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Coulomb disintegration of weak electron fluxes and the photocounts

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Contents

Abstract. Formation of electron bunches in the interelectrode space in vacuum detectors of optical radiation is discussed. Such bunches give rise to pulses of the electric current, usually interpreted as photocounts, in the external circuit. It is shown that the traditional theory of photocounts is inconsistent and, in particular, it violates causality. Calculations based on the variational method are used to show that a distributed low-density electron cloud is unstable in the presence of the Coulomb forces and that it splits into bunches. The electron bunches moving in the interelectrode space experience peaking, which is easiest to understand on the basis of the catastrophe theory. Spatial (caustics) and temporal (overtaking) catastrophes may occur in an electron flux. Numerical simulation is used to consider spherical and linear expansion of electron bunches under the action of the Coulomb forces. It is shown that sharp electron density maxima are formed and that their properties resemble those of point-like particles capable of inducing electric current peaks (photocounts) in the external circuit of a detector when they travel across the interelectrode space. Circumstances leading to a higher probability of formation of one-electron bunches are pointed out. The analysis as a whole is intended to help the understanding of the discrete nature of photocounts when a photocathode is excited by a continuous high-energy laser radiation train.

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1. Introduction

The situation in the theory of photocounts seems to be fully satisfactory. Experiments are carried out, they are analysed theoretically, and there are no contradictions between the theory and experiment. However, from the logical point of view the existing ideas on photocounts are far from perfect. We shall consider mainly the physical processes that occur in vacuum photodetectors (photocells, photomultipliers, image converters, etc.) and attempt to analyse the logical contradictions inherent in the existing theory of photocounts. We shall propose a physical mechanism which can explain the appearance of discrete photocounts when a photocathode is excited by a continuous highly coherent laser radiation train.

The existing theory of photocounts is largely phenomenological, i.e. it is not a theory that follows from fundamental equations of physics (Maxwell, Schrodinger, or Dirac equations). For example, the discrete nature of photocounts does not follow from the theory, but is postulated: some theoretically continuous quantities are interpreted as the average values of quantities describing discrete fluxes and the concept of discrete photon and electron fluxes dates back to the founders of quantum theory. In the case of the photoelectric effect the discrete nature is expressed in the postulate that the knocking out of one electron from a photo-cathode requires one quantum of light, i.e. a photon. Both an electron and a photon are regarded as certain localised objects travelling in, respectively, electron and electromagnetic radiation fluxes. In the photoelectric effect one particle (photon) is absorbed and another (electron) is emitted.

In the theory of quantised electromagnetic and electron fields there is not even a hint of the existence of such localised objects. The absence of these localised objects is particularly clear in highly coherent laser radiation. A high-quality laser source can emit a continuous train of laser radiation lasting seconds, i.e. the length of a train is

$\approx 3 \times 10^5$ km and it contains $\approx 10^{15}$ perfectly identical waves. In this time a detector records a large number of photocounts.

However, the concepts of localised photons and electrons are used widely and in the laboratory jargon these concepts are called the 'theory of peas', i.e. localised objects resemble peas.

This point of view is particularly prominent in the description of the shot noise (which is not typical of photodetectors alone), put forward by Schottky [1, 2]. The photo-electric effect and the shot noise have by now been investigated extensively and they continue to be studied on the basis of this approach [3–5].

However, from the modern point of view, this approach suffers from some inconsistency. For example, the process of photon absorption and the appearance of photoelectrons inside the cathode is considered quantum-mechanically. It is usual to assume that a quantised plane optical wave interacts with a quantised plane electron wave, which belongs to the valence band. The result of this interaction is a plane electron wave in the conduction band. This new conduction-band wave is scattered by the interface between the cathode material and the vacuum. The amplitude of the electron wave which passes into the vacuum determines the probability of the appearance of photoelectrons near the cathode. The inconsistency of the theory lies in the termination, at this stage, of the quantum-mechanical description of the process. The subsequent evolution of electrons is considered classically [6] and electrons are regarded as particles and not waves.

This lack of consistency of the theory is frequently attributed to fundamental features of quantum-mechanical measurements. A quantum-mechanical object is always observed with the aid of a macroscopic instrument which has the properties of a classical object. It is therefore essential to draw a boundary between the observed object and the instrument. After crossing this boundary in the direction from the object to the instrument all the phenomena should be considered from the classical point of view. This is why a photoelectron knocked out from a cathode and then moving towards an anode is usually regarded as a classical point particle. In other words, the boundary between the investigated object and the instrument is then located on the cathode surface.

This approach cannot generally be objected to or overturned logically, but its validity is not as wide as is usually assumed. In fact, this approach can be used only if the observed scenario is independent of the position of the boundary between the quantum-mechanical (object) and classical (instrument) parts of the whole observation–measurement system. This is an imaginary boundary and the results of measurements should be independent of its position! If we now go back to photocounts, we can — for example — ask whether the results of a theoretical analysis of experiments on the photoelectric effect will change if the boundary between the quantum-mechanical and the classical parts of the system is drawn near the anode and not near the cathode, contrary to the usual approach. Therefore, an additional study is needed to consider the problem of such interpretation of photocounts.

It would have been more consistent to continue to apply quantum mechanics to the motion of electrons between the cathode and anode and to find the field and the current induced by an electron wave in the external circuit of a photodetector. However, if this more consistent approach is adopted, the theory does not predict photocount pulses, but more or less smooth solutions for the fields and currents induced in the circuit, depending on time to the same extent as the amplitude of the incident optical wave. The absence of short pulses in this theoretically more consistent approach is of course in conflict with experimental observations, i.e. with the existence of photocounts. This conflict is particularly severe when we consider laser light sources with longer coherence times, extending up to seconds. Naturally, an electron flux in a photodetector should also have a smooth amplitude over time intervals comparable with the coherence time. However, the photoelectric effect shows no special features when laser sources are used. The random nature of light emitted by thermal and luminescence sources, used before the advent of lasers, could mask this conflict. Therefore, the existence of photocounts, at least in the case of laser light, has to be explained.

Our purpose is to consider a physical mechanism for the appearance of photocounts different from that presented in the published literature in which a localised electron is emitted by a cathode after the absorption of a localised photon. The proposed mechanism is based on the idea that an electron flux emitted by a cathode under the action of, specifically, laser light is initially, i.e. immediately after leaving the cathode, a plane wave or some entity close to a plane wave (for example, an electron cloud). However, this flux is unstable and it tends to disintegrate into bunches under the action of the interelectrode Coulomb field. This instability is readily understood if we bear in mind the well-known Wigner crystallisation process, i.e. the decay of an electron plasma in a solid into bunches when the plasma density is low [7, 8]. Therefore, an electron flux emerging from a cathode should split near the cathode into separate bunches ('electrons') which are subsequently accelerated in the interelectrode electric field and give rise to current pulses in the electric circuit of the investigated detector. These pulses are detected by instruments and by an observer as photocounts.

The mechanism under discussion naturally obeys all the main laws governing the photoelectric effect, including the Einstein law with the prediction of the red edge of the effect, because such laws act during the first stage when an electron wave is formed in the conduction band.

We were the first to propose this mechanism [9]. The mechanism leads to certain questions and some of them are discussed below. The main questions are as follows: what are the manifestations of an instability of an electron flux, is there, and of what nature, a mechanism responsible for peaking of electron bunches, what is the mechanism of appearance of mainly one-electron bunches, and finally what physical conclusions can be drawn from this approach if it is confirmed experimentally? It is quite difficult to answer these questions. We shall show that at this stage the answers to these questions are not in conflict with the proposed approach.

It should also be pointed out that the existing theory suffers additionally from minor shortcomings. For example, some of the photocount characteristics such as the photocount recording rate (number of photocounts per second) do not satisfy the principle of causality [10–13], i.e. the photocounts are not delayed in the expected manner.

We shall begin considering this problem with a brief account of violation of causality. Then we shall use a

quantum-mechanical calculation to show that at low densities an electron cloud splits into bunches. Next, we shall investigate the process of peaking of the bunches, based on the appearance of temporal and spatial catastrophes in an electron flux. We shall carry out this investigation using classical language, outside the framework of quantum mechanics, although later we shall propose a quantum-mechanical generalisation. We shall also use the classical language to introduce the concept of transparent and nontransparent bunches. We shall conclude by considering the reason for preferential formation and observation of one-electron bunches. Here again we shall use a quantum-mechanical approach, although in general this will be of qualitative nature. Let us begin with violation of causality.

2. Violation of causality in the theory of photocounts

We can demonstrate violation of causality [14] in the theory of photocounts by considering a quantity usually regarded as the photocount recording rate or, more precisely, the probability of detection of photons at a point $\boldsymbol{r}$ during a time interval from t to $t + \mathrm{d}t$:

$$\mathrm{d}R(\boldsymbol{r},t) = \eta G(\boldsymbol{r},t)\,\mathrm{d}t\,, \tag{1}$$

where $G(\boldsymbol{r},t)$ is a correlation function described by the relationship

$$G(\boldsymbol{r},t) = \mathrm{Tr}\left[\rho E^{(+)}(\boldsymbol{r},t)\,E^{(-)}(\boldsymbol{r},t)\right], \tag{2}$$

ρ is the density matrix of the electromagnetic field,

$$E^{(-)}(\boldsymbol{r},t) = \mathrm{i}\sum_n \left(\frac{2\pi\hbar\omega_n}{V}\right)^{1/2} \alpha_n a_n \exp\left[\mathrm{i}(\boldsymbol{k}_n\cdot\boldsymbol{r} - \omega_n t)\right] \tag{3}$$

is the negative frequency part of the electric field, and α_n is the cosine of the angle between the direction of the polarisation of the mode field and the detector.

Let us assume that the field is in a coherent state described by the system of equations

$$a_n|\psi\rangle = \left(\frac{Z_n}{V^{1/2}}\right)|\psi\rangle. \tag{4}$$

Then, it follows from expression (3) that

$$E^{(-)}(\boldsymbol{r},t)|\psi\rangle = W(\boldsymbol{r},t)|\psi\rangle \tag{5}$$

where

$$W(\boldsymbol{r},t) = \mathrm{i}V^{-1}(2\pi\hbar)^{1/2}\sum_n \omega_n^{1/2}\alpha_n Z_n \exp\left[\mathrm{i}(\boldsymbol{k}_n\cdot\boldsymbol{r} - \omega_n t)\right], \tag{6}$$

or, in terms of integrals

$$W(\boldsymbol{r},t) = \frac{2\mathrm{i}(2\pi\hbar)^{1/2}}{(2\pi\hbar)^3}\int \mathrm{d}^3\boldsymbol{k}\,\alpha(\omega)\omega^{1/2}Z(\omega)\exp\left[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\right]. \tag{7}$$

The above quantity is usually called an analytic signal. The integral in the above expression contains a factor $\exp(-\mathrm{i}\omega t)$ which corresponds to $\omega < 0$. If t is a complex variable, $t = t_1 - \mathrm{i}t_2$, then expression (7) contains the factor $\exp(-\omega t_2)$ and the function $W(\boldsymbol{r},t)$ is thus an analytic function in the lower half-plane of the complex variable t. The imaginary and real parts of $W(\boldsymbol{r},t)$ are then related by

$$\mathrm{Im}\,W(\boldsymbol{r},t) = \frac{1}{\pi}\int \mathrm{d}t'\,\frac{\mathrm{Re}\,W(\boldsymbol{r},t')}{t-t'}, \tag{8}$$

i.e. by the Hilbert transformation.

It follows from expressions (2) and (5) that, for a coherent state of the field, we have

$$G(\boldsymbol{r},t) = W^*(\boldsymbol{r},t)\,W(\boldsymbol{r},t)\,, \tag{9}$$

Therefore, the photocount recording rate is

$$\frac{\mathrm{d}R(\boldsymbol{r},t)}{\mathrm{d}t} = \eta\left[(\mathrm{Re}\,W)^2 + (\mathrm{Im}\,W)^2\right]. \tag{10}$$

However, according to expressions (4) and (5), the average field is

$$\langle\psi|E(\boldsymbol{r},t)|\psi\rangle = W^*(\boldsymbol{r},t) + W(\boldsymbol{r},t) = 2\mathrm{Re}\,W(\boldsymbol{r},t)\,. \tag{11}$$

It follows from expression (5) that the function W satisfies a homogeneous wave equation. Therefore, we shall consider a plane wave travelling in the positive direction on the z axis and characterised by an abrupt front:

$$\mathrm{Re}\,W(\boldsymbol{r},t) = \theta(ct - z)\,F(z - ct)\,, \tag{12}$$

where θ is the Heaviside step function. The average value of the field itself $\langle\psi|E(\boldsymbol{r},t)|\psi\rangle$ and all of its powers $\langle\psi|{:}\,E^n(\boldsymbol{r},t)\,{:}\,|\psi\rangle$ also have abrupt fronts; normal ordering is used here to avoid an infinite contribution of the vacuum fluctuations, which are independent of the state $|\psi\rangle$.

The investigated signal reaches a detector at the moment $t = z/c$ and it vanishes at $t < z/c$, but the probability of recording photocounts, given by expression (10), does not vanish up to the moment $t = z/c$, i.e. until the signal reaches the detector. The nonzero photocount recording rate follows from the second term in expression (10). In fact, we can see from relationship (8) that $\mathrm{Im}\,W \neq 0$ for all t, even if $\mathrm{Re}\,W(\boldsymbol{r},t) = 0$ for $t < z/c$. By way of example, we shall consider a wave

$$\mathrm{Re}\,W(\boldsymbol{r},t) = \theta(ct - z)\,\theta(l + z - ct)\sin(kz - \omega t + \phi)\,. \tag{13}$$

According to relationship (8), we have

$$\mathrm{Im}\,W(z,t) = \\ = \frac{1}{\pi}\left\{\sin(\zeta+\phi)\left[\ln\left|\frac{\zeta+kl}{\zeta}\right| + \mathrm{Cin}|\zeta| - \mathrm{Cin}|\zeta+kl|\right] - \right. \\ \left. \cos(\zeta+\psi)\left[\mathrm{Si}(\zeta+kl) - \mathrm{Si}(\zeta)\right]\right\},\quad \zeta = kl - ct\,, \tag{14}$$

where

$$\mathrm{Cin}\,x = \int_0^x \mathrm{d}t\,\frac{1-\cos t}{t}, \qquad \mathrm{Si}\,x = \int_0^x \mathrm{d}t\,\frac{\sin t}{t}. \tag{15}$$

The field $\langle E\rangle$ and the function $\mathrm{Re}\,W$ are nonzero only in the interval $ct - l < z < ct$, whereas $\mathrm{Im}\,W$ is nonzero also outside this interval. Fig. 1 shows the dependence $\mathrm{d}R(\zeta = kz - \omega t)/\mathrm{d}t$ for the values $kl = \pi$, 4π and $\psi = 0$, $\pi/2$. It is evident from Fig. 1 that the photocount recording rate does not vanish outside the interval where the signal field is concentrated. This virtual recording rate increases on reduction in the pulse duration. In the case of femtosecond pulses, which contain just a few waves, the distortion of the true pattern due to an active precursor in the photocount recording rate becomes significant.

Apart from relationship (8), there is a similar relationship between the real part of W and its imaginary part. Consequently, localisation of one part of W makes the other part distributed over the whole of space. Therefore, for a coherent state of the field the correlation function described by expression (2) cannot

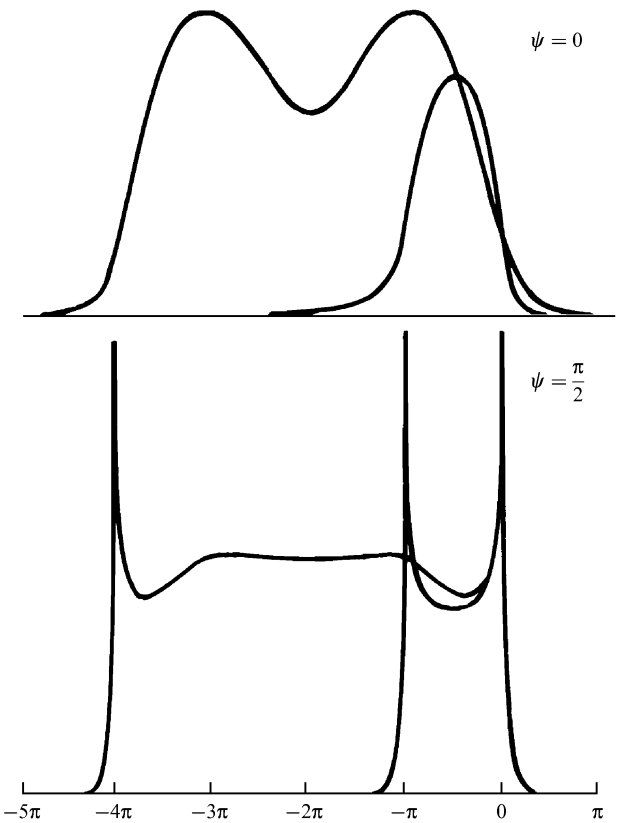

Figure 1. Time dependence of the theoretical photocount recording rate. The two upper curves represent signals without discontinuities and the two lower curves correspond to signals with discontinuities at $\zeta = 0$ and $\zeta = \pi, 4\pi$. The nonzero photocount recording rate at positive values of ζ represents a physically meaningless precursor.

be localised. Only the electric field $\langle E \rangle = 2\mathrm{Re}\, W$ and the functions of this field can be localised.

Relationship (8) and its conjugate resemble the Kramers–Kronig relationships. However, the meaning of our relationships is different. The Kramers–Kronig relationships impose limits on the spectral properties of the permittivity $\varepsilon(\omega)$ and these limits are the result of causality. On the other hand, relationship (8) and its conjugate indicate violation of causality, which is due to artificial limits imposed on the spectrum of $E^{(+)}$ and $E^{(-)}$.

This violation of causality by the correlation function described by expression (2) requires modification of the definition of the correlation functions as a whole and, in particular, a change in the definition of the photocount recording rate.

It is sometimes said that violation of causality is quantitatively slight and, consequently, it does not play a major role. However, this is not true. First, the quantitative criteria are inapplicable to such fundamental concepts as the causality. There are only two possibilities: either causality is violated and the theory is incorrect, or causality is not violated and the theory may be correct. Second, attempts have been made to improve the theory of photocounts. These attempts have been partly successful [14, 15], but the theory then predicts a photocount recording rate which depends on the properties of a photodetector, whereas the photocount recording rate is a quantity which in principle represents only the properties of the field and is independent of the detector parameters. In fact, this independence of the photocount recording rate from the detector parameters is an additional requirement which the rate has to satisfy (in addition to causality). It is not easy to satisfy both these requirements and it may even be impossible.

Having thus recalled the causal shortcomings of the traditional theory of photocounts, we shall return to the approach described above and consider the problems in a consistent manner.

3. Instability of an electron cloud at low densities

Decay of an electron cloud in two bunches at low electron densities [9] will now be illustrated by considering a simple example: two electrons in a square potential well interact with one another in accordance with the Coulomb law. Changes in the parameters of the potential well make it possible to control the electron density of such a system to find the moment when an electron cloud begins to split into bunches.

The electrons in this potential well are described by the Hamiltonian

$$H = -\frac{1}{2}\nabla_1^2 - \frac{1}{2}\nabla_2^2 + \alpha(\rho_1^2 + \rho_2^2) - \frac{1}{\rho_{12}},$$

where ∇_1^2 and ∇_2^2 are the Laplace operators for the first and second electrons, respectively; ρ_1, ρ_2, and ρ_{12} are, respectively, distances from the two electrons to the origin of the coordinate system and the distance between the electrons, all expressed in terms of the Bohr radii; the parameter α of a square potential well determines the field in this well, which acts on the electrons. Calculations of the stationary wave functions and of the corresponding energies have been carried out [16, 17] by a basis-free variational method. Fig. 2 gives the dependences of the energies of the symmetric (which does not decay into bunches) and asymmetric (which decays into two bunches) states on the parameter α, i.e. effectively only on the forces which confine these electrons. We can see that at values of the parameter $\alpha < 0.82$ the asymmetric state is preferable from

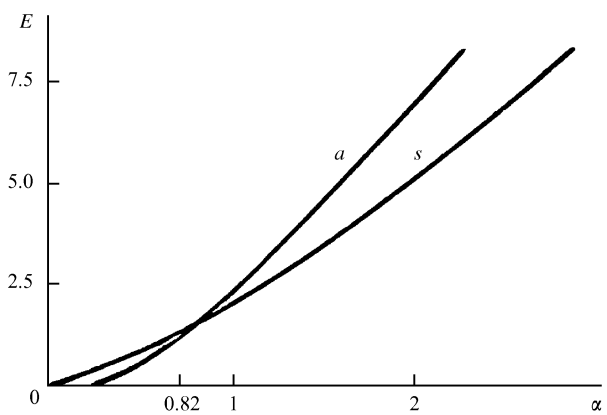

Figure 2. Energy E (in atomic units) of symmetric (s) and asymmetric (a) states, plotted as a function of the parameter α of a square potential well. In a wide potential well the asymmetric state which decays into two bunches is preferred from the energy point of view.

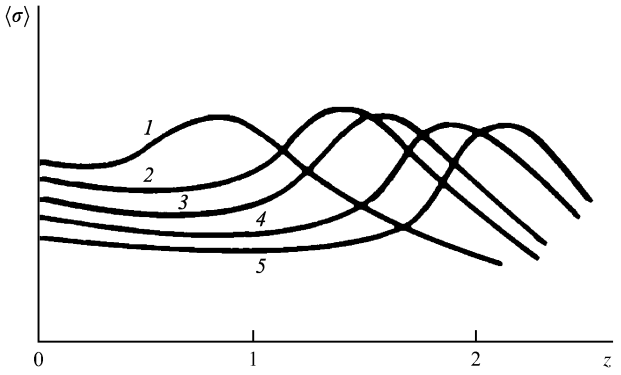

Figure 3. Distribution of the charge along the z axis for an asymmetric state, plotted for the following values of the parameter α: (*1*) 0.4; (*2*) 0.25; (*3*) 0.2; (*4*) 0.15; (*5*) 0.1. The distance between the bunches increases with the width of the potential well.

the energy point of view. The decay of an electron cloud into bunches is shown in Fig. 3, where the charge density of the asymmetric state is shown as a function of the distance along the z axis. When the forces that confine the electrons are strong, the Coulomb repulsion plays no significant role and the symmetric state is energy-preferred. As the compressive forces (i.e. the parameter α) decrease, the relative contribution of the Coulomb energy increases and, beginning from $\alpha = 0.82$, the asymmetric state — which splits into two bunches — becomes preferable.

As pointed out above, the photocount pulses can be explained by what is known as the 'pea theory' which has no mathematical basis. If the above approach is adopted, there are no discrete objects in the field and there is no need for these objects. Bunches in a low-density electron flux are formed because of its Coulomb instability. However, these bunches represent objects which are smeared out to a considerable extent. Therefore, there should be some mechanism that causes peaking of the bunches, so that they can give rise to abrupt pulses of the current in the external circuit of a detector.

4. Spatial and temporal catastrophes of an inhomogeneous electron flux

We shall show later that catastrophes occur in an electron flux and they should be considered in the same sense as in the theory of catastrophes [18–20]. Catastrophes are of interest in relation to our mechanism because as a result of them the electron density becomes infinite at some points. Thus, catastrophes are a natural mechanism that compresses electron bunches. Catastrophes can be spatial or temporal. Spatial catastrophes are well known in geometric optics. They are simply caustics and foci: they are the result of focusing of rays. The intensity of light on the caustics and at the foci tends to infinity in the geometric-optics approximation. The situation in an electron flux is fully analogous.

Temporal catastrophes represent overtaking of some electron layers by others. The electron density again tends to infinity.

4.1 Temporal catastrophes (overtaking)

We can gain an idea of how the charge density increases in a time catastrophe, i.e. as a result of overtaking, by considering two examples of expansion of electron bunches under the action of Coulomb forces: spherically symmetric and linear expansion. The spherically symmetric case is convenient because the equation of motion of an electron cloud is integrated over a time interval from the beginning of motion to the beginning of overtaking; therefore, an increase in the electron density can be investigated analytically.

4.1.1 Spherically symmetric expansion of an electron bunch. Let us assume that initially we have a spherically symmetric distribution of the charge density $\sigma(\boldsymbol{r})$ and that a bunch is at rest, i.e. the velocities at all points in the bunch are zero. Then, the electric field $E(R)$ on a sphere of radius R is

$$E(R) = \frac{Q(R)}{R^2}, \tag{16}$$

where $Q(R)$ is the total charge enclosed by this sphere:

$$Q(R) = 4\pi \int_0^R \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{r}^2\sigma(\boldsymbol{r}). \tag{17}$$

It should be pointed out that, up to the moment when overtaking begins, the quantity $Q(R)$ is a constant, i.e. $Q(R) = Q(R_0)$, if R_0 is the initial value of R. Therefore, the law of motion of charges located on a sphere of radius R is

$$m\ddot{R} = \frac{eQ(R_0)}{R^2}. \tag{18}$$

We shall also assume that initially the charge distribution is Gaussian:

$$\sigma(\boldsymbol{r}) = \frac{Q_0}{\pi^{3/2}\boldsymbol{r}_0^3}\exp\left(-\frac{\boldsymbol{r}^2}{\boldsymbol{r}_0^2}\right), \tag{19}$$

where Q_0 is the total charge in the distribution. Then, integration of the above equation gives the dependence

$$\left\{\left[\rho(\rho-\rho_0)\right]^{1/2} + \rho_0 \ln\frac{(\rho-\rho_0)^{1/2}+\rho^{1/2}}{\rho_0^{1/2}}\right\}\left(\frac{\rho_0}{J(\rho_0)}\right)^{1/2} = \tau, \tag{20}$$

where

$$\rho = \frac{R}{\boldsymbol{r}_0}; \quad \rho_0 = \frac{R_0}{\boldsymbol{r}_0}; \quad \tau = \frac{t}{t_0}; \tag{21}$$

$$t_0 = \left(\frac{m}{e}\frac{\boldsymbol{r}_0^3}{Q_0}\right)^{1/2}; \quad J(\rho_0) = \frac{4}{\pi^{1/2}}\int_0^{\rho_0}\mathrm{d}\rho\rho^2\exp(-\rho^2). \tag{22}$$

Let us now consider how the charge density $\sigma(R)$ varies with time. We note that up to the beginning of overtaking the total charge in a thin spherical layer of thickness $\mathrm{d}R$ is conserved with time. Therefore, we have

$$\sigma(R)R^2\,\mathrm{d}R = \sigma(R_0)R_0^2\,\mathrm{d}R_0$$

or

$$\sigma(R) = \sigma(R_0)\frac{R_0^2}{R^2}\frac{\mathrm{d}R_0}{\mathrm{d}R}. \tag{23}$$

It follows from the above relationship that the charge density can become infinite only if the derivative $\mathrm{d}R_0/\mathrm{d}R$ becomes infinite or, which is equivalent, if the derivative $\mathrm{d}R/\mathrm{d}R_0$ vanishes.

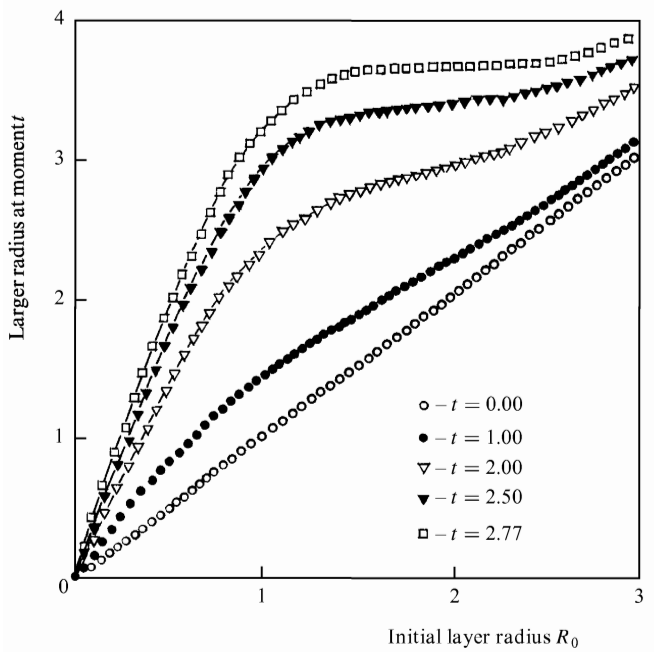

Figure 4. Spherical expansion of an electron bunch under the action of the Coulomb field. At the beginning of overtaking of some layers by others the derivative of the dependence of R on R_0, considered at a fixed value of t, vanishes at the inflection point.

We can find the derivative $\mathrm{d}R/\mathrm{d}R_0$ by differentiating dependence (20) with respect to R_0:

$$\dot{R} = \rho - \frac{1}{2}\left\{(\rho - 1) + \left(\frac{\rho - 1}{\rho}\right)^{1/2} \times \ln\left[(\rho - 1)^{1/2} + \rho^{1/2}\right]\right\} \frac{3 - R_0 Q}{Q}. \quad (24)$$

The dependence of R on R_0 was calculated on a computer and the results of this calculation are plotted in Fig. 4. It is quite clear from this figure that at the point where $\mathrm{d}R/\mathrm{d}R_0$ first vanishes, the dependence is an inflection and, consequently, the dependence can be represented in the form

$$R = B + \varepsilon(R_0 - A)^3 + \ldots, \quad (25)$$

where A and B are certain constants. Then, R_0 depends on R as follows:

$$R_0 = A + \left(\frac{R - B}{\varepsilon}\right)^{1/3}. \quad (26)$$

The derivative $\mathrm{d}R/\mathrm{d}R_0$ near this point is

$$R = 3\varepsilon(R_0 - A)^2 = 3\varepsilon^{1/2}(R - B)^{2/3}. \quad (27)$$

Therefore, the dependence of the charge density on R near the point where this density becomes infinite is of the form:

$$\sigma(R) = \frac{1}{3}\sigma(A)\varepsilon^{-1/2}\left(\frac{A}{B}\right)^2 (R - B)^{-2/3}, \quad (28)$$

It can be seen from Fig. 4 that initially the values of R and R_0 are identical. A bunch then begins to expand and the charges on the slopes of the charge distribution have the highest velocities and, consequently, are displaced by the largest distances. Consequently, the following layers should overtake the preceding layers. For example, displacements corresponding to $R_0 \approx 1.5$ are considerably greater than the displacements for $R_0 \approx 2.0$. For $t = 2.77$ the values of R corresponding to $1.5 < R_0 < 2.0$ become equalised and this implies the beginning of overtaking and an increase in the charge density until it becomes infinite. Fig. 5 demonstrates the formation of a maximum in the charge density distribution and peaking of this maximum with time near $R \approx 3.65$. For $t = 2.77$ the charge density at the maximum of the distribution is almost four orders of magnitude higher than at the centre of the distribution.

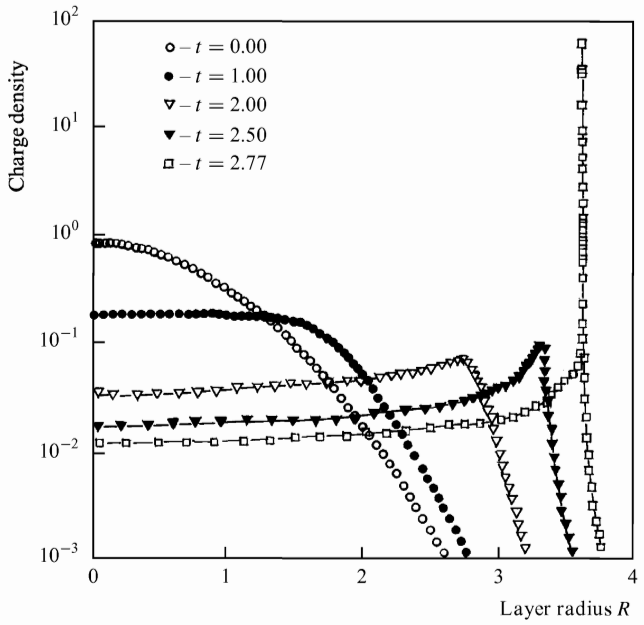

Figure 5. Formation of an electron density maximum and its peaking with time under spherical expansion conditions. The density at the maximum is almost four orders of magnitude higher than the density at the centre of a distribution.

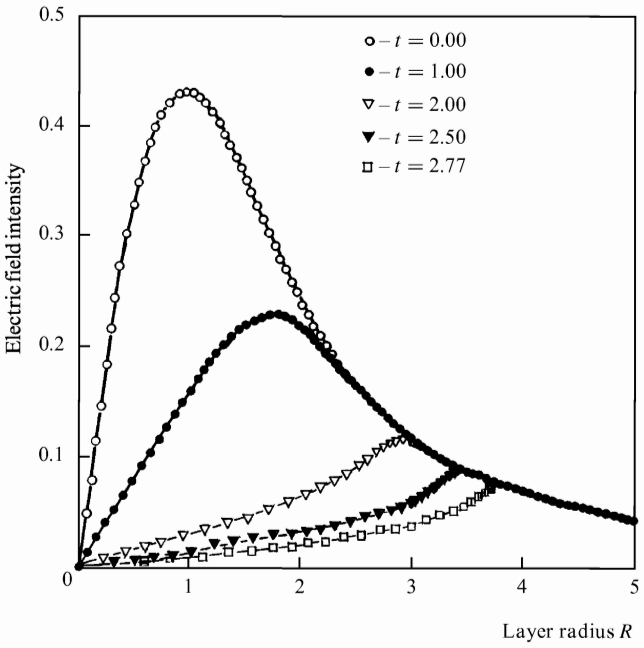

Figure 6. Jump of the electric field intensity near an electron density maximum.

Fig. 6 shows the dependence of the electric field on the coordinate R. We can see that a discontinuity of the electric field forms at the point where the charge density in the distribution becomes infinite, as expected in the region of a charged layer. Fig. 7 shows the dependence of the total charge inside a certain sphere on its radius R. It follows from this dependence that at least 20% of the total charge is concentrated in the charged layer ($3.6 < R < 3.7$).

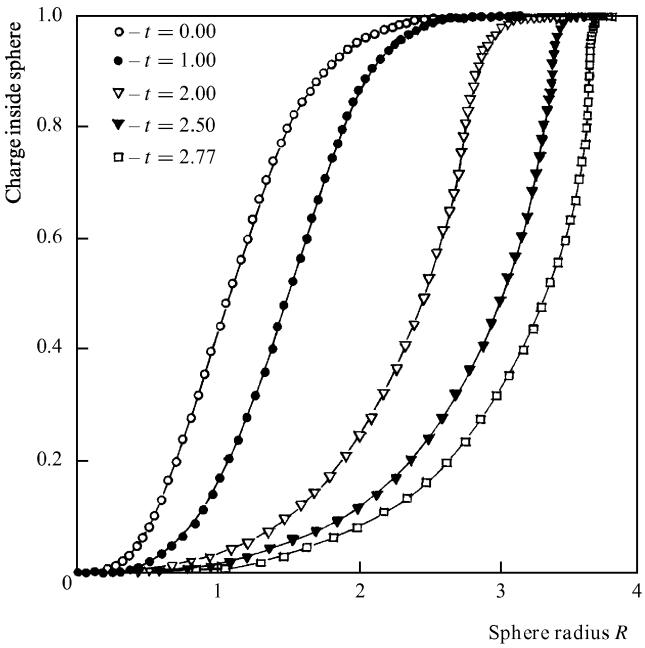

Figure 7. Dependence of the total charge inside a sphere on its radius. The layer near the maximum contains between 20% and 50% of the total charge.

4.1.2. Linear expansion of an electron bunch. Let us assume that a charge is confined by external fields acting near the z axis. The linear charge density is given by the distribution $\sigma(z)$, which for the sake of simplicity we shall regard as symmetric relative to the origin of the coordinate system. As before, it is assumed that initially the charge velocity is zero. The equations of motion of an element of charge are

$$\frac{\mathrm{d}z(z_0,t)}{\mathrm{d}t} = V(z_0,t), \qquad \frac{\mathrm{d}V(z_0,t)}{\mathrm{d}t} = a(z_0,t)\,, \tag{29}$$

where z is the coordinate of the charge element located at the point z_0; V is the velocity of this element;

$$a(z_0,t) = \frac{e}{m}\,E(z_0,t) \tag{30}$$

is the acceleration of the charge; E is the intensity of the electric field acting on the charge.

In a transverse direction the distribution is assumed to be uniform inside a circle of radius r, i.e. the whole distribution is inside a cylinder of radius r and the cylinder axis coincides with the z axis. The hypothesis of an infinitely thin distribution in a transverse direction leads to diverging expressions for the fields. The intensity of the electric field on the axis of a cylindrical charged disk is

$$E = 2\pi\rho\left[1 - \frac{l}{(l^2+r^2)^{1/2}}\right]\mathrm{d}z\,, \tag{31}$$

where r is the disk radius; ρ is the charge density in the disk; l is the distance, along the disk axis, from the disk to a point in the field; $\mathrm{d}z$ is the disk thickness. Summation of the above expression over all values of l gives the field of the charge distribution at the point z:

$$E(z) = 2\pi\left[\int_{-\infty}^{z}\mathrm{d}z'\rho(z')\left(1 - \frac{z-z'}{\left[(z-z')^2+r^2\right]^{1/2}}\right) - \int_{z}^{+\infty}\mathrm{d}z''\rho(z'')\left(1 - \frac{z''-z}{\left[(z''-z)^2+r^2\right]^{1/2}}\right)\right]. \tag{32}$$

We shall find it convenient to consider the total charge to the left of the point z. This charge is

$$Q(z) = \pi r^2\int_{-\infty}^{z}\mathrm{d}z'\rho(z'); \qquad Q(z=+\infty) = Q_0\,. \tag{33}$$

The relationship

$$\mathrm{d}Q = \pi r^2\rho(z)\,\mathrm{d}z \tag{34}$$

yields

$$E(z) = \frac{2}{r^2}\left\{\int_0^{Q}\mathrm{d}Q'\left[1 - \frac{z(Q)-z(Q')}{\left[r^2+\left(z(Q)-z(Q')\right)^2\right]^{1/2}}\right] - \int_{Q}^{Q_0}\mathrm{d}Q''\left[1 - \frac{z(Q'')-z(Q)}{\left[r^2+\left(z(Q'')-z(Q)\right)^2\right]^{1/2}}\right]\right\}. \tag{35}$$

The equations of motion (29) and (30) can now be written in the form

$$\frac{\mathrm{d}z(Q,t)}{\mathrm{d}t} = V(Q,t); \qquad \frac{\mathrm{d}V(Q,t)}{\mathrm{d}t} = a(Q,t)\,, \tag{36}$$

$$a(Q,t) = \frac{2e}{mr^2}\int_0^{Q_0}\mathrm{d}Q''W(Q,Q')\,, \tag{37}$$

where

$$W(Q,Q') = 1 - \frac{z(Q)-z(Q')}{\left\{r^2+\left[z(Q)-z(Q')\right]^2\right\}^{1/2}} \text{ for } Q' < Q\,, \tag{38}$$

and

$$W(Q,Q') = \frac{z(Q')-z(Q)}{\left\{r^2+\left[z(Q)-z(Q')\right]^2\right\}^{1/2}} - 1 \text{ for } Q' > Q\,, \tag{39}$$

where the dependence $z(Q)$ is given by relationship (33).

Before the onset of overtaking, the charge located between two sections of the distribution is conserved. Therefore, the relationship

$$\rho(z,t)\mathrm{d}z|_t = \rho(z_0,0)\mathrm{d}z_0|_{t=0}$$

or

$$\rho(z,t) = \rho(z_0,0)\left(\frac{\mathrm{d}z}{\mathrm{d}z_0}\right)^{-1}_{t=\mathrm{const}} \tag{40}$$

is obeyed. The charge density may thus become infinite if the derivative $(\mathrm{d}z/\mathrm{d}z_0)_{t=\mathrm{const}}$ vanishes.

The results of calculations of the charge density are plotted in Fig. 8. As in the spherical case, an infinite maximum of the charge distribution forms with time. However, we can now investigate the dependence of the pattern of formation of bunches on the diameter of the charge distribution ($2r$). Fig. 9 shows such patterns for $r = 0.05$, 0.15, and 0.5. We can easily see that the

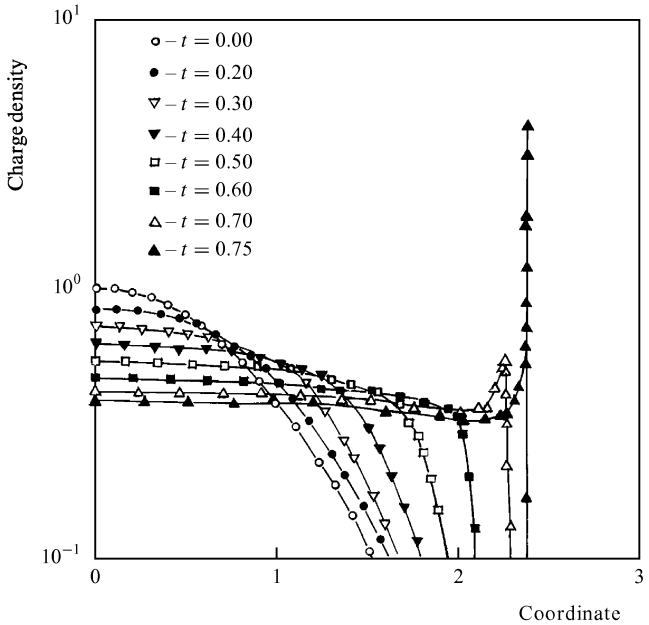

Figure 8. Formation of an electron density maximum and its peaking under linear expansion conditions.

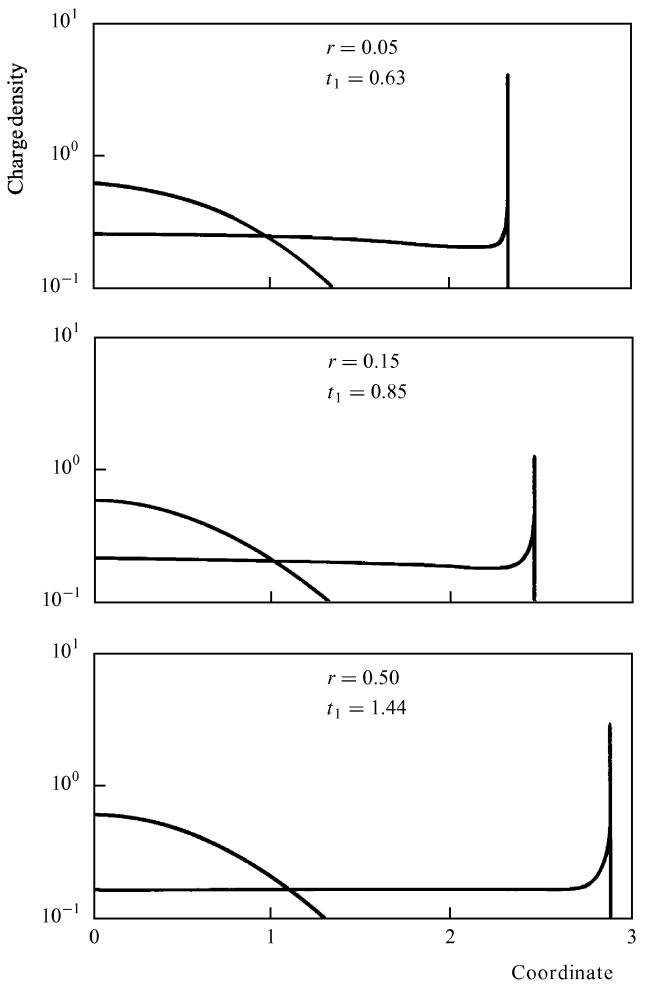

Fig. 9. Formation of electron density maxima shown for different dimensions of the transverse distribution.

formation of bunches occurs earlier for thinner distributions in a transverse direction, i.e. for higher charge densities in the distribution. This enhances the importance of spatial focusing discussed above.

4.2. Spatial catastrophes (focusing)

The spatial catastrophes associated with focusing are well known from, for example, geometric optics [21]. We shall therefore discuss only qualitatively the role of focusing in compression of electron bunches.

As a rule, the motion of an electron cloud in the cathode–anode space occurs in a focusing static field. The Coulomb field of the electron distribution itself can also be sometimes focusing. Focusing can give rise to spatial catastrophes of two types: foci and caustics. At the foci the density of the charge distribution (considered ignoring the wave nature of electrons) increases as $1/R^2$ and on the caustics it increases as $1/R$. However, in contrast to a caustic, a focus is not a catastrophe of general form. In other words, the radii of curvature of the electron wavefront are only exceptionally identical, which would have been necessary for the sphericity of this front and the formation of a focus. However, the differences between these radii are not too large, and certainly they do not amount to orders of magnitude. Therefore, an increase in the charge density near a caustic most probably obeys the law $1/R^n$, where $1 < n < 2$.

A reduction in the diameter of the distribution (discussed in the preceding section) induces a temporal catastrophe earlier. This means that the overtaking processes become more likely on approach of an electron bunch to a caustic, and consequently, on increase in the charge density. In other words, there is a high probability that temporal and spatial catastrophes occur close to one another. An increase in the charge density then approaches the dependence $1/R^2$, i.e. an electron bunch becomes nontransparent.

5. Transparent and nontransparent electron bunches

As shown above, the catastrophes that occur in an electron flux increase the charge density to infinity at some points and the charge bunches near these points behave like point particles. Such bunches can obviously be of two types: transparent and nontransparent. Nontransparent bunches are very similar to point particles, because such bunches are more stable. Consequently, they are more important in our discussion. We shall explain this by considering the example of spherically symmetric bunches.

We shall analyse the law which should be obeyed by the charge density in the spherically symmetric bunch so as to ensure that the potential at the centre of the bunch tends to infinity. Under this condition, electron bunches incident from outside the bunch in question cannot penetrate to its centre, i.e. in this case the bunch is nontransparent.

Let us assume that the charge density in a bunch is described by the distribution

$$\sigma(r) = \frac{G}{r^n}. \tag{41}$$

The total charge in a bunch $Q(R)$ in its central part (in a sphere of small radius R) should be finite:

$$Q(R) = 4\pi \int_0^R \mathrm{d}r\, r^2 \sigma(r) = 4\pi G \int_0^R \mathrm{d}r\, r^{2-n} = \frac{4\pi G}{3-n}\, r^{3-n}\Big|_0^R. \tag{42}$$

This is possible only if

$$n < 3 .$$

The charge is then given by

$$Q(R) = \frac{4\pi G}{3-n} R^{3-n} . \tag{43}$$

The electric field intensity is

$$E(R) = \frac{Q(R)}{R^2} = \frac{4\pi G}{3-n} R^{1-n} . \tag{44}$$

The corresponding expression for the potential is

$$U(R) = \int_R^\infty \mathrm{d}R' E(R') = \frac{4\pi G}{3-n} \int_R^\infty \mathrm{d}R' R'^{(1-n)}$$

$$= \frac{4\pi G}{(2-n)(3-n)} R'^{(2-n)} \Big|_R^\infty . \tag{45}$$

The behaviour of the potential at the upper limit is unimportant for the purposes of our further discussion because far from a singularity of the distribution described by expression (41) the charge density may decrease faster. The main point is the behaviour of the potential in the core of a bunch. The core potential tends to infinity if $n > 2$. A bunch therefore becomes nontransparent if the charge distribution inside it obeys the law given by expression (41) when $2 < n < 3$. In reality, n cannot exceed 2, since the energy needed to introduce new charges into the central part of the investigated bunch tends to infinity for $n \to 2$.

6. One-electron bunches

One of the more difficult questions relating to the proposed mechanism of the appearance of discrete photocounts is the need to explain the one-electron nature of the bunches. It is generally accepted in the literature that one photocount corresponds to one localised electron knocked out by photons from a cathode. However, we found no experimental confirmation of this view in the published work. Moreover, we did not find even the proof that the pulses corresponding to photocounts are identical. Nevertheless, the generally accepted view is close to the truth. Therefore, in developing the approach proposed above we must identify the reasons why the observation of one-electron bunches is the preferred mechanism.

In investigation of the state of a two-electron system in a square potential well we have shown that a low-density cloud splits into two bunches. Let us assume that this is not accidental and that three-electron, four-electron, and so on, up to n-electron clouds will split into three, four, and n bunches.

This statement can be justified as follows. It is known that the Hamiltonian which takes account of the Coulomb interaction

$$H = \sum_n \frac{1}{2m} \bar{p}_n^2 + \sum_{n,l} \frac{e^2}{r_{nl}} , \tag{46}$$

does not include the self-interaction, i.e. the action of the Coulomb field of one or other electron on itself; in expression (46), the first term sums the kinetic energies of electrons and the second sums the mutual Coulomb energies of electron pairs. There is no specific term representing the self-interaction. Consequently, if a narrow wave packet consisting of one electron is formed in some way, there will be no Coulomb repulsion in the packet; it will spread out nearly in the same way as a neutral wave packet, i.e. relatively slowly. Therefore, one-electron wave packets will be relatively longer-lived than the packets containing more electrons; in actual observations, one-electron packets will be encountered most frequently.

However, the problem under discussion is fundamentally of the many-electron type and a many-electron wave function is known to have certain symmetry properties under transposition operations (for example, the coordinate wave functions, i.e. a wave packet, should be completely antisymmetric under the operation of transposition of electrons if their spins are directed in the same way). In this sense all the electron wave packets are of many-electron nature and at first sight it seems that it is fundamentally impossible to form a one-electron wave packet.

Nevertheless, among many-electron configurations with the appropriate transposition symmetry there are some which are equivalent to one-electron configurations. Let us assume that $\psi(\boldsymbol{r})$ describes a lumped electron wave packet, for example a Gaussian one. Then, the wave function

$$\psi(\boldsymbol{r}_1, \boldsymbol{r}_2) = 2^{-1/2} \left[\psi(\boldsymbol{r}_1)\psi(\boldsymbol{r}_2 - \boldsymbol{a}) - \psi(\boldsymbol{r}_2)\psi(\boldsymbol{r}_1 - \boldsymbol{a}) \right] \tag{47}$$

describes a two-electron system concentrated in two packets: one at the origin of the coordinate system and the other at a point with the radius vector $\boldsymbol{a}$. One can show that there is almost no Coulomb interaction inside each packet or at least this is true if the distance $|\boldsymbol{a}|$ between the packets is much greater than their width (for simplicity all the spins are assumed to be identically oriented and the spin–spin and spin–orbit interactions are negligible). In fact, the average value of the Coulomb energy in a state described by the function $\Psi(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is

$$\begin{aligned} \langle U_{\mathrm{Coul}} \rangle &= e^2 \iint \mathrm{d}\boldsymbol{r}_1 \, \mathrm{d}\boldsymbol{r}_2 |\psi(\boldsymbol{r}_1, \boldsymbol{r}_2)| \cdot |\boldsymbol{r}_1 - \boldsymbol{r}_2|^{-1} \\ &= \frac{1}{2} e^2 \iint \mathrm{d}\boldsymbol{r}_1 \, \mathrm{d}\boldsymbol{r}_2 \big[|\psi(\boldsymbol{r}_1)|^2 |\psi(\boldsymbol{r}_2 - \boldsymbol{a})|^2 \\ &\quad + |\psi(\boldsymbol{r}_2)|^2 |\psi(\boldsymbol{r}_1 - \boldsymbol{a})|^2 \\ &\quad - \psi^*(\boldsymbol{r}_1)\, \psi(\boldsymbol{r}_1 - \boldsymbol{a}) \psi(\boldsymbol{r}_2)\, \psi^*(\boldsymbol{r}_2 - \boldsymbol{a}) \\ &\quad - \psi(\boldsymbol{r}_1) \psi^*(\boldsymbol{r}_1 - \boldsymbol{a}) \psi^*(\boldsymbol{r}_2)\, \psi(\boldsymbol{r}_2 - \boldsymbol{a}) \big] |\boldsymbol{r}_1 - \boldsymbol{r}_2|^{-1} . \end{aligned} \tag{48}$$

The first two terms in the above expression represent the Coulomb energy of the interaction of two electrons located at, respectively, the origin of the coordinate system and the point with the radius vector $\boldsymbol{a}$. This part of the Coulomb energy is approximately

$$\langle U_{\mathrm{Coul}} \rangle' = \frac{e^2}{|\boldsymbol{a}|} , \tag{49}$$

since the factor $1/|\boldsymbol{r}_1 - \boldsymbol{r}_2|$ can be regarded as practically constant within the limits of the distributions $|\psi(\boldsymbol{r}_1)|^2$ and $|\psi(\boldsymbol{r}_2 - \boldsymbol{a})|^2$, and equal to $1/|\boldsymbol{a}|$.

The last two terms in expression (48) are known as integrals of overlapping, which under the conditions considered here can be only very small. In fact, these integrals contain the combinations

$$\psi^*(\boldsymbol{r}_1)\, \psi(\boldsymbol{r}_1 - \boldsymbol{a}), \quad \psi(\boldsymbol{r}_2) \psi^*(\boldsymbol{r}_2 - \boldsymbol{a})$$

and conjugates of them. However, these combinations are necessarily small because if the width of a Gaussian packet is much less than $|\boldsymbol{a}|$, then the function $\psi(\boldsymbol{r}_2 - \boldsymbol{a})$ is very small if the function $\psi^*(\boldsymbol{r}_1)$ is close to its maximum value and vice versa.

It follows that although the probability of finding both electrons in either of the two wave packets is the same, there is no Coulomb interaction inside the packets. The situation is the same as if each electron had been in its own packet. These considerations are easily generalised to the case of several electrons: it is sufficient to replace the function described by expression (47) with the Slater determinant. We can easily see that these considerations are identical with those used to justify ignoring the existence of electrons far outside an atom of interest (for example, in another atom).

Consequently, if a many-electron bunch is formed initially, it decays rapidly under the action of the intra-packet Coulomb interaction and tends to form finally one-electron bunches in the sense described above. Such one-electron bunches then spread out relatively slowly, in the same way as a neutral wave packet spreads in free space. Therefore, one-electron bunches have a longer lifetime than many-electron bunches and, on the average, the observation of one-electron bunches is more likely.

7. Conclusions

The discussion of the problem given above is in no way exhaustive and does not provide a proof of the validity of the adopted approach. The reader undoubtedly will be able to formulate questions which are not answered in this note. It is however surprising how many proofs can be cited in support of the adopted approach and the range of known phenomena with which this approach is compatible is fairly wide. Among the proofs in support of the proposed mechanism the main one is that an electron bunch forms in a natural physically justified manner and not as a result of some wonderful secret transformation of an electron wave into a particle. Therefore, we decided to publish this note so as to attract, if possible, the attention of both theoreticians and experimentalists.

Even at this early stage the above analysis provides a qualitative picture of the appearance of photocounts when coherent (laser) radiation is detected. The electron system in a metal or semiconductor cathode is characterised by a high density and electrons are distributed uniformly (apart from fluctuations) over the cathode volume. The action of coherent radiation excites this electron system, which remains more or less uniformly distributed, so that an electron cloud can escape through the cathode surface into the vacuum space between the cathode and anode. Outside the cathode the density of this electron cloud falls considerably and this makes it likely that the cloud splits into separate bunches, possibly even consisting of one electron. However, calculations show that these bunches spread out quite readily, so that the distance between them becomes of the order of their width. Acceleration of the bunches by the static electric field in the cathode–anode space focuses the bunches (spatial catastrophe), which increases the transverse density. The Coulomb field in the bunches gives rise to the phenomenon of overtaking (temporal catastrophe) and their density then increases in the longitudinal direction to such an extent that their width becomes much less than the distance between them. Such a dense bunch resembles essentially a point-like particle and it flies across the cathode–anode space giving rise to a sharp and intense current pulse in the external circuit of a detector. An observer sees this pulse as a photocount. The possibility of applying the theory of catastrophes with its topological methods to the problem of compression of electron bunches indicates that the process is not random. It would be of interest to consider the application of the theory of catastrophes to the wave function in the $3N$-dimensional configurational space of N electrons.

It follows from the proposed approach that the statistics of the current pulses in a photodetector, which is usually identified with the statistics of photons, represents primarily the statistical nature of the process of decay of an electron cloud into bunches and only secondarily does it represent the properties of light causing the photoelectric effect. Essentially, the photocounts are the detection noise which strongly distorts the real properties of the light flux.

From the experimental point of view the most interesting consequence of the proposed mechanism is the possibility that, in principle, a photocathode can operate without generating current pulses. This may be possible in sufficiently strong fields when an electron cloud does not decay in the available time into bunches. This should alter greatly the spectral composition of the photocurrent: the high-frequency components, representing separate bunches, should disappear and only the low-frequency components, representing the change in the amplitude of the optical signal, should remain. As a rule, amplifiers receiving a photodetector signal are less sensitive in the low-frequency range. This reduction in the sensitivity under the conditions when there are no current pulses should be corrected, because otherwise the disappearance of the pulses will be received as a reduction in the photocathode efficiency. Unfortunately, at this stage it is not possible to consider quantitatively the operation without current pulses.

On the one hand, the above analysis confirms the statistical nature of the appearance of photocounts and, on the other, it shows that this statistical nature should not be regarded as a fundamental phenomenon additional to quantum mechanics, but should be regarded as nonfundamental and following from quantum mechanics. In this respect the proposed approach goes back to the old discussions on localisation and delocalisation of particles (one should mention particularly here the book of de Broglie [22]). It may give us a chance to have a fresh look at the old questions (it is worth noting here an interesting paper by Klyshko [23]).

It should also be pointed out that the photocounts regarded as elementary events have not been investigated thoroughly. To what extent does one photocount correspond exactly to the passage of one elementary charge through the cathode–anode space? What is the spatial size of an electron wave packet creating a photocount? Are the current pulses in the device circuit corresponding to single photocounts all identical? It would be desirable to investigate these questions not under the conditions in a photomultiplier, which are complicated by side effects due to secondary electrons, but under 'purer' conditions.

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N K Flavitskii, V P Karasev, I I Tugov, Ch K Mukhtarov, G Lochak, S L Chin, E G Rudashevskii, Yu E Lozovik, A S Chirkin, V A Shcheglov, V V Savranskii, and V B Fedorov for numerous discussions. This work was supported by the Russian Fund for Fundamental Research.

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