

METHODOLOGICAL NOTES

PACS number: 03.65.Bz; 03.40.–t

Bell's inequalities and EPR – Bohm correlations: working classical radiofrequency model

N V Evdokimov, D N Klyshko, V P Komolov, V A Yarochkin

Contents

Abstract. **A description is given of an electron model which makes it possible to simulate pair correlation of random dichotomous signals of the Einstein – Podolsky – Rosen (EPR) type in the Bohm variant. This model can be used to demonstrate that Bell's inequality is satisfied in classical physics. Some features of parametric rf oscillators are used in the model. An analysis of the operation of the model helps one to understand the difference between quantum and classical correlations in EPR experiments. A specific mechanism is suggested for 'nonlocal' control of EPR-type correlations between distant observers, which applies to classical and quantum models. A controlled correlation between two random telegraphic signals is ensured by transmission, to two observers, of a sequence of pulses with the same random phase. An elementary derivation of Bell's inequality is given and an analysis is made of the logic of the use of the popular term 'quantum nonlocality' employed usually in the description of the quantum variants of the investigated experiment.**

1. Introduction

The fundamental difference between the postulates of quantum and classical physics is demonstrated in the starkest contrast when we consider experiments demonstrating the Einstein – Podolsky – Rosen (EPR) paradox [1] in the Bohm variant [2], and failure to satisfy Bell's inequality [3 – 5] (for reviews, see Refs [6 – 10]). Numerous optical experiments of this type, carried out in the last 25 years [6 – 10], have confirmed sufficiently reliably that the quantum models are satisfactory.

N V Evdokimov, D N Klyshko, V P Komolov, V A Yarochkin
Physics Department, M V Lomonosov Moscow State University, Vorob'evy gory, 119899 Moscow, Russia
Tel. (7-095) 939 11 04. E-mail: dnk@klyshko.phys.mu.ru

Received 8 June 1995
Uspekhi Fizicheskikh Nauk **166** (1) 91 – 107 (1996)
Translated by A Tybulewicz, edited by S D Danilov

In spite of the importance of this range of topics for physics in general and numerous attempts to popularise them [11 – 14], the essence of the problem is not very well known. This is partly due to the lack of clarity of the investigated experimental schemes and the complex logic structure of the relevant proofs.

It is desirable to distinguish two separate questions or 'paradoxes':

(1) the existence of 'nonlocally' controlled correlations of the EPR – Bohm type between the readings of distant instruments;

(2) Bell's paradox, that is the violation of certain inequalities which follow from these readings.

In 1935, Einstein, Podolsky, and Rosen [1] drew attention to the existence of perfect (100%) controlled quantum correlations between the observable properties of two distant particles. On this basis they reached the conclusion that, by measuring the properties of one particle, one could carry out indirectly and without any perturbation, precision measurements of noncommuting observables of the other particle, which is in conflict with quantum theory. However, if the possibility of measuring noncommuting observables is admitted, then these observables should have definite values also before measurement, i.e. *a priori*. Consequently, quantum theory may be supplemented, i.e. the properties of both particles and their correlations can be described by certain 'hidden' variables, for example in terms of classical statistical physics, which would reduce quantum to classical physics.

Approximately 30 years later, Bell used very general ideas to show that this EPR programme is possible only if classical physics is supplemented by certain new long-range interactions, i.e. by introducing *nonlocality*. Experimental confirmation of failure to satisfy Bell's inequality (in exact agreement with the predictions of quantum theory) has shown, in accordance with the widely used modern terminology, the inapplicability of *local realism*. Two main alternatives remain: to admit 'nonlocal realism' or to accept quantum theory in its Copenhagen (or different) interpretation. Some writers speak also (fairly inconsistently) of *nonlocality of quantum theory*.

It should be stressed that the proof of Bell's inequality does not depend simply on the existence of controlled correlations or their value (the proof in fact relies on partial correlations), but on a certain finer property of the correlations. Therefore, we are faced with the natural question: are controlled correlations of the EPR – Bohm type between the readings of two instruments themselves essentially a quantum phenomenon? A negative answer to this question has long been known [3, 7, 11, 15, 16], but not very widely. It is frequently assumed that the existence of these correlations is a sufficient proof of nonlocality and that the EPR paradox consists specifically in their existence.

Recently several quantum models with more than two 'observers' (i.e. more than two correlated spin-$\frac{1}{2}$ particles or photons), characterised by a greater difference between quantum and classical predictions [17, 18], have been considered. For example, a model with three observers is proposed in Ref. [17] (see also Refs [9, 14, 16, 19]) and the results of quantum and classical calculations are shown to differ in *sign* for a certain combination of observables. It is remarkable that this property (like the violation of the relevant modified Bell's inequality [19, 20]) is associated with *perfect* correlation of three observables, in contrast to Bell's inequalities in the case of two observers. In some models with N observers ($N \geqslant 2$), the ratio of the quantum and classical limits for a certain observable S_N increases as $2^{(N-1)/2}$ [18] (see also Ref. [16]). Once again there are corresponding classical models with a perfect controlled correlation between N signals [9, 16].

We shall describe a working classical radio-frequency model which has much in common with quantum EPR models. It demonstrates the first of the paradoxes listed above, i.e. it simulates a controlled correlation or anticorrelation of the EPR – Bohm type, including a perfect (100%) correlation. A brief description of the operation of this model can be found in Refs [9, 16]. Its analysis reveals a simple classical mechanism of the appearance of such correlations and of the influence on them of distant observers, and it thus lifts the aura of mystery surrounding them. In this way the first paradox is stripped of its status as the *true* quantum paradox, because it has a classical analogue. (It should be pointed out that other classical models are known and they demonstrate a perfect correlation of the EPR – Bohm type [3, 7, 11, 15], but they are of gedanken type.)

The second paradox, which is the violation of Bell's inequality is a true paradox and it demonstrates the fundamental difference between quantum and classical representations. In the experiments carried out so far, this paradox is revealed only under partial correlation conditions and only in the results of a statistical analysis of a sufficiently large ensemble of experimental results. In our model, Bell's inequality is naturally confirmed.

Section 2 describes a general scheme of both classical and quantum experiments which demonstrate controlled correlations. Section 3 deals with the procedure of measurements that revealed Bell's paradox and gives the results of a numerical experiment demonstrating the statistical nature of this paradox. Section 4 provides a brief analysis of the EPR – Bohm correlations in the case of three correlated signals. A specific example is used in Section 5 to elucidate the general principle of the appearance of 'nonlocally' controlled correlations and Section 6 describes specific realisation of this principle in our model. A discussion of the topics considered can be found in the Conclusions (Section 7).

The mathematical treatments are basically contained in Appendices I and II. The first of them provides an elementary derivation of Bell's inequality and explains the former logical meaning of the popular term 'nonlocality', whereas the second discusses the statistics of double-valued (telegraphic) signals that occur in experiments of the EPR – Bohm type, and gives simple examples of the appearance of negative (see Refs [7, 21 – 23]) and multivalued 'probabilities' when an attempt is made to describe quantum correlations in terms of classical probabilities. Appendix III gives some technical details of our rf experiment.

A transparent model described below and the related discussion should help significantly in the understanding by uninitiated readers of the essence of the EPR – Bohm correlations and of Bell's inequality based on them. It would be desirable to use similar models in practical work which is carried out in the physics departments of higher educational establishments.

2. Demonstration of EPR – Bohm correlations

Our model implements the experimental situation illustrated in Fig. 1, which is frequently used to provide clear descriptions of the EPR – Bohm type of experiment [14]. From time to time a signal transmitter S sends simultaneous messages to two distant addresses A and B. The messages contain commands to switch on green or red lamps. Symbolically, the four possible commands can be represented thus: (++), (−+), (+−) or (−−); here the plus sign corresponds to the green lamp and the minus sign represents the red lamp.

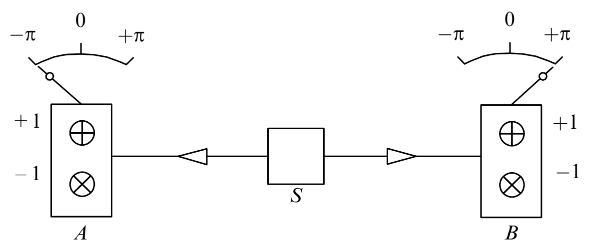

Figure 1. General experimental setup used to demonstrate the EPR – Bohm type of correlation and to reveal whether Bell's inequality is obeyed.

Each of the observers A and B has one signal arm which he controls and this arm has three distinct positions labelled $-\pi$, 0 and $+\pi$.

The messages are repeated many times and the result of each of these messages is random: either the green or the red lamp is switched on with an equal probability ($\frac{1}{2}$) which is independent of the arm positions α and β. In fact, in our demonstration experiment the lamps are lit up approximately every second and they are on for about half a second at a time.

A third observer C, who can see simultaneously both pairs of lamps at A and B, notes that there is a definite correlation between the colours of the simultaneously lit lamps, and the degree of this correlation and its sign depend on the arm positions. Three situations are the most typical and they are listed below.

(1) If both hands stand, for example, at identical average 0 positions, then the colour of the lamps is every time the same, i.e. only the results (++) and (−−) are observed. Therefore, the colours are fully correlated. Such a perfect correlation is observed also in the more general case on condition that the sum of the coordinates of the signal arms $\gamma = \alpha + \beta$ is 0. (In practice, because of technical limitations, the correlation is not perfect but only about 90%).

(2) If only one of the observers sets his signal arm at one of the extreme positions so that $\gamma = \pm\pi$, the result is then (+−) or (−+), i.e. the colours become anticorrelated.

(3) Finally, if the signal arms stand at, for example, the position 0 and $\pi/2$ (or, in general, are set so that $\gamma = \pm\pi/2$), there is no correlation at all between the colours of the lamps at A and B.

A natural question to ask is: how can the position of the arm at A influence the colour of the lamp which is lit as a result of the next message at B? If the arms at A and B are in the middle 0 position, the observer at A seeing the green light is certain that the green lamp was lit at this moment at B. If the arm is then set in one of the extremal positions and he again sees the green light, he can be certain that at this moment the red lamp is lit at B. How do the lamps at B 'learn' of the position of the arm at A? It would seem that this is action at a distance.

These three series of observations based on our model reproduce exactly a quantum correlation, an anticorrelation, or its absence in 'real' EPR – Bohm experiments. In the quantum case the information carriers are pairs of photons or spin-$\frac{1}{2}$ particles with correlated properties. Lighting of the lamps corresponds to simultaneous detection of a pair of particles in two out of four detectors. The positions of the arms α and β in Fig. 1 correspond, in optical experiments, to the lengths of optical paths or orientations of polarisation prisms in intensity interferometers [9], whereas in experiments on fermions they correspond to orientations of the magnets in the Stern – Gerlach experiments.

True, in quantum experiments the moments of transmission of the next 'message' (i.e. the moments of detection of particle pairs) are random, whereas in our model they are regular. However, the latter can be readily made random with a random number generator. Quantum stochasticity of the intervals between messages are of no importance in the topics under discussion.

The answer to the question asked above is often, in the context of quantum experiments, as follows: this is a manifestation of *quantum nonlocality*. This implies some mysterious influence, travelling at a superluminal velocity, of the apparatus at A on the events at B (or vice versa). Sometimes it is said that quantum mechanics predicts stronger correlations than those permitted by *local* classical theories or *local realism*. The term *locality* is understood here to be the absence of action at a distance implemented by some unknown interactions between apparatus at A and B. (However, the term *nonlocality* is used most frequently in connection with violation of Bell's inequality in quantum models and experiments of the kind described below; the formal logic for the use of this term in such situations is demonstrated in Appendix I.) Another popular term is the *inseparability* of quantum objects. In this context this means the requirement to describe simultaneously pairs of correlated particles, even if they fly apart to large distances, as well as impossibility of separate individual descriptions (in this sense the pairs of electric signals in our experiments can also be regarded as inseparable).

A more specific and rigorous answer is that the positions of the signal arms influence only their 'own' lamps and that this is quite sufficient for the change in the degree and sign of the correlations between distant signals. This statement is obviously not quite trivial and the model under discussion should help to overcome the existing prejudices. In our model, as well as in 'genuine' quantum EPR models, use is made of a general principle of 'remote' control of pair correlations. This control is performed by selection of specific subsets from the overall information set at A and B. This will be explained later by a simple specific example (see Figs 5 and 6). The difference between the classical and quantum models is only in the method describing this general a priori set. In the classical case this can be done by joint probabilities of all possible 'messages', whereas in the quantum case this can be done only with the aid of the wave function of the information carrier, which can be a pair of fermions or a two-photon four-mode electromagnetic field (for details, see Ref. [9]).

It follows that the effect of distant observers on their mutual correlations does not imply, as one might expect, any mysterious action at a distance: it is fully local. It should be noted that observation of a correlation requires information transfer (it requires protocols of tests in which the serial numbers of the transmissions are fixed) from A to B to the third observer (C) by the usual channels; however, A and B themselves do not see any visible influence of the positions of the arms (their own signal arm or that of the other observer) on the nature of the observed light flashes. This exclude the frequently discussed possibility of superluminal exchange of information between A and B with the aid of the EPR correlation.

3. Verification of Bell's inequality

The fundamental difference between the classical and quantum experiments with two observers of the EPR – Bohm type can be seen only for certain intermediate positions of the signal arms, different from those discussed above and causing partial correlations. This difference can be revealed if at least four series of measurements, with four positions of the arms α and β, are carried out and four correlation coefficients are calculated. Therefore, this difference can only be of statistical nature.

We shall now introduce the following parametrisation of the experimental results obtained by means of the setup shown in Fig. 1. When the green or red lamp is lit by the observer on the left, we shall assign, respectively, the values $a = +1$ or -1 to a discrete random quantity A (we shall use capitals to denote a random quantity and lower-case letters for the values assumed by this quantity). Similarly, a dichotomous (double-valued) random quantity B with the values $b = \pm 1$ represents the light of the lamps lit by the receiver on the right. Let us assume that the variables α and β have values in the range $(-\pi, \pi)$ and denote the positions of the controlled signal arms which influence in some way the combined statistics of A and B, and the variable $\gamma = \alpha + \beta$ is their sum. There are thus two 'telegraphic' random processes $a_i(\alpha)$ and $b_i(\beta)$, and the correlation between them depends on two controlled parameters α and β (i is the serial number of the test).

It is convenient to describe the degree of correlation by the average value of the product of the observed random numbers: $M(\gamma) = \langle A(\alpha)\, B(\beta) \rangle$. Therefore, $M = +1$ and $M = -1$ represent full correlation and anticorrelation, respectively, and $M = 0$ represents the absence of correlation. In this connection we shall call the parameter M simply the *correlation*.

Fig. 2 shows three types of the dependence: $M(\gamma)$: (*a*) the dependence described in Appendix II and representing the classical theory of our model

$$M_c(\gamma) = 1 - \frac{2|\gamma|}{\pi} \qquad (-\pi \leqslant \gamma \leqslant +\pi) \tag{1a}$$

(*b*) the dependence in accordance with the quantum theory of 'genuine' experiments of the EPR type, which is (see Ref. [9])

$$M_q(\gamma) = \cos\gamma, \tag{1b}$$

and (*c*), which represents the results of experiments in our rf model. The rhombs represent the points of intersection of the quantum and classical dependences discussed above, whereas the circles and squares are the points usually employed in the formulation and verification of Bell's inequality (this is discussed below). The well-known classical EPR model, in which there are two particles with correlated angular momenta [7, 11], also leads to the dependence described by expression (1a).

It follows that the observed difference between the specific models under discussion reduces to the difference between the nature of the dependence of the correlation $M(\gamma) = \langle AB \rangle$ on the sum of the coordinates of the signal arms $\gamma = \alpha + \beta$, which is a cosinusoidal curve or a broken straight line. The absolute value of the excess of the quantum above the classical correlation (in the case of our specific models) reaches 41% for $\gamma = \pm\pi/4, \pm 3\pi/4$ (see Fig. 2). Therefore, if we exclude the shared points 0, $\pm\pi/2$, the quantum correlations in our models are indeed somewhat stronger than the classical correlations, but this by itself is not very surprising. We shall show later that the cosinusoidal dependence $M(\gamma)$ is incompatible with any classical probabilistic models.

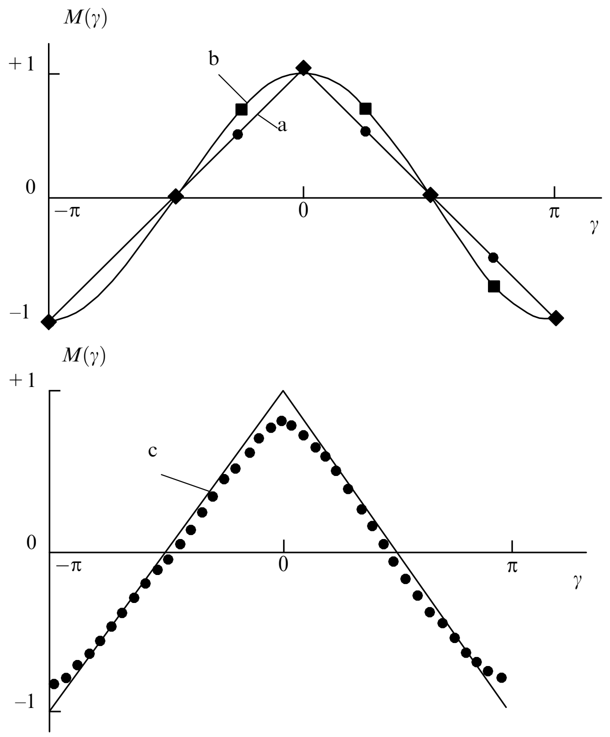

Figure 2. Dependence of the correlation $M = \langle AB \rangle$ of the colours of the lit lamps (see Fig. 1) on the sum of the coordinates of the controlled arms $\gamma = \alpha + \beta$: (a) according to the classical theory; (b) according to the quantum theory; (c) according to the experiment described in the text.

It is quite easy to show, as demonstrated by expression (II.2), that the average value $M = \langle AB \rangle$ determines also the joint distribution of the probabilities $P(a, b)$ for the random quantities A and B described by formula $P(a, b) = (1 + abM)/4$ (here, $a, b = \pm 1$). In accordance with (1), this distribution has the following forms in the classical and quantum models:

$$P(++) = P(--) = \frac{1}{2} - \frac{|\gamma|}{2\pi},$$
$$P(+-) = P(-+) = \frac{|\gamma|}{2\pi}, \tag{2a}$$
$$P(++) = P(--) = \frac{1}{2}\cos^2\frac{\gamma}{2},$$
$$P(+-) = P(-+) = \frac{1}{2}\sin^2\frac{\gamma}{2}. \tag{2b}$$

Here, for example, $P(+-)$ is the relative frequency of the event $(+-)$ when the number of tests is made sufficiently large. The usual definition of the average value gives the inverse transformation

$$M = \sum_{a,b} abP(a, b) = P(++) + P(--) - P(-+) - P(+-) = 4P(++) - 1.$$

It is natural to assume that we can construct some more sophisticated classical model which would reproduce the quantum dependences $M(\gamma)$ and $P(a, b, \gamma)$ for all the values of γ. However, Bell's inequality (see Appendix I) excludes this possibility, irrespective of the nature of the model devices which are constructed. This universality of Bell's approach, which excludes the possibility of carrying out a whole class of experiments, is the remarkable property which distinguishes it from the other more specific criteria of classical behaviour [9].

The protocol for a series of tests with fixed values of α and β can be written in the form of a table of the following type:

Table 1.

i	a_i, $\alpha = \pi/2$	b_i, $\beta = \pi/4$	$m_i = a_i b_i$, $\gamma = 3\pi/4$
1	+1	−1	−1
2	−1	−1	+1
3	−1	+1	−1
4	+1	−1	−1
5	+1	+1	+1

The last column lists the products $m_i = a_i b_i$ representing, after averaging, the correlation between the colours of the lamps. A statistical analysis of such products, obtained for different pairs of α and β, is presented in Fig. 2 (curve c). The average value $M = \langle m_i \rangle$ depends, in accordance with the theoretical dependence given by expression (1a), only on the sum γ of the coordinates of the controlled arms.

We shall now assume that four series of tests (with N tests in each series) are carried out successively and they involve the following four sets of parameters:

$$(\alpha, \beta) = \left(0, -\frac{\pi}{4}\right), \quad \left(\frac{\pi}{2}, -\frac{\pi}{4}\right), \quad \left(0, \frac{\pi}{4}\right), \quad \left(\frac{\pi}{2}, \frac{\pi}{4}\right).$$

We then have $\gamma_1 = \alpha + \beta = -\pi/4$, $\gamma_2 = \alpha' + \beta = \pi/4$, $\gamma_3 = \alpha + \beta' = \pi/4$, $\gamma_4 = \alpha' + \beta' = 3\pi/4$, (Fig. 2). We shall introduce the notation $a \equiv a(0)$, $a' \equiv a(\pi/2)$, $b \equiv b(-\pi/4)$, $b' \equiv b(\pi/4)$ and form N numbers

$$s_k \equiv \frac{1}{2}\left(a_k b_k + a'_{N+k} b_{N+k} + a_{2N+k} b'_{2N+k} - a'_{3N+k} b'_{3N+k}\right). \tag{3}$$

These numbers assume, with certain probabilities (Appendix II), the values 0, ±1, ±2, i.e. they realise a certain random quantity S.

It should be stressed that a single realisation s_k of this quantity is calculated from the results of four different tests with four sets (α, β) given above.

Averaging of the resultant numbers s_k for a sufficiently large value of N shows that $\langle S\rangle_{\text{exp}} \approx 1$. This result follows also from expression (1a) or from the graph a in Fig. 2:

$$\begin{aligned}\langle S\rangle &= \frac{1}{2}\left(\langle AB\rangle + \langle A'B\rangle + \langle AB'\rangle - \langle A'B'\rangle\right) \\ &= \frac{1}{2}\left[M\left(-\frac{\pi}{4}\right) + M\left(\frac{\pi}{4}\right) + M\left(\frac{\pi}{4}\right) - M\left(\frac{3\pi}{4}\right)\right] \\ &= \frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right) = 1.\end{aligned}$$

It is shown in Appendix I that this value of $\langle S\rangle$ is the maximum for any classical model, since these models should satisfy the following Bell's inequality [3–5]:

$$|\langle S\rangle_{\text{c}}| \leqslant 1. \tag{4a}$$

On the other hand, the corresponding quantum models are subject to a weaker inequality:

$$|\langle S\rangle_{\text{q}}| \leqslant \sqrt{2}. \tag{4b}$$

This limiting value is in fact given by function (1b):

$$\begin{aligned}\langle S\rangle_{\text{q}} = \frac{1}{2}\bigg[&\cos\left(-\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) \\ &+ \cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{3\pi}{4}\right)\bigg] = \sqrt{2}.\end{aligned}$$

The resultant difference of 41% between the quantum and classical values of $\langle S\rangle$ is the largest for the models under consideration; it will be smaller for any other set of the parameters α and β. Let us assume that, for example, the first series of tests establishes certain phases α, β, the second gives the phases α', β, and the third and the fourth give the phases, α, β' and α', β', respectively; here, $\alpha' = \alpha + \pi/2$, $\beta' = \beta + \pi/2$. According to expression (1b), the parameter $\langle S\rangle$ depends only on the combined phase $\gamma_1 = \alpha + \beta$, established in the first series of tests: $\langle S\rangle_{\text{q}} = \sqrt{2}\cos(\gamma_1 + \pi/4)$. Consequently, Bell's inequality (4a) is violated everywhere in the interval $-\pi/2 < \gamma_1 < 0$ (Fig. 3). Let us now assume that $\gamma_1 = -\varepsilon$, $0 < \varepsilon \ll 1$. We then find that, in the four series, the combined phases are successively $-\varepsilon$, $\pi/2 - \varepsilon$, $\pi/2 - \varepsilon$ and $\pi - \varepsilon$, which yield an almost complete (anti)-correlation and almost complete absence of correlation (Fig. 3). We have here $\langle S\rangle_{\text{c}} = 1$ and $\langle S\rangle_{\text{q}} = 1 + \varepsilon$.

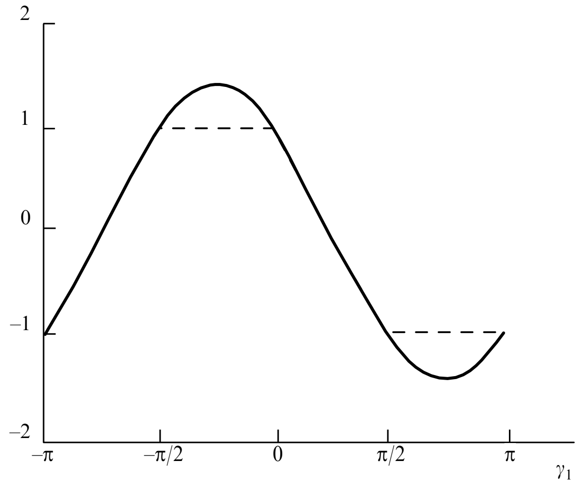

Figure 3. Dependence of $\langle S\rangle$ on the sum phase γ_1, established in the first series of tests, in accordance with the quantum (continuous curve) and classical (dashed curve) models. It is assumed that the conditions $\gamma_2 = \gamma_3 = \gamma_1 + \pi/2$, $\gamma_4 = \gamma_1 + \pi$ are satisfied.

Individual realisations s_k and their average values found for a *finite N* can exceed, in the absolute sense, the limits set by inequalities (4a) and (4b), i.e. these inequalities are only of statistical nature. Fig. 4 gives the results of a numerical experiment which demonstrates this aspect in the classical case. In this partial realisation the average values $\langle S\rangle_N$ exceed, in the first few N tests, even the quantum limit $\sqrt{2}$ and after the next approximately 20 tests they still exceed the classical limit 1. The quantity S behaves similarly in the quantum model, but the values $s = 2$ are countered 68% more frequently,

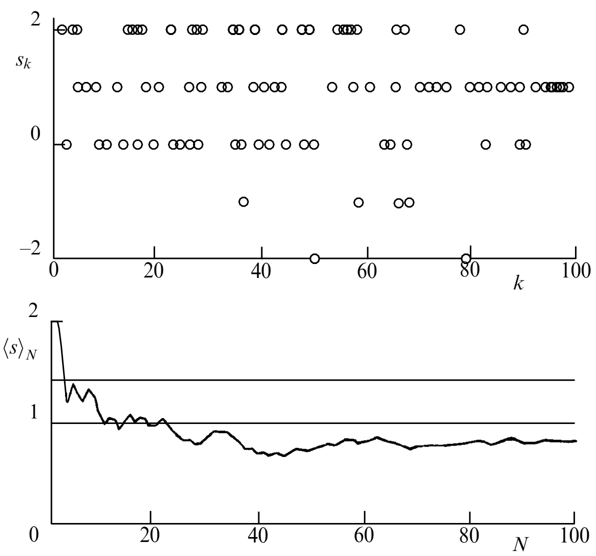

Figure 4. Statistical nature of Bell's inequality. The top part of the figure shows a sequence of 100 values of the variable s_k, deduced from expressions (3) and (11) with the aid of a random number generator. The lower part of the figure gives the averages of the first N values of s_k, shown in the upper part of the figure, as a function of N. The horizontal lines identify the maximum classical (1) and quantum ($\sqrt{2}$) levels.

wheres the remaining values, 1, 0, -1, -2 are found somewhat less frequently (Appendix II).

It should be stressed that the inequalities (4a) and (4b) do not limit the actual value of the correlation $M(\gamma)$ of the observables A and B, but only the average of a certain bilinear combination of these observables corresponding to different values of α and β, i.e. the inequalities in question limit a certain functional $S(\alpha, \beta, \alpha', \beta')$ of $M(\alpha, \beta)$.

Possible logical consequences of violation of Bell's inequality (4a) in quantum models and experiments are discussed in the Conclusions. Here, we shall mention simply that Bell's inequality is proved in Appendix I starting only from the concept of joint 'four-dimensional' probabilities for all four observables $P(a, b, a', b')$, which determine — in accordance with the probability theory — all the other statistical characteristics of the experiments under discussion. Violation of this inequality can be explained in a natural manner by the fact that the concept of the joint probability $P(a, b, a', b')$ is invalid in accordance with of the principle of complementarity.

If someone prefers to retain this concept, nevertheless, it is then necessary either to admit the possibility of negative probabilities [8, 21 – 23] or to introduce 'nonlocality', that is, the interaction of distant instruments as a result of some unknown superluminal forces (see Appendix I).

4. EPR – Bohm correlations in the case of three observers

The proof that the hidden variables cannot be used to describe some predictions of quantum models is sometimes called Bell's theorem. Violation of Bell's inequality (4a) in the EPR – Bohm model, discussed above, is a partial form of this theorem. An interesting new form of Bell's theorem was proposed by Greenberger, Horne, Shimony, and Zeilinger (GHSZ) [17] (see also Refs [9, 14 (1990), 16, 19]). This new form is also based on the EPR – Bohm model, but with the addition of one more particle and, consequently, a third observer. The transmitter in Fig. 1 then sends messages not to two but to three receivers, and each of them can again operate in two regimes, depending on the signal arm position. Therefore, each test now involves measurement of three quantities, for example A, B, C, or A', B, C, and so on (the prime means that the arm is set in a different position: $\alpha \to \alpha'$), and three lamps are lit.

According to quantum theory, we can define a state of three particles, which are information carriers (three spin-$\frac{1}{2}$ fermions or three photons, each characterised by two modes) with the following observable properties::

$$\langle A'BC\rangle = -1, \quad \langle AB'C\rangle = -1,$$
$$\langle ABC'\rangle = -1, \quad \langle A'B'C'\rangle = 1\,. \tag{5}$$

All the first and second moments vanish: $\langle A\rangle = \langle AA'\rangle = \langle AB\rangle = \langle AC\rangle = \ldots = 0$. Formulas (5) describe full correlation or anticorrelation between sets of three readings. This means that, for example, an observable $A'BC$ does not fluctuate, i.e. in each test with the corresponding position of the signal arms the product of three numbers $\alpha' bc$ is always -1. When tests are repeated with the same position of the arms, only an even number of the green lamps is lit, i.e. the following four sets of three readings are observed with equal probabilities: $(a'bc) = (++-), (+-+), (-++), (---)$. This also applies to the tests in which A, B', C, and A, B, C' are observed. When three quantities identified with a prime are observed, the number of the green lamps is always odd: $(a'b'c') = (--+), (-+-), (+--), (+++)$.

We shall try to describe these experiments (which are still speculative) from the 'common sense' point of view, i.e. within the framework of the classical probabilities. We shall do this by considering six operators A, A', ... in the set of expressions (5) as classical random quantities with the values $a = \pm 1$, $a' = \pm 1, \ldots$. We shall take the symbol $\langle \ldots \rangle$ as representing classical averaging over some six-dimensional distribution of probabilities $P(a, b, c, a', b', c')$. [It should be pointed out straight away that these 'probabilities' are defined fully by the properties of the set of expressions (5) and that some of them are negative (Appendix II).]

In classical theory in each test a transmitter sends full information which is a set of six numbers ± 1. All six quantities a, b, c, a', b', c' have certain definite values $+1$ or -1, irrespective of whether they are observed or not. (This can be called the postulate of the a priori existence of observables.) In *each* test these six numbers should satisfy quantum predictions described by the set of expressions (5) of perfect (anti)correlation of the sets of three observables (since the positions of the signal arms can be selected in an arbitrary manner *after* sending of a message, when it is still on its way; this is known as the delayed choice experiment, proposed by John Wheeler in 1978). Therefore, the transmitted signals should every time satisfy the following system of equations:

$$a'bc = -1, \quad ab'c = -1, \quad abc' = -1, \quad a'b'c' = 1 \tag{6a}$$

Here, the averaging symbols $\langle \ldots \rangle$ are omitted, because the products $a'bc$, ... do not fluctuate (although a', b, c are random numbers).

We can easily show that inequalities (6a) are incompatible. Let us assume that, for example, $(a,b,c,a',b',c') = (+++---)$. Then the first three equalities in expression (6a) are satisfied, but not the last one. We can consider the general case by multiplying all four equalities. Then, on the left-hand side each factor is encountered twice:

$$a'bcab'cabc'a'b'c' = (abca'b'c')^2 = +1. \tag{7a}$$

On the other hand, the product of the right-hand sides of equalities (6a) gives $(-1)^3(+1) = -1$. This is the GHSZ paradox of the $+1 = -1$ type and it proves Bell's theorem: the postulate of the a priori existence of the observables as invalid or, if one prefers, the existence of 'nonlocality'.

In principle, in classical models one can improve the apparatus and eliminate the switches in Fig. 1 so that all six quantities a, a', b, b', c, c' are measured in one test and four products $a'bc$, $ab'c$, abc' and $a'b'c'$ can be formed directly as a result of such a test. These products cannot have the signs $(---+)$, predicted by quantum theory and described by expression (6a), but only the signs of the $(--++)$ or $(----)$ type and so on, which are such that they satisfy equality (7a). This model with perfectly controlled correlations between three signals can be implemented by means of parametric oscillators [9, 16].

How can one solve the GHSZ paradox within the framework of the orthodox quantum theory? The problem is that, according to quantum mechanics, the four predictions of measurements described by expression (5) apply to four different positions of the signal arms in Fig. 1. This can be taken into account by rewriting expression (6a) as follows:

$$a_1' b_1 c_1 = -1, \quad a_2 b_2' c_2 = -1,$$
$$a_3 b_3 c_3' = -1, \quad a_4' b_4' c_4' = 1. \tag{6b}$$

In the above expression the subscripts identify the number of the experiment. Now, expression (7a) becomes

$$a_2 a_3 a_1' a_4' b_1 b_3 b_2' b_4' c_1 c_2 c_3' c_4' = \pm 1. \tag{7b}$$

This expression is no longer a quadratic form and, therefore, it may be equal to -1. In other words, all four equalities in expression (6b) are compatible, in contrast to the case represented by expression (6a). For example, in the first experiment, when the three observables A', B, and C are acquired, the remaining quantities A, B', C' do not have — according to quantum mechanics — definite values (here the quantities with and without primes can represent, respectively, the x and z spatial components of the particle spin vectors).

We can rewrite equalities (5) in the operator form [9]

$$A'BC = -I, \quad AB'C = -I, \quad ABC' = -I, \quad A'B'C' = I, \tag{8}$$

where I is the operator unity. The transition to expression (6a) can be considered also from a different point of view, namely as a replacement of the operators $A, A', \ldots, I$ in the operator identities (8) with their eigenvalues $a, a', \ldots, 1$. The incompatibility of the equalities (6a) obtained in this way shows that this replacement is not permissible. A similar conclusion has also been demonstrated for an *arbitrary* state of three spin-$\frac{1}{2}$ particles and also for one spin-1 particle; this is called the Kochen – Specker theorem [24] (see also Refs [9, 25]).

An experimental demonstration of the GHSZ paradox in the ideal case should consist, say, of 40 tests: 10 tests for each signal arm position. If in each test the equalities of expression (6a) are confirmed, it becomes obvious that it is meaningless to speak of hidden variables in such a test: they do not have definite values. Such an experiment would evidently be the most direct proof of the main quantum theory paradox, which is the principle of complementarity.

This experiment with three observers can also be used to demonstrate the validity of Bell's theorem in its usual form, i.e. by violation of some classical inequality [20]. In this case, the following combinations are derived by analogy with expression (3): $S \equiv (A'BC + AB'C + ABC' - A'B'C')/2$. It follows from expression (5) that in this specific quantum model we have $\langle S \rangle_q = -2$, whereas in an arbitrary classical model we always have $\langle S \rangle_c \leqslant 1$ (Appendix II).

It should be pointed out that the GHSZ paradox cannot be resolved formally by means of negative probabilities, which is in contrast to violation of Bell's inequalities.

5. Control of correlations at a distance

We shall first consider a simple method for the generation, from one continuous random process, of two dichotomous random signals with a controlled degree of mutual correlation. Let $\phi(t)$ be a stationary random process characterised by $\langle \phi \rangle = 0$, and governing a random phase distributed uniformly in the reduced interval $0 - 2\pi$. We shall form a random telegraphic signal from this process by means of the algorithm

$$a(\alpha, t) \equiv \operatorname{sign}\{\cos[\phi(t) + \alpha]\}, \tag{9}$$

where α is an arbitrary parameter. It follows from this definition that $\langle a(\alpha) \rangle = 0$ and $a(\alpha \pm \pi) = -a(\alpha)$, i.e. that the signals $a(\alpha)$ and $a(\alpha \pm \pi)$ are anticorrelated. On the other hand, $a \equiv a(\alpha)$ and

$$a' \equiv a\left(\alpha \pm \frac{\pi}{2}\right) = \mp \operatorname{sign}\{\sin[\phi(t) + \alpha]\} \tag{9a}$$

are uncorrelated signals (since $\cos x$ and $\sin x$ are orthogonal functions, see Fig. 5). The correlation between the signals $a(\alpha)$ and $a(\alpha + \Delta\alpha)$ can readily be found from expression (9) (Appendix II):

$$M(\Delta\alpha) \equiv \langle a(\alpha)\, a(\alpha + \Delta\alpha) \rangle = 1 - 2\frac{|\Delta\alpha|}{\pi} \quad (-\pi \leqslant \Delta\alpha \leqslant \pi). \tag{10}$$

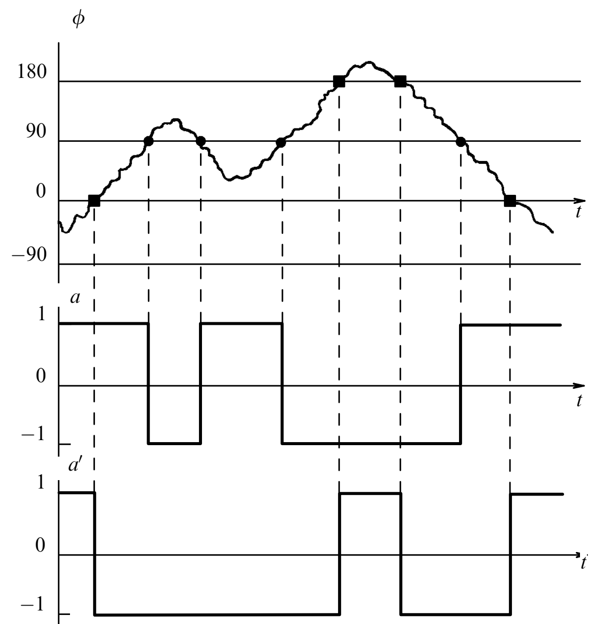

Figure 5. Formation of two independent telegraphic signals $a(t)$ and $a'(t)$ from one random process $\phi(t)$ in accordance with rule (9).

In this way we can form, from one random process $\phi(t)$, two or more random dichotomous signals $a(\alpha)$ and $a(\alpha + \Delta\alpha)$ with an arbitrary degree of mutual correlation $M(\Delta\alpha)$, lying between -1 and $+1$. Consequently, if the same process $\phi(t)$ is transmitted, for example as an rf signal, to several observers, each of them can influence 'at a distance', by his 'own' local parameter α_n, a mutual pair correlation between the observables (Fig. 6). This is the classical analogue of the EPR – - Bohm correlation. However, the nature of the dependence of this correlation on the parameters is now different (Fig. 2). In 'genuine' EPR experiments the telegraphic nature of the signals is due to the dichotomous spectrum of the observed operators: the number of spin-$\frac{1}{2}$ projections along a selected direction and the number of types of polarisation of a photon is 2. The dichotomous nature of the spectrum is responsible for the double-valued function sign (x).

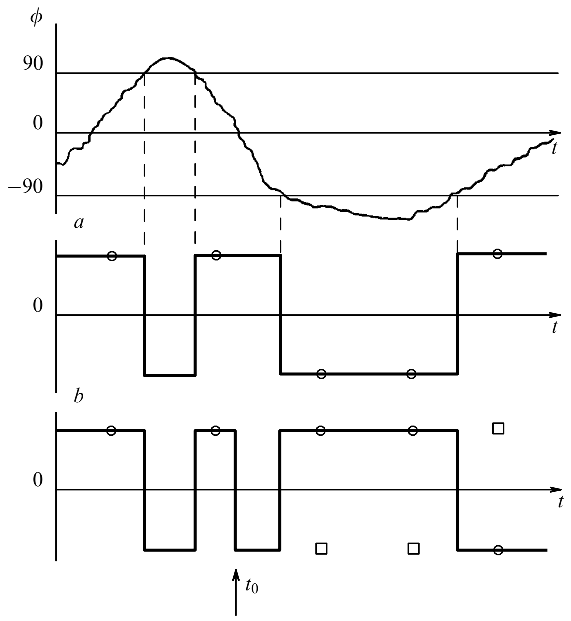

Figure 6. Mechanism of the 'nonlocal' influence on the correlations between the readings of two distant instruments A and B. At the moment t_0 the observer at B switches the controlling phase: $\beta = 0 \to \beta = \pi$. The random dichotomous process $b(t)$ observed at B shows a sign reversal in accordance with expression (11): $b(t) \to -b(t)$. Consequently, an anticorrelation, instead of correlation, occurs in the next observation. The circles identify the values which determine the colour of the lamps obtained as a result of interrogation when the phase switching is taken into account; the squares represent the values obtained ignoring such switching.

The procedure described can be implemented by modulating, with a random signal $\phi(t)$, the phase of a monochromatic 'carrier' oscillation $X(t)$ of frequency ω_a: $X(t) = \cos\left[\omega_a t + \phi(t)\right]$ [it is assumed that the correlation time τ_ϕ of the process $\phi(t)$ is much longer than the oscillation period $2\pi/\omega_a$]. The next stage is introduction, into the oscillation $X(t)$, of a controlled phase shift α and mixing of this oscillation with a 'homodyne' oscillation $\cos(\omega_a t)$ which has a stable phase. The result is a superposition

$$Z(t) = \cos\left[\omega_a t + \phi(t) + \alpha\right] + \cos(\omega_a t)$$

and this superposition is subjected to a detection procedure. The output from a square-law detector (after filtering off the hf component of frequency $2\omega_a$) is in the form of an lf signal $\left(Z(t)\right)^2 \approx 2 + 2\cos\left[\phi(t) + \alpha\right]$. Hence, nonlinear amplification subject to limits makes it possible to form readily the required signal $a(\alpha, t)$ of the type described by expression (9).

Let us now consider the effects of modulation, by the same random process $\phi(t)$, of two oscillations (with possibly different average frequencies $\omega_a \neq \omega_b$), which can be transmitted along wires or by radio links to two addressees A and B, where homodyne phase detectors can be used to transform these oscillations, by the method described above, into two telegraphic signals $a(\alpha, t)$ and $b(\beta, t)$ (here, α and β are the phases added at the receiver positions A and B). We then have $\langle a \rangle = \langle b \rangle = 0$ irrespective of the values of α and β, but an observer C which receives realisations of both processes detects the correlation described above [cf. (10)] and this correlation depends on the phase difference:

$$M(\alpha, \beta) = \langle ab \rangle = 1 - 2\,\frac{|\alpha - \beta|}{\pi}\,. \tag{10a}$$

It is important to note that the correlation depends on both parameters α and β, i.e. each of the observers A and B can alter its magnitude and sign, but these observers will not detect any changes. For example, a change in β results in generation, from the initial general process $\phi(t)$, of a different signal $b(\beta) \to b(\beta + \Delta\beta)$, which is anticorrelated with the initial signal if $\Delta\beta = \pi$ (Fig. 6).

This example demonstrates clearly that it is possible to influence the correlation of two stochastic processes at distant regions by altering only one of them. Consequently, also in the context of quantum EPR experiments, there is no need to speak of some mysterious 'nonlocal influence' of measurements at detectors distant from one another in order to explain the dependence of the correlations on the positions of controlled signal arms shown in Fig. 1.

We should mention the possibility of transmission to A and B of a random process $\phi(t)$ directly (by telephone) without the use of 'carrier' oscillations, which may be followed by the subsequent processing at the receiver positions, and also possible variants with a controlled correlation $\langle a_1 a_2 \ldots a_N \rangle$ between N receivers [9, 16].

6. Parametric oscillations and controlled correlations

In our model experiment a classical rf analogue of the EPR correlation is obtained in double-loop parametric oscillators operating in the megahertz range [27].

In the nondegenerate (two-frequency) case such oscillators have two special features: the phases $\phi(t)$ and $\phi'(t)$ of each of the two oscillations with average frequencies ω_a and ω_b fluctuate freely (for technical reasons), but there is full anticorrelation between them: $\phi(t) + \phi'(t) = \phi_0$. Here, $\phi_0 = \text{const}$ is the phase of the pump oscillator (with the average frequency $\omega_0 = \omega_a + \omega_b$), which for the sake of simplicity is assumed to be stable and equal to zero, so that $\phi'(t) = -\phi(t)$. In other words, the instantaneous frequencies $\omega_a(t) = \omega_a + \mathrm{d}\phi/\mathrm{d}t$ and $\omega_\beta(t) = \omega_b + \mathrm{d}\phi'/\mathrm{d}t$ of the two oscillations at the oscillator output always drift in opposite directions: if the signal frequency increases, then the idler frequency decreases, and vice versa.

As a result of these operations, the output oscillations are $X(t) = \cos\left[\omega_a t + \phi(t)\right]$ and $Y(t) = \cos\left[\omega_b t + \phi'(t)\right] = \cos\left[\omega_b t - \phi(t)\right]$. Therefore, the same random process $\phi(t)$ phase-modulates both output oscillations. Both of them are transmitted (for simplicity, along wires) to two receivers A and B. The difference between the signs of ϕ and ϕ', typical of parametric oscillators, has the effect that the correla-tion depends on the *sum* of the parameters $\gamma = \alpha + \beta$ and not on their difference, in contrast to expression (10a).

In the proposed scheme the stable local homodyne oscillations with frequencies ω_a and ω_b are replaced by signals from a second identical parametric oscillator, excited by the same pump signal [9]. Thus, the observer A receives two oscillations $X_1(t) = \cos\left[\omega_a t + \phi_1(t)\right]$ and $X_2(t) = \cos\left[\omega_a t + \phi_2(t)\right]$, where $\phi_1(t)$ and $\phi_2(t)$ are independent random phases of the two parametric oscillators. A controlled phase α is added to one of these oscillations and the result is the sum and the difference: $X_\pm \equiv X_1 \pm X_2$. Two square-law detectors produce lf signals proportional to $|X_\pm|^2 \approx 2 \pm 2\cos\left[\phi(t) + \alpha\right]$, where $\phi(t) \equiv \phi_1(t) - \phi_2(t)$. Subtraction of the signals yields, after restriction to the function of the required type described by Eqn (9), the following result

$$a(\alpha, t) = \mathrm{sign}\{\cos[\phi(t) + \alpha]\}\,. \tag{11a}$$

Similarly, a second random telegraphic signal is formed at the receiver B:

$$b(\beta, t) = \mathrm{sign}\{\cos[-\phi(t) + \beta]\}\,. \tag{11b}$$

A property $\phi_k + \phi'_k = \phi_0$ of parametric oscillators (ϕ_0 is the pump phase) is used here and it leads to $\phi + \phi' = \phi_1 - \phi_2 + \phi'_1 - \phi'_2 = 0$.

The actual analysis of signals was carried out by a digital method. A description of this analysis and some technical details are given in Appendix III.

In our model the hidden variables λ (see Appendix I) are represented by the full set of variables which describe the instantaneous state of the whole system, including the power sources. Evolution of these variables is manifested by fluctuations of the phases of parametric oscillators $\phi_k(\lambda(t))$ with a characteristic time τ_ϕ, which in turn lead — in accordance with the set of expressions (11) — to fluctuations of the signs of the signals a_i and b_i. These signs are observed at certain moments t_i separated by an interval much greater than τ_ϕ.

The characteristic time τ_ϕ of the 'natural' reversal of the signs of the signals a and b, related to the coherence time of parametric oscillators, is of the order of 10^{-4} s. The moments at which changes of the sign occur in a and b are not always identical (Fig. 5). Consequently, in visual observation of the correlation in accordance with the colours of the lamps one should use a clock-signal generator operating at a frequency of about 1 Hz which sets the periodicity of sampling of the state of the output stages of the detectors and of the corresponding switching between the lamps. In automatic measurement of the correlation M one would use a higher sampling frequency.

An even closer analogy with quantum experiments would be achieved if oscillations with a random phase were transmitted by radio links at random moments in the form of short pulses of duration shorter than the phase coherence time.

7. Conclusions

(1) It should be stressed that the classical simulator of quantum effects with two double-loop parametric oscillators described above is in many respects similar to 'genuine' EPR systems. Three main types of such systems are known: systems based on beams of spin-$\frac{1}{2}$ fermions and optical systems utilising either conventional or polarisation intensity interferometers.

Optical parametric oscillators have been used recently in optical EPR experiments, but this has been done in an amplification regime (below the self-excitation threshold) when the emitted light represents the intrinsic quantum noise of an optical parametric amplifier (*parametric scattering of light* [9]). In the variant of an EPR experiment described in Ref. [9], use is made, as in our case, of two two-mode parametric amplifiers. The operators of the observables $A = A(0)$, $A' = A(\pi/2)$ can then be interpreted as noncommuting operators $\mathrm{Cos}(\phi)$ and $\mathrm{Sin}(\phi)$, where ϕ is the difference between the phases of the two oscillations (Fig. 5). Phase detection in optics is performed by beam splitters. In classical and quantum-optical models the whole information transmitted in the ith message to observers is encoded in the random phase ϕ_i, which enables the observers to form double-valued telegraphic signals with a controlled degree of correlation.

Formal descriptions of the other two types of EPR experiments (optical and spin polarisation) also have much in common with the model considered in Ref. [9]. In the polarisation experiments the role of the shared random phase is played by the linear polarisation angle and information selection control is performed by a polarising prism. In experiments on fermions the role of a random phase is played by the angle between the spin and the Stern – Gerlach magnetic field.

The fundamental distinction of quantum experiments, which leads to the possibility of violation of Bell's inequalities, reduces to 'just' the inability to measure simultaneously, in one test, both $A_i(\alpha)$ $A_i(\alpha')$ when $\alpha \neq \alpha'$, since in each receiving channel in each test there is only one photon or one fermion.

It therefore clearly follows from the working classical model described above that the correlation effect of the EPR – Bohm type controlled at a distance has a close classical analogue and the effect itself is no evidence of any mysterious superluminal effects.

(2) All the classical models which are at least minimally reasonable should however satisfy Bell's inequality and violation of this inequality in quantum models is a true paradox. It is clear from the discussion following expressions (1) and (2) that three possible 'explanations' of violation of Bell's inequality can be put forward: one can reject either the concept of joint probabilities or the positive nature of these probabilities or the locality.

We are thus faced with three possibilities:

(a) rejection of joint probabilities;

(b) acceptance of joint probabilities and the possibility of their negative value;

(c) acceptance of joint positive probabilities and unknown superluminal forces.

[Introduction of hidden variables, which ensure classical determinism, implies — in accordance with expression (I.6) — the existence of joint probabilities and is therefore a special case of the second or third possibility.]

In fact, violation of Bell's inequality which is a feature of quantum theory could, if desired, be explained also on the basis of classical joint probabilities, but only at the price of introduction of unknown long-range forces between measuring instruments or rejection of a probability non-negativity [see expression (I.2)]. Hence it is frequently concluded (in conflict even with the rules of formal logic) that quantum theory and quantum phenomena are *nonlocal*. It is sometimes also said that violation of Bell's inequalities is evidence of invalidity of *local realism*, i.e. realism is equated with the validity of the concept of joint probabilities and a priori properties.

In addition to general considerations, there is also a specific objection to such a 'nonlocal' explanation of Bell's paradox: after all, the interaction between distant instruments should not simply lift the restriction $|\langle S\rangle| \leqslant 1$, but should ensure the exact quantum value $\langle S\rangle = \sqrt{2}$, irrespective of the detailed structure of the measuring apparatus.

Introduction of negative probabilities is a purely formal hypothesis which has no operational meaning.

It therefore follows that out of these three possibilities the first is the least objectionable and it represents a rejection both of the concept of joint probabilities and of the possibility of attributing quantum objects certain *a priori* properties (corresponding to noncommuting operators). This conclusion is related directly to the principle of complementarity, i.e. it belongs to the current paradigm of quantum physics and, therefore, it is natural to regard violation of Bell's inequalities as one more (perhaps the most direct) proof of the principle of complementarity and not a proof of the existence of ad hoc assumed unknown interactions or of physically meaningless negative probabilities.

The conclusion that the concept of joint probabilities is unacceptable can also be made independently of the inequality $|\langle S\rangle| \leqslant 1$, simply starting from formal expressions for the average values of products of noncommuting operators, leading to negative and multivalued 'probabilities' (see Appendix II). Therefore, quantum EPR models differ in respect of two significant properties from classical models: in the case of quantum models the inequality $|\langle S\rangle| \leqslant 1$ is not obeyed and joint 'probabilities' are negative for noncommuting observables. The first difference can be detected directly in experiments. Rejection of the concept of joint probabilities resolves both contradictions.

We are still left with the 'eternal' general problem of interpretation of quantum formalism. Its Copenhagen form, accepted explicitly or implicitly by the majority of physicists, simply forbids asking 'unnecessary' questions of nature, i.e. it is of positivistic and pragmatic nature. In the context of the EPR experiments, this problem obviously becomes more acute: it is necessary to face the circumstance that although before measurement some real properties are only potentially existing, nevertheless there is a correlation between them. Taking into account the simple classical analogue described above, Bell's paradox seems to be not more or less mysterious than, for example, the inability of simultaneous exact measurement of the coordinate and momentum of a particle.

(3) Selection of an interpretation is largely a matter of personal taste. The only experimental statistical fact in the case of quantum models is the cosinusoidal dependence of the correlation function $M(\gamma)$, with extremal values ± 1, on the sum of the coordinates of the controlled arms, which is plotted in Fig. 2 (curve *b*). This is precisely the dependence observed in experiments on the intensity interference of light when sources of 'nonclassical' (two-photon) light are used [9, 26]. If a classical source of light is employed, the same apparatus can give the same cosinusoidal dependence, but the extremal values are now $\pm 1/2$ [9], which makes the value of $|\langle S\rangle|$ half as large and, therefore, does not violate Bell's inequality.

If experiments of this type are described in the Heisenberg representation, it becomes obvious that the difference between quantum and classical experiments is not the effect of the characteristic features of one or another specific optical system for the transformation and detection of signals, but of the specifics of the initial quantum state of an electromagnetic field entering an optical system, which has no classical statistical equivalent [9, 26]. This general Heisenberg approach to optical EPR experiments makes it possible to draw clear classical parallels and it is convenient for the classification of different variants of such experiments.

Acknowledgements. We are grateful to A V Belinskiiÿ for numerous fruitful discussions and to the Russian Fund for Fundamental Research (Project No. 93-02-14848) for financial support. One of us (D N Klyshko) is also grateful to A Zeilinger for his continuous interest in the topics discussed here.

8. Appendices

I. Proof of Bell's inequality

In the experiment shown schematically in Fig. 1, the results of repeated tests on the same apparatus under constant macroscopic conditions are averaged. It is natural to assume that in a theoretical description, the process of time averaging can be replaced with ensemble averaging with the aid of a probability distribution $P_{ABA'B'}(a,b,a',b') \equiv P(a,b,a',b')$ for all four observable random quantities $A = A(\alpha)$, $A' = A(\alpha')$, $B = B(\beta)$, $B' = B(\beta')$. Here, $P(a,b,a',b') \geqslant 0$, $\sum P(a,b,a',b') = 1$, and the variables a, a', b, b' assume the values ± 1; α and β are arbitrary parameters which affect the statistics at A and B. The function $P(a,b,a',b')$ is defined for a set of $2^4 = 16$ possible results of one 'complete' test and it determines the *elementary* or *primary probabilities* in our probabilistic model. For example, $P(1,1,1,-1)$ is the probability of observation of the event $\{a=+1,\ b=+1,\ a'=+1,\ b'=-1\}$.

Although according to Fig. 1 each test involves measurements not of four but of two quantities (for example, A and B, or A' and B), there would seem to be no obstacle to the use in each receiver of two parallel instruments controlled by one shared signal from an information transmitter and recording simultaneously two observables, for example, A and A'. There would then be no need for the controlled arms in Fig. 1 and each receiver would have four indicator lights. (Moreover, we could obviously measure A also for more than two values of the argument $\alpha, \alpha', \alpha'', \ldots$) Since the parameters α and β vary arbitrarily at the points of signal reception, it is natural to assume the elementary probabilities $P(a,b,a',b')$ of all possible results to be a priori given properties of the signal source, irrespective of which quantities are measured and which are not.

The elementary probabilities and the rule of addition of the probabilities of independent events can be used to find the probabilities of all other events. For example, the probability that two green lamps are lit in the system shown in Fig. 1, i.e. the probability of the event $\{a=+1,\ b=+1\}$, is determined by the following sum of four elementary probabilities:

$$P_{AB}(+,+) = \sum_{a',\,b'=\pm 1} P_{ABA'B'}(+,+,a',b')\,.$$

The moments of the distribution are also defined in terms of the elementary probabilities, for example,

$$M \equiv \langle AB\rangle = \sum_{a,b,a',b'} ab\, P_{ABA'B'}(a,b,a',b') = \sum_{a,b} ab\, P_{AB}(a,b).$$

We can also solve the inverse problem, i.e. we can express the elementary probabilities P in terms of the set of moments M (Appendix II). In our classical model, the moments are readily calculated by means of expressions (11) on the assumption that the phase distribution is uniform, ensuring stationary situation; there is no difficulty either in calculation of the moments in the case of quantum EPR models [9].

We shall express the average value of a random quantity S, defined by expression (3), directly in terms of the elementary probabilities:

$$\langle S\rangle_{\mathrm{c}} = \sum_{a,b,a',b'=\pm 1} P(a,b,a',b')\,\sigma(a,b,a',b')\,. \tag{I.1}$$

Here, $\sigma(a,b,a',b') \equiv (ab + a'b + ab' - a'b')/2$, in accordance with the postulate of ergodicity and with the experimental procedure [see expression (3)]. The above function contains only four different factors and, therefore, it assumes only two values ± 1 (in contrast to eight factors and five values $0, \pm 1, \pm 2$ in the case of the experimental quantity s; see Fig. 4). In fact, we can group the terms as follows: $\sigma = \left[a(b+b') + a'(b-b')\right]/2$; if, for example, $b = b'$, then $\sigma = ab = \pm 1$; however, if $b = -b'$, then $\sigma = a'b = \pm 1$.

The modulus of the sum does not exceed the sum of the moduli, so that expression (I.1) and the conditions

$$\sigma = \pm 1, \quad P \geqslant 0, \quad \Sigma P = 1$$

lead us directly to Bell's inequality (4a):

$$|\langle S\rangle_{\mathrm{c}}| \leqslant \sum |\sigma P| = \sum |\sigma| P = \sum P = 1. \tag{I.2}$$

Why does the same elementary conclusion cease to be valid in a quantum description? Introduction of joint probabilities $P(a,b,a',b')$ was made on the basis of an implicit but natural assumption: the a priori existence and the possibility of measuring in one test of all four observables A', A, B, and B'. This assumption is not always justified in quantum probabilistic models in which information carried by, for example, a one-photon state of the optical field cannot be branched ('cloned') and it cannot control two recording instruments. For example, one photon cannot be absorbed in two detectors. In experiments on fermions, the observables A and A' describe the projections of the spin along different directions and these projections are measured for different orientations of the magnets. As a result, A and A' cannot, like B and B', be measured in one test. This paradoxical property of quantum models is related to the principle of complementarity.

In quantum theory the inability to measure simultaneously some observables is formally related to the noncommuting properties of the relevant operators, which in this case are A and A', B and B'. Since these operators do not commute, they cannot be attributed any a priori values, including the eigenvalues ± 1. Consequently, the equality $\sigma = \pm 1$ is meaningless in quantum theory. Moreover, the elementary probabilities $P(a,b,a',b')$ are also meaningless. It is shown in Appendix II that the formally defined quantum moments $\langle ABA'\rangle$, $\langle ABA'B'\rangle, \ldots$ lead to negative and multivalued elementary probabilities.

How can one therefore remove the restriction $|\langle S\rangle| \leqslant 1$, and yet remain within the framework of classical ideas on the a priori probabilities, i.e. how can one use expression (I.1)? According to expression (I.2), there are two obvious (and equally unacceptable) formal possibilities: we can either drop the condition that the probabilities should be non-negative, $P \geqslant 0$, or we can reject the equality $\sigma = \pm 1$. This equality is violated if we postulate the existence of some interaction between instruments. In accordance with the widely used interpretation of the EPR correlations (Section 1), we shall assume that the parameter α of the apparatus at A influences by some unknown 'nonlocal' manner the readings of the apparatus at B, and β influences the readings of the apparatus at A. It is then necessary to replace $a(\alpha)$ everywhere with $\alpha(\alpha,\beta)$ and $b(\beta)$ with $b(\alpha,\beta)$. As a result, σ depends no longer on four but on eight different factors, like s in expression (3):

$$\sigma = \frac{1}{2}\left[a(\alpha,\beta)\,b(\alpha,b) + a(\alpha',\beta)\,b(\alpha',\beta) + a(\alpha,\beta')\,b(\alpha,\beta') - a(\alpha'\beta')\,b(\alpha',\beta')\right]. \tag{I.3}$$

As a result, the quantity σ may, like s, assume the values $0, \pm 1, \pm 2$ and expression (I.1) should contain other elementary probabilities which describe the statistics of all eight factors.

However it is now meaningless to define S in terms of the elementary probabilities and it is clear that all four terms in expression (I.3) can be statistically independent. Since $\langle S\rangle = \langle AB + A'B' + A''B'' - A'''B'''\rangle/2$, it follows that the universal model-independent restriction $|\langle S\rangle| \leqslant 1$ no longer applies and, in the absence of any additional conditions, the value of $\langle S\rangle$ is constrained only by the natural limits ± 2.

Frequently in a discussion of Bell's inequalities the elementary probabilities are not $P(a,b,a',b')$ but are of the $P(\lambda)$ type, where $\lambda \equiv \{\lambda_1, \lambda_2, \ldots\}$ is a set of hidden variables which govern in a causal manner (for example, in accordance with the laws of classical dynamics and electrodynamics) all the properties of the transmitted messages. Consequently, there are some single-valued functional dependences of the $a = a(\lambda,\alpha)$ and $b = b(\lambda,\beta)$ types (in the absence of nonlocality). The essence of the proof given above does not change. The averaging procedure becomes [compare with expression (I.1)]

$$\langle S\rangle_{\mathrm{c}} = \int \mathrm{d}\lambda\, P(\lambda)\,\sigma(\lambda)\,, \tag{I.4}$$

where

$$\sigma(\lambda) \equiv \frac{1}{2}\left[a(\lambda,\alpha)\,b(\lambda,\beta) + a(\lambda,\alpha')\,b(\lambda,\beta) + a(\lambda,\alpha)\,b(\lambda,\beta') - a(\lambda,\alpha')\,b(\lambda,\beta')\right] \tag{I.5}$$

and the quantity $\sigma(\lambda)$ is defined again in terms of four variables such that $\sigma(\lambda) = \pm 1$. Hence, since $\int \mathrm{d}\lambda\, P(\lambda) = 1$, $P(\lambda) \geqslant 0$, we again obtain $|\langle S\rangle| \leqslant 1$.

Once more, we can avoid this restriction by the hypothesis of nonlocality which violates the condition $\sigma(\lambda) = \pm 1$.

It should be noted that the assumption of the existence of a distribution density of the hidden variables $P(\lambda)$ and single-valued causal links $a(\lambda,\alpha)$, $b(\lambda,\beta)$ implies also the existence of the joint distribution $P_{ABA'B'}(a,b,a',b')$:

$$P_{ABA'B'}(a,b,a',b') = \int_{\Lambda(a,b,a',b')} \mathrm{d}\lambda\, P(\lambda)\,. \tag{I.6}$$

Here, $\Lambda(\alpha, \beta, \alpha', \beta')$ is one of the $2^4 = 16$ nonintersecting subsets of the whole set of hidden variables $\Lambda \equiv \{\lambda\}$, which lead in a causal manner to a specific combination of the signs of a, b, a', and b'.

It therefore follows that both the above derivations of the inequality $|\langle S\rangle| \leqslant 1$ postulate the possibility of describing the observed effects in terms of the elementary probabilities $P_{ABA'B'}(a,b,a',b')$, and violation of this inequality in quantum models can be explained logically by the fact that such probabilities cannot be used in the quantum case. It is shown in Appendix II that the same conclusion follows also from the appearance of negative and multivalued probabilities, which can be calculated from expression

(II.2) with the aid of quantum averages of the products of noncommuting operators.

II. Statistics of telegraphic signals and negative 'probabilities'

Let us assume that there are N double-valued ($a_i = \pm 1$) random quantities $A_1, \ldots, A_N$ defined by a set of 2^N elementary probabilities $P(a_1, a_2, \ldots, a_N)$. The moments of this distribution of probabilities can be calculated, by definition, in accordance with the following rule:

$$M_{ij\ldots} \equiv \langle A_i A_{j\ldots} \rangle \equiv \sum_{a_1 \ldots a_N} (a_i a_{j\ldots}) P(a_1, a_2, \ldots, a_N)$$
$$(1 \leqslant i < j, \ldots, \leqslant N) \qquad \text{(II.1)}$$

(it is assumed here that all the indices $i, j, \ldots$ are different). Therefore, these moments are invariant under all transpositions of their indices: $M_{12} = M_{21}, \ldots$.

If equalities in the above rule (II.1) are regarded as a system of equations for the probabilities P, we can find the inverse transformations:

$$P(a_1, a_2, \ldots, a_N) = 2^{-N}\Bigg[1 + \sum_{i=1}^{N} a_i M_i + \sum_{i<j} a_i a_j M_{ij} + \sum_{i<j<k} a_i a_j a_k M_{ijk} + \ldots + (a_1 a_2 \ldots a_N) M_{12\ldots N}\Bigg]. \qquad \text{(II.2)}$$

Here, $\sum_{i<j}$ is the sum of C_N^2 terms with all possible combinations of the indices i and j, $\sum_{i<j<k}$ is the sum of C_N^3 terms, etc.

For example, if $N = 2$, $A_1 = A$, $A_2 = B$, and $\langle A \rangle = \langle B \rangle = 0$, we find that $P(a, b) = (1 + ab\langle AB \rangle)/4$. This relationship has been used in the system of equations (2). We can easily check that consistency conditions of the $\sum_{a_i = \pm 1} P_N = P_{N-1}$ type are satisfied and that the substitution of expression (II.2) into the rule described by (II.1) leads to identities of the $M = M$ types (one should then use equalities $\sum_{a_i} a_i a_k = 2\delta_{ik}$, $\sum_{a_1 \ldots a_N} 1 = 2^N$).

In this way we can describe the statistics of telegraphic signals A_n not in terms of the elementary probabilities P but in terms of their moments M which — according to expression (II.2) — determine the probabilities. However, these moments cannot be selected arbitrarily, since both expression (II.2) and the condition $0 \leqslant P(a_1, a_2, \ldots, a_N)$ impose certain constraints: with all sets $\{a\}$ the inequality

$$-1 \leqslant \sum_{i=1}^{N} a_i M_i + \sum_{i<j} a_i a_j M_{ij} + \sum_{i<j<k} a_i a_j a_k M_{ijk} + \ldots + (a_1 a_2 \ldots a_N) M_{12\ldots N} \qquad \text{(II.2a)}$$

should be satisfied.

These conditions are not always satisfied by the moments calculated with the aid of quantum models, which indicates that the description in terms of the elementary probabilities is invalid in such cases. If quantum calculations are confirmed by measurements of some of the moments, it is possible to demonstrate experimentally this 'forbiddenness' of the concept of the elementary (joint) probabilities. This is one possible view on the meaning of the experiments demonstrating violation of Bell's inequalities.

It should also be mentioned that in quantum theory the moments generally change as a result of transposition of the operators, i.e. we have to distinguish between M_{12} and M_{21}, between M_{123} and M_{132}, and so on. This asymmetry is not reflected in expressions (II.1) and (II.2).

Let us now consider, by way of example, first the classical probabilistic model based on the set of expressions (11) and implemented by our rf system. For symmetry, we shall introduce the notation $A_1 = A$, $A_2 = B$, $A_3 = A'$, $A_4 = B'$, $\alpha_1 = \alpha$, $\alpha_2 = -\beta$, $\alpha_3 = \alpha'$, $\alpha_4 = -\beta'$, etc. In the case of stationary processes the phase is distributed uniformly and the odd moments are zero. The moment of the N-th order is given by the formula

$$M(\alpha_1, \ldots, \alpha_N) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \times \, \mathrm{d}\phi \, \mathrm{sign}\big\{\cos(\phi + \alpha_1) \ldots \cos(\phi + \alpha_N)\big\}. \qquad \text{(II.3)}$$

Hence it follows that the function $M(\alpha_1, \ldots, \alpha_N)$ is invariant under all $N!$ transpositions of its arguments $\{\alpha_i\}$ and under simultaneous reversal of the signs of these arguments. A shift of one of the arguments by $\pm\pi$ reverses the sign of the function $M(\alpha_1, \ldots, \alpha_N)$. In view of such a high symmetry it is sufficient to find $M(\alpha_1, \ldots, \alpha_N)$ only under the conditions $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \alpha_3 \leqslant \alpha_4 \leqslant \pi$. A calculation of the above integral reduces then to determination of the relative fractions of the sections along the ϕ axis where the products of the cosines is positive. As a result, the even moments become

$$M(\alpha_1, \ldots, \alpha_N) = 1 - \frac{2\gamma}{\pi}, \quad \gamma \equiv \alpha_2 - \alpha_1 + \ldots + \alpha_N - \alpha_{N-1} \qquad \text{(II.4a)}$$

(here, γ also belongs to the interval $[0, \pi]$).

If $N = 2$, then formulas (1a) and (2a) follow from expressions (II.2) and (II.4a). If $N = 3$, we obtain

$$P(a, b, a') = \frac{1}{8}\big(1 + aa'\langle AA' \rangle + ab\langle AB \rangle + a'b\langle A'B \rangle\big) = \frac{1}{4}\left[2 + \left(\frac{aa'|\alpha - \alpha'| + ab|\alpha + \beta| + a'b|\alpha' + \beta|}{\pi}\right)\right]. \qquad \text{(II.5a)}$$

The above expression is non-negative (the sums and differences of the phases should be reduced to the interval $[0, \pi]$). Let us assume that, as in the case of expression (3), we have $\alpha = 0$, $\beta = -u\pi/4$, $\alpha' = \pi/2$; then

$$P(+++) = P(---) = \frac{1}{4},$$
$$P(+-+) = P(-+-) = \frac{1}{8}, \qquad \text{(II.6a)}$$
$$P(-++) = P(+--) = P(++-) = P(--+) = \frac{1}{16}.$$

If $N = 4$ then we can assume directly, in agreement with expression (3), that $\alpha_1 = \alpha = 0$, $\alpha_2 = -\beta = \pi/4$, $\alpha_3 = \beta' + \pi = \pi/4$, $\alpha_4 = \alpha' = \pi/2$. The fourth moment $\langle ABA'B' \rangle$, as well as the moments $\langle AA' \rangle$ and $\langle BB' \rangle$, all vanish. Since $\langle AB \rangle = \langle A'B \rangle = \langle AB' \rangle = -\langle A'B' \rangle = 1/2$, we can find from expressions (II.2) the elementary probabilities in the form

$$P(a, b, a', b') = \frac{1}{16}\big(1 + ab\langle AB \rangle + a'b\langle A'B \rangle + ab'\langle AB' \rangle + a'b'\langle A'B' \rangle\big) = \frac{1}{16}(1 + \sigma), \qquad \text{(II.7a)}$$

where $\sigma \equiv (ab + a'b + ab' - a'b')/2$. Hence, we obtain

$$\begin{aligned} P(++++) &= P(+--+) = P(-++-) = P(----) \\ &= P(+++-) = P(++-+) = P(---+) \\ &= P(--+-) = \frac{1}{8}, \\ P(++--) &= P(--++) = P(+-+-) = P(-+-+) \\ &= P(-+++) = P(+-++) = P(+---) \\ &= P(-+--) = 0. \end{aligned} \quad \text{(II.8a)}$$

It follows from expression (II.7a) or (II.8a) that $\langle S\rangle = \sum \sigma P = \sum \sigma(1+\sigma)/16 = 1$; here, all the terms in the sum are positive since $\sum P = 1$ and $P \geqslant 0$. The equality $\sum \sigma P = 1$ follows from the fact that, according to expression (II.7a), the probabilities $P(a,b,a',b')$ with the arguments corresponding to $\sigma = -1$ all vanish. Consequently, the distribution described by expression (II.8a) yields the distribution $P_\Sigma(+) = 1$, $P_\Sigma(-) = 0$ for the random quantity Σ with the values $\sigma = \pm 1$, i.e. Σ is a deterministic quantity with zero variance ($\Sigma = \sigma = 1$). The vanishing of some of the elementary probabilities in expression (II.8a) has a number of consequences such as $P_{ABA'}(+-+) = P_{ABA'}(-+-) = 0$ and $P_{AB}(+-) = P_{ABA'}(+--)$, which originate from the condition $\sigma = 1$.

The quantity S observed in reality assumes, with certain probabilities, the values $s = 0, \pm 1, \pm 2$ (see Fig. 3). Our ergodic model ensures equality of only the average values of two random quantities Σ and S.

Let us now consider the EPR – Bohm quantum model described by (see Ref. [9])

$$\langle AA'\rangle = \cos(\alpha - \alpha'), \quad \langle AB\rangle = \cos(\alpha + \beta),$$
$$\langle ABA'\rangle = 0, \quad \langle ABA'B'\rangle = \cos(\alpha + \beta - \alpha' - \beta'). \quad \text{(II.4b)}$$

Instead of expression (II.5a), we now have

$$\begin{aligned} P(a,b,a') &= \frac{1}{8}\big[1 + aa'\cos(\alpha - \alpha') \\ &\quad + ab\cos(\alpha+\beta) + a'b\cos(\alpha'+\beta)\big] \quad \text{(II.5b)} \\ &= \frac{1}{8}\big[1 + aa'\cos(x-y) + ab\cos(x) + a'b\cos(y)\big], \end{aligned}$$

where $x = \alpha + \beta$, $y = \alpha' + \beta$. Two components $P(a,b,a')$ out of eight are negative for nearly all values of x and y [the exceptions are the subsets of the $(x,0)$, $(0,y)$, and (x,x) types]. Since $P(a,b,a') = P(a,b,a',+) + P(a,b,a',-)$, it follows that some of the components of the four-dimensional distribution $P(a,b,a',b')$ are negative again for almost all the values of the parameters α, β, α', β'.

Let us now assume that $\alpha = 0$, $\alpha' = \pi/2$, $\beta = -\pi/4$, in accordance with expression (3). We then have

$$\begin{aligned} P(+++) &= P(---) = \frac{1}{8}(1+\sqrt{2}), \\ P(-+-) &= P(+-+) = \frac{1}{8}(1-\sqrt{2}), \quad \text{(II.6b)} \\ P(-++) &= P(+--) = P(++-) = P(--+) = \frac{1}{8}. \end{aligned}$$

We shall now use the three-dimensional (quasi)-probabilities $P_{ABA'}(a,b,a') \equiv P(a,b,a')$ to form the following combination of two-dimensional probabilities: $P_{AA'}(++) + P_{BA'}(+-) - P_{AB}(++)$. The probability addition rule shows that this combination is equal to $P(-+-) + P(+-+)$; according to expression (II.6b), this quantity is negative for $\alpha = 0$, $\alpha' = \pi/2$, $\beta = -\pi/4$.

If the phases are arbitrary, then

$$\begin{aligned} &P_{AA'}(++) + P_{BA'}(+-) - P_{AB}(++) \\ &\quad = \frac{1}{2}\left[\sin^2\left(\frac{\alpha+\beta}{2}\right) + \sin^2\left(\frac{\alpha'+\beta}{2}\right) - \sin^2\left(\frac{\alpha-\alpha'}{2}\right)\right]. \end{aligned}$$

The above expression assumes the minimum value $-1/8$ for $\alpha + \beta = \pi/3$, $\alpha' + \beta = -\pi/3$.

On the other hand, in classical theory we always have $P(-+-) + P(+-+) \geqslant 0$ and, therefore, $P_{AA'}(++) + P_{BA'}(+-) \geqslant P_{AB}(++)$. One of the initial Bell's inequalities is of similar form [3, 7, 12]. We can see that violation of this inequality in quantum models is due to the negative nature of the three-dimensional 'probabilities' $P(-+-)$, $P(+-+)$.

Similarly, in the case when $N = 4$, we find that, instead of expression (II.7a), we now have

$$\begin{aligned} P(a,b,a',b') &= 2^{-4}\big[1 + aa'\cos(\alpha-\alpha') + bb'\cos(\beta-\beta') \\ &\quad + ab\cos(\alpha+\beta) + a'b\cos(\alpha'+\beta) + ab'\cos(\alpha+\beta') \\ &\quad + a'b'\cos(\alpha'+\beta') + aa'bb'\cos(\alpha+\beta-\alpha'-\beta')\big]. \quad \text{(II.7b)} \end{aligned}$$

Let us assume that $\alpha' - \alpha = \beta' - \beta = \pi/2$, $\alpha + \beta \equiv \gamma_1$. We then obtain

$$\begin{aligned} P(a,b,a',b') &= \frac{1}{16}\big[1 + (ab - a'b')\cos\gamma_1 \\ &\quad - (a'b + ab')\sin\gamma_1 - aba'b'\big]. \end{aligned}$$

For some values of a, b, a', and b', the above expression has negative values for all γ_1 (apart from 0 and π). Let us assume that, for example, in accordance with expression (3), we have $\gamma_1 = -\pi/4$. The result is then

$$P(a,b,a',b') = \frac{1}{16}\big[1 + \sqrt{2}\,\sigma - aa'bb'\big],$$

where $\sigma = (ab + a'b + ab' - a'b')/2$. Some of the 'probabilities' are then negative:

$$\begin{aligned} P(++++) &= P(+--+) = P(-++-) \\ &= P(----) = \frac{\sqrt{2}}{16}, \\ P(++--) &= P(--++) = P(+-+-) \\ &= P(-+-+) = -\frac{\sqrt{2}}{16}, \\ P(+++-) &= P(++-+) = P(---+) \\ &= P(--+-) = \frac{2+\sqrt{2}}{16}, \\ P(-+++) &= P(+-++) = P(+---) \\ &= P(-+--) = \frac{2-\sqrt{2}}{16}. \end{aligned} \quad \text{(II.8b)}$$

The above expressions agree with the set of expressions (II.6b): for example, $P(+-++) + P(+-+-) = P(+-+) = (1-\sqrt{2})/8$.

It follows from the set of expressions (II.8b) that $\langle S\rangle = \sum \sigma P = \sum \sigma[1 + \sqrt{2}\,\sigma - aa'bb']/16 = \sqrt{2}$, i.e. that Bell's inequality $|\langle S\rangle| \leqslant 1$ is violated. Formally, this can be regarded as the result that not all the elementary 'probabilities' P in the sum $\sum \sigma P$ are positive; this is shown also in Ref. [23] where formulas (II.5b)–(II.8b) are obtained by a different method. It should be pointed out that the reverse is not true: the existence of the negative components $P(a, b, a', b')$ does not always lead to the inequality $|\langle S\rangle| > 1$.

We have ignored here the ambiguities in the selection of the fourth moment associated with the noncommutative nature of the operators and leading to a multivalued solution of the inverse problem represented by expression (II.2). Let us transpose A and A' in the moment $\langle ABA'B'\rangle$. Then, $\cos(\alpha + \beta - \alpha' - \beta')$ in expressions (II.4b) and (II.7b) is replaced with $\cos(\alpha' + \beta - \alpha - \beta')$, and the probabilities change. For example, if $\alpha = 0$, $\beta = -\pi/4$, $\alpha' = \pi/2$, $\beta' = \pi/4$ then $P(a, b, a', b') = 2^{-4}[1 + \sqrt{2}\,\sigma + aa'bb']$. As a result, the first and third expressions, as well as the second and fourth, are interchanged on the right of the set (II.8b). One further set of probabilities is obtained if symmetrised moments, analogous to the Wigner distribution, are used. However, this ambiguity does not affect the average values observed in reality.

Therefore, although in quantum models the moments such as $\langle ABA'\rangle$, $\langle ABA'B'\rangle, \ldots$, composed of noncommuting operators and therefore without operational meaning, can be calculated formally, the corresponding — according to classical formula (II.2) — elementary 'probabilities' can be negative and unambiguously defined, so that they have no physical meaning.

We shall now find the distribution of the probabilities $P_S(s)$ for the composite random quantity S, which is measured in the process of verification of Bell's inequalities [the definition is given by expression (3) and illustrated in Fig. 4]. In the classical case, this quantity is again defined fully by the elementary probabilities $P(a, b, a', b')$. However, we at once can take into account that this quantity is measured in the course of independent repeated tests and, therefore, it can be described by the binomial distribution. Let p be the probability that in one test the product AB is positive and let q be the probability that AB is negative. It follows from the set of expressions (2) that

$$p(\gamma) = P_{AB}(++) + P_{AB}(--) = \frac{1}{2}\left[1 + M(|\gamma|)\right],$$
$$q(\gamma) = P_{AB}(+-) + P_{AB}(-+) = \frac{1}{2}\left[1 - M(|\gamma|)\right]. \qquad \text{(II.9)}$$

According to expression (3), the parameter $|\gamma| = \pi/4$ is identical in three series of observations, but in one series it is $|\gamma| = 3\pi/4$. In the last case the product ab is taken with its sign reversed and, therefore, the roles of p and q are interchanged. Since, in accordance with the set of expressions (1), we have $M(3\pi/4) = M(\pi/4)$, we can assume that in all four series of observations the probability $p \equiv p(\pi/4)$ is the same. Consequently, the quantity S assumes the values $s = (2, 1, 0, -1, -2)$ with the binomial probabilities

$$P_S(s) = (p^4,\ 4p^3q,\ 6p^2q^2,\ 4pq^3,\ q^4). \qquad \text{(II.10)}$$

According to expressions (1a) and (II.9), in the classical model we have $p = 0.75$ and $q = 0.25$, which gives

$$P_S(s) = (0.316;\ 0.422;\ 0.211;\ 0.047;\ 0.004). \qquad \text{(II.11)}$$

These values are confirmed by the results of a numerical experiment (see Fig. 4). Hence, we again find that

$$\langle S\rangle_c = \sum_{s=-2}^{2} sP_S(s) = 1.$$

In the quantum case, it follows from expressions (1b) and (II.9) that $p = \cos^2(\pi/8) = 0.853$, $q = \sin^2(\pi/8) = 0.146$, so that all the probabilities apart from $P_S(2)$ are smaller than the corresponding classical values:

$$P_S(s) = (0.531;\ 0.364;\ 0.093;\ 0.011;\ 0.0005). \qquad \text{(II.12)}$$

Hence, we have $\langle S\rangle_q = \sqrt{2}$.

Let us finally consider the case of three observers (Section 4). The quantum model predicts the following operator identities [9]:

$$F_1 \equiv A'BC = -I, \quad F_2 \equiv AB'C = -I,$$
$$F_3 \equiv ABC' = -I, \quad F_4 \equiv A'B'C' = I. \qquad \text{(II.13)}$$

Hence, we have

$$Z \equiv F_1F_2F_3F_4 = -I. \qquad \text{(II.14)}$$

Here, $A^2 = A'^2 = I$, $AB = BA$, $F_1F_2 = F_2F_1$, etc. Therefore, the operator Z can be represented in the form $Z = AA'AA'B'BBB'CCC'C' = I + [A, A']\,AA'$. If we now ignore the noncommutative nature of the operators A and A', we find that $Z = I$, which is in accordance with classical expectations. Therefore, the GHSZ paradox [17], like the Kochen–Specker paradox [24], is formally related to the noncommutative nature of the algebra of observables (for details, see Ref. [9]).

We shall show that some of the probabilities corresponding — in accordance with rule (II.2) — to relationships (II.13) are negative. The set of expressions (II.13) yields the moments

$$\langle A'BC\rangle = -1, \quad \langle AB'C\rangle = -1,$$
$$\langle ABC'\rangle = -1, \quad \langle A'B'C'\rangle = 1. \qquad \text{(II.15)}$$

Substitution of these values in expression (II.2) gives, by analogy to expression (II.7b), the following quasiprobabilities:

$$P(a, b, c, a', b', c') = 2^{-6}[1 - a'bc - ab'c - abc' + a'b'c'] \equiv 2^{-6}[1 - 2\sigma], \qquad \text{(II.16)}$$

where we now have $\sigma \equiv (a'bc + ab'c + abc' - a'b'c')/2$. We can readily show that $\sigma = \pm 1$, so that for all the combinations of the numbers a, b, c, a', b', c' that give $\sigma = 1$ and $\sigma = -1$, we have, respectively, $P = -1/64$ and $P = 3/64$. The number of these combinations is 32, so that $\sum P = 1$.

By analogy with expression (3), we shall now consider a composite observable

$$S_3 \equiv \frac{1}{2}\left(A'BC + AB'C + ABC' - A'B'C'\right). \qquad \text{(II.17)}$$

In the classical case the absolute average value of this observable does not exceed 1 [compare with expression (I.2)]. The resultant inequality

$$|\langle S_3\rangle_c| \equiv \frac{1}{2}\left\langle A'BC + AB'C + ABC' - A'B'C'\right\rangle \leqslant 1 \qquad \text{(II.18)}$$

is an analogue of the usual Bell's inequality (4a) for three observers [20]. It is remarkable that all four observations are now carried out under the conditions of perfect (100%) correlation or anticorrelation. In the quantum model the above inequality is violated 'by 100%': according to expressions (II.15) or (II.16), we have $|\langle S_3\rangle_q| = 2$. Further iteration gives $\langle S_N\rangle_q = 2^{(N-1)/2}$ [18] (see also Ref. [16]).

III. Description of the model of an EPR simulator

The EPR simulator, shown in Fig. 7, contains (like the system in Fig. 1) a transmitter source S and two receivers A and B with discretely controlled delay lines (DL) and LED indicators (I) of the correlation between the signals, one of which is red (R) and the other green (G). The transmitter contains two double-loop parametric oscillators (PO1 and PO2), operating in a nondegenerate manner with a shared pump oscillator (PP) tuned to the frequency $\omega_0 = 3\text{MHz}$. The frequencies $\omega_{a1,2}$ and $\omega_{b1,2}$ of combination parametric oscillations at the PO outputs are close to nonmultiple frequencies of the order of 1 and 2 MHz (the letter subscripts identify nonsynchronous signals with similar frequencies and the number subscripts identify the parametric oscillators which generate the signals).

A continuous pump voltage free of random modulation in time is used. As pointed out already, this is not of fundamental importance in simulation of the EPR correlations, because the stochasticity of the signals is determined by free fluctuations of their phases. In fact, only the sum of the phases of the combination oscillations is fixed. For each PO, this sum is equal to the phase of pump oscillator: $\phi_{1,2} + \phi'_{1,2} = \phi_0 = 0$. Therefore, instantaneous values of the difference between the phases of two oscillations with similar frequencies are always identical and opposite in sign: $\phi' = -\phi$, i.e. they are anticorrelated (see, for example, Ref. [27]). This is demonstrated in Fig. 8 which shows clearly that a change in the phase of any of the signals ($\phi_{1,2}$, $\phi'_{1,2}$, where the subscript identifies the parametric oscillator) can be used to control their correlations.

The input devices in the receivers D act as comparators and transform harmonic signals into pulses of logic levels with an off-duty factor 2, i.e. they transform them into a symmetric telegraphic signal. This removes the undesirable amplitude modulation, but retains fully all the phase information. The output signals obtained from each of the four comparators have the same energy characteristics. If the logic level 1 is attributed the value +1 and the logic level 0 is attributed the value −1, the transformation of the received signals in the comparators can be represented in the following form in the case of the receiver *A*:

$$X_{1,2}(t) = \cos\left[\omega_a t + \phi_{1,2}(t)\right] \to \operatorname{sign}\left[X(t)\right] = \operatorname{sign}\left\{\cos\left[\omega_a t + \phi_{1,2}(t)\right]\right\}, \qquad \text{(III.1a)}$$

whereas for the receiver *B*, we have

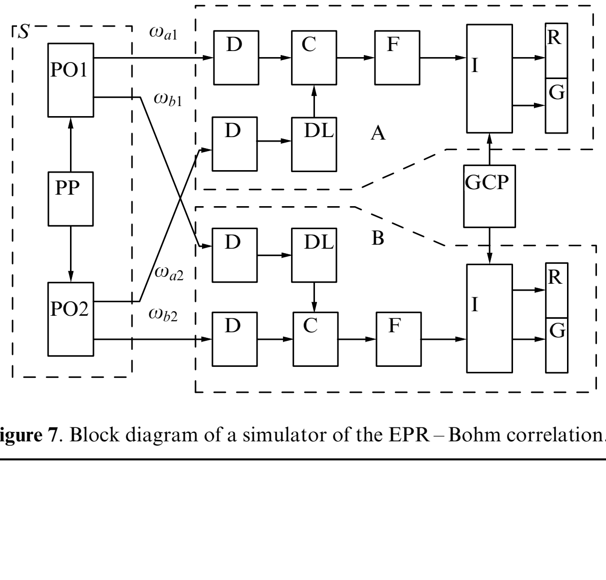

Figure 7. Block diagram of a simulator of the EPR – Bohm correlation.

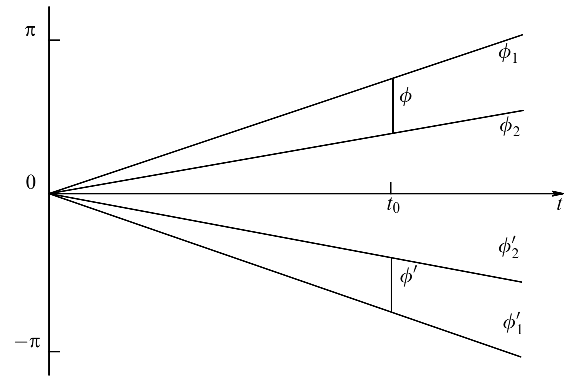

Figure 8. Free fluctuations of the phases of asynchronous signals from parametric oscillators.

$$Y_{1,2}(t) = \cos\left[\omega_b t - \phi_{1,2}(t)\right] \to \operatorname{sign}\left[Y(t)\right] = \operatorname{sign}\left\{\cos\left[\omega_b t - \phi_{1,2}(t)\right]\right\}. \qquad \text{(III.1b)}$$

The use of logic level signals makes it possible to employ discrete delay lines (DL) which are characterised by sufficiently high precision and are based on digital logic elements. This ensures uniformity of the discrete phase delay step. In our case the step is 0.16 rad. The delay lines are connected to the outputs of the signal comparators in each receiver in order to introduce controlled phase shifts α and β into these signals.

Multiplication of the logic signals in the receivers is carried out in sign correlators (C), which operate as anticoincidence ('exclusive-OR') circuits. The sum and difference frequencies are obtained from the correlator outputs. In the case of the receiver fa, the output is

$$\begin{aligned}
&\operatorname{sign}\left[\cos\left(\omega_a t + \phi_1(t) + \alpha\right)\right] \times \operatorname{sign}\left[\cos\left(\omega_a t + \phi_2(t)\right)\right] \\
&= \operatorname{sign}\left[\cos\left(\omega_a t + \phi_1(t) + \alpha\right) \times \cos\left(\omega_a t + \phi_2(t)\right)\right] \\
&= \operatorname{sign}\left[\cos\left(\phi_1 - \phi_2 + \alpha\right) + \cos\left(2\omega_a t + \phi_1 + \phi_2 + \alpha\right)\right] \\
&= \operatorname{sign}\left[\cos\left(\phi(t) + \alpha\right) + \cos\left(2\omega_a t + \phi_1 + \phi_2 + \alpha\right)\right],
\end{aligned} \qquad \text{(III.2a)}$$

and for the receiver *B*, it is

$$\begin{aligned}
&\operatorname{sign}\left[\cos\left(\omega_b t - \phi_1(t) + \beta\right)\right] \times \operatorname{sign}\left[\cos\left(\omega_b t - \phi_2(t)\right)\right] \\
&= \operatorname{sign}\left[\cos(\phi_2 - \phi_1 + \beta) + \cos\left(2\omega_b t - \phi_1 - \phi_2 + \beta\right)\right] \\
&= \operatorname{sign}\left[\cos\left(-\phi(t) + \beta\right) + \cos\left(2\omega_b t - \phi_1 - \phi_2 + \beta\right)\right].
\end{aligned} \qquad \text{(III.2b)}$$

The signals of the difference frequencies are separated in each receiver by low-frequency filters (F). The outputs from the filters are sawtooth signals of the type

$$\left\langle \operatorname{sign}\left[\cos\left(\phi + \alpha\right) + \cos\left(2\omega_a t + \phi_1 + \phi_2 + \alpha\right)\right]\right\rangle = 1 - 2\,\frac{|\phi + \alpha|}{\pi},$$

$$\left\langle \operatorname{sign}\left[\cos\left(-\phi + \beta\right) + \cos\left(2\omega_b t - \phi_1 - \phi_2 + \beta\right)\right]\right\rangle = 1 - 2\,\frac{|-\phi + \beta|}{\pi}, \qquad \text{(III.3)}$$

where $(\phi + \alpha)$, $(-\phi + \beta)$ assumes the values from $-\pi$ to $+\pi$.

In the LED indicator circuits (I) of each receiver, which is controlled by a shared generator of clock pulses (GCP) repeated at a frequency of 1 Hz, the signals described by the set of expressions (III.3) are transformed into rectangular signals at the 0 level and the results are similar to those described by expressions (7a) and (7b):

$$\operatorname{sign}\left[1-2\,\frac{|\phi+\alpha|}{\pi}\right]=\operatorname{sign}\left[\cos(\phi+\alpha)\right]=\begin{cases}+1, & -\dfrac{\pi}{2}<(\phi+\alpha)<+\dfrac{\pi}{2},\\ 0, & (\phi+\alpha)=\pm\dfrac{\pi}{2},\\ -1, & +\dfrac{\pi}{2}<(\phi+\alpha)<+\dfrac{3\pi}{2},\end{cases} \tag{III.4a}$$

$$\operatorname{sign}\left[1-2\,\frac{|-\phi+\beta|}{\pi}\right]=\operatorname{sign}\left[\cos(-\phi+\beta)\right]=\begin{cases}+1, & -\dfrac{\pi}{2}<(-\phi+\beta)<+\dfrac{\pi}{2},\\ 0, & (-\phi+\beta)=\pm\dfrac{\pi}{2},\\ -1, & +\dfrac{\pi}{2}<(-\phi+\beta)<+\dfrac{3\pi}{2}.\end{cases} \tag{III.4b}$$

The value $+1$ corresponds to illumination of the green indicator and the value -1 means that the red indicator is lit. Therefore, in recording the instantaneous phase difference between the signals in the interval $[-\pi/2,\ +\pi/2]$, a green LED lights up in each of the receivers, but when the instantaneous phase difference is in the interval $[+\pi/2,\ +3\pi/2]$, a red LED lights up.

The sign of the difference between the phases (or the cosine of this difference) is determined from the front of a clock pulse produced by the generator (GCP) simultaneously in each receiver and this is done during a clock pulse in 0.5 s.

In statistical measurements the clock pulse frequency is 10 kHz. The size of the sample represented by one experimental point is 10^4. The measurements are performed not by visual detection of lighting up of the indicators, but with a frequency meter of the Ch3-54 type. The experimental results obtained in these measurements are plotted in Fig. 2. The absence of 100% correlations is explained by natural thermodynamic fluctuations, particularly fluctuations of the reference voltages of the comparators, which cause fluctuations at the fronts of the signals being compared. In the case of statistical measurements of uncorrelated signals the errors due to these fluctuations are averaged out and they affect the results most in the case of perfect (100%) correlation and anticorrelation. It is evident from Fig. 2 that in our experiments the maximum measurement error associated with these fluctuations is 8% at the points of perfect (anti)correlation.

References

1. Einstein A, Podolsky B, Rosen N *Phys. Rev.* **47** 777 (1935); Fock V A, Einstein A, Podolsky B, Rosen N, Bohr N *Usp. Fiz. Nauk* **16** 436 (1936) [in Russian]
2. Bohm D *Quantum Theory* (Englewood Cliffs, New York: Prentice-Hall, 1951)
3. Bell J S *Physics* (Long Island city, New York) **1** 195 (1964)
4. Clauser J F et al. *Phys. Rev. Lett.* **23** (15) 880 (1969)
5. Clauser J F, Horne M A *Phys. Rev. D* **10** (2) 526 (1974)
6. Clauser J F, Shimony A *Rep. Prog. Phys.* **41** 1881 (1978)
7. Grib A A *Usp. Fiz. Nauk* **142** 619 (1984) [*Sov. Phys. Usp.* **27** 284 (1984)]
8. Home D, Selleri F *Riv. Nuovo Cimento* **14** (9) 1 (1991)
9. Belinskiĭ A V, Klyshko D N *Usp. Fiz. Nauk* **163** (8) 1 (1993) [*Phys. Usp.* **36** 653 (1993)]
10. Zukowski M *Laser Phys.* **4** 690 (1994)
11. Peres A *Am. J. Phys.* **46** 745 (1978)
12. d'Espagnat B *Sci. Am.* **241** (5) 128 (1979)
13. Shimony A *Sci. Am.* **258** (1) 36 (1988)
14. Mermin N D *Am. J. Phys.* **49** 940 (1981); **58** 731 (1990); **62** 880 (1994)
15. Mizrahi S S, Moussa M H Y *Int. J. Mod. Phys. B* **7** 1321 (1993)
16. Belinsky A V, Klyshko D N *Phys. Lett. A* **176** 415 (1993)
17. Greenberger D M et al. *Am. J. Phys.* **58** 1131 (1990)
18. Mermin N D *Phys. Rev. Lett.* **65** 1838 (1990)
19. Klyshko D N *Phys. Lett. A* **172** 399 (1993)
20. Hardy L *Phys. Lett. A* **160** 1 (1991)
21. Wodkiewicz K *Phys. Lett. A* **129** 1 (1988)
22. Agarwal G S, Home D, Schleich W *Phys. Lett. A* **170** 359 (1992)
23. Belinskiĭ A V *Usp. Fiz. Nauk* **164** 435 (1994) [*Phys. Usp.* **37** 413 (1994)]; *Pis'ma Zh. Eksp. Teor. Fiz.* **59** 278 (1994) [*JETP Lett.* **59** 301 (1994)]
24. Kochen S, Specker E P *J. Math. Med.* **17** (1) 59 (1967)
25. Mermin N D *Rev. Mod. Phys.* **65** 803 (1993)
26. Klyshko D N *Usp. Fiz. Nauk* **164** 1187 (1994) [*Phys. Usp.* **37** 1097 (1994)]
27. Kaplan A E, Kravtsov Yu A, Rylov V A *Parametricheskie Generatory i Deliteli Chastoty* (Parametric Oscillators and Frequency Dividers) (Moscow: Sovetskoe Radio, 1966)