teor. Fiz.* **59** 546 (1994) [*JETP Lett.* **59** 577 (1994)]
121. Antonov N V *Vestn. St. Petersburg Univ. Fiz. Khim.* (2) 74 (1996)
122. Avellaneda M, Majda A J *Commun. Math. Phys.* (West Germ.) **131** 381 (1990)