

# De Haas – van Alphen effect as a first-order electronic topological transition

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**Abstract.** The de Haas–van Alphen effect and the behaviour of a superlattice in a quantising magnetic field can be described in terms of an electronic topological transition. Near the transition, the thermodynamic stability condition is shown to break down, thus eliminating the  $1\frac{1}{2}$ -order transition and giving rise to a first-order phase transition. The latter leads to the formation of diamagnetic Condon domains.

**1.** The purpose of the present Note is to suggest a new perspective for a well-studied phenomenon known as the de Haas–van Alphen (dHvA) effect. As is well known, the dependence of the thermodynamic properties (e.g., the magnetic susceptibility) of metal single crystals on the magnetic field  $H$  at low temperatures is oscillatory in nature (an exhaustive review of the relevant results is given in Ref. [1]). The reason is the quantisation of the motion of electrons in a magnetic field. The traditional approach to the dHvA effect is [1] that a physical property (say, susceptibility) is represented as a sum over the quantised energy states; after this, Poisson's summation formula is applied and the property in question emerges as an expansion in terms of functions periodic in  $H^{-1}$ . This is discussed in somewhat more detail below. In the standard approach, all occupied quantisation levels are 'in service'. Our intention here is to look at how the physical properties of interest are affected by one level, namely, that which has just begun, or is about, to be filled. This means that we change the 'scale' of the problem: rather than investigate the behaviour of the thermodynamic function as the inverse magnetic field is changed by a period, we will examine the neighbourhood of a magnetic field value corresponding to just one particular level. This approach reveals an electronic topological transition (ETT, see Refs. [2, 3]) in the system of conduction electrons.

We begin by discussing the basic concepts of the problem, which are of course common to both approaches. The motion of conduction electrons in a

magnetic field is determined by the Lorentz force acting on them. The path of an electron in  $\mathbf{r}$  space is similar to its path in  $\mathbf{p}$  space. The nature of electronic motion is greatly influenced by the shape of the cross section of the energy surface in a plane normal to the magnetic field direction. If the cross section is closed, the path is finite, and if open, infinite. In the former case, that part of the electronic energy associated with the motion perpendicular to the field is quantised. In the quasiclassical case, which is of the greatest importance for analysis, the quantisation conditions can be written down explicitly (Lifshitz–Onsager quantisation condition; see, for example, Ref. [4]):

$$S(\varepsilon, p_z) = \frac{2\pi\hbar e H}{c} (n + \gamma), \quad n = 0, 1, 2, \dots \quad (1)$$

Here  $H$  is the strength of the magnetic field directed along the  $z$  axis;  $S(\varepsilon, p_z)$  is the cross-sectional area of the energy surface  $\varepsilon(\mathbf{p}) = \varepsilon$  in the plane  $p_z = \text{const}$  normal to the magnetic field direction;  $\gamma$  is a constant which in most cases is equal to  $1/2$ . There is also the spin electron quantisation, which splits in two each energy level given by Eqn (1). In the simplest case, when the spin–orbital coupling is ignored, the magnitude of the splitting is  $eH\hbar/m_0c$ , where  $m_0$  is the free electron mass.

The quantisation-induced rearrangement of the electronic spectrum clearly changes the electron density of states, thus substantially affecting the physical properties of the metal. Suppose the electronic dispersion and the magnetic field direction are such that the electronic orbits are closed. The Lifshitz–Onsager quantisation, Eqn (1), transforms the three-dimensional spectrum into a one-dimensional one. To each particular specification of the quantum number  $n$  and spin projection  $\sigma = \pm 1/2$ , there corresponds a one-dimensional band (more frequently referred to as Landau subband) of the form

$$\varepsilon = \varepsilon_{n\sigma}(p_z). \quad (2)$$

The structure of the Landau subband is determined by the dependence of the cross-sectional area  $S(\varepsilon, p_z)$  on  $p_z$ . At this point, some simple general statements can be made. The cross-sectional area  $S(\varepsilon, p_z)$  is bound to have a maximum at a certain value of  $p_z$ . The greatest of the maxima determines the lower edge (bottom) of the Landau subband. The cross-sectional area  $S(\varepsilon, p_z)$  may have several extrema as a function of  $p_z$ . For each of these, there is a corresponding maximum in the  $\varepsilon_n = \varepsilon_n(p_z)$  dependence. The quasimomentum  $p_z$  which maximises  $S(\varepsilon, p_z)$  will be denoted by  $p_{zn}$ ; thus  $\varepsilon(p_{zn}) = \varepsilon_n$ . According to Eqn (1),  $p_{zn}$  and  $\varepsilon_n$  are determined by the equations

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$$S(\varepsilon_n, p_{zn}) = \frac{2\pi\hbar eH}{c} (n + \gamma),$$

$$\left. \frac{\partial S(\varepsilon_n, p_z)}{\partial p_z} \right|_{p_z=p_{zn}} = 0. \quad (3)$$

Expanding  $S(\varepsilon, p_z)$  in powers of  $p_z - p_{zn}$  yields

$$\varepsilon_{n\sigma}(p_z) \approx \varepsilon_{n\sigma}(p_{zn}) + \frac{(p_z - p_{zn})^2}{2m_{\parallel}}, \quad p_z \simeq p_{zn}, \quad (4)$$

where  $\varepsilon_{n\sigma}$  differs from  $\varepsilon_n$  by the spin splitting term,

$$\varepsilon_{n\sigma} = \varepsilon_n \pm \frac{e\hbar H \sigma}{m_0 c}, \quad \sigma = \pm \frac{1}{2},$$

and where, in terms of the cyclotron mass  $m^* = (2\pi)^{-1} [\partial S(\varepsilon, p_z) / \partial \varepsilon]$ , the longitudinal effective mass  $m_{\parallel}$  is

$$m_{\parallel} = -2\pi m^* \left( \frac{\partial^2 S}{\partial p_z^2} \right)^{-1} \Big|_{\varepsilon=\varepsilon_{n\sigma}, p_z=p_{n\sigma}}. \quad (5)$$

The sign of  $m_{\parallel}$  is determined by that of the cyclotron mass and by the structure of the energy surface (see Table 1). For the free electron gas,  $m^* = m_0$ ,  $\partial^2 S / \partial p_z^2 = -2\pi$  and  $m_{\parallel} = m_0$ . Eqns (2) and (4) are not limited to the quasiclassical approximation. The magnetic field renders the electronic motion one-dimensional, with the quasimomentum  $p_z$  retaining its meaning. Close to the edges of the one-dimensional band, Eqn (4) is valid as before (see Ref. [4], § 19, Ref. [5], and the discussion below).

**Table 1.**

Carriers	Cross-sectional area $S$ has a	
	maximum	minimum
Electrons $m^* > 0$	$m_{\parallel} > 0$	$m_{\parallel} < 0$
Holes $m^* < 0$	$m_{\parallel} < 0$	$m_{\parallel} > 0$

To illustrate the above ideas, two examples will be considered:

(a) the dispersion away from Brillouin band edges is†

$$\varepsilon = \frac{p_{\perp}^2}{2m_{\perp}} - \frac{p_z^2}{2m_{\parallel}} + \frac{p_z^4}{4p_0^2 m_{\parallel}} + \varepsilon_c, \quad (6)$$

with energy surfaces as shown schematically in Fig. 1.

(b) energy surfaces are warped cylinders (Fig. 2).

In the former case, the quantisation condition becomes

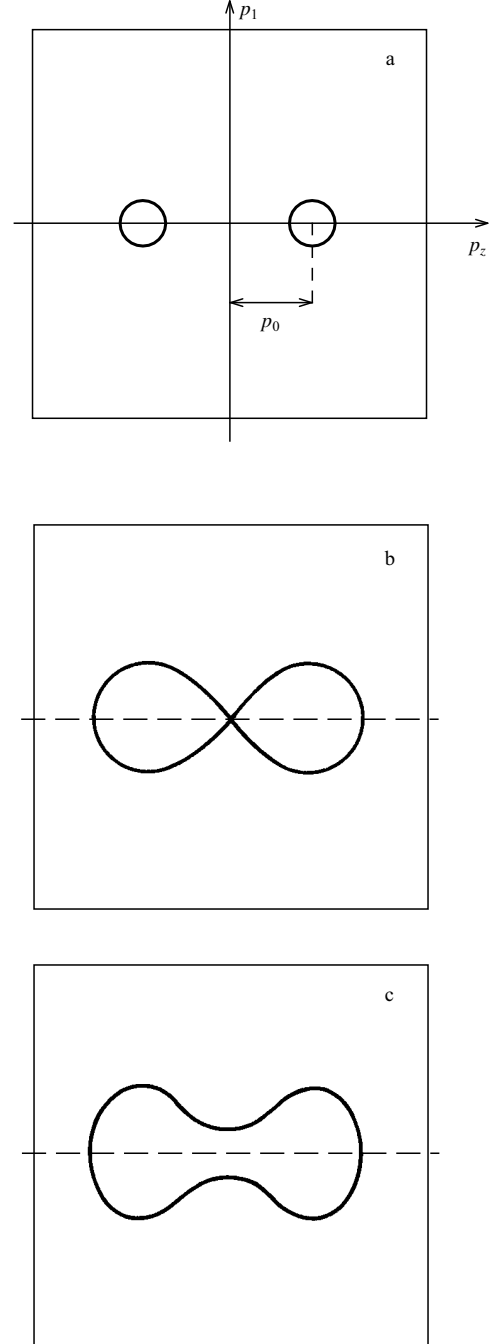
$$S(\varepsilon, p_z) \equiv \pi p_{\perp}^2 = 2\pi m_{\perp} \left[ \varepsilon - \varepsilon_c + \frac{p_z^2}{2m_{\parallel}} - \frac{p_z^4}{4p_0^2 m_{\parallel}} \right]$$

$$= \frac{2\pi\hbar eH}{c} (n + \gamma), \quad (7)$$

giving the quantised energy levels

$$\varepsilon_n(p_z) = \varepsilon_c + \frac{eH\hbar}{m_{\perp} c} (n + \gamma) - \frac{p_z^2}{2m_{\parallel}} + \frac{p_z^4}{4p_0^2 m_{\parallel}}. \quad (8)$$

†By assumption, the points  $p_z = \pm p_0$  and  $p_z = 0$  are away from the edges of the Brillouin zone.



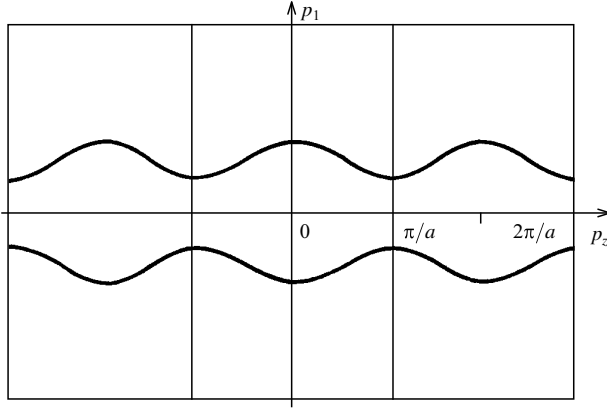
**Figure 1.** Energy surface of the spectrum given by Eqn (6) for energy values  $\varepsilon < \varepsilon_c - p_0^2/2m_{\parallel}$  (a),  $\varepsilon = \varepsilon_c$  (b), and  $\varepsilon > \varepsilon_c$  (c).

The energy band structure, i.e., the energy as a function of  $p_z$ , is shown in Fig. 3a for two values of  $n$ . As seen from Eqn (7), the cross-sectional area  $S(\varepsilon, p_z)$  at fixed  $n$  and  $\varepsilon$  shows one minimum (at  $p_z = 0$ ) and two maxima (at  $p_z = \pm p_0$ ) and, as predicted for the general case, the energy  $\varepsilon_n(p_z)$  also has extrema at these values of  $p_z$ : at  $p_z = 0$  we have a maximum, and at  $p_z = \pm p_0$ , we have minima.

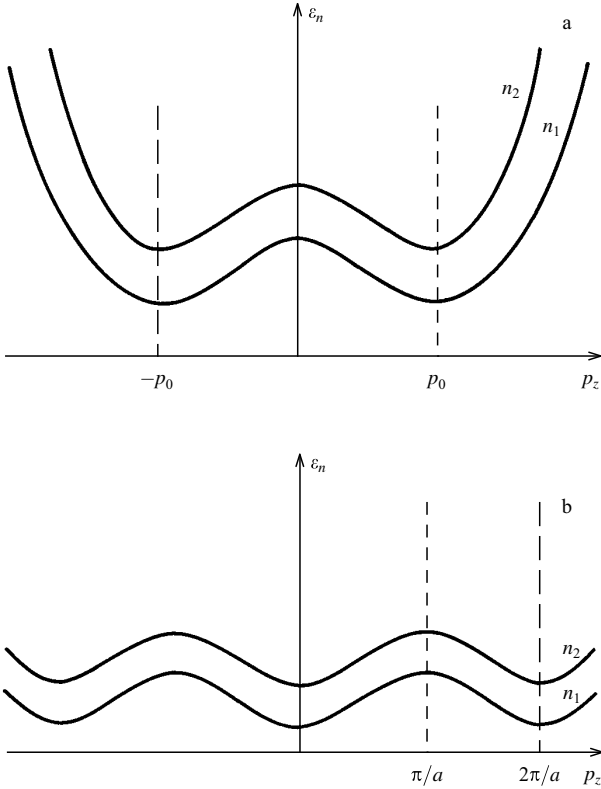
In the latter case,  $\varepsilon = \varepsilon_n(p_z)$  is a periodic function of the quasimomentum (Fig. 3b), each of the reciprocal space cells necessarily having one minimum and one maximum.

Let us now return to the general case. The density of electronic states in a magnetic field is [4]

$$v(\varepsilon) = \frac{eH}{(2\pi\hbar)^2 c} \sum_{n,\sigma} \int dp_z \delta[\varepsilon - \varepsilon_{n\sigma}(p_z)]. \quad (9)$$



**Figure 2.** Energy surfaces of a warped-cylinder spectrum.



**Figure 3.** Energy band structure for the spectrum given by Eqn (6) (a) and for the warped-cylinder spectrum (b). In both cases  $n_2 > n_1$ .

For  $\varepsilon = \varepsilon_{n\sigma}$ , the electron velocity component  $v_z$  vanishes and hence  $v(\varepsilon)$  has a square root singularity at  $\varepsilon = \varepsilon_{n\sigma}$ .

In the usual approach, the sum over  $n$  is transformed by the Poisson formula and the density of states becomes a sum of  $H^{-1}$ -harmonic functions whose periods are inversely proportional to the maximum (in  $p_z$ ) cross-sectional area  $S_m$  of the Fermi surface in planes perpendicular to the  $p_z$  axis:  $\Delta H^{-1} = 2\pi\hbar/cS_m$ . Knowledge of the density of states allows one to find whatever thermodynamic quantities are of interest. In strong magnetic fields,  $T \ll \hbar\omega_c$  ( $\omega_c = eH/m^*c$  is the cyclotron frequency of the electronic orbit), the constant component of the magnetic susceptibility is less than the oscillatory one, and it is the latter

which dominates. The magnetic susceptibility is [6]

$$\begin{aligned} \tilde{\chi}_{zz} = & \frac{1}{\hbar^3} \frac{1}{2^{3/2}\pi^{7/2}} \sqrt{\frac{e\hbar}{c}} \frac{1}{H^{3/2}} \sum_m \frac{S_m^2}{m^* \sqrt{|\partial^2 S / \partial p_z^2|_m}} \\ & \times \sum_{k=1}^{\infty} \frac{\psi(2\pi^2 kT / \hbar\omega_c)}{\sqrt{k}} \cos \left[ k \left( \frac{cS_m}{e\hbar H} - \pi \right) \pm \frac{\pi}{4} \right] \\ & \times \cos \left( \pi k \frac{m^*}{m_0} \right), \quad \psi(z) = \frac{z}{\sinh z}. \end{aligned} \quad (10)$$

From Eqn (10), and letting  $T \rightarrow 0$ , it follows that, since  $\psi(0) = 1$ , the amplitude of the susceptibility oscillations is given by the sum of the series in  $1/\sqrt{k}$  and hence tends to infinity. The increase in susceptibility makes the electronic system unstable and results in separating the metal into regions of different magnetic induction (Condon domains, see, for example, Refs [1, 7]). However, the growth of  $\chi_{zz}$  with temperature or magnetic field is very difficult to determine since this requires a summation of the series (10).

We shall proceed in a different way. When the energy  $\varepsilon$  is close enough to  $\varepsilon_{n\sigma}(p_n)$  to permit the expansion (2) (from this point on, this value is denoted by  $\varepsilon_n$  and referred to as an energy level), then the density of states may be expressed in the form  $v(\varepsilon) = v_0(\varepsilon) + \delta v_n(\varepsilon)$ , where  $v_0(\varepsilon)$  results from the summation in Eqn (9) over all the levels except for those near  $\varepsilon_n$ , and where

$$\delta v_n(\varepsilon) = \frac{eH}{2(2\pi\hbar)^2 c} \left[ \frac{|2m_{\parallel}|}{|\varepsilon - \varepsilon_n|} \right]^{1/2} \theta[\pm(\varepsilon - \varepsilon_n)] \quad (11)$$

is the contribution from the level considered. The upper sign in the argument of the  $\theta$  function refers to the case where the longitudinal effective mass is positive. To be specific, let  $m_{\parallel} > 0$ . Note that as  $\varepsilon \rightarrow \varepsilon_n$ , the density of states  $v_0(\varepsilon)$  remains finite whereas  $\delta v_n(\varepsilon)$  goes to infinity, so that near  $\varepsilon = \varepsilon_n$  there is a range of values for which  $\delta v_n(\varepsilon) \gg v_0(\varepsilon)$ .

The total number of electronic states below the energy  $\varepsilon$  has an anomalous part,

$$\delta N_n(\varepsilon) = \frac{eH\sqrt{2m_{\parallel}}}{(2\pi\hbar)^2 c} \sqrt{|\varepsilon - \varepsilon_n|} \theta(\varepsilon - \varepsilon_n), \quad (12)$$

and the thermodynamic potential is represented in the form  $\Omega = \Omega_0 + \delta\Omega_n$ , where

$$\delta\Omega_n = -\frac{eH_n/2m_{\parallel}}{(2\pi\hbar)^2 c} \int_{\varepsilon_n}^{\varepsilon} \frac{\sqrt{\varepsilon - \varepsilon_n} d\varepsilon}{\exp[(\varepsilon - \mu)/T] + 1}. \quad (13)$$

The field  $H_n$  is determined from the condition  $\varepsilon_n = \mu$  ( $\mu$  is the chemical potential). At zero temperature

$$\delta\Omega_n = -\frac{2}{3} \frac{eH_n\sqrt{2m_{\parallel}}}{(2\pi\hbar)^2 c} z^{3/2} \theta(z). \quad (14)$$

The energy parameter  $z = \mu - \varepsilon_n$  we have introduced will be important in the discussion below.

Formally, the situation is reminiscent of ETT [2] (see Ref. [3] for a review). If the Fermi surface of a metal changes its topology by, say, developing a pocket or losing a neck, it is known that the Fermi-surface density of states in a pure crystal acquires a singular term. This term is zero on that side of the transition which has fewer Fermi-surface

pockets, while on the other side it depends in a square root fashion on the parameter  $z = \mu - \varepsilon_c$ , where  $\varepsilon_c$  is the Fermi topology change energy. The thermodynamic potential at zero temperature also acquires a singular term: zero on one side of the transition and proportional to  $|z|^{5/2}$  on the other. Thus, in Ehrenfest's terminology, a three-dimensional ETT at zero temperature is a  $2\frac{1}{2}$ -order phase transition. To avoid misunderstanding, this is not a temperature-induced phase transition, but one due to external parameters (such as pressure, impurity concentration, or magnetic field). At finite temperature, as well as in a nonideal crystal, the thermodynamic potential singularity is washed out, and the ETT is no longer a phase transition but manifests itself in anomalies. In particular, the anomalous part of the electronic specific heat increases as a square root of  $z$ , and the thermal coefficient of expansion tends to infinity as  $|z|^{-1/2}$ .

A similar argument suggests that, for the case under consideration, the intersection of Fermi levels  $\varepsilon_{n\sigma}$  gives rise to a  $1\frac{1}{2}$ -order phase transition, or one-dimensional ETT<sup>†</sup>, which is the appearance (or disappearance) of a 'one-dimensional pocket' at the Fermi level [the appearance of electrons in a new Landau subband, when  $\varepsilon_n$  corresponds to the greatest of the maxima of  $S(\varepsilon, p_z)$  as a function of  $p_z$ ].

At finite temperature, as well as in a nonideal crystal, the singularity is washed out and the phase transition transforms into an anomaly (see below). What thermodynamic singularities will then occur? Suppose  $T \ll \hbar\omega_c$ ,  $|z| \ll \hbar\omega_c$ , and the number of occupied Landau subbands is large enough, so that the chemical potential may be considered as weakly dependent on the magnetic field (a typical dHvA situation). Let us start by determining the dependence of the parameter  $z$  on the temperature, volume, and magnetic field. From the condition that the number of electrons  $N$  is constant we obtain, for  $T \ll |z|$ , that

$$\frac{\partial z}{\partial T} = -\frac{\pi^2 T}{3} \frac{v'(z)}{v(z)}, \quad \frac{\partial z}{\partial V} = -\frac{N}{V^2 v(z)}. \quad (15)$$

Since, from Eqn (11),  $v(z) \propto 1/\sqrt{z}$  as  $z \rightarrow 0$ ,

$$\frac{\partial z}{\partial T} = \frac{\pi^2 T}{6z} \theta(z), \quad \frac{\partial z}{\partial V} \propto \frac{1}{\sqrt{z}}, \quad z \rightarrow 0. \quad (16)$$

Since  $v(z)$  approaches infinity as  $1/\sqrt{z}$ , it follows that for  $|z| \rightarrow 0$  the anomalous part of the density of states is much larger than the normal part:  $v_0 \ll \delta v_n$  and  $|v'_0| \ll |\delta v'_n|$ .

Further, at  $T = 0$  and

$$0 < \mu_0 - \varepsilon_n \ll \left[ \frac{eH_n \sqrt{m_{\parallel}}}{c\hbar^2 v^2(\mu_0)} \right]^2,$$

we have

$$\begin{aligned} z(H) &= \mu(H) - \varepsilon_n \\ &= (\mu_0 - \varepsilon_n)^2 \frac{v^2(\mu_0) \sqrt{2} \pi^2 c \hbar^3}{m_{\parallel} e H_n} \propto (H_n - H)^2, \end{aligned} \quad (17)$$

where  $\mu_0$  is the chemical potential in the absence of a magnetic field.

The thermodynamic quantities of interest—electronic specific heat ( $C$ ), thermal coefficient of electron pressure

( $\alpha = T^{-1} \partial P / \partial T$ ), and compressibility ( $\kappa = \partial P / \partial V$ ) are conveniently expressed by the use of Eqns (15) and (16) to give

$$C = \frac{\pi^2 T}{3} v(z), \quad (18)$$

$$\alpha = \frac{\pi^2 T}{3} \left[ v(z) - \frac{N v'(z)}{V v(z)} \right], \quad (19)$$

$$\kappa = -\frac{N^2}{V^2} \frac{1}{v(z)}. \quad (20)$$

Thus, the specific heat has a singularity which turns to zero as  $z \rightarrow 0$ :

$$\frac{\delta C}{T} = \frac{\pi^2 a_1}{3} \frac{1}{\sqrt{z}} \theta(z). \quad (21)$$

We have introduced the notation  $a_1 = [eH_n / 2(2\pi\hbar)c] \sqrt{|2m_{\parallel}|}$ . The thermal coefficient of pressure (and similarly the thermal coefficient of expansion) is still more singular:

$$\delta\alpha = \frac{\pi^2 N}{6Vz} \theta(z), \quad (22)$$

and the electronic component of the compressibility at  $z > 0$  vanishes as a square root:

$$\kappa = -\frac{N^2}{V^2} a_1 \sqrt{z}, \quad z > 0. \quad (23)$$

For  $Z < 0$ , all the electron characteristics assume their background values involving the quantity  $v_0$ . Treating the magnetic field as a variable, we find that the specific heat and the thermal coefficient of expansion diverge as  $(H_n - H)^{-1}$  and  $(H_n - H)^{-2}$ , respectively, and that the electronic compressibility is proportional to  $H_n - H$ .

We note, however, that the  $1\frac{1}{2}$ -order transition gives rise to a phenomenon which is not found in ETT. This is seen by calculating the magnetic susceptibility anomaly<sup>‡</sup>  $\delta\chi = -\partial^2 \delta\Omega / \partial H^2$ . Retaining only the term divergent as  $z \rightarrow 0$ , we find

$$\delta\chi = -\frac{eH_n}{c} \left( \frac{\partial \varepsilon_n}{\partial H} \right)^2 \frac{\sqrt{2m_{\parallel}}}{(2\pi\hbar)^2} \int_0^\infty \frac{\partial n_F(\varepsilon + \varepsilon_n)}{\partial \varepsilon} \frac{d\varepsilon}{2\sqrt{\varepsilon}}. \quad (24)$$

In the limiting cases we have: for  $T \ll z$ ,

$$\delta\chi = \frac{eH_n}{2c} \left( \frac{\partial \varepsilon_n}{\partial H} \right)^2 \frac{\sqrt{2m_{\parallel}}}{(2\pi\hbar)^2 \sqrt{z}} \theta(z) \left( 1 + \frac{\pi^2}{8} \sqrt{\frac{T}{z}} \right), \quad (25)$$

and for  $z \ll T$ ,

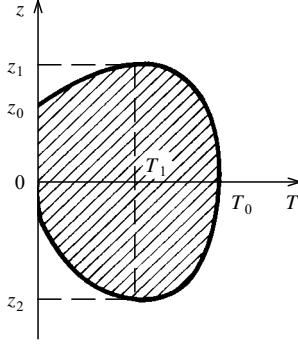
$$\delta\chi = \frac{eH_n}{2c} \left( \frac{\partial \varepsilon_n}{\partial H} \right)^2 \frac{\sqrt{2m_{\parallel}}}{(2\pi\hbar)^2 \sqrt{2T}} \left( A_1 + \frac{z}{T} A_2 \right), \quad (26)$$

$$A_1 = \int_0^\infty \frac{d\xi}{\cosh^2 \xi^2} \approx 0.953,$$

$$A_2 = \int_0^\infty \frac{\sinh \xi^2 d\xi}{\cosh^3 \xi^2} \approx 0.298. \quad (27)$$

<sup>†</sup>This is of course because the electron spectrum becomes one-dimensional in a magnetic field.

<sup>‡</sup>For simplicity we neglect anisotropy effects.



**Figure 4.** Phase diagram in  $z, T$  coordinates. The dashed part is the Condon domain region, the enclosing line being that of first-order phase transitions.

In the low-temperature low- $z$  region (i.e., for the chemical potential  $\mu$  close enough to  $\varepsilon_n$ ), the susceptibility is highly positive and tends to infinity. In particular, in the dashed region of the  $z$ – $T$  phase diagram of Fig. 4† the magnetic susceptibility exceeds  $(4\pi)^{-1}$ .

The large value of the magnetic susceptibility raises the question of whether it is the magnetic field  $H$  or the magnetic induction  $B$  which determines the thermodynamic potential  $\Omega$  and other thermodynamic characteristics of the conduction electrons. Normally (in nonmagnetic media) this question does not arise because the magnetic susceptibility (of both diamagnetics and paramagnetics) is very limited, so that  $B \simeq H$ . The most important — oscillatory — component of the thermodynamic potential is dominated by macroscopically large electronic orbits (of size larger than the crystal cell). In following such an orbit, an electron feels the average magnetic field, that is, the induction  $B$ . Therefore the argument of the thermodynamic potential  $\Omega$  is  $B$ , not  $H$ , and hence we must replace  $H$  by  $B$  in all the formulas above. Using

$$H = B - 4\pi M(B), \quad (28)$$

where  $M(B) = -\partial\Omega(B)/\partial B$  is the sample magnetisation, we can readily show that the thermodynamic stability condition  $\partial H/\partial B > 0$  (see Ref. [8]) becomes the inequality

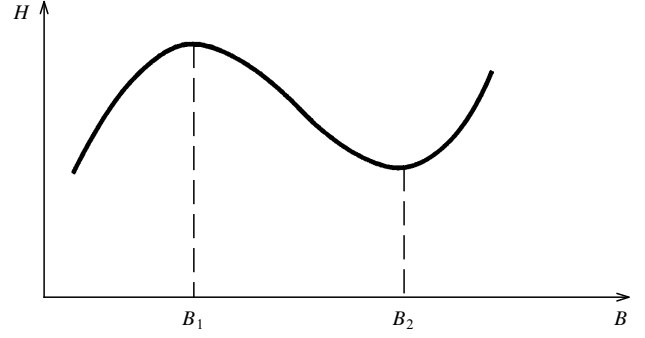
$$\chi > \frac{1}{4\pi}. \quad (29)$$

Thus, the dashed region in Fig. 4 is one of instability. The existence of an instability region in the dependence  $H = H(B)$  (Fig. 5) suggests that the electronic subsystem of the metal must necessarily undergo a first-order phase transition in which the system breaks down into two phases differing in induction (Shoenberg effect, see Refs [1, 17]).

Thus, at a given temperature  $T_1$  (see Fig. 4) and  $z_1 < z < z_2$ , the system consists of two phases whose inductions correspond to  $B(z_1)$  and  $B(z_2)$ ; these are diamagnetic Condon domains [9]. The entire situation is very similar to that of a liquid–vapour transition in a Van der Waals system.

The region of existence of the domains is easy to estimate. Thus, at zero temperature the susceptibility

†The quantities  $z_0$  and  $T_0$  are found from (25), (26) and the equality  $\delta\chi = (4\pi)^{-1}$  for  $T = 0$  and  $z = 0$ , respectively (the term  $\chi_0 \ll 1$  may of course be neglected).



**Figure 5.** Typical dependence of the magnetic field  $H$  upon the induction  $B$  close to the phase transition. For  $B_1 < B < B_2$ , the thermodynamic inequality  $\partial H/\partial B > 0$  fails, and the system is unstable to Condon domain formation.

becomes  $(4\pi)^{-1}$  for  $z = z_0$  (see Fig. 4), which from Eqn (17) implies the induction

$$H_0 = H_n \left[ 1 - \frac{v(\mu_0)e^2\hbar^2}{\pi c^2 m_{\parallel}^2} \right]. \quad (30)$$

The infinite dHvA amplitude and the appearance of Condon domains may be interpreted as an electronic topological transition, a first-order phase transition that precedes (and thereby eliminates) the  $1\frac{1}{2}$ -order transition.

Lifshitz [2] notes that in an ETT (without a magnetic field) the compressibility  $\partial P/\partial V$  acquires a finite positive term, which can render the crystal unstable. We emphasise that at  $H = 0$  the additional term is finite, and the total compressibility can be negative only for  $|z| \neq 0$ . In a magnetic field (for  $T \rightarrow 0$ ), an instability is bound to occur, and so also is a first-order phase transition. This latter fact accounts for the repeatedly observed effect of metal domains [1] or, in other words, for the repeatedly observed first-order electronic topological transition.

**2.** There is currently considerable interest in artificial (man-made) objects. In particular, these include crystals with a superstructure — property-periodic superlattices with a period greatly exceeding the lattice constant. Among such objects are systems of alternating ‘degenerate semiconductor–dielectric’ layers. The motion in a semiconductor layer may be considered quasifree. Coupling between the layers is generally weak. If the tunnel penetrability of the dielectric barriers is not too weak, and if the periodicity is strict enough, band motion across the layers (i.e., along the  $z$  axis) is possible. In actual fact, one cannot go beyond the weak coupling condition (see, for example, Ref. [10]), and the carrier dispersion relation has a simple form (for  $\mathbf{H} = 0$ ) of

$$\varepsilon(\mathbf{p}) = \frac{p_x^2 + p_y^2}{2m_{\perp}} + \Delta \sin^2 \left( \frac{p_z d}{2\hbar} \right), \quad (31)$$

where  $m_{\perp}$  denotes the effective mass for the motion in a conducting layer,  $\Delta$  is the miniband width related to the dielectric-layer tunnel transparency, and  $d$  is the superlattice period. In a magnetic field along the  $z$  axis (i.e., perpendicular to the layers), the spectrum gets quantised (the spin splitting is for simplicity neglected):

$$\varepsilon_n(p_z) = \hbar\omega_c \left( n + \frac{1}{2} \right) + \Delta \sin^2 \left( \frac{p_z d}{2\hbar} \right), \quad n = 0, 1, 2, \dots, \quad (32)$$

the electron cyclotron frequency being  $\omega_c = eH/m_\perp c$ . The density of states of such a system is, from Eqn (9),

$$\nu(\varepsilon) = \frac{eH}{\pi^2 c d \hbar} \sum_n \left[ \varepsilon - \hbar\omega_c \left( n + \frac{1}{2} \right) \right]^{-1/2} \times \left[ \hbar\omega_c \left( n + \frac{1}{2} \right) + \Delta - \varepsilon \right]^{-1/2}, \quad (33)$$

where the energy lies of course within the intervals

$$\hbar\omega_c \left( n + \frac{1}{2} \right) < \varepsilon < \hbar\omega_c \left( n + \frac{1}{2} \right) + \Delta, \quad (34)$$

while outside these intervals  $\nu(\varepsilon) \equiv 0$  provided  $\hbar\omega_c > \Delta$ . It is seen that in strong magnetic fields (for  $\hbar\omega_c > \Delta$ ) there are gaps in the energy spectrum. The allowed and forbidden bands alternate but do not overlap. The width of each allowed band, in a superlattice of volume  $1 \text{ cm}^3$ , is  $\Delta$ , and the total capacity of the band is  $N_H = eH/\pi\hbar c d$  ( $\text{cm}^{-3}$ ). Thus, if the carrier density  $N$  is a multiple of  $N_H$  ( $N = sN_H$ ,  $s = 1, 2, \dots$ ), then at  $T = 0$  the superlattice is a dielectric: there are only filled and empty bands in the system. It is true that this is a rather peculiar dielectric. All the dissipative components of its conductivity are zero, whereas the Hall component  $\sigma_{xy} = -\sigma_{yx}$  is not and equals  $e^2 s / \pi \hbar d$ . It is easy to realise that we are dealing with a three-dimensional analogue of the quantum Hall effect (see Refs [11, 12]). Introducing two-dimensional conductivity  $\sigma_{ik}^{(2)} = d\sigma_{ik}$ , the steps in question assume their usual form of  $\sigma_{ik}^{(2)} = 2e^2 s / h$ .

For  $N = sN_H$  and  $T = 0$ , the electron chemical potential,  $\mu$ , lies in the middle of the forbidden band. For  $T \ll \hbar\omega_c$ , the number of electrons in the first ‘empty’ (conduction) band and the number of holes in the last ‘filled’ (valence) band are exponentially small:  $N_e = N_h \propto \exp(-\hbar\omega_c/2T)$ . The occupation of the bands at a constant carrier density can be controlled by varying the magnetic field. For  $H \neq H_s = \pi\hbar c N d / es$ , there is necessarily a partly filled band, and the system is a ‘metal’ (degenerate semiconductor). Thus, varying the magnetic field enables a metal–dielectric transition to be modelled [11]. This latter is also an example of a  $1\frac{1}{2}$ -order topological phase transition. However, as in the case above, a  $1\frac{1}{2}$ -order transition does not actually occur. As the magnetic field approaches  $H_s$ , the susceptibility increases causing the loss of stability. As a result, a first-order phase transition takes place. Such a situation arises, at least for  $T = 0$ , when the magnetic susceptibility is given by

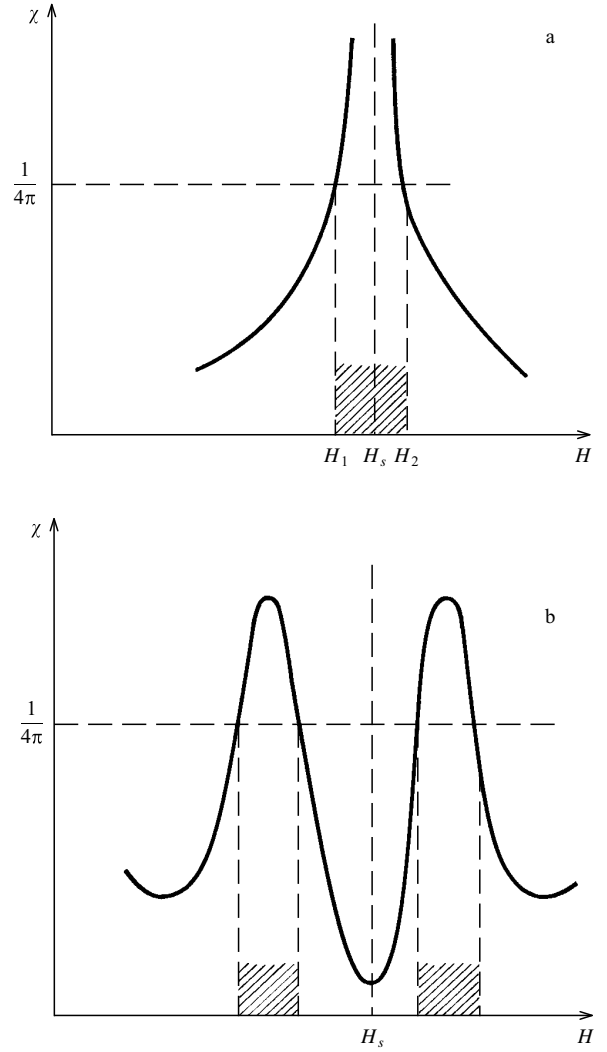
$$\chi = \frac{2e^2}{\pi c^2 d m} \frac{s + 1/2}{\sqrt{1 - X^2}}, \quad X = 2 \sin^2 \left( \frac{\pi N}{N_H} \right) - 1. \quad (35)$$

This formula may also be considered true for  $T \ll \mu'$ , where  $\mu'$  is the chemical potential measured from the band edge.

Sufficiently close to the metal–dielectric transition, when the inequality

$$\hbar\omega_c \gg T \gg \mu'$$

holds (or for  $H \rightarrow H_s$ :  $|H - H_s| \ll H_s \sqrt{\pi T / \Delta}$ ), the degeneracy of the system is removed and the electron gas becomes a Boltzmann one. In this case, the chemical potential lies in the centre of the forbidden gap, and the magnetic susceptibility is given by



**Figure 6.** Magnetic susceptibility versus magnetic field for a superlattice for  $T = 0$  (a) and  $T \neq 0$  (b). Dashed regions correspond to magnetic fields for which  $\chi > (4\pi)^{-1}$  (Condon domain region). For  $T \rightarrow 0$ , the maxima move off to infinity, and the minimum disappears.

$$\chi = \frac{2e^2}{\pi c^2 d m} \hbar\omega_c T \exp \left( -\frac{\hbar\omega_c - \Delta}{2T} \right) \left[ \left( s + \frac{1}{2} \right)^2 + \left( s + \frac{3}{2} \right)^2 \right], \quad \hbar\omega_c - \Delta \gg T. \quad (36)$$

It is seen that in this region the susceptibility is exponentially small. Figs 6a and 6b present the variation of the magnetic susceptibility with the field for  $T = 0$  and  $T \neq 0$ , respectively. At zero temperature, an increase (or decrease) of the magnetic field is accompanied by two first-order phase transitions (i.e., breaking of the system into Condon domains). The fields  $H_1$  and  $H_2$  (Fig. 6a) are readily found to be

$$H_{1,2} = H_s \left[ 1 \mp \frac{4e^2}{\pi c^2 d m} \left( s + \frac{1}{2} \right) \right]. \quad (37)$$

At finite temperature the maximum value of the susceptibility occurs at the fields

$$H \approx H_s \left( 1 \mp \sqrt{\frac{\pi T}{\Delta}} \right). \quad (38)$$

At temperatures

$$\sqrt{\frac{\pi T}{\Delta}} > \frac{4e^2}{\pi c^2 d m} \left( s + \frac{1}{2} \right) \quad (39)$$

the susceptibility never exceeds  $1/4\pi$ , and the Condon-domain instability region is absent: the system exhibits only a metal–dielectric transition (although clearly a smeared one since  $T \neq 0$ ).

It should be noted that, in contrast to the three-dimensional case, in superlattices (at low temperatures) equilibrium electrons appear only as a result of doping. Because of the collisions of electrons with impurities, Landau levels are broadened. This reduces the amplitude of susceptibility oscillations, the amount of reduction being characterised by the effective Dingle temperature  $T_D$ , where  $T_D \simeq \hbar/\tau$ , and  $\tau$  is the characteristic time for electron scattering by impurities. In reference [13] it was found that the electron scattering time in the point-defect field  $U(\mathbf{r}) = U_0 \sum_i \delta(\mathbf{r} - \mathbf{r}_i)$  is

$$\tau^{-1} = \frac{U_0^2 N_{\text{im}} e H}{2\pi^2 \hbar c d \Delta} [x(1-x)]^{1/2}, \quad x = \frac{\mu'}{\Delta}, \quad (40)$$

where  $N_{\text{im}}$  is the impurity concentration. Its value controls the existence (or otherwise) of the first-order phase transition.

It should be remembered, however, that populating Landau subbands can also be achieved without introducing impurities (say, by light pumping). This may prove convenient for the observation of phase transitions in superlattices.

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