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# Critical phenomena and quenched disorder

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## Contents



**Abstract.** Theoretical ideas for deriving singularities of thermodynamical functions at the second-order phase transitions in spin systems with weak quenched disorder are considered. In particular, $p$-component vector magnets and the two-dimensional Ising model with disorder in spin – spin interactions are studied. Generalisation of the traditional renormalisation-group scheme, which takes into account nonperturbative spin-glass degrees of freedom, is proposed. Low-temperature properties and the phase transition in the Ising systems with quenched random fields are also considered.

## 1. Introduction

This review is devoted to the theory of critical phenomena at the phase transitions of the second order. It is generally believed that this part of statistical mechanics is well understood, at least at the qualitative level. Moreover, it is one of the few fields where close cooperation with experiments is possible and in certain cases one is able to think in terms of the *quantitative* correspondence of experimental results with a theory, and vice versa. In this sense, it is tempting to conclude that nothing very new could be expected any more in the world of second-order phase transitions. One of the main points of the present review is to demonstrate that this is not so. Life is not so boring.

The theory of the critical phenomena deals with macroscopic statistical systems which are close to the phase transition point, where spontaneous symmetry breaking takes place. This situation is characterised by large-scale instabilities, or fluctuations (unfortunately, these can be observed even in everyday life). According to the traditional scaling theory of the second-order phase transitions, the large-scale fluctuations are characterised by a certain dominant scale, or the correlation length, $R_c$. The correlation length grows as the critical point is approached, where it becomes infinite. The large-scale fluctuations lead to singularities in the macroscopic characteristics of the system as a whole. These singularities are the main subject of the theory.

In the studies of the second-order phase transitions, the systems considered were usually assumed to be perfectly homogeneous. In real life, however, some defects or impurities are always present. Therefore, it is natural to

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consider the effect impurities might have on the phase transition phenomena.

Originally, many years ago, it was generally believed that impurities either completely destroy the long-range fluctuations, such that the singularities of the thermodynamic functions are smoothed out, or can produce only a shift of a critical point but cannot affect the critical behaviour itself. Later it was realised that an intermediate situation is also possible, in which a new critical behaviour, with new universal critical exponents, is established sufficiently close to the phase transition point. However, according to the most recent developments in this field, the situation could appear to be much more sophisticated, such that completely new types of critical phenomena of the spin-glass nature could be established near the critical point. The consideration of the disorder-induced phenomena near the second-order phase transitions is the main subject of the present review.

To a certain extent, the review is assumed to be of pedagogical character (it is based on the lecture course for the students of the Landau Institute). It is supposed to be self-contained so that the reader is not required to go through all the references. One of the purposes of the present review is to introduce the reader to the subject. To do that, the reader need only understand the basic principles of statistical mechanics. The only exception is Section 4, where in certain places the elements of the theory of replica symmetry breaking of spin-glasses are used without detailed explanation (the reader may refer, for example, to Ref. [9]). The calculations, although sometimes rather tedious (where this cannot be avoided), are made as elementary as possible, while the emphasis is placed on the qualitative physical ideas.

Section 2 is devoted to the systematic consideration of the traditional scaling theory of the critical phenomena, including $\varepsilon$-expansion. This topic has been a part of the textbooks [1–3] for many years now, so the reader who is familiar with the subject may easily skip this section.

In Section 3 the concept of quenched disorder is introduced. Here the physical idea of self-averaging is discussed and the technical method, known as the 'replica method', is considered in general terms. Besides, in terms of the renormalisation-group (RG) approach, the traditional procedure for obtaining a new universal impurity-induced critical regime is considered for the ferromagnetic spin vector systems.

In Section 4 we study the renormalisation-group theory generalised to take into account essentially nonperturbative phenomena of the spin-glass (SG) nature. It is shown that, whenever the disorder is relevant for the critical behaviour, the traditional RG flows, which are usually considered as describing the disorder-induced universal critical behaviour, are unstable with respect to the SG-type perturbations. It is demonstrated that in general there exist no stable fixed points, and the RG flows lead to the so-called *strong-coupling regime* at a finite spatial scale. The physical consequences of the RG solutions obtained are discussed. In particular, we argue that the strong-coupling phenomena discovered indicate the onset of a completely new SG-type critical behaviour in the close vicinity of the phase transition.

In Section 5 we consider the critical properties of the two-dimensional Ising model with impurities. Here the exact solution for the critical behaviour of the specific heat is derived, and the phase diagram as well as the results of the recent numerical simulations are discussed.

Finally, in Section 6 the Ising spin systems with quenched random fields are considered. These are a type of statistical model which exhibits qualitatively different properties from those considered before. The random-field Ising systems are of special interest for two reasons. First, because they have many experimentally accessible realisations and, second, because despite extensive theoretical and experimental efforts during the last twenty years very little is understood about their basic properties even at the qualitative level.

The Conclusions are written to inform the reader of the main idea of the present review, which is encoded in the text. It cannot be decoded, however, until the whole review is carefully studied.

## 2. Scaling theory of the critical phenomena

### 2.1 General principles of statistical mechanics

In the most simple terms, the basic statements of statistical mechanics could be introduced in the following way. Let the microscopic state of some *macroscopic* system having many degrees of freedom be described by the configurations of $N$ variables $\{s_i\}$, where $i = 1, 2, \ldots, N$. The basic quantity characterising the microscopic states is called the *energy* $H$, and it is defined as a function of all the microscopic variables $\{s_i\}$:

$$H = H(s_1, s_2, \ldots, s_N) \equiv H[s] \ .$$

The microscopic dynamic behaviour of the system is defined by dynamic differential equations such that, in general, the energy of the system tends to a minimum. Besides, it is assumed that no observable system can be perfectly isolated from the surrounding world, and the effect of the interaction with the surroundings (the thermal bath) is believed to produce the so-called *thermal noise* in the exact dynamical equations. The thermal (white) noise acts as random and uncorrelated fluctuations which produce the randomisation and mixing of the exact dynamical trajectories of the system.

Let $A[s]$ be some observable quantity. The quantities which are of interest in statistical mechanics are the *averaged* values of the observables. In other words, instead of studying the exact change in time values $A[s(t)]$, one introduces the averaged quantity

$$\langle A \rangle = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathrm{d}t' A\left[s(t')\right] \ , \tag{2.1}$$

which could be formally obtained after the observations during an infinite time period.

The fundamental hypothesis of equilibrium statistical mechanics lies in the following. It is believed that, owing to the mixing of the dynamic trajectories, after an infinitely long observation time, the system in general 'visits' its different microscopic states many times, and therefore the averaged quantity in Eqn (2.1) could be obtained by averaging over the *ensemble* of the states instead of that over the time:

$$\langle A \rangle = \int \mathrm{d}s_1 \, \mathrm{d}s_2 \ldots \mathrm{d}s_N \, A[s] P(s_1, s_2, \ldots, s_N) \ . \tag{2.2}$$

Here $P[s]$ is the *probability distribution function* of the microscopic states of the system. In other words, it is believed that because of the mixing of the dynamical

trajectories, instead of solving the exact dynamics, the system could be statistically described in terms of the probabilities of its microscopic states given by the function $P[s]$. The probability distribution function, whatever it is, must be normalised:

$$\int \mathrm{d}s_1\,\mathrm{d}s_2 \ldots \mathrm{d}s_N\, P(s_1, s_2, \ldots, s_N) = 1 \ . \tag{2.3}$$

The fundamental quantity of statistical mechanics which characterises the probability distribution itself is called the entropy. It is defined as the average of the logarithm of the distribution function:

$$S = -\langle \log(P[s]) \rangle \equiv -\int \mathrm{d}s_1\,\mathrm{d}s_2 \ldots \mathrm{d}s_N\, P[s] \log(P[s]) \ . \tag{2.4}$$

First of all, it is obvious from the above definition that because of the normalisation (2.3), the entropy is at least nonnegative. In general, the value of the entropy could tell to what extent the probability distribution of the system is 'ordered'. Consider a simple illustrative example. Let the (discrete) microscopic states of the system be labeled by an index $\alpha$, and let us assume that the probability distribution is such that only $L$ (among all) states have nonzero and equal probability. Then, as a result of the normalisation (2.3), the probability of any of these $L$ states must be equal to $1/L$. According to the definition of the entropy, one gets

$$S = -\sum_{\alpha}^{L} P_\alpha \log P_\alpha = \log L \ .$$

Therefore, the broader the distribution (the larger $L$), the larger the value of the entropy. On the other hand, the more concentrated the distribution function is, the smaller the value of the entropy. In the extreme case, when there is only one microscopic state occupied by the system, the entropy is equal to zero. In general, the value of $\exp(S)$ could be interpreted as the average number of the states occupied by the system with a fairly large probability.

Now let us consider what the general form of the probability distribution function must be. According to the basic hypothesis, the average value of the energy of the system is

$$E \equiv \langle H \rangle = \sum_{\alpha} P_\alpha H_\alpha \ . \tag{2.5}$$

The interaction of the system with the surrounding world produces the following fundamental effects. First, the average value of its energy in thermal equilibrium is conserved. Second, for some reason Nature is constructed in such a way that, irrespective of the internal structure of the system, the value of its entropy in the equilibrium state tries to attain a maximum (bounded by the condition that the average energy is constant). In a sense, it is natural: random noise makes the system as disordered as possible. Let us now consider the value of the probability distribution function which would maximise the entropy. To take into account the two constraints — the conservation of the average energy, Eqn (2.5), and the normalisation $\sum_\alpha P_\alpha = 1$ — one can use the method of the Lagrangian multipliers. Therefore, the following expression must be maximised with respect to all possible distributions $P_\alpha$:

$$S_{\beta,\gamma}[P] = -\sum_{\alpha} P_\alpha \log(P_\alpha) - \beta\left(\sum_{\alpha} P_\alpha H_\alpha - E\right) - \gamma\left(\sum_{\alpha} P_\alpha - 1\right) , \tag{2.6}$$

where $\beta$ and $\gamma$ are the Lagrangian multipliers. Variation with respect to $P_\alpha$ gives

$$P_\alpha = \frac{1}{Z} \exp(-\beta H_\alpha) \ , \tag{2.7}$$

where

$$Z = \sum_{\alpha} \exp(-\beta H_\alpha) = \exp(\gamma + 1) \tag{2.8}$$

is called the partition function, and the parameter $\beta$, which is called the inverse temperature, is defined by the condition

$$\frac{1}{Z} \sum_{\alpha} H_\alpha \exp(-\beta H_\alpha) = E \ . \tag{2.9}$$

In practice, however, it is the temperature which is usually taken as an independent parameter, whereas the average energy is obtained as a function of the temperature by Eqn (2.9).

The other fundamental quantity of statistical mechanics is the free energy defined as follows:

$$F = E - TS \ , \tag{2.10}$$

where $T = 1/\beta$ is the temperature. Using Eqn (2.7), one can easily derive the following basic relations among the free energy, the partition function, the entropy, and the average energy

$$F = -T \log(Z) \ , \tag{2.11}$$

$$S = \beta^2 \frac{\partial F}{\partial \beta} \ , \tag{2.12}$$

$$E = -\frac{\partial}{\partial \beta} \log(Z) = F + \beta \frac{\partial F}{\partial \beta} \ . \tag{2.13}$$

Note that according to the definition given by Eqn (2.10), the principle of maximum entropy is equivalent to that of the minimum of the free energy. One can easily confirm that taking the free energy (instead of the entropy) as the fundamental quantity, which must be minimal with respect to all possible distribution functions, the same form of the probability distribution as given by Eqn (2.7) is obtained.

### 2.2 The mean-field approximation

In magnetic materials, the microscopic state of the system is supposed to be defined by the values of the local spin magnetisations. In many magnetic systems, the electrons responsible for the magnetic behaviour are localised near the atoms of the crystal lattice, and the force which tends to orient the spins is the (short-range) exchange interaction.

The most popular models which describe this situation qualitatively are called the Ising models. The microscopic variables in these systems are the Ising spins $\sigma_i$ which by definition can take only two values: $+1$ or $-1$. The traditional form for the microscopic energy (which from now on will be called the Hamiltonian) as the function of all the Ising spins is the following:

$$H = -\frac{1}{2} \sum_{\langle i,j \rangle} J_{ij} \sigma_i \sigma_j - h \sum_i \sigma_i \ . \tag{2.14}$$

Here the notation $\langle i, j \rangle$ indicates the summation over all the lattice sites of the nearest neighbours, $J_{ij}$ are the values of the spin–spin interactions, and $h$ is the external magnetic

field. If all the values of $J_{ij}$ are equal to some positive constant, then one gets the ferromagnetic Ising model; otherwise, if all the $J_{ij}$ values are equal to some negative constant, one gets the antiferromagnetic Ising model.

In spite of the apparent simplicity of the Ising model, an exact solution (which means the calculation of the partition function and the correlation functions) has been found only for the one- and two-dimensional systems in the zero external magnetic field. In all other cases, one needs to use approximate methods. One of the simplest methods is called the mean-field approximation. In many cases, this method gives results which are not too far from the correct ones, and very often it makes it possible to get some qualitative understanding of what is going on in the system under consideration.

The starting point of the mean-field approximation is the assumption about the structure of the probability distribution function. It is assumed that the distribution function in the equilibrium state can be factorised as the product of the independent distribution functions in the lattice sites:

$$P[\sigma] = \frac{1}{Z}\exp\left(-\beta H[\sigma]\right) \simeq \prod_i P_i(\sigma_i) \ . \tag{2.15}$$

The normalised site distribution functions are taken in the form

$$P_i(\sigma_i) = \frac{1+\phi_i}{2}\,\delta(\sigma_i - 1) + \frac{1-\phi_i}{2}\,\delta(\sigma_i + 1) \ , \tag{2.16}$$

where $\phi_i$ are the parameters which have to be specified.

The factorisation of the distribution function, Eqn (2.15), means that the average of any product of any functions at different sites is also factorising on the product of the independent averages

$$\left\langle f(\sigma_i) g(\sigma_j)\right\rangle = \left\langle f(\sigma_i)\right\rangle\left\langle g(\sigma_j)\right\rangle \ , \tag{2.17}$$

where, according to the initial statement (2.15),

$$\left\langle f(\sigma_i)\right\rangle = \frac{1+\phi_i}{2}\,f(1) \ + \ \frac{1-\phi_i}{2}\,f(-1) \ . \tag{2.18}$$

In particular, for the average site magnetisations, one easily gets

$$\langle\sigma_i\rangle = \phi_i \ . \tag{2.19}$$

Therefore, the physical meaning of the parameters $\{\phi_i\}$ in the trial distribution function is that they describe the average site spin magnetisations. According to the general principles of statistical mechanics, these parameters must be such that they would minimise the free energy of the system.

Using Eqns (2.15) and (2.16) for the entropy and for the average energy, we get

$$S = -\left\langle\log\left(P[\sigma]\right)\right\rangle \simeq -\sum_i\left\langle\log\left[P_i(\sigma_i)\right]\right\rangle$$
$$= -\sum_i\left[\frac{1+\phi_i}{2}\log\left(\frac{1+\phi_i}{2}\right) + \frac{1-\phi_i}{2}\log\left(\frac{1-\phi_i}{2}\right)\right] , \tag{2.20}$$

$$E = -\frac{1}{2}\sum_{\langle i,j\rangle} J_{ij}\,\phi_i\phi_j - h\sum_i \phi_i \ . \tag{2.21}$$

For the free energy, Eqn (2.10), one obtains

$$F = -\frac{1}{2}\sum_{\langle i,j\rangle} J_{ij}\,\phi_i\phi_j - h\sum_i \phi_i$$
$$+T\sum_i\left[\frac{1+\phi_i}{2}\log\left(\frac{1+\phi_i}{2}\right) + \frac{1-\phi_i}{2}\log\left(\frac{1-\phi_i}{2}\right)\right] . \tag{2.22}$$

To be more specific, consider the ferromagnetic system on the $D$-dimensional cubic lattice. In this case, all the spin–spin couplings are equal to some positive constant $J_{ij} = J/2D > 0$ (the factor $1/2D$ is inserted just for convenience), and each site has $2D$ nearest neighbors. Since the system is homogeneous, it is natural to expect that all the $\phi_i$ values must be equal to some constant $\phi$. Then, for the free energy [Eqn (2.22)], one gets

$$\frac{F}{V} \equiv f(\phi) = -\frac{1}{2}J\phi^2 - h\phi$$
$$+T\left[\frac{1+\phi}{2}\log\left(\frac{1+\phi}{2}\right) \ + \ \frac{1-\phi}{2}\log\left(\frac{1-\phi}{2}\right)\right] , \tag{2.23}$$

where $V$ is the volume of the system and $f$ is the density of the free energy.

The necessary condition for the minimum of $f$ is

$$\frac{\mathrm{d}f(\phi)}{\mathrm{d}\phi} = 0 \ ,$$

or

$$-J\phi - h + T\,\mathrm{arctanh}(\phi) = 0 \ . \tag{2.24}$$

The resulting equation, which defines the order parameter $\phi$, is

$$\phi = \tanh\left[\beta(J\phi + h)\right] \ . \tag{2.25}$$

Note that the minimum of the free energy is conditioned by $\mathrm{d}^2 f/\mathrm{d}\phi^2 > 0$. Using Eqn (2.24), we can reduce this condition to

$$\frac{1}{1-\phi^2} > \beta J \ . \tag{2.26}$$

Consider first the case of a zero external magnetic field ($h = 0$). One can easily see that, if $T > T_c = J$, the only solution of Eqn (2.25) is $\phi = 0$, and this solution satisfies condition (2.26). Therefore, at all temperatures higher than $T_c$, the minimum of the free energy is achieved in the state in which all the site spin magnetisations are zeros.

However, if $T < T_c$, then in addition to the solution $\phi = 0$, Eqn (2.25) (with $h = 0$) has two nontrivial solutions $\phi = \pm\phi(T) \neq 0$. One can easily check that in this temperature region the solution $\phi = 0$ becomes the maximum and not the minimum of the free energy, while the true minima are achieved at $\phi = \pm\phi(T)$. Therefore, in the low-temperature region $T < T_c$, the free energy has two minima, which are characterised by nonzero site magnetisations with opposite signs.

Near $T_c$, the magnetisation $\phi(T)$ is small. In this case, the expansion in powers of $\phi$ in Eqn (2.25) can be made. In the leading order in $\tau \equiv (T/T_c - 1)$, $|\tau| \ll 1$, one gets

$$\phi(T) = \mathrm{const}\,|\tau|^{1/2} , \quad \tau < 0 \ . \tag{2.27}$$

Thus, as $T \to T_c$, $\phi(T) \to 0$.

The expansion of the free energy Eqn (2.23) as the function of a small value of $\phi$ yields

$$f(\phi) = \frac{1}{2}\tau\phi^2 + \frac{1}{4}g\phi^4 - h\phi\,, \tag{2.28}$$

where $g = T/3$, and for simplicity we have taken $J = 1$. The qualitative shape of $f(\phi)$ at $T > T_c$ ($\tau > 0$) and at $T < T_c$ ($\tau < 0$) is shown in Fig. 1 Note that, since the total free energy $F$ is proportional to the volume of the system, the value of the free energy barrier separating the states with $\phi = \pm\phi(T)$ at $T < T_c$ is also proportional to the volume of the system. Therefore, in the *thermodynamic limit* $V \to \infty$ (which corresponds to the consideration of the *macroscopic* systems) the barrier separating the two states approaches infinity.

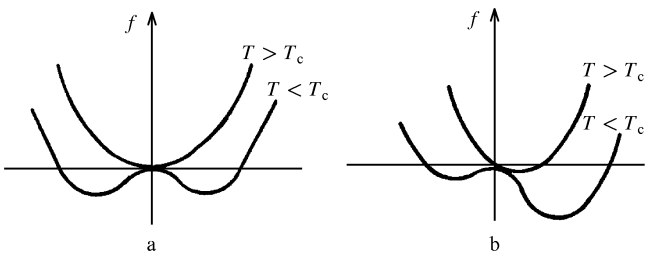


**Figure 1.** Free energy of the ferromagnetic Ising magnet: (a) in the zero external magnetic field; (b) in the nonzero magnetic field.

The simple considerations described above demonstrate on a qualitative level the fundamental phenomenon called spontaneous symmetry breaking. At the temperature $T = T_c$, the phase transition of the second order occurs, such that in the low-temperature region $T < T_c$ the symmetry with respect to the global change of the signs of the spins is broken, and the *two* (instead of one) ground states appear. These two states differ by the sign of the average spin magnetisation, and they are separated by the macroscopic barrier of the free energy.

In a small nonzero magnetic field ($h \ll 1$), the qualitative shape of the free energy is shown in Fig. 1b Near the phase transition point for the equation $\mathrm{d}f/\mathrm{d}\phi = 0$, one gets

$$\tau\phi + g\phi^3 = h\,. \tag{2.29}$$

This equation always has nonzero solutions for the order parameter $\phi$ at all temperatures. In particular, in the low-temperature region ($\tau < 0$), one finds that

$$\phi \simeq \begin{cases} \left(\dfrac{|\tau|}{g}\right)^{1/2} + \dfrac{h}{2\tau}\,, & h \ll h_c(\tau)\,, \\ \left(\dfrac{h}{g}\right)^{1/3}, & h \gg h_c(\tau)\,, \end{cases} \tag{2.30}$$

where

$$h_c(\tau) = \frac{1}{\sqrt{g}}\,|\tau|^{3/2}\,. \tag{2.31}$$

For the high-temperature region ($\tau > 0$), one finds that

$$\phi \simeq \begin{cases} \dfrac{h}{\tau}\,, & h \ll h_c(\tau)\,, \\ \left(\dfrac{h}{g}\right)^{1/3}, & h \gg h_c(\tau)\,. \end{cases} \tag{2.32}$$

Therefore, at $h \neq 0$ the phase transition is 'smoothed out' in the temperature interval $|\tau| \sim h^{2/3}$ [Eqn (2.31)] near $T_c$. Thus in systems in a nonzero external field, the sharp phase transition does not exist.

The physical quantity which describes the reaction of the system on the infinitely small magnetic field is called susceptibility. It is defined as follows:

$$\chi = \left.\frac{\partial\phi}{\partial h}\right|_{h=0}\,. \tag{2.33}$$

According to Eqns (2.30)–(2.32), one finds that near the critical point the susceptibility becomes divergent:

$$\chi \simeq \begin{cases} \tau^{-1}\,, & T > T_c\,, \\ \dfrac{1}{2}|\tau|^{-1}\,, & T < T_c\,. \end{cases} \tag{2.34}$$

For the nonlinear susceptibility $\chi(h) = \partial\phi/\partial h$ at the critical point ($g^{-1/2}|\tau|^{3/2} \ll h$), we get

$$\chi(h) \simeq h^{-2/3}\,. \tag{2.35}$$

The other basic physical quantity is the specific heat, which is defined as follows:

$$C = -T\,\frac{\partial^2 f}{\partial T^2}\,. \tag{2.36}$$

For the specific heat near the critical point (in the zero magnetic field), according to the Eqns (2.27) and (2.28), one obtains

$$C \simeq \begin{cases} \mathrm{const} = \dfrac{1}{2g}\,, & T > T_c\,, \\ 0\,, & T < T_c\,. \end{cases} \tag{2.37}$$

Of course, all the above cases which were considered in terms of very primitive mean-field approximations cannot pretend to give reliable results. Nevertheless, on a qualitative level they demonstrate a very important physical phenomenon: near the point of the second-order phase transition, at least some of the physical quantities become singular (or nonanalytic).

Now let us consider one simple and natural improvement of the mean-field theory discussed above.

### 2.3 The Ginzburg – Landau theory

The apparent defect of the mean-field approximation given above is that it does not take into account correlations among spins. This can be easily amended if we are interested in the studies of only *large-scale* phenomena, which will be shown to be responsible for the leading singularities in the thermodynamical functions. In this case, the order parameters $\phi_i$ are almost spatially homogeneous, and they can be represented as slowly varying (with small gradients) functions of the continuous space coordinates. Then the interaction term in the Hamiltonian (2.14) can be represented as follows:

$$\frac{1}{2}\sum_{\langle i,j\rangle}\phi_i\phi_j \to \frac{1}{2}\int \mathrm{d}^D x\,\left\{\phi^2(x) + \left[\nabla\phi(x)\right]^2\right\}\,. \tag{2.38}$$

The Hamiltonian in which only small spatial fluctuations of the order parameter are taken into account can be written as follows:

$$H = \int \mathrm{d}^D x\,\left\{\frac{1}{2}\left[\nabla\phi(x)\right]^2 + \frac{1}{2}\tau\phi^2(x) + \frac{1}{4}g\phi^4(x) - h\phi(x)\right\}\,. \tag{2.39}$$

The theory which is based on the above Hamiltonian is called the Ginzburg–Landau approach. In fact the Ginzburg–Landau Hamiltonian is nothing but the first few terms of the expansion in powers of $\phi$ and $\nabla\phi$. In the vicinity of the (second-order) phase transition point, where the order parameter is small and the leading contributions come from large-scale fluctuations, such an approach appears to be quite natural.

Consider the contributions caused by small fluctuations at the background of the homogeneous order parameter $\phi_0 = \sqrt{|\tau|/g}$,

$$\phi(x) = \phi_0 + \varphi(x)\,, \tag{2.40}$$

where $\varphi(x) \ll \phi_0$.

For simplicity, let us consider the case of the zero magnetic field. Then the expansion of the Hamiltonian (2.39) to the second order in $\varphi$ yields

$$H = H_0 + \int \mathrm{d}^D x \left\{ \frac{1}{2}\left[\nabla\varphi(x)\right]^2 + |\tau|\varphi^2(x) \right\}\,. \tag{2.41}$$

In terms of the Fourier representation

$$\varphi(x) = \int \frac{\mathrm{d}^D k}{(2\pi)^D}\, \varphi(k) \exp(-\mathrm{i}kx)\,, \tag{2.42}$$

one gets

$$H = \frac{1}{2}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\, \left(k^2 + 2|\tau|\right)\left|\varphi(k)\right|^2 + H_0\,. \tag{2.43}$$

Therefore, for the correlation function

$$G_0(k) \equiv \left\langle \left|\varphi(k)\right|^2 \right\rangle = \frac{\int \mathrm{D}\varphi(k)|\varphi(k)|^2 \exp(-H[\varphi])}{\int \mathrm{D}\varphi(k) \exp(-H[\varphi])}\,, \tag{2.44}$$

one obtains the following result:

$$G_0(k) = \frac{1}{k^2 + 2|\tau|}\,. \tag{2.45}$$

Besides, it is obvious that

$$\left\langle \varphi(k)\varphi(k') \right\rangle = G_0(k)\delta(k + k')\,. \tag{2.46}$$

Therefore, for the spatial correlation function

$$G_0(x) = \left\langle\left\langle \phi(0)\phi(x) \right\rangle\right\rangle \equiv \left\langle \phi(0)\phi(x) \right\rangle - \left\langle \phi(0) \right\rangle\left\langle \phi(x) \right\rangle$$

$$= \left\langle \varphi(0)\varphi(x) \right\rangle = \int \frac{\mathrm{d}^D k}{(2\pi)^D}\, \left\langle \left|\varphi(k)\right|^2 \right\rangle \exp(\mathrm{i}kx)\,, \tag{2.47}$$

we obtain

$$G_0(x) \sim \begin{cases} |x|^{-(D-2)}\,, & |x| \ll R_\mathrm{c}(\tau) = \dfrac{1}{\sqrt{2|\tau|}}\,, & \text{(2.48a)} \\ \exp\dfrac{-|x|}{R_\mathrm{c}}\,, & |x| \gg R_\mathrm{c}(\tau)\,. & \text{(2.48b)} \end{cases}$$

Here the quantity

$$R_\mathrm{c}(\tau) \sim |\tau|^{-1/2} \tag{2.49}$$

is called the *correlation length*.

Thus, the situation near $T_\mathrm{c}$ ($|\tau| \ll 1$) is as follows. At scales much greater than the correlation length $R_\mathrm{c}(\tau) \gg 1$, the fluctuations of the field $\phi(x)$ around its equilibrium value $\phi_0$ ($\phi_0 = 0$ at $T > T_\mathrm{c}$, and $\phi_0 = \sqrt{|\tau|/g}$ at $T < T_\mathrm{c}$) become effectively independent (their correlations decay exponentially [see Eqn (2.48b)]. On the other hand, at scales much smaller than $R_\mathrm{c}(\tau)$, in the so-called fluctuation region, the fluctuations of the order parameter are strongly correlated, and their correlation functions exhibit weak power-law decay [see Eqn (2.48a)]. Therefore, inside the fluctuation region at scales $\ll R_c(\tau)$, the gradient or the fluctuation term of the Hamiltonian (2.39) becomes crucial for the theory. At the critical point, the fluctuation region becomes infinite.

Let us estimate to what extent the above simple considerations are correct. Expansion (2.41) could be used and the result [Eqn (2.48)] justified only if the characteristic value of the fluctuations $\varphi$ are small in comparison with the equilibrium value of the order parameter $\phi_0$. Since the correlation length $R_\mathrm{c}$ is the only relevant spatial scale which exists in the system near the phase transition point, the characteristic value of the fluctuations of the order parameter could be estimated as follows:

$$\overline{\varphi^2} \equiv \frac{1}{R_\mathrm{c}^D}\int_{|x|<R_\mathrm{c}} \mathrm{d}^D x \left\langle \varphi(0)\varphi(x) \right\rangle \sim R_\mathrm{c}^{-(D-2)}\,. \tag{2.50}$$

The above simple mean-field estimates for the critical behaviour are justified only if the value of $\overline{\varphi^2}$ is much smaller than the corresponding value of the order parameter $\phi_0^2$ at equilibrium:

$$R_\mathrm{c}^{-D+2} \ll \frac{|\tau|}{g}\,. \tag{2.51}$$

Using Eqn (2.49) we find that this condition is satisfied if

$$g|\tau|^{(D-4)/2} \ll 1\,. \tag{2.52}$$

Therefore if the dimensions of the system are greater than 4, near the phase transition point, $\tau \to 0$, condition (2.52) is always satisfied. On the other hand, if the dimensions $D$ are less than 4, this condition is always violated near the critical point.

Thus, these simple estimates reveal the following important points:

(1) If the dimensions $D$ of the considered system are greater than 4, then its critical behaviour in the vicinity of the second-order phase transition is described successfully by the mean-field theory.

(2) If the dimensions of the system are smaller than 4, then, according to Eqn (2.52), the mean-field approach gives correct results only in the range of temperatures not too close to $T_\mathrm{c}$:

$$\tau \gg \tau_*(D, g) \equiv g^{2/(4-D)} \qquad (\tau \ll 1) \tag{2.53}$$

(here it is assumed that $g \ll 1$, otherwise there would be no mean-field critical region $|\tau| \ll 1$ at all). In the close vicinity of $T_\mathrm{c}$, $|\tau| \ll \tau_*$, the other (non-Gaussian) type of critical behaviour can be expected to occur.

### 2.4 Critical exponents

In general, it is believed that critical behaviour of the physical quantities near the phase transition point can be described in terms of the so-called *critical exponents*. In particular, for the quantities considered above, the critical exponents are defined as follows:

— order parameter

$$\phi_0 \sim |\tau|^\beta\,, \qquad h \ll h_\mathrm{c}(\tau); \qquad \tau < 0;$$

$$\phi_0 \sim h^{1/\delta}\,, \qquad h \gg h_\mathrm{c}(\tau)\,;$$

— specific heat

$$C \sim |\tau|^{-\alpha}, \quad h \ll h_c(\tau) ;$$

— susceptibility

$$\chi \sim |\tau|^{-\gamma}, \quad h \ll h_c(\tau);$$

$$\chi \sim h^{1/\delta - 1}, \quad h \gg h_c(\tau) ;$$

— correlation function

$$G(x) \sim |x|^{-D+2-\eta}, \quad |x| \ll R_c ; \tag{2.54}$$

where the value of the critical field is $h_c(\tau) \sim |\tau|^{\nu/\mu}$ (this estimate follows from the comparison of the correlation lengths in small and in large fields).

In fact, not all the critical exponents listed in Eqn (2.54) are independent. One could easily derive (see below) the following relations among them:

$$\alpha = 2 - D\nu , \tag{2.55}$$

$$\delta = \frac{D+2-\eta}{D-2+\eta} , \tag{2.56}$$

$$\gamma = (2-\eta)\nu , \tag{2.57}$$

$$2\beta = 2 - \gamma - \alpha , \tag{2.58}$$

$$\mu = \frac{2}{D+2-\eta} . \tag{2.59}$$

For 7 exponents there are exist 5 equations, which means that only two exponents are independent. In other words, to find all the critical exponents one needs to calculate only two of them.

In particular, the Ginzburg – Landau mean-field theory considered above gives $\nu = 1/2$ and $\eta = 0$ [see Eqns (2.48), (2.49)]. Using Eqns (2.55) – (2.59) we can easily find the rest of the exponents: $\alpha = -(D-4)/2$; $\delta = (D+2)/(D-2)$; $\gamma = 1$; $\beta = (D-2)/4$; and $\mu = 1/3$. These critical exponents fully describe the critical behaviour of any scalar field, $D$-dimensional system with $D \geqslant 4$.

Let us now calculate the relations (2.55) – (2.59).

According to the definition of specific heat,

$$C = -T \frac{\partial^2 f}{\partial T^2} , \tag{2.60}$$

one gets

$$C = \frac{1}{V} \int \mathrm{d}^D x \int \mathrm{d}^D x' \left[ \langle \phi^2(x) \phi^2(x') \rangle - \langle \phi^2(x) \rangle \langle \phi^2(x') \rangle \right] \sim \frac{1}{R_c^D} \langle \Phi \rangle^2 , \tag{2.61}$$

where

$$\Phi = \int_{|x|<R_c} \mathrm{d}^D x \phi^2(x) . \tag{2.62}$$

According to Eqn (2.39), the equilibrium energy density of the system (at scales greater than $R_c$) is proportional to $|\tau|\Phi$. Thus, the equilibrium value of $\langle \Phi \rangle$ is defined by the condition $|\tau| \langle \Phi \rangle \sim T$ ($T \simeq T_c = 1$ in our case). Therefore, from Eqn (2.61) we get

$$C \sim R_c^{-D} |\tau|^{-2} \sim |\tau|^{D\nu - 2} . \tag{2.63}$$

On the other hand, according to the definition of the critical exponent $\alpha$, $C \sim |\tau|^{-\alpha}$, and one obtains Eqn (2.55).

Using the definitions of the susceptibility as well as the critical exponents of the correlation function $\eta$ and that of the correlation length $\nu$ [see Eqn (2.54)], one obtains

$$\chi = \left. \frac{\partial \langle \phi \rangle}{\partial h} \right|_{h=0} = \int \mathrm{d}^D x \langle \langle \phi(0) \phi(x) \rangle \rangle \sim R_c^D R_c^{2-D-\eta} \sim |\tau|^{-\nu(2-\eta)} . \tag{2.64}$$

On the other hand, $\chi \sim |\tau|^{-\gamma}$, which provides Eqn (2.57).

The value of the susceptibility, Eqn (2.64), can be estimated in another way:

$$\chi \sim R_c^D \phi_0^2 \sim |\tau|^{-D\nu + 2\beta} . \tag{2.65}$$

This yields $\gamma = D\nu - 2\beta$. Using Eqn (2.55), one gets Eqn (2.58).

Now let us define the value of the order parameter in the region, which is less than the correlation length

$$\psi \equiv \int_{|x|<R_c} \mathrm{d}^D x \phi(x) . \tag{2.66}$$

The characteristic value of the field $\psi$ is

$$\psi_c \equiv \sqrt{\langle \psi^2 \rangle} \sim \left( R_c^D \int_{|x|<R_c} \mathrm{d}^D x \langle \phi(0) \phi(x) \rangle \right)^{1/2} \sim R_c^{(D+2-\eta)/2} . \tag{2.67}$$

The critical value of the external field $h_c(\tau)$ is defined by the condition

$$\psi_c h_c \sim T \ (= 1) . \tag{2.68}$$

Therefore, at this value of the field

$$R_c(h) \sim h^{-2/(D+2-\eta)} , \tag{2.69}$$

which yields Eqn (2.59).

On the other hand, $\psi_c \sim \phi_0 R_c$. Using condition (2.68), the result (2.69), and the definition $\phi_0 \sim h^{1/\delta}$, one gets

$$\psi_c \sim \frac{1}{h} \sim h^{1/\delta} h^{-2D/(D+2-\eta)} . \tag{2.70}$$

Simple algebra gives the result, which is the same as Eqn (2.56).

In actual calculations, one usually obtains the critical exponent of the correlation length $\nu$ and that of the correlation function $\eta$, while the rest of the exponents are derived from the relations (2.55) – (2.59) automatically.

## 2.5 Scaling

The concepts of the critical exponents and the correlation length are crucial for the theory of the second-order phase transitions. In the scaling theory of the critical phenomena, it is implied that $R_c$ is the only relevant spatial scale which exists in the system near $T_c$. As we have seen in the GL mean-field approach discussed above, at scales smaller than $R_c$ all the spatial correlations are power-like, which means that at scales much smaller than the correlation length everything must be scale-invariant. On the other hand, at the phase transition point the correlation length becomes infinite. Therefore, the properties of the system at scales smaller then $R_c$ must be equivalent to those of the whole system at the phase transition point.

The other important consequence of scale invariance is that the microscopic details of a system (lattice struc-ture, etc.) should not be expected to affect the critical behaviour. What may appear to be relevant for the critical properties of a system are only its 'global' characteristics, such as space dimensionality, topology of the order parameter, etc.

All the above arguments make a basis for the so-called *scaling hypothesis*, according to which the macroscopic properties of a system at the critical point do not change after a global change in the spatial scale.

Let us consider, in brief, what the immediate general consequences of such a statement would be. Let the Hamiltonian of a system be the following:

$$H = \int \mathrm{d}^D x \left\{ \frac{1}{2} \left[\nabla\phi(x)\right]^2 + \sum_{n=1} h_n \phi^n(x) \right\} . \tag{2.71}$$

Here the parameters $h_n$ describe a concrete system under consideration. In particular: $h_1 \equiv -h$ is the external field; $h_2 \equiv \tau$ is the 'mass' in the Ginzburg – Landau theory; $h_4 \equiv g/4$; and the rest of the parameters could describe some other types of interactions.

After the scale transformation

$$x \to \lambda x, \quad \lambda > 1 , \tag{2.72}$$

one gets

$$\frac{1}{2} \int \mathrm{d}^D x \left[\nabla\phi(x)\right]^2 \to \frac{1}{2} \lambda^{D-2} \int \mathrm{d}^D x \left[\nabla\phi(\lambda x)\right]^2 ,$$

$$\int \mathrm{d}^D x \, \phi^n(x) \to \lambda^D \int \mathrm{d}^D x \, \phi^n(\lambda x) . \tag{2.73}$$

To leave the gradient term of the Hamiltonian (which is responsible for the scaling of the correlation functions) unchanged, one has to rescale the fields

$$\phi(\lambda x) \to \lambda^{-\Delta_\phi} \phi(x) \tag{2.74}$$

with

$$\Delta_\phi = \frac{D-2}{2} . \tag{2.75}$$

The *scale dimensions* $\Delta_\phi$ define the critical exponent of the correlation function

$$G(x) = \langle \phi(0)\phi(x) \rangle \sim |x|^{-2\Delta_\phi} . \tag{2.76}$$

To leave the Hamiltonian (2.71) unchanged after these transformations, one must also rescale the parameters $h_n$

$$h_n \to \lambda^{-\Delta_n} h_n , \tag{2.77}$$

where

$$\Delta_n = \frac{1}{2}(2-n)D + n . \tag{2.78}$$

The quantities $\Delta_n$ are called the *scale dimensions* of the corresponding parameters $h_n$. In particular

$$\Delta_1 \equiv \Delta_h = \frac{1}{2}D + 1 , \tag{2.79}$$

$$\Delta_2 \equiv \Delta_\tau = 2 , \tag{2.80}$$

$$\Delta_4 \equiv \Delta_g = 4 - D . \tag{2.81}$$

Correspondingly, the rescaled parameters $h_\lambda$, $\tau_\lambda$, and $g_\lambda$ of the Ginzburg–Landau Hamiltonian are

$$h_\lambda = \lambda^{\Delta_h} h , \tag{2.82}$$

$$\tau_\lambda = \lambda^{\Delta_\tau} \tau , \tag{2.83}$$

$$g_\lambda = \lambda^{\Delta_g} g . \tag{2.84}$$

These equations demonstrate the following points.

(1) If the initial value of the 'mass' $\tau$ is nonzero, then the scale transformations make the value of the rescaled $\tau_\lambda$ grow, and at the scale

$$\lambda_c \equiv R_c = |\tau|^{-1/\Delta_\tau} \tag{2.85}$$

the value of $\tau_\lambda$ becomes of the order of 1. This indicates that at $\lambda > R_c$ we are getting out of the scaling region, and the value $R_c$ must be called the correlation length. Moreover, according to Eqn (2.85), for the critical exponent of the correlation length we find that

$$\nu = \frac{1}{\Delta_\tau} . \tag{2.86}$$

(2) The value (and the critical exponent) of the critical field $h_c(\tau)$ can be obtained from Eqns (2.79) and (2.82) along the same lines:

$$h_\lambda \Big|_{\lambda = R_c} = R_c^{\Delta_h} h_c \sim 1$$

$$\Rightarrow h_c \sim R_c^{-\Delta_h} \sim |\tau|^{\Delta_h/\Delta_\tau} . \tag{2.87}$$

(3) If the dimensions $D$ of the system are greater than 4, then according to Eqns (2.81) and (2.84), $\Delta_g < 0$, and the rescaled value of the parameter $g_\lambda$ tends to zero at infinite scales. Therefore, the theory becomes asymptotically Gaussian in this case. That is why the systems with dimensions $D > 4$ are described correctly by the Ginzburg–Landau theory.

On the other hand, at dimensions $D < 4$, $\Delta_g > 0$, and the rescaled value of $g_\lambda$ grows as the scale increases. In this case, the situation becomes highly nontrivial because the asymptotic (infinite scale) theory becomes non-Gaussian. Nevertheless, if the dimensions $D$ are formally taken to be close to 4, such that the value of $\varepsilon = 4 - D$ is treated as the small parameter, then the deviation from the Gaussian theory is also small in $\varepsilon$, and this allows us to treat such systems in terms of the perturbation theory (see Section 2.6). In the lucky case, if for some reason the series in $\varepsilon$ would appear to be 'good' and quickly converging, then one could hope to get the critical exponents close to the real ones if we set $\varepsilon = 1$ in the final results.

It is a miracle, but although the actual series in $\varepsilon$ can by no means be considered as 'good' (it is not even converging), the results for the critical exponents given by the first three terms of the series at $\varepsilon = 1$ ($D = 3$) appear to be very close to the real ones.

### 2.6 Renormalisation-group approach and $\varepsilon$-expansion

Let us assume that at large scales the asymptotic theory is described by the Hamiltonian (2.39) (for simplicity, the external field $h$ is taken to be zero):

$$H = \int \mathrm{d}^D x \left\{ \frac{1}{2} \left[\nabla\phi(x)\right]^2 + \frac{1}{2} \tau\phi^2(x) + \frac{1}{4} g\phi^4(x) \right\}, \tag{2.88}$$

where the field $\phi(x)$ is supposed to be slowly-varying in space, such that the Fourier-transformed field $\phi(k)$,

$$\phi(x) = \int_{|k|<k_0} \frac{\mathrm{d}^D k}{(2\pi)^D} \, \phi(k) \exp(\mathrm{i}kx) , \tag{2.89}$$

has only long-wave components: $|k| < k_0 \ll 1$. The parameters of the Hamiltonian are also assumed to be small:

$|\tau| \ll 1$; $g \ll 1$. Correspondingly, the Fourier-transformed Hamiltonian is

$$H_{k_0} = \frac{1}{2}\int_{|k|<k_0} \frac{\mathrm{d}^D k}{(2\pi)^D}\, k^2 |\phi(k)|^2 + \frac{1}{2}\,\tau \int_{|k|<k_0} \frac{\mathrm{d}^D k}{(2\pi)^D}\, |\phi(k)|^2 + \frac{1}{4}\, g \int_{|k|<k_0} \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2\, \mathrm{d}^D k_3\, \mathrm{d}^D k_4}{(2\pi)^{4D}} \times \phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\, \delta(k_1 + k_2 + k_3 + k_4). \quad (2.90)$$

In the most general terms, the problem is to calculate the partition function

$$Z = \left[\prod_{k=0}^{k_0} \int \mathrm{d}\phi(k)\right] \exp\left(-H_{k_0}[\phi]\right) , \quad (2.91)$$

and the corresponding free energy $F = -\ln Z$.

The idea of the renormalisation-group (RG) approach is described below.

In the *first step* one integrates only over the components of the field $\phi(k)$ in the limited wave band $\lambda k_0 < k < k_0$, where $\lambda \ll 1$. In the result, we get a new Hamiltonian which would depend on the new cutoff$\lambda k_0$:

$$\exp\left(-\tilde{H}_{\lambda k_0}[\phi]\right) \equiv \left[\prod_{k=\lambda k_0}^{k_0} \int \mathrm{d}\phi(k)\right] \exp\left(-H_{k_0}[\phi]\right) . \quad (2.92)$$

It is expected that under certain conditions the new Hamiltonian $\tilde{H}_{\lambda k_0}[\phi]$ would have a structure similar to the original one, given by Eqn (2.90):

$$\tilde{H}_{\lambda k_0} = \frac{1}{2}\,\tilde{a}(\lambda) \int_{|k|<\lambda k_0} \frac{\mathrm{d}^D k}{(2\pi)^D}\, k^2 |\phi(k)|^2 + \frac{1}{2}\,\tilde{\tau}(\lambda) \times \int_{|k|<\lambda k_0} \frac{\mathrm{d}^D k}{(2\pi)^D}\, |\phi(k)|^2 + \frac{1}{4}\,\tilde{g}(\lambda) \times \int_{|k|<\lambda k_0} \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2\, \mathrm{d}^D k_3\, \mathrm{d}^D k_4}{(2\pi)^{4D}} \times \phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\delta(k_1 + k_2 + k_3 + k_4) + (\ldots) . \quad (2.93)$$

All the additional terms which could appear in $\tilde{H}_{\lambda k_0}[\phi]$ after the integration in Eqn (2.92) [denoted by '(...)'] will be shown to be irrelevant for $\tau \ll 1$, $g \ll 1$, $\lambda \ll 1$, and $\varepsilon = (4 - D) \ll 1$. In fact, the leading terms in Eqn (2.93) will be shown to be large with respect to the parameter $\xi \equiv \ln(1/\lambda) \gg 1$, conditioned such that $\varepsilon \ln(1/\lambda) \ll 1$.

In the *second step* one makes the inverse scaling transformation (see Section 2.5) with the aim of restoring the original cutoffscale $k_0$:

$$k \to \lambda k ,$$

$$\phi(\lambda k) \to \theta(\lambda)\phi(k) . \quad (2.94)$$

The parameter $\theta(\lambda)$ should be chosen such that the coefficient of the $k^2|\phi(k)|^2$ term remains the same as in the original Hamiltonian (2.90):

$$\theta = \lambda^{-(D+2)/2}\left[\tilde{a}(\lambda)\right]^{-1/2} . \quad (2.95)$$

The two steps given above compose the so-called *renormalisation transformation*. The renormalised Hamiltonian is

$$H_{k_0}^{(R)} = \frac{1}{2}\int_{|k|<k_0} \frac{\mathrm{d}^D k}{(2\pi)^D}\, k^2 |\phi(k)|^2 + \frac{1}{2}\,\tau^{(R)}(\lambda) \int_{|k|<k_0} \frac{\mathrm{d}^D k}{(2\pi)^D}\, |\phi(k)|^2 + \frac{1}{4}\, g^{(R)}(\lambda) \int_{|k|<k_0} \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2\, \mathrm{d}^D k_3\, \mathrm{d}^D k_4}{(2\pi)^{4D}} \times \phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\delta(k_1 + k_2 + k_3 + k_4) . \quad (2.96)$$

This Hamiltonian depends once again on the original cutoff$k_0$, whereas its parameters are renormalised:

$$\tau^{(R)}(\lambda) = \lambda^{-2}\, \tilde{a}(\lambda)^{-1}\, \tilde{\tau}(\lambda) , \quad (2.97)$$

$$g^{(R)}(\lambda) = \lambda^{-(4-D)}\, \tilde{a}(\lambda)^{-2}\, \tilde{g}(\lambda) . \quad (2.98)$$

The above RG transformation must be applied (infinitely) many times, and then the problem is to study the limiting properties of the renormalised Hamiltonian, which is expected to describe the asymptotic (infinite scale) properties of the system. In particular, it is hoped that the limiting Hamiltonian would arrive at some fixed-point Hamiltonian $H^*$ which would be invariant with respect to the above RG transformation. The hypothesis about the existence of the fixed-point (non-Gaussian) Hamiltonian $H^*$, which would be invariant with respect to the scale transformations in the critical point, is nothing but a more conventional formulation of the scaling hypothesis discussed in the Section 2.5.

Let us consider the RG procedure in some more detail. To get the RG Eqns (2.97) and (2.98) in explicit form, one has to obtain the parameters $\tilde{a}(\lambda)$, and $\tilde{\tau}(\lambda), \tilde{g}(\lambda)$ by integrating over the 'fast' degrees of freedom in Eqn (2.92).

Let us separate the 'fast' fields (with $\lambda k_0 < |k| < k_0$) and the 'slow' fields (with $|k| < \lambda k_0$) explicitly:

$$\phi(x) = \tilde{\phi}(x) + \varphi(x) ,$$

$$\tilde{\phi}(x) = \int_{|k|<\lambda k_0} \frac{\mathrm{d}^D k}{(2\pi)^D}\, \tilde{\phi}(k) \exp(\mathrm{i}kx) ,$$

$$\varphi(x) = \int_{\lambda k_0<|k|<k_0} \frac{\mathrm{d}^D k}{(2\pi)^D}\, \varphi(k) \exp(\mathrm{i}kx) . \quad (2.99)$$

Then the Hamiltonian (2.90) can be represented as follows:

$$H_{k_0}[\tilde{\phi}, \varphi] = H_{\lambda k_0}[\tilde{\phi}] + \frac{1}{2}\int_{\lambda k_0<|k|<k_0} \frac{\mathrm{d}^D k}{(2\pi)^D}\, G_0^{-1}(k) |\varphi(k)|^2 + V[\tilde{\phi}, \varphi] , \quad (2.100)$$

where

$$G_0(k) = k^{-2} , \quad (2.101)$$

$$V[\tilde{\phi},\varphi] = \frac{1}{2}\tau \int_{\lambda k_0<|k|<k_0} \frac{\mathrm{d}^D k}{(2\pi)^D}\, |\varphi(k)|^2$$
$$+g \int \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2\, \mathrm{d}^D k_3\, \mathrm{d}^D k_4}{(2\pi)^{4D}} \Big[\frac{3}{2}\tilde{\phi}(k_1)\tilde{\phi}(k_2)\varphi(k_3)\varphi(k_4)$$
$$+\tilde{\phi}(k_1)\varphi(k_2)\varphi(k_3)\varphi(k_4) + \tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\varphi(k_4)$$
$$+\frac{1}{4}\,\varphi(k_1)\varphi(k_2)\varphi(k_3)\varphi(k_4)\Big]\delta(k_1+k_2+k_3+k_4)\ . \tag{2.102}$$

In standard diagram notations, the interaction term $V[\tilde{\phi},\varphi]$ is shown in Fig. 2, where the wavy lines represent the 'slow' fields $\tilde{\phi}$, the straight lines represent the 'fast' fields $\varphi$, the solid circle represents the 'mass' $\tau$, the open circle represents the interaction vertex $g$, and at each vertex the sum of entering 'impulses' $k$ is zero.

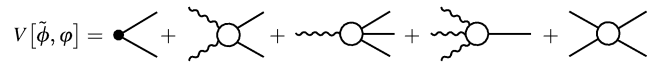


**Figure 2.** Diagram representation of the interaction energy $V[\tilde{\phi},\varphi]$.

Then, the integration over the $\varphi$ values, Eqn (2.92), yields:

$$\exp\big(-\tilde{H}_{\lambda k_0}[\tilde{\phi}]\big) = \exp\big(-H_{\lambda k_0}[\tilde{\phi}]\big)\big\langle \exp\big(-V[\tilde{\phi},\varphi]\big)\big\rangle\,, \tag{2.103}$$

where the averaging $\langle(\ldots)\rangle$ is performed as follows:

$$\langle(\ldots)\rangle \equiv \left[\prod_{k=\lambda k_0}^{k=k_0} \int \mathrm{d}\varphi(k)\right]$$
$$\times \exp\left[-\frac{1}{2}\int_{\lambda k_0<|k|<k_0} \frac{\mathrm{d}^D k}{(2\pi)^D}\, G_0^{-1}(k)\big|\varphi(k)\big|^2\right](\ldots)\ . \tag{2.104}$$

Standard perturbation expansion in $V$ gives

$$\tilde{H}_{\lambda k_0}[\tilde{\phi}] = H_{\lambda k_0}[\tilde{\phi}] + \langle V\rangle - \frac{1}{2}\left[\langle V^2\rangle - \langle V\rangle^2\right]\ . \tag{2.105}$$

In terms of the diagrams (Fig. 2) the averaging $\langle\ldots\rangle$ is just the pairing of the straight lines. The nonzero contribution to $\langle V\rangle$ is shown in Fig. 3, where each closed loop is

$$\int_{\lambda k_0<|k|<k_0} \frac{\mathrm{d}^D k}{(2\pi)^D}\, G_0(k) = \frac{S_D}{(2\pi)^D (D-2)}\, k_0^{(D-2)}\big(1-\lambda^{(D-2)}\big) \tag{2.106}$$

(here $S_D$ is the surface area of a unit $D$-dimensional sphere).

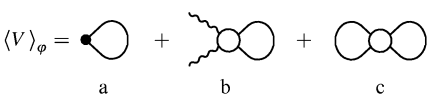


**Figure 3.** Diagrammatic representation of the first-order perturbation contribution $\langle V\rangle$.

In what follows we are going to study the limiting case of the small cutoff $k_0$ (large spatial scales). Besides, at each RG step the rescaling parameter $\lambda$ will also be assumed to be small, such that in all the integrations over the 'internal' $k$ values ($\lambda k_0 < |k| < k_0$), the 'external' $k$ values ($|k| < \lambda k_0$) could be considered as negligibly small.

The result for the first-order perturbation expansion $\langle V\rangle$ consists of three contributions. The diagrams (a) and (c) in Fig. 3 produce only irrelevant constants (they do not depend on $\tilde{\phi}$). The diagram (b) is proportional to $|\tilde{\phi}(k)|^2$ and gives the contribution to the mass term, but since this contribution is proportional to $k_0^{(D-2)}$, in the asymptotic region $k_0 \to 0$ it could be ignored as well. In fact we are going to look for the contributions, which: (1) do not depend on the value of the cutoff $k_0$; and (2) are large in the RG parameter $\xi \equiv \ln(1/\lambda) \gg 1$.

Consider the second-order perturbation contribution $\langle\langle V^2\rangle\rangle \equiv \langle V^2\rangle - \langle V\rangle^2$ (Fig. 4). Here the diagrams (a), (c), and (i) give irrelevant constants. The diagrams (d), (g), and (h) are proportional to the positive power of the cutoff $k_0$ and therefore their contribution is small.

The relevant diagrams are (b), (e), and (f). The diagram (e) is proportional to

$$\int_{|k|<\lambda k_0} \frac{\mathrm{d}^D k}{(2\pi)^D}\, |\tilde{\phi}(k)|^2$$
$$\times \int_{\lambda k_0<|k_{1,2}|<k_0} \mathrm{d}^D k_1\, \mathrm{d}^D k_2\, G_0(k_1)G_0(k_2)G_0(k+k_1+k_2)$$
$$= \int_{|k|<\lambda k_0} \frac{\mathrm{d}^D k}{(2\pi)^D}\, |\tilde{\phi}(k)|^2 \int_{\lambda k_0<|k_{1,2}|<k_0} \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2}{k_1^2 k_2^2 (k+k_1+k_2)^2}\ . \tag{2.107}$$

Since $k \ll k_{1,2}$, the principal contribution in Eqn (2.107) is given by the first terms of the expansion in $k/k_{1,2}$

$$g^2 \int_{|k|<\lambda k_0} \frac{\mathrm{d}^D k}{(2\pi)^D}\, |\tilde{\phi}(k)|^2 \int_{\lambda k_0<|k_{1,2}|<k_0} \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2}{k_1^2 k_2^2 (k_1+k_2)^2}$$
$$+3g^2 \int_{|k|<\lambda k_0} \frac{\mathrm{d}^D k}{(2\pi)^D}\, |\tilde{\phi}(k)|^2 k^2$$
$$\times \int_{\lambda k_0<|k_{1,2}|<k_0} \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2}{k_1^2 k_2^2 (k_1+k_2)^4}\ . \tag{2.108}$$

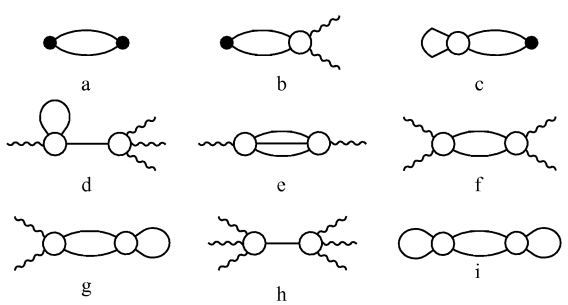


**Figure 4.** Diagrammatic representation of the second-order perturbation contribution $\langle\langle V^2\rangle\rangle$.

The first contribution in Eqn (2.108) is of the order of $k_0^{(D-2)}$ and is therefore irrelevant. As for the second contribution, it could be easily checked that at dimensions $D = 4 - \varepsilon$, where $\varepsilon \ll 1$, the integration over $k_1$ and $k_2$ does yield the factor proportional to $\ln(1/\lambda) \gg 1$ independent of the cutoff$k_0$. Therefore this diagram gives a finite contribution of the order of $g^2 \ln(1/\lambda)$ into $\tilde{a}$ [Eqn (2.93)]. However, as will be demonstrated below, the renormalised fixed-point value of $g$ appears to be of the order of $\varepsilon$. It means that the diagram in Fig. 3e gives the contribution of the order of $\varepsilon^2 \ln(1/\lambda)$ in $\tilde{a}$ (which provides the correction of the order of $\varepsilon^2$ into the critical exponents). Therefore until we study only the first order in $\varepsilon$ corrections, the contribution of the diagram (e) should not be taken into account:

$$\tilde{a} = 1 + O(g^2)\xi\,, \tag{2.109}$$

where $\xi \equiv \ln(1/\lambda)$.

Fig. 4b gives the following contribution:

$$\frac{3}{2}\, g\tau \int_{\lambda k_0 < |k| < k_0} \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{k^4} \int_{|k| < \lambda k_0} \frac{\mathrm{d}^D k}{(2\pi)^D} \left|\tilde{\phi}(k)\right|^2$$
$$= \frac{3}{2}\, g\tau \frac{S_D}{(2\pi)^D} \frac{k_0^{(D-4)}(1 - \lambda^{(D-4)})}{D-4} \int_{|k| < \lambda k_0} \frac{\mathrm{d}^D k}{(2\pi)^D} \left|\tilde{\phi}(k)\right|^2 . \tag{2.110}$$

For $D = 4 - \varepsilon$, where $\varepsilon \ll 1$, this gives the finite contribution to the parameter $\tilde{\tau}$,

$$\tilde{\tau} = \tau - \frac{3}{8\pi^2}\, \tau g \xi \tag{2.111}$$

(we have taken $S_{D=4} = 2\pi^2$).

For Fig. 4f one gets

$$\frac{9}{4}\, g^2 \int_{\lambda k_0 < |k| < k_0} \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{k^4}$$
$$\times \int_{|k| < \lambda k_0} \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2\, \mathrm{d}^D k_3\, \mathrm{d}^D k_4}{(2\pi)^{4D}}\, \tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4)$$
$$= \frac{9}{4}\, g^2 \frac{S_D}{(2\pi)^D} \frac{k_0^{(D-4)}(1 - \lambda^{(D-4)})}{D-4}$$
$$\times \int_{|k| < \lambda k_0} \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2\, \mathrm{d}^D k_3\, \mathrm{d}^D k_4}{(2\pi)^{4D}}\, \tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4) . \tag{2.112}$$

For $D = 4 - \varepsilon$, this gives the following contribution:

$$\tilde{g} = g - \frac{9}{8\pi^2}\, g^2 \xi\,. \tag{2.113}$$

After the operation of rescaling to the original cutoff$k_0$, according to the Eqns (2.97) and (2.98) for the renormalised parameters $\tau^{(R)}$ and $g^{(R)}$, we get

$$\tau^{(R)} = \left(\tau - \frac{3}{8\pi^2}\, \tau g \xi\right) \exp(2\xi)\,,$$
$$g^{(R)} = \left(g - \frac{9}{8\pi^2}\, g^2 \xi\right) \exp(\varepsilon\xi)\,. \tag{2.114}$$

When $g\xi \ll 1$ and $\varepsilon\xi \ll 1$, these equations can be written as follows:

$$\ln(\tau^{(R)}) - \ln(\tau) = 2\xi - \frac{3}{8\pi^2}\, g\xi\,,$$
$$g^{(R)} - g = \varepsilon g \xi - \frac{9}{8\pi^2}\, g^2 \xi\,. \tag{2.115}$$

Assuming that the RG procedure is performed continuously, the evolution (as the scale changes) of the renormalised parameters could be described in terms of the differential equations. From Eqns (2.115), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \ln|\tau| = 2 - \frac{3}{8\pi^2}\, g\,, \tag{2.116}$$

$$\frac{\mathrm{d}g}{\mathrm{d}\xi} = \varepsilon g - \frac{9}{8\pi^2}\, g^2\,. \tag{2.117}$$

The fixed-point solution $g^*$ is defined by the condition $\mathrm{d}g/\mathrm{d}\xi = 0$, which yields

$$g^* = \frac{8\pi^2}{9}\, \varepsilon\,. \tag{2.118}$$

Then, from Eqn (2.116) for the scale dimensions $\Delta_\tau$ we find that

$$\Delta_\tau = 2 - \frac{1}{3}\, \varepsilon\,. \tag{2.119}$$

Correspondingly, according to Eqn (2.86) for the critical exponent $\nu$, we obtain

$$\nu = \frac{1}{2} + \frac{1}{12}\, \varepsilon\,. \tag{2.120}$$

Since the fixed-point value $g^*$ is of the order of $\varepsilon$, according to Eqns (2.109), (2.94), and (2.95) there are no corrections in the first order in $\varepsilon$ to the scale dimensions $\Delta_\phi$ of the field $\phi$. Accordingly [see Eqns (2.76), and (2.75)], in the first order in $\varepsilon$ the critical exponent $\eta$ [see Eqn (2.54)] of the correlation function $\langle\phi(0)\phi(x)\rangle$ remains zero, as in the Ginzburg–Landau theory.

Using relations (2.55) – (2.59), one can now easily find all the other critical exponents

$$\alpha = \frac{1}{6}\, \varepsilon\,, \quad \gamma = 1 + \frac{1}{6}\, \varepsilon\,, \quad \beta = \frac{1}{2} - \frac{1}{6}\, \varepsilon\,,$$
$$\delta = 3 + \varepsilon\,, \quad \mu = \frac{1}{3}\,. \tag{2.121}$$

In Table 1 we give the values of the critical exponents in the first order in $\varepsilon$ formally continued for dimensions $D = 3$ ($\varepsilon = 1$). These are compared with the corresponding values given by numerical simulations and the Ginzburg–Landau theory.

**Table 1**

| Physical quantities and corresponding critical exponents | | $\varepsilon$-expansion | Numerical simulations | Ginzburg–Landau theory |
|---|---|---|---|---|
| Specific heat | $\alpha$ | 0.167 | $0.125 \pm 0.015$ | 0 |
| Susceptibility | $\gamma$ | 1.167 | $1.250 \pm 0.003$ | 1 |
| Correlation length | $\nu$ | 0.583 | $0.642 \pm 0.003$ | 0.5 |
| Correlation function | $\eta$ | 0 | $0.055 \pm 0.010$ | 0 |
| Order parameters | $\beta$ | 0.333 | $0.312 \pm 0.003$ | 0.5 |
| | $\delta$ | 4 | $5.15 \pm 0.02$ | 3 |

To obtain results in the second order in $\varepsilon$, one proceeds in a similar way taking into account $g$ diagrams of the next order (see, for example, Refs [2], [3]).

It is interesting to note that although the RG $\varepsilon$-expansion procedure discussed above is mathematically not well grounded, it provides rather accurate values for the critical exponents.

### 2.7 Specific heat singularity in four dimensions

Note also that although in dimensions $D = 4$ the critical exponent $\alpha$ is zero, it does not necessarily mean that the specific heat is not singular at the critical point. Actually in this case, the specific heat is logarithmically (and not power-law) divergent. As a useful exercise, let us calculate the specific heat singularity for the four dimensions.

According to the definition of the specific heat [see Eqns (2.60), (2.61)] we have

$$C = -T\,\frac{\partial^2 f}{\partial T^2} = \frac{1}{V}\int \mathrm{d}^4x \int \mathrm{d}^4x'\,\langle\langle \phi^2(x)\phi^2(x')\rangle\rangle$$

$$= \int_{|x|<R_c(\tau)} \mathrm{d}^4x\,\langle\langle \phi^2(0)\phi^2(x)\rangle\rangle\,. \tag{2.122}$$

Here the upper cutoffin the spatial integration is taken to be the correlation length $R_c(\tau) \sim |\tau|^{-1/2}$, since for larger scales all the correlations decay exponentially. The integral in Eqn (2.122) can be calculated by summing up the so-called 'parquette' diagrams [4] shown in Fig. 5. The idea of the 'parquette' calculations is that all the contributions from the $\phi^4$ interactions in the correlation function $\langle\langle \phi^2(x)\phi^2(x')\rangle\rangle$ can be collected into the mass-like vertex $m(\xi)$

$$C \simeq \int_{|k|>\sqrt{\tau}} \frac{\mathrm{d}^4k}{(2\pi)^4}\,G_0^2(k)\left[\frac{m(k)}{\tau}\right]^2 \sim \int_{|k|>\sqrt{\tau}} \frac{\mathrm{d}k}{k}\left[\frac{m(k)}{\tau}\right]^2$$

$$\sim \int_{\xi<\ln(1/\tau)} \mathrm{d}\xi\left[\frac{m(\xi)}{\tau}\right]^2. \tag{2.123}$$

Here the renormalisation of the 'dressed' mass $m(\xi)$ is defined by the diagram shown in Fig. 5b [also see Eqns (2.110) – (2.113)]

$$m^{(R)} = m - 3mg\int_{\lambda k_0<|k|<k_0} \frac{\mathrm{d}^Dk}{(2\pi)^D}\,G_0^2(k) \to m - \frac{3}{8\pi^2}\,mg\xi\,, \tag{2.124}$$

where, as usual, $\xi \equiv \ln(1/\lambda)$. In differential form

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\,m(\xi) = -\frac{3}{8\pi^2}\,m(\xi)g(\xi) \tag{2.125}$$

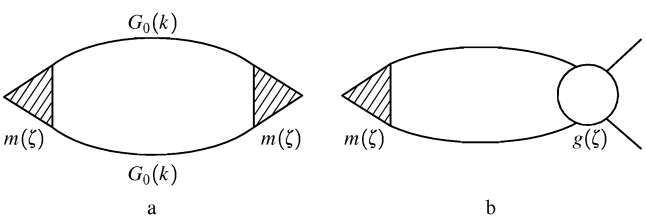


**Figure 5.** (a) Diagrammatic representation of the specific heat. (b) The diagram which contributes to the renormalisation of the 'dressed' mass $m(\xi)$.

with the initial conditions $m(\xi = 0) = \tau$. The renormalisation of the interaction parameter $g(\xi)$ for the dimensions $D = 4$ is defined by the RG Eqn (2.117) with $\varepsilon = 0$:

$$\frac{\mathrm{d}g(\xi)}{\mathrm{d}\xi} = -\frac{9}{8\pi^2}\,g^2(\xi)\,. \tag{2.126}$$

The solutions of Eqns (2.125) and (2.126) are

$$m(\xi) = \tau\left(1 + \frac{9g}{8\pi^2}\,\xi\right)^{-1/3},$$

$$g(\xi) = g\left(1 + \frac{9g}{8\pi^2}\,\xi\right)^{-1}, \tag{2.127}$$

where $g \equiv g(\xi = 0)$. Then for the specific heat, Eqn (2.123), one gets

$$C(\tau) \simeq \int_{\xi<\ln(1/\tau)} \frac{\mathrm{d}\xi}{\left[1 + (9g/8\pi^2)\xi\right]^{2/3}}$$

$$= \frac{8\pi^2}{3g}\left[\left(1 + \frac{9g}{8\pi^2}\ln\frac{1}{\tau}\right)^{1/3} - 1\right]. \tag{2.128}$$

This result demonstrates that there exists a characteristic temperature interval

$$\tau_g \sim \exp\left(-\frac{8\pi^2}{9g}\right) \ll 1\,, \tag{2.129}$$

such that at temperatures not too close to $T_c$, $\tau_g \ll |\tau| \ll 1$, the system is Gaussian (it does not depend on the non-Gaussian interaction parameter $g$):

$$C(\tau) \sim \ln\frac{1}{\tau}\,. \tag{2.130}$$

This result could be easily obtained just in the framework of the Gaussian Ginzburg – Landau theory:

$$C(\tau) \sim \int \mathrm{d}^4x\,\langle\langle \phi^2(0)\phi^2(x)\rangle\rangle \sim \int_{|k|<1} \mathrm{d}^4k\,(k^2+\tau)^{-2}$$

$$\sim \int_{\sqrt{\tau}}^{1} \frac{\mathrm{d}^4k}{k^4} \sim \ln\frac{1}{\tau}\,. \tag{2.131}$$

On the other hand, in the close vicinity of the critical point ($\tau \ll \tau_g$) the theory becomes non-Gaussian, and the result for the specific heat becomes less trivial

$$C(\tau) \sim \frac{1}{g}\left(g\ln\frac{1}{\tau}\right)^{1/3}. \tag{2.132}$$

Thus although the critical exponent $\alpha$ is zero for the 4-dimensional system, the specific heat still remains (logarithmically) divergent at the critical point.

## 3. Critical behaviour in systems with impurities

### 3.1 Harris criterion

In the studies of the phase transition phenomena, the systems considered before were assumed to be perfectly homogeneous. In real physical systems, however, some defects or impurities are always present. Therefore, it is natural to consider what effect the impurities might have on the phase transition phenomena. As we have seen in Section 2, the thermodynamics of the second-order phase transition is dominated by large-scale fluctuations. The dominant scale, or the correlation length, $R_c \sim |T/T_c - 1|^{-\nu}$ grows as $T$ approaches the critical temperature $T_c$, where it becomes infinite. The large-

scale fluctuations lead to singularities in the thermodynamical functions as $|\tau| \equiv |T/T_c - 1| \to 0$. These singularities are the main subject of the theory.

If the concentration of impurities is small, their effect on the critical behaviour remains negligible so long as $R_c$ is not too large, i.e. for $T$ not too close to $T_c$. In this regime, the critical behaviour will be essentially the same as in the perfect system. However, as $|\tau| \to 0$ $(T \to T_c)$ and $R_c$ becomes greater than the average distance between impurities, their influence can become crucial.

As $T_c$ is approached, the following change of the scale takes place. First, the correlation length of the fluctuations becomes much larger than the lattice spacing, and the system 'forgets' about the lattice. The only relevant scale that remains in the system in this regime is the correlation length $R_c(\tau)$. When we move close to the critical point, $R_c$ grows and becomes larger than the average distance between the impurities, so that the effective concentration of impurities, measured with respect to the correlation length, becomes large. It should be stressed that such a situation is reached for an arbitrary small initial concentration $u$. The value of $u$ affects only the width of the temperature region near $T_c$ in which the effective concentration becomes effectively large. If $uR_c^D \gg 1$, there are no grounds for believing that the effect of impurities will be small.

Originally, many years ago, it was generally believed that impurities either completely destroy the long-range fluctuations, such that the singularities of the thermodynamical functions are smoothed out [5, 6], or can produce only a shift of a critical point but cannot affect the critical behaviour itself, so that the critical exponents remain the same as in the pure system [7]. Later it was realised that an intermediate situation is also possible, in which a new critical behaviour, with new critical exponents, is established sufficiently close to the phase transition point [8]. Moreover, a criterion, the so-called Harris criterion, has also been developed, which makes it possible to predict qualitatively the effect of impurities by using the critical exponents of the pure system only [6, 8]. According to this criterion, the impurities change the critical behaviour only if the specific heat exponent $\alpha$ of the pure system is greater than zero (the specific heat of the pure system is divergent at the critical point). In the opposite case, $\alpha < 0$ (the specific heat if finite), the impurities appear to be irrelevant, i.e. their presence does not affect the critical behaviour.

Let us consider this point in more detail. It would be natural to assume that, in the $\phi^4$ Hamiltonian (Section 2.6), the presence of impurities manifests itself as small random spatial fluctuations of the reduced transition temperature $\tau$. Then near the phase transition point, the $D$-dimensional Ising-like systems can be described in terms of, for a scalar field, the Ginzburg–Landau Hamiltonian with a double-well potential

$$H = \int \mathrm{d}^D x \left\{ \frac{1}{2} \left[\nabla\phi(x)\right]^2 + \frac{1}{2}\left[\tau - \delta\tau(x)\right]\phi^2(x) + \frac{1}{4} g\phi^4(x) \right\} . \tag{3.1}$$

Here the quenched disorder is described by random fluctuations of the effective transition temperature $\delta\tau(x)$ whose probability distribution is taken to be symmetric and Gaussian:

$$P[\delta\tau] = p_0 \exp\left\{ -\frac{1}{4u} \int \mathrm{d}^D x \left[\delta\tau(x)\right]^2 \right\} , \tag{3.2}$$

where $u \ll 1$ is the small parameter which describes the disorder, and $p_0$ is the normalisation constant. For notational simplicity, we define the sign of $\delta\tau(x)$ in Eqn (3.1) so that positive fluctuations lead to locally ordered regions, whose effects are the object of our study.

Configurations of the fields $\phi(x)$ which correspond to local minima in $H$ satisfy the saddle-point equation

$$-\Delta\phi(x) + \tau\phi(x) + g\phi^3(x) = \delta\tau(x)\phi(x) . \tag{3.3}$$

Such localised solutions exist in regions of space where $\tau - \delta\tau(x)$ assumes negative values. Clearly, the solutions of Eqn (3.3) depend on a particular configuration of the function $\delta\tau(x)$ being inhomogeneous. Let us estimate under which conditions the quenched fluctuations of the effective transition temperature are the dominant factor for the field configurations of the local minima.

Let us consider a large region $\Omega_L$ of a linear size $L \gg 1$. The spatially averaged value of the function $\delta\tau(x)$ in this region could be defined as follows:

$$\delta\tau(\Omega_L) = \frac{1}{L^D} \int_{x \in \Omega_L} \mathrm{d}^D x \, \delta\tau(x) . \tag{3.4}$$

Correspondingly, for the characteristic value of the temperature fluctuations (averaged over realisations) in this region, we get

$$\delta\tau_L = \left[ \overline{\delta\tau^2(\Omega_L)} \right]^{1/2} = \sqrt{2u}\, L^{-D/2} . \tag{3.5}$$

Then, the average value of the order parameter $\phi(\Omega_L)$ in this region can be estimated from the equation

$$\tau + g\phi^2 = \delta\tau(\Omega_L) . \tag{3.6}$$

One can easily see that if the value of $\tau$ is sufficiently small, i.e. if

$$\delta\tau(\Omega_L) \gg \tau , \tag{3.7}$$

then the solutions of Eqn (3.6) are defined only by the value of the random temperature

$$\phi(\Omega_L) \simeq \pm \left[ \frac{\delta\tau(\Omega_L)}{g} \right]^{1/2} . \tag{3.8}$$

Now let us estimate up to which sizes of locally ordered regions this may occur. According to Eqn (3.5), the condition $\delta\tau_L \gg \tau$ yields

$$L \ll \frac{u^{1/D}}{\tau^{2/D}} . \tag{3.9}$$

On the other hand, the estimation of the order parameter in terms of the saddle-point equation (3.6) could be correct only at scales much larger than the correlation length $R_c \sim \tau^{-\nu}$. Thus, one has the lower bound for $L$:

$$L \gg \tau^{-\nu} . \tag{3.10}$$

Therefore, quenched temperature fluctuations are relevant when

$$\tau^{-\nu} \ll \frac{u^{1/D}}{\tau^{2/D}} \tag{3.11}$$

or

$$\tau^{2-\nu D} \ll u . \tag{3.12}$$

According to the scaling relations [see Eqn (2.55)], one has $2 - \nu D = \alpha$. Thus one recovers the Harris criterion: if the

critical exponent of the specific heat of the pure system is positive, then in the temperature interval,

$$\tau < \tau_* \equiv u^{1/\alpha} , \tag{3.13}$$

the disorder becomes relevant. This argument identifies $1/\alpha$ as the cross-over exponent associated with randomness.

A special consideration is required in the marginal situation $\alpha = 0$. This is the case, for instance, for the four-dimensional $\phi^4$-model (Section 3.3), or for the two-dimensional Ising model to be studied in Section 5. The calculations show that, although the critical exponent of the specific heat remains zero in the impurity models, the logarithmic singularities are affected by the disorder.

### 3.2 Self-averaging and the replica method

The main problem in dealing with impurity systems is that the disorder in their interaction parameters is *quenched*. Formally, all the results one may hope to get for the observable quantities for a given concrete system must depend on the concrete interaction matrix $J_{ij}$, i.e. the result would be defined by a macroscopic number of random parameters. Apparently, the results of this type are impossible to calculate and, moreover, they are useless. Intuitively it is clear, however, that the quantities which are called the observables should depend on some general averaged characteristics of the random interactions. This brings us to the concept of the *self-averaging*.

Traditional way of reasoning why the self-averaging phenomenon should occur is as follows. The free energy of the system is known to be proportional to the volume $V$ of the system. Therefore, in the thermodynamic limit $V \to \infty$, the main contribution to the free energy must come from the volume, and not from the boundary, which usually produces the effects of the next orders in the small parameter $1/V$.

Any macroscopic system could be divided into a macroscopic number of macroscopic subsystems. Then the total free energy of the system would consist of the sum of the free energies of the subsystems, plus the contribution which comes from the interactions of the subsystems, at their boundaries. If all the interactions in the system are short range (which takes place in any normal system), then the contributions from the mutual interactions of the subsystems are just the boundary effects which vanish in the thermodynamic limit. Therefore, the total free energy could be represented as a sum of the macroscopic number of terms. Each of these terms would be a random quenched quantity since it contains, as the parameters, the elements of the random spin–spin interaction matrix. Next, in accordance with the law of large numbers, the sum of many random quantities can be represented as their average value, obtained from their statistical distribution, times their number (all this is true, of course, only under certain requirements on the characteristics of the statistical distribution). Therefore, the total free energy of a macroscopic system must be self-averaging over the *realisations* of the random interactions in accordance with their statistical distribution.

The free energy is known to be given by the logarithm of the partition function. Thus in order to calculate the observable thermodynamics, one has to average the logarithm of the partition function over the given distribution of random $J_{ij}$ values after the calculation of the partition function itself. To perform such a program, the following technical trick, which is called the replica method, is used.

Formally, the replicas are introduced as follows. In order to obtain the physical (self-averaging) free energy of the quenched random system we have to average the logarithm of the partition function

$$F \equiv \overline{F_J} = -\frac{1}{\beta}\,\overline{\ln Z_J}, \tag{3.14}$$

where $\overline{(\ldots)}$ denotes the averaging over random interactions $(J_{ij})$ with a given distribution function $P[J]$:

$$\overline{\ln Z_J} \equiv \left(\prod_{\langle i,j\rangle}\int \mathrm{d}J_{ij}\right) P[J]\,\ln Z_J\,, \tag{3.15}$$

and the partition function is

$$Z_J = \sum_{\sigma} \exp\left(-\beta H[\sigma]\right)\,. \tag{3.16}$$

To perform this procedure of the averaging, the following trick is invented. Let us consider the *integer* power $n$ of the partition function (3.16). This quantity is the partition function of the $n$ noninteracting *identical* replicas of the original system (i.e. having identical fixed spin–spin couplings $J_{ij}$)

$$Z_J^n = \left(\prod_{a=1}^{n}\sum_{\sigma^a}\right)\exp\left(\beta\sum_{a=1}^{n}\sum_{i<j}^{N} J_{ij}\sigma_i^a\sigma_j^a\right)\,. \tag{3.17}$$

Here the subscript $a$ denotes the replicas. Let us introduce the quantity

$$F_n = -\frac{1}{\beta n}\ln Z_n\,, \tag{3.18}$$

where

$$Z_n \equiv \overline{Z_J^n}\,. \tag{3.19}$$

Now, if a *formal* limit $n \to 0$ is taken in expression (3.18), the original expression for the physical free energy (3.14) will be recovered:

$$\begin{aligned}\lim_{n\to 0} F_n &= -\lim_{n\to 0}\frac{1}{\beta n}\ln(Z_n) = -\lim_{n\to 0}\frac{1}{\beta n}\ln\left[\overline{\exp(n\ln Z_J)}\right]\\ &= -\frac{1}{\beta}\,\overline{\ln Z_J} = F\,.\end{aligned} \tag{3.20}$$

Thus, the scheme of the replica method can be described in the following steps. First, the quantity $F_n$ for the integer $n$ must be calculated. Second, the analytical continuation of the obtained function of the parameter $n$ should be made for an arbitrary noninteger $n$. Finally, the limit $n \to 0$ has to be taken.

Although this procedure may look rather doubtful at first, it is actually quite logical. First, if the free energy appears to be an analytic function of the temperature and the other parameters ( so that it can be represented as a series in powers of $\beta$), then the replica method can be easily proved to be correct in a strict sense. Second, in all cases, when the calculations can be performed by some other method, the results of the replica method are confirmed.

One could also introduce replicas in another way as described in Refs [9], [10]. Let us consider a general spin system described by some Hamiltonian $H[J;\sigma]$, which depends on the spin variables $\{\sigma_i\}$ $(i = 1,\ldots,N)$ and the

spin–spin interactions $J_{ij}$ (the concrete form of the Hamiltonian is irrelevant). If the interactions $J_{ij}$ are quenched, the free energy of the system would depend on the concrete realisation of the $J_{ij}$ values

$$F[J] = -\frac{1}{\beta}\log Z[J] \ . \tag{3.21}$$

Now, let us assume that the spin–spin interactions are *partially annealed* (i.e. not perfectly quenched), so that they can also change their values, but the characteristic timescale of their changes is much larger than the timescale at which the degrees of freedom of the spin reach thermal equilibrium. In this case, the free energy given by Eqn (3.21) would still make sense, and it would become the energy function (the Hamiltonian) for the degrees of freedom of $J_{ij}$.

Besides, the space in which the interactions $J_{ij}$ take their values should be specified separately. The interactions $J_{ij}$ could be discrete variables taking on the values $\pm J_0$, or they could be the continuous variables taking on values in some restricted interval, or they could be something else. In the quenched case, this space of $J_{ij}$ values is defined by a statistical distribution function $P[J]$. In the case of the partial annealing, this function $P[J]$ has a meaning of the internal potential for the interactions $J_{ij}$, which restricts the space of their values.

Let us now assume that the spin and the degrees of freedom of the interaction *are not thermally equilibrated*, so that the degrees of freedom of the interaction have their own temperature $T'$, which is different from the temperature $T$ of the degrees of freedom of the spin. In this case for the total partition function of the system, we get

$$\mathcal{Z} = \int \mathrm{D}J\, P[J]\exp\left(-\beta' F[J]\right) = \int \mathrm{D}J\, P[J]\exp\left(\frac{\beta'}{\beta}\log Z_J\right)$$
$$= \int \mathrm{D}J\, P[J](Z_J)^n \equiv \overline{(Z_J)^n}\,, \tag{3.22}$$

where $n = T/T'$. Correspondingly, the total free energy of the system would be

$$\mathcal{F} = -T'\log\left\{\overline{(Z[J])^n}\right\} \ . \tag{3.23}$$

In this way we have arrived at the replica formalism again, in which the 'number of replicas' $n = T/T'$ is still the *finite* parameter.

To obtain the physical (self-averaging) free energy in the case of the quenched random interactions $J_{ij}$, one takes the limit $n \to 0$. From the point of view of partial annealing, this situation corresponds to the limit of the infinite temperature $T'$ in the system of $J_{ij}$ values. This is natural in that in this case the thermodynamics of the degrees of freedom of the spin has no effect on the distribution of the spin–spin interactions.

In the case where the degrees of freedom of the spin and the interaction are thermally equilibrated, $T' = T$ $(n = 1)$, we arrive at the trivial case of the purely annealed disorder, irrespective of the difference between the characteristic timescales of the $J_{ij}$ interactions and the spins. This is also natural because the thermodynamic description formally corresponds to the infinite times, and the characteristic timescales of the dynamics of the internal degrees of freedom become irrelevant.

If $n \neq 0$ and $n \neq 1$, one gets a situation which could be called the partial annealing and which is the intermediate case between quenched disorder and annealed disorder.

### 3.3 Critical exponents in the $\phi^4$-theory with impurities

Consider a general case of a weakly disordered $p$-component spin system which, near the critical point in the continuous limit, can be described by the Hamiltonian [cf. Eqn (3.1)]:

$$H[\delta\tau,\phi] = \int \mathrm{d}^D x \left\{\frac{1}{2}\sum_{i=1}^{p}\left[\nabla\phi_i(x)\right]^2 + \frac{1}{2}\left[\tau - \delta\tau(x)\right]\sum_{i=1}^{p}\phi_i^2(x) + \frac{1}{4}g\sum_{i,j=1}^{p}\phi_i^2(x)\phi_j^2(x)\right\}, \tag{3.24}$$

where the random quantity $\delta\tau(x)$ is described by the Gaussian distribution [Eqn (3.2)].

In terms of the replica approach, we have to calculate the following replica partition function

$$Z_n = \overline{\left[\int \mathrm{D}\phi_i(x)\exp\left(-H[\delta\tau,\phi]\right)\right]^n}$$
$$= \int \mathrm{D}\delta\tau(x)\int \mathrm{D}\phi_i^a(x)\exp\left(-\frac{1}{4u}\int \mathrm{d}^D x\,\left[\delta\tau(x)\right]^2 - \int \mathrm{d}^D x\left\{\frac{1}{2}\sum_{i=1}^{p}\sum_{a=1}^{n}\left[\nabla\phi_i^a(x)\right]^2 + \frac{1}{2}\left[\tau - \delta\tau(x)\right]\sum_{i=1}^{p}\sum_{a=1}^{n}\left[\phi_i^a(x)\right]^2 + \frac{1}{4}g\sum_{i,j=1}^{p}\sum_{a=1}^{n}\left[\phi_i^a(x)\right]^2\left[\phi_j^a(x)\right]^2\right\}\right), \tag{3.25}$$

where the superscript $a$ labels the replicas. (Here and in what follows all irrelevant preexponential factors are omitted.) After Gaussian integration over $\delta\tau(x)$, one gets

$$Z_n = \int \mathrm{D}\phi_i^a(x)\exp\left(-\int \mathrm{d}^D x\left\{\frac{1}{2}\sum_{i=1}^{p}\sum_{a=1}^{n}\left[\nabla\phi_i^a(x)\right]^2 + \frac{1}{2}\tau\sum_{i=1}^{p}\sum_{a=1}^{n}\left[\phi_i^a(x)\right]^2 + \frac{1}{4}\sum_{i,j=1}^{p}\sum_{a,\,b=1}^{n}g_{ab}\left[\phi_i^a(x)\right]^2\left[\phi_j^b(x)\right]^2\right\}\right), \tag{3.26}$$

where

$$g_{ab} = g\delta_{ab} - u \ . \tag{3.27}$$

Now we shall calculate the critical exponents by using the RG procedure developed in Section 2.6 for dimensions $D = 4 - \varepsilon$ assuming that $\varepsilon \ll 1$. Taking into account the vector and the replica components, the $\phi^4$ interaction terms in the Hamiltonian (3.26) can represented in terms of the diagram shown in Fig. 6.

If we proceed similarly to the calculations of Section 2.6, we find that the (one-loop) renormalisation of the interaction parameters $g_{ab}$ (Fig. 6) are given by the diagrams shown in Fig. 7. Taking into account corresponding combinational factors, one obtains the following contributions:

$$(a) \to g_{ab}^2\int_{\lambda k_0<|k|<k_0}\frac{\mathrm{d}^D k}{(2\pi)^D}\,G_0^2(k)\Big|_{\varepsilon\ll 1} \simeq g_{ab}^2\,\frac{1}{8\pi^2}\ln\frac{1}{\lambda}\ ;$$

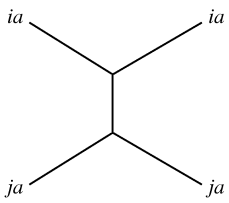


**Figure 6.** Diagrammatic representation of the interaction term $g_{ab}[\phi_i^a(x)]^2[\phi_j^b(x)]^2$.

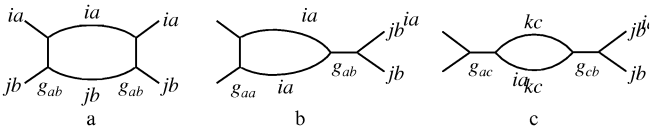


**Figure 7.** The diagrams which contribute to the renormalisation of the interaction terms $g_{ab}[\phi_i^a(x)]^2[\phi_j^b(x)]^2$.

$$(b) \to \frac{1}{2}(g_{aa}+g_{bb})g_{ab}\int_{\lambda k_0<|k|<k_0}\frac{\mathrm{d}^D k}{(2\pi)^D}\,G_0^2(k)\Big|_{\varepsilon\ll 1}$$

$$\simeq \frac{1}{2}(g_{aa}+g_{bb})g_{ab}\,\frac{1}{8\pi^2}\ln\frac{1}{\lambda}\,;$$

$$(c) \to \frac{p}{4}\sum_{c=1}^{n} g_{ac}g_{cb}\int_{\lambda k_0<|k|<k_0}\frac{\mathrm{d}^D k}{(2\pi)^D}\,G_0^2(k)\Big|_{\varepsilon\ll 1}$$

$$\simeq \frac{p}{4}\sum_{c=1}^{n} g_{ac}g_{cb}\,\frac{1}{8\pi^2}\ln\frac{1}{\lambda}\,. \tag{3.28}$$

The corresponding RG equations are

$$\frac{\mathrm{d}g_{ab}}{\mathrm{d}\xi} = \varepsilon g_{ab} - \frac{1}{8\pi^2}\left[4g_{ab}^2 + 2(g_{aa}+g_{bb})g_{ab} + p\sum_{c=1}^{n} g_{ac}g_{cb}\right]. \tag{3.29}$$

Taking into account the definition (3.27), one easily gets two RG equations for two interaction parameters $\tilde{g} \equiv g_{aa} = g - u$ and $g_{a\neq b} = -u$

$$\frac{\mathrm{d}\tilde{g}}{\mathrm{d}\xi} = \varepsilon\tilde{g} - \frac{1}{8\pi^2}\left[(8+p)\tilde{g}^2 + p(n-1)u^2\right],$$

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = \varepsilon u - \frac{1}{8\pi^2}\left\{(4+2p)\tilde{g}u - \left[4+p(n-2)\right]u^2\right\}. \tag{3.30}$$

In the limit $n \to 0$ we obtain

$$\frac{\mathrm{d}\tilde{g}}{\mathrm{d}\xi} = \varepsilon\tilde{g} - \frac{1}{8\pi^2}\left[(8+p)\tilde{g}^2 - pu^2\right],$$

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = \varepsilon u - \frac{1}{8\pi^2}\left[(4+2p)\tilde{g}u - (4-2p)u^2\right]. \tag{3.31}$$

Similarly, the renormalisation of the 'mass' term $\tau[\phi_i^a(x)]^2$ is given by the diagrams shown in Fig. 8. Their contributions are

$$(a) \to \tau g_{aa}\int_{\lambda k_0<|k|<k_0}\frac{\mathrm{d}^D k}{(2\pi)^D}\,G_0^2(k)\Big|_{\varepsilon\ll 1} \simeq \tau g_{aa}\,\frac{1}{8\pi^2}\ln\frac{1}{\lambda}\,,$$

$$(b) \to \frac{1}{2}p\tau\sum_{c=1}^{n} g_{ca}\int_{\lambda k_0<|k|<k_0}\frac{\mathrm{d}^D k}{(2\pi)^D}\,G_0^2(k)\Big|_{\varepsilon\ll 1}$$

$$\simeq \frac{1}{2}p\tau\sum_{c=1}^{n} g_{ca}\,\frac{p}{8\pi^2}\ln\frac{1}{\lambda}\,. \tag{3.32}$$

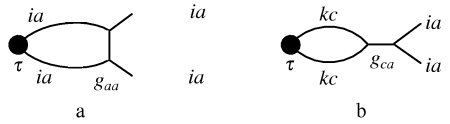


**Figure 8.** The diagrams which contribute to the renormalisation of the 'mass' term $\tau[\phi_i^a(x)]^2$.

Note that the above contributions do not depend on the replica index $a$ (which for simplicity can be taken to be, for example, 1).

The corresponding RG equation for the renormalised 'mass' $\tau$ is

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\ln|\tau| = 2 - \frac{1}{8\pi^2}\left(2g_{aa} + p\sum_{c=1}^{n} g_{ca}\right). \tag{3.33}$$

In the limit $n \to 0$ we finally obtain

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\ln\tau = 2 - \frac{1}{8\pi^2}\left[(2+p)\tilde{g}(\xi) + pu(\xi)\right], \tag{3.34}$$

where the renormalised interaction parameters $\tilde{g}(\xi)$ and $u(\xi)$ are defined by the Eqns (3.31).

The fixed-point values $\tilde{g}^*$ and $u^*$ are defined by the conditions $\mathrm{d}\tilde{g}^*/\mathrm{d}\xi = 0$ and $\mathrm{d}u^*/\mathrm{d}\xi = 0$, which according to Eqns (3.31) yield:

$$(8+p)\tilde{g}^2 - pu^2 = 8\pi^2\varepsilon g\,,$$

$$(4+2p)\tilde{g}u - (4-2p)u^2 = 8\pi^2\varepsilon u\,. \tag{3.35}$$

These equations have two nontrivial solutions:

$$\tilde{g}^* = \frac{8\pi^2}{p+8}\,\varepsilon\,, \qquad u^* = 0\,; \tag{3.36}$$

$$\tilde{g}^* = \varepsilon\pi^2\,\frac{p}{2(p-1)}\,, \qquad u^* = \varepsilon\pi^2\,\frac{4-p}{2(p-1)} \qquad (p \neq 1)\,. \tag{3.37}$$

The first solution, Eqn (3.36), describes the pure system without disorder. Using Eqn (3.33) and the relations (2.86) and (2.55) for the critical exponents of the pure system [we mark them by the label '(0)'] one gets

$$\Delta_\tau^{(0)} = 2 - \frac{1}{8\pi^2}(2+p)\tilde{g}^*_{(0)} = 2 - \frac{2+p}{8+p}\,\varepsilon;$$

$$\Rightarrow \nu_{(0)} = \frac{1}{\Delta_\tau^{(0)}} \simeq \frac{1}{2} + \frac{2+p}{4(8+p)}\,\varepsilon\,, \tag{3.38}$$

$$\alpha_{(0)} = 2 - (4-\varepsilon)\nu_{(0)} \simeq \frac{4-p}{2(8+p)}\,\varepsilon\,. \tag{3.39}$$

By using relations (2.55) – (2.59), we can automatically obtain the rest of the exponents.

Simple analysis of the evolution trajectories defined by RG Eqns (3.31) near the fixed points given by Eqns (3.36) and (3.37) shows that the 'pure' fixed point [Eqn (3.36)] is stable only for $p > 4$. Note that the value of $u^*$ in the other fixed point [Eqn (3.37)] becomes negative for $p > 4$, which means that this fixed point becomes essentially nonphysical, since the parameter $u$ being a mean square value of the quenched disorder fluctuations is only positively defined.

Thus, the critical behaviour of the $p$-component vector system with $p > 4$ is not modified by the presence of quenched disorder. It should be stressed that this is just the case when the critical exponent $\alpha$ of the specific heat is negative [Eqn (3.39)], in accordance with the Harris criterion (Section 3.1).

For $p < 4$, the 'pure' fixed point given by Eqn (3.36) becomes unstable and the critical properties of the system are defined by the 'impure' fixed point given by Eqn (3.37). Using Eqn (3.33), one gets

$$\Delta_\tau = 2 - \frac{1}{8\pi^2}\left[(2+p)\tilde{g}^* + pu^*\right] = 2 - \frac{3p}{8(p-1)}\,\varepsilon\,;$$

$$\Rightarrow \nu = \frac{1}{\Delta_\tau} \simeq \frac{1}{2} + \frac{3p}{32(p-1)}\,\varepsilon\,, \tag{3.40}$$

$$\alpha = 2 - (4-\varepsilon)\nu \simeq -\frac{4-p}{8(p-1)}\,\varepsilon\,, \tag{3.41}$$

where $p$ must be greater than 1. The rest of the exponents are obtained automatically.

The case of the one-component system, $p = 1$, requires more detailed consideration, because for $p = 1$ the equations (3.31) become degenerate. However, such degeneracy is the property only of the first order in the $\varepsilon$ approximation. It could be proved that by taking into account the next order in the $\varepsilon$ diagrams the degeneracy of the RG equations is removed. It could be shown then that a new 'impure' fixed point of the RG equations exists for $p = 1$ as well, and in this case the corrections to the critical exponents appear to be of the order of $\sqrt{\varepsilon}$ [8]. We omit this analysis here because it is technically much more cumbersome, and on a qualitative level provides results similar to those obtained above.

Thus, in agreement with the Harris criterion (Section 3.1) in the $p$-component vector system with $p < 4$, the critical behaviour is modified by the presence of quenched disorder. In the vicinity of the critical point, a new critical regime appears, and it is described by a new set of (universal) critical exponents. Note that the 'impure' critical exponent of the specific heat [Eqn (3.41)] appears to be negative, unlike that of the pure system. Therefore, the disorder makes the specific heat finite (although still singular) at the critical point, unlike the divergent specific heat of the corresponding pure system.

It should be stressed, however, that because of nonperturbative spin-glass phenomena, the relevance to real physics of the approach considered in this Section, although it is quite elegant and clear, may be questioned (see next Chapter).

### 3.4 Critical behaviour of the specific heat in four dimensions

In the full analogy, with the corresponding considerations for the pure systems [Eqns (2.122) and (2.123), and Section 2.6], for the singular part of the specific heat at $D = 4$, we get

$$C \simeq \int_{|k|>\sqrt{\tau}} \frac{d^4k}{(2\pi)^4}\, G_0^2(k)\left[\frac{m(k)}{\tau}\right]^2 \sim \int_{\xi<\ln(1/\tau)} d\xi \left[\frac{m(\xi)}{\tau}\right]^2 . \tag{3.42}$$

Here the renormalisation of the 'dressed' mass $m(\xi)$ is defined by the 'parquette' diagrams of Fig. 8. Accordingly, the renormalisations of the interaction parameters $\tilde{g}(\xi)$ and $u(\xi)$ are defined by the RG Eqns (3.31) with $\varepsilon = 0$:

$$\frac{d}{d\xi}\ln|m| = -\frac{1}{8\pi^2}\left[(2+p)\tilde{g} + pu\right], \tag{3.43}$$

$$\frac{d\tilde{g}}{d\xi} = -\frac{1}{8\pi^2}\left[(8+p)\tilde{g}^2 - pu^2\right], \tag{3.44}$$

$$\frac{du}{d\xi} = -\frac{1}{8\pi^2}\left[(4+2p)\tilde{g}u - (4-2p)u^2\right]. \tag{3.45}$$

The initial conditions are: $m(\xi = 0) = \tau$, $\tilde{g}(\xi = 0) = g_0$, and $u(\xi = 0) = u_0$.

In the pure system, $u = 0$, and the solutions for $m(\xi)$ and $\tilde{g}(\xi) \equiv g(\xi)$ are

$$g(\xi) = g_0\left[1 + \frac{(8+p)g_0}{8\pi^2}\,\xi\right]^{-1}\Bigg|_{\xi\to\infty} \to \sim \frac{8\pi^2}{8+p}\,\xi^{-1}\,,$$

$$m(\xi \to \infty) \sim \xi^{-(2+p)/(8+p)}\,. \tag{3.46}$$

Integration in Eqn (3.42) yields the following specific heat singularity:

$$C \sim \left(\ln\frac{1}{\tau}\right)^{(4-p)/(8+p)}. \tag{3.47}$$

For the system with a nonzero impurity interaction parameter $u$, one finds the following asymptotic (for $\xi \to \infty$) solutions of the Eqns (3.43) – (3.45)

$$\tilde{g}(\xi) \sim \pi^2\frac{p}{2(p-1)}\,\xi^{-1}\,; \qquad u(\xi) \sim \pi^2\frac{(4-p)}{2(p-1)}\,\xi^{-1}\,;$$

$$m(\xi) \sim \xi^{-3p/[8(p-1)]}\,. \tag{3.48}$$

Such solutions exist only for $p < 4$, otherwise $u$ becomes formally negative, which is the nonphysical situation. Actually, in this case the vertex $u(\xi)$ becomes zero at a finite scale $\xi$, and then the asymptotic solutions for $m(\xi)$ and $\tilde{g}(\xi)$ coincide with those of the pure system.

The case of the one-component field, $p = 1$, requires special consideration. As in the case of dimensions $D = 4 - \varepsilon$ (see above), one has to take into account second-order loop terms, which makes the analysis rather cumbersome, and we will not consider it here. On a qualitative level, however, the results for the specific heat appear to be similar to those for $p < 4$: the one-component system with impurities exhibits a new type of (logarithmic) singularity.

For $p < 4$, the integration in Eqn (3.42) yields

$$C \sim \left(\ln\frac{1}{\tau}\right)^{-(4-p)/[4(p-1)]}. \tag{3.49}$$

It is interesting to note that, although at dimensions $D = 4$ the critical exponent $\alpha$ of the specific heat is zero, the Harris criterion, taken in the generalised form, still works. Namely, if the specific heat of the pure system is divergent at the critical point [the case of $p < 4$, Eqn (3.47)], the impurities appear to be relevant for the critical behaviour, and change the behaviour of the specific heat into a new type of (universal) singularity [Eqn (3.49)]. Otherwise, if the specific heat of the pure system is finite at the critical point [$p > 4$, Eqn (3.47)], then the presence of impurities does not modify the critical behaviour.

## 4. Spin-glass effects in critical phenomena

### 4.1 Nonperturbative degrees of freedom

In this section we consider nontrivial spin-glass (SG) effects produced by weak quenched disorder, which have been ignored in the previous Section. It will be shown that these effects could dramatically change the whole physical scenario of the critical phenomena.

According to the traditional point of view (considered in the previous Section), the effects produced by weak

quenched disorder in the critical region could be summarised as follows. If $\alpha$, the specific heat exponent of the pure system, is greater than zero (i.e. the specific heat of the pure system is divergent at the critical point), the disorder is relevant for critical behaviour, and a new universal critical regime, with new critical exponents, is established sufficiently close to the phase transition point $\tau < \tau_u \equiv u^{1/\alpha}$. In contrast, when $\alpha < 0$ (the specific heat is finite), the disorder appears to be irrelevant, i.e. its presence does not affect the critical behaviour.

Actually, if the disorder is relevant for the critical behaviour, the situation could appear to be much more sophisticated. Let us consider the physical motivation of the traditional RG approach in some more detail.

Near the phase transition point the $D$-dimensional Ising-like systems are described in terms of the Ginzburg – Landau Hamiltonian, of a scalar field, with a double-well potential

$$H = \int \mathrm{d}^D x \left\{ \frac{1}{2} \left[\nabla\phi(x)\right]^2 + \frac{1}{2}\left[\tau - \delta\tau(x)\right]\phi^2(x) + \frac{1}{4} g\phi^4(x) \right\} . \tag{4.1}$$

Here, as usual, the quenched disorder is described by random fluctuations of the effective transition temperature $\delta\tau(x)$, whose probability distribution is taken to be symmetric and Gaussian:

$$P[\delta\tau] = p_0 \exp\left\{ -\frac{1}{4u} \int \mathrm{d}^D x \, \left[\delta\tau(x)\right]^2 \right\}, \tag{4.2}$$

where $u \ll 1$ is the small parameter which describes the disorder, and $p_0$ is the normalisation constant.

Now, if one is interested in the critical properties of the system, it is necessary to integrate over all local field configurations up to the scale of the correlation length. This type of calculation is usually performed with a renormalisation-group (RG) scheme, which self-consistently takes into account all the fluctuations of the field on scale lengths up to $R_c$.

In order to derive the traditional results for the critical properties of this system, one can use the usual RG procedure developed for dimensions $D = 4 - \varepsilon$, where $\varepsilon \ll 1$. Then we find that in the presence of the quenched disorder the fixed point of the pure system becomes unstable, and the RG rescaling trajectories arrive at another (universal) fixed point $g_* \neq 0$; $u_* \neq 0$, which yields the new critical exponents describing the critical properties of the system with disorder.

However, there exists an important point which is missing in the traditional approach. Consider the ground state properties of the system described by the Hamiltonian Eqn (4.1). Configurations of the fields $\phi(x)$ which correspond to local minima in $H$ satisfy the saddle-point equation

$$-\Delta\phi(x) + \left[\tau - \delta\tau(x)\right]\phi(x) + g\phi^3(x) = 0 . \tag{4.3}$$

Clearly, the solutions of these equations depend on a particular configuration of the function $\delta\tau(x)$ being inhomogeneous. The localised solutions with nonzero values of $\phi$ exist in regions of space where $\tau - \delta\tau(x)$ has negative values. Moreover, one finds a *macroscopic* number of local minimum solutions of the saddle-point equation (4.3). Indeed, for a given realisation of the random function $\delta\tau(x)$, there exist a macroscopic number of spatial 'islands' where $\tau - \delta\tau(x)$ is negative (so that the local effective temperature is below $T_c$), and in each of these 'islands' one finds two local minimum configurations of the field: one which is 'up', and another which is 'down'. These local minimum energy configurations are separated by finite energy barriers, whose heights increase as the size of the 'islands' are increased.

The problem is that the traditional RG approach is only a perturbative theory in which the deviations of the field around the ground state configuration are treated, but it cannot take into account other local minimum configurations which are 'beyond barriers'. This problem does not arise in pure systems, where the solution of the saddle-point equation is unique. However, in a situation such as that discussed above, when one gets numerous local minimum configurations separated by finite barriers, the direct application of the traditional RG scheme may be questioned.

In a systematic approach one would like to integrate in an RG way over fluctuations around the local minimum configurations. Furthermore, one also has to sum over all these local minima up to the scale of the correlation length. In view of the fact that the local minimum configurations are defined by the random quenched function $\delta\tau(x)$ in an essentially nonlocal way, the possibility of implementing such a systematic approach successfully seems rather hopeless.

On the other hand, there exists another technique which has been developed specifically for dealing with systems which exhibit numerous local minimum states. It is the Parisi replica symmetry breaking (RSB) scheme which has proved to be crucial in the mean-field theory of spin-glasses (for example, see Ref. [11]). Recent studies show that in certain cases the RSB approach can also be generalised for situations where one has to deal with fluctuations as well [12 – 14]. Moreover, it has recently been shown that the RSB technique can be applied successfully for the RG studies of the critical phenomena in the Sine – Gordon model, where remarkable instability of the RG flows with respect to the RSB modes has been discovered [15].

It can be argued that the summation over multiple local minimum configurations in the present problem could provide additional nontrivial RSB interaction potentials for the fluctuating fields [16]. Let us consider this point in more detail.

To carry out the appropriate average over quenched disorder, one can use the standard replica approach (Section 3.2). This is accomplished by introducing the replicated partition function $Z_n \equiv \overline{Z^n[\delta\tau]}$ [see Eqn (3.26)]

$$Z_n = \int \mathrm{D}\phi_a(x) \exp\left\{ -\int \mathrm{d}^D x \left\{ \frac{1}{2}\sum_{a=1}^{n} \left[\nabla\phi_a(x)\right]^2 + \frac{1}{2}\tau \sum_{a=1}^{n} \phi_a^2(x) + \frac{1}{4}\sum_{a,\,b=1}^{n} g_{ab}\phi_a^2(x)\phi_b^2(x) \right\} \right\}, \tag{4.4}$$

where

$$g_{ab} = g\delta_{ab} - u \tag{4.5}$$

is the *replica-symmetric* (RS) interaction parameter. If one would start the usual RG procedure for the above replica Hamiltonian (as is done in the previous Section), then it would correspond to the perturbation theory around the homogeneous ground state $\phi = 0$.

However, in the situation when there exist numerous local minimum solutions of the saddle-point equation (4.3),

we have to be more careful. Let us denote the local solutions of Eqn (4.3) by $\psi^{(i)}(x)$, where $i = 1, 2, \ldots, N_0$ denotes the 'islands' where $\delta\tau(x) > \tau$. If the size $L_0$ of an 'island', where $[\delta\tau(x) - \tau] > 0$ is not too small, then the value of $\psi^{(i)}(x)$ in this 'island' should be $\sim \pm[\delta\tau(x) - \tau)/g]^{1/2}$, where $\delta\tau(x)$ should now be interpreted as the value of $\delta\tau$ averaged over the region of size $L_0$. Such 'islands' occur at a certain finite density per unit volume. Thus the value of $N_0$ is macroscopic: $N_0 = \kappa V$, where $V$ is the volume of the system and $\kappa$ is a constant. An approximate global extremal solution $\Phi(x)$ is constructed as the union of all these local solutions, and each local solution can occur with either sign:

$$\Phi_{(\alpha)}[x; \delta\tau(x)] = \sum_{i=1}^{\kappa V} \sigma_i \psi^{(i)}(x), \tag{4.6}$$

where each $\sigma_i = \pm 1$. Accordingly, the total number of global solutions must be $2^{\kappa V}$. We label these solutions with $\alpha = 1, 2, \ldots, K = 2^{\kappa V}$. As mentioned earlier, it seems unlikely that an integration over fluctuations around $\phi(x) = 0$ will include the contributions from the configurations of $\phi(x)$ which are near $\Phi(x)$, since $\Phi(x)$ is 'beyond a barrier', so to speak. Therefore, it seems appropriate to include separately the contribu-tions from small fluctuations about each of the many $\Phi_{(\alpha)}(x; \delta\tau)$. Thus we have to sum over the $K$ global minimum solutions (nonperturbative degrees of freedom) $\Phi_{(\alpha)}(x; \delta\tau)$ and also to integrate over 'smooth' fluctuations $\varphi(x)$ around them

$$\begin{aligned} Z[\delta\tau] &= \int \mathrm{D}\varphi(x) \sum_{\alpha}^{K} \exp\left(-H[\Phi_{(\alpha)} + \varphi; \delta\tau]\right) \\ &= \int \mathrm{D}\varphi(x) \exp\left(-H[\varphi; \delta\tau]\right) \times \tilde{Z}[\varphi; \delta\tau], \end{aligned} \tag{4.7}$$

where

$$\tilde{Z}[\varphi; \delta\tau] = \sum_{\alpha}^{K} \exp\left\{-H_\alpha - \int \mathrm{d}^D x \left[\frac{3}{2} g\Phi^2_{(\alpha)}(x; \delta\tau)\varphi^2(x) + g\Phi_{(\alpha)}(x; \delta\tau)\varphi^3(x)\right]\right\}, \tag{4.8}$$

and $H_\alpha$ is the energy of the $\alpha$th solution.

Next we carry out the appropriate average over quenched disorder, and for the replica partition function $Z_n$, we get

$$Z_n = \int \mathrm{D}\delta\tau P[\delta\tau] \int \mathrm{D}\varphi_a \exp\left\{-\sum_{a=1}^{n} H[\varphi_a; \delta\tau]\right\} \times \tilde{Z}_n[\varphi_a; \delta\tau], \tag{4.9}$$

where the subscript $a$ is a replica index and

$$\tilde{Z}_n[\varphi_a; \delta\tau] = \sum_{\alpha_1 \ldots \alpha_n}^{K} \exp\left\{-\sum_{a}^{n} H_{\alpha_a} - \int \mathrm{d}^D x \sum_{a}^{n} \left[\frac{3}{2} g\Phi^2_{(\alpha_a)}(x; \delta\tau)\varphi_a^2(x) + g\Phi_{(\alpha_a)}(x; \delta\tau)\varphi_a^3(x)\right]\right\}. \tag{4.10}$$

It is clear that, if the saddle-point solution is unique, from Eqns (4.9) and (4.10) one would obtain the usual RS representation given by Eqns (4.4) and (4.5). However, in the case of a macroscopic number of local minimum solutions, the problem becomes highly nontrivial.

It is obviously hopeless to try and evaluate the replicated partition function given above systematically. The global solutions $\Phi_{(\alpha)}$ are complicated implicit functions of $\delta\tau(x)$. These quantities have fluctuations of two different types. In the first instance, they depend on the stochastic variables $\delta\tau(x)$. But even when the $\delta\tau(x)$ variables are completely fixed, $\Phi_{(\alpha)}(x)$ will depend on $\alpha$ (which denotes the possible ways of constructing the global minimum out of the choices for the signs $\{\sigma\}$ of the local minima). A crude way of treating this situation is to regard the local solutions $\psi^{(i)}(x)$ as if they were random variables, even though $\delta\tau(x)$ has been specified. This randomness, which one can see is not all that different from that which exists in spin glasses, is the crucial one. It can be then be shown that, owing to the interaction of the fluctuating fields with the local minimum configurations [the term $\Phi^2_{(\alpha_a)}\varphi_a^2$ in the Eqn (4.10)], the summation over solutions in the replica partition function $\tilde{Z}_n[\varphi_a]$, Eqn (4.10), could provide the additional nontrivial RSB potential

$$\sum_{a,\, b} g_{ab}\varphi_a^2\varphi_b^2,$$

in which the matrix $g_{ab}$ has the Parisi RSB structure [16].

In this Section we are going to study the critical properties of weakly disordered systems in terms of the RG approach, taking into account the possibility of a general type of RSB potentials for the fluctuating fields. The idea is that hopefully, as in spin-glasses, this type of generalised RG scheme self-consistently takes into account the relevant degrees of freedom coming from the numerous local minima. In particular, the instability of the traditional replica-symmetric (RS) fixed points with respect to RSB indicates that the multiplicity of the local minima can be relevant for the critical properties in the fluctuation region.

It will be shown (in Section 4.2) that, whenever the disorder appears to be relevant for the critical behaviour, the usual RS fixed points (which used to be considered as providing new universal disorder-induced critical exponents) are unstable with respect to 'turning on' an RSB potential. Moreover, it will be shown that, in the presence of a general type of RSB potentials, the RG flows actually lead to the so-called *strong-coupling regime* at the finite spatial scale $R_* \sim \exp(1/u)$ [which corresponds to the temperature scale $\tau_* \sim \exp(-1/u)$]. At this scale, the renormalised matrix $g_{ab}$ develops strong RSB, and the values of the interaction parameters are no longer small [17].

Usually the strong-coupling situation indicates that certain essentially nonperturbative excitations have to be taken into account, and it could be argued that in the present model these are due to exponentially rare 'instantons' in the spatial regions, where the value of $\delta\tau(x) \sim 1$, and the local value of the field $\varphi(x)$ must be $\sim \pm 1$. (A distant analog of this situation exists in the two-dimensional Heisenberg model where the Polyakov renormalisation develops into the strong-coupling regime at a finite (exponentially large) scale which is known to be due to the nonlinear localised instanton solutions [18].)

In Section 4.3 , the physical consequences of the RG solutions obtained above will be discussed. In particular we show that, because of the absence of fixed points at the disorder-dominated scales $R \gg u^{-\nu/\alpha}$ (or at the corresponding temperature scales $\tau \ll u^{1/\alpha}$), there must be no simple scaling of the correlation functions or of other physical

quantities. Besides, it is shown that the structure of the SG-type two-points correlation functions is characterised by the strong RSB, which indicates the onset of a new type of critical behaviour of the SG type.

The remaining problems as well as future perspectives are discussed in Section 4.4. Particular attention is given to the possible relevance of the considered RSB phenomena for the so-called Griffith phase which is known to exist in a finite temperature interval above $T_c$ [21].

### 4.2 Replica symmetry breaking in the renormalisation-group theory

Let us again consider again the $p$-component ferromagnet with quenched, random effective temperature fluctuations described by the usual Ginzburg–Landau Hamiltonian, Eqn (3.21). In terms of the standard replica approach, after integration over the disorder variable $\delta\tau(x)$ for the corresponding replica Hamiltonian, we get [see Eqns (3.26) and (3.27)]

$$H_n = \int \mathrm{d}^D x \left\{ \frac{1}{2}\sum_{i=1}^{p}\sum_{a=1}^{n}\left[\nabla\phi_i^a(x)\right]^2 + \frac{1}{2}\tau\sum_{i=1}^{p}\sum_{a=1}^{n}\left[\phi_i^a(x)\right]^2 + \frac{1}{4}\sum_{i,j=1}^{p}\sum_{a,\,b=1}^{n} g_{ab}\left[\phi_i^a(x)\right]^2\left[\phi_j^b(x)\right]^2 \right\}, \quad (4.11)$$

where $g_{ab} = g\delta_{ab} - u$.

Along the lines of the usual rescaling scheme for dimensions $D = 4 - \varepsilon$ (Section 3.3), one can obtain the following (one-loop) RG equations for the interaction parameters $g_{ab}$ [see Eqns (3.9)]

$$\frac{\mathrm{d}g_{ab}}{\mathrm{d}\xi} = \varepsilon g_{ab} - \frac{1}{8\pi^2}\left[4g_{ab}^2 + 2(g_{aa} + g_{bb})g_{ab} + p\sum_{c=1}^{n} g_{ac}g_{cb}\right], \quad (4.12)$$

where $\xi$ is the standard rescaling parameter.

Changing $g_{ab} \to 8\pi^2 g_{ab}$, and $g_{a\neq b} \to -g_{a\neq b}$ (so that the off-diagonal elements would be positively defined), and introducing $\tilde{g} \equiv g_{aa}$, we get the following RG equations:

$$\frac{\mathrm{d}g_{ab}}{\mathrm{d}\xi} = \varepsilon g_{ab} - (4+2p)\tilde{g}g_{ab} + 4g_{ab}^2 + p\sum_{c\neq a,\,b}^{n} g_{ac}g_{cb} \quad (a \neq b), \quad (4.13)$$

$$\frac{\mathrm{d}\tilde{g}}{\mathrm{d}\xi} = \varepsilon\tilde{g} - (8+p)\tilde{g}^2 - p\sum_{c\neq 1}^{n} g_{1c}^2 . \quad (4.14)$$

If one takes the matrix $g_{ab}$ to be replica symmetric, as in the starting form of Eqn (4.5), then we can recover the usual RG equations (3.31) for the parameters $\tilde{g}$ and $u$, and eventually obtain the old results of Section 3.3 for the fixed points and the critical exponents. Here we leave aside the question of how perturbations could arise out of the RS subspace (see the discussion in Ref. [16]) and formally consider the RG Eqns (4.13) and (4.14) assuming that the matrix $g_{ab}$ has a general Parisi RSB structure.

According to the standard technique of the Parisi RSB algebra (for example, see Refs [9, 11]), in the limit $n \to 0$ the matrix $g_{ab}$ is parametrised in terms of its diagonal elements $\tilde{g}$ and the off-diagonal *function* $g(x)$ defined in the interval $0 < x < 1$. All the operations with the matrices in this algebra can be performed according to the following simple rules [12, 19]:

$$g_{ab}^k \to \left[\tilde{g}^k; g^k(x)\right], \quad (4.15)$$

$$(\hat{g}^2)_{ab} \equiv \sum_{c=1}^{n} g_{ac}g_{cb} \to \left[\tilde{c}; c(x)\right], \quad (4.16)$$

where

$$\tilde{c} = \tilde{g}^2 - \int_0^1 \mathrm{d}x\, g^2(x),$$

$$c(x) = 2\left[\tilde{g} - \int_0^1 \mathrm{d}y\, g(y)\right]g(x) - \int_0^x \mathrm{d}y\,\left[g(x) - g(y)\right]^2 . \quad (4.17)$$

The RS situation corresponds to the case $g(x) = \text{const}$, independent of $x$.

Using the above rules from Eqns (4.13) and (4.14), one gets

$$\frac{\mathrm{d}}{\mathrm{d}\xi}g(x) = \left[\varepsilon - (4+2p)\tilde{g}\right]g(x) + 4g^2(x) - 2pg(x)\int_0^1 \mathrm{d}y\, g(y) - p\int_0^x \mathrm{d}y\,\left[g(x) - g(y)\right]^2, \quad (4.18)$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\tilde{g} = \varepsilon\tilde{g} - (8+p)\tilde{g}^2 + p\overline{g^2}, \quad (4.19)$$

where

$$\overline{g^2} \equiv \int_0^1 \mathrm{d}x\, g^2(x) .$$

Usually in the studies of critical behaviour, one tries to determine the stable fixed-point solutions of the RG equations. The fixed-point values of the renormalised interaction parameters are believed to describe the structure of the asymptotic Hamiltonian which allows us to calculate the singular part of the free energy, as well as the other thermodynamic quantities.

From Eqn (4.18), one can easily determine the structure of the function $g(x)$ at the fixed point, $\mathrm{d}g(x)/\mathrm{d}\xi = 0$, $\mathrm{d}\tilde{g}/\mathrm{d}\xi = 0$. Taking the derivative over $x$ twice we get, from Eqn (4.18), $g'(x) = 0$. This means that either the function $g(x)$ is constant (which is the RS situation), or it has a step-like structure. It is interesting to note that the structure of fixed-point equations is similar to that of the Parisi function $q(x)$ near $T_c$ in the Potts spin-glasses [20], and it is the term $g^2(x)$ in Eqn (4.18) which is known to produce the 1-step RSB solution there. The numerical solution of the RG equations given above demonstrates convincingly that, whenever the trial function $g(x)$ has the many-step RSB structure, it quickly develops into the 1-step one, with the coordinate of the step being the most correct one of the original many-step function.

Let us consider the 1-step RSB statement for the function $g(x)$:

$$g(x) = \begin{cases} g_0, & 0 \leqslant x < x_0, \\ g_1, & x_0 < x \leqslant 1, \end{cases} \quad (4.20)$$

where $0 \leqslant x_0 \leqslant 1$ is the coordinate of the step.

In terms of this statement,from Eqns (4.18) and (4.19) one easily gets the following fixed-point equations for the parameters $g_1$, $g_0$, and $\tilde{g}$:

$$(4 - 2px_0)g_0^2 - 2p(1 - x_0)g_1g_0 - (4+2p)\tilde{g}g_0 + \varepsilon g_0 = 0,$$

$$-px_0g_0^2 + (4 - 2p + px_0)g_1^2 - (4+2p)\tilde{g}g_1 + \varepsilon g_1 = 0,$$

$$-px_0g_0^2 - p(1 - x_0)g_1^2 + (8+p)\tilde{g}^2 - \varepsilon\tilde{g} = 0 . \quad (4.21)$$

These equations have several nontrivial solutions.

(1) The RS fixed point which corresponds to the pure system defined by Eqn (3.36):

$$g_0 = g_1 = 0\,; \qquad \tilde{g} = \frac{1}{8+p}\,\varepsilon\,. \tag{4.22}$$

This fixed point (in accordance with the Harris criterion) is stable for the spin components with $p > 4$, and it becomes unstable for those with $p < 4$.

(2) The disorder-induced RS fixed point given by Eqns (3.37) (for $p > 1$):

$$g_0 = g_1 = \varepsilon\,\frac{4-p}{16(p-1)}; \qquad \tilde{g} = \varepsilon\,\frac{p}{16(p-1)}\,. \tag{4.23}$$

This point was usually considered to be the one which describes the new universal critical behaviour in systems with impurities. This fixed point has been shown to be stable (with respect to the RS deviations!) for $p < 4$, which is consistent with the Harris criterion. (For $p = 1$ this fixed point involves an expansion in powers of $(\varepsilon)^{1/2}$, and this structure is only revealed within a two-loop approximation.) However, the stability analysis with respect to the RSB deviations shows that this fixed point is *always unstable* [16]. The three eigenvalues of the corresponding linearised equations near this point are

$$\lambda_1 = -\frac{1}{2}\,, \quad \lambda_2 = -\frac{(4-p)}{8(p-1)}\,, \quad \lambda_3 = +\frac{(4-p)}{8(p-1)}\,,$$

so that one of the eigenvalues is always positive. Therefore whenever the disorder is relevant for the critical behaviour, the RSB perturbations must be the dominant factor in the asymptotic large-scale limit.

(3) The 1-step RSB fixed point [16]:

$$g_0 = 0\,; \qquad g_1 = \varepsilon\,\frac{4-p}{16(p-1) - px_0(8+p)}\,;$$

$$\tilde{g} = \varepsilon\,\frac{p(1-x_0)}{16(p-1) - px_0(8+p)}\,. \tag{4.24}$$

This fixed point can be shown to be stable (within 1-step RSB subspace!) for

$$1 < p < 4\,,$$

$$0 < x_0 < x_c(p) \equiv \frac{16(p-1)}{p(8+p)}\,. \tag{4.25}$$

In particular, $x_c(p=2) = 4/5$; $x_c(p=3) = 32/33$; and $x_c(p=4) = 1$. Using the result given by Eqn (4.24), one can easily obtain the corresponding critical exponents, which become nonuniversal, as they are dependent on the starting parameter $x_0$ (see Section 4.3). Note that in addition to the fixed points listed above, there exist several other 1-step RSB solutions which are either unstable or unphysical.

The problem, however, is that if the parameter $x_0$ of the starting function $g(x;\xi=0)$ (or more generally, the coordinate of the most correct step of the many-step starting function) is taken to be beyond the stability interval, such that $x_c(p) < x_0 < 1$, then there exist *no stable fixed points* of the RG Eqns (4.18) and (4.19). One faces the same situation also in the case of a general continuous starting function $g(x;\xi=0)$. Moreover, according to Eqn (4.25) there exist no stable fixed points out of the RS subspace in the most interesting Ising case, where $p = 1$.

Unlike in the RS situation for $p = 1$, where one finds the stable $(\sim\sqrt{\varepsilon})$ fixed point in the two-loop RG equations, in the case adding from the next order terms in the RG equations does not cure the problem. In the RSB case considered above, one finds that in the two-loop RG equations the values of the parameters in the fixed point are formally of the order of one, and this indicates that we are entering the strong-coupling regime where all the orders of the RG become relevant.

Nevertheless, to get at least some information about the physics behind this instability phenomena, one can proceed to analyse the actual evolution of the above one-loop RG equations given above. The scale evolution of the parameters of the Hamiltonian would still adequately describe the properties of the system until we reach a critical scale $\xi_*$, at which the strong-coupling regime begins.

The evolution of the renormalised function $g(x;\xi)$ can be analysed both numerically and analytically. It can be shown (see Ref. [17]) that, when $p < 4$, for a general continuous starting function $g(x;\xi=0) \equiv g_0(x)$ the renormalised function $g(x;\xi)$ tends to zero everywhere in the interval $0 \leqslant x < [1-\Delta\xi]$ whereas in the narrow (scale-dependent) interval $\Delta\xi$ near $x = 1$ the values of the function $g(x;\xi)$ increase:

$$g(x;\xi) \sim \begin{cases} a\,\dfrac{u}{1-u\xi}\,, & (1-x) \ll \Delta\xi\,, \\ 0\,, & (1-x) \gg \Delta\xi\,, \end{cases} \tag{4.26}$$

$$\tilde{g}(\xi) \sim u \ln \frac{1}{1-u\xi}\,, \tag{4.27}$$

where

$$\Delta\xi \simeq (1-u\xi)\,. \tag{4.28}$$

Here $a$ is a positive nonuniversal constant, and the critical scale $\xi_*$ is defined by the condition that the values of the renormalised parameters are of the order of 1: $(1-u\xi_*) \sim u$, or $\xi_* \sim 1/u$. Correspondingly, the spatial scale at which the system enters the strong-coupling regime is

$$R_* \sim \exp\frac{1}{u}\,. \tag{4.29}$$

Note that the value of this scale is much greater than the usual crossover scale $\sim u^{-\alpha/\nu}$ (where $\alpha$ and $\nu$ are the specific heat and the correlation length of the pure system, respectively), at which the disorder becomes relevant for the critical behaviour.

According to the above result, the value of the narrow band near $x = 1$, where the function $g(x;\xi)$ is formally divergent, is $\Delta\xi \simeq (1-u\xi) \to u \ll 1$ as $\xi \to \xi_*$.

Besides, it can also be shown that the value of the integral

$$\overline{g}(\xi) \equiv \int_0^1 g(x;\xi)$$

becomes formally divergent logarithmically as $\xi \to \xi_*$:

$$\overline{g}(\xi) \sim u \ln \frac{1}{1-u\xi}\,. \tag{4.30}$$

Qualitatively similar asymptotic behaviour for $g(x;\xi)$ is obtained for the case when the starting function $g_0(x)$ has the 1-step RSB structure [Eqn (4.20)], and the coordinate of the step $x_0$ is in the instability region (or for any $x_0$ in the Ising case $p = 1$):

$$g(x;\xi) \sim \begin{cases} \dfrac{g_1(0)}{1-(4-2p+px_0)g_1(0)\xi}\,, & x_0 < x < 1\,, \\ 0\,, \quad 0 \leqslant x < x_0\,. \end{cases} \tag{4.31}$$

Here $g_1(0) \equiv g_1(\xi=0) \sim u$, and the coefficient $(4-2p+px_0)$ is always positive. In this case again, the system enters into the strong-coupling regime at scales $\xi \sim 1/u$.

Note that the above asymptotics do not explicitly involve $\varepsilon$. In fact, the role of the parameter $\varepsilon > 0$ is to 'push' the RG trajectories out of the trivial Gaussian fixed point $g=0$, $\tilde{g}=0$. Thus the value of $\varepsilon$, as well as the values of the starting parameters $g_0(x)$, $\tilde{g}_0$, defines a scale at which the solutions finally enter the above asymptotic regime. When $\varepsilon < 0$ (above dimensions 4) the Gaussian fixed point is stable; on the other hand, the strong-coupling asymptotics still exist in this case as well, separated from the trivial one by a finite (depending on the value of $\varepsilon$) barrier. Therefore although *infinitely small* disorder remains irrelevant for the critical behaviour above dimensionality of 4, if the disorder is strong enough (bigger than some value depending on the $\varepsilon$ threshold) the RG trajectories could enter the strong-coupling regime again.

### 4.3 Scaling properties and replica symmetry breaking

#### 4.3.1 Spatial and temperature scales

The renormalisation of the mass term

$$\tau(\xi)\sum_{a=1}^{n}\phi_a^2$$

is described by the following RG equation [see Eqn (3.33)]:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\ln\tau = 2 - \frac{1}{8\pi^2}\left[(2+p)\tilde{g} + p\sum_{a\neq 1}^{n} g_{1a}\right]\,. \tag{4.32}$$

Changing (as in the Section 4.2) $g_{ab} \to 8\pi^2 g_{ab}$, and $g_{a\neq b} \to -g_{a\neq b}$ in the Parisi representation, we get:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\ln\tau = 2 - \left[(2+p)\tilde{g}(\xi) + p\int_0^1 \mathrm{d}x\, g(x;\xi)\right] \tag{4.33}$$

or

$$\tau(\xi) = \tau_0 \exp\left\{2\xi - \int_0^{\xi}\mathrm{d}\eta\,\left[(2+p)\tilde{g}(\eta) + p\overline{g}(\eta)\right]\right\}, \tag{4.34}$$

where $\tilde{g}(\eta)$ and $\overline{g}(\eta) \equiv \int_0^1 \mathrm{d}x\, g(x;\eta)$ are the solutions of the RG equations given in the the previous Section.

Let us first consider the traditional (replica-symmetric) situation. The RS interaction parameters $\tilde{g}(\xi)$ and $g(\xi)$ approach the fixed-point values $\tilde{g}_*$ and $g_*$ (which are of the order of $\varepsilon$), and then for the dependence of the renormalised mass $\tau(\xi)$, according to Eqn (4.34), one gets

$$\tau(\xi) = \tau_0 \exp(\Delta_\tau \xi)\,, \tag{4.35}$$

where

$$\Delta_\tau = 2 - \left[(2+p)\tilde{g}_* + pg_*\right]\,. \tag{4.36}$$

At the scale $\xi_c$, such that $\tau(\xi_c)$ approaches the order of 1, the system gets out of the scaling region. Since the RG scale parameter $\xi = \ln R$, where $R$ is the spatial scale, this defines the correlation length $R_c$ as a function of the reduced temperature $\tau_0$. According to Eqn (4.35), one obtains

$$R_c(\tau_0) \sim \tau_0^{-\nu}\,, \tag{4.37}$$

where $\nu = 1/\Delta_\tau$ is the critical exponent of the correlation length.

Actually if the starting value of the disorder parameter $g(\xi=0) \equiv u$ is much smaller than the starting value of the pure system interaction $\tilde{g}(\xi=0) \equiv g_0$, the situation is a little bit more complicated. In this case, the RG flow for $\tilde{g}(\xi)$ first arrives at the fixed point $\tilde{g}_*^{(\mathrm{pure})}$ of the pure system as if the disorder perturbation did not exist. Then, since the fixed point of the pure system is unstable with respect to the disorder perturbations, at scales bigger than a certain disorder-dependent scale $\xi_u$ the RG trajectories eventually arrive at the stable (universal) disorder-induced fixed point $(\tilde{g}_*, g_*)$. According to the traditional theory [8], it is known that $\xi_u \sim (\nu/\alpha)\ln(1/u)$. The corresponding spatial scale is $R_u \sim u^{-\nu/\alpha} \gg 1$, and it is big in comparison with the small parameter $u$. Coming back to the scaling behaviour of the mass parameter $\tau(\xi)$, Eqn (4.35), we see that if the value of the temperature $\tau_0$ is such that $\tau(\xi)$ reaches the order of 1 before the crossover scale $\xi_u$ is reached, then for the scaling behaviour of the correlation length (as well as for other thermodynamic quantities) one finds essentially the result $R_c(\tau_0) \sim \tau_0^{-\nu_{(\mathrm{pure})}}$ of the pure system. However, the critical behaviour of the pure system is observed only until $R_c \ll R_u$, which imposes the following restriction on the temperature parameter: $\tau_0 \gg u^{1/\alpha} \equiv \tau_u$. In other words, at temperatures not too close to $T_c$, $\tau_u \ll \tau_0 \ll 1$, the presence of disorder is irrelevant for the critical behaviour.

On the other hand, if $\tau_0 \ll \tau_u$ (in the close vicinity of $T_c$) the RG trajectories for $\tilde{g}(\xi)$ and $g(\xi)$ arrive (after crossover) at a new (universal) disorder-induced fixed point $(\tilde{g}_*, g_*)$, and the scaling of the correlation length (as well as other thermodynamic quantities), according to Eqns (4.36), and (4.37), is controlled by a new universal critical exponent $\nu$ which is defined by the RS fixed point $(\tilde{g}_*, g_*)$, Eqn (3.39), of the random system.

Consider now the situation if the RSB scenario occurred. Again if the disorder parameter $u$ is small, in the temperature interval $\tau_u \ll \tau_0 \ll 1$, the critical behaviour is controlled essentially by the fixed point of the pure system, and the presence of disorder is irrelevant. For the same reasons as discussed above, the system exits the scaling regime [$\tau(\xi)$ approaches the order of one] before the disorder parameters start 'pushing' the RG trajectories out of the fixed point of the pure system.

However, at temperatures $\tau_0 \ll \tau_u$ the situation is completely different from the RS case. If the RG trajectories arrive at the 1-step RSB fixed point, Eqn (4.24) (in the $1 < p < 4$ case), then according to the standard scaling relations for the critical exponent of the correlation length, one finds:

$$\nu(x_0) = \frac{1}{2} + \frac{1}{2}\varepsilon\frac{3p(1-x_0)}{16(p-1) - px_0(p+8)}\,. \tag{4.38}$$

Thus depending on the value of the starting parameter $x_0$, one finds a whole *spectrum* of the critical exponents. Therefore, unlike in the traditional point of view described in Section 3.3, the critical properties become *nonuniversal*, as they are dependent on the concrete statistical properties of the disorder involved. However, this result is not the only consequence of RSB. More essential effects can be observed in the scaling properties of the spatial correlation functions (see below).

In the Ising case, $p=1$, as well as in the systems with $1 < p < 4$ for a general starting RSB function $g_0(x)$, the

consequences of RSB appear to be much more dramatic. Here, at scales $\xi \gg \xi_u$ (although still $\xi \ll \xi_* \sim 1/u$) according to the solutions (4.26) and (4.31) the parameters $\tilde{g}(\xi)$ and $g(x;\xi)$ do not arrive at any fixed point, and they keep evolving as the scale $\xi$ increases. Therefore, in this case, according to Eqn (4.34), the correlation length (defined, as usual, by the condition that the renormalised $\tau(\xi)$ approaches the order of 1) is defined by the following nontrivial equation:

$$2\ln R_c - \int_0^{\ln R_c} d\eta\left[(2+p)\tilde{g}(\eta) + p\bar{g}(\eta)\right] = \ln\frac{1}{\tau_0} . \quad (4.39)$$

Thus as the temperature becomes sufficiently close to $T_c$ (in the disorder-dominated region $\tau_0 \ll \tau_u$), there will be *no usual scaling dependence* of the correlation length (as well as of other thermodynamic quantities).

Finally, as the temperature parameter $\tau_0$ becomes smaller and smaller, what happens is that at scale $\xi_* \equiv \ln R_* \sim 1/u$ we enter the strong-coupling regime [such that the parameters $\tilde{g}(\xi)$ and $g(x;\xi)$ are no longer small], while the renormalised mass $\tau(\xi)$ still remains small. The corresponding crossover temperature scale is

$$\tau_* \sim \exp\left(-\frac{\text{const}}{u}\right) . \quad (4.40)$$

In the close vicinity of $T_c$, at $\tau \ll \tau_*$, the situation is that at large scales the interaction parameters of the asymptotic (zero-mass) Hamiltonian are no longer small, and the properties of the system cannot be analysed in terms of the simple one-loop RG approach. Nevertheless, the qualitative structure of the asymptotic Hamiltonian allows us to argue that in the temperature interval $\tau \ll \tau_*$ near $T_c$ the properties of the system should be essentially SG-like. The point is that it is the parameter describing the disorder, $g(x;\xi)$, which is the most divergent.

In a sense, here the problem is qualitatively reduced back to the original one with *strong* disorder at the critical point. It does not seem probable, however, that the state of the system will be described by the nonzero true SG order parameter $Q_{ab} = \langle\phi_a\phi_b\rangle$ (which would mean real SG freezing). Otherwise there must exist a finite value of $\tau$ at which a real thermodynamic phase transition into the SG phase takes place, whereas we observe only the *crossover* temperature $\tau_*$, at which a change of critical regime occurs.

It seems more realistic to expect that at scales $\sim \xi_*$ the RG trajectories finally arrive at a fixed point characterised by values of the interaction parameters which are no longer small and by strong RSB. Then, the SG-like behaviour of the system near $T_c$ will be characterised by highly nontrivial critical properties exhibiting strong RSB phenomena.

### 4.3.2 Correlation functions

Consider the scaling properties of the spin-glass-type connected correlation function:

$$K(R) = \overline{\left(\langle\phi(0)\phi(R)\rangle - \langle\phi(0)\rangle\langle\phi(R)\rangle\right)^2}$$
$$\equiv \overline{\langle\langle\phi(0)\phi(R)\rangle\rangle^2} . \quad (4.41)$$

In terms of the replica formalism we get:

$$K(R) = \lim_{n\to 0}\frac{1}{n(n-1)}\sum_{a\neq b}^{n} K_{ab}(R) , \quad (4.42)$$

where

$$K_{ab}(R) = \langle\langle\phi_a(0)\phi_b(0)\phi_a(R)\phi_b(R)\rangle\rangle . \quad (4.43)$$

In terms of the standard RG formalism for the replica correlation function $K_{ab}(R)$, we find that

$$K_{ab}(R) \sim \left[G_0(R)\right]^2\left[Z_{ab}(R)\right]^2 , \quad (4.44)$$

where

$$G_0(R) = R^{-(D-2)} \quad (4.45)$$

is the free-field correlation function, and in the one-loop approximation the scaling of the mass-like object $Z_{ab}(R)$ (with $a \neq b$) is defined by the RG equation:

$$\frac{d}{d\xi}\ln Z_{ab}(\xi) = 2g_{ab}(\xi) . \quad (4.46)$$

Here $g_{a\neq b}(\xi) > 0$ is the solution of the corresponding RG equations (4.13) and (4.14); $\xi = \ln R$; and $Z_{ab}(0) \equiv 1$.

For the correlation function (4.44) one finds:

$$K_{ab}(R) \sim \left[G_0(R)\right]^2 \exp\left[4\int_0^{\ln R} d\xi\, g_{ab}(\xi)\right] . \quad (4.47)$$

Similarly, in the Parisi representation, where $g_{a\neq b}(\xi) \to g(x;\xi)$ and $K_{a\neq b}(R) \to K(x;R)$, one gets

$$K(x;R) \sim \left[G_0(R)\right]^2 \exp\left[4\int_0^{\ln R} d\xi\, g(x;\xi)\right] . \quad (4.48)$$

To understand the effects of RSB more clearly let us again consider the situation in the traditional RS case. Here (for $p < 4$) one finds that the interaction parameter $g_{a\neq b}(\xi) \equiv u(\xi)$ arrives at the RS fixed point

$$u_* = \varepsilon\frac{4-p}{16(p-1)} ,$$

and according to Eqns (4.42) and (4.47), one obtains the simple scaling

$$K_{rs}(R) \sim R^{-2(D-2)+\theta} , \quad (4.49)$$

with the universal disorder-induced critical exponent

$$\theta = \varepsilon\frac{4-p}{4(p-1)} . \quad (4.50)$$

In the case of the 1-step RSB fixed point, Eqn (4.24), the situation is somewhat more complicated. Here we find that the correlation function $K(x;R)$ also has the 1-RSB structure:

$$K(x;R) \sim \begin{cases} K_0(R); & 0 \leqslant x < x_0 , \\ K_1(R); & x_0 < x \leqslant 1 , \end{cases} \quad (4.51)$$

where (in the first order in $\varepsilon$)

$$K_0(R) \sim R^{-2(D-2)} = G_0^2(R) ,$$
$$K_1(R) \sim R^{-2(D-2)+\theta_{\text{1-RSB}}} , \quad (4.52)$$

with the *nonuniversal* critical exponent $\theta_{\text{1-RSB}}$ explicitly depending on the coordinate of the step $x_0$:

$$\theta_{\text{1-RSB}} = \varepsilon\frac{4(4-p)}{16(p-1) - px_0(8+p)} . \quad (4.53)$$

Since the critical exponent $\theta_{\text{1-RSB}}$ is positive, the leading contribution to the 'observable' quantity $K(R) = \overline{\langle\langle\phi(0)\phi(R)\rangle\rangle^2}$, Eqn (4.42), is given by $K_1(R)$:

$$K(R) \sim (1-x_0)K_1(R) + x_0K_0(R) \sim R^{-2(D-2)+\theta_{\text{1-RSB}}} . \quad (4.54)$$

But the difference between the 1-RSB and the RS cases manifests itself not only in the result that the critical exponent $\theta$ of the correlation function $K(R)$ ceases to be universal. According to the traditional SG philosophy [11], the result that the scaling of the RSB correlation function $K_{ab}(R)$ or $K(x;R)$ does depend on the replica indices $(a,b)$ or the replica parameter $x$, Eqn (4.51), indicates that, in different measurements of the correlation function for *the same* realisation of the quenched disorder, one is going to obtain *different* results, $K_0(R)$ or $K_1(R)$, with the probabilities defined by the value of $x_0$.

In real experiments, however, one is dealing with the quantities averaged in space. In particular, for the two-point correlation functions the measurable quantity is obtained by integration over the two points, such that the distance $R$ between them is fixed. Of course, the result obtained in this way must be equivalent simply to $K(R)$, Eqn (4.54), found by formal averaging over different realisations of disorder, and different scalings $K_0(R)$ and $K_1(R)$ cannot be observed in this way.

Nevertheless, for a somewhat different scheme of the measurements the qualitative difference with the RS situation can be observed. In spin glasses it is generally believed that RSB can be interpreted as factorisation of the phase space into a (ultrametric) hierarchy of 'valleys', or pure states of local minima separated by macroscopic barriers. Although in the present case the local minimum configurations responsible for the RSB cannot be separated by infinite barriers, it would be natural to interpret the phenomenon observed as effective factorisation of the phase space into a hierarchy of valleys separated by *finite* barriers. In this situation, one could expect that besides the usual critical slowing down (corresponding to the relaxation inside one valley), relaxation times which are qualitatively much bigger would be required for overcoming barriers separated by different valleys. Therefore, the traditional measurements of the observables in 'thermal equilibrium' can in fact correspond to the equilibration within one valley only, and not to the true thermal equilibrium. Then in different measurements (for the same sample) one could be effectively 'trapped' in different valleys.

To check whether the above speculations are correct or not, as in spin-glasses, one can invent traditional 'overlap' quantities which could hopefully reveal the existence of the multiple valley structures. For instance, one can introduce the spatially averaged quantity for *pairs* of different realisations of the disorder:

$$K_{ij}(R) \equiv \frac{1}{V}\int \mathrm{d}^D r \langle\phi(r)\phi(r+R)\rangle_i \langle\phi(r)\phi(r+R)\rangle_j \,, \quad (4.55)$$

where $i$ and $j$ denote different realisations, and it is assumed that the measurable thermal average corresponds to a particular valley, and not to the true thermal average. If the RS situation occurs (so that only one global valley exists), then for different pairs of realisations one will obtain the same result given by Eqn (4.49). On the other hand, in the case of the 1-RSB, according to the general theory of RSB [11], after obtaining statistics over pairs of realisations for $K_{ij}(R)$ one has to get the result $K_0(R)$ with the probability $x_0$, and $K_1(R)$ with the probability $(1-x_0)$.

Consider finally what would be the situation if a general type of RSB takes place. According to the qualitative solution given by Eqns (4.26) and (4.27), the function $g(x;\xi)$ does not arrive at any fixed point at scales $\xi \gg \xi_u \sim (\nu/\alpha)\ln(1/u)$. Therefore, at the disorder-dominated scales $R \gg R_u \sim u^{-\nu/\alpha} \gg 1$, there must be no scaling behaviour of the correlation function $K(R)$. Near the critical scale $\xi_* \sim 1/u$, the qualitative behaviour of the solution $g(x;\xi)$ is given by Eqn (4.26). Therefore, according to Eqn (4.48), near the critical scale $R_* \sim \exp(1/u)$, for the correlation function $K(x;R)$ one obtains:

$$K(x;R) \sim \begin{cases} R^{-2(D-2)}(1-u\ln R)^{-4a} \equiv K_1(R)\,, & (1-x) \ll \Delta R\,, \\ R^{-2(D-2)} = G_0^2(R) \equiv K_0\,, & (1-x) \gg \Delta R\,, \end{cases} \quad (4.56)$$

where $\Delta R = (1-u\ln R) \to u \ll 1$ as $R \to R_*$.

At the critical scale we have $(1-u\ln R_*) \sim u$, and according to Eqn (4.56) the shape of the replica function $K(x;R)$ must be 'quasi-1-step':

$$K(x;R_*) \sim \begin{cases} u^{-4a}\exp\left[-\dfrac{2(D-2)}{u}\right] \equiv K_1^*\,, & (1-x) \ll u\,, \\ \exp\left[-\dfrac{2(D-2)}{u}\right] \equiv K_0^*\,, & (1-x) \gg u\,. \end{cases} \quad (4.57)$$

According to the above discussion of the observable quantities for the 1-step RSB case, the result given by Eqn (4.57) could be measured for the spatially averaged overlaps of the correlation functions $K_{ij}(R)$, Eqn (4.55), for the statistics of pairs of realisations of the disorder. Then, for the correlation function $K_{ij}(R)$ one is expected to obtain the value $K_1$ with the small probability $u$, and the value $K_0$ with the probability $(1-u)$. Although both values $K_1^*$ and $K_0^*$ are expected to be exponentially small, their ratio $K_1^*/K_0^* \sim u^{-4a}$ must be large.

Finally, at scales $R \gg R_*$ we enter the strong-coupling regime, where the simple one-loop RG approach can no longer be used.

### 4.3.3 Specific heat

According to the standard procedure, the leading singularity of the specific heat can be calculated as follows:

$$C \sim \int \mathrm{d}^D R\left[\overline{\langle\phi^2(0)\phi^2(R)\rangle} - \overline{\langle\phi^2(0)\rangle\langle\phi^2(R)\rangle}\right] . \quad (4.58)$$

In terms of the RG scheme for the correlation function,

$$W(R) \equiv \overline{\langle\phi^2(0)\phi^2(R)\rangle} - \overline{\langle\phi^2(0)\rangle\langle\phi^2(R)\rangle} \quad (4.59)$$

we get

$$W(R) = \left(G_0(R)\right)^2 m^2(R)\,, \quad (4.60)$$

where $G_0(R) = R^{-(D-2)}$ is the free-field two-point correlation function, and the mass-like object $m(R)$ is given by the solution of the following (one-loop) RG equation [compare with Eqn (4.33)]:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\ln m(\xi) = -\left[(2+p)\tilde{g}(\xi) - p\sum_{a\neq 1}^{n} g_{a1}(\xi)\right] . \quad (4.61)$$

Here, as usual, $\xi = \ln R$, and the renormalised interaction parameters $\tilde{g}(\xi)$ and $g_{a\neq b}(\xi)$ are the solutions of the replica

RG equations (4.13) and (4.14). In the Parisi representation, $g_{a\neq b}(\xi) \to g(x;\xi)$, one gets:

$$m(R) = \exp\left[-(2+p)\int_0^{\ln R} \mathrm{d}\xi\, \tilde{g}(\xi) - p\int_0^{\ln R} \mathrm{d}\xi \int_0^1 \mathrm{d}x\, g(x;\xi)\right] . \tag{4.62}$$

Then, after simple transformations for the singular part of the specific heat, Eqn (4.58), we get:

$$C \sim \int_0^{\xi_{\max}} \mathrm{d}\xi \exp\left[\varepsilon\xi - 2(2+p)\int_0^{\xi} \mathrm{d}\eta\, \tilde{g}(\eta) - 2p\int_0^{\xi} \mathrm{d}\eta\, \overline{g}(\eta)\right] , \tag{4.63}$$

where $\overline{g}(\eta) \equiv \int_0^1 \mathrm{d}x\, g(x;\eta)$. The infrared cutoff $\xi_{\max}$ in Eqn (4.63) is the scale at which the system comes out of the scaling regime.

Usually $\xi_{\max}$ is the scale at which the renormalised mass $\tau(\xi)$, Eqn (4.34), approaches the order of 1, and if the traditional scaling situation takes place, one finds that $\xi_{\max} \sim \ln(1/\tau_0)$.

Again, let us first consider the situation in the traditional RS case. Here at scales $\xi \gg \xi_u \sim \ln(1/u)$ (which correspond to the temperature region $\tau_0 \ll \tau_u \sim u^{\nu/\alpha}$) the renormalised parameters $\tilde{g}(\eta)$ and $g(\xi)$ arrive at the universal fixed point $\tilde{g}_* = \varepsilon[p/16(p-1)]$, $g_* = \varepsilon[4 - p/16(p-1)]$ given by Eqn (4.23); and according to Eqn (4.63) for the singular part of the specific heat we find that

$$C(\tau_0) \sim \int_0^{\ln(1/\tau_0)} \mathrm{d}\xi \exp\{\xi[\varepsilon - 2(2+p)\tilde{g}_* - 2pg_*]\} \sim \tau_0^{\varepsilon(4-p)/4(p-1)} . \tag{4.64}$$

So that in the close vicinity of $T_c$ one would expect to observe new universal disorder-induced critical behaviour with a negative specific heat critical exponent $\alpha = -\varepsilon(4-p)/4(p-1)$, Eqn (3.40) (unlike positive $\alpha$ in the corresponding pure system).

Similarly, if the scenario with the stable 1-step RSB fixed points takes place, then one finds that the specific heat critical exponent $\alpha(x_0)$ becomes nonuniversal, and depends explicitly on the coordinate of the step $x_0$ [16]:

$$\alpha(x_0) = -\frac{1}{2}\,\varepsilon\,\frac{(4-p)(4-px_0)}{16(p-1) - px_0(p+8)} . \tag{4.65}$$

Again (as for the critical exponent of the correlation length), depending on the value of the parameter $x_0$, one finds a whole *spectrum* of the critical exponents. In particular, the possible values of the specific heat critical exponent appear to be in the following band:

$$-\infty < \alpha(x_0) < -\varepsilon\,\frac{(4-p)}{8(p-1)} . \tag{4.66}$$

The upper limit for $\alpha(x_0)$ is achieved in the RS limit $x_0 \to 0$, and it coincides with the usual RS result, Eqn (3.41). On the other hand, as $x_0$ tends to the 'border of stability' $x_c(p)$ of the 1-step RSB fixed point, formally the specific heat critical exponent tends to $-\infty$.

In the general RSB case, the situation is completely different. Here in the disorder-dominated region $\tau_* \ll \tau_0 \ll u^{\nu/\alpha}$ (which corresponds to scales $\xi_u \ll \xi \ll \xi_*$), the RG trajectories of the interaction parameters $\tilde{g}(\xi)$ and $\overline{g}(\xi)$ do not arrive at any fixed point, and according to Eqn (4.64) one finds that the specific heat becomes a complicated function of the temperature parameter $\tau_0$ which does not have the traditional scaling form.

Finally, in the SG-like region in the close vicinity of $T_c$, where the interaction parameters $\tilde{g}$ and $\overline{g}$ are finite, one finds that the integral over $\xi$ in Eqn (4.63) is convergent (so that the upper cutoff scale $\xi_{\max}$ becomes irrelevant). Thus, in this case one obtains the result that the 'would-be singular part' of the specific heat remains finite in the temperature interval $\sim \tau_*$ around $T_c$, so that the specific heat becomes *nonsingular* at the phase transition point.

### 4.4 Discussion

According to the results obtained in this Section, we can conclude that spontaneous replica symmetry breaking coming from the interaction of the fluctuations with the multiple local minimum solutions of the mean-field equations has a dramatic effect on the renormalisation group flows and on the critical properties. In systems with the number of spin components $p < 4$, the traditional RG flows at dimensions $D = 4 - \varepsilon$, which are usually considered as describing the disorder-induced universal critical behaviour, appear to be unstable with respect to the RSB potentials as found in spin glasses. For a general type of the Parisi RSB structures, there exist no stable fixed points, and the RG flows lead to the *strong-coupling regime* at the finite scale $R_* \sim \exp(1/u)$, where $u$ is the small parameter describing the disorder. Unlike the systems with $1 < p < 4$, where there exist stable fixed points having 1-step RSB structures [Eqn (4.24)], in the Ising case, $p = 1$, there exist no stable fixed points, and any RSB interactions lead to the strong-coupling regime.

There exists another general problem which may appear to be interconnected with the RSB phenomena considered in this Section. The problem is related to the existence of the so-called Griffith phase [21] in a finite temperature interval above $T_c$. Numerous experiments for various disordered systems [22], as well as numerical simulations for the three-dimensional random bonds Ising model [23], clearly demonstrate that in the temperature interval $T_c < T < T_0$ (in the high temperature phase), the time correlations decay as $\sim \exp[-(t/\tau)^{\lambda}]$ instead of the usual exponential relaxation law $\sim \exp(-t/\tau)$, as it should be in the ordinary paramagnetic phase. Moreover, it is claimed that the parameter $\lambda$ is the temperature-dependent exponent, as it is a finite value (less than unity) at $T = T_c$, and increases monotonically up to $\lambda = 1$ at $T = T_0$. The temperature $T_0$ is claimed to coincide with the phase transition point of the corresponding pure system.

This phenomenon clearly demonstrates the existence of numerous metastable states separated by finite barriers, their values forming infinite continuous spectra, and it could be interconnected with a general idea that the critical phenomena should be described in terms of an infinite hierarchy of correlation lengths and critical exponents [24].

On the other hand, if there is RSB in the fourth-order potential in the problem considered in this Section, one could identify a phase with a different symmetry than the conventional paramagnetic phase, and thus there would have to be a temperature $T_{\mathrm{RSB}}$ at which this change in symmetry occurs. Actually, the RSB situation is the property of the statistics of the saddle-point solutions

only, and it is clear that for sufficiently large values of $\tau$ there must be no RSB. Therefore, one can try to solve the problem of summing over saddle-point solutions for arbitrary $\tau$, with the aim of finding a finite value of $\tau_c$ at which the RSB solution for this problem disappears. Of course, in general this problem is very difficult to solve, but one can easily obtain an estimate for the value of $\tau_c$ (assuming that at $\tau = 0$ the RSB situation takes place). According to the qualitative study of this problem in Ref. [16], the RSB solution can occur only when the effective interactions between the 'islands' (where the system is effectively below $T_c$) are no longer small. The 'islands' are the regions where $\delta\tau(r) > \tau$. Because the distribution of the random function $\delta\tau(r)$ is Gaussian, the average distance between the 'islands' will be of the order of $\exp(-\tau^2/u)$, so that the 'islands' become sufficiently remote at $\tau > \sqrt{u}$. The interaction between the 'islands' decreases exponentially with their separation. Therefore at $\tau > \sqrt{u}$ they must be interacting weakly, and there must be no RSB.

Note now that the shift of $T_c$ with respect to the corresponding pure system is also of the order of $\sqrt{u}$. On the other hand, the existence of local solutions to the mean-field equations is reminiscent of the Griffith phase which is claimed to be observed in the temperature interval between the temperature $T_c$ of the disordered system and $T_c$ of the corresponding pure system. On these grounds it is tempting to associate the (hypothetical) RSB transition in the statistics of the saddle-point solutions with the Griffith transition. Correspondingly, it would also be natural to suggest that RSB phenomena discovered in the scaling properties of weakly disordered systems could be associated with the Griffith effects.

The other key question which remains unanswered is whether or not the obtained strong-coupling phenomena in the RG flows can be interpreted as the onset of a kind of spin-glass phase near $T_c$. Since it is the RSB interaction parameter describing disorder, $g(x;\xi)$, which is the most divergent, it is tempting to argue that, in the temperature interval $\tau \ll \tau_* \sim \exp(-1/u)$ near $T_c$, the properties of the system should be essentially SG-like.

It should be stressed, however, that in the present study we observe only the *crossover* temperature $\tau_*$, at which the change in the critical regime occurs, and it is hardly possible to associate this temperature with any kind of phase transition. Therefore, if the RSB effects could indeed provide any kind of true thermodynamic order parameter, then this must be true in a complete temperature interval where the RSB potentials exist.

The true spin-glass order (in the traditional sense) arises from the onset of the nonzero order parameter $Q_{ab}(x) = \langle \phi_a(x)\phi_b(x)\rangle$; $a \neq b$, and, at least for the infinite-range model, $Q_{ab}$ develops the hierarchical dependence on replica indices [11]. In the present problem we find that only the coupling matrix $g_{ab}$ for the fluctuating fields develops a strong RSB structure and its elements are no longer small at the finite scale. Therefore, it seems more realistic to interpret RSB strong-coupling phenomena discovered in the RG as a completely new type of the critical behaviour characterised by strong SG effects in the scaling properties rather than in the ground state.

## 5. Two-dimensional Ising model with impurities

### 5.1 Two-dimensional Ising systems

In the general theory of phase transitions the two-dimensional (2D) Ising model plays the prominent role, as it is the simplest nontrivial lattice model with a known exact solution [25]. It is natural to ask, therefore, what the effects of the impurities are in this particular case. As for the Harris criterion (Section 3.1), the 2D Ising model constitutes a special case, because the specific heat exponent $\alpha = 0$ in this model. However, speaking intuitively, we could expect that, as in the case of the vector field model in four dimensions (Section 3.3), the effect of impurities could be predicted on a qualitative level. Although the critical exponent $\alpha$ is zero, the specific heat of the 2D Ising model is (logarithmically) divergent at the critical point. Therefore, we should expect the critical behaviour of this system to be strongly affected by the impurities.

Indeed, the exact solution for the critical behaviour of the specific heat of the 2D Ising model with a small concentration $c \ll 1$ of impurities [26] (see Section 4.3 below) yields the following result for the singular part of the specific heat:

$$C(\tau) \sim \begin{cases} \ln\dfrac{1}{\tau}, & \tau^* \ll |\tau| \ll 1, \\ \dfrac{1}{c}\ln\left(\ln\dfrac{1}{\tau}\right), & \tau \ll \tau^*, \end{cases} \tag{5.1}$$

where $\tau^* \sim \exp(-\text{const}/c)$ is the temperature scale at which a crossover from one critical behaviour to another takes place.

Thus, in the 2D Ising model, as well as in the four-dimentional vector field system, the impurities are in fact 'relevant variables'. Unlike the vector field model, the specific heat of the 2D impurity Ising magnet remains divergent at $T_c$, though the singularity is weakened by impurities. Another important property of the 2D Ising model is that, unlike in the $\phi^4$-theory near four dimensions (Section 4), the spin-glass RSB phenomena appear to be irrelevant for the critical behaviour. Thus, the result given by Eqn (5.1) for the leading singularity of the specific heat of the weakly disordered 2D Ising system must be exact.

In this section the emphasis is laid not on the exact lattice expressions, but on their large-scale asymptotics, i.e. we will be interested mainly in the critical long-range behaviour because only that is interesting for the general theory of phase transitions. It is well known that in the critical region the 2D Ising model can be reduced to the free-fermion theory [27]. In Section 4.2 this reduction will be demonstrated in very simple terms by means of the Grassman variables technique. The operator language or the transfer matrix formalism will not be used, as they are not symmetric enough to be applied to the model with impurities. The resulting continuum theory, to which the exact lattice impurity model is equivalent in the critical region, appears to be simple enough, and in it the critical behaviour of the specific heat can be determined exactly (Section 4.3).

The results of the recent numerical simulations are briefly described in Section 4.4. The general structure of the disordered 2D Ising model is considered in Section 4.5.

### 5.2 The fermion solution

The partition function of the pure 2D Ising model is given by

$$Z = \sum_{\sigma} \exp\left(\beta \sum_{x,\,\mu} \sigma_x \sigma_{x+\mu}\right) . \tag{5.2}$$

Here $\{\sigma_x = \pm 1\}$ are the Ising spins defined at lattice sites of a simple square lattice; $x$ are integer-valued coordinates of the lattice sites; and $\mu = 1, 2$ are basic vectors of the lattice.

This partition function can be rewritten as follows:

$$\begin{aligned} Z &= \sum_{\sigma} \prod_{x,\,\mu} \exp\{\beta\, \sigma_x \sigma_{x+\mu}\} \\ &= \sum_{\sigma} \prod_{x,\,\mu} (\cosh\beta + \sigma_x \sigma_{x+\mu} \sinh\beta) \\ &= (\cosh\beta)^V \sum_{\sigma} \prod_{x,\,\mu} (1 + \sigma_x \sigma_{x+\mu} \lambda) , \end{aligned} \tag{5.3}$$

where $V$ is the total number of the lattice bonds, and $\lambda \equiv \tanh\beta$. Expanding the product over the lattice bonds in Eqn (5.3) and averaging over the $\sigma$ values, we obtain the following representation for the partition function (the high-temperature expansion):

$$Z = (\cosh\beta)^V \sum_{\mathcal{P}} (\lambda)^{L_{\mathcal{P}}} . \tag{5.4}$$

The summation here is over configurations of closed paths $\mathcal{P}$ drawn on lattice links (Fig. 9), and $L_{\mathcal{P}}$ is the total length of paths in a particular configuration $\mathcal{P}$.

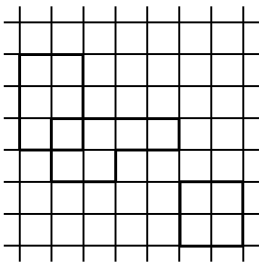

**Figure 9.** Closed paths entering the high-temperature expansion of the partition function of the 2D Ising model.

The summation in Eqn (5.4) could be performed exactly, and these calculations constitute the classical exact solution for the 2D Ising model found by Sherman and Vdovichenko [28]. This solution is well described in detail in textbooks (for example, see Ref. [29]), and we will not discuss it here.

Let us now consider an alternative approach to the calculations of the partition function in terms of the so-called Grassmann variables (for detailed treatment of this new mathematics see Ref. [30]). The Grassmann variables were first used for the 2D Ising model by Hurst and Green [31], and this approach was later developed by a number of authors [32] (see also Ref. [26]). It appears that technically this method enables the equations to be obtained in a very simple way. We shall describe this formalism, recover the equation for the partition function [Eqn (5.4)], and introduce some new notations which will be useful for the problem with impurities.

Let us introduce the four-component Grassmann variables $\{\psi^\alpha(x)\}$ defined at the lattice sites $\{x\}$, where the superscript $\alpha = 1, 2, 3, 4$ indicates the four directions on the 2D square lattice (such that $3 \equiv -1$ and $4 \equiv -2$). All the $\{\psi^\alpha(x)\}$ variables and all their differentials $\{\mathrm{d}\psi^\alpha(x)\}$ are anticommutative variables. By definition,

$$\begin{aligned} &\psi^\alpha(x)\psi^\beta(y) = -\psi^\beta(y)\psi^\alpha(x) , \\ &\left[\psi^\alpha(x)\right]^2 = 0 , \\ &\mathrm{d}\psi^\alpha(x)\,\mathrm{d}\psi^\beta(y) = -\,\mathrm{d}\psi^\beta(y)\,\mathrm{d}\psi^\alpha(x) , \\ &\mathrm{d}\psi^\alpha(x)\psi^\beta(y) = -\psi^\beta(y)\,\mathrm{d}\psi^\alpha(x) , \end{aligned} \tag{5.5}$$

and the integration rules are defined as follows:

$$\begin{aligned} &\int \mathrm{d}\psi^\alpha(x) = 0 , \\ &\int \mathrm{d}\psi^\alpha(x)\,\psi^\alpha(x) = -\int \psi^\alpha(x)\,\mathrm{d}\psi^\alpha(x) = 1 . \end{aligned} \tag{5.6}$$

Let us consider the following partition function defined as an integral over all the Grassmann variables of the 2D lattice system:

$$Z = \int \mathrm{D}\psi \exp\left(A[\psi]\right) . \tag{5.7}$$

In this case, the integration measure $D\psi$ and the action $A[\psi]$ are defined as follows:

$$\mathrm{D}\psi = \prod_x \left[-\,\mathrm{d}\psi^1(x)\,\mathrm{d}\psi^2(x)\,\mathrm{d}\psi^3(x)\,\mathrm{d}\psi^4(x)\right] , \tag{5.8}$$

$$A[\psi] = -\frac{1}{2}\sum_x \bar{\psi}(x)\,\psi(x) + \frac{1}{2}\,\lambda \sum_{x,\alpha} \bar{\psi}(x+\alpha)\,\hat{p}_\alpha \psi(x) . \tag{5.9}$$

The 'conjugated' variables $\overline{\psi}(x)$ are defined as follows:

$$\overline{\psi^\alpha} = \psi^\gamma (\hat{C}^{-1})^{\gamma\alpha} , \tag{5.10}$$

where

$$\hat{C} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix} ,$$

$$\hat{C}^{-1} = \begin{bmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix} . \tag{5.11}$$

The vector matrix $\hat{\boldsymbol{p}}$ in Eqn (5.9) is defined as follows:

$$\hat{\boldsymbol{p}} = \left( \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) . \tag{5.12}$$

More explicitly for the action $A[\psi]$, Eqn (5.9), one gets:

$$A[\psi] = -\frac{1}{2}\sum_x \psi(x)\hat{C}^{-1}\psi(x) + \frac{1}{2}\lambda\sum_{x,\,\alpha}\psi(x+\alpha)\hat{C}^{-1}\hat{p}_\alpha\psi(x)$$

$$\equiv \sum_x \big[\psi^3(x)\psi^1(x) + \psi^4(x)\psi^2(x) + \psi^1(x)\psi^2(x) + \psi^3(x)\psi^4(x) + \psi^2(x)\psi^3(x) + \psi^1(x)\psi^4(x)\big] + \lambda\sum_x\big[\psi^3(x+1)\psi^1(x) + \psi^4(x+2)\psi^2(x)\big]. \quad (5.13)$$

Using the rules (5.5) and (5.6), we can easily check by direct calculations that the integration in Eqn (5.7) with the integration measure [Eqn (5.8)] reproduces the high-temperature expansion of the partition function [Eqn (5.4)] 2D Ising model with $\lambda = \tanh\beta$.

Let us consider the Green function:

$$G^{\alpha\beta}(x,x') = Z^{-1}\int \mathrm{D}\psi\,\exp\big(A[\psi]\big)\,\psi^\alpha(x)\,\bar{\psi}^\beta(x')\,. \quad (5.14)$$

Simple (although a little cumbersome) calculations yield:

$$G^{\alpha\beta}(x,x') = \lambda\sum_\gamma \Lambda^{\alpha\gamma}G^{\gamma\beta}(x-\gamma,\,x') + \delta_{x,\,x'}\delta^{\alpha\beta}\,, \quad (5.15)$$

where $\hat{\Lambda} \equiv \sum_\alpha \hat{p}_\alpha$,

$$\hat{\Lambda} = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix}. \quad (5.16)$$

If we perform a Fourier transform of equation (5.15), it acquires the following matrix form:

$$\hat{G}(\boldsymbol{k}) = \big[\hat{1} - \lambda\hat{\Lambda}(\boldsymbol{k})\big]^{-1}\,, \quad (5.17)$$

where

$$\hat{\Lambda}(\boldsymbol{k}) = \sum_\alpha \exp(-\mathrm{i}\boldsymbol{k}\alpha)\hat{p}_\alpha = \begin{bmatrix} \exp(-\mathrm{i}k_1) & \exp(-\mathrm{i}k_2) & 0 & -\exp(\mathrm{i}k_2) \\ \exp(-\mathrm{i}k_1) & \exp(-\mathrm{i}k_2) & \exp(\mathrm{i}k_1) & 0 \\ 0 & \exp(-\mathrm{i}k_2) & \exp(\mathrm{i}k_1) & \exp(\mathrm{i}k_2) \\ -\exp(-\mathrm{i}k_1) & 0 & \exp(\mathrm{i}k_1) & \exp(\mathrm{i}k_2) \end{bmatrix}. \quad (5.18)$$

It is obvious from Eqn (5.17) that, if one of the eigenvalues of the matrix $\lambda\hat{\Lambda}(\boldsymbol{k})$ becomes unity, it signals a singularity. To find this point we first put the space momentum $\boldsymbol{k} = 0$ (which corresponds to the infinite spatial scale).

The four-valued indices of the Green function $G^{\alpha\beta}$ are related to four possible directions on a square lattice. Therefore, the idea is to perform the Fourier transform over these angular degrees of freedom. One can easily check that the matrix $\hat{\Lambda}(0)$ is diagonalised in the following representation:

$$\psi_{\pm 1/2} = \frac{1}{2}\begin{bmatrix} 1 \\ \exp\left(\pm\mathrm{i}\,\frac{\pi}{4}\right) \\ \exp\left(\pm\mathrm{i}\,\frac{\pi}{2}\right) \\ \exp\left(\pm\mathrm{i}\,\frac{3\pi}{4}\right) \end{bmatrix}, \quad \psi_{\pm 3/2} = \frac{1}{2}\begin{bmatrix} 1 \\ \exp\left(\pm\mathrm{i}\,\frac{3\pi}{4}\right) \\ \exp\left(\pm\mathrm{i}\,\frac{3\pi}{2}\right) \\ \exp\left(\pm\mathrm{i}\,\frac{9\pi}{4}\right) \end{bmatrix}. \quad (5.19)$$

The transform matrix from the initial representation to the angular momentum (or spinor) representation, with the basic vectors given above, has the form:

$$\hat{U} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ E & \bar{E} & E^3 & \bar{E}^3 \\ E^2 & \bar{E}^2 & E^6 & \bar{E}^6 \\ E^3 & \bar{E}^3 & E^9 & \bar{E}^9 \end{bmatrix},$$

$$E = \exp\left(\mathrm{i}\,\frac{\pi}{4}\right), \quad \bar{E} = \exp\left(-\mathrm{i}\,\frac{\pi}{4}\right). \quad (5.20)$$

In this representation, we get

$$\lambda\hat{\Lambda}'(0) = \lambda\hat{U}^{-1}\hat{\Lambda}(0)\hat{U} = \lambda\begin{bmatrix} \sqrt{2}+1 & 0 & 0 & 0 \\ 0 & \sqrt{2}+1 & 0 & 0 \\ 0 & 0 & -\sqrt{2}+1 & 0 \\ 0 & 0 & 0 & -\sqrt{2}+1 \end{bmatrix}. \quad (5.21)$$

There is a singularity in Eqn (5.17) (at $k \to 0$) when one of the eigenvalues of $\lambda\hat{\Lambda}'$ becomes unity. From Eqn (5.21) we can easily find the critical point of the 2D Ising model:

$$\lambda_c \equiv \tanh\beta_c = \frac{1}{\sqrt{2}+1}\,. \quad (5.22)$$

Another important point which follows from these considerations is that, for the critical fluctuations in the vicinity of the critical point, only states $\psi_{\pm 1/2}$ (with the eigenvalues $\simeq 1$) are important. Indeed it is easily checked (see below) that the correlation radius for $\psi_{\pm 1/2}$ goes to infinity as $\lambda \to \lambda_c$, whereas the correlations for $\psi_{\pm 3/2}$ are confined to lattice sizes.

Now, to describe the critical long-range fluctuations, which are responsible for the singularities in the thermodynamic functions, we can expand Eqn (5.17) near the point $\lambda = \lambda_c$. Using the explicit expression (5.18), and retaining only the first powers of $|\boldsymbol{k}|$ and $(\lambda - \lambda_c)/\lambda_c$, one gets:

$$\hat{G}(\boldsymbol{k}) \simeq \frac{2\lambda_c^2}{\Delta}\begin{bmatrix} \tau - \mathrm{i}k_1 & \frac{\tau - \mathrm{i}k_1 - \mathrm{i}k_2}{\sqrt{2}} & -\mathrm{i}k_2 & -\frac{\tau - \mathrm{i}k_1 + \mathrm{i}k_2}{\sqrt{2}} \\ \frac{\tau - \mathrm{i}k_1 - \mathrm{i}k_2}{\sqrt{2}} & \tau - \mathrm{i}k_2 & \frac{\tau + \mathrm{i}k_1 - \mathrm{i}k_2}{\sqrt{2}} & \mathrm{i}k_1 \\ -\mathrm{i}k_2 & \frac{\tau + \mathrm{i}k_1 - \mathrm{i}k_2}{\sqrt{2}} & \tau + \mathrm{i}k_1 & \frac{\tau + \mathrm{i}k_1 + \mathrm{i}k_2}{\sqrt{2}} \\ -\frac{\tau - \mathrm{i}k_1 + \mathrm{i}k_2}{\sqrt{2}} & \mathrm{i}k_1 & \frac{\tau + \mathrm{i}k_1 + \mathrm{i}k_2}{\sqrt{2}} & \tau + \mathrm{i}k_2 \end{bmatrix}. \quad (5.23)$$

Here

$$\Delta = \det\big[\hat{1} - \lambda\hat{\Lambda}(\boldsymbol{k})\big] \simeq 2\lambda_c^2(\tau^2 + |\boldsymbol{k}|^2) \quad (5.24)$$

and

$$\tau \equiv 2\,\frac{(\lambda - \lambda_c)}{\lambda_c}\,. \quad (5.25)$$

In the spinor representation given by Eqn (5.19), the asymptotic expression for Eqn (5.23) simplifies to the following compact form:

$$\hat{G}_{\mathrm{sp}}(\boldsymbol{k}) = \hat{U}^{-1}\hat{G}(\boldsymbol{k})\hat{U} \simeq \frac{2}{\tau^2 + |\boldsymbol{k}|^2}\begin{bmatrix} \tau & \mathrm{i}k_1 - k_2 & 0 & 0 \\ \mathrm{i}k_1 + k_2 & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (5.26)$$

The zero components here are $\sim |\boldsymbol{k}|^2, \tau^2$. The nonzero block can be represented as:

$$\hat{S}(\boldsymbol{k}) = 2\frac{\tau - \mathrm{i}\hat{k}}{\tau^2 + \boldsymbol{k}^2}\,, \tag{5.27}$$

Here

$$\hat{k} = k_1\hat{\gamma}_1 + k_2\hat{\gamma}_2\,; \tag{5.28}$$

$$\gamma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}. \tag{5.29}$$

Eqn (5.27) is the Green function of the free (real) spinor field in two Euclidean dimensions described by the Lagrangian:

$$A_{\mathrm{sp}}[\psi] = -\frac{1}{4}\int \mathrm{d}^2x\left(\overline{\psi}\,\hat{\partial}\,\psi + \tau\overline{\psi}\psi\right) \tag{5.30}$$

where $\overline{\psi} = \psi\hat{\gamma}_5$, and $\hat{\gamma}_5 = \hat{\gamma}_1\hat{\gamma}_2$.

Using Eqn (5.30), one immediately finds the logarithmic singularity of the specific heat of the 2D Ising model:

$$\begin{aligned} Z &\simeq \int \mathrm{D}\psi \exp\left(A_{\mathrm{sp}}[\psi]\right) \simeq \left[\det(\tau + \hat{\partial})\right]^{1/2}\,; \\ F &\simeq -\ln Z \simeq -\mathrm{Tr}\ln(\tau + \hat{\partial}) \\ &\simeq -\int \mathrm{d}^2k \ln(\tau^2 + |\boldsymbol{k}|^2) \sim -\tau^2 \ln\frac{1}{|\tau|}\,. \end{aligned} \tag{5.31}$$

Hence

$$C \sim -\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}F(\tau) \sim \ln\frac{1}{|\tau|}\,. \tag{5.32}$$

### 5.3 Critical behaviour in the impurity model

We turn now to the model with impurity bonds. In this Section we shall determine the critical behaviour of the specific heat in the critical regime of the impurity.

The partition function of the 2D Ising model with impurity bonds is given by:

$$Z(\beta) = \sum_{\sigma} \exp\left\{\beta \sum_{x,\,\mu} J_{x\,\mu}\,\sigma_x\sigma_{x+\mu}\right\}\,, \tag{5.33}$$

where the coupling constant $J_{x\mu}$ on a particular lattice bond $(x, \mu)$ is equal to the regular value $J$ with probability $(1-c)$, and to the impurity value $J' \neq J$ with probability $c$. We impose no restriction on $J'$ but we shall require $c \ll 1$, so that the concentration of impurities is assumed to be small.

The Grassmann variables technique described in the previous Section can be applied to the model with random lattice couplings as well. In this representation, the partition function [Eqn (5.33)] is given by:

$$Z(\beta) = \int \mathrm{D}\psi \exp\left\{-\frac{1}{2}\sum_x \overline{\psi}(x)\,\psi(x) + \frac{1}{2}\sum_{x,\,\mu}\lambda_{x\,\mu}\overline{\psi}(x+\mu)\,\hat{p}_\mu\psi(x)\right\}\,, \tag{5.34}$$

where

$$\lambda_{x\,\mu} = \begin{cases} \lambda = \tanh(\beta J)\,, & \text{with probability } (1-c)\,, \\ \lambda' = \tanh(\beta J')\,, & \text{with probability } c\,. \end{cases} \tag{5.35}$$

It is easy to check by direct expansion in powers of the second term in Eqn (5.34) that the partition function can be represented as a sum over configurations of closed loops, each loop entering with a weight

$$\prod_{\mathcal{P}} \lambda_{x\,\mu}\Phi(\mathcal{P})\,, \tag{5.36}$$

where $\Phi(\mathcal{P})$ is a product ordered along the path $\mathcal{P}$ of matrices $\{\hat{p}\}$:

$$\Phi(\mathcal{P}) = \prod_{\mathcal{P}} \hat{p}\,. \tag{5.37}$$

The same representation for the partition function follows from the high-temperature expansion of Eqn (5.33).

Proceeding along these lines and averaging over the disorder in the couplings, one could finally obtain the exact continuum-limit representation for the free energy of the impurity model (see Ref. [26]). Here, however, we shall consider a more intuitive and much more simplified approach, which nevertheless provides the same results as the exact one. This approach is based on the natural assumption that in the continuum limit representation in terms of the free-fermion fields (see previous Section), the disorder in the couplings manifests itself as a small spatial disorder in the effective critical temperature $\tau$ in the mass term of the spinor Lagrangian Eqn (5.30). Therefore, the starting point for further considerations of the impurity model will be the assumption that its continuum limit representation is described by the following spinor Lagrangian:

$$A_{\mathrm{imp}}[\psi;\,\delta\tau(x)] = -\frac{1}{4}\int \mathrm{d}^2x\left\{\overline{\psi}\,\hat{\partial}\psi + [\tau + \delta\tau(x)]\overline{\psi}\psi\right\}\,. \tag{5.38}$$

Here the quenched random variable $\delta\tau(x)$ is assumed to be described by simple Gaussian distribution:

$$P[\delta\tau(x)] = \prod_x \left\{\frac{1}{\sqrt{8\pi u}}\exp\left[-\frac{[\delta\tau(x)]^2}{8u}\right]\right\}\,, \tag{5.39}$$

where the small parameter $u \ll 1$ is proportional to the concentration of impurities.

Then, the self-averaging free energy can be obtained in terms of the traditional replica approach (Section 3.2):

$$F \equiv \overline{F[\delta\tau(x)]} = -\frac{1}{\beta}\lim_{n\to 0}\frac{1}{n}\ln(Z_n)\,, \tag{5.40}$$

where

$$\begin{aligned} Z_n &\equiv \overline{Z^n} \\ &= \int \mathrm{D}\,\delta\tau(x)\int \mathrm{D}\,\psi^a P[\delta\tau(x)] \\ &\quad\times \exp\left(-\frac{1}{4}\int \mathrm{d}^2x\sum_{a=1}^{n}\left\{\overline{\psi^a}\,\hat{\partial}\,\psi^a + [\tau + \delta\tau(x)]\overline{\psi^a}\,\psi^a\right\}\right) \end{aligned} \tag{5.41}$$

is the replica partition function and the superscript $a = 1, 2, \ldots, n$ denotes the replicas.

Simple Gaussian integration over $\delta\tau(x)$ yields:

$$Z_n = \int \mathrm{D}\,\psi^a \exp\left(A_n[\psi]\right)\,, \tag{5.42}$$

where

$$A_n[\psi] = -\int \mathrm{d}^2x\left[\frac{1}{4}\sum_{a=1}^{n}\overline{\psi^a}(\hat{\partial} + \tau)\,\psi^a - \frac{1}{4}\,u\sum_{a,b=1}^{n}\overline{\psi^a}\,\psi^a\,\overline{\psi^b}\,\psi^b\right]\,. \tag{5.43}$$

Note that rigorous perturbative consideration of the original lattice problem [26] yields the same result for the effective continuous limit Lagrangian [Eqn (5.43)], in which

$$u = c\,\frac{\left[(\lambda_c' - \lambda_c)/\lambda_c\right]^2}{\left[1 + 1/2\sqrt{2}(\lambda_c' - \lambda_c)\right]^2}\,, \tag{5.44}$$

where

$$\lambda_c = \tanh\beta_c J = \sqrt{2} - 1\,,$$
$$\lambda_c' = \tanh\beta_c J'\,. \tag{5.45}$$

The spinor field theory with the four-fermion interaction [Eqn (5.43)] obtained above can be renormalised in two dimensions, just as the vector field theory with the interaction $\phi^4$ can be renormalised in four dimensions (Section 2.6).

Indeed, after the scale transformation (see Section 2.5),

$$x \to \lambda x \quad (\lambda > 1)\,, \tag{5.46}$$

one gets

$$\int \mathrm{d}^D x\,\overline{\psi}(x)\hat{\partial}\psi(x) \to \lambda^{D-1}\int \mathrm{d}^D x\,\overline{\psi}(\lambda x)\hat{\partial}\psi(\lambda x)\,,$$
$$u\int \mathrm{d}^D x\left[\overline{\psi}(x)\psi(x)\right]\left[\overline{\psi}(x)\psi(x)\right]$$
$$\to \lambda^D u\int \mathrm{d}^D x\left[\overline{\psi}(\lambda x)\psi(\lambda x)\right]\left[\overline{\psi}(\lambda x)\psi(\lambda x)\right]\,. \tag{5.47}$$

To leave the gradient term of the Hamiltonian (which is responsible for the scaling of the correlation functions) unchanged, one has to rescale the fields:

$$\psi(\lambda x) \to \lambda^{-\Delta_\psi}\psi(x)\,, \tag{5.48}$$

with

$$\Delta_\psi = \frac{D-1}{2}\,. \tag{5.49}$$

The scale dimension $\Delta_\psi$ defines the critical exponent of the correlation function:

$$G(x) = \langle\overline{\psi}(0)\psi(x)\rangle \sim |x|^{-2\Delta_\psi}\big|_{D=2} = |x|^{-1}\,. \tag{5.50}$$

To leave the Hamiltonian Eqn (5.43) unchanged after these transformations, one has to rescale the parameter $u$:

$$u \to \lambda^{-\Delta_u} u\,, \tag{5.51}$$

where

$$\Delta_u = 2 - D\,. \tag{5.52}$$

Therefore, the scale dimension $\Delta_u$ of the four-fermion interaction term is zero in two dimensions, just as the scale dimension of the $\phi^4$ interaction term is zero in four dimensions.

We shall see below that the renormalisation equations lead to the 'zero-charge' asymptotics for the charge $u$ and the mass $\tau$. In this lucky case, the critical behaviour can be found by the renormalisation-group methods or, in the same way, the main singularities of the thermodynamic functions can be found by summing up the 'parquette' diagrams of the theory [Eqn (5.43)] (cf. Section 2.6)

Let us renormalise the charge $u$ and the mass $\tau$. The diagrammatic representation of the interaction $u\left[\overline{\psi^a}(x)\psi^a(x)\right]\left[\overline{\psi^b}(x)\psi^b(x)\right]$ and the mass $\tau\left[\overline{\psi^a}(x)\psi^a(x)\right]$ terms are shown in Fig. 10.

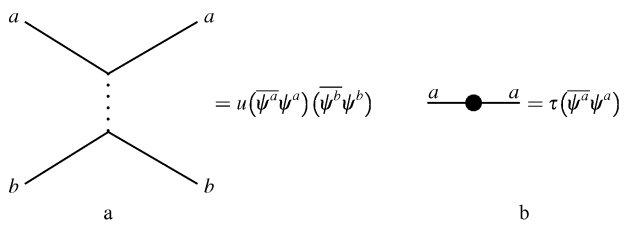


**Figure 10.** Diagrammatic representation of the interaction $u[\overline{\psi^a}(x)\psi^a(x)][\overline{\psi^b}(x)\psi^b(x)]$ and the mass $\tau[\overline{\psi^a}(x)\psi^a(x)]$ terms.

It should be stressed that the model under consideration is described in terms of *real* fermions, and although we are using (just for convenience) the notation of the conjugated fields $\overline{\psi}$, they are not independent variables: $\overline{\psi} = \psi\hat{\gamma}_5$. For this reason the fermion lines in the diagrammatic representation are not 'directed'. Actually, the interaction term, see Fig. 10, can be represented explicitly in terms of only one two-component fermion (anticommuting) field: $u\psi_1^a\psi_2^a\psi_1^b\psi_2^b$. Therefore, the diagonals in the replica ($a = b$) interaction terms are identical and equal to zero.

Proceeding in a similar way to the calculations of Section 3.3, one then finds that the renormalisations of the parameter $u$ are provided only by the diagram shown in Fig. 11c, whereas the first two diagrams, Figs 11a and 11b, are identical and equal to zero. For the same reason, the renormalisation of the mass term is provided only by the diagram shown in Fig. 12b, while the diagram in Fig. 12a is zero.

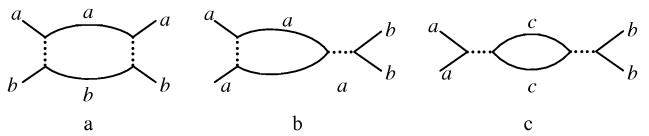


**Figure 11.** Diagrams which contribute to the renormalisation of the interaction term $u[\overline{\psi^a}(x)\psi^a(x)][[\overline{\psi^b}(x)\psi^b(x)]$.

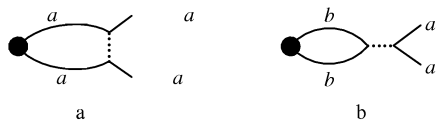


**Figure 12.** Diagrams which contribute to the renormalisation of the mass term $\tau[\overline{\psi^a}(x)\psi^a(x)]$.

The internal lines in Figs 11 and 12 represent the massless free-fermion Green function [cf. Eqns (5.27), (5.28)]:

$$\hat{S}_{ab} = -\mathrm{i}\,\frac{\hat{k}}{|\boldsymbol{k}|^2}\,\delta_{ab}\,. \tag{5.53}$$

Taking into account the corresponding combinatorial factors, one easily obtains the following RG transformation for the scale-dependent interaction parameter $u(\lambda)$ and mass parameter $\tau(\lambda)$:

$$u^{(R)}(\lambda) = u + 2(n-2)u^2\int_{\lambda k_0<|\boldsymbol{k}|<k_0}\frac{\mathrm{d}^2k}{(2\pi)^2}\,\mathrm{Tr}\,\hat{S}^2(\boldsymbol{k})\,, \tag{5.54}$$

$$\tau^{(R)}(\lambda) = \tau + 2(n-1)u\tau\int_{\lambda k_0<|\boldsymbol{k}|<k_0}\frac{\mathrm{d}^2k}{(2\pi)^2}\,\mathrm{Tr}\,\hat{S}^2(\boldsymbol{k})\,. \tag{5.55}$$

After a simple integration of Eqn (5.55), one gets the following RG equations (in the limit $n \to 0$):

$$\frac{\mathrm{d}}{\mathrm{d}\xi} u(\xi) = -\frac{2}{\pi} u^2(\xi) , \tag{5.56}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \ln \tau(\xi) = -\frac{1}{\pi} u(\xi) , \tag{5.57}$$

where, as usual, $\xi \equiv \ln(1/\lambda)$ is the RG parameter.

These equations can be solved easily and yield:

$$u(\xi) = \frac{u}{1 + (2u/\pi)\xi} , \tag{5.58}$$

$$\tau(\xi) = \frac{\tau}{\left[1 + (2u/\pi)\,\xi\right]^{1/2}} , \tag{5.59}$$

where $u \equiv u(\xi = 0)$ and $\tau \equiv \tau(\xi = 0)$. For large scales ($\xi \to \infty$),

$$u(\xi) \sim \frac{1}{\xi} \to 0 , \quad \tau(\xi) \sim \frac{1}{\sqrt{\xi}} \to 0 . \tag{5.60}$$

The critical behaviour of a model with the 'zero-charge' renormalisation can be studied exactly by the RG methods. Using standard procedure one obtains for the singular part of the specific heat (cf. Section 3.3):

$$C(\tau) \simeq -\frac{1}{2} \int_{|k|>|\tau|} \frac{\mathrm{d}^2 k}{(2\pi)^2} \operatorname{Tr} \hat{S}^2(k) \left[\frac{\tau(k)}{\tau}\right]^2$$
$$= \frac{1}{4\pi} \int_{\xi<\ln(1/|\tau|)} \mathrm{d}\xi \left[\frac{\tau(\xi)}{\tau}\right]^2 . \tag{5.61}$$

Here the mass is taken to be dependent on the scale, in accordance with Eqn (5.59):

$$\left[\frac{\tau(\xi)}{\tau}\right]^2 = \left(1 + \frac{2u}{\pi}\xi\right)^{-1} . \tag{5.62}$$

Simple calculations yield

$$C(\tau) \simeq \frac{1}{8u} \ln \left[1 + \frac{2u}{\pi} \ln \frac{1}{|\tau|}\right] . \tag{5.63}$$

From Eqn (5.63) we see that in the temperature range $\tau_u \ll \tau \ll 1$, where

$$\tau_u \sim \exp\left(-\frac{\pi}{2u}\right) , \tag{5.64}$$

the specific heat has the well-known logarithmic behaviour of the pure 2D Ising model: $C(\tau) \sim \ln(1/|\tau|)$. However, in the vicinity of the phase transition point, at $|\tau| \ll \tau_u$, the specific heat exhibits different (universal) behaviour,

$$C(\tau) \sim \frac{1}{u} \ln \left(\ln \frac{1}{|\tau|}\right) , \tag{5.65}$$

which is still singular, although the singularity is now weaker.

Note that the critical exponent of the two-point correlation function in the 2D Ising model is not modified by the presence of disorder [33]:

$$\overline{\langle \sigma_0 \sigma_x \rangle} \sim |x|^{-1/4} . \tag{5.66}$$

This result is also confirmed convincingly by recent numerical simulations [36]–[38].

Note finally that the effects of replica symmetry breaking (Section 4) in the present case appear to be irrelevant. The corresponding calculations, although straightforward, are rather cumbersome and we will not reproduce them here. On the other hand, in the 2D Potts systems the disorder-induced RSB effects can be shown to be relevant and provide the existence of a nontrivial stable fixed point with a continuous RSB (for details see Ref. [34]).

### 5.4 Numerical simulations

In recent years, extensive numerical investigations on special-purpose computers [35] have been conducted, with the aim of checking the theoretical results derived for the 2D Ising model with impurity bonds [36, 37, 38].

In these studies, the calculations were performed for the model defined on a square lattice of $L \times L$ spins with the Hamiltonian

$$H = -\sum_{\langle i,\, j\rangle} J_{ij} \sigma_i \sigma_j , \tag{5.67}$$

where the ferromagnetic couplings $J_{ij}$ between spins $\sigma_i$ and $\sigma_j$ of the nearest neighbour are independent random variables taking two values $J$ and $J'$ with probabilities $1 - u$ and $u$, repectively.

Since the critical behaviour of the impurity is believed to be universal and independent of the concentration of impurities, it is much more convenient in numerical experiments to assume the concentration $u$ to be large. The point is that, according to the theory discussed in the previous section, the parameter $u$ defines the temperature $\tau_*(u)$ and correspondingly the spatial scale $L_*(u) \sim \exp(\mathrm{const}/u)$, Eqn (5.64), at which the crossover to the critical behaviour of the impurity takes place. At small concentrations, the crossover scale $L_*$ is exponentially large and it becomes inaccessible in numerical experiments for finite systems. On the other hand, if both coupling constants $J$ and $J'$ are ferromagnetic, then even for a finite concentration of impurity bonds the ferromagnetic ground state (and the ferromagnetic phase transition) is not destroyed, whereas the crossover scale $L_*$ can be expected not to be very large.

Here we shall review only one set of numerical studies in which quite convincing results for the specific heat singularity have been obtained [37]. The model with the concentration of the impurities $u = 1/2$ has been studied. In this particular case, the model given by Eqn (5.67) appears to be 'self-dual', and its critical temperature can be determined exactly from the equation [39]:

$$\tanh(\beta_c J) = \exp(-2\beta_c J') . \tag{5.68}$$

In the Monte Carlo simulations a cluster-flip algorithm formulated by Swendsen and Wang [40] was used; this algorithm overcomes the difficulty of critical slowing down. In one Monte Carlo sweep, the spin configuration is decomposed into clusters constructed stochastically by connecting neighboring spins of equal sign with the probability $[1 - \exp(-2\beta J_{ij})]$. Each cluster is then flipped with a probability 1/2. At $T_c$ and for large lattices, the relaxation to equilibrium for this algorithm appears to be much faster than for the standard single-spin-flip dynamics.

Technically it is much more convenient to calculate the maximum value of the specific heat as the function of the size of the system, instead of the direct dependence of the specific heat on the reduced temperature $\tau$. Since the temperature and spatial scales are in one-to-one correspondence [$R_c(\tau) \sim \tau^{-1}$ in the 2D Ising model], the minimum

possible value for $\tau$ in a finite system of size $L$ is $\tau_{\min} \sim L^{-1}$. Therefore, the maximum value of the specific heat in the system which exhibits the critical behaviour $C(\tau)$ must be of the order of $C(L^{-1})$. Then, according to Eqn (5.63), the size dependence of the specific heat in the critical regime of the impurity, in the case of the 2D Ising model, can be expected to be as follows:

$$C(L) = C_0 + C_1 \ln\left(1 + b \ln L\right) , \tag{5.69}$$

where $C_0$ and $C_1$ are some constants, and $b = 1/\ln L_*$, where $L_*$ is the impurity crossover length, which is of a finite size.

In general terms, the calculation procedure is as follows. First, one calculates the energy:

$$\overline{\langle H \rangle} = -\frac{1}{L^2} \overline{\left( \sum_{\langle i,\, j \rangle} J_{ij} \langle \sigma_i \sigma_j \rangle \right)} , \tag{5.70}$$

where $\langle ... \rangle$ denotes the thermal (Monte Carlo) average. Then the specific heat is obtained from the energy fluctuations:

$$C(L) = L^2 \overline{\left( \langle H^2 \rangle - \langle H \rangle^2 \right)} . \tag{5.71}$$

The simulations were performed for various ratios $r = J'/J = 1/10$, $1/4$, $1/2$, and 1. The system sizes ranged up to $600 \times 600$.

Figure 13 displays the data for the critical specific heat, as determined from Eqn (5.71) at $r = 1/10$, $1/4$, $1/2$, and 1, plotted against the logarithm of $L$. For the sake of clarity, the vertical axis has been scaled differently for various $r$.

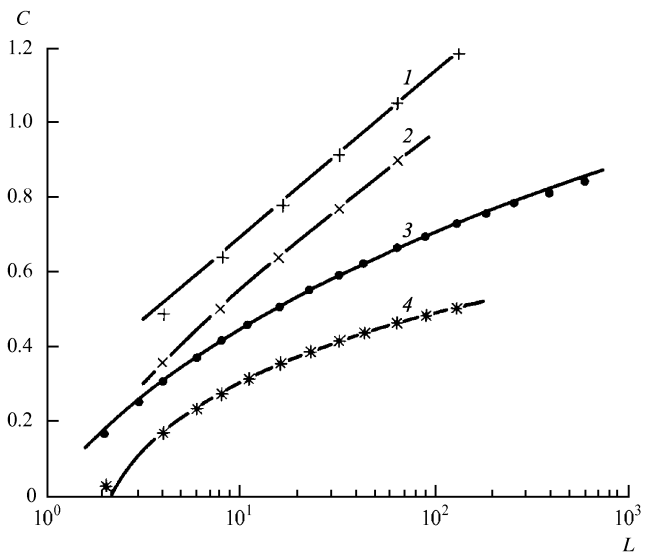


**Figure 13.** The specific heat $C$ at the critical temperature plotted as a function of the logarithm of size $L$. The curves are: (1) the exact asymptotic result for the pure system $r = 1$; (2) $r = 1/2$ with fitting parameters $C_0 = 0.048$, $C_1 = 15.7, b = 0.085$; (3) $r = 1/4$ with fitting parameters $C_0 = 0.048$, $C_1 = 2.04, b = 0.35$; (4) $r = 1/10$ with fitting parameters $C_0 = -0.28$, $C_1 = 0.224, b = 8.8$.

For the perfect model, $r = 1$, the deviations from the exactly known asymptotic behaviour are obviously rather small for $L \geqslant 16$, in agreement with the analytic results on the corrections to scaling [41]. At $r = 1/2$, the size dependence data for $L \leqslant 128$ are still in the perfect Ising regime, where $C \sim \ln L$. At $r = 1/4$ and $r = 1/10$, strong deviations from the logarithmic size dependence occur, reflecting the crossover to the randomness-dominated region for sufficiently large values of $L$.

In Fig. 14 the same data are shown plotted against $\ln(\ln L)$. A strong upwards curvature is evident for $r = 1$ and $1/2$, indicating the logarithmic increase. In notable contrast, the data for $r = 1/4$ approach a straight line for moderate values of $L$, and those for $r = 1/10$ seem to satisfy such behaviour even for small sizes, $L \geqslant 4$. From fits to Eqn (5.69), one obtains $L_* = 16 \pm 4$ at $r = 1/4$, and $L_* = 2 \pm 1$ at $r = 1/10$. The general trends are certainly clear, and confirm the expected crossover to a doubly logarithmic increase of $C$ in the randomness-dominated region sets for smaller sizes $L_*$ as $r$ decreases.

Finally, in Fig. 15 the same data for $r = 1/4$ are plotted against $\ln(1 + b \ln L)$ and exhibit a perfectly straight line for all values of $L$.

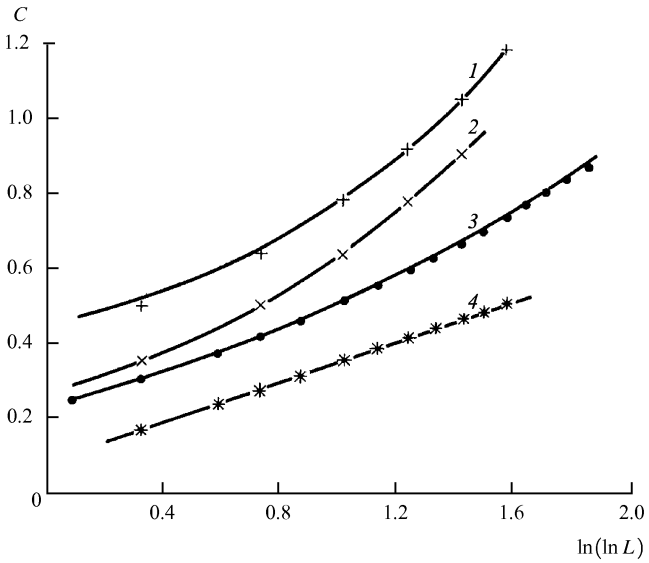


**Figure 14.** The same set of data as in Fig. 13, but in this case plotted against $\ln(\ln L)$.

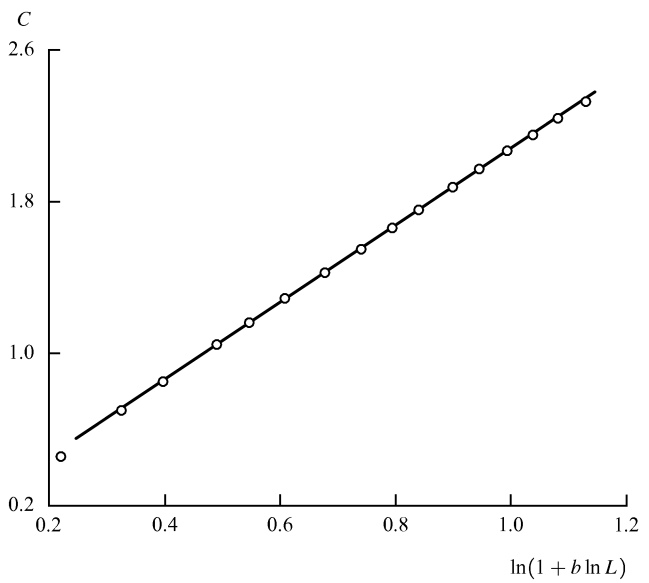


**Figure 15.** The same set of data as in Fig. 13 for $r = 1/4$, in this case plotted against $\ln(1 + b \ln L)$ with $b = 0.35$.

Therefore, in accordance with the analytical predictions of the renormalisation-group calculations (section 4.3), the results obtained in the Monte Carlo simulations provide convincing evidence for the onset of a new randomness-dominated critical regime. Besides, evidence is provided for a $\ln(\ln L)$ dependence in the behaviour of the specific heat at the critical point for sufficiently large system sizes.

### 5.5 General structure of the phase diagram

Let us consider a general structure of the phase diagram of the Ising spin systems with impurities. Apparently, in a ferromagnetic system with antiferromagnetic or broken impurity bonds, as the concentration $u$ of the impurities increases, the ferromagnetic phase transition temperature $T_c(u)$ decreases. Then, at some finite concentration $u_c$ the ferromagnetic ground state could be completely destroyed, and correspondingly the phase transition temperature should turn to zero: $T_c(u_c) = 0$. On the basis of these general arguments, one could guess that the qualitative phase diagram of such systems looks like that shown in Fig. 16 (for details, see Refs [42], [43]). To the right of the line $T_c(u)$, the system is either in the paramagnetic state (at temperatures which are high enough) or in the spin-glass state [44]. The second possibility depends, however, on the dimensionality of the system; at $D = 2$ the spin-glass state is believed to be unstable at any nonzero temperature [45].

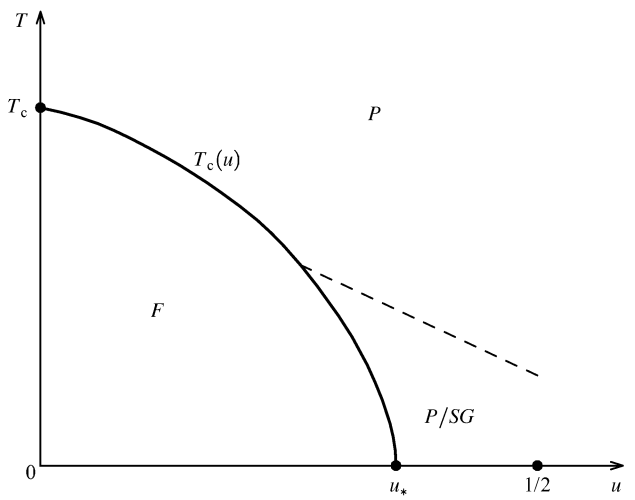


**Figure 16.** A naive phase diagram of a ferromagnetic system diluted by antiferromagnetic or broken couplings.

The critical phenomena considered in Section 4.3 formally correspond to the limit of small concentrations of impurities, i.e. they describe the properties of the phase transition near the upper left-hand side of the line $T_c(u)$ in Fig. 16. Nevertheless, the results obtained for the impurity-dominated critical regime appear to be universal, as they are independent of the concentration of impurities (as well as of the values of the impurity bonds). They make it possible to believe that the critical phenomena in the vicinity of the phase transition line $T_c(u)$ must be the same for other concentrations which are not small. The only parameter which does depend on the impurity concentration is the value of the temperature interval near $T_c(u)$, $\tau_*(u)$, where the impurity-dominated critical phenomena occur. According to the analytic theory of Section 5.3, the value of this interval shrinks to zero as $u \to 0$: $\tau_*(u) \sim \exp(-\text{const}/u) \to 0$. At finite concentrations, this temperature interval becomes formally finite, which indicates that the whole critical region near $T_c(u)$ must be described by the critical regime of the impurity.

On the other hand, it is generally believed [42] that the bottom-right part of the phase transition line $T_c(u)$ (the region near the critical concentration $u = u_c$, $T \ll 1$) belongs to another universality class, which is different from the ferromagnetic phase transition at $u \ll 1$. For example, it is obvious that in magnets with broken impurity bonds the phase transition as a function of the concentration (at $T \ll 1$) at $u = u_c$ must be of the kind of percolation transition which has nothing to do with the temperature of the ferromagnetic transition. It means that there must be a special point $(T^*, u^*)$ on the line $T_c(u)$ which separates two different critical regimes.

Actually, there does exist a special line, the so-called Nishimori line $T_N(u)$ [46], which crosses the line $T_c(u)$ at the point $(T^*, c^*)$ (Fig. 17). There is no real phase transition at the Nishimori line. Formally it is special only in a sense that everywhere on the line the free energy as well as some other thermodynamic quantities appear to be analytic functions of the temperature and the concentration. Moreover, an explicit expression for free energy in the case of the Nishimori line can be obtained for arbitrary $T$ and $u$ for any dimensions. In fact, it makes the structure of the phase diagram much less trivial than that shown in Fig. 16. Let us consider this point in more detail.

For the sake of simplicity, let us consider the Ising ferromagnet,

$$H = -\sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j \,, \tag{5.72}$$

defined at a lattice with arbitrary structure. The ferromagnetic spin–spin couplings $J_{ij}$ are equal to 1, while the impurity antiferromagnetic ones are equal to $-1$, so that the statistical distribution of the $J_{ij}$ values can be defined as follows:

$$P\,[J_{ij}] = \prod_{\langle i,j\rangle} \left[(1-u)\,\delta(J_{ij}-1) + u\delta\,(J_{ij}+1)\right] \,, \tag{5.73}$$

where $u$ is the concentration of the impurity bonds.

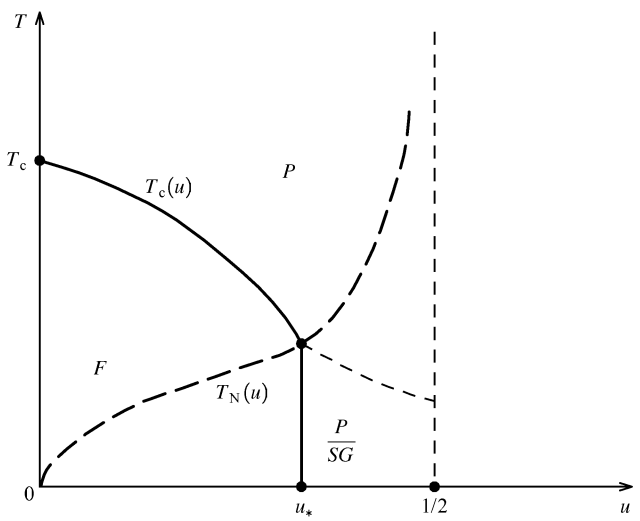


**Figure 17.** Phase diagram of the Ising ferromagnet diluted by antiferromagnetic couplings; $T_N(u)$ is the Nishimori line.

One can easily check that the statistical averaging over configurations of the $J_{ij}$ values,

$$\overline{(\ldots)} = \sum_{J_{ij}=\pm 1} \prod_{\langle i,\, j\rangle} \left[(1-u)\,\delta(J_{ij}-1) + u\delta(J_{ij}+1)\right](\ldots)\ , \tag{5.74}$$

can be rewritten as follows:

$$\overline{(\ldots)} = \sum_{J_{ij}=\pm 1} \left[2\cosh\tilde{\beta}(u)\right]^{-N_{\mathrm{b}}} \exp\left[\tilde{\beta}(u)\sum_{\langle i,\, j\rangle} J_{ij}\right](\ldots)\ , \tag{5.75}$$

where $N_{\mathrm{b}}$ is the total number of bonds in the system, and the impurity parameter $\tilde{\beta}(u)$ is defined by the equation

$$\exp\left[-2\tilde{\beta}(u)\right] = \frac{u}{1-u}\ . \tag{5.76}$$

For given values of the temperature $T$ and the concentration $u$, the average energy of the system is defined as follows:

$$\begin{aligned} E(u,T) &= \overline{\langle H\rangle} \\ &= -\left[2\cosh\tilde{\beta}(u)\right]^{-N_{\mathrm{b}}} \sum_{J_{ij}=\pm 1} \exp\left[\tilde{\beta}(u)\sum_{\langle i,\, j\rangle} J_{ij}\right] \\ &\quad \times \frac{\sum_{\sigma=\pm 1}\left(\sum_{\langle i,\, j\rangle} J_{ij}\,\sigma_i\,\sigma_j\right)\exp\left(\beta\sum_{\langle i,\, j\rangle} J_{ij}\,\sigma_i\,\sigma_j\right)}{\sum_{\sigma=\pm 1}\exp\left(\beta\sum_{\langle i,\, j\rangle} J_{ij}\,\sigma_i\,\sigma_j\right)}\ . \end{aligned} \tag{5.77}$$

It is obvious that the system under consideration is invariant under the local 'gauge' transformations:

$$\begin{aligned} &\sigma_i \to \sigma_i\, s_i\ , \\ &J_{ij} \to J_{ij}\, s_i\, s_j\ , \end{aligned} \tag{5.78}$$

for arbitrary $s_i = \pm 1$. Using the above gauge invariance, the following trick can be performed. Let us redefine the variables in Eqn (5.77) according to Eqn (5.78) (which should leave the value of $E$ unchanged), and then let us 'average' the obtained expression obtained for $E$ over all configurations of $s_i$:

$$\begin{aligned} E(c,T) &= -\left[2\cosh\tilde{\beta}(u)\right]^{-N_{\mathrm{b}}} 2^{-N} \\ &\quad \times \sum_{J_{ij}=\pm 1}\left\{\sum_{s=\pm 1}\exp\left[\tilde{\beta}(u)\sum_{\langle i,\, j\rangle} J_{ij}\, s_i\, s_j\right]\right\} \\ &\quad \times \frac{\sum_{\sigma=\pm 1}\left(\sum_{\langle i,\, j\rangle} J_{ij}\sigma_i\sigma_j\right)\exp\left(\beta\sum_{\langle i,\, j\rangle} J_{ij}\sigma_i\sigma_j\right)}{\sum_{\sigma=\pm 1}\exp\left(\beta\sum_{\langle i,\, j\rangle} J_{ij}\sigma_i\sigma_j\right)}\ . \end{aligned} \tag{5.79}$$

One can easily see that the expression in Eqn (5.79),

$$\left\{\sum_{s=\pm 1}\exp\left[\tilde{\beta}(u)\sum_{\langle i,\, j\rangle} J_{ij}\, s_i\, s_j\right]\right\} \equiv Z\left[\tilde{\beta}(u),\, J_{ij}\right]\ , \tag{5.80}$$

is the partition function of the system at the temperature $\tilde{\beta}(u)$. Therefore, if $\tilde{\beta}(u) = \beta$ the partition function (at the temperature $\beta$) in the denominator in Eqn (5.79) is cancelled by the partition function [Eqn (5.80)]. In this case, the value of the average energy $E$ (as well as the free energy) can be calculated explicitly:

$$\begin{aligned} E(c,\,T) &= -\left[2\cosh\tilde{\beta}(u)\right]^{-N_{\mathrm{b}}} 2^{-N} \sum_{J_{ij}=\pm 1}\sum_{\sigma=\pm 1}\left(\sum_{\langle i,\, j\rangle} J_{ij}\,\sigma_i\,\sigma_j\right) \\ &\quad \times \exp\left(\beta\sum_{\langle i,\, j\rangle} J_{ij}\sigma_i\sigma_j\right) = -\left[2\cosh\tilde{\beta}(u)\right]^{N_{\mathrm{b}}} 2^{-N} \\ &\quad \times \frac{\partial}{\partial\beta}\left[\sum_{J_{ij}=\pm 1}\sum_{\sigma=\pm 1}\exp\left(\beta\sum_{\langle i,\, j\rangle} J_{ij}\sigma_i\sigma_j\right)\right] \\ &= -N_{\mathrm{b}}\tanh\beta(u) = -N_{\mathrm{b}}\left[1-2u(T)\right]\ . \end{aligned} \tag{5.81}$$

The internal energy obtained is analytic for all values of the temperature and the concentration.

The above result is valid at the Nishimori line $T_{\mathrm{N}}(u)$ defined by the condition $\tilde{\beta}(u) = \beta$:

$$T_{\mathrm{N}}(u) = \frac{2}{\ln\left[(1-\mathrm{u})/\mathrm{u}\right]}\ . \tag{5.82}$$

This line is shown qualitatively in Fig. 17. It starts for the zero concentration (pure system) at $T = 0$, and for $u \to 1/2$ (completely disordered system) $T_{\mathrm{N}} \to \infty$.

Apparently, the Nishimori line must cross the phase transition line $T_{\mathrm{c}}(u)$. This creates rather a peculiar situation, because at the line of the phase transition the thermodynamic functions should be nonanalytic (for details, see Ref. [46]). Actually, this crossection point, $(T_*, u_*)$, is argued to be the multicritical point at which the paramagnetic, ferromagnetic, and spin-glass phases merge [47]

For the Ising models of this type it can also be proved rigorously [46] that the ferromagnetic phase does not exist for $u > u_*$, where $u_*$ is the point at which the Nishimori line crosses the boundary between the paramagnetic and the ordered phases $T_{\mathrm{c}}(u)$ (Fig. 17). (It means that the structure of the naive phase diagram shown in Fig.16 is in general not quite correct.)

To prove this statement let us consider the following two-point correlation function:

$$G(x) = \overline{\langle\sigma_0\sigma_x\rangle_\beta}\,, \tag{5.83}$$

where $\langle\ldots\rangle_\beta$ denotes the thermal average for a given temperature $\beta$.

Using once again the trick with the gauge transformation [Eqn (5.78)] for the correlation function [Eqn (5.83)] one gets:

$$\begin{aligned} G(x) &= \left[2\cosh\tilde{\beta}(u)\right]^{-N_{\mathrm{b}}} \sum_{J_{ij}=\pm 1}\exp\left[\tilde{\beta}(u)\sum_{\langle i,\, j\rangle} J_{ij}\right] \\ &\quad \times \frac{\sum_{\sigma=\pm 1}(\sigma_0\sigma_x)\exp\left(\beta\sum_{\langle i,\, j\rangle} J_{ij}\,\sigma_i\,\sigma_j\right)}{\sum_{\sigma=\pm 1}\exp\left(\beta\sum_{\langle i,\, j\rangle} J_{ij}\sigma_i\,\sigma_j\right)} \\ &= \left[2\cosh\tilde{\beta}(u)\right]^{-N_{\mathrm{b}}} 2^{-N} \\ &\quad \times \sum_{J_{ij}=\pm 1}\sum_{s=\pm 1}(s_0\,s_x)\exp\left[\tilde{\beta}(u)\sum_{\langle i,\, j\rangle} J_{ij}\,s_i\,s_j\right] \\ &\quad \times \frac{\sum_{\sigma=\pm 1}(\sigma_0\sigma_x)\exp\left(\beta\sum_{\langle i,j\rangle} J_{ij}\,\sigma_i\,\sigma_j\right)}{\sum_{\sigma=\pm 1}\exp\left(\beta\sum_{\langle i,\, j\rangle} J_{ij}\,\sigma_i\,\sigma_j\right)} \\ &= \left[2\cosh\tilde{\beta}(u)\right]^{-N_{\mathrm{b}}} 2^{-N} \\ &\quad \times \sum_{s'=\pm 1}\sum_{J_{ij}=\pm 1}\exp\left[\tilde{\beta}(u)\sum_{\langle i,\, j\rangle} J_{ij}\,s'_i\,s'_j\right] \\ &\quad \times\left\langle(s_0 s_x)\right\rangle_{\tilde{\beta}(u)}\left\langle(\sigma_0\sigma_x)\right\rangle_\beta = \overline{\langle s_0 s_x\rangle_{\tilde{\beta}(u)}\langle\sigma_0\sigma_x\rangle_\beta}\ . \end{aligned} \tag{5.84}$$

Thus, the absolute value of the correlation function given by Eqn (5.83) satisfies the condition

$$\left|G(x)\right| = \left|\overline{\langle\sigma_0\sigma_x\rangle_\beta}\right| \leqslant \overline{\left|\langle s_0 s_x\rangle_{\tilde{\beta}(u)}\right|} \,, \tag{5.85}$$

since the absolute value of any Ising ($|\sigma| = 1$) correlation function does not exceed one.

Therefore the absolute value of the two-point correlation function calculated at the temperature $T$ and the impurity concentration $u$ do not exceed the average of the absolute value of the corresponding correlation function calculated at the Nishimori line at the same concentration. This quantity in the long-range limit $|x| \to \infty$ vanishes if the corresponding point on the Nishimori line is in the paramagnetic phase, which takes place for all concentrations $u > u_*$. On the other hand, the value of the correlation function $G(x)$ in the limit $|x| \to \infty$ becomes the square of the ferromagnetic magnetisation: $G(|x| \to \infty) = m^2(T, u)$. Thus, the above simple arguments prove that $m(T, u) \equiv 0$ for $u > u_*$.

The boundary line between the ferromagnetic and nonferromagnetic (spin-glass) phases is vertical to the concentration axis, as in Fig. 17 [46], although the existence of the reentrant phenomena cannot in general be excluded.

## 6. The Ising systems with quenched random fields

### 6.1 The model

In the previous sections we have considered the spin systems in which the quenched disorder was introduced in the form of random fluctuations in the spin–spin interactions. There exists another class of statistical models in which the disorder is present in the form of random magnetic fields. This type of disorder is essentially different from that with fluctuating interactions, since external magnetic fields break the symmetry with respect to the change of the signs of the spins.

In the most simplified form, the random-field spin systems could be qualitatively described by the following Ising Hamiltonian:

$$H = -\sum_{<i\neq j>}^{N} \sigma_i \sigma_j - \sum_i h_i \sigma_i \,, \tag{6.1}$$

where the Ising spins $\{\sigma_i = \pm 1\}$ are placed in the vertices of a $D$-dimensional lattice with the ferromagnetic interactions between the nearest neighbours, and the quenched random fields $\{h_i\}$ are described by the symmetric Gaussian distribution:

$$P[h_i] = \prod_i^N \left[\frac{1}{\left(2\pi h_0^2\right)^{1/2}} \exp\left(-\frac{h_i^2}{2h_0^2}\right)\right], \quad h_0 \ll 1 \,. \tag{6.2}$$

The best-studied experimentally accessible realisations of systems of this type are the site-diluted antiferromagnets in a homogeneous magnetic field [48]. On a qualitative level, this could be understood as follows. An ordinary ordered antiferromagnetic system in the ground state is described by the two sublattices A and B, with magnetisations which are equal in magnitude and opposite in sign. Dilution means that some of the spins chosen at random are removed from both sublattices. In the zero external magnetic field, the dilution alone does not break symmetry between the two ground states $\sigma_A = -\sigma_B = \pm 1$. However, if the external magnetic field $h$ is nonzero, then an isolated missing spin on the sublattice A provides the energy difference $2h$ between the two ground states $\sigma_A = -\sigma_B = +1$, and $\sigma_A = -\sigma_B = -1$.

Another example is absorbed monolayers with two ground states on impure substrates [49]. Here if one of the substrate lattice sites is occupied by a quenched impurity, it prevents additional occupation of this site, which effectively acts as a local symmetry breaking field.

Other realisations are binary liquids in porous media [50], and diluted frustrated antiferromagnets [51].

### 6.2 General arguments

Despite extensive theoretical and experimental efforts during the last twenty years (for reviews see, for example, Ref. [52]), there are few reliable statements for the problem of the random-field Ising model.

According to simple physical arguments by Imry and Ma [53], one would expect that the dimensions above which the ferromagnetic ground state is stable at low temperatures (it is called the lower critical dimension) must be equal to 2. (Note that, for the Ising systems without random fields, the lower critical dimension is 1.) Indeed, if we try to reverse a large region $\Omega$ of linear size $L$, there are two competing effects: the gain in energy due to the alignment with the random magnetic field $E_h$, and the loss of energy due to the creation of an interface $E_f$. The first effect scales as follows:

$$E_h \sim \left[\overline{\left(\sum_{i\in\Omega} h_i\right)^2}\right]^{1/2} = \left(\sum_{i,\,j\in\Omega} \overline{h_i h_j}\right) \sim h_0 L^{D/2} \,. \tag{6.3}$$

The second effect is the energy of a domain wall, which is proportional to the square of the boundary of the region $\Omega$:

$$E_f \sim L^{(D-1)} \,. \tag{6.4}$$

These estimates show that at dimensions 2 or lower for arbitrary small (but nonzero) values of the field $h_0$, the two energies become comparable for sufficiently large sizes $L$, and no spontaneous magnetisation should be present. On the other hand, at dimensions greater than 2, the energy at the interface $E_f$ is always greater than that at $E_h$. Therefore this effect should not destroy the long-range order and a ferromagnetic transition should be present. This naive (but physically correct) argument was later confirmed by a rigorous proof by Imbrie [54].

On the other hand, a perturbative study of the phase transition shows that, as far as the leading large-scale divergences are concerned, the strange phenomenon of a dimensional reduction is present, such that the critical exponents of the system in dimensions $D$ are the same as those of the ferromagnetic system without random fields in dimension $D-2$ [55] This result would imply that the lower critical dimension is 3, in contradiction with the results obtained rigorously.

In fact, the procedure of summing up the leading large-scale divergences could give the correct result only if the Hamiltonian in the presence of the magnetic field has one minimum. In this case, the dimensional reduction can be

rigorously shown to be exact by the use of supersymmetric arguments [56].

However, as soon as the temperature is close enough to the critical point, as well as in a low temperature region, there are values of the magnetic field for which the free energy has more than one minimum (this phenomenon is similar to that considered in Section 4). In this situation, there is no reason to believe that the supersymmetric approach should give the correct results and therefore the dimensional reduction is not justified. This is not surprising because the dimensional reduction completely misses the appearance of Griffith's singularities [21].

Recently it has also been shown that the existence of more that one solution of the stationary equations in the presence of random fields is related, in the replica approach, to the existence of new instanton-type solutions of the mean-field equations which are not invariant under translations in replica space [57].

### 6.3 Griffith phenomena in the low-temperature region

In this section, simple physical arguments will be used to demonstrate the origin of the Griffith singularities in the thermodynamical functions in the low-temperature (ordered) phase in the temperature region $h_o^2 \ll T \ll 1$ for the dimensions $D < 3$ [58]. This nonperturbative contribution to the thermodynamics will be shown to come from rare, large spin clusters having a characteristic size $\sim \sqrt{T}/h_0$ with magnetisation opposite to the ferromagnetic background, and which are the *local* minima of the free energy.

If the dimensions of the system are greater than 2, then the spin configuration of the ground state is ferromagnetic. The thermal excitations are the spin clusters with the magnetisation opposite to the background. If the linear size $L$ of such a cluster is large, then (in the continuous limit) the energy of this thermal excitation could be estimated as follows:

$$E(L) \simeq L^{D-1} - V(L) \,, \tag{6.5}$$

where

$$V(L) = \int_{|x|<L} \mathrm{d}^D x \, h(x) \,. \tag{6.6}$$

The statistical distribution of the energy function $V(L)$ [which is the energy of the spin cluster of the size $L$ in the random field $h(x)$] is:

$$P\big[V(L)\big] = \int \mathrm{D}h(x) \exp\left[-\frac{1}{2h_0^2}\int \mathrm{d}^D x \, h^2(x)\right] \times \prod_L \left\{\delta\left[\int_{|x|<L} \mathrm{d}^D x \, h(x) - V(L)\right]\right\} \tag{6.7}$$

(here and in what follows all types of preexponential factors are omitted). For future calculations it will be more convenient to deal with the quenched function $V(L)$ instead of with $h(x)$. One can easily derive an explicit expression for the distribution function $P[V(L)]$, Eqn (6.7) (for the sake of simplicity, the parameter $L$ is first taken to be discrete):

$$\begin{aligned}
P\big[V(L)\big] &= \left[\prod_x \int_{-\infty}^{+\infty} \mathrm{d}h(x)\right]\left(\prod_i \int_{-\infty}^{+\infty} \mathrm{d}\xi_i\right) \\
&\quad \times \exp\left\{-\frac{1}{2h_0^2}\int \mathrm{d}^D x \, h^2(x) \right. \\
&\quad \left. + i\sum_i \xi_i \left[\int_{|x|<L_i} \mathrm{d}^D x \, h(x) - V(L_i)\right]\right\} \\
&= \left(\prod_i \int_{-\infty}^{+\infty} \mathrm{d}\xi_i\right) \exp\left[-\mathrm{i}\sum_i \xi_i V(L_i)\right] \\
&\quad \times \left[\prod_x \int_{-\infty}^{+\infty} \mathrm{d}h(x)\right] \exp\left\{-\frac{1}{2h_0^2}\int \mathrm{d}^D x h^2(x) \right. \\
&\quad \left. + i\sum_{i=1}^{\infty} \int_{L_i<|x|<L_{i+1}} \mathrm{d}^D x h(x) \left(\sum_{j=i}^{\infty} \xi_j\right)\right\} \\
&= \left(\prod_i \int_{-\infty}^{+\infty} \mathrm{d}\xi_i\right) \exp\left\{-\mathrm{i}\sum_i \xi_i V(L_i) \right. \\
&\quad \left. - \frac{1}{2} h_0^2 \sum_{i=1}^{\infty}\left(L_{i+1}^D - L_i^D\right)\left(\sum_{j=i}^{\infty} \xi_j\right)^2\right\} \\
&= \exp\left\{-\frac{1}{2h_0^2}\sum_i \frac{\big[V(L_{i+1}) - V(L_i)\big]^2}{L_{i+1}^D - L_i^D}\right\} .
\end{aligned} \tag{6.8}$$

Making $L$ continuous again, we finally get:

$$P\big[V(L)\big] \simeq \exp\left\{-\frac{1}{2h_0^2}\int \mathrm{d}L \, \frac{1}{L^{D-1}}\left[\frac{\mathrm{d}V(L)}{\mathrm{d}L}\right]^2\right\} . \tag{6.9}$$

Since the probability of the flips of big spin clusters is exponentially small, their contributions to the partition function could be assumed to be independent (it is assumed that such clusters are noninteracting, as they are very far from each other). Then, their contribution to the total free energy could be obtained from the statistical averaging of the free energy of one isolated cluster:

$$\Delta F = -T\left[\prod_L \int \mathrm{d}V(L)\right] P\big[V(L)\big] \times \log\left(1 + \int_1^{\infty} \mathrm{d}L \exp\left\{\beta\big[V(L) - L^{D-1}\big]\right\}\right) . \tag{6.10}$$

Here the factor under the logarithm is the partition function obtained as a sum over all the sizes of the flipped cluster (the factor 1 is the contribution of the ordered state, which is the state without the flipped cluster).

The idea of the calculations of the free energy given above is described below. Since at dimensions $D > 2$ the energy $E(L) = L^{D-1} - V(L)$ is on average a function that increases with $L$, it would be reasonable to expect that the deep local minima (if any) of this function are well separated and the values of the energies at these minima increase with the size $L$. For this reason, let us assume that the leading contribution in the integration over the sizes of the clusters in Eqn (6.10) comes only from *one* (if any) deepest local minimum of the function $L^{D-1} - V(L)$ [for a given realisation of the quenched function $V(L)$].

Again, in view of the fact that the energy $E(L) = L^{D-1} - V(L)$ is a function of $L$, the sufficiency condition for existence of a minimum somewhere above

a given size $L$ is:

$$\frac{\mathrm{d}V(L)}{\mathrm{d}L} > (D-1)L^{D-2} \ . \tag{6.11}$$

By the use of the above assumptions, the contribution to the free energy from the flipped clusters, Eqn (6.10), could be estimated as follows:

$$\Delta F \simeq -T \int_1^\infty \mathrm{d}L \int_{-\infty}^{+\infty} \mathrm{d}V P_L(V) P\left[\frac{\mathrm{d}V(L)}{\mathrm{d}L} > (D-1)L^{D-2}\right] \times \log\left\{1 + \exp\left[\beta(V - L^{D-1})\right]\right\} , \tag{6.12}$$

where $P_L(V)$ is the probability of a given value of the energy $V$ at a given size $L$, and $P[\mathrm{d}V(L)/\mathrm{d}L > (D-1)L^{D-2}]$ is the probability that condition (6.11) is satisfied at the unit length for the given size $L$.

According to Eqn (6.6), $\overline{V^2(L)} \simeq h_0^2 L^D$ (for large values of $L$). Since the distribution $P_L(V)$ is expected to be Gaussian, one gets:

$$P_L(V) \simeq \exp\left(-\frac{V^2}{2h_0^2 L^D}\right) \ . \tag{6.13}$$

Note that the above result could also be obtained by integrating the general distribution function $P[V(L)]$, Eqn (6.9), over all the 'trajectories' $V(L)$ with the fixed value $V(L) = V$ at the given length $L$.

The value of the probability $P\left[\mathrm{d}V(L)/\mathrm{d}L > (D-1)L^{D-2}\right]$ could also be obtained by integrating $P[V(L)]$ over all the functions $V(L)$ conditioned by $\mathrm{d}V(L)/\mathrm{d}L > (D-1)L^{D-2}$ (at the given value of $L$ ). It is clear, however, that — with exponential accuracy — the result of such an integration is defined only by the lower bound $(D-1)L^{D-2}$ for the derivative $\mathrm{d}V(L)/\mathrm{d}L$ (at the given length $L$) in Eqn (6.9). Therefore, one gets:

$$P\left[\frac{\mathrm{d}V(L)}{\mathrm{d}L} > (D-1)L^{D-2}\right] \simeq \exp\left\{-\frac{1}{2h_0^2 L^{D-1}} \times \left[(D-1)L^{D-2}\right]^2\right\} = \exp\left[-\frac{(D-1)^2 L^{D-3}}{2h_0^2}\right] \ . \tag{6.14}$$

Note the important property of the energy $E(L)$, which follows from Eqns (6.13) – (6.14): although at dimensions $D > 2$ the function $E(L)$ increases with $L$, the probability of finding a local minimum of this function at dimensions $D < 3$ also increases with $L$. It is the competition of these two effects which produces the nontrivial contribution to be calculated below.

In the low-temperature limit, $T \ll 1$ (although still $T \gg h_0^2$), the contribution to the free energy, Eqn (6.12), could be divided into two separate parts:

$$\Delta F = \Delta F_1 + \Delta F_2 \ ,$$

$$\Delta F_1 \simeq -T \int_1^\infty \mathrm{d}L \int_{V > L^{D-1}} \mathrm{d}V \exp\left[-\frac{V^2}{2h_0^2 L^D} - \frac{(D-1)^2 L^{D-3}}{2h_0^2}\right] \times \log\left\{1 + \exp\left[\beta(V - L^{D-1})\right]\right\} ,$$

$$\Delta F_2 \simeq -T \int_1^\infty \mathrm{d}L \int_{V < L^{D-1}} \mathrm{d}V \exp\left[-\frac{V^2}{2h_0^2 L^D} - \frac{(D-1)^2 L^{D-3}}{2h_0^2}\right] \times \log\left\{1 + \exp\left[\beta(V - L^{D-1})\right]\right\} . \tag{6.15}$$

The first part is the contribution from the minima which have negative energies (the excitations which produce the *gain* in energy with respect to the ordered state). Here the main contribution in the integration over $V$ comes from the limit $V = L^{D-1}$, and in the leading order one gets:

$$\Delta F_1 \sim -T \int_1^\infty \mathrm{d}L \exp\left[-\frac{L^{D-2}}{2h_0^2} - \frac{(D-1)^2 L^{D-3}}{2h_0^2}\right] \ . \tag{6.16}$$

For dimensions $D > 2$, the leading contribution to $\Delta F_1$ comes from $L \sim 1$, and this takes us back to the Imry and Ma [53] arguments that there are no flipped large spin clusters which would produce the gain in energy with respect to the ordered state.

The second contribution in Eqn (6.15) comes from the local minima which have positive energies. These could contribute to the free energy only as thermal excitations at nonzero temperatures. In the low-temperature limit $\beta \gg 1$, one could approximate:

$$\log\left\{1 + \exp\left[\beta(V - L^{D-1})\right]\right\} \simeq \exp\left[-\beta(L^{D-1} - V)\right], \tag{6.17}$$

where $L^{D-1} > V$. Then for $\Delta F_2$, one gets:

$$\Delta F_2 \simeq -T \int_1^\infty \mathrm{d}L \int_{-\infty}^{L^{D-1}} \mathrm{d}V \times \exp\left[-\frac{V^2}{2h_0^2 L^D} - \frac{(D-1)^2 L^{D-3}}{2h_0^2} + \beta V - \beta L^{D-1}\right] . \tag{6.18}$$

The main contribution in this integral also comes from the 'trivial' region $L \sim 1$, $V \sim \beta h_0^2$, which corresponds to the 'elementary excitations' at scales of the lattice spacing.

However, if the temperature is not too low, $\beta h_0^2 \ll 1$ and $D < 3$, there exists another nontrivial contribution which comes from the vicinity of the saddle point:

$$V_* = (\beta h_0^2) L_*^D \ ,$$
$$L_* = \left[\frac{(D-1)(3-D)}{2\beta h_0^2}\right]^{1/2} \gg 1 \ , \tag{6.19}$$

which is separated from the region $L \sim 1, V \sim \beta h_0^2$ by a large barrier. Note that the condition of integration in Eqn (6.18), $V_* \ll L_*^{D-1}$, according to Eqn (6.19) is satisfied for $L_* \ll 1/\beta h_0^2$, which is correct only if $\beta h_0^2 \ll 1$.

For the contribution to the free energy at this saddle-point, one gets:

$$\Delta F_2 \sim \exp\left[-\frac{\mathrm{const}}{2h_0^2} (\beta h_0^2)^{(3-D)/2}\right] , \tag{6.20}$$

where

$$\mathrm{const} = \frac{1}{2}(D+1)(D-1)^{(D-1)/2} \left(\frac{2}{3-D}\right)^{(3-D)/2} \ . \tag{6.21}$$

The result [Eqn (6.20)] demonstrates that, in addition to the usual thermal excitations in the vicinity of the ordered state (which could be taken into account by the traditional perturbation theory), owing to the interaction with the random fields there exist essentially nonperturbative large-scale thermal excitations which produce exponentially small nonanalytic contributions to the thermodynamics. These excitations are large spin clusters with the magnetisation opposite to the background which are the *local* energy minima. At finite temperatures, such that $h_o^2 \ll T \ll 1$, the characteristic size of the clusters giving

the leading contribution to the free energy is $L_* \sim \sqrt{T}/h_0 \gg 1$.

This phenomenon, although it seems to produce a negligibly small contribution to the thermodynamical functions, could be extremely important for understanding the dynamical relaxation processes. The large clusters with reversed magnetisation are the local minima and are separated from the ground state by large energy barriers, and this could produce the essential slowing down of the relaxation (see, for example, Ref. [59]). In particular, the characteristic 'saddle-point' clusters [Eqn (6.19)] with the size $L_*(T) \sim \sqrt{T}/h_0 \gg 1$ are separated from the ground state by the energy barrier of the order of $V_* \sim (\beta h_0^2)^{-(D-2)/2} \gg 1$, and the corresponding characteristic relaxation time at low temperatures can be expected to be exponentially large:

$$\tau(T) \sim \exp\left[\beta(\beta h_0^2)^{-(D-2)/2}\right] \gg 1 \ . \tag{6.22}$$

However, in order to describe the temporal asymptotics of the relaxation processes, one needs to know the *spectrum* of the relaxation times (or the energy barriers), and this would require more special consideration.

Unfortunately, the results obtained in this Section cannot be applied directly for dimensions $D = 3$, which appears to be marginal for the considered phenomena (at dimensions $D > 3$ this type of nonperturbative effect is absent). At $D = 3$ all those simple estimates for the energies and probabilities of the cluster excitations which have been used in this section [in particular, Eqn (6.14)] do not work, and a much more detailed analysis is required.

On the other hand, it seems quite reasonable to expect that the results obtained are correct for dimensions $D = 2$, regardless of the fact that the long-range order in not stable there. The point is that at $D = 2$ the correlation length at which the long-range order is destroyed is exponentially large in the parameter $1/h_0$, whereas the characteristic size of the spin clusters considered here is only the power of the parameter $1/h_0$. Therefore, at the scales at which the Griffith singularities [Eqn (6.20)] appear, the system is still effectively ordered at $D = 2$.

### 6.4 The phase transition

The nature of the phase transition in the random-field Ising model is still a mystery. The only reliable fact about it is that the upper critical dimensionality (the dimensionality above which the critical phenomena are described by the mean-field theory, Section 2.3) for this phase transition is equal to 6 (unlike in pure systems where it is equal to 4). Let us consider this point in some more detail.

Near the phase transition, the random-field Ising model can be described in terms of the scalar field Ginzburg – Landau Hamiltonian with the double-well potential:

$$H = \int \mathrm{d}^D x \left\{ \frac{1}{2}\left[\nabla\phi(x)\right]^2 + \frac{1}{2}\tau\phi^2(x) - h(x)\phi(x) + \frac{1}{4}g\phi^4(x) \right\}, \tag{6.23}$$

where quenched random fields $h(x)$ are assumed to be described by the symmetric Gaussian distribution with the mean square equal to $h_0^2$.

Ground state configurations of the fields $\phi(x)$ are defined by the saddle-point equation:

$$-\Delta\phi(x) + \tau\phi(x) + g\phi^3(x) = h(x) \ . \tag{6.24}$$

In the usual RG approach for the phase transition in the pure systems $[h(x) = 0]$, one constructs the perturbation theory over large-scale deviations of the background homogeneous solution of the above equation, $\phi_0 = \sqrt{|\tau|/g}$, $\tau < 0$ or $\phi_0 = 0$, $\tau > 0$ (Section 2.6).

Apparently, the solutions of equation (6.24) with nonzero $h(x)$ may essentially depend on a particular configuration of the quenched fields being nonhomogeneous. Let us estimate the conditions under which the external fields become the dominant factor for the ground state configurations.

Let us consider a large region $\Omega_L$ of a linear size $L \gg 1$. An average value of the field in this region could be defined as follows:

$$h(\Omega_L) = \frac{1}{L^D}\int_{x\in\Omega_L} \mathrm{d}^D x h(x) \ . \tag{6.25}$$

Correspondingly, for the characteristic value of the field $h(\Omega_L)$ (averaged over realisations) one gets:

$$h_L = \left[\overline{h^2(\Omega_L)}\right]^{1/2} = \frac{1}{L^D}\left[\int_{x,\,x'\in\Omega_L} \mathrm{d}^D x\, \mathrm{d}^D x'\, \overline{h(x)h(x')}\right]^{1/2} = \frac{h_0}{L^{D/2}} \ . \tag{6.26}$$

The average value of the order parameter $\phi(\Omega_L)$ in a given region $\Omega_L$ can be estimated from the following equation:

$$\tau\phi + g\phi^3 = h(\Omega_L) \ . \tag{6.27}$$

The solutions of this equation are:

$$\phi(\Omega_L) \simeq \phi_0 + \frac{h(\Omega_L)}{2|\tau|} \ , \qquad h(\Omega_L) \ll |\tau|^{3/2} \ , \tag{6.28}$$

$$\phi(\Omega_L) \simeq \left[\frac{h(\Omega_L)}{g}\right]^{1/3} \ , \qquad h(\Omega_L) \gg |\tau|^{3/2} \ . \tag{6.29}$$

In the first case, Eqn (6.28), the external fields can be considered as small perturbations, whereas in the second case, Eqn (6.29), the external fields are the dominant factor and the solution for the order parameter does not depend on the temperature parameter $\tau$.

Now let us estimate up to which characteristic sizes of the clusters the external fields could dominate. According to Eqn (6.26), the condition $h(\Omega_L) \gg \tau^{3/2}$, Eqn (6.29), yields:

$$L \ll \frac{h_0^{2/D}}{|\tau|^{3/D}} \ . \tag{6.30}$$

On the other hand, the estimation of the order parameter in terms of the equilibrium equation (6.27) could be correct only for scales much greater than the size of the fluctuation region, which is equal to the correlation length $R_c \sim |\tau|^{-\nu}$. Thus, one has the lower bound for $L$:

$$L \gg |\tau|^{-\nu} \ . \tag{6.31}$$

Therefore the situation when the external fields become the dominant factor could exist in the region of parameters defined by the condition

$$|\tau|^{-\nu} \ll \frac{h_0^{2/D}}{|\tau|^{3/D}} \tag{6.32}$$

or

$$|\tau|^{3-\nu D} \ll h_0^2 \ . \tag{6.33}$$

Such a region of temperatures near $T_c$ exists only if:

$$\nu D < 3 \ . \tag{6.34}$$

In this case, the temperature interval near $T_c$ in which the order parameter configurations are defined mainly by the random fields is

$$\tau_*(h_0) \sim h_0^{2/(3-\nu D)} \ . \tag{6.35}$$

Outside this interval, $\tau \gg \tau_*$, the external fields can be considered as small perturbations to the usual critical phenomena.

In the mean-field theory (which correctly describes the phase transition in the pure system for $D > 4$), $\nu = 1/2$. Thus, according to condition (6.34), the above nontrivial temperature interval $\tau_*$ exists only at dimensions $D < 6$. Correspondingly, at dimensions $D > 6$ the phase transition is correctly described by the usual mean-field theory.

What is going on in the close vicinity of the phase transition point, $\tau \ll \tau_*(h_0)$, at dimensions $D < 6$ is not known. The only concrete statement for the critical behaviour in the random field $D$-dimensional Ising model worked out some years ago claims that its critical exponents coincide with those of the pure $(D-2)$-dimensional system [55]. Unfortunately, although it is very elegant, this statement is wrong for the reasons mentioned in Section 6.2.

Indeed, let us turn back to the order parameter saddle-point equation (6.24). There exist strong indications both theoretical [57, 58, 60] and numerical [61] in favour of the possibility of the existence of many (macroscopic number) solutions of this equation. Moreover, according to the numerical studies [61] there exists another critical temperature $T_*$ above $T_c$ such that at temperatures $T > T_*$ the solution of the saddle-point equation (6.24) is unique (this region corresponds to the usual paramagnetic phase), while at $T < T_*$ multiple solutions appear, and only below $T_c$ the onset of the long-range magnetic order takes place. All these solutions must essentially depend on a particular configuration of the quenched fields being nonhomogeneous. In such a situation the usual RG approach, at least in its traditional form (which is nothing else but the perturbation theory) cannot be used.

It seems probable that we could find here again a completely new type of critical phenomenon of the spin-glass nature similar to that discussed in Section 4. As in spin glasses [9, 11] one could find here numerous disorder-dependent local energy minima. Unlike in spin glasses, however, these minima are most probably separated by *finite* energy barriers. Therefore, it is hardly possible to expect the existence of the real spin-glass phase near $T_c$. Nevertheless, it is widely believed that there must be a kind of 'glassy' phase in a small finite temperature interval, which separates the real paramagnetic state at high temperatures from the ferromagnetic one at low temperatures [62, 63].

One can also consider the following qualitative arguments. Actually, multiple global solutions of the saddle-point equation can appear due to the double-well structure of the local potential:

$$U(\phi) = \frac{1}{2}\tau\phi^2 + \frac{1}{4}g\phi^4 - h\phi(x) \tag{6.36}$$

This potential has two minima only: one for $\tau < 0$ and the other for the values of the field $h$ which are not too large. Therefore

$$h < h_c(\tau) = \frac{2}{3\sqrt{3}}\sqrt{\frac{|\tau|^3}{g}} \ . \tag{6.37}$$

At $\tau < 0$ the global solutions corresponding to the ferromagnetic state appear. The spatial density of the 'islands' where the condition (6.37) is fulfilled can be estimated as follows:

$$\rho = \frac{1}{\sqrt{2\pi}h_0}\int_{-h_c}^{+h_c} \mathrm{d}h \exp\left(-\frac{h^2}{2h_0^2}\right) \ . \tag{6.38}$$

Taking into account Eqn (6.37) one gets:

$$\rho \sim \begin{cases} \dfrac{1}{h_0}\sqrt{\dfrac{|\tau|^3}{g}} \ll 1 \ , & |\tau| \ll (gh_0^2)^{1/3} \ , \\ 1 - \exp\left(-\dfrac{|\tau|^3}{2gh_0^2}\right) \sim 1 \ , & |\tau| \gg (gh_0^2)^{1/3} \ . \end{cases} \tag{6.39}$$

In the second case in Eqn (6.39), the average distance between the 'islands' approaches the order of one, such that they interact strongly. It is only in this situation that it would be natural to expect that the ferromagnetic solution must be the global minimum. Indeed, for the energy [Eqn (6.23)] of the ferromagnetic state $\phi_f = \sqrt{|\tau|/g}$, one obtains $E_f = -\tau^2/4g$. On the other hand, a simple estimate for the characteristic energy of the disordered (field-defined) solution $\phi(x) \simeq [h(x)/g]^{1/3}$ yields: $E_h \sim -(h_0^4/g)^{1/3}$. Thus, the ferromagnetic solution attains the global minimum only for $|\tau| \leqslant \tau_h \sim (gh_0^2)^{1/3}$.

According to the above qualitative arguments, when the temperature is lowered the following physical phenomenon is expected to take place. At temperatures above $\tau = 0$, the disordered local minimum solution must be unique. It is only below $\tau = 0$ that multiple local minimum solutions appear. Simultaneously the ferromagnetic solutions appear below this point, although its energy is higher than the typical energy of the disordered solutions. According to these speculations, the point $\tau = 0$ should be associated to the temperature $T_*$ mentioned above and not to the ferromagnetic transition temperature $T_c$. On further lowering the temperature down to $\sim (gh_0^2)^{1/3}$, the interaction among the local minimum solutions is no longer small, and this may produce the nontrivial spin-glass effects discussed in Section 4. At temperatures of the same order, the ferromagnetic state attains the global minimum, so that somewhere in that temperature region at $T_c \sim T_* - (gh_0^2)^{1/3}$ the ferromagnetic phase transition takes place. In the low-temperature ferromagnetic phase, random fields also produce multiple local minimum states (due to large spin cluster flips), although here these are higher in energy than the ordered state, so that they can produce only Griffith singularities (considered in Section 6.3) and anomalously slow relaxations.

In the situation when the thermodynamics is defined by numerous disorder-dependent local energy minima, the most developed technique, which makes it possible to perform actual calculations, is the Parisi replica symmetry breaking (RSB) scheme (see Section 4). It is now many years since the possibility of RSB in the random-field Ising systems was first discussed [63, 64]. Recently the RSB technique has been successfully applied for the statistics of random manifolds [12], as well as for the $m$-component ($m \gg 1$) spin systems with random fields [13]. In the second case, it has been rigorously proved that the usual scaling replica-symmetric solution is unstable with respect to RSB

at the phase transition point. Moreover, recent studies of the $D$-dimensional random-field Ising systems, made in terms of the Legendre transforms and the general scaling arguments, demonstrate that for $D < 6$ in a finite temperature interval near $T_c$ a new type of critical regime is established, which is characterised by explicit RSB in the scaling of the correlation functions [65].

## 7. Conclusions

In this extensive review I have considered the problem of the effects produced by weak quenched disorder in statistical spin systems. The idea was to demonstrate on a qualitative rather than quantitative level the existing basic theoretical approaches and concepts. That is why the considerations were restricted by the simplest statistical models, and most of the details of the theoretical and experimental studies were not discussed.

The key problem which still remains unsolved is whether or not the obtained strong-coupling phenomena in the RG flows could be interpreted as the onset of a type of spin-glass phase in a narrow temperature interval near $T_c$. In spin glasses it is generally believed that RSB phenomena can be interpreted as a factorisation of the phase space into the (ultrametric) hierarchy of 'valleys', or local minimum pure states, separated by macroscopic (infinite) barriers. Although in the systems considered here the local minimum configurations responsible for RSB are not likely to be separated by infinite barriers, it would be natural to interpret the phenomenon obtained as effective factorisation of the phase space into a hierarchy of valleys separated by *finite* barriers. Since the only relevant scale in the critical region is the correlation length, the maximum energy barriers must be proportional to $R_c^D(\tau)$, and they become divergent as the critical temperature is approached. In this situation, one could expect that besides the usual critical slowing down (corresponding to the relaxation inside one valley) qualitatively much greater (exponentially large) relaxation times would be required for overcoming the barriers separating different valleys. Therefore the traditional measurements (made at finite equilibration times) can actually correspond to the equilibration within one valley only, and not to the true thermal equilibrium. Then in a close vicinity of the critical point, different measurements of the critical properties of, for example, spatial correlation functions (in the same sample) would exhibit different results, as if the state of the system become effectively 'trapped' in different valleys.

In any case, this phenomenon clearly demonstrates the existence of numerous metastable states forming infinite continuous spectra, and it could be interconnected with a general idea that the critical phenomena should be described in terms of an infinite hierarchy of correlation lengths and critical exponents. Unfortunately owing to the present state of knowledge in this field it is very difficult to hypothesise what the systematic approach for solving this type of problem should be.

It is now many years since, after the work of L D Landau and K G Wilson, the theory of the second-order phase transitions has become quite respectable and well-established science. It is generally believed that no bright qualitative breakthrough can be expected in this field any more, and that the only remaining problems relate to the progressively more exact calculations of the critical exponents.

In a sense, the theory of the disorder-induced critical phenomena has tried to attain a similar status. However, recent developments in this field clearly indicate the existence of a qualitatively new physical phenomenon, which goes well beyond the traditional concepts of the scaling theory. It seems as if we are close to a breakthrough to a new level of understanding of the critical phenomena in weakly disordered materials. I do believe so. This is in fact the main reason why the present review has been written.

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