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Fluid flow in tubes with elastic walls

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Contents

Abstract. Physico-mathematical aspects of fluid flow in elastic thin-walled tubes have been examined and nonlinear equations describing it have been derived and solved. It is shown that an auto-oscillating flow regime (flutter) can occur and mathematical models of solitons (pulse waves) based on the Korteveg – de Vries equation and on the modified nonlinear Schrodinger equation have been developed. The appropriateness of using the elastic thin-walled tube model for the description of blood flow in major blood vessels is discussed.

1. Introduction

Fluid flow in elastic thin-walled tubes is an interesting and complex physical problem. Numerous attempts to find its solution have been reported, especially in the context of blood circulation biomechanics, since the French scientist Poiseuille first attacked this problem [1 – 3]. Some interesting results have been obtained, with major progress having been made in the studies of the effect of rheological properties of the fluid on flow hydrodynamics and the use of various electric analogues in which characteristics of the flow and the elastic walls are simulated by electrical parameters, e.g. potential difference, capacity, inductivity, etc. [4 – 6].

The present study is largely focused on two aspects:

— the relationship between the nature of fluid flow and the geometry of the elastic tube which varies under the effect of the flow;

— the use of nonlinear hydrodynamic equations for the description of fluid flow in elastic tubes.

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How well the passive elastic tube model employed in the present study approximates a real blood vessel will be discussed further on, but here it should be mentioned that in this paper certain geometric relations inherent in the cardiovascular system of living organisms are used.

With the objective of the study in mind, the hydrodynamic model has been considerably simplified. First, the ideal fluid approximation has been adopted; second, the momentum-free elastic tube theory is being used.

Fluid flow in elastic thin-walled tubes can be arbitrarily divided into three relatively independent hydrodynamic phenomena: (1) fluid transport along the tube, (2) pressure wave propagation at a speed higher than that of the fluid (in blood circulation biodynamics such a wave is referred to as the pulse wave), and (3) high-frequency oscillations caused by the loss of stability of the 'flow – wall' type (flutter). Each of these phenomena is described by appropriate equations ensuing from the Navier – Stokes equations and the continuity condition.

Nonlinearity of the Navier – Stokes equations implies that at least some of the above three phenomena lack linearity. It should be emphasised that the nonlinear turbulence problems are beyond the scope of the present paper.

2. Hooke's law for an elastic tube

To obtain a closed system of hydrodynamic equations, it is necessary to define the relation between deformation of the elastic thin-walled tube and the excess pressure within the tube. In the simplest case of small deformations, this relationship can be described by Hooke's law. However, formulation of this law depends on the nature of the problem and can to be different for different rates of change of the tube cross-section area. For relatively low velocities of fluid flow and change of the tube cross-section area, it is appropriate to use a formula that relates excess pressure inside the tube to its cross-section area:

$$P_t = C \frac{\Delta S}{S_0}, \qquad (2.1)$$

where $\Delta S = S - S_0$, S is the tube cross-section area at a given site, S_0 is the tube cross-section area at zero excess pressure P_t, and C is the elasticity of tube walls. It follows from the theory of thin shells that $C = Eh/d$, where E is the effective modulus of tube wall elasticity, h is wall thickness, and d is the mean tube diameter.

When the changes in the tube cross-section area are greater, Hooke's law can be written down in the following form [1]:

$$dP_t = C\,\frac{dS}{S}\,. \qquad (2.2)$$

In this formula, the linear relationship between changes in excess pressure and deformation of the tube cross-section area is retained, but the dependence of area on excess pressure is nonlinear:

$$S = S_0 \exp\left(\frac{P_t}{C}\right)\,. \qquad (2.3)$$

If we retain only the first term in the expansion, formula (2.3) reduces to Eqn (2.1). When changes in the tube cross-section area are very rapid, Hooke's law must include parameters describing fluid flow within the elastic tube. In this case, Hooke's law may be applied in the following form [7]:

$$P_t = C\,\frac{\Delta(dV)}{dV}\,, \qquad (2.4)$$

where $dV = S\,dX$ is the tube volume element (Fig. 1). When the fluid is stationary, $dV = S\,dl$ where dl is a fixed length of the tube. In this case, formula (2.4) reduces to Eqn (2.2). The transformation is more involved when the fluid flows in the tube (see Fig. 1):

$$\varepsilon = \frac{\Delta(dV)}{dV} = \frac{V_1^* - V^*}{dV} = \left(V^* + \frac{dV^*}{dV}\,dV - V^*\right)\frac{1}{dV} = \frac{dV^*}{dV} = \frac{\partial(uS)}{S\,\partial x}\,, \qquad (2.5)$$

where ε is the deformation of the tube volume element dV as the fluid is displaced by distance u within the tube, and $V_1^* - V^*$ is the accompanying change in the tube volume element.

At $S = \text{const}$, formula (2.5) becomes the conventional expression for the strain of a solid: $\varepsilon = \partial u/\partial x$.

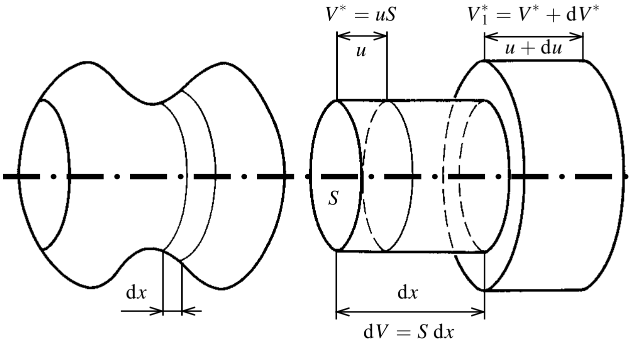

Figure 1. Deformation of the elastic tube volume element dV during fluid flow ($V_1^* - V^*$ is the change of the tube volume element during fluid flow).

The following comment should be made with regard to the displacement u. If high-frequency auto-oscillations of both the flow and the walls occur in an elastic tube, they may be described as oscillations with velocity v_{osc} superimposed on average fluid flow at a velocity of v_0, so that the resulting velocity is

$$v = v_0 + v_{osc}\,. \qquad (2.6)$$

The quantity u is the displacement of the tube volume element in the course of longitudinal oscillations relative to a point moving with velocity v_0, so that $v_{osc} = \partial u/\partial t$ (where t is time).

Therefore, it follows from Eqn (2.4) that

$$P_t = C\,\frac{\partial(uS)}{S\,\partial x}\,. \qquad (2.7)$$

Formulas (2.1)–(2.4) and (2.7) relate excess pressure to tube geometry. However, if we wish to derive an equation for the oscillations, it is necessary to use the relationship between the reaction of the elastic tube wall and its geometry. Reaction of the tube wall P is the force with which the unit of the inner tube surface area acts on the fluid. Since $P = -P_t$, formulas (2.1)–(2.4) and (2.7) must take into account the minus sign.

3. Auto-oscillating (flutter) regime of fluid flow in an elastic tube

Pressure waves (pulse waves) are not necessary for the flutter regime to occur. This regime is quite possible at a constant discharge rate in the elastic tube, that is at a constant pressure difference along the tube.

Fluid flow in an elastic tube is an essentially unstable process. Any accidental increase in flow velocity causes a decrease of static pressure in the fluid, in accordance with the 'Bernoulli law'. This results in a reduction of the tube cross-section area which in turn leads to a further rise in the flow velocity. This process progresses as an avalanche (owing to positive feedback) and finally leads to tube occlusion. Nonetheless, a certain amount of fluid has to be pumped through the tube, thus opening it again. This accounts for the generation of auto-oscillations of the 'flow–wall' instability type or surface mode flutter. This process is not directly dependent on fluid viscosity which allows us to examine initiation of the auto-oscillating (autowave) regime in an ideal fluid and thereafter consider the effect of viscosity.

Let us derive the impulse equation for this process. A change in the tube cross-section area at the half-wave of the newly-generated oscillations comprises two components: those of inflation and deflation (Fig. 2). Projection of Newton's second law on the x axis for a fluid volume element dV can be written down as

$$dF_i + N - (N + dN) - dF_d = \rho\,dV\,\frac{dV}{dt}\,, \qquad (3.1)$$

where $dF_i = dF_d = P\,dS$ are longitudinal constituents of the forces that act on volume element dV from the inflation and the deflation sides of the tube (according to Newton's third law, these forces are equal); $N = PS$ is the longitudinal component of pressure forces that maintains fluid flow; ρ is fluid density.

The force F_i that acts on the tube volume element dV over the entire inner surface on the inflation side is

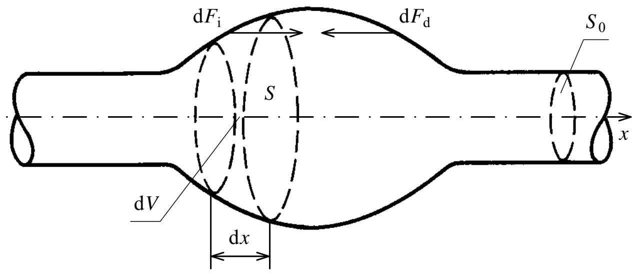

Figure 2. Schematic representation of direct components of the forces that act on volume element dV from the inflation and the deflation sides of the tube.

counterbalanced by the force F_d which acts from the deflation side. At $dF_d = 0$, i.e. when the tube is open, Eqn (3.1) assumes the form

$$-\frac{dP}{dx} = \rho \frac{dV}{dt} . \tag{3.2}$$

The impulse equation at $S = \text{const}$ has a similar form. However, in the case of symmetric geometry of the deflation and inflation sides, the impulse equation (3.1) is transformed to

$$-\frac{\partial(PS)}{S\,\partial x} = \rho \frac{dV}{dt} . \tag{3.3}$$

Therefore, the Euler equation (3.2) is equally applicable to widening and narrowing tubes and to those with a uniform cross section. Conversely, Eqn (3.3) holds for tubes in which expansion is followed by contraction. Equations of similar form are used to solve Zhukovsky's problem of hydraulic shock in an elastic tube [8].

By substituting Hooke's law (2.7) into Eqn (3.3) with due regard for the sign, and using Eqn (2.6) and taking into consideration that $\partial v_0/\partial t = 0$, we obtain

$$C \frac{\partial^2(uS)}{S\,\partial x^2} = \rho\left(\frac{\partial v_{osc}}{\partial t} + v_{osc}\frac{\partial v_{osc}}{\partial x}\right) . \tag{3.4}$$

When the amplitude of oscillations is low, the nonlinear convective term in the right side of Eqn (3.4) can be neglected [9]. The velocity of autowave propagation along the tube is given by

$$a = \sqrt{\frac{C}{\rho}} = \sqrt{\frac{Eh}{\rho d}} . \tag{3.5}$$

The use of the relation $v_{osc} = \partial u/\partial t$ and Eqn (3.5) yields

$$a^2 \frac{\partial^2(uS)}{\partial x^2} = S\frac{\partial^2 u}{\partial t^2} , \tag{3.6}$$

which is a two-parameter wave equation. The second equation that relates the unknown variables u and S is the continuity equation [1]:

$$\frac{\partial S}{\partial t} + \frac{\partial(v_{osc}S)}{\partial x} = 0 . \tag{3.7}$$

This equation includes only the oscillation component v_{osc} of the velocity since variations of both the area and the discharge ($Q = vS$) are due solely to v_{osc}.

The system of equations (3.6) and (3.7) can be solved by the Fourier method of separation of variables. We put

$$u = F_1(x)\,\Phi_1(t) , \qquad S = F_2(x)\,\Phi_2(t) ,$$

$$uS = F_1(x)\,F_2(x)\,\Phi_1(t)\,\Phi_2(t) = F(x)\,\Phi(t) . \tag{3.8}$$

Substituting expressions (3.8) into Eqn (3.6) and cancelling factors $\Phi_2(t)$ we separate the variables:

$$\frac{1}{\Phi_1}\frac{d^2\Phi_1}{dt^2} = a^2\frac{1}{F}\frac{d^2F}{dx^2} = -\omega^2 = \text{const} . \tag{3.9}$$

This yields two equations:

$$\frac{d^2\Phi_1}{dt^2} + \omega^2\Phi_1 = 0 , \qquad \frac{d^2F}{dx^2} + \left(\frac{\omega}{a}\right)^2 F = 0 . \tag{3.10}$$

Their solution has the form of

$$\Phi_1(t) = A_0\sin(\omega t + \varphi_0) , \tag{3.11}$$

$$F(x) = A_1\sin\left(\frac{\omega}{a}x + \varphi_1\right) , \tag{3.12}$$

where ω is the angular frequency of oscillation and A_i and φ_i are integration constants. Functions $\Phi_2(t)$, $F_1(x)$, and $F_2(x)$ can be obtained from the continuity equation (3.7). Recalling that $v_{osc} = \partial u/\partial t$, we obtain

$$F_2\frac{d\Phi_2}{dt} + \Phi_2\frac{d\Phi_1}{dt}\frac{dF}{dx} = 0 . \tag{3.13}$$

Therefore, the continuity equation splits into two equations

$$\frac{1}{F_2}\frac{dF}{dx} = \frac{1}{\Phi_2}\frac{\partial\Phi_2}{\partial\Phi_1} = \Theta , \tag{3.14}$$

where Θ is a constant. Solution of these equations, with account of Eqn (3.12), has the form

$$\Phi_2 = A_2\exp(-\Theta\Phi_1) ,$$

$$F_2 = \frac{A_1\omega}{\Theta a}\cos\left(\frac{\omega}{a}x + \varphi_1\right) . \tag{3.15}$$

The function F_1 is given by the condition

$$F_1 = \frac{F}{F_2} = \frac{a\Theta}{\omega}\tan\left(\frac{\omega}{a}x + \varphi_1\right) . \tag{3.16}$$

The system of functions (3.11), (3.15), and (3.16) allows the solution of Eqns (3.6) and (3.7):

$$S = F_2\Phi_2 = \frac{A_1A_2\,\omega}{\Theta a}\left|\cos\left(\frac{\omega}{a}x + \varphi_1\right)\right| \times \exp\left[-\Theta A_0\sin(\omega t + \varphi_0)\right] , \tag{3.17}$$

$$u = F_1\Phi_1 = \frac{a\Theta A_0}{\omega}\left|\tan\left(\frac{\omega}{a}x + \varphi_1\right)\right|\sin(\omega t + \varphi_0) . \tag{3.18}$$

In dealing with Eqns (3.6) and (3.7), it is necessary to remember that these equations show weak nonlinearity. They are linear in each of the individual parameters but also contain products of these parameters. This imposes some additional constraints on the solution of the system of equations (3.6), (3.7). The area of the elastic tube is a positive quantity which makes it necessary to take the absolute value of the function $\cos(\omega x/a + \varphi_1)$. The absolute value of the function $\tan(\omega x/a + \varphi_1)$ should be used, according to the continuity equation (3.7), also in expression (3.18).

Let us now determine the flow velocity and fluid pressure:

$$v = v_0 + \frac{\partial u}{\partial t} = v_0 + a\Theta A_0\left|\tan\left(\frac{\omega}{a}x + \varphi_1\right)\right|\cos(\omega t + \varphi_0) . \tag{3.19}$$

The pressure is deduced from relation (2.7):

$$P_t = C\,\frac{\partial(uS)}{S\,\partial x} = C\,\frac{\partial(F\Phi)}{F_2\Phi_2\,\partial x} = C\,\Theta\Phi_1\ ,$$

$$P_t = C\,\Theta A_0 \sin(\omega t + \varphi_0)\ . \qquad (3.20)$$

Not surprisingly, the pressure gradient is zero, the fluid being ideal one. Therefore, auto-oscillations of the tube cross-section area, flow velocity, and fluid pressure may occur in elastic tubes. Such auto-oscillations (flutter) are known to develop in blood vessels during arterial pressure reading with a sphygmomanometer. They are referred to as Korotkov's sounds.

If the maximum strain is denoted by ε_{max} and the amplitude of pressure oscillations $P_{t\,max} = C\varepsilon_{max}$, it follows from Eqn (3.20) that $\varepsilon_{max} = \Theta A_0$.

Take the following boundary conditions based on the model of arterial pressure measurement by the method of Korotkov (Fig. 3). At $x = 0$, we have $S = S_0$, $u = u_0 = 0$, and $v = v_0$ where S_0, u_0, and v_0 stand for the cross-section area, displacement, and flow velocity outside the cuff of the sphygmomanometer, respectively. It follows from Eqn (3.19) that $\varphi_1 = 0$. Taking as initial conditions $u|_{t=0} = u_0 = 0$ and $S|_{t=0} = S_0$, we obtain $\varphi_0 = 0$ from Eqn (3.18) and $A_1A_2\omega/(\Theta a) = S_0$ from Eqn (3.17). Hence, the final solution of the system of equations (3.6), (3.7) at the given initial and boundary conditions has the form

$$u = \varepsilon_{max}\,\frac{a}{\omega}\left|\tan\left(\frac{\omega}{a}\,x\right)\right|\,\sin(\omega t)\ ,$$

$$v = v_0 + a\varepsilon_{max}\left|\tan\left(\frac{\omega}{a}\,x\right)\right|\,\cos(\omega t)\ ,$$

$$S = S_0\left|\cos\left(\frac{\omega}{a}\,x\right)\right|\,\exp\left[-\varepsilon_{max}\sin(\omega t)\right]\ ,$$

$$P_t = C\varepsilon_{max}\,\sin(\omega t)\ . \qquad (3.21)$$

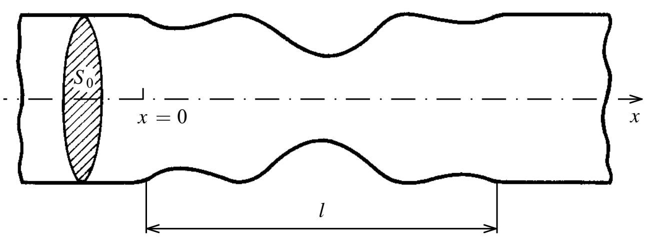

Figure 3. Auto-oscillations in elastic tube walls (see diagram of arterial pressure measurement by the method of Korotkov. The cuff of the sphygmomanometer is not shown).

Fig. 4 shows the variation of the elastic tube cross-section area and the fluid flow velocity along the tube, at the moment of time corresponding to $\sin(\omega t) = 0$ and $\cos(\omega t) = 1$, i.e. at $t = 2k\pi/\omega$ where $k = 0, 1, 2, \ldots$ The plots are based on formulas (3.21). For the sake of convenience, $\omega x/a$ is shown on the axis of abscissas.

Fig. 4 demonstrates that the flow velocity increases to infinity on tube occlusion, when $S = 0$. This is due to the fact that the formulas for fluid velocity and tube cross-section area pertain to inviscid fluid. A similar effect is encountered in the case of forced oscillations when their amplitude at resonance tends to infinity owing to the

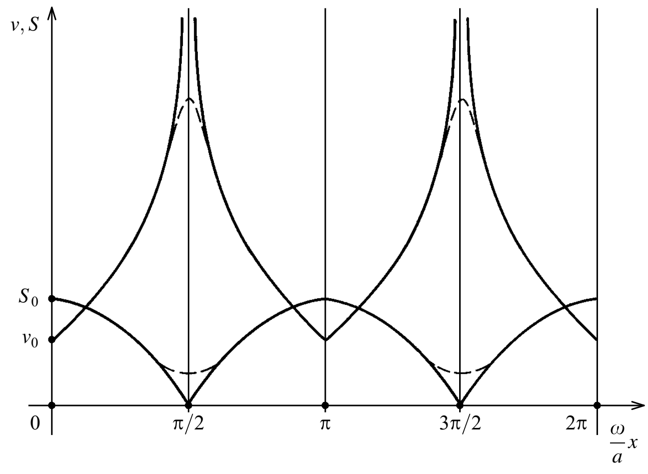

Figure 4. Calculated curves showing the variation of the elastic tube cross-section area and flow velocity along the tube. Solid and dashed lines relate to ideal and viscous fluids, respectively.

absence of damping. Dashed lines show real behaviour after correction for fluid viscosity.

Let us determine the possible auto-oscillation frequencies during arterial pressure measurement by the method of Korotkov. With the length of the sphygmomanometer cuff denoted by l, the additional boundary condition is that at $x = l$ at any instant of time $v = v_0$. It follows from Eqn (3.21) that $\omega l/a = k\pi$, where $k = 0, 1, 2, \ldots$, i.e. k is an integer. Hence, the auto-oscillation frequency is

$$\nu = \frac{ka}{2l} = \frac{k}{2l}\sqrt{\frac{Eh}{\rho d}}\ . \qquad (3.22)$$

The latter equation allows effective modulus of elasticity of the vascular wall to be determined in vivo by measuring the frequency of Korotkov's sounds.

4. Modelling solitons in an elastic tube with the use of the Korteveg – de Vries equation

Soliton propagation will be examined by using the model of sudden injection of a fluid volume into an elastic thin-walled tube. The continuity equation for this case with axial-symmetry may be written as follows [10]:

$$\frac{\partial(vr)}{\partial x} + \frac{\partial(wr)}{\partial r} = 0\ , \qquad (4.1)$$

where v and w are the longitudinal and transversal components of fluid velocity, and x and r are the longitudinal and radial coordinates, respectively (Fig. 5).

The impulse equation has the form

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x} + w\,\frac{\partial w}{\partial x} = -\frac{1}{\rho(S_0 + S)}\,\frac{\partial[P(S_0 + S)]}{\partial x}\ . \qquad (4.2)$$

S_0 in Eqn (4.2) is the cross-section area of the undisturbed elastic tube and S is the additional area due to inflation of the tube by the propagating wave. It is also assumed that no vortices are formed in the flow, i.e. $\operatorname{curl} v = 0$ where v is the velocity vector.

Solution of the system of equations (4.1), (4.2) will be sought by using the velocity potential $\varphi = \varphi(x, r, t)$:

$$v = \frac{\partial\varphi}{\partial x}\ , \quad w = \frac{\partial\varphi}{\partial r}\ . \qquad (4.3)$$

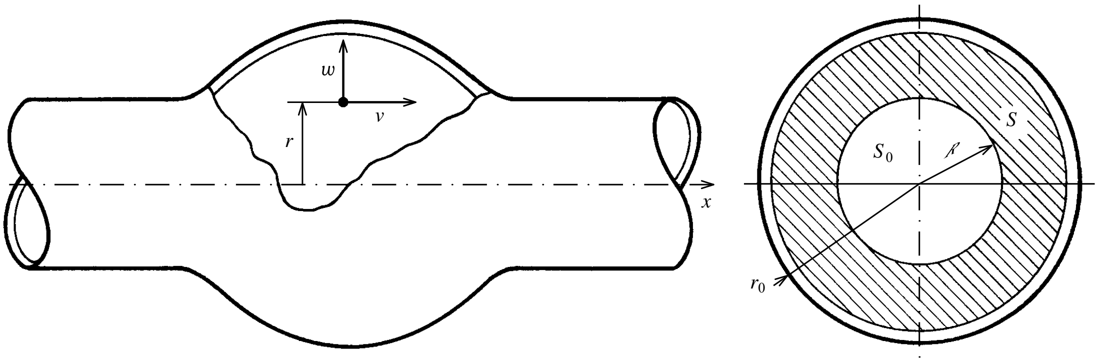

Figure 5. Schematic representation of soliton (pulse wave) propagation in an elastic tube.

We shall introduce dimensionless variables:

$$x = M_x x^* , \quad r = M_r r^* , \quad S = M_S S^* , \quad P = M_P P^* ,$$
$$\varphi = M_\varphi \varphi^* , \quad t = M_t t^* , \quad v = M_v v^* . \tag{4.4}$$

Let $S_{\max}/S = \varepsilon$, where $S_{\max}$ is the maximum incremental increase of area S produced by the soliton. We shall choose dimensions that correspond to those of real functioning blood vessels in the body. In other words, we shall assume that the amplitude of the soliton (pulse wave) is much less than its diameter:

$$M_S = \varepsilon S_0 = S_{\max} , \quad M_x = \sqrt{\frac{S_0}{\varepsilon}} , \quad M_r = \sqrt{S_0} .$$

Let us now expand the velocity potential into a series in powers of r^* :

$$\varphi^* = \sum_{n=0}^{\infty} r^{*n} \varphi_n^*(x^*, t^*) . \tag{4.5}$$

Substitution of Eqn (4.5) into expression (4.1) with the use of Eqns (4.3) and (4.4) and comparison of coefficients for equal powers of r^* yields:

$$\varphi_{n+2}^* = -\frac{\varepsilon}{(n+2)^2} \frac{\partial^2 \varphi_n^*}{\partial x^{*2}} . \tag{4.6}$$

From the symmetry condition we have that at $r^* = 0$ on the flow axis $w = 0$ and $\partial\varphi^*/\partial r^* = 0$. From Eqn (4.5) we find $\varphi_1^* = 0$ and taking into account Eqn (4.6) we obtain $\varphi_{2n+1}^* = 0$. Moreover, if φ_0^* is the dimensionless velocity potential at the flow axis, then

$$\varphi^* = \varphi_0^* - r^{*2} \frac{\varepsilon}{2^2} \frac{\partial^2 \varphi_0^*}{\partial x^{*2}} + r^{*4} \frac{\varepsilon^2}{2^2 \times 4^2} \frac{\partial^4 \varphi_0^*}{\partial x^{*4}} - \ldots \tag{4.7}$$

To find expansion of the impulse equation (4.2) in dimensionless form, we shall make values of the parameters obey the conditions

$$\frac{M_\varphi M_t}{S_0} = \frac{M_P M_t}{\rho M_\varphi} = 1 . \tag{4.8}$$

Introducing dimensionless velocity at the flow axis $v_0^* = \partial\varphi_0^*/\partial x^*$, then performing rather simple but cumbersome transformations using Eqn (4.7), and neglecting terms of the order ε^2 or higher we obtain

$$\frac{\partial v_0^*}{\partial t^*} + \frac{\partial P^*}{\partial x^*} + \varepsilon \left[S^* \frac{\partial v_0^*}{\partial t^*} + \frac{\partial}{\partial x^*} \left(\frac{v_0^{*2}}{2} \right) + \frac{\partial P^* S^*}{\partial x^*} - \frac{r^{*2}}{2^2} \frac{\partial^3 v_0^*}{\partial t^* \partial x^{*2}} \right] = \mathrm{O}(\varepsilon^2) . \tag{4.9}$$

In deriving Eqn (4.9), we took into consideration that

$$v^2 + w^2 = \frac{M_\varphi^2}{S_0} \left[\varepsilon \left(\frac{\partial \varphi^*}{\partial x} \right)^2 + \left(\frac{\partial \varphi^*}{\partial r^*} \right)^2 \right] = \frac{M_\varphi^2}{S_0} \varepsilon \left(\frac{\partial \varphi^*}{\partial x} \right)^2 ,$$

since $(\partial\varphi^*/\partial r^*)^2 \sim \varepsilon^2$.

Let us apply Eqn (4.9) to the inner surface of the elastic tube. The dimensionless radius of the tube is $r_0^* = r_0/M_r = r_0/\sqrt{S_0}$. In the unperturbed state, $r_0^* = 1/\sqrt{\pi}$. When the amplitude of the soliton is small, i.e. $\varepsilon = S_{\max}/S_0 \ll 1$, it may be assumed that $r^* \approx r_0^*$. Moreover, it should be borne in mind that the time scale for blood flow is substantially different from that for the pulse wave. For this reason, correct description of soliton propagation requires the introduction of 'slow time' [11]

$$\tau = \frac{\varepsilon}{8\pi} t^* ,$$

where the coefficient $1/8\pi$ is introduced to facilitate further transformations. Taking into account that $v_0^* = v_0^*(t^*, \tau)$, we obtain

$$\frac{\partial v_0^*}{\partial t^*} + \frac{\varepsilon}{8\pi} \frac{\partial v_0^*}{\partial \tau}$$

instead of $\partial v_0^*/\partial t^*$, and then from Eqn (4.9)

$$\frac{\partial v_0^*}{\partial t^*} + \frac{\partial P^*}{\partial x^*} + \varepsilon \left[\frac{1}{8\pi} \frac{\partial v_0^*}{\partial \tau} + S^* \frac{\partial v_0^*}{\partial t^*} + v_0^* \frac{\partial v_0^*}{\partial x^*} + \frac{\partial P^* S^*}{\partial x^*} - \frac{1}{4\pi} \frac{\partial^3 v_0^*}{\partial t^* \partial x^{*2}} \right] = \mathrm{O}(\varepsilon^2) . \tag{4.10}$$

When the cross-section of the elastic tube is not changed very quickly under the action of the propagating soliton, the relationship between excess pressure in the tube and its cross-section area S can be found from Eqn (2.1):

$$P = C \frac{S}{S_0} . \tag{4.11}$$

For the sake of convenience, in Eqn (4.11) and further on in this section the index t is omitted, and the additional area produced by inflation is denoted by S instead of the symbol for incremental increase ΔS [see Eqn (4.2)]. Hooke's law in dimensionless form may be represented as

$$P^* = \frac{C M_S}{S_0 M_P} S^* = \frac{C\varepsilon}{M_P} S^* = S^* . \tag{4.12}$$

In Eqn (4.12), for the pressure scale we have chosen $M_P = C\varepsilon$. Substitution of this expression into Eqn (4.10) gives

$$\frac{\partial v_0^*}{\partial t^*} + \frac{\partial S^*}{\partial x^*} + \varepsilon\left[\frac{1}{8\pi}\frac{\partial v_0^*}{\partial \tau} + v_0^*\frac{\partial v_0^*}{\partial x^*} + S^*\left(\frac{\partial v_0^*}{\partial t^*} + 2\frac{\partial S^*}{\partial x^*}\right) - \frac{1}{4\pi}\frac{\partial^3 v_0^*}{\partial t^*\,\partial x^{*2}}\right] = \mathrm{O}(\varepsilon^2)\,. \quad (4.13)$$

In Eqn (4.13), the variables can be separated by introducing new, independent ones:

$$r_- = x^* - t^*\,, \qquad l_+ = x^* + t^*\,. \quad (4.14)$$

We shall look for solution of Eqn (4.13) in the form

$$S^* = \frac{3}{4\pi}\left[f(r_-;\,\tau) + g(l_+;\,\tau)\right],$$

$$v_0^* = \frac{3}{4\pi}\left[f(r_-;\,\tau) - g(l_+;\,\tau)\right]. \quad (4.15)$$

Substitution of Eqns (4.14) and (4.15) into Eqn (4.13) yields after some simple but cumbersome transformations

$$\varepsilon\left[\left(\frac{\partial f}{\partial \tau} + 6f\frac{\partial f}{\partial r_-} + \frac{\partial^3 f}{\partial r_-^3}\right) + \left(-\frac{\partial g}{\partial \tau} + 6g\frac{\partial g}{\partial l_+} + \frac{\partial^3 g}{\partial l_+^3}\right)\right] = \mathrm{O}(\varepsilon^2)\,. \quad (4.16)$$

Since the terms in brackets include a number of dependent and independent variables, Eqn (4.16) is in fact a sum of two independent Korteveg–de Vries equations in which the first expression in parentheses describes a soliton travelling to the right and the second a soliton propagating to the left. These waves are separated in space and do not affect each other:

$$\frac{\partial f}{\partial \tau} + 6f\frac{\partial f}{\partial r_-} + \frac{\partial^3 f}{\partial r_-^3} = 0\,,$$

$$-\frac{\partial g}{\partial \tau} + 6g\frac{\partial g}{\partial l_+} + \frac{\partial^3 g}{\partial l_+^3} = 0\,. \quad (4.17)$$

Pulse waves described by the Korteveg–de Vries equations are solitons because they have some properties that are characteristic of particles [11].

Soliton g travelling to the left has no physical sense as far as the cardiovascular system is concerned. Removal of function g from formulas (4.15) indicates similarity of behaviour of flow velocity and tube cross-section area which is true of the pulse wave travelling in a blood vessel: where the blood flow starts, the cross-section area of the tube increases. Therefore, solution of Eqn (4.13) to the first order in ε does not obey the 'Bernoulli law' postulating a fall in static pressure of the fluid with increasing flow velocity. What happens is the reverse: as the fluid pressure increases so do the cross-section area of the elastic tube and the flow velocity within it.

Solution of Eqn (4.17) for f has the form [11]:

$$f = \frac{2k^{*2}}{\cosh^2\left[k^*(r_- - 4k^{*2}\,\tau - r_{-0})\right]}\,, \quad (4.18)$$

where $2k^{*2}$ is the amplitude of the pulse wave and r_{-0} is a constant. Changing over to variables x^* and t^* gives

$$f = \frac{2k^{*2}}{\cosh^2\left[k^*(x^* - t^* - k^{*2}\,\varepsilon t^*/2\pi - r_{-0})\right]}\,. \quad (4.19)$$

In accordance with formula (4.15), solution of Eqn (4.19) is valid for both v_0^* and S^*. First, let us examine a soliton with area S^*. Dimensionless amplitude of the wave may be defined as

$$2k^{*2} = f_{\max} = \frac{4\pi}{3}\,S^*_{\max} = \frac{4\pi}{3}\,\frac{S_{\max}}{M_S} = \frac{4\pi}{3}\,. \quad (4.20)$$

Hence, the dimensionless wavenumber $k^* = \sqrt{2\pi/3}$ and the total cross-section area of the soliton is

$$S_s = S_0 + S = S_0 + S_{\max}\left\{\cosh\left[\sqrt{\frac{2\pi S_{\max}}{3S_0^2}}\left(x - x_0 - a\left(1 + \frac{S_{\max}}{3S_0}\right)t\right)\right]\right\}^{-2}, \quad (4.21)$$

where $a = M_x/M_t$. It follows from Eqn (4.21) that the wave propagates with a velocity

$$a_s = a\left(1 + \frac{S_{\max}}{3S_0}\right). \quad (4.22)$$

Let us examine the velocity soliton. Its amplitude is

$$2k^{*2} = f_{\max} = \frac{4\pi}{3}\,v^*_{0\max} = \frac{4\pi}{3}\,\frac{v_{0\max}}{M_v}\,,$$

where $v_{0\max}$ is the maximal velocity at the flow axis. Hence:

$$v_0 = v_{0\max}\left\{\cosh\left[\sqrt{\frac{2\pi S_{\max}}{3S_0^2}}\left(x - x_0 - a\left(1 + \frac{S_{\max}}{3S_0}\right)t\right)\right]\right\}^{-2}. \quad (4.23)$$

Note that formula (4.23) specifies the dependence of longitudinal fluid velocity at the flow axis on the coordinate x. However, in the soliton, because of the local increase of the elastic tube diameter, there is a distribution of ideal fluid velocities across the tube section which, to the first order in ε, can be found from formula (4.7). Bearing in mind Eqn (4.3), we have

$$v^* = v_0^* - r^{*2}\,\frac{\varepsilon}{2^2}\,\frac{\partial^2 v_0^*}{\partial x^{*2}}\,. \quad (4.24)$$

It follows from Eqn (4.24) that the velocity profile has a parabolic shape. Moreover, the ideal fluid is known to slip along the tube wall with a velocity v_{wall} which can be calculated with the use of condition $r = r_0$, where $r_0(x)$ is the internal radius of the zone where wave motion occurs and is a function of x. Changing back to variables with dimensions we have

$$v = v_0 - \frac{r_0^2}{4}\,\frac{\partial^2 v_0}{\partial x^2}\,.$$

Putting $x_0 = 0$ as the origin of the coordinates, we can calculate the additional volume of the fluid sucked in by the soliton (Fig. 6). In the case of aorta, this volume may be considered to be equivalent to the heartbeat volume, V_{hb}. Integration of Eqn (4.21) at $t = 0$ gives:

$$V_{\text{hb}} = 2\int_0^\infty \frac{S_{\max}}{\cosh^2(kx)}\,\mathrm{d}x = \frac{2S_{\max}}{k}\,\tanh(kx)\Big|_0^\infty = \frac{2S_{\max}}{k}\,.$$

Hence, $k = 2S_{\max}/V_{\text{hb}}$. Therefore, the shape of the soliton (Fig. 7) is

$$S_s = S_0 + \frac{S_{\max}}{\cosh^2\left[2(S_{\max}/V_{\text{hb}})(x - a_s t)\right]}\,. \quad (4.25)$$

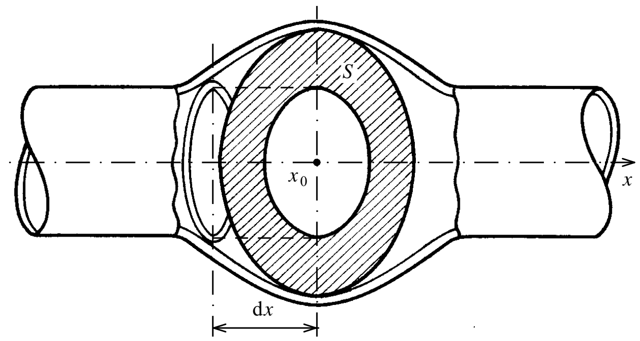

Figure 6. Calculation of the additional fluid volume $V_{\rm hb}$, sucked in by the soliton.

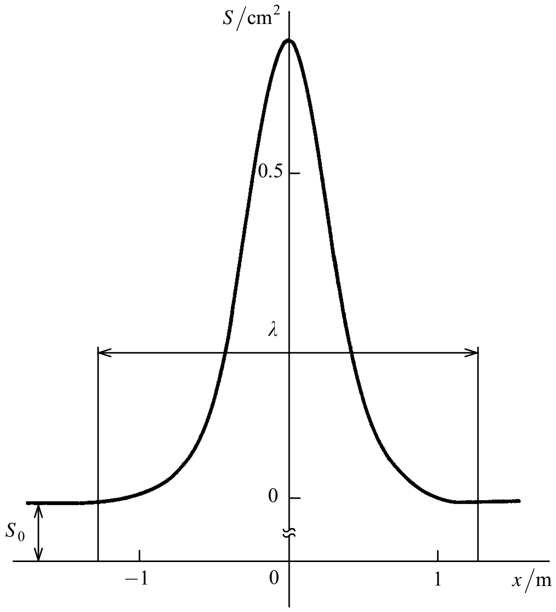

Figure 7. Variation of the cross-section area of an elastic tube caused by a soliton travelling along the tube.

The length of the soliton (pulse wave in the case of the artery) can be deduced from the condition $k = 2S_{\max}/V_{\rm hb} = 2\pi/\lambda$. Hence, $\lambda = \pi V_{\rm hb}/S_{\max}$. Assuming that for humans $V_{\rm hb} = 60$ cm^3 and $S_{\max} = 0.7$ cm^2, we obtain $\lambda = 2.7$ m, which is in good agreement with experimental findings [1].

According to Eqn (4.11), the expression for the pressure wave is fully analogous to the expression for the cross-section area of the elastic tube during soliton propagation:

$$P = \frac{P_{\max}}{\cosh^2\left[2(S_{\max}/V_{\rm hb})(x - a_s t)\right]}\,, \tag{4.26}$$

where $P_{\max} = CS_{\max}/S_0$.

We can now find the final velocity of the pressure wave (pulse wave) propagation. Making use of Eqn (4.8) and equality $M_P = C\varepsilon$ we obtain $M_\varphi = \sqrt{C\varepsilon S_0/\rho}$. Also, it follows from Eqn (4.8) that $M_t = S_0/M_\varphi = \sqrt{\rho S_0/(C\varepsilon)}$. Taking into consideration that

$$a = \frac{M_x}{M_t} = \frac{\sqrt{S_0/\varepsilon}}{\sqrt{\rho S_0/(C\varepsilon)}} = \sqrt{\frac{C}{\rho}}\,,$$

we find from Eqn (4.22)

$$a_s = \sqrt{\frac{C}{\rho}}\left(1 + \frac{S_{\max}}{3S_0}\right) = \sqrt{\frac{C}{\rho}}\left(1 + \frac{P_{\max}}{3C}\right). \tag{4.27}$$

To the leading order at $S_{\max} = 0$, expression (4.27) coincides with the approximate Moens–Korteveg formula for pulse wave velocity [1] if arterial wall elasticity is assumed to be $C = Eh/d$, where E is the modulus of elasticity, h is the wall thickness, and d is the average diameter of the vessel.

To conclude this section, let us derive the dispersion relation for a soliton, starting with the definition of group velocity:

$$v_{\rm gr} = \frac{d\omega}{dk} = a_s = a\left(1 + \frac{S_{\max}}{3S_0}\right),$$

where ω is the angular frequency of the oscillation. Bearing in mind that the wavenumber $k = (2\pi S_{\max}/3S_0^2)^{1/2}$ and that $\omega = 0$ at $k = 0$ we obtain

$$\omega = a\left(k + \frac{S_0}{6\pi}\,k^3\right). \tag{4.28}$$

5. Modelling solitons in an elastic tube with the use of the nonlinear Schrodinger equation

Let us examine the impulse equation in the form of expression (3.3) assuming that the transversal component of velocity w is small. Then, the impulse equation has the form

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x} + \frac{1}{\rho}\,\frac{\partial(PS)}{S\,\partial x} = 0\,. \tag{5.1}$$

The continuity equation can be written in the form of Eqn (3.7) without distinguishing between the main and oscillatory components of velocity, that is ignoring what was done in Eqn (2.6):

$$\frac{\partial S}{\partial t} + \frac{\partial(vS)}{\partial x} = 0\,. \tag{5.2}$$

The relationship between excess pressure in the elastic tube and its cross-section area can be defined by analogy with Eqn (2.2):

$$P_{\rm t} = C\,\frac{\Delta S}{S}\,. \tag{5.3}$$

In Eqn (5.3), the area differential dS is replaced by the area increment $\Delta S = S - S_0$ to facilitate the use of Hooke's law in the impulse equation (5.1).

Solution of the system of equations (5.1), (5.2) is attempted by using the complex velocity potential $\varphi = \varphi(x, t)$. This is expanded in powers of the small parameter λ by analogy with the transition from the Schrödinger equation to the Hamilton–Jacobi equation in quantum mechanics [12]:

$$\varphi = \varphi_0 + \frac{\lambda}{\rm i}\,\varphi_1 + \left(\frac{\lambda}{\rm i}\right)^2 \varphi_2 + \ldots\,, \tag{5.4}$$

where i is the square root of -1. We now define function Φ

$$\Phi = \exp\left(\frac{\rm i}{\lambda}\,\varphi\right). \tag{5.5}$$

Use of the first two terms of expansion (5.4) leads to

$$\Phi = |\Phi| \exp\left(\frac{\mathrm{i}}{\lambda}\,\varphi_0\right) . \tag{5.6}$$

In this expression, $|\Phi| = \exp(\varphi_1)$ is the modulus of function Φ; $V = \partial\varphi_0/\partial x$ is fluid velocity in the tube since φ_0 is (to within λ^2) the real part of the velocity potential.

Putting $|\Phi| = \sqrt{S/S_0}$ and using Eqn (5.3) we obtain

$$PS = -C(S - S_0) = -CS_0\left(|\Phi|^2 - 1\right) . \tag{5.7}$$

As mentioned in Section 2 the minus sign in Eqn (5.7) reflects the well-known fact that the reaction of the elastic element has to be substituted in the impulse equation (Newton's second law) to obtain the oscillation equation. In our case the reaction of the tube wall is equal to the excess internal pressure with a change of the sign.

Taking into account Eqn (5.7), we can write the last term in Eqn (5.1) in the form

$$\frac{1}{\rho}\,\frac{\partial(PS)}{S\,\partial x} = -2a^2\,\frac{\partial\left(\ln|\Phi|\right)}{\partial x} = -2a^2\,\frac{\partial\varphi_1}{\partial x} , \tag{5.8}$$

where $a = \sqrt{C/\rho}$ is the pressure wave velocity in the elastic tube and $\varphi_1 = \ln|\Phi|$.

Using the velocity potential, we can integrate once the impulse equation (5.1). Then the system of equations (5.1), (5.2) takes on the form

$$\frac{\partial\varphi_0}{\partial t} + \frac{v^2}{2} - 2a^2\varphi_1 = \mathrm{O}(\lambda^2) , \tag{5.9}$$

$$\frac{\partial|\Phi|^2}{\partial t} + \frac{\partial\left(v|\Phi|^2\right)}{\partial x} = 0 . \tag{5.10}$$

The integration constant in Eqn (5.9) is made zero by choosing an appropriate initial value of the potential φ_0 [9].

Now we shall demonstrate that the system of equations (5.9), (5.10) is equivalent to the nonlinear Schrödinger equation

$$\mathrm{i}\,\frac{\partial\Phi}{\partial t} + \frac{\lambda}{2}\,\frac{\partial^2\Phi}{\partial x^2} = -\frac{2a^2}{\lambda}\,\varphi_1\Phi . \tag{5.11}$$

Taking into account, in accordance with Eqn (5.5), that

$$\frac{\partial\Phi}{\partial t} = \frac{\mathrm{i}}{\lambda}\,\Phi\,\frac{\partial\varphi}{\partial t} ,$$

$$\frac{\partial^2\Phi}{\partial x^2} = -\frac{1}{\lambda^2}\,\Phi\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{\mathrm{i}}{\lambda}\,\Phi\,\frac{\partial^2\varphi}{\partial x^2} , \tag{5.12}$$

and separating the real and imaginary parts of Eqn (5.11), we obtain:

$$\frac{\partial\varphi_0}{\partial t} + \frac{1}{2}\left(\frac{\partial\varphi_0}{\partial x}\right)^2 - 2a^2\varphi_1 = \frac{\lambda^2}{2}\left[\left(\frac{\partial\varphi_1}{\partial x}\right)^2 + \frac{\partial^2\varphi_1}{\partial x^2}\right] = \mathrm{O}(\lambda^2) , \tag{5.13}$$

$$\frac{\partial\varphi_1}{\partial t} + \frac{\partial\varphi_0}{\partial x}\,\frac{\partial\varphi_1}{\partial x} + \frac{1}{2}\,\frac{\partial^2\varphi_0}{\partial x^2} = 0 . \tag{5.14}$$

Eqn (5.14) is fully equivalent to Eqn (5.10) since

$$\frac{\partial\varphi_1}{\partial t} = \frac{1}{2|\Phi|^2}\,\frac{\partial|\Phi|^2}{\partial t} , \qquad \frac{\partial\varphi_1}{\partial x} = \frac{1}{2|\Phi|^2}\,\frac{\partial|\Phi|^2}{\partial x} .$$

As with expansion (5.4), only terms linear in the small parameter λ are left in Eqn (5.13). Therefore, Eqns (5.13), (5.14), and hence (5.11) are fully equivalent to the system of equations (5.9), (5.10).

The nonlinear Schrodinger equation can be written in the form

$$\mathrm{i}\,\frac{\partial\Phi}{\partial t} + \frac{a}{k}\,\frac{\partial^2\Phi}{\partial x^2} = -\omega\,\Phi\ln|\Phi| , \tag{5.15}$$

where $\omega = 2a^2/\lambda$ is the angular frequency and $k = 2a/\lambda$ is the wavenumber. Solution of Eqn (5.15) is sought, as described in Ref. [13], in the form

$$\Phi = f(kx - \omega t)\exp\left[\mathrm{i}(rx - \delta t)\right] , \tag{5.16}$$

where the constants r and δ and the function $f(kx - \omega t)$ are as yet unknown. Taking into account Eqn (5.6), we conclude that $|\Phi| = f(kx - \omega t)$. Substitution of Eqn (5.16) into expression (5.15) gives

$$ak\,\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} + \mathrm{i}\,\frac{\mathrm{d}f}{\mathrm{d}\xi}\,(-\omega + 2ar) + f\left(\delta - \frac{ar^2}{k}\right) + \omega f\ln f = 0 . \tag{5.17}$$

The derivatives in Eqn (5.17) are with respect to the variable $\xi = kx - \omega t$. Since the function $f = |\Phi|$ is a real variable, Eqn (5.17) cannot contain imaginary terms. Putting $r = \omega/(2a)$ and recalling that $\omega = ak$ we find

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} + f\left(\frac{\delta}{\omega} - \frac{1}{4}\right) + f\ln f = 0 . \tag{5.18}$$

A similar equation was first examined in Ref. [14] where vortex structures in an ideal fluid were investigated.

Solution of Eqn (5.18) may be found in the form

$$f = C_1\exp\left[\frac{C_2}{2}\,(kx - \omega t)^2\right] , \tag{5.19}$$

where C_1 and C_2 are constants. Substitution of Eqn (5.19) into Eqn (5.18) gives

$$C_2 + \left(\frac{\delta}{\omega} - \frac{1}{4}\right) + \ln C_1 + (kx - \omega t)^2\left(C_2^2 + \frac{C_2}{2}\right) = 0 . \tag{5.20}$$

The last term in Eqn (5.20) must be independent of coordinate x and time t, i.e. $C_2 = -1/2$, and then $C_1 = \exp(3/4 - \delta/\omega)$. Hence,

$$f = |\Phi| = \exp\left(\frac{3}{4} - \frac{\delta}{\omega}\right)\exp\left[-\frac{(kx - \omega t)^2}{4}\right] . \tag{5.21}$$

Taking into account that the cross-section area $S = S_0|\Phi|^2$ with the boundary condition $S|_{x\to\pm\infty} = S_0$, we finally obtain the shape of the soliton propagating in the elastic tube:

$$S = S_0 + \Delta S_{\mathrm{a}}\exp\left[-\frac{(kx - \omega t)^2}{2}\right] , \tag{5.22}$$

where ΔS_{a} corresponds to the maximum value of the additional area in Eqn (5.3) and may be represented as

$$\Delta S_{\mathrm{a}} = S_0\exp\left(\frac{3}{2} - \frac{2\delta}{\omega}\right) .$$

In this equation, δ is a characteristic of the elastic tube material. At high values of δ, which are characteristic for rigid tubes, ΔS_a tends to zero, whereas low values of δ are peculiar to elastic tubes.

The excess pressure in the tube is determined from Eqn (5.3):

$$P_t = C\,\frac{S - S_0}{S} = C\,\frac{\Delta S_a}{S}\exp\left[-\frac{(kx - \omega t)^2}{2}\right].$$

Taking into account that $P_{max} = C\,\Delta S_a/S$, we finally obtain the maximum excess pressure in the soliton:

$$P_t = P_{max}\exp\left[-\frac{(kx - \omega t)^2}{2}\right]. \qquad (5.23)$$

Thus, it follows from Eqns (5.22) and (5.23) that the laws describing the variation of the cross-section area of the elastic tube and of the pressure are similar, while the shape of the soliton corresponds to a Gaussian curve.

Fig. 8 shows the difference between the soliton shapes as described by the Korteveg–de Vries equation and the Gaussian curve.

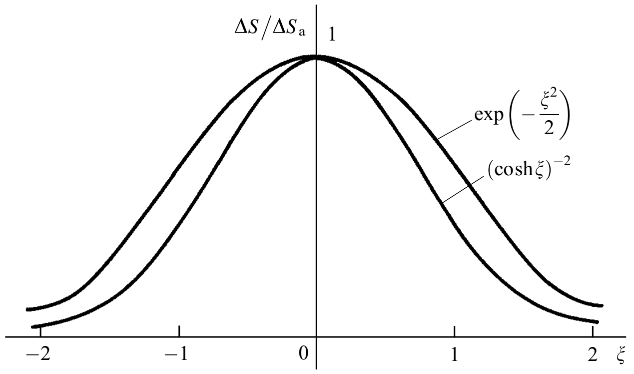

Figure 8. Difference between the shapes of solitons described by the Korteveg–de Vries model and the nonlinear Schrodinger equation ($\xi = kx - \omega t$).

6. Use of the passive elastic tube model for the approximate description of a blood vessel

Investigations into basic aspects of circulation, that is blood flow in arteries and veins of the body, have a long history. They have been largely developing along two major lines: one involves extensive physiological studies with theoretical interpretation of numerous experimental findings (see, for example, Refs [15, 16]), the other is based on a more specific biophysical approach and its aim is to roughly calculate the blood flow velocity in the vessels [1, 2]. The two lines overlap, and many authors have not infrequently used both approaches in one study (see Refs [3, 17]).

The flow of blood in blood vessels of a living organism is a complicated process, largely dependent on mechanisms of vascular regulation. This regulation takes place at several levels including self-regulation, central and autonomous neural reflex regulation, hormonal control, etc. [18]. Therefore, the blood vessel can by no means be regarded as a passive elastic tube.

Another important feature of blood flow is that blood behaves like a structured non-Newtonian visco-plastic fluid [6]. The use of equations for the ideal fluid does not bring modelling as described above closer to reality. Therefore it should be pointed out that the existing biomechanical models of blood flow in major vessels can describe only some phenomena peculiar to the cardiovascular system. Let us briefly review a few of them.

It has been demonstrated in Section 3 that an auto-oscillating regime of fluid flow may develop in a passive elastic tube. This is in fact known to occur under some conditions in blood vessels of the living organism. For example, high-frequency oscillations of up to 10 Hz have been reported to occur, imposed on the pulse wave, in equine coronary arteries [19]. A similar observation was made in experiments on canine aorta when the mechanisms controlling the blood flow were disturbed [7]. However, the most common example of an auto-oscillating regime in the circulatory system is probably provided by high-frequency oscillations generated in arterial vessels during blood pressure measurement by the method of Korotkov. Biomechanical solution of this problem is given in Section 3. Analysis of the variations of the frequencies of the Korotkov sounds allows changes in the modulus of elasticity of the vascular wall to be estimated in vivo.

Another important haemodynamic phenomenon is the occurrence of the pulse wave. Results of modelling reported in Sections 4 and 5 indicate that this phenomenon is entirely due to the nonlinearity of hydrodynamic equations the solution of which is in the form of solitons. Since it is the first order term of the expansion of hydrodynamic equations that is of primary importance in the description of the pulse wave, the 'Bernoulli law' associated with the zero order term does not play a major role in pulse propagation in circulating blood. Synchronous changes of intravascular pressure, cross-section area of the vessel, and blood flow velocity are a well-known physiological phenomenon.

7. Conclusion

In spite of the limitations of biomechanical models of blood flow in major vessels discussed in the previous sections, these models allow certain well-known physiological phenomena to be explained, such as the occurrence of high frequency oscillations in blood vessels and specific features of pulse wave propagation related to the nonlinearity of the equations of hydrodynamics.

Modelling ideal fluid flow in an elastic tube requires the application of the modified Euler equation apparently first suggested by Zhukovsky in the form of expression (3.3).

The possibility of establishment of an auto-oscillating regime, and hence enhanced hydraulic resistance, in passive elastic tubes places additional demands on vascular prostheses which must be made of sufficiently hard materials. Manifestations of the auto-oscillating regime are diverse and may include noise, murmur, etc.

In designing instruments and equipment incorporating elastic thin-walled tubes, e.g. pulse pumps for biotechnology or artificial blood circulation machines, it is necessary to take into account resonance phenomena in fluid flow which may cause tube occlusion and may produce reflected waves interfering with the normal functioning of the equipment.

Nonlinear modelling of solitons in elastic tubes confirms the hypothesis that nonlinear processes may give rise to qualitatively new features even after very minor quantitative changes in the parameters of the model. This suggestion is supported by the results of comparison between the pulse wave models based on the Korteveg – de Vries equation (see Section 4) and the nonlinear Schrodinger equation (Section 5). The use of Hooke's law in the form of Eqn (5.3), rather than (2.1), accounts for the appearance of a non-dispersing soliton that propagates at a velocity independent of the excess pressure in the elastic tube.

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