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Nonlinear long waves on water and solitons†

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Abstract. The water wave problem has been pivotal in the history of nonlinear wave theory. This problem is one of the most interesting and successful applications of nonlinear hydrodynamics. Waves on the free surface of a body of water (perfect liquid) have always been a fascinating subject, for they represent a familiar yet complex phenomenon, easy to observe but very difficult to describe! The archetypical model equations of Kordeweg and de Vries and of Boussinesq, for example, were originally derived as approximations for water waves, and research into the problem has been sustained vigorously up to the present day. In the present paper, the derivation of the model equations is given in depth and rational use is made of asymptotic methods. Indeed, it is important to understand that in some cases the derivation of these approximate equations is intuitive and heuristic. In fact, it is not clear how to insert the model equation under consideration into a hierarchy of rational approximations, which in turn result from the exact formulation of the selected water wave problem.

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†The Editorial Board of *Uspekhi Fizicheskikh Nauk* dedicates this review to the 30th anniversary of the publication of the paper by N B Zabusky and M D Kruskal [25], in which the term 'soliton' was mentioned for the first time in scientific literature. (Editorial Board's note.) The present paper is an 'extended' version of the author's recent review: "A quasi-one-dimensional asymptotic theory for non-linear water waves" published in *Journal of Engineering Mathematics* **28** 261 (1994). A more complete theory of "Nonlinear long surface waves in shallow water" is given in the author's preprint (Universite de Lille 1, Laboratoire de Mecanique de Lille, 224 pages, 1993). (Author's note.)

1. Introduction

The wave motion, under the force of gravity, of a moving body of water with a free surface in a channel with an uneven bottom is one of the most interesting and successful applications of nonlinear hydrodynamics.

Studies of water waves have always been enriched by the interest coming from diverse fields of science, including applied mathematics and singular perturbation techniques. Indeed, waves on the free surface of a body of water have always been a fascinating subject, for they represent a familar yet complex phenomenon, easy to observe but very difficult to describe!

Given that the wave motion of an inviscid and incompressible liquid (such as water) is irrotational, it would be the obvious choice to derive the classical Laplace equation for the velocity potential $\phi(t,x,y,z)$. However, the Laplace (elliptic) equation has little to do with waves and this choice would be wrong, because of curious effects of the free-surface conditions. Indeed, there is one boundary condition for the Laplace equation, but only when the boundary is known ('classical' Dirichlet or Neumann problems).

In fact, two conditions are needed for a free (unknown) surface, $z=\zeta(t,x,y)$, because the surface position $\zeta(t,x,y)$, has to be determined as well as $\phi(t,x,y,z)$.

Moreover, although the Laplace equation is linear, the two boundary free-surface conditions are unfortunately nonlinear.

However, it is necessary to note that in the presence of a free surface, the vorticity of an inviscid and incompressible body of water does not necessarily remain zero if it is zero initially! Indeed, the free surface can intersect itself, which happens when a wave breaks and vortex sheets are formed. In this case, instead of the Laplace equation, it is necessary to consider the full Euler equations (this is always the case for an incompressible fluid). I shall not consider here this important, but very difficult, question and I shall analyse only the classical nonlinear problem for ϕ and ζ, when the effects of the surface tension and an uneven bottom are included.

Naturally, some degree of mathematical intractability seems inevitable in the initial-value water wave problem relating to ϕ and ζ. We recognise the probability that the initial-value problem cannot be correctly (well) posed: water waves may break! The (rotational) motion may become turbulent and so the continuous dependence on the initial data may be lost. In this case the emergence of chaos via a strange attractor is possible. This aspect of the subject still remains largely mysterious and caution regarding it is essential in order to put any theoretical work on water waves into a proper scientific perspective. The fact that most of the existing theory — dealing with linearised, long waves or with weakly nonlinear approximations — is essentially tentative does not, of course, impair its practical value.

1.1 Some historical notes

The first rigorous demonstration of the existence of a velocity-potential $\phi(t,x,y,z)$ for an inviscid fluid motion (Lagrange 1781 theorem [1]) is due to Cauchy (1815) [2]. Another proof was given by Stokes (1849) [3]. An excellent historical and critical account of the whole matter (was provided by Lamb (1932) [4].† A fuller proof of the general surface free slip condition

$$\frac{\mathrm{D}}{\mathrm{D}t}(z-\zeta)=0 \quad \text{on} \quad z=\zeta(t,x,y)\ ,$$

where $\mathrm{D}/\mathrm{D}t=\partial/\partial t+\phi_x\,\partial/\partial x+\phi_y\,\partial/\partial y+\phi_z\,\partial/\partial z$, is due to Lord Kelvin [see W Thomson (1848)] [6]. For the first investigation of progressive waves in a canal see Green (1839) [7] and also Airy's (1845) [8] treatise.

The theory of the (infinitesimally small) waves produced in deep water by a local disturbance of a free surface was investigated in two classical memoirs by Cauchy (1815) [2] and Poisson (1816) [9].‡

The determination of the waveforms which satisfy the conditions of uniform propagation without change of type, when the restriction to 'infinitesimally small' amplitude of waves is abandoned, forms the subject of the classical research by Stokes [3] and of many subsequent investigations (*Stokes expansion*). For this problem, see also Rayleigh's (1876) results [10]. The validity of the Stokes expansion requires that:

(a) the amplitude must be smaller than the wavelength;

(b) amplitude of water waves must be less than the depth or *the wave properties must vary little over a distance of the same order as the depth*.

It is interesting to note also that the convergence proofs of the Stokes expansion were given by Levi-Civita (1925) [12] and Struik (1926) [13]. But convergence does not imply stability (!) and the Stokes waves in deep water are unstable!

A system of *exact* equations, expressing a possible form of wave motion when the depth of the fluid is infinite, was given so long ago as 1802 by Gerstner [14], and at a later period independently by Rankine (1863) [15].

The '*shallow-water theory*' is governed by a system of equations favoured by Airy [8], who first formulated the limiting equations for the analysis of very long waves of finite amplitude in shallow water. However, the effects of the dispersion do not appear in the Airy equations. These dispersion effects are present in the Boussinesq (1871, 1872 and 1877) equations [16–19]. In the one-dimensional case, these Airy equations are the Saint-Venant (1871) [20] *hydraulic* equations.

Russell (1844) [21]§ in his interesting experimental investigations paid great attention to a particular type of wave which he called the *solitary wave*. This is a wave consisting of a *single elevation*, of height not necessarily small compared with the depth of the fluid, which (if properly started) *may travel for a considerable distance along a uniform canal with little or no change*. But his description of the wave as a solitary elevation of finite amplitude and constant profile contradicts Airy's shallow water theory prediction that a wave of finite amplitude cannot propagate without change of profile!

The conflict between Russell's observations and Airy's shallow water theory (and also Stokes' expansion, for oscillatory waves of constant profile) was resolved independently by Boussinesq [16–19] and Rayleigh [10], who showed that appropriate allowance for the vertical acceleration — which is ultimately responsible for dispersion, but

† For the steady-state version of the Bernoulli equation, see Bernoulli (1738) [5].

‡Concerning this problem see, also, the papers by Rayleigh (1883) [10] and Popoff (1858) [11].

§Many writers (see, for instance, Lamb [4], Section 252) identify Russell as *Scott Russell*, but the correct surname is simply *Russell* according to: Encyclopaedia Britannica (11th edition).

is neglected in the Airy's shallow water theory [see Miles (1980)] [22]—as well as for the finite amplitude, leads to the solution:

$$\zeta = a_0 \operatorname{sech}^2 \frac{x - ct}{\lambda_0}, \quad \varepsilon = \frac{a_0}{h_0} \ll 1, \quad \delta^2 = \left(\frac{h_0}{\lambda_0}\right)^2 = O(\varepsilon), \tag{1.1.1}$$

where a_0 is a characteristic amplitude [for the initial elevation of a free surface characterised by the function $\zeta^0(x/\lambda_0)$] and λ_0 is the characteristic wavelength, in the horizontal x-direction. Finally, $c = [g(h_0 + a_0)]^{1/2}$ is the wave velocity. If we introduce the *Froude number*, $Fr = c/(gh_0)^{1/2}$, then $(Fr)^2 = 1 + \varepsilon$.

The characteristic length λ_0 is determined by the Ursell criterion:

$$U = \frac{3\varepsilon}{4\delta^2} = 1, \tag{1.1.2}$$

and the essential quality of the solitary wave is then the *balance between nonlinearity and dispersion.*

The dimensionless parameter U appears in the work of Stokes [3]. However, its full significance as a measure of the nonlinearity/dispersion balance was first enunciated by Ursell (1953) [23]. Rayleigh's derivation [10] of the equivalents of Eqns (1.1.1) and (1.1.2) is reproduced by Lamb ([4], Section 252): it is more direct but less penetrating than that of Boussinesq (according to Miles [22]). As noted by Miles [22], Boussinesq, in his first paper on the solitary wave [16], only sketches the derivation of Eqn (1.1.1) for the profile of this wave. It is necessary to look into his 1871 supplementary paper [17] and into either his 1872 paper [18] or his 1877 essay [19] to obtain a fuller appreciation of his contributions. Lamb [4] refers only to the 1871a paper [16] and, at least in retrospect, appears to have underestimated the significance of Boussinesq's work! The *Boussinesq equations*, which in their conventional form are evolution equations for the free surface displacement and the mean horizontal velocity and are not restricted to unidirectional propagation, do not appear explicitly in the 1871 and 1872 Boussinesq papers. However, the Boussinesq equation (19) in Ref. [18] or equation (280) in the essay [19] are, after dropping several higher-order terms, equivalent equations for the free surface displacement and the horizontal velocity at the (flat) bottom of the channel. In fact, in place of these two equations, it is possible to derive the following *single Boussinesq equation* for $\zeta(t, x)$:

$$\zeta_{tt} = c_0^2\left(\zeta_{xx} + \frac{3}{2h_0}\zeta_{xx}^2 + \frac{1}{3}h_0^2\zeta_{xxxx}\right), \tag{1.1.3}$$

where $c_0^2 = gh_0$. The above equation is reduced below [Eqn (1.1.4)], to the classical Kordeweg and de Vries (KdV) equation [24] by factoring the operator $c_0^2\,\partial^2/\partial x^2 - \partial^2/\partial t^2$, invoking the prior assumption of unidirectional propagation, and integrating with respect to x:

$$\zeta_t + c_0\left(\frac{3}{2h_0}\zeta\zeta_x + \frac{1}{6}h_0^2\zeta_{xxx}\right) = 0. \tag{1.1.4}$$

This KdV equation admits only wave solutions moving to the right.

Interest waned after the resolution of the Airy–Stokes paradox by Boussinesq and Rayleigh and was sporadic prior to Zabusky and Kruskal's (1965) discovery that the *solitary waves typically dominate the asymptotic solution of the KdV equation* [25]. Current interest stems from that discovery and is intense (see Section 7.4). In fact the original form of the KdV equation (see p. 423 in the paper of Korteweg and de Vries [24]) is of the following form:

$$\zeta_t = \frac{3}{2}c_0\left(\zeta\zeta_x + \frac{2}{3}\alpha\zeta_x + \frac{1}{3}\sigma\zeta_{xxx}\right), \tag{1.1.5}$$

where α is a small but arbitrary constant, which is related closely to the exact velocity of the uniform motion imparted to the liquid; the parameter σ is of the form:

$$\sigma = \frac{1}{3}h_0^3 - \frac{h_0 T}{\rho_0 g},$$

and depends of the surface tension T of the liquid (of constant density ρ_0). Korteweg and de Vries, who apparently did not know of the work of Boussinesq and Rayleigh and who were still trying to answer the objections of Airy and Stokes,† derived in 1895 the unidirectional equation (1.1.5)—the *KdV equation.*

As is noted in Newell's book [27] (Chapter 1: "The history of the soliton"): "... In this first stage of discovery, the primary thrust was to establish the existence and resilience of the wave. The discovery of its universal nature and its additional properties was to await another day and an unexpected result from another experiment designed to answer a totally different question [*Fermi–Pasta–Ulam* (FPU) *experiment*]; see Newell's book [27], Section 1b...". Kruskal and Zabusky [25, 28–30] approached the FPU problem‡ from the continuum viewpoint and demonstrated that it is sufficient to consider the following *KdV equation*

$$u_t + uu_x + \kappa^2 u_{xxx} = 0. \tag{1.1.6}$$

They solved Eqn (1.1.6) with $u(x, 0) = \cos \pi x$, $0 \leqslant x \leqslant 2$, and u, u_x, u_{xx}, periodic in the interval $[0, 2]$ for all t; they chose $\kappa = 0.022$. A set of their results is shown in Fig. 1 ([31], p. 14).

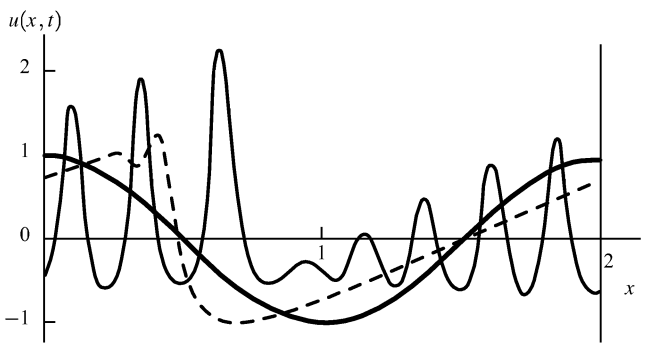

Figure 1. Solution of the periodic boundary-value problem for the KdV equation (1.1.6) [Zabusky and Kruskal (1965)] [25]. Showing the initial profile at $t = 0$ (thick line) the profile at $t = 1/\pi$ (broken line) and the profile at $t = 3.6/\pi$ (full line).

After a short time the wave steepens and almost produces a shock, but the dispersive term ($\kappa^2 u_{xxx}$) then becomes significant and some sort of local balance between nonlinearity and dispersion ensues. At later times the solution develops '*a train of eight well- defined waves*'

†Later on, Stokes (1891) was to recognise and admit his errors [26].

‡Why do solids have finite thermal conductivity? A solid is modelled by a one-dimensional *lattice*, a set of masses coupled by springs!

(see Fig. 1), each like sech^2 functions, with the faster (taller) waves continuously catching-up and overtaking the slower (shorter) waves. At the *heart* of these observations is the discovery that these nonlinear waves can *interact strongly* and then continue thereafter *almost* as if there had been no interaction at all. This persistence of the wave led Zabusky and Kruskal to coin the name *soliton* to emphasise the particle-like character of these waves *which seem to retain their identities in a collision*. The discovery has led, in turn, to an intense study over the last twenty five years. The *theory of solitons* is attractive and exciting: it brings together many branches of mathematics, some of which touch upon *profound* ideas and several of its aspects are amazing and beautiful (for instance I can mention the following important topics: the *conservation laws* and the *Miura transformation*, the *inverse scattering transform* (IST), the *Lax equation,* the *Backlund transformation,* the *Hirota method,*...).†

Naturally, when the nonlinear surface waves in weakly dispersing shallow water are *not strictly one-dimensional*, the KdV equation no longer applies! In fact, it is necessary to derive a new approximate model equation for this case.‡ This Kadomtsev–Petviashvili equation is of the following form:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+\frac{3c_0}{2h_0}\,u\frac{\partial u}{\partial x}+\frac{1}{6}\,c_0h_0^2\,\frac{\partial^3 u}{\partial x^3}\right)+\frac{1}{2}\,c_0\,\frac{\partial^2 u}{\partial y^2}=0\ . \quad (1.1.7)$$

According to the 1970 paper by Kadomtsev and Petviashvili (KP) [34], if the y dependence is *weak*, the KdV equation can be easily corrected by adding a small term. In their paper [34], KP deduced the form of this additional linear (!) term from consideration of the two-dimensional long-wave dispersion relation, but they did not verify that there were no additional nonlinear terms!

Now, it is necessary to note that, the IST and the structure of the KdV equation would have remained a mathematical curiosity, if further important model equations (for water waves!) had not been found to be solvable in this way. However, in 1972, in a paper of fundamental importance [35], Zakharov and Shabat showed that the nonlinear Schrodinger (NLS) equation,

$$-\mathrm{i}A_t+\alpha A_{xx}+\beta|A|^2A=0\ , \quad (1.1.8)$$

could also be solved by the IST for initial data which decayed sufficiently fast as $|x|\to\infty$. The NLS equation (1.1.8) for the water wave problem was derived first for the *finite* depth (classical problem) by Hasimoto and Ono (1972) [36]. A similar NLS equation was deduced earlier, but for the *infinite* depth, by Zakharov (1968) [37]. For two-dimensional surface water waves, in place of the NLS equation (1.1.8), Benney and Roskes (1969) [38] and Davey and Stewartson (1974) [39], derived a system of two equations, the *NLS–Poisson system of two equations*:

$$\begin{aligned}&\mathrm{i}A_t+\lambda A_{xx}+\mu A_{yy}=\chi|A|^2A+\chi_1AB_x\ ,\\&aB_{xx}+B_{yy}=-b|A|^2_x\ .\end{aligned} \quad (1.1.9)$$

For the capillary–gravity water waves (when we take into account the surface tension in a classical problem), expressions for the various constant coefficients in Eqn (1.1.9) were given by Djordjevic and Redekopp (1977) [40] and Ablowitz and Segur (1979) [41] (see, also the book by Craik [42], Chapter 6).

For the long waves (in shallow water), Freeman and Davey [33] derived a generalisation of the KP equation, which is valid as $\delta\to 0$ for finite (fixed) $\delta^2/\varepsilon=\kappa_0$.

If now $1/\kappa_0\to 0$, the long-wave limit (for $\delta\to 0$) of the system of equations (1.1.9) is recovered for $O(1/\kappa_0)$ after a further slight rescaling (*matching* between KP and NLS–Poisson equations, in long-wave limit). In fact, the double limit, $\varepsilon\to 0$ and $\delta\to 0$, is nonuniform(!) and the result depends on the sequence in which these limits are taken. However, Freeman and Davey [33] showed that the introduction of a similarity parameter $\Delta=1/k_0$, in place of ε, leads to a uniform double limit $\Delta\to 0$, $\delta\to 0$.§

For an *uneven* bottom of the channel it is also possible to derive the Boussinesq, KdV and KP equations.¶ In this case, Eqn (1.1.6) is replaced by the following equation:

$$u_t+uu_x+\kappa^2u_{xxx}=v(h)u\ , \quad (1.1.10)$$

where the function $v(h)$ represents the effects of variable depth. It has been found numerically and confirmed experimentally that a KdV soliton travelling from one constant depth to another constant but smaller depth, disintegrates into several solitons of varying sizes, trailed by an oscillatory tail. This 'fission' is clearly related to the result of the IST [see, Gardner et al. (1974)] [49] and the 'perturbed' KdV equation (1.1.10) predicts the soliton fission that occurs as a solitary wave moves into a shelving region [Madsen and Mei (1969)] [50]. In particular, the phenomenon of the shelf that appears behind the solitary wave is now well-understood [Krickerbocker and Newell (1980)] [51]. The soliton interactions in two dimensions are discussed in the review paper of Freeman (1980) [52].

The case of free-surface water waves in a channel with a *rough bottom*:

$$z=-h(x^*)\,,\qquad x^*=\frac{x}{\varepsilon^{1/2}}\,,\qquad \varepsilon=\frac{a_0}{h_0}\ll 1$$

is very interesting in the relation to the application of the multiple scale asymptotic method [Rosales and Papanicolaou (1983)] [53] and this method gives a surprising result: a KdV equation governing again the evolution of free-surface one-dimensional disturbances, as in the usual flat bottom case, but the coefficients in this KdV equation are not given explicitly! The determination of these coefficients requires solution of four auxiliary problems. In a recent paper by Benilov (1992) [54], three types of bottom topography are distinguished, allowing a simplification of the basic (two-dimensional) shallow-water wave equations and for two of them, asymptotic equations of KdV type are derived. In a paper by Xue-Nong Chen (1989) [55], a unified KP equation is derived asymptotically, in which *viscous* (when the effects of viscosity can be considered only in the boundary layer on the bottom), topographic, and transverse modulational effects are incorporated.

†I can recommend three books on soliton mathematics: Refs [27, 31, 32].

‡For a formal self-consistent derivation of the KP equation see the paper by Freeman and Davey (1975) [33].

§For the derivation of these evolution equations (KdV, KP, NLS and NLS–Poisson), see the books by Newell [27], Craik [42], Mei [43], Infeld and Rowlands [44] and the author's review paper [45, 64a].

¶The Boussinesq equations for a variable depth, are discussed by Peregrine (1967) [46]. The modified (by a variable depth) KdV equation is considered by Ono (1972) [47] and Johnson (1973) [48].

Finally, in Ref. [45], quasi-one-dimensional generalisations of different forms of the Boussinesq equations are asymptotically derived, the influence of the bottom topography on the KP equation is elucidated and a significant second-order approximation for the quasi-one-dimensional long nonlinear waves in shallow water is obtained. In this case it is possible to introduce the notion of a '*dressed KP soliton*'; for the notion of a 'dressed KdV soliton', that is a KdV soliton involving higher-order corrections, see the paper by Sugimoto and Kakutani (1977) [56] and the references cited in Jeffrey and Kawahara's book, Section 7.2 [57].

The boundary-value classical problem is extremely difficult, mostly because the boundary conditions on a free (unknown) surface are nonlinear and are imposed on an unknown boundary.

Some idea of the difficulty of the problem may be obtained by asking what is known about it. The simplest nontrivial statement that a mathematician can make about a physical problem that it has a solution!

According to Shinbrot's (1973) book [58], at the end of sixty years, there are only five situations in which this statement can be made about our classical problem. These situations are as follows.†

(1) $h = \infty$. In 1925, Levi-Civita proved that in water of infinite depth, there is a periodic wave that travels without a change in shape. This means that the velocity potential $\phi(t,x,z)$ does not depend on x and t separately, but only on a combination $(x - ct)$ for some constant c. The free-surface elevation $\zeta(t,x)$ also depends only on $(x - ct)$, while ϕ and ζ are both periodic functions of $(x - ct)$.

(2) $h = h_0 = \text{const}$. Shortly after Levi-Civita proved his result, Struik (1926) [13] showed that it could be generalised to the case of a flat horizontal bottom. Again, Struik proved the existence of a periodic wave travelling without a change in shape.

(3) *A solitary wave*. There was a long gap between Struik's result and the next step. In 1954, Friedrichs and Hyers [59] proved, again for $h = h_0 = \text{const}$, the existence of another type of wave, again travelling without a change in shape at a constant speed. This solitary wave can be looked on as a periodic wave, a la Struik, but with an infinite wavelength.

(4) *Waves over a periodic bottom*. If the bottom is periodic and has only one maximum and one minimum per period, Gerber (1955) [60] proved that there is a steady flow in which a free surface has the same properties. In addition, the troughs of the free surface lie directly over the troughs of the bottom, and the crests lie over the crests of the bottom.

(5) *Flow over a monotonic bottom*. In the same paper [60], Gerber proved also that over a monotonic bottom, there is a flow with a monotonic free surface. Again this can be looked upon as a flow over a periodic bottom with an infinite period.

All these examples are essentially examples of *steady* flows. The last two are steady to begin with. The first three become steady when observed in a coordinate system moving at a velocity c and all the above flows are *two-dimensional*. After sixty years, there *are no known unsteady or three-dimensional flows or theorems about existence of flows over 'general' bottoms*!

†Later, we shall consider additional information on the 'correctness' of our classical problem.

At last, it is also necessary to mention the *Whitham theory* of nonlinear dispersive systems. For a first account, the reader can see the "Epilogue" in Lighthill's book [61]. Whitham's book [62] includes a full account of nonlinear dispersive waves.

The nonlinear instability and bifurcation of water waves with special reference to the *Benjamin–Feir resonance mechanism* are discussed in Section 13 of a review paper by Debnath ([63]; pp 233–255). This section of Debnath's paper also includes, the *Whitham instability theory* of *deep water waves*, the nonlinear problem of the *instability of a finite-amplitude uniform wavetrain,* from the NLS equation, the FPU *recurrence phenomenon* and Longuet-Higgin's *bifurcation analysis* of gravity waves on deep water.

The problem of the *wave interactions*, is discussed in the book by Craik [42]: these interactions are represented mainly by the *three-wave resonance* driven by nonlinearities which are quadratic in waves amplitudes.

2. Mathematical formulation of the nonlinear theory of water waves

2.1 Master equations

In the absence of viscosity, tangential stresses in the fluid are zero everywhere and the stress tensor reduces to $-p\delta_{ij}$, where δ_{ij} is the Kronecker delta.

Therefore, the equation of motion becomes:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \frac{1}{\rho}\nabla p = \boldsymbol{g}\ , \tag{2.1.1}$$

where we assume that the body force acting on a liquid is only due to gravity $\boldsymbol{g} = (0, 0, -g)$. The momentum equation (2.1.1) and the mass conservation equation

$$\nabla\cdot\boldsymbol{u} = 0 \tag{2.1.2}$$

provide four scalar equations for the determination of $\boldsymbol{u}$, p and ρ as functions of independent variables x and t.

In general, one further scalar equation is needed, and it is usually the equation of state of a liquid. However, if the liquid behaves as if it were incompressible, we then have the additional equation

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0\ , \tag{2.1.3}$$

which is of course simply a particular form of the equation of state for our liquid.

The explicit use of Eqn (2.1.3) is often rendered unnecessary by a statement that the density is initially (at $t = 0$) homogeneous and consequently remains homogeneous.

Finally, for a 'really' (when $\rho = \rho_0 = \text{const}$) incompressible inviscid liquid, the set of equations

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \frac{1}{\rho_0}\nabla p = \boldsymbol{g}\ , \tag{2.1.4a}$$

$$\nabla\cdot\boldsymbol{u} = 0 \tag{2.1.4b}$$

is now sufficient for determination of the functions $\boldsymbol{u}$, p, provided adequate boundary and initial conditions are known (see Section 2.2).

If we start from the above incompressible system of equations (2.1.4), we can easily derive a single equation for the *vorticity vector* $\operatorname{curl}\boldsymbol{u}$. In view of the vector identities

$\nabla \times \boldsymbol{u} = \operatorname{curl} \boldsymbol{u}$ and $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nabla(q^2/2) - \boldsymbol{u} \times \operatorname{curl} \boldsymbol{u}$, where $q^2 = \boldsymbol{u}\cdot\boldsymbol{u}$, Eqn (2.1.4a) can be written as:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u} \times \operatorname{curl} \boldsymbol{u} + \nabla\left[\frac{p}{\rho_0} + \frac{q^2}{2}\right] = \boldsymbol{g} \ . \tag{2.1.5}$$

Then if $\boldsymbol{g}$ is of the form ∇G, as is indeed the case when $\boldsymbol{g}$ represents the force of gravity, and if the curl of the above equation is assumed to eliminate the term $\nabla H = \nabla[p/\rho_0 + q^2/2 + G]$, we obtain the following Helmholtz's equation:

$$\frac{\partial \operatorname{curl} \boldsymbol{u}}{\partial t} + \nabla \times (\operatorname{curl} \boldsymbol{u} \times \boldsymbol{u}) = 0 \tag{2.1.6a}$$

or

$$\frac{\mathrm{D}}{\mathrm{D}t} \operatorname{curl} \boldsymbol{u} = (\operatorname{curl} \boldsymbol{u}\cdot\nabla)\boldsymbol{u} \ , \tag{2.1.6b}$$

in which use has been made of the auxiliary relations: $\nabla\cdot\boldsymbol{u} = 0$ and $\nabla\cdot\operatorname{curl} \boldsymbol{u} = 0$.

Now, we can use the Lagrangian type of specification, by noting that $\boldsymbol{a}$ and $\boldsymbol{X}(\boldsymbol{a}, t)$ are the position vectors of one end of a material line element at times $t = 0$ and t respectively. In this case we can derive the following Cauchy formula at a time t:

$$\operatorname{curl} \boldsymbol{u} = \omega_i^0 \frac{\partial \boldsymbol{X}}{\partial a_i} \ , \tag{2.1.6c}$$

where the ω_i are the components of $\operatorname{curl} \boldsymbol{u}$ and a_i are the components of $\boldsymbol{a}$; ω_i^0 is the initial value (for $t = 0$) of ω_i (see, [64b], p. 65).

From Eqn (2.1.6b), we see that $\operatorname{curl} \boldsymbol{u} = 0$ is a possible solution! Therefore if $\omega_i^0 = 0$, this remains true at all times. In the case of water waves, typical free-surface problems are propagation into water at rest or through a uniform stream: in both cases we have: $\omega_i^0 = 0$ and the above argument applies. Naturally, the solution is unique provided that all components of $\nabla\boldsymbol{u}$ are bounded.

However, it is necessary to note also that in the presence of a free surface the vorticity of an inviscid incompressible liquid is not necessarily zero, if it is zero initially! Indeed, a free surface can intersect itself, as it happens when a water wave breaks and *vortex sheets* are formed. Of course, in this case, in place of the *Laplace equation*:

$$\nabla\cdot(\nabla\phi) = \nabla^2\phi = \Delta\phi = 0 \ , \tag{2.1.7}$$

where ϕ is the *velocity potential*, Δ is the three-dimensional *Laplace operator*, it is necessary to consider, for the rotational and inviscid flows, the full Euler incompressible system of equations (2.1.4) for $\boldsymbol{u}$ and p.

But here, in most cases, we shall restrict the discussion to irrotational potential flows, when $\operatorname{curl} \boldsymbol{u} = 0$ and $\boldsymbol{u} = \nabla\phi$.

In such cases, it follows from, Eqn (2.1.5) that

$$\frac{\partial(\nabla\phi)}{\partial t} + \nabla H = \nabla\left(\frac{\partial\phi}{\partial t} + \frac{p}{\rho_0} + \frac{q^2}{2} + G\right) = 0 \ . \tag{2.1.8}$$

This shows that the quantity in the parentheses must be a function of t alone, say, $B(t)$.

The form of this unknown function is without significance, because we could define a new velocity potential ϕ' such that $\phi' = \phi - \int B(t)\,\mathrm{d}t$, $\nabla\phi' = \nabla\phi$, and thereby remove the function of t without affecting the velocity distribution. It is customary to ignore the arbitrary function of t and to write the integral of Eqn (2.1.8) as the *Bernoulli equation*:

$$p = p_0 - \rho_0\left(\frac{\partial\phi}{\partial t} + \frac{q^2}{2} - \boldsymbol{g}\cdot\boldsymbol{x}\right) \ , \tag{2.1.9}$$

throughout the liquid, where p_0 is an arbitrary constant and $G = -\boldsymbol{g}\cdot\boldsymbol{x} = gz$.

The Bernoulli integral of the above relationship provides an explicit expression for the pressure p, when the velocity distribution is known. It is particularly useful in the free-surface problem, because ϕ satisfies the Laplace equation (2.1.7) and is determined uniquely by certain types of boundary conditions (see Section 2.2), and can therefore be determined without regard for the pressure (since $p_0 = p_\mathrm{a}$ on a free surface, assuming that the atmospheric pressure p_a is independent of position on the free surface).

When the solution of Eqn (2.1.7) is found for the relevant boundary conditions, the interesting physical quantities $\boldsymbol{u}$ and p are given by $\boldsymbol{u} = \nabla\phi$ and by Eqn (2.1.9).

2.2 Boundary and initial conditions

Various *initial and boundary conditions* may be specified for the Euler equations (2.1.1)–(2.1.3). But for the water wave motion, those encountered most frequently are the following:

(a) a complete set of initial conditions is obtained if $\boldsymbol{u}$, p and ρ are specified initially (for $t = 0$);

(b) at a solid boundary, a liquid does not penetrate the boundary, i.e., the normal component of the liquid velocity must be zero relative to the boundary (*slip condition*);†

(c) at a boundary between two immiscible liquids, the condition to be satisfied is that the *pressure shall be continuous* at the boundary as we pass from on side to the other (assuming that there is *no surface tension*!);‡

(d) there is no condition on the density ρ at the solid boundary.

Usually, for the Laplace equation (2.1.7), one boundary condition is given (on the contour line containing the liquid), but only when the boundary is known! Two conditions are needed for a free surface, $z = \zeta(t, x, y)$, because the surface position $\zeta(t, x, y)$ has to be determined as well as the potential $\phi(t, x, y, z)$.

On a free surface, the first boundary condition is the *kinematic condition*. This condition can be derived most readily by requiring that the substantial derivative $\mathrm{D}/\mathrm{D}t$ of the quantity $f = z - \zeta$ should vanish on the free surface. The result of this constraint is that:

$$\phi_z = \zeta_t + \phi_x\zeta_x + \phi_y\zeta_y \quad \text{on} \quad z = \zeta(x, y, t) \ . \tag{2.2.1}$$

We shall generally ignore the motion of the atmospheric air above a free surface, but the kinematic boundary condition (2.2.1) is not affected by this choice.

The second condition on the free surface is the *dynamic condition*. In the derivation of this condition, we shall assume that for no motion of the air, the pressure is constant (p_a). The pressure on the free water surface then depends on the surface tension. If we draw a line on this free surface, the liquid on the right of the line is found to exert a tension T (per unit length of line) on the liquid to the left; T is the *surface tension*: it differs for different liquids and it also depends on temperature. For

†For an inviscid Eulerian liquid, there is no restriction on the velocity component tangential to a solid boundary.

‡ All the available evidence does show that, under conditions common in moving liquids, both the tangential and normal components of velocity are continuous across a material boundary between a liquid and another medium.

instance, for an interface separating air and 'pure' water at 15 °C: $T = 73.5$ dyn cm^{-1} (or erg cm^{-2}). For a three-dimensional free surface we can show that, for an Eulerian liquid, we have,

$$p = p_a + TJ \ , \tag{2.2.2}$$

where $J = \nabla\cdot\boldsymbol{n} = 1/R_1 + 1/R_2$ is called the sum of the principal curvatures on the free surface (R_1 and R_2 are the principal radii of curvature of the sections of the interface formed by two orthogonal planes containing the vertical axis Oz which is opposite in direction to the gravitational force). Thus, at any point on a free surface, there must be a jump in the liquid pressure when passing towards the side of the surface on which the centre of curvature lies. We note that at a point near O the unit normal vector $\boldsymbol{n}$ to the free surface $z = \zeta(t, x, y)$, expressed in terms of rectangular coordinate axes, has the components $(-\zeta_x/N; -\zeta_y/N; +1/N)$, where $N^2 = 1 + \zeta_x^2 + \zeta_y^2$. The *exact* expression for $\nabla\cdot\boldsymbol{n}$ is:

$$\nabla\cdot\boldsymbol{n} = -\frac{1}{N^3}\left[(1+\zeta_x^2)\zeta_{yy} - 2\zeta_x\zeta_y\zeta_{xy} + (1+\zeta_y^2)\zeta_{xx}\right] . \tag{2.2.3}$$

To apply the surface pressure condition (2.2.2), in connection with expression (2.2.3), we go back to the Bernoulli equation (2.1.9), where $p_0 = p_a$ and $-\boldsymbol{g}\cdot\boldsymbol{x} = gz$; since on a free surface we have $z = \zeta$ and p is given by (2.1.9), it follows that

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + \frac{T}{\rho_0}(\nabla\cdot\boldsymbol{n}) + g\zeta = 0 \quad \text{on} \quad z = \zeta(x, y, t) \tag{2.2.4}$$

is our *dynamic free-surface nonlinear boundary condition*.

Now, if we assume that the liquid rests on a horizontal and impermeable bottom of infinite extent ($z = -h_0$), where $h_0 = \text{const}$ is finite, we have the following simple (*flat*) bottom boundary condition:

$$\phi_z = 0 \quad \text{at} \quad z = -h_0 \ . \tag{2.2.5}$$

Naturally, if we take into account the bottom topography (but assume that it is independent of time t), then in place of this simple condition (2.2.5), we must write (for inviscid liquid, we have $\boldsymbol{n}\cdot\nabla\phi = 0$) an *uneven* bottom condition:

$$\phi_z = g_0(\phi_x G_x + \phi_y G_y) \quad \text{at} \quad z = -h_0 + g_0 G\left(\frac{x}{l_0}; \frac{y}{m_0}\right) , \tag{2.2.6}$$

where g_0 is a typical elevation of the bottom topography $[g_0 = G(0,0)]$ and l_0, m_0 are the scale lengths associated with the variations in the channel bottom in the x and y directions.

For the *deep-water waves*, in place of conditions (2.2.5) or (2.2.6), we can write the following behavioural condition for ϕ_z:

$$\phi_z \to 0, \quad \text{when} \quad z \to -\infty \ . \tag{2.2.7}$$

In the case of the conditions relating to x and y, since we suppose that the liquid rests on a bottom of infinite extent, it is necessary to impose some behavioural conditions at infinity in the x and y directions. In fact, usually it is sufficient to suppose that the water wave motion is *periodic* in x and y.

The Laplace equation

$$\Delta\phi = \phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \text{for} \quad -h_0 \leqslant z \leqslant \zeta(x, y, t) , \tag{2.2.8}$$

with the three boundary conditions (2.2.1), (2.2.4) and (2.2.5) is our [simplified because we usually have also $T = 0$ in (2.2.4)] *classical three-dimensional nonlinear water wave problem*.

In the derivation of the model equations from the classical problem, a more expedient approach is to replace the kinematic boundary condition (2.2.1) by the statement that *the substantial derivative of the pressure p is zero on a free surface*. This is a rather pragmatic mixture of the dynamic and kinematic boundary conditions, since the statement that $\mathrm{D}p/\mathrm{D}t = 0$ on $z = \zeta(x, y, t)$ implies that this is precisely the appropriate moving surface on which the pressure p is constant ($p = p_a$). But, from the Bernoulli equation (2.1.9), we also have

$$-\frac{p - p_a}{\rho_0} = \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + gz \ .$$

Hence, we obtain the desired boundary condition on a free surface:

$$\begin{aligned} 0 &= \left(\frac{\partial}{\partial t} + \phi_x\frac{\partial}{\partial x} + \phi_y\frac{\partial}{\partial y} + \phi_z\frac{\partial}{\partial z}\right) \times \left[\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + gz\right] \\ &= \phi_{tt} + g\phi_z + 2\phi_x\phi_{xt} + 2\phi_y\phi_{yt} + 2\phi_z\phi_{zt} \\ &\quad + \frac{1}{2}\left(\phi_x\frac{\partial}{\partial x} + \phi_y\frac{\partial}{\partial y} + \phi_z\frac{\partial}{\partial z}\right)(\phi_x^2 + \phi_y^2 + \phi_z^2) = 0 \\ &\qquad \text{on} \quad z = \zeta(x, y, t) \ . \end{aligned} \tag{2.2.9}$$

The above boundary condition can also be derived directly from the two boundary conditions (2.2.1) and (2.2.4) with $T = 0$.

Not only do nonlinear terms appear in the classical water wave problem, but the position of a free surface is also an unknown quantity—an exact analytical theory of the water wave problem is therefore almost impossible!

In dealing with the free-surface classical pivotal problem we can first consider a '*signaling*' (*two-dimensional*) *problem*, when the liquid is initially at rest in a semi-infinite channel $x > 0$. We then have

$$\phi(0, x, z) = \zeta(0, x) = 0 , \quad \text{for} \quad x > 0 , \quad t = 0 , \tag{2.2.10}$$

and at time $t = 0$ an *idealised wave-maker* at $x = 0$ generates a horizontal velocity disturbance:

$$\phi_x(t, 0, z) = W_0 B\left(\frac{t}{t_0}\right) \quad \text{at} \quad t > 0 , \tag{2.2.11}$$

where W_0 and t_0 are the characteristic velocity and time scales associated with the wave-maker idealised by the function $B(t/t_0)$.

A second category of classical problem of water waves in an infinite channel is encountered by *specifying an initial surface shape* but zero velocity (one could also specify an arbitrary initial velocity distribution, for example, $\phi_t, \ldots$) for $t = 0$:

$$\zeta = a_0\zeta_0\left(\frac{x}{\lambda_0}, \frac{y}{\mu_0}\right), \quad \phi(0, x, y, z) = 0 , \tag{2.2.12}$$

where λ_0 and μ_0 are the characteristic wavelengths (in the x and y directions) for our three-dimensional water wave motion and a_0 is the characteristic amplitude for the *initial elevation* of a free surface represented by the function $\zeta_0(x/\lambda_0, y/\mu_0)$.

A *variational* derivation of the classical problem is dealt with in a paper by Luke (1967) [65] and also in a book by Whitham (1974, [62], p. 435). In this derivation it is necessary to take into account the following variational principle:

$$\delta \iiint_{\mathcal{R}} \mathcal{L}\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y = 0\,, \tag{2.2.13}$$

where

$$\mathcal{L} = -\rho \int_{-h}^{\zeta} \left[\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + gz\right] \mathrm{d}z\,. \tag{2.2.14}$$

Variation with respect to ϕ yields the Laplace equation (2.2.8) inside a liquid, the bottom boundary condition (2.2.6) (with $h = h_0 - g_0 G$), as well as the kinematic boundary condition (2.2.1) on a free surface $z = \zeta$.

Variation with respect to ζ yields the dynamic boundary condition (2.2.4), but with $T = 0$, on a free surface $z = \zeta$.

We note that here R in (2.2.13) is an arbitrary region in the (t, x, y) space. When the expression (2.2.14) is substituted in Eqn (2.2.13), the integration is over a region R^+ of the (t, x, y, z) time–space consisting of points with (t, x, y) in $\mathcal{R}$ for $-h < z < \zeta$.

The extra terms gz and ϕ_t in the expression (2.2.14), compared with the Dirichlet principle

$$\delta \iiint_{\mathcal{R}^+} \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2)\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = 0\,,$$

which yields the Laplace equation, affect only the boundary conditions, since they may be integrated out and contribute only to the terms originating from the boundary of $\mathcal{R}^+$.

2.3 Dimensionless problem

The *dimensionless independent variables* (with the primes) x', y', z', and t' are defined by:

$$x' = \frac{x}{\lambda_0}\,, \quad y' = \frac{y}{\mu_0}\,, \quad z' = \frac{z}{h_0}\,, \quad t' = \frac{t}{t_0}\,, \tag{2.3.1}$$

with $t_0 = \lambda_0/c_0$ and $c_0 = (gh_0)^{1/2}$ and in this case the Strouhal number is $S = \lambda_0/c_0 t_0 = 1$. Now, we scale the functions ϕ and ζ:

$$\phi' = \frac{\phi}{\varepsilon c_0 \lambda_0}\,, \quad \zeta' = \frac{\zeta}{a_0}\,, \quad \varepsilon = \frac{a_0}{h_0}\,. \tag{2.3.2}$$

When we drop the primes, we can write, for the dimensionless velocity potential $\phi(t, x, y, z)$, the following dimensionless Laplace equation, in place of Eqn (2.2.8)

$$\phi_{zz} + \delta^2 \phi_{xx} + \Delta^2 \phi_{yy} = 0\,, \quad -1 \leqslant z \leqslant \varepsilon\zeta(x, y, t)\,. \tag{2.3.3}$$

In the Laplace equation (2.3.3), we have the following two nondimensional parameters:

$$\delta = \frac{h_0}{\lambda_0}\,, \quad \Delta = \frac{h_0}{\mu_0}\,, \quad \frac{\lambda_0}{\mu_0} = \frac{\Delta}{\delta}\,. \tag{2.3.4}$$

In place of the boundary condition (2.2.5), we find the following simple dimensionless horizontal (flat) bottom condition:

$$\phi_z = 0\,, \quad \text{at} \quad z = -1\,. \tag{2.3.5}$$

In place of the boundary conditions (2.2.1) and (2.2.4), with $T = 0$, on $z = \varepsilon\zeta(t, x, y)$, we obtain the following two dimensionless free surface conditions:

$$\phi_z = \delta^2 \zeta_t + \varepsilon(\delta^2 \phi_x \zeta_x + \Delta^2 \phi_y \zeta_y)\,, \tag{2.3.6}$$

$$\phi_t + \frac{1}{2}\left(\varepsilon\phi_x^2 + \varepsilon\frac{\Delta^2}{\delta^2}\phi_y^2 + \frac{\varepsilon}{\delta^2}\phi_z^2\right) + \zeta = 0\,, \tag{2.3.7}$$

along the free surface $z = \varepsilon\zeta(t, x, y)$. We note that in the condition (2.3.7) the Froude number is $Fr = c_0/(gh_0)^{1/2} = 1$.

Eqn (2.3.3) with conditions (2.3.5)–(2.3.7) represents our *main dimensionless classical problem*.

If we take into account expression (2.2.3) for $\nabla\cdot\boldsymbol{n}$, then we can also write down the full dynamic boundary condition (2.2.4) in the following dimensionless form:

$$\begin{aligned}\phi_t &+ \frac{1}{2}\left(\varepsilon\phi_x^2 + \varepsilon\frac{\Delta^2}{\delta^2}\phi_y^2 + \frac{\varepsilon}{\delta^2}\phi_z^2\right) + \zeta \\ &= \delta^2\,\mathrm{We}(1 + \varepsilon^2\delta^2\zeta_x^2 + \varepsilon^2\Delta^2\zeta_y^2)^{-3/2} \\ &\times\left[\frac{\Delta^2}{\delta^2}(1 + \varepsilon^2\delta^2\zeta_x^2)\zeta_{yy} - 2\varepsilon^2\Delta^2\zeta_x\zeta_y\zeta_{xy} + (1 + \varepsilon^2\Delta^2\zeta_y^2)\zeta_{xx}\right]\,, \\ &\qquad\qquad \text{on} \quad z = \varepsilon\zeta(t, x, y)\,,\end{aligned} \tag{2.3.8}$$

where the dimensionless parameter

$$\mathrm{We} = \frac{T}{g\rho_0 h_0^2} \tag{2.3.9}$$

is the *Bond–Weber* number. The dimensionless form of the boundary condition (2.2.9) is

$$\begin{aligned}\phi_{tt} &+ \frac{1}{\delta^2}\phi_z + 2\varepsilon\left(\phi_x\phi_{xt} + \frac{\Delta^2}{\delta^2}\phi_y\phi_{yt} + \frac{1}{\delta^2}\phi_z\phi_{zt}\right) \\ &+ \frac{\varepsilon^2}{2}\left(\phi_x\frac{\partial}{\partial x} + \frac{\Delta^2}{\delta^2}\phi_y\frac{\partial}{\partial y} + \frac{1}{\delta^2}\phi_z\frac{\partial}{\partial z}\right) \\ &\times\left(\phi_x^2 + \frac{\Delta^2}{\delta^2}\phi_y^2 + \frac{1}{\delta^2}\phi_z^2\right) = 0\,, \quad \text{on} \quad z = \varepsilon\zeta(t, x, y)\,,\end{aligned} \tag{2.3.10}$$

when $\mathrm{We} = 0$.

Finally, in place of the uneven bottom condition (2.2.6), we can obtain the following dimensionless condition:

$$\phi_z = \alpha\left[\delta^2\phi_x G_x + \Delta^2\phi_y G_y\right] \quad \text{on} \quad z = -1 + \alpha G(x^*, y^*) \tag{2.3.11}$$

with the following three dimensionless parameters:

$$\alpha = \frac{g_0}{h_0}\,, \quad \beta = \frac{\lambda_0}{l_0}\,, \quad \gamma = \frac{\mu_0}{m_0}\,, \tag{2.3.12}$$

and with the bottom variables: $x^* = \beta x$ and $y^* = \gamma y$.

In the signaling problem, in place of the boundary condition (2.2.11), we obtain the following dimensionless condition:

$$\phi_x(t, 0, z) = \omega B(t) \quad \text{for} \quad t > 0\,, \tag{2.3.13}$$

when we assume that t_0, in expression (2.2.11), is just λ_0/c_0. In condition (2.3.13) we have a new dimensionless parameter

$$\omega = \frac{W_0}{\varepsilon c_0} = \frac{W_0/c_0}{\varepsilon} . \tag{2.3.14}$$

When dealing with the initial surface shape problem we can write, in place of the first of conditions (2.2.12), the following initial dimensionless condition for $t = 0$:

$$\zeta = \zeta_0(x, y) . \tag{2.3.15}$$

In the above dimensionless pivotal problem the parameter $\varepsilon = a_0/\lambda_0$ is the *nonlinearity parameter* and, for $\varepsilon \to 0$, with x, y, z, and t fixed and also for fixed values of δ and Δ, we have — in place of expressions (2.3.3), (2.3.5)–(2.3.7) — the *classical linear water wave pivotal problem* for ϕ_0:

$$(\phi_0)_{zz} + \delta^2(\phi_0)_{xx} + \Delta^2(\phi_0)_{yy} = 0 , \quad \text{for} \quad -1 \leqslant z \leqslant 0 ,$$

$$(\phi_0)_z = 0 \quad \text{for} \quad z = -1 ,$$

$$(\phi_0)_z + \delta^2(\phi_0)_{tt} = 0 \quad \text{for} \quad z = 0 , \tag{2.3.16}$$

where $\lim^1 \phi = \phi_0$, with: $\lim^1 = [\varepsilon \to 0$; with x, y, z, t, δ and Δ fixed]. The parameter δ is the *long longitudinal* (x direction) water wave parameter and Δ is the *long transverse* (y direction) water wave parameter.

In the next sections we shall consider mainly the following asymptotic situation:

$$\varepsilon \ll 1 , \quad \delta \ll 1 , \quad \Delta \ll 1 , \tag{2.3.17}$$

with two similarity relationships

$$\delta^2 = \kappa_0 \varepsilon , \quad \Delta = \nu_0 \varepsilon , \tag{2.3.18}$$

where κ_0 and ν_0 are of the order of unity when $\varepsilon \to 0$. In fact, we assume that:

(a) the water wave amplitudes are *small*;

(b) the water is *shallow*, compared with typical horizontal wavelengths;

(c) the water waves are *nearly* one-dimensional;

(d) these three small effects all have *comparable* influence (all three effects balance, according to the Urcell criterion [23]).

When $\alpha \ll 1$, the *effects of the elevation of the uneven bottom topography are small*. We note also that when $\beta \gg 1$ and $\gamma \gg 1$, we have a *rough* bottom and for $\beta \ll 1$ and $\gamma \ll 1$, a *slowly varying* bottom.

If we now consider the more complete dynamic free-surface condition (2.3.8), we encounter two cases. In the first case we suppose that $\mathrm{We} = O(1)$ is of the order of unity, and then, in the linear problem (2.3.16), the last boundary condition for $z = 0$ must be replaced by the following condition

$$(\phi_0)_z + \delta^2(\phi_0)_{tt} + \mathrm{We}\,(\phi_0)_{zzz} = 0 , \quad \text{for} \quad z = 0 . \tag{2.3.19}$$

In the second case we assume that

$$\mathrm{We} \gg 1 , \quad \text{but} \quad \delta^2\,\mathrm{We} = \mathrm{We}^* = O(1) . \tag{2.3.20}$$

3. Boussinesq equations

3.1 Two-dimensional Boussinesq equations

In the derivation of the two-dimensional Boussinesq equations for two-dimensional waves on the surface of water, we take

$$\phi = \phi_0(t, x, y, z') + \delta^2\phi_1(t, x, y, z') + \delta^4\phi_2(t, x, y, z') + O(\delta^6) \quad \text{for} \quad z' = z + 1 , \tag{3.1.1}$$

and substitute this into the dimensionless three-dimensional pivotal problem formulated in Section 2.3 (we assume that $\mathrm{We} = 0$ and $\Delta^2 \equiv \delta^2$) for ζ and ϕ.

The lowest-order term, according to the dimensionless Laplace equation (2.3.3)

$$\phi_{z'z'} + \delta^2(\phi_{xx} + \phi_{yy}) = 0$$

is

$$\phi_0 = F(t, x, y) , \tag{3.1.2}$$

equivalent to assumption that the horizontal velocity components are independent of the depth:

$$u_0(t, x, y) = \frac{\partial F}{\partial x} , \quad v_0(t, x, y) = \frac{\partial F}{\partial y} . \tag{3.1.3}$$

If ϕ_1 vanishes at the bottom ($z' = 0$), and u_0, v_0 are the horizontal components of the velocity at the bottom, we can drop a further arbitrary function and find ϕ_1:

$$\phi_1 = -\frac{1}{2} z'^2 \left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) , \tag{3.1.4}$$

since $\partial\phi_1/\partial z' = 0$ at $z' = 0$. Substituting the solution for ϕ_1 into the equation for ϕ_2

$$\frac{\partial^2\phi_2}{\partial z'^2} = -\left(\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2}\right) \equiv \frac{z'^2}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) ,$$

and integrating, and again using the condition $\partial\phi_2/\partial z' = 0$ at $z' = 0$, we can determine the function ϕ_2

$$\phi_2 = \frac{1}{24} z'^4 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) , \tag{3.1.5}$$

since we assume that ϕ_2 vanishes at $z' = 0$.

Now, we turn to the two dimensionless boundary conditions on the free surface $z' = 1 + \varepsilon\zeta$ [see expressions (2.3.7) and (2.3.6)]:

$$\phi_t + \frac{1}{2}\varepsilon\left(\phi_x^2 + \phi_y^2 + \frac{1}{\delta^2}\phi_{z'}^2\right) + \zeta = 0 , \tag{3.1.6a}$$

$$\phi_{z'} = \delta^2\left[\zeta_t + \varepsilon(\phi_x\zeta_x + \phi_y\zeta_y)\right] . \tag{3.1.6b}$$

As before, we shall retain up to order δ^4, ε^2 and $\delta^2\varepsilon$ in Eqn (3.1.6b) and δ^2, ε in Eqn (3.1.6a). In this case, in place of Eqns (3.1.6a) and (3.1.6b), we obtain, making use of solutions (3.1.2), (3.1.4) and (3.1.5), the following two equations:

$$\delta^2\frac{\partial\zeta}{\partial t} + \delta^2\varepsilon\left(u_0\frac{\partial\zeta}{\partial x} + v_0\frac{\partial\zeta}{\partial y}\right) + \delta^2(1 + \varepsilon\zeta)\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) - \frac{1}{6}\delta^4\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) = 0 , \tag{3.1.7}$$

$$\frac{\partial F}{\partial t} - \frac{1}{2}\delta^2\frac{\partial}{\partial t}\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) + \zeta + \frac{1}{2}\varepsilon(u_0^2 + v_0^2) = 0 . \tag{3.1.8}$$

We can immediately rearrange Eqn (3.1.7) as:

$$\frac{\partial\zeta}{\partial t} + \frac{\partial}{\partial x}\left[(1 + \varepsilon\zeta)u_0\right] + \frac{\partial}{\partial y}\left[(1 + \varepsilon\zeta)v_0\right] - \frac{\varepsilon\kappa_0}{6}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) = 0 , \tag{3.1.9}$$

since we are assuming that $\delta^2 = \kappa_0\varepsilon$.

In Eqn (3.1.8) we differentiate first with respect to x, and then with respect to y, to get two equations:

$$\frac{\partial u_0}{\partial x}+\frac{\partial \zeta}{\partial x}+\varepsilon\left(u_0\frac{\partial u_0}{\partial x}+v_0\frac{\partial u_0}{\partial y}\right)-\frac{\varepsilon\kappa_0}{2}\frac{\partial^2}{\partial t\,\partial x}\left(\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)=0\,, \quad (3.1.10a)$$

$$\frac{\partial v_0}{\partial x}+\frac{\partial \zeta}{\partial y}+\varepsilon\left(u_0\frac{\partial v_0}{\partial x}+v_0\frac{\partial v_0}{\partial y}\right)-\frac{\varepsilon\kappa_0}{2}\frac{\partial^2}{\partial t\,\partial y}\left(\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)=0\,, \quad (3.1.10b)$$

with

$$\frac{\partial u_0}{\partial y}=\frac{\partial v_0}{\partial x}\,, \quad (3.1.11)$$

since the 'flow F' is irrotational.

Finally, for our three unknown functions u_0, v_0 and ζ we find three *approximate two-dimensional Boussinesq equations* (3.1.9), (3.1.10a) and (3.1.10b).

We specify that, in these above Boussinesq equations, u_0 and v_0 are the nonaveraged components of the horizontal velocity at the bottom $z=-1$, satisfying also the irrotationality condition (3.1.11).

When $\kappa_0\to 0$, in the Boussinesq system (3.1.9), (3.1.10a) and (3.1.10b), we obtain the nonlinear Airy's *shallow water equations*:

$$\frac{\partial \zeta}{\partial t}+\frac{\partial}{\partial x}\left[(1+\varepsilon\zeta)u_0\right]+\frac{\partial}{\partial y}\left[(1+\varepsilon\zeta)v_0\right]=0\,,$$

$$\frac{\partial u_0}{\partial x}+\varepsilon\left(u_0\frac{\partial u_0}{\partial x}+v_0\frac{\partial u_0}{\partial y}\right)+\frac{\partial \zeta}{\partial x}=0\,, \quad (3.1.12)$$

$$\frac{\partial v_0}{\partial x}+\varepsilon\left(u_0\frac{\partial v_0}{\partial x}+v_0\frac{\partial v_0}{\partial y}\right)+\frac{\partial \zeta}{\partial y}=0\,.$$

The corresponding velocity potential ϕ for these two-dimensional Boussinesq equations is of the following form, according to Eqns (3.1.1), with (3.1.2), (3.1.4) and (3.1.5):

$$\phi=F(t,x,y)-\frac{1}{2}\delta^2 z'^2\left(\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)+\frac{1}{24}\delta^4 z'^4\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)+O(\delta^6)\,. \quad (3.1.13)$$

3.2 Quasi-one-dimensional Boussinesq equations

In the derivation of these equations, let us consider, according to Zeytounian [45, 64a], the following dimensionless problem (see Section 2.3), but with an error of $O(\varepsilon^2\Delta^2)$,

$$\phi_{zz}+\delta^2\phi_{xx}+\Delta^2\phi_{yy}=0 \quad \text{for} \quad -1\leqslant z\leqslant \varepsilon\zeta(x,y,t)\,, \quad (3.2.1a)$$

$$\phi_z=0 \quad \text{for} \quad z=-1\,, \quad (3.2.1b)$$

$$\phi_z=\delta^2\zeta_t+\varepsilon(\delta^2\phi_x\zeta_x+\Delta^2\phi_y\zeta_y) \quad \text{on} \quad z=\varepsilon\zeta(x,y,t)\,, \quad (3.2.1c)$$

$$\phi_t+\frac{1}{2}\left(\varepsilon\phi_x^2+\varepsilon\frac{\Delta^2}{\delta^2}\phi_y^2+\frac{\varepsilon}{\delta^2}\phi_z^2\right)+\zeta=\delta^2\,\mathrm{We}\left[\zeta_{xx}+\frac{\Delta^2}{\delta^2}\zeta_{yy}-\frac{3}{2}\varepsilon^2\delta^2\zeta_x^2\zeta_{xx}+O(\varepsilon^2\Delta^2)\right] \quad \text{on} \quad z=\varepsilon\zeta(x,y,t)\,. \quad (3.2.1d)$$

The Laplace equation (3.2.1a) is the only equation which contains z in its solution and this variant may be made explicit by formally expanding its solution in powers of δ^2 and Δ^2 and writing:

$$\phi=\phi_{00}+\delta^2\phi_{10}+\delta^4\phi_{20}+\Delta^2\phi_{01}+\delta^6\phi_{30}+\delta^2\Delta^2\phi_{11}+\ldots\,. \quad (3.2.2)$$

This above asymptotic representation is consistent with our main hypothesis described by expressions (2.3.17) and (2.3.18). Now, for ϕ_{00} we must resolve the following trivial problem:

$$(\phi_{00})_{zz}=0 \quad \text{with} \quad (\phi_{00})_z=0 \quad \text{for} \quad z=-1\,,$$

and we find

$$\phi_{00}=F(x,y,t)\,. \quad (3.2.3)$$

Below, for the simplicity we assume again that *the arbitrary function F is the (unknown) value of velocity potential ϕ on $z=-1$*, and in this case we can write also the following boundary conditions for the terms of expansion (3.2.2):

$$\phi_{10}=\phi_{20}=\phi_{01}=\phi_{30}=\phi_{11}=0 \quad \text{for} \quad z=-1\,. \quad (3.2.4)$$

But, according to Eqn (3.2.1b), we have also, as the boundary conditions on $z=-1$, the following bottom conditions:

$$(\phi_{10})_z=(\phi_{20})_z=(\phi_{01})_z=(\phi_{30})_z=(\phi_{11})_z=0 \quad \text{for} \quad z=-1\,. \quad (3.2.5)$$

We can write immediately the solution of the equation for ϕ_{10}, $(\phi_{10})_{zz}=-(F)_{xx}$:

$$\phi_{10}=-\frac{1}{2}(z+1)^2(F)_{xx}\,. \quad (3.2.6)$$

By analogy, for the functions ϕ_{20}, ϕ_{01}, ϕ_{30} and ϕ_{11}, which are the solutions of equations $(\phi_{20})_{zz}=-(\phi_{10})_{xx}$, $(\phi_{01})_{zz}=-(F)_{yy}$, $(\phi_{30})_{zz}=-(\phi_{20})_{xx}$, and $(\phi_{11})_{zz}=-(\phi_{10})_{yy}$, we obtain, respectively, the following explicit solutions in terms of z:

$$\phi_{20}=\frac{1}{24}(z+1)^4(F)_{xx\,xx}\,, \quad (3.2.7a)$$

$$\phi_{01}=-\frac{1}{2}(z+1)^2(F)_{yy}\,, \quad (3.2.7b)$$

$$\phi_{30}=-\frac{1}{720}(z+1)^6(F)_{xx\,xx\,xx}\,, \quad (3.2.7c)$$

$$\phi_{11}=\frac{1}{24}(z+1)^4(F)_{xx\,yy}\,. \quad (3.2.7d)$$

Finally, we obtain, in place of Eqn (3.2.2), the following asymptotic representation for the velocity potential ϕ, as the solution of the Laplace equation with the bottom condition for $z=-1$:

$$\phi(x,y,z,t)=F(x,y,t)-\frac{1}{2}\delta^2(z+1)^2(F)_{xx}+\frac{1}{24}\delta^4(z+1)^4(F)_{xx\,xx}-\frac{1}{2}\Delta^2(z+1)^2(F)_{yy}-\frac{1}{720}\delta^6(z+1)^6(F)_{xx\,xxxx}+\frac{1}{24}\delta^2\Delta^2(z+1)^4(F)_{xx\,yy}\,. \quad (3.2.8)$$

Now, by means of the Taylor expansions, we can calculate the derivatives ϕ_s, with $s=(t,x,y)$, and ϕ_z, on $z=\varepsilon\zeta(s)$:

$$[\phi_s]_{z=\varepsilon\zeta(s)} = (F)_s - \frac{1}{2}\delta^2\Big\{(F)_{xxs} + \frac{1}{2}\delta^2(F)_{xxxxs} - \frac{\Delta^2}{\delta^2}(F)_{yys} - 2\varepsilon\big[\zeta(F)_{xxs} + \zeta_s(F)_{xx}\big]\Big\}\,, \quad (3.2.9a)$$

$$[\phi_z]_{z=\varepsilon\zeta(s)} = -\delta^2\Big\{(F)_{xx} + \frac{1}{6}\delta^2(F)_{xxxx} - \frac{\Delta^2}{\delta^2}(F)_{yy} - \varepsilon\zeta(F)_{xx} - \frac{1}{120}\delta^4(F)_{xxxxxx} + \frac{1}{6}\Delta^2(F)_{xxyy} - \frac{\Delta^2}{\delta^2}\varepsilon\zeta(F)_{yy} + \frac{1}{2}\varepsilon\delta^2\zeta(F)_{xxxx}\Big\}\,. \quad (3.2.9b)$$

Finally, if we take into account our two boundary conditions (3.2.1c) and (3.2.1d) on the free surface $z=\varepsilon\zeta(s)$, $s=(t,x,y)$, relations (3.2.9) and two similarity relations (2.3.18), $\delta^2=\kappa_0\varepsilon$ and $\Delta=\nu_0\varepsilon$, and also the hypothesis (2.3.20), $\mathrm{We}\gg 1$ but $\delta^2\,\mathrm{We}=\mathrm{We}^*=O(1)$, we obtain the following two approximate equations [with an error of $O(\varepsilon^3)$] for the two unknown functions $\zeta(t,x,y)$ and $F(t,x,y)$:

$$(F)_t + \zeta - \mathrm{We}^*\zeta_{xx} + \varepsilon\Big\{\frac{1}{2}(F)_{xx}^2 - \frac{\kappa_0}{2}(F)_{xxt} - \frac{\nu_0^2}{\kappa_0}\mathrm{We}^*\zeta_{yy}\Big\} + \varepsilon^2\Big\{\frac{\kappa_0^2}{24}(F)_{xxxxt} + \frac{\kappa_0}{2}(F)_{xx}^2 - \frac{\nu_0^2}{2}(F)_{yyt} + \frac{\nu_0^2}{2\kappa_0}(F)_y^2 - \frac{\kappa_0}{2}(F)_{xxx}(F)_x - \kappa_0\big[\zeta(F)_{xx}\big]_t\Big\} = O(\varepsilon^3)\,, \quad (3.2.10a)$$

$$\zeta_t + (F)_{xx} + \varepsilon\Big\{\frac{\nu_0^2}{\kappa_0}(F)_{yy} + \big[\zeta(F)_x\big]_x - \frac{\kappa_0}{6}(F)_{xxxx}\Big\} + \varepsilon^2\Big\{\frac{\kappa_0^2}{120}(F)_{xxxxxx} - \frac{\kappa_0}{2}\big[\zeta(F)_{xxx}\big]_x + \frac{\nu_0^2}{\kappa_0}\big[\zeta(F)_y\big]_y - \frac{\nu_0^2}{6}(F)_{xxyy}\Big\} = O(\varepsilon^3)\,. \quad (3.2.10b)$$

Here, approximate equations (3.2.10a) and (3.2.10b), which include the terms of order $O(\varepsilon)$ and also $O(\varepsilon^2)$, are called the '*quasi-one-dimensional generalised Boussinesq* (Q1DGB) *equations*'.

Naturally, in Eqns (3.2.10a) and (3.2.10b) the unknown functions $\zeta(x,y,t)$ and $F(x,y,t)$ are implicit functions of ε, and we can write:

$$F = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \ldots\,, \qquad \zeta = \zeta_0 + \varepsilon\zeta_1 + \varepsilon^2\zeta_2 + \ldots \quad (3.2.11)$$

Using the above expressions and Eqns (3.2.10a) and (3.2.10b), we derive successively the following limiting equations for F_0 and ζ_0, F_1 and ζ_1 and also for F_2 and ζ_2:

$$(F_0)_t + \zeta_0 - \mathrm{We}^*(\zeta_0)_{xx} = 0\,,$$
$$(\zeta_0)_t + (F_0)_{xx} = 0\,; \quad (3.2.12a)$$

.....

$$(F_1)_t + \zeta_1 - \mathrm{We}^*(\zeta_1)_{xx} = -\frac{1}{2}(F_0)_{xx}^2 + \frac{\kappa_0}{2}(F_0)_{xxt} + \frac{\nu_0^2}{\kappa_0}\mathrm{We}^*(\zeta_0)_{yy}\,,$$
$$(\zeta_1)_t + (F_1)_{xx} = -\frac{\nu_0^2}{\kappa_0}(F_0)_{yy} - \big[\zeta_0(F_0)_x\big]_x + \frac{\kappa_0}{6}(F_0)_{xxxx}\,; \quad (3.2.12b)$$

.....

$$(F_2)_t + \zeta_2 - \mathrm{We}^*(\zeta_2)_{xx} = -(F_0)_x(F_1)_x + \frac{\kappa_0}{2}(F_1)_{xxt} + \frac{\nu_0^2}{\kappa_0}\mathrm{We}^*(\zeta_1)_{yy} - \frac{\kappa_0^2}{24}(F_0)_{xxxxt} - \frac{\kappa_0}{2}(F_0)_{xx}^2 + \frac{\nu_0^2}{2}(F_0)_{yyt} - \frac{\nu_0^2}{2\kappa_0}(F_0)_{yy}^2 + \frac{\kappa_0}{2}(F_0)_{xxx}(F_0)_x + \kappa_0\big[\zeta_0(F_0)_{xx}\big]_t\,,$$

$$(\zeta_2)_t + (F_2)_{xx} = -\frac{\nu_0^2}{\kappa_0}(F_1)_{yy} - \big[\zeta_0(F_1)_x + \zeta_1(F_0)_x\big]_x + \frac{\kappa_0}{6}(F_1)_{xxxx} - \frac{\kappa_0^2}{120}(F_0)_{xxxxxx} + \frac{\kappa_0}{2}\big[\zeta_0(F_0)_{xxx}\big]_x - \frac{\nu_0^2}{\kappa_0}\big[\zeta_0(F_0)_y\big]_y + \frac{\nu_0^2}{6}(F_0)_{xxyy}\,. \quad (3.2.12c)$$

.....

In the derivation of the properly called quasi-one-dimensional Boussinesq (Q1DB) equations, we return to general equations (3.2.12a) and (3.2.12b), but we suppose that the Bond–Weber number is $\mathrm{We}=O(1)$. In this case, according to the first of similarity relations (2.3.18), all the terms are proportional to $\mathrm{We}^*=\varepsilon\kappa_0\mathrm{We}$, and are therefore of the order of ε.

Hence, in place of Eqns (3.2.12a) and (3.2.12b), we find the following system of two equations for F_0 and F_1:

$$(F_0)_{tt} - (F_0)_{xx} = 0\,, \quad (3.2.13)$$

$$(F_1)_{tt} - (F_1)_{xx} = \frac{\kappa_0}{2}(F_0)_{xxtt} - \frac{\kappa_0}{6}(F_0)_{xxxx} - \kappa_0\mathrm{We}(F_0)_{xxtt} - (F_0)_x(F_0)_{xt} - \big[(F_0)_x(F_0)_t\big]_x + \frac{\nu_0^2}{\kappa_0}(F_0)_{yy}\,, \quad (3.2.14)$$

after the elimination of the functions ζ_0 and ζ_1.

Now, if we introduce the following composite function $F^*=F_0+\varepsilon F_1$, we can derive, from Eqns (3.2.13) and (3.2.14), the following Q1DB single equation for $F^*(x,y,t)$:

$$(F^*)_{tt} - (F^*)_{xx} - \varepsilon\frac{\nu_0^2}{\kappa_0}(F^*)_{yy} + \varepsilon\Big[(F^*)_x^2 + \frac{1}{2}(F^*)_t^2\Big]_t + \varepsilon\kappa_0\Big(\mathrm{We} - \frac{1}{3}\Big)(F^*)_{xxtt} = 0\,, \quad (3.2.15)$$

when we take into account that

$$\big[(F_0)_x(F_0)_t\big]_x = \frac{1}{2}\big[(F_0)_x^2 + (F_0)_t^2\big]_t\,, \qquad (F_0)_{tt} = (F_0)_{xx}\,.$$

This last, single Boussinesq equation (3.2.15), is a generalisation of the classical Boussinesq equation

$$(F^*)_{tt} - (F^*)_{xx} + \varepsilon\Big[(F^*)_x^2 + \frac{1}{2}(F^*)_t^2\Big]_t - \frac{\varepsilon\kappa_0}{3}(F^*)_{xxtt} = 0\,, \quad (3.2.16)$$

for the *nearly* two-dimensional long waves in shallow water.

The above Q1DB equation (3.2.15) is also directly obtained from the initial dimensionless problem described by the system of equation (3.2.1), with expression (2.3.18),

if we take into account, in place of representation (3.2.8), the following representation for $\phi(t,x,y,z)$:

$$\phi(t,x,y,z) = F(x,\,y,\,t) + \varepsilon\left[G - \frac{\kappa_0}{2}(z+1)^2(F)_{xx}\right] + \varepsilon^2\left[H + \frac{1}{24}\kappa_0^2(z+1)^4(F)_{xxxx} - \frac{1}{2}v_0^2(z+1)^2(F)_{yy} - \frac{\kappa_0}{2}(z+1)^2(G)_{xx}\right] + O(\varepsilon^3)\,, \tag{3.2.17}$$

where F, G, $H,\ldots$ are unknown functions of the independent variables x, y and t. Naturally, in this case, in Eqn (3.2.17), $F(x,\,y,\,t)$ *is not the value of* ϕ *on the bottom* where $z=-1$.

We assume now that

$$(F,G,H) = (F,G,H)_0 + \varepsilon(F,G,H)_1 + \ldots\,,\qquad \zeta = \zeta_0 + \varepsilon\zeta_1 + \ldots \tag{3.2.18}$$

and, making use of Eqns (3.2.17) and (3.2.18), we can calculate [as in Eqn (3.2.9)] the derivatives ϕ_s, with $s=(t,x,y)$, and ϕ_z, on $z=\varepsilon\zeta(s)$.

Finally, from the two boundary conditions (3.2.1c) and (3.2.1d) for $z=\varepsilon\zeta(x,\,y,\,t)$, we derive the following equations for the functions F_0, ζ_0, F_1+H_0 and ζ_1:

$$(F_0)_t + \zeta_0 = 0\,,\qquad (F_0)_{xx} + (\zeta_0)_t = 0\,, \tag{3.2.19a}$$

$$\left[F_1+H_0\right]_t + \zeta_1 + \frac{1}{2}(F_0)_x^2 - \frac{\kappa_0}{2}(F_0)_{xxt} - \kappa_0\mathrm{We}(\zeta_0)_{xx} = 0\,, \tag{3.2.19b}$$

$$\left[F_1+H_0\right]_{xx} + (\zeta_1)_t + (F_0)_x(\zeta_0)_x + \zeta_0(F_0)_{xx} - \frac{\kappa_0}{6}(F_0)_{xxxx} + \frac{v_0^2}{\kappa_0}(F_0)_{yy} = 0\,. \tag{3.2.19c}$$

From the system of equations (3.2.19a)–(3.2.19c) we derive immediately the same Q1DB equation (3.2.15), but for the function $F^{**} = F_0 + \varepsilon[F_1+H_0]$.

At last, from Eqns (3.2.12a) and (3.2.12b), we can also obtain a system of *quasi-one-dimensional Boussinesq equations* for the free surface position, $\zeta(x,\,y,\,t)$ and for the horizontal velocity components, $u(x,\,y,\,t)=(F)_x$ and $v(x,\,y,\,t)=(F)_y$, in the following form (if we assume that $\mathrm{We}^*\equiv 0$):

$$\frac{\partial\zeta}{\partial t} + \frac{\partial}{\partial x}\left[(1+\varepsilon\zeta)u\right] + \frac{\varepsilon v_0^2}{\kappa_0}\frac{\partial v}{\partial y} - \frac{\varepsilon\kappa_0}{6}\frac{\partial^3 u}{\partial x^3} = 0\,, \tag{3.2.20}$$

$$\frac{\partial\zeta}{\partial x} + \frac{\partial u}{\partial t} + \varepsilon u\frac{\partial u}{\partial x} - \frac{\varepsilon\kappa_0}{2}\frac{\partial^3 u}{\partial x^2\,\partial t} = 0\,, \tag{3.2.21}$$

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}\,, \tag{3.2.22}$$

with an error of $O(\varepsilon^2)$.

Averaging

$$\frac{\partial\phi}{\partial x} = u - \frac{\varepsilon\kappa_0}{2}(z+1)^2\frac{\partial^2 u}{\partial x^2} + O(\varepsilon^2)\,,$$

$$\frac{\partial\phi}{\partial y} = v - \frac{\varepsilon\kappa_0}{2}(z+1)^2\frac{\partial^2 v}{\partial x^2} + O(\varepsilon^2)$$

over the depth yields we obtain:

$$u = U + \frac{\varepsilon\kappa_0}{6}\frac{\partial^2 U}{\partial x^2} + O(\varepsilon^2)\,, \tag{3.2.23a}$$

$$v = V + \frac{\varepsilon\kappa_0}{6}\frac{\partial^2 V}{\partial x^2} + O(\varepsilon^2)\,. \tag{3.2.23b}$$

When Eqns (3.2.23a) and (3.2.23b) are used in the Q1DB equations (3.2.20)–(3.2.22), we obtain the following form of our Q1DB equations:

$$\frac{\partial\zeta}{\partial t} + \frac{\partial}{\partial x}\left[(1+\varepsilon\zeta)U\right] + \frac{\varepsilon v_0^2}{\kappa_0}\frac{\partial V}{\partial y} = 0\,, \tag{3.2.24a}$$

$$\frac{\partial\zeta}{\partial x} + \frac{\partial U}{\partial t} + \varepsilon U\frac{\partial U}{\partial x} - \frac{\varepsilon\kappa_0}{3}\frac{\partial^3 U}{\partial x^2\,\partial t} = 0\,, \tag{3.2.24b}$$

$$\frac{\partial V}{\partial x} = \frac{\partial U}{\partial y} \tag{3.2.24c}$$

for the average horizontal velocity components $U(x,\,y,\,t)$, $V(x,\,y,\,t)$ and $\zeta(x,\,y,\,t)$.

We note that our Q1DB equations (3.2.24) are not similar to 'three-dimensional generalisation of the Boussinesq equations', derived by Infeld in 1980 ([44], Appendix 1, B1 equations). Apparently, these B1 Infeld equations, are inconsistent from the point of view of asymptotic methodology. Finally, instead of Eqns (3.2.24), we can also derive two equations for $\zeta(x,\,y,\,t)$ and $U(x,\,y,\,t)$, if we differentiate Eqns (3.2.24a) with respect to x and utilise Eqn (3.2.24c):

$$\frac{\partial}{\partial t}\frac{\partial\zeta}{\partial x} + \frac{\partial^2}{\partial x^2}\left[(1+\varepsilon\zeta)U\right] + \frac{\varepsilon v_0^2}{\kappa_0}\frac{\partial^2 U}{\partial y^2} = 0\,, \tag{3.2.25a}$$

$$\frac{\partial\zeta}{\partial x} + \frac{\partial U}{\partial t} + \varepsilon U\frac{\partial U}{\partial x} - \frac{\varepsilon\kappa_0}{3}\frac{\partial^3 U}{\partial x^2\,\partial t} = 0\,. \tag{3.2.25b}$$

3.3 Boussinesq solitary and cnoidal waves

We shall now consider a particular solution of the single one-dimensional Boussinesq equation [see the equation (3.2.16)]:

$$\frac{\partial^2 F}{\partial t^2} - \frac{\partial^2 F}{\partial x^2} + \varepsilon\frac{\partial}{\partial t}\left[\left(\frac{\partial F}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial F}{\partial t}\right)^2\right] - \frac{\varepsilon\kappa_0}{3}\frac{\partial^4 F}{\partial x^2\,\partial t^2} = 0\,, \tag{3.3.1}$$

in the following form:

$$F = \Phi(\xi)\,,\qquad \xi = x - Ct\,,\qquad \frac{\partial}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}\xi}\,,\qquad \frac{\partial}{\partial t} = -C\frac{\mathrm{d}}{\mathrm{d}\xi}\,. \tag{3.3.2}$$

In this case we obtain for $\Phi(\xi)$ the following ordinary differential equation:

$$(C^2-1)\frac{\mathrm{d}^2\Phi}{\mathrm{d}\xi^2} = C^2\frac{\varepsilon\kappa_0}{3}\frac{\mathrm{d}^4\Phi}{\mathrm{d}\xi^4} - \varepsilon C\left(1+\frac{C^2}{2}\right)\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\right)^2\,,$$

and, in fact, we have $C=1+O(\varepsilon)$, therefore the terms on the right-hand side of the above equation may be approximated with $C=1$, without affecting the accuracy. Integrating once with respect to ξ, we get:

$$(C^2-1)\frac{\mathrm{d}\Phi}{\mathrm{d}\xi} + A = \frac{\varepsilon\kappa_0}{3}\frac{\mathrm{d}^3\Phi}{\mathrm{d}\xi^3} + \frac{3\varepsilon}{2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\right)^2\,.$$

But, to the leading order, we have $\partial F_0/\partial t = -\zeta = -\mathrm{d}\Phi/\mathrm{d}\xi$. Thus

$$(C^2-1)\zeta + A = \frac{\varepsilon\kappa_0}{3}\frac{\mathrm{d}^2\zeta}{\mathrm{d}\xi^2} + \frac{3}{2}\varepsilon\zeta^2\,.$$

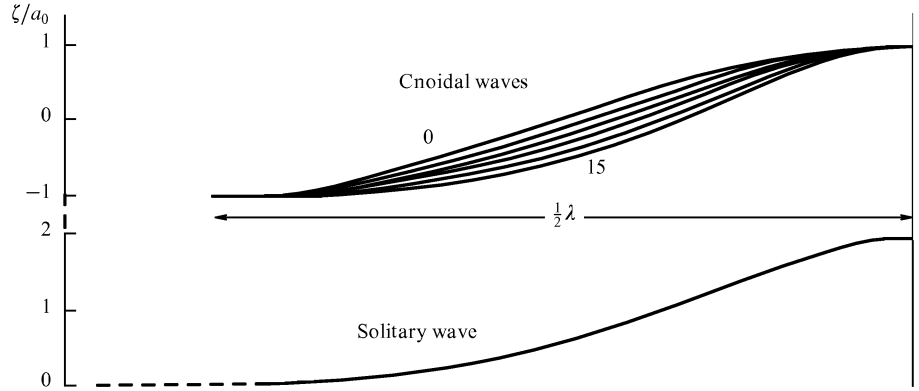

Figure 2. Waveforms of cnoidal waves of length λ and amplitude a_0 on water of depth h_0 for six values of $a_0\lambda^2/h_0^3$. The solitary wave is the limit of the cnoidal waves with infinite wavelength.

Finally, we multiply the above equation for ζ by $\mathrm{d}\zeta/\mathrm{d}\xi$ and integrate once more to get:

$$-\frac{1}{2}\varepsilon\zeta^3+\frac{1}{2}(C^2-1)\zeta^2+A\zeta+B=\frac{\varepsilon\kappa_0}{6}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}\xi}\right)^2, \quad (3.3.3)$$

where the integration constants A and B are both of order $O(\varepsilon)$.

Two cases will now be discussed.

3.3.1 Solitary waves. A solitary wave, discovered by John Scott Russell in 1834 (and published in 1844) [21], has a single crest whose amplitude diminishes to zero as $|\xi|\to\infty$.

Since ζ, $\mathrm{d}\zeta/\mathrm{d}\xi$ and $\mathrm{d}^2\zeta/\mathrm{d}\xi^2$ vanish at infinity, so should the constants A and B. In this case Eqn (3.3.3) becomes simply:

$$\kappa_0\left(\frac{\mathrm{d}\zeta}{\mathrm{d}\xi}\right)^2=3\zeta^2\left(\frac{C^2-1}{\varepsilon}-\zeta\right), \quad (3.3.4)$$

and for the right-hand side to be positive we must have $C>1$ or, in physical variables, $C>(gh_0)^{1/2}$. This wave speed is called *supercritical*. Furthermore, we must insist that $\zeta\leqslant(C^2-1)/\varepsilon$. Hence, $(C^2-1)/\varepsilon$ is just the maximum amplitude of the crest which is *unity* because of the normalisation that specifies $C^2=1+\varepsilon$. In this case Eqn (3.3.4) can be written as

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\xi}=\left(\frac{3}{\kappa_0}\right)^{1/2}\zeta[1-\zeta]^{1/2}, \quad (3.3.5)$$

which can be integrated to give $[3/\kappa_0]^{1/2}[\xi-\xi_0]=-2\operatorname{arctanh}[(1-\zeta)^{1/2}]$, or

$$\zeta=\operatorname{sech}^2\left[\frac{1}{2}\left(\frac{3}{\kappa_0}\right)^{1/2}(\xi-\xi_0)\right]. \quad (3.3.6)$$

The corresponding profile is a *solitary hill* with its crest at $\xi=\xi_0$, but the integration constant ξ_0 may be taken to be zero. In terms of dimensional physical variables, the *surface wave profile* is:

$$\zeta(x,t)=a_0\operatorname{sech}^2[b_0(x-Ct)], \quad (3.3.7)$$

with $C^2=g(a_0+h_0)$ and $b_0=(3a_0/4h_0^3)^{1/2}$. Thus, the higher the crest, the narrower the profile. Solitary waves can be easily generated in a long tank by almost any kind of impulse.

3.3.2 Cnoidal waves. In addition to the solitary wave just discussed, periodic persistent waves are also possible in the framework of the single one-dimensional Boussinesq equation (3.3.1).

When A differs from zero but $B=0$, we can rewrite Eqn (3.3.3) as follows:

$$\frac{\kappa_0}{3}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}\xi}\right)^2=\zeta(1-\zeta)(\zeta-1+\beta) \quad (3.3.8)$$

where $C^2=1+2\varepsilon[1-(\beta/2)]$ and $2A/\varepsilon=\beta-1$, $0<1<\beta$.

This time, ζ has the minimum value of zero, the maximum value of 1 and oscillates between the two. In this range, we have $(\mathrm{d}\zeta/\mathrm{d}\xi)^2>0$; ζ cannot oscillate between zero and $-(\beta-1)$ since $(\mathrm{d}\zeta/\mathrm{d}\xi)^2<0$, although $\zeta=-(\beta-1)$ would give uniform supercritical flow. The full equation (3.3.8) has solutions which can be expressed in terms of the Jacobian elliptic function 'Cn', hence the name *cnoidal waves*:

$$\zeta=\mathrm{Cn}^2\left\{\frac{1}{2}\left(\frac{3\beta}{\kappa_0}\right)^{1/2}(\xi-\xi_0)|m\right\}, \quad m=\left(\frac{1}{\beta}\right)^{1/2}, \quad (3.3.9)$$

where m is the modulus of the elliptic function. The wavelength is:

$$\lambda_0=\frac{4h_0^3}{(3\beta)^{1/2}}\mathrm{K}(m), \quad (3.3.10)$$

where $\mathrm{K}(m)$ is a complete elliptic integral of the first kind. The reader is unlikely to be familiar with the elliptic functions. This is not particularly important and we simply observe that $\mathrm{Cn}(\mathrm{v}|m)$ is periodic, so we can now have a train of periodic waves in shallow water.†

† First, we define the integral:

$$\mathrm{v}=\int_0^{\phi}(1-m\sin^2\theta)^{-1/2}\,\mathrm{d}\theta, \quad 0\leqslant m\leqslant 1.$$

We can also derive a pair (Jacobi and Abel) of inverse functions from this integral:

$$\mathrm{Sn}(\mathrm{v}|m)=\sin\phi, \quad \mathrm{Cn}(\mathrm{v}|m)=\cos\phi.$$

There are two Jacobian elliptic functions. If $m=0$, then $\mathrm{v}=\phi$, so that $\mathrm{Cn}(\mathrm{v}|0)=\cos\phi=\cos\mathrm{v}$ and, if $m=1$, the integral can be evaluated to yield $\mathrm{v}=\operatorname{arcsech}(\cos\phi)$ and so $\mathrm{Cn}(\mathrm{v}|1)=\operatorname{sech}\mathrm{v}$. Now the period of Cn and Sn corresponds to the period 2π of cos and sin, and so the period of these elliptic functions can be written as $4\mathrm{K}(m)$, where

$$\mathrm{K}(m)=\int_0^{\pi/2}(1-m\sin^2\theta)^{-1/2}\,\mathrm{d}\theta$$

and $\mathrm{K}(m)\to\infty$ as $m\to 1$; of course, this just demonstrates the infinite 'period' of the $\mathrm{Cn}(\mathrm{v}|1)=\operatorname{sech}\mathrm{v}$ function.

Note that as $\beta \to 1$, we get the solitary waves. The wave train following an undulating bore can be regarded as a train of cnoidal waves.The cnoidal waveforms are plotted in the Fig. 2.

4. Korteweg – de Vries and Kadomtsev – Petviashvili equations

4.1 Direct asymptotic derivation of the Korteweg – de Vries equation

Here we start from the two-dimensional physical problem for a free-surface water wave with a *horizontal bottom* in the plane $z = 0$. We neglect the surface tension $(T/\rho_0 = 0)$ and we introduce the following dimensionless quantities using the depth of water h_0 and the velocity $c_0 = (gh_0)^{1/2}$

$$\phi^* = \frac{\phi}{h_0 c_0}\,, \quad (x^*, z^*) = \frac{(x,z)}{h_0}\,, \quad t^* = \frac{c_0}{h_0}\,t\,, \quad \zeta^* = \frac{\zeta}{h_0}\,. \tag{4.1.1}$$

In this case, we are dealing with the following dimensionless master problem (dropping the asterisks)

$$\phi_{xx} + \phi_{zz} = 0\,, \quad 0 < z < 1 + \zeta(t,x)\,, \tag{4.1.2a}$$

$$\phi_z = 0\,, \quad z = 0\,, \tag{4.1.2b}$$

$$\phi_z = \zeta_t + \phi_x \zeta_x\,, \quad z = 1 + \zeta(t,x)\,, \tag{4.1.2c}$$

$$\phi_t + \zeta + \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0\,, \quad z = 1 + \zeta(t,x)\,. \tag{4.1.2d}$$

If we want to obtain the KdV equation with respect to the water surface displacement, then it is necessary to introduce (from a classical dimensional analysis) the following asymptotic representation

$$\phi = \varepsilon^{1/2}\left[\phi_1(\xi,z,\tau) + \varepsilon\phi_2(\xi,z,\tau) + \varepsilon^2\phi_3(\xi,z,\tau) + \ldots\right]\,, \tag{4.1.3a}$$

$$\zeta = \varepsilon\zeta_1(\xi,\tau) + \varepsilon^2\zeta_2(\xi,\tau) + \ldots \tag{4.1.3b}$$

with the new variables

$$\xi = \varepsilon^{1/2}(x - t)\,, \quad \tau = \varepsilon^{3/2} t\,, \tag{4.1.4}$$

where $\varepsilon = (h_0/\lambda_0)^2 \ll 1$ and λ_0 is the wavelength for the long waves.

First, in the new variables (ξ, z, τ), the problem is rewritten in the following dimensionless form:

$$\varepsilon\phi_{\xi\xi} + \phi_{zz} = 0\,, \tag{4.1.5a}$$

$$\phi_z = 0 \quad \text{on} \quad z = 0\,, \tag{4.1.5b}$$

$$\phi_z = \varepsilon^{3/2}\zeta_\tau + \varepsilon\phi_\xi\zeta_\xi - \varepsilon^{1/2}\zeta_\xi\,; \tag{4.1.5c}$$

$$\varepsilon^{3/2}\phi_\tau + \frac{1}{2}\,\varepsilon\phi_\xi^2 - \frac{1}{2}\,\varepsilon^{1/2}\phi_\xi + \frac{1}{2}\,\phi_z^2 + \zeta = 0\,, \tag{4.1.5d}$$

$$\text{on} \quad z = 1 + \varepsilon\zeta_1(\xi,\tau) + \varepsilon^2\zeta_2(\xi,\tau) + \ldots\,,$$

since

$$\frac{\partial}{\partial x} = \varepsilon^{1/2}\,\frac{\partial}{\partial\xi}\,, \quad \frac{\partial}{\partial t} = \varepsilon^{3/2}\,\frac{\partial}{\partial\tau} - \varepsilon^{1/2}\,\frac{\partial}{\partial\xi}\,.$$

At a *free surface* we have $z = 1 + \varepsilon\zeta_1(\xi,\tau) + \varepsilon^2\zeta_2(\xi,\tau) + \ldots$, and, as consequence, the derivatives ϕ_σ, $\sigma = (\xi,\tau)$, and ϕ_z may be expressed by means of the Taylor expansions as:

$$\begin{aligned}\phi_\sigma = \varepsilon^{1/2}\Big\{&(\phi_1)_\sigma + \varepsilon\left[(\phi_2)_\sigma + \zeta_1(\phi_1)_{\sigma z} + (\zeta_1)_\sigma(\phi_1)_z\right] \\ &+ \varepsilon^2\left[(\phi_3)_\sigma + \zeta_1(\phi_2)_{\sigma z} + (\zeta_1)_\sigma(\phi_2)_z + \zeta_2(\phi_1)_{z\sigma}\right. \\ &\left. + (\zeta_2)_\sigma(\phi_1)_z + \frac{1}{2}\zeta_1^2(\phi_1)_{\sigma zz} + \zeta_1(\zeta_1)_\sigma(\phi_1)_{zz}\right] + O(\varepsilon^3)\Big\}_{z=1}\,,\end{aligned} \tag{4.1.6a}$$

$$\begin{aligned}\phi_z = \varepsilon^{1/2}\Big\{&(\phi_1)_z + \varepsilon\left[(\phi_2)_z + \zeta_1(\phi_1)_{zz}\right] + \varepsilon^2\left[(\phi_3)_z\right. \\ &\left. + \zeta_1(\phi_2)_{zz} + \zeta_2(\phi_1)_{zz} + \frac{1}{2}(\zeta_1)^2(\phi_1)_{zzz}\right] + O(\varepsilon^3)\Big\}_{z=1}\,.\end{aligned} \tag{4.1.6b}$$

Now, by substitution, in place of the system of equations (4.1.5) we obtain for different orders of ε: for the order ε^0

$$(\phi_1)_{zz} = 0\,, \quad (\phi_1)_z = 0\,, \quad z = 0\,,$$

$$(\phi_1)_z + \zeta_1 = (\phi_1)_\xi\,, \quad (\phi_1)_z = 0\,, \quad z = 1\,, \tag{4.1.7a}$$

and consequently

$$\phi_1 = F(\xi,\tau)\,, \quad \zeta_1 = F_\xi\,; \tag{4.1.7b}$$

for the *order* ε^1

$$(\phi_2)_{zz} = -F_{\xi\xi}\,, \quad (\phi_2)_z = 0\,, \quad z = 0\,,$$

$$(\phi_2)_z + (\zeta_1)_\xi = 0\,, \quad z = 1\,,$$

$$\zeta_2 - (\phi_2)_\xi + \frac{1}{2}\,\zeta_1^2 + (\phi_1)_\tau = 0\,, \quad z = 1\,, \tag{4.1.8a}$$

and consequently

$$\phi_2 = -\frac{1}{2}\,z^2 F_{\xi\xi} + G(\xi,\tau)\,, \tag{4.1.8b}$$

$$G_\xi = \zeta_2 + \left[(\phi_1)_\tau\right]_{z=1} + \frac{1}{2}\,\zeta_1^2 + \frac{1}{2}\,F_{\xi\xi\xi}\,;$$

for the *order* ε^2, we have for ϕ_3 the following equation:

$$(\phi_3)_{zz} = -(\phi_2)_{\xi\xi} \equiv \frac{1}{2}\,z^2 F_{\xi\xi\xi\xi} - G_{\xi\xi}\,,$$

and consequently

$$\phi_3 = \frac{1}{4!}\,z^4 F_{\xi\xi\xi\xi} - \frac{1}{2!}\,z^2 G_{\xi\xi} + H(\xi,\tau)\,, \tag{4.1.9}$$

since $(\phi_3)_z = 0$ on $z = 0$.

But, for this *order* ε^2, the free-surface condition (4.1.5b) yields also the following relationship between ϕ_1, ϕ_2, ϕ_3, ζ_1 and ζ_2

$$\left[(\phi_3)_z\right]_{z=1} + \zeta_1\left[(\phi_2)_{zz}\right]_{z=1} - (\zeta_1)_\xi\left[(\phi_1)_\xi\right]_{z=1} = (\zeta_1)_\tau - (\zeta_2)_\xi\,. \tag{4.1.10}$$

As direct consequence of relationship (4.1.10), when we utilise solutions (4.1.7b), (4.1.8b) and (4.1.9) for ϕ_1, ϕ_2 and ϕ_3, and also the third relationship (4.1.8a) for $(\zeta_2)_{z=1}$, we obtain the following reduced KdV equation for ζ_1, which is, in fact, a compatibility condition for the consistency of our asymptotic derivation connected with expansions (4.1.3a) and (4.1.3b):

$$(\zeta_1)_\tau + \frac{3}{2}\,\zeta_1(\zeta_1)_\xi + \frac{1}{6}(\zeta_1)_{\xi\xi\xi} = 0\,. \tag{4.1.11}$$

Hence, we confirm that the KdV equation emerges 'very naturally' from a consistent asymptotic expansion, with respect to the small parameter ε, when we start from the free-surface dimensionless problem (4.1.5) with expansions (4.1.3). For dimensional physical variables, in place of the KdV equation (4.1.11), we obtain:

$$(\zeta_1)_\tau + \frac{3c_0}{2h_0}\,\zeta_1(\zeta_1)_\xi = -\frac{c_0}{6}\,h_0^2(\zeta_1)_{\xi\xi\xi}\ . \tag{4.1.12}$$

The linear dispersive relationship for water waves is: $\omega^2 = gk\tanh kh_0$, from which, for *shallow-water waves*, we have

$$kh_0 \ll 1 \Rightarrow \omega = kc_0\left[1 - \frac{1}{6}\,h_0^2k^2 + \ldots\right]\ , \tag{4.1.13}$$

and we observe that the coefficient of the term on the right-hand side of the KdV equation (4.1.12) is the same as the coefficient of k^2 in the above (nonlinear) dispersion relationship (concerning the 'relation' between the model equations and the dispersion relationships, see [62], Section 11.1).

4.2 From the Boussinesq to the Korteweg – de Vries equations

Naturally, it is possible to derive the KdV equation directly from the single Boussinesq equation (3.3.1). Indeed, for the more general transient evolution of nonlinear and dispersive long water waves propagating in the positive x direction, we introduce in Eqn (3.3.1) the following new variables

$$\sigma = x - t\ , \qquad \theta = \varepsilon t\ , \tag{4.2.1}$$

since the dimensionless scale of the slow time is $(1/\varepsilon)$. In terms of these variables (σ, θ), the derivatives become $\partial/\partial x \to \partial/\partial\sigma$ and $\partial/\partial t \to \varepsilon\,\partial/\partial\theta - \partial/\partial\sigma$, and by substituting these into Eqn (3.3.1) for $F(t,x)$, we get immediately the following equation for $F(\theta/\varepsilon, \sigma + t) = f(\theta, \sigma)$:

$$\frac{\partial^2 f}{\partial\sigma\,\partial\theta} + \frac{3}{4}\,\frac{\partial}{\partial\sigma}\left(\frac{\partial f}{\partial\sigma}\right)^2 + \frac{\kappa_0}{6}\,\frac{\partial^4 f}{\partial\sigma^4} = O(\varepsilon)\ , \tag{4.2.2}$$

and we see that the similarity parameter $\kappa_0 = \delta^2/\varepsilon$ plays the central role in the consistency of Eqn (4.2.2). But the leading order $\partial f/\partial\sigma \sim \zeta + O(\varepsilon)$ is

$$\frac{\partial\zeta}{\partial\theta} + \frac{3}{2}\,\zeta\,\frac{\partial\zeta}{\partial\sigma} + \frac{\kappa_0}{6}\,\frac{\partial^3\zeta}{\partial\sigma^3} = 0\ , \tag{4.2.3}$$

while for our KdV equation in physical variables and for stationary coordinates, Eqn (4.2.3) takes the following form:

$$\frac{\partial\zeta}{\partial t} + c_0\left(1 + \frac{3}{2h_0}\,\zeta\right)\frac{\partial\zeta}{\partial x} + \frac{c_0}{6}\,h_0^2\,\frac{\partial^3\zeta}{\partial x^3} = 0\ , \tag{4.2.4}$$

where $c_0 = (gh_0)^{1/2}$.

4.3 A more complete Korteweg – de Vries equation

Here we assume that $\mathrm{We} = 0$ in the classical *two-dimensional* problem, (3.2.1a)–(3.2.1d), where $\Delta^2 = 0$. In this last problem the parameter is $\delta = kh_0$, where k is the wave number, and it may be regarded as small for either small depths or long wavelengths. We note that when δ and ε are both small in the problem (3.2.1), the nonlinearity is exactly balanced by dispersion when $\kappa_0 = \delta^2/\varepsilon = O(1)$ and $\chi_0 = 1/\kappa_0$ is small when the dispersion exceeds the nonlinearity and is of order of unity when they are balanced!

For $\delta \ll 1$ the solution is formally

$$\phi = F(x,t;\delta,\chi_0) + \delta^2\left[G - \frac{1}{2}(z+1)^2F_{xx}\right] + \delta^4\left[H - \frac{1}{2}(z+1)^2G_{xx} + \frac{1}{24}(z+1)^4F_{xxxx}\right] + O(\delta^6)\ . \tag{4.3.1}$$

It should be observed that this is only a formal expansion in powers of δ^2, since $F, G, H, \ldots$ and hence the coefficients are themselves dependent on δ; we note that $\chi_0 = 1/\kappa_0 = \varepsilon/\delta^2$. This dependence will be removed by further expansion at a later stage. In general, we might expect (with Freeman and Davey [33]) the double limit $\delta \to 0$ and $\chi_0 \to 0$ to be uniform! This expectation motivates the expansion procedure which is used below where the problem with finite χ_0 and small δ is considered, first, to derive a 'generalised KdV' equation (GKdV-equation). If δ *is small with* χ_0 *of the order of unity*, we can write the following expansion for the functions ϕ and ζ, in the two-dimensional classical problem with (4.3.1):

$$\begin{aligned} F &= F_0(\xi,\sigma,\tau) + \chi_0\delta^2F_1 + \ldots\ , \\ \zeta &= \zeta_0(\xi,\sigma,\tau) + \chi_0\delta^2\zeta_1 + \ldots\ , \end{aligned} \tag{4.3.2}$$

where we define, according to [33]

$$\xi = x - c_pt\ , \qquad \sigma = \chi_0(x - c_gt)\ , \qquad \tau = \chi_0^2\delta^2t\ , \tag{4.3.3}$$

for $\delta \ll 1$: $c_p = 1 - (1/6)\delta^2 + \ldots$, $c_g = 1 - \frac{1}{2}\delta^2 + \ldots$ are the phase velocity and the group velocity of linearised theory (see, for instance, Zeytounian [64a], pp 37–39). It is assumed that a wave packet propagates in the x-positive direction, so that the motion is unidirectional. Naturally, when χ_0 is small, then σ will be a slow variable modulating the rapid variation characterised by ξ.

Now, according to (4.3.3), we have the following formulae for the new derivatives:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial\xi} + \chi_0\,\frac{\partial}{\partial\sigma} \equiv \mathrm{D}\ , \\ \frac{\partial}{\partial t} &= \chi_0^2\delta^2\,\frac{\partial}{\partial\tau} + \frac{1}{6}\,\delta^2\,\frac{\partial}{\partial\xi} + \frac{1}{2}\,\chi_0\delta^2\,\frac{\partial}{\partial\sigma}\ . \end{aligned} \tag{4.3.4}$$

Substituting Eqns (4.3.2)–(4.3.4) into Eqns (3.2.1a)–(3.2.1d), with $\Delta^2 = 0$ and expansion (4.3.1), by equating terms of order δ^2 and for fixed χ_0, we obtain

$$\zeta_0 = \mathrm{D}F_0\ , \tag{4.3.5}$$

$$\chi_0(\mathrm{D}\zeta_1 - \mathrm{D}^2F_0) - \mathrm{D}^2G = \chi_0^2\,\frac{\partial\zeta_0}{\partial\tau} + \frac{1}{6}\,\frac{\partial\zeta_0}{\partial\xi} + \frac{1}{2}\,\chi\,\frac{\partial\zeta_0}{\partial\sigma} + \chi_0\,\mathrm{D}F_0\,\mathrm{D}\zeta_0 + \chi_0\zeta_0\,\mathrm{D}^2F_0 - \frac{1}{6}\,\mathrm{D}^4F_0\ , \tag{4.3.6}$$

$$\chi_0(\zeta_1 - \mathrm{D}F_0) - \mathrm{D}G = -\chi_0^2\,\frac{\partial F_0}{\partial\tau} - \frac{1}{6}\,\frac{\partial F_0}{\partial\xi} - \frac{1}{2}\,\chi_0\,\frac{\partial F_0}{\partial\sigma} - \frac{1}{2}\,\mathrm{D}^3F_0 - \frac{1}{2}\,\chi_0\,(\mathrm{D}F_0)^2\ . \tag{4.3.7}$$

As expected, the first equation (4.3.5) is insufficient to determine both functions F_0 and ζ_0 and it is necessary to go to the second order [Eqns (4.3.6) and (4.3.7)] to obtain a consistency condition to do this. In fact, it is sufficient to differentiate Eqn (4.3.7) with respect to D defined by the first equation in system (4.3.4) and to subtract from

Eqn (4.3.6). The result is the following GKdV equation for ζ_0:

$$\chi_0^2 \frac{\partial \zeta_0}{\partial \tau} + \frac{1}{3} \frac{\partial \zeta_0}{\partial \xi} + \chi_0 \frac{\partial \zeta_0}{\partial \sigma} + 3\chi_0 \zeta_0 \, \mathrm{D}\zeta_0 + \frac{1}{3} \mathrm{D}^3 \zeta_0 = 0 \,, \qquad \mathrm{D} \equiv \frac{\partial}{\partial \xi} + \chi_0 \frac{\partial}{\partial \sigma} \,, \tag{4.3.8}$$

which with Eqn (4.3.5) suffices to determine ζ_0 and F_0, given the appropriate boundary conditions.†

4.4 Phase plane analysis

We start with the following KdV equation

$$\beta \frac{\partial^3 u}{\partial x^3} + u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0 \,, \tag{4.4.1}$$

and we look, here, for a solution which is a function $f(\eta)$ of the form

$$u = f(\eta) \,, \qquad \eta = x - ct \,, \tag{4.4.2}$$

where c is a constant representing the propagation velocity of a stationary wave.

Substitution of $u = f(\eta)$ into Eqn (4.4.1) gives an ordinary differential equation for the function $f(\eta)$, namely:

$$\beta \frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} + \frac{1}{2} f^2 - cf = 0 \,. \tag{4.4.3}$$

The integration constant is assumed to be zero in Eqn (4.4.3) since, using the substitution $f = f^* + f^0$ and $c = c^* + f^0$, we can choose const $= -f^0(c^* + f^0/2)$.

We must bear in mind, therefore, that *an arbitrary constant can be added to any solution of Eqn* (4.4.3) *if the same constant is added to c*, which is equivalent to transfer to a moving coordinate system. Now, we note that Eqn (4.4.3) is equivalent to:

$$\beta \frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} = -\frac{\mathrm{d}U}{\mathrm{d}f} \,, \qquad U = U(f) = \frac{1}{6} f^3 - \frac{c}{2} f^2 \,, \tag{4.4.4}$$

which is the equation of motion for a particle of a mass β in a field of force with a potential $U(f)$ (see, for instance, Brekhovskikh and Goncharova [66], pp. 296–299). But, it is well known that *a bounded solution $f(\eta)$ exists only if the total energy of the particle $E = (\beta/2)(\mathrm{d}f/\mathrm{d}\eta)^2 + U(f)$ is located inside a potential hole*. The dependence $U(f)$ (for $\beta > 0$ and $c > 0$) is plotted in Fig. 3a. A bounded solution exists if $E(f) \leqslant 0$. We can write Eqn (4.4.4) also as

$$\frac{\mathrm{d}f}{\mathrm{d}\eta} = \pm \left[\frac{2}{\beta} (E - U) \right]^{1/2} \tag{4.4.5}$$

† It is also interesting to note that, in fact, the above GKdV equation (4.3.8) follows also easily from the classical KdV equation. Indeed, if in the classical KdV equation

$$\frac{\partial \zeta}{\partial t} + \frac{3}{2} \zeta \frac{\partial \zeta}{\partial x} + \frac{1}{6\chi_0} \frac{\partial^3 \zeta}{\partial x^3} = 0 \,, \qquad \chi_0 = \frac{1}{\kappa_0} = \frac{\varepsilon}{\delta^2} \,,$$

for $\zeta(x, t; \chi_0)$, we introduce the variables τ, ξ and σ via the following transformation

$$\xi = x + \frac{1}{6\chi_0^2} \tau \,, \qquad \sigma = \chi_0 x + \frac{1}{2\chi_0} \tau \,, \qquad \tau = \chi_0 t \,, \tag{4.3.9}$$

we obtain again the GKdV equation (4.3.8), but for the function $\zeta(\tau, \xi, \sigma)$. Finally, we note that this GKdV equation (4.3.8) is a very convenient equation for the derivation of the one-dimensional classical *nonlinear Schrodinger* (NLS) equation in the shallow-water limit, when $\delta \to 0$ (see Section 5).

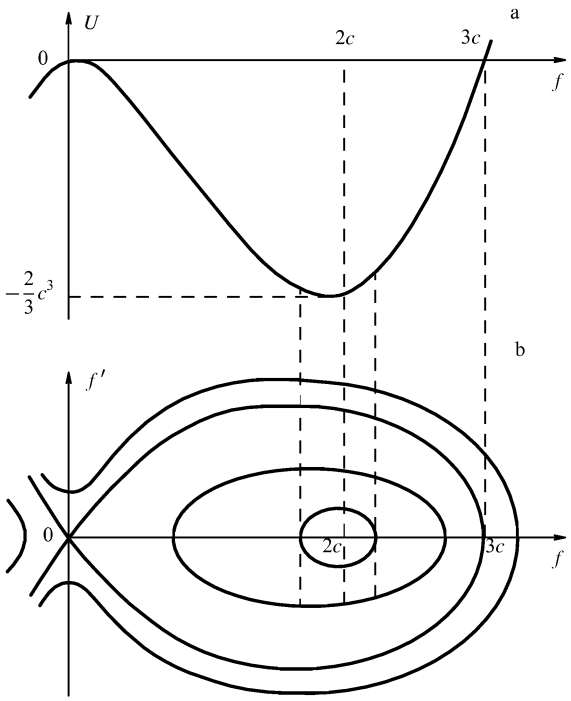

Figure 3. Equivalent potential $U(f)$ (a) and phase trajectories in the (f', f) phase plane for the KdV equation (b).

after integration. In Eqn (4.4.5), the integration constant E describes the total energy.

Using the value of $U(f)$ from Eqns (4.4.4) and (4.4.5), we can find the corresponding $\mathrm{d}f/\mathrm{d}\eta = f'$ for each value of f. The corresponding curves on the *phase plane* (f', f) at different but fixed E are called *phase trajectories*. For our KdV equation, these trajectories are shown in Fig. 3b. They intersect the f axis at a point which can be found from the equation $E = U(f)$.The most interesting trajectory corresponds to $E = 0$, in which case this equation has a double root $f_{1,2} = 0$ and a simple root $f_3 = 3c$. This trajectory separates that part of the (f', f) plane where the trajectories are *closed* (*periodic motion*) from the part where the trajectories *go to infinity* (*nonperiodic motion*). That is why it is called *the separating trajectory*. It can be proved easily by substitution that the corresponding solution of Eqn (4.4.5) $(E = 0)$ is

$$f(\eta) = 3c \cosh^{-2} \frac{\eta + \eta^0}{d} \,, \qquad d = \left(\frac{12\beta}{3c} \right)^{1/2} \,, \tag{4.4.6}$$

which is called a *soliton* and $A = 3c$ is its amplitude. Its velocity is determined in terms of its amplitude $c = A/3$. The quantity d is the length scale of the soliton and the integration constant η^0 determines the soliton position in space. The soliton form $f(\eta)$ is shown in Fig. 4a. Periodic solutions for $E < 0$ are called the *cnoidal waves*. A typical form of a cnoidal wave is shown in the Fig. 4b (the root f_2 can even be negative since an arbitrary constant can always be added to it!). As $E \to 0$, the phase trajectory approaches the separating trajectory. A representative point moving along this trajectory spends most of its time near the root f_2 of the equation $E = U(f)$ and the solution becomes a *periodic sequence of solitons*. We note, that the soliton with the largest amplitude (and the greatest speed) is the first one and that a sequence of solitons is arranged in order of their strength.

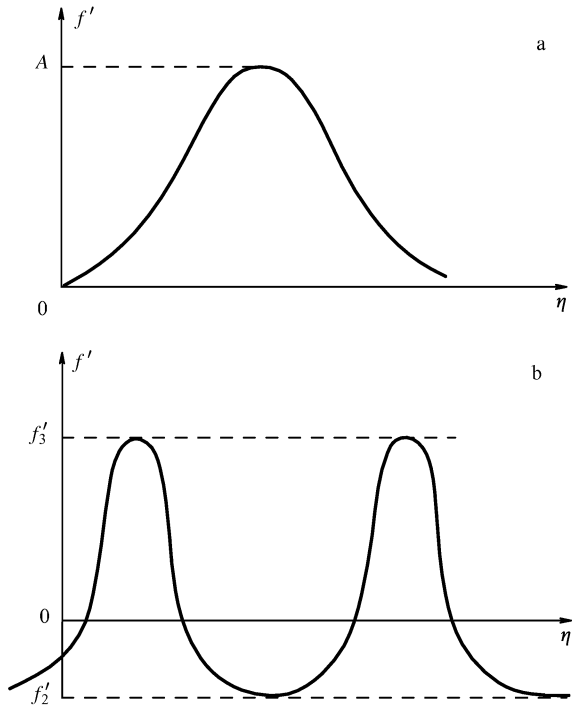

Figure 4. Two simple solutions of the KdV equation: (a) soliton, (b) cnoidal wave.

If we have two solitons and the first has a smaller amplitude, then the other soliton which was initially behind will overtake the first in the course of time. We now have a wave disturbance which is not just the sum of two solitons during some of the time. Later, however, this disturbance again breaks into separate solitons but now that with the greater amplitude is in front! Hence, in a sense, solitons behave as noninteracting linear waves. It turns out that the soliton positions after interaction are somewhat different from those in the absence of an interaction.

We note finally that introduction of Eqn (4.4.2) into the KdV equation (4.4.1), followed by two integration respect to η, leads to

$$3\beta\left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^2 = -f^3 + 3cf^2 + 6Bf + 6C \equiv F(f) \ ,$$

with $F(f) \equiv (f-a)(f-b)(e-f)$, $a < b < e$, $c = (a+b+e)/3$, $B = -(ab+be+ea)/6$, $C = abe/6$.

Here we have assumed that $F(f) = 0$ has three real roots in order to ensure real bounded solutions. If $F(f) = 0$ has three distinct roots, the solutions are *uniform wavetrains* or cnoidal waves (Fig. 4b). If $F(f) = 0$ has a double root, say $a = b$, then the cnoidal wave solutions reduce to solitary waves (Fig. 4a), while if $b = e$, only a constant state is obtained: $f = e = f^0$ is a possible solution. The cnoidal wave solutions can be expressed in terms of the Jacobian elliptic functions as follows:

$$f = b + (e-b)\,\mathrm{Cn}^2\left\{\left(\frac{e-a}{12\beta}\right)^{1/2}\left[x - \frac{1}{3}(a+b+e)t\right] | m\right\} , \tag{4.4.7}$$

with $m^2 = (e-b)/(e-a)$, $1 \geqslant m \geqslant 0$, where m represents the modulus of an elliptic function. In the limit as $b \to a$ $(m \to 1)$, Eqn (4.4.7) reduces to the solitary wave solution

$$f = f^0 + a_0\,\mathrm{sech}^2\left\{\left(\frac{a_0}{12\beta}\right)^{1/2}\left[x - \left(f^0 + \frac{1}{3}a_0\right)t\right]\right\} . \tag{4.4.8}$$

This shows that the wave velocity relative to the constant state f^0 is proportional to the amplitude. The width of the solitary wave is inversely proportional to the square root of the wave amplitude and, therefore, taller solitary waves are narrower and travel faster than shorter ones. The fact that the KdV equation is of the first order in time means that it only characterises unidirectional wave motion, so that all solitary waves represented by solution (4.4.8) will propagate in the direction of increasing x. Consequently, if two solitary waves are propagating, with the larger one initially on the left, then this wave will eventually overtake the smaller one which was initially on the right. It is well known that the solitary waves described by expression (4.4.8) show the following remarkable properties, which were first demonstrated by computer studies of Zabusky and Kruskal [25]. A single solitary wave travels without any change in shape, which means that a solitary wave is stable. When two solitary waves are well separated initially, with the larger one on the left, the faster solitary wave overtakes the slower one, they interact nonlinearly, and when this process is completed their positions are interchanged with the larger one to the right. The structure of each solitary wave is exactly the same after the nonlinear interaction as it was before: only their relative positions are interchanged. Thus a solitary wave is stable even when subjected to nonlinear interactions. This remakable stability of solitary waves in which they exhibit a *particle-like behaviour* led Zabusky and Kruskal to coin the name *soliton* [25]. A more thorough discussion of the solitary wave–soliton phenomenon is given in Section 7.

4.5 Kadomtsev–Petviashvili limit

We start with the *dimensional* equations. As before, a liquid has a velocity potential $\phi(t,x,y,z)$, which satisfies the Laplace equation

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 , \qquad -h_0 \leqslant z \leqslant \zeta(x,\,y,\,t) \ . \tag{4.5.1}$$

The flat-bottom condition is

$$\phi_z = 0 , \qquad z = -h_0 \ . \tag{4.5.2}$$

Finally, we have the following two boundary conditions on a free surface $z = \zeta(t,x,y)$

$$\phi_z = \zeta_t + \phi_x\zeta_x + \phi_y\zeta_y \ , \tag{4.5.3a}$$

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\zeta = \frac{T}{\rho_0 N^3}\left[(1+\zeta_x^2)\zeta_{yy} - 2\zeta_x\zeta_y\zeta_{xy} + (1+\zeta_y^2)\zeta_{xx}\right] , \tag{4.5.3b}$$

where $N^2 = 1 + \zeta_x^2 + \zeta_y^2$.

Now, we shall consider the full nonlinear pivotal problem described by Eqns (4.5.1)–(4.5.3) in the following KP *limiting* case:

$\varepsilon = a_0/h_0 \ll 1$ — small amplitude,
$h_0^2(k^2 + l^2) \ll 1$ — long waves,
$(l/k)^2 \ll 1$ — nearly one-dimensional waves,

where (k,l) is the horizontal (x,y) wavenumber characteristic of disturbances. We shall orient the horizontal coordinates so that the x direction is the *principal* direction of wave propagation. Finally, a_0 denotes, as before, the characteristic amplitude of the disturbances.

The KP equation results when all three above-mentioned small effects balance

$$h_0^2(k^2+l^2)=O(\varepsilon)\,,\qquad \left(\frac{l}{k}\right)^2=O(\varepsilon)\,. \tag{4.5.4}$$

Under the above assumption, the first approximation to our problem (4.5.1)–(4.5.3) reduces to the following classical linear wave equation for the elevation of a free surface

$$\frac{\partial^2\zeta}{\partial t^2}-c_0^2\frac{\partial^2\zeta}{\partial x^2}=O(\varepsilon)\,, \tag{4.5.5}$$

where $c_0^2=gh_0$.

Thus, to the lowest order, the solution of the hyperbolic equation (4.5.5) for $\zeta(t,x,y)$ may be approximated by

$$\zeta\approx\varepsilon h_0\left[F(x-c_0t,y)+G(x+c_0t,y)\right]\,, \tag{4.5.6}$$

where F and G are known in terms of the initial data (for $t=0$).

To go to a higher order (in ε), we shall define scaled dimensionless variables

$$\xi=\varepsilon^{1/2}\frac{x-c_0t}{h_0}\,,\qquad \sigma=\varepsilon^{1/2}\frac{x+c_0t}{h_0}\,,\qquad \eta=\varepsilon\,\frac{y}{h_0}\,,$$
$$\tau=\varepsilon^{3/2}t\,\frac{c_0}{h_0}\,,\qquad \mathrm{We}=\frac{T}{g\rho_0h_0^2}\,. \tag{4.5.7}$$

Let us now look at the following solution for the elevation of a free surface

$$\zeta\approx\varepsilon h_0\left[U(\xi,\eta,\tau)+V(\xi,\eta,\tau)\right]\,, \tag{4.5.8}$$

and let us apply the MS method — in this case to eliminate the secular terms of the next order in ε. We then obtain automatically the following two *KP equations* for the unknown functions U and V, respectively:

$$\frac{\partial}{\partial\xi}\left[2\frac{\partial U}{\partial\tau}+3U\,\frac{\partial U}{\partial\xi}+\left(\frac{1}{3}-\mathrm{We}^*\right)\frac{\partial^3U}{\partial\xi^3}\right]+\frac{\partial^2U}{\partial\eta^2}=0\,, \tag{4.5.9a}$$

$$\frac{\partial}{\partial\sigma}\left[2\frac{\partial V}{\partial\tau}-3V\,\frac{\partial V}{\partial\sigma}-\left(\frac{1}{3}-\mathrm{We}^*\right)\frac{\partial^3V}{\partial\sigma^3}\right]-\frac{\partial^2V}{\partial\eta^2}=0\,, \tag{4.5.9b}$$

where $U=F$ and $V=G$, if $\tau=0$.

In most cases of interest, we have $1/3<\mathrm{We}^*$ for water waves, and it follows from the linearised dispersion relationship for the initial problem (4.5.1)–(4.5.3) that the linearised phase velocity has a (local) maximum at $k=0$ and $l=0$. Thus, the waves described by the system(4.5.9) travel *faster* than their neighbours in the (k,l) plane and there should be no disturbances as $\xi\to+\infty$ or $\sigma\to-\infty$. Consequently, for example, Eqn (4.5.9a) may be integrated with respect to ξ, which gives

$$2\frac{\partial U}{\partial\tau}+3U\,\frac{\partial U}{\partial\xi}+\left(\frac{1}{3}-\mathrm{We}^*\right)\frac{\partial^3U}{\partial\xi^3}=\int_\xi^{+\infty}\frac{\partial^2U}{\partial\eta^2}\,\mathrm{d}\xi^*\,, \tag{4.5.10}$$

and this is now in the form of an *evolution equation for* $U(\xi,\eta,\tau)$.

If U and all of its derivatives vanish initially as $\xi\to-\infty$, it is evident from Eqn (4.5.10) that U will not remain zero at infinity unless:

$$\int_{-\infty}^{+\infty}\frac{\partial^2U}{\partial\eta^2}\,\mathrm{d}\xi^*=0\,. \tag{4.5.11}$$

Since U is a derivative of the velocity potential, Eqn (4.5.11) is automatically satisfied at the initial instant. Indeed, for the linearised form of Eqn (4.5.10), Eqn (4.5.11) is a constant of motion and it is sufficient to know it initially!

The constraint (4.5.11) has a simple physical interpretation. One can identify $\int U(\xi,\eta,\tau)\,\mathrm{d}\xi^*$ as the total mass of a wave in a thin strip at η. Then, condition (4.5.11) assures that the transverse derivative of mass is constant, and this prevents net flow of mass to (or from) any particular strip.

4.6 A direct asymptotic derivation of the Kadomtsev–Petviashvili equation

The asymptotic derivation of the KP equation follows in fact closely that of the KdV equation (see Section 4.1) and, therefore, we merely review only the main point of this asymptotic derivation here. First, as in Eqn (4.1.4), we employ the new variables

$$\xi=\varepsilon^{1/2}(x-t)\,,\qquad \tau=\varepsilon^{3/2}t\,,\qquad \eta=\varepsilon y\,. \tag{4.6.1a}$$

In this case, instead of Eqn (4.1.5), we are faced with the following dimensionless problem:

$$\varepsilon^2\phi_{\eta\eta}+\varepsilon\phi_{\xi\xi}+\phi_{zz}=0\,, \tag{4.6.2a}$$

$$\phi_z=0\qquad\text{for}\qquad z=0\,, \tag{4.6.2b}$$

$$\phi_z=\varepsilon^2\phi_\eta\zeta_\eta+\varepsilon^{3/2}\zeta_\tau+\varepsilon\phi_\xi\zeta_\xi-\varepsilon^{1/2}\zeta_\xi\,, \tag{4.6.2c}$$

$$\frac{1}{2}\,\varepsilon^2\phi_\eta^2+\varepsilon^{3/2}\phi_\tau+\frac{1}{2}\,\varepsilon\phi_\xi^2-\frac{1}{2}\,\varepsilon^{1/2}\phi_\xi+\frac{1}{2}\,\phi_z^2+\zeta=0\,, \tag{4.6.2d}$$

where Eqns (4.6.2c) and (4.6.2d) are satisfied on $z=1+\varepsilon\zeta_1(\xi,\eta,\tau)+\varepsilon^2\zeta_2(\xi,\eta,\tau)+\ldots$.

For ϕ we have:

$$\phi=\varepsilon^{1/2}\left[F(\xi,\eta,\tau,z)+\varepsilon\phi_2+\varepsilon^2\phi_3+\ldots\right]\,. \tag{4.6.3}$$

We can see from Eqns (4.6.2a), (4.6.2c) and (4.6.2d) that the dependence on η appears explicitly only at the order ε^2 and, as a consequence, the results of Section 4.1 remain unchanged up to Eqn (4.1.8b). In place of solution (4.1.9), we obtain now:

$$\phi_3=\frac{1}{4!}\,z^4F_{\xi\xi\xi\xi}-\frac{1}{2!}\,z^2G_{\xi\xi}-\frac{1}{2!}\,z^2F_{\eta\eta}+H(\xi,\eta,\tau)\,, \tag{4.6.4}$$

where $F=F(\xi,\eta,\tau)$ and $\zeta_1=F_\xi$.

At the order ε^2, relationship (4.1.10) is also unchanged [at this order the term $\varepsilon^2\phi_\eta\zeta_\eta\sim\varepsilon^{7/2}(\zeta_1)_\eta F_\eta$ cannot appear]. But, if we now utilise the new solution (4.6.4) for ϕ_3, in place of solution (4.1.9), we derive from relationship (4.1.10) the following new 'two-dimensional' relationship for $F(\xi,\eta,\tau)$:

$$\frac{1}{3}\,F_{\xi\xi\xi\xi}+F_{\eta\eta}+3F_\xi F_{\xi\xi}+2F_{\tau\xi}=0\,. \tag{4.6.5}$$

Finally, if we introduce the function $f(\xi,\eta,\theta)=\frac{2}{3}F_\xi(6\theta,\xi,\eta)$, where $\theta=\tau/6$, we obtain the classical KP equation in the form

$$\frac{\partial}{\partial\xi}\left(\frac{\partial f}{\partial\theta}+6f\,\frac{\partial f}{\partial\xi}+\frac{\partial^3f}{\partial\xi^3}\right)+3\,\frac{\partial^2f}{\partial\eta^2}=0\,, \tag{4.6.6}$$

$$\zeta=\frac{3}{2}\,\varepsilon f+O(\varepsilon^2)\,. \tag{4.6.7}$$

To make the model quite explicit, we can also write the KP equation for the waves travelling to the right in the dimensional form for the elevation of a free surface:

$$\frac{\partial}{\partial x}\left(\frac{1}{c_0}\frac{\partial\zeta}{\partial t}+\frac{\partial\zeta}{\partial x}+\frac{3}{2h_0}\zeta\frac{\partial\zeta}{\partial x}+\frac{h_0^2}{6}\frac{\partial^3\zeta}{\partial x^3}\right)+\frac{1}{2}\frac{\partial^2\zeta}{\partial y^2}=0\,. \quad (4.6.8)$$

4.7 Generalised Kadomtsev–Petviashvili equation

In the three-dimensional classical problem, in place of expansion (4.3.2), we have

$$F=F_0(\xi,\sigma,\eta,\tau)+\chi_0\delta^2F_1+\ldots\,,$$
$$\zeta=\zeta_0(\xi,\sigma,\eta,\tau)+\chi_0\delta^2\zeta_1+\ldots\,, \quad (4.7.1)$$

with [see definitions (4.3.3)]

$$\xi=x-c_pt\,,\quad \sigma=\chi_0(x-c_gt)\,,\quad \tau=\chi_0^2\delta^2t\,,\quad \eta=\frac{y}{v_0}\,, \quad (4.7.2)$$

where $\chi_0=1/\kappa_0$ is small if the dispersion *exceeds* the nonlinearity and is of *order of unity* when they are *balanced*.

In this case, as in Section 4.3, we can derive easily the following generalised Kadomtsev–Petviashvili (GKP) equation for the elevation of a free surface $\zeta_0(\xi,\sigma,\eta,\tau)$:

$$2\chi_0^2\frac{\partial\zeta_0}{\partial\tau}+\frac{1}{3}\frac{\partial\zeta_0}{\partial\xi}+\chi_0\frac{\partial\zeta_0}{\partial\sigma}+3\chi_0\zeta_0\,\mathrm{D}\zeta_0+\frac{1}{3}\mathrm{D}^3\zeta_0=-\chi_0^2\frac{\partial^2F_0}{\partial\eta^2}\,, \quad (4.7.3)$$

$$\zeta_0=\mathrm{D}F_0\,,\quad \mathrm{D}\equiv\frac{\partial}{\partial\xi}+\chi_0\frac{\partial}{\partial\sigma}\,. \quad (4.7.4)$$

According to Freeman and Davey [33], this GKP equation, with definitions (4.7.4), is very convenient for the derivation of the two-dimensional NLS–Poisson system of two equations, obtained first by Davey and Stewartson (1974) [39] in the long-wave limit (see Section 5).

4.8 Second-order Kadomtsev–Petviashvili equation

We return now to the Q1DGB equations (3.2.10a) and (3.2.10b) for the functions $F(x,y,t)$ and $\zeta(x,y,t)$, where $\mathrm{We}^*=\varepsilon\kappa_0\mathrm{We}$, according to expressions (2.3.18) and (2.3.20). Again, in Eqns (3.2.10a) and (3.2.10b), we introduce a slow time $\tau=\varepsilon t$ and we suppose that $F=F(x,y,t,\tau)$ and $\zeta=\zeta(x,y,t,\tau)$, and that $\partial/\partial t=\partial/\partial t+\,+\varepsilon\,\partial/\partial\tau$.

Then, in place of Eqns (3.2.10a) and (3.2.10b), we obtain the following system of two equations for $F(x,y,t,\tau)$ and $\zeta(x,y,t,\tau)$

$$\begin{aligned}&(F)_t+\zeta+\varepsilon\left\{\frac{1}{2}(F)_{xx}^2-\frac{\kappa_0}{2}(F)_{xxt}-\kappa_0\mathrm{We}\,\zeta_{xx}+(F)_\tau\right\}\\&+\varepsilon^2\left\{-\frac{\kappa_0}{2}(F)_{xx\tau}+\frac{\kappa_0^2}{24}(F)_{xxxxt}+\frac{\kappa_0}{2}(F)_{xx}^2\right.\\&-\frac{v_0^2}{2}(F)_{yyt}+\frac{v_0^2}{2\kappa_0}(F)_y^2-\frac{\kappa_0}{2}(F)_{xxx}(F)_x\\&\left.-\kappa_0\left[\zeta(F)_{xx}\right]_t-v_0^2\mathrm{We}\,\zeta_{yy}\right\}=O(\varepsilon^3)\,,\end{aligned} \quad (4.8.1a)$$

$$\begin{aligned}&\zeta_t+(F)_{xx}+\varepsilon\left\{\frac{v_0^2}{\kappa_0}(F)_{yy}+\left[\zeta(F)_x\right]_x-\frac{\kappa_0}{6}(F)_{xxxx}+\zeta_\tau\right\}\\&+\varepsilon^2\left\{\frac{\kappa_0^2}{120}(F)_{xxxxxx}-\frac{\kappa_0}{2}\left[\zeta(F)_{xxx}\right]_x+\frac{v_0^2}{\kappa_0}\left[\zeta(F)_y\right]_y\right.\\&\left.-\frac{v_0^2}{6}(F)_{xxyy}\right\}=O(\varepsilon^3)\,.\end{aligned} \quad (4.8.1b)$$

If the appropriate asymptotic expansions of F and ζ are

$$F=F_0+\varepsilon F_1+\varepsilon^2F_2+\ldots\,,\qquad \zeta=\zeta_0+\varepsilon\zeta_1+\varepsilon^2\zeta_2+\ldots\,, \quad (4.8.2)$$

we obtain the following set of equations to different powers of ε

$$O(\varepsilon^0):\quad (F_0)_t+\zeta_0=0\,,\qquad (\zeta_0)_t+(F_0)_{xx}=0\,; \quad (4.8.3a)$$

$$\begin{aligned}O(\varepsilon^1):\quad &(F_1)_t+\zeta_1+\frac{1}{2}(F_0)_x^2-\frac{\kappa_0}{2}(F_0)_{xxt}\\&-\kappa_0\mathrm{We}(\zeta_0)_{xx}+(F_0)_\tau=0\,,\\&(\zeta_1)_t+(F_1)_{xx}-\frac{v_0^2}{\kappa_0}(F_0)_{yy}+\left[\zeta_0(F_0)_x\right]_x\\&-\frac{\kappa_0}{6}(F_0)_{xxxx}+(\zeta_0)_\tau=0\,;\end{aligned} \quad (4.8.3b)$$

$$\begin{aligned}O(\varepsilon^2):\quad &(F_2)_t+\zeta_2+(F_0)_x(F_1)_x-\frac{\kappa_0}{2}(F_1)_{xxt}+(F_1)_\tau\\&-\kappa_0\mathrm{We}(\zeta_1)_{xx}-\frac{\kappa_0}{2}(F_0)_{xx\tau}+\frac{\kappa_0^2}{24}(F_0)_{xxxxt}\\&+\frac{\kappa_0}{2}(F_0)_{xx}^2-\frac{\kappa_0}{2}(F_0)_x(F_0)_{xxx}\\&-\kappa_0\left[\zeta_0(F_0)_{xx}\right]_t+\frac{v_0^2}{2\kappa_0}(F_0)_y^2\\&-\frac{v_0^2}{2}(F_0)_{yyt}-v_0^2\mathrm{We}(\zeta_0)_{yy}=0\,,\\&(\zeta_2)_t+(F_2)_{xx}-\frac{\kappa_0}{6}(F_1)_{xxxx}+\frac{v_0^2}{\kappa_0}(F_1)_{yy}\\&+\left[\zeta_0(F_1)_x+\zeta_1(F_0)_x\right]_x+(\zeta_1)_\tau\\&+\frac{\kappa_0}{120}(F_0)_{xxxxxx}-\frac{\kappa_0}{2}\left[\zeta_0(F_0)_{xxx}\right]_x\\&+\frac{v_0^2}{\kappa_0}\left[\zeta_0(F_0)_y\right]_y-\frac{v_0^2}{6}(F_0)_{xxyy}=0\,.\end{aligned} \quad (4.8.3c)$$

From Eqn (4.8.3a) it follows that F_0 and ζ_0 depend on x and t either through $x-t$ or $x+t$. Here we shall only consider the wave propagating to the right, and hence we shall assume that the dependences of F_0 and ζ_0 on x and t appear only through the variable $\xi=x-t$ and in this case we have $\partial/\partial x=\partial/\partial\xi$ and $\partial/\partial t=-\partial/\partial\xi$. Furthermore, when F_1 and ζ_1 depend *also only* on ξ, τ and y, Eqn (4.8.3b) can be reduced to

$$\begin{aligned}&\frac{\partial F_1}{\partial\xi}=\zeta_1+\frac{1}{2}\left(\frac{\partial F_0}{\partial\xi}\right)^2+\frac{\kappa_0}{2}\frac{\partial^3F_0}{\partial\xi^3}-\kappa_0\mathrm{We}\frac{\partial^3F_0}{\partial\xi^3}+\frac{\partial F_0}{\partial\tau}\,,\\&\frac{\partial\zeta_1}{\partial\xi}=\frac{\partial^2F_1}{\partial\xi^2}-\frac{\kappa_0}{6}\frac{\partial^4F_0}{\partial\xi^4}+\frac{\partial^2F_0}{\partial\xi\,\partial\tau}+\frac{\partial}{\partial\xi}\left(\frac{\partial F_0}{\partial\xi}\right)^2+\frac{v_0^2}{\kappa_0}\frac{\partial^2F_0}{\partial y^2}\,,\end{aligned} \quad (4.8.4)$$

and we derive, again, the classical KP equation for the function $F_0(\xi,\tau,y)$, as a compatibility condition for the system of two equations (4.8.4), namely

$$\frac{\partial}{\partial\xi}\left[2\frac{\partial F_0}{\partial\tau}+\frac{3}{2}\left(\frac{\partial F_0}{\partial\xi}\right)^2+\kappa_0\left(\frac{1}{3}-\mathrm{We}\right)\frac{\partial^3F_0}{\partial\xi^3}\right]+\frac{v_0^2}{\kappa_0}\frac{\partial^2F_0}{\partial y^2}=0\,. \quad (4.8.5)$$

In addition, we note that

$$\zeta_0=\frac{\partial F_0}{\partial\xi}\,. \quad (4.8.6)$$

But, before determination of $\zeta_1(\xi,\tau,y)$ it is necessary to find first the function $F_1(\xi,\tau,y)$, since

$$\zeta_1 = \frac{\partial F_1}{\partial \xi} - A(F_0)\,, \tag{4.8.7}$$

with

$$A(F_0) = \frac{1}{2}\left(\frac{\partial F_0}{\partial \xi}\right)^2 + \kappa_0\left(\frac{1}{2} - \mathrm{We}\right)\frac{\partial^3 F_0}{\partial \xi^3} + \frac{\partial F_0}{\partial \tau}\,. \tag{4.8.8}$$

Now, if again $F_2 = F_2(\xi,\tau,y)$ and $\zeta_2 = \zeta_2(\xi,\tau,y)$, we obtain the following result from Eqn (4.8.3c):

$$\frac{\partial F_2}{\partial \xi} = \zeta_2 + \frac{\partial F_0}{\partial \xi}\frac{\partial F_1}{\partial \xi} + \kappa_0\left(\frac{1}{2} - \mathrm{We}\right)\frac{\partial^3 F_1}{\partial \xi^3} + \frac{\partial F_1}{\partial \tau} + B(F_0) - \kappa_0\,\mathrm{We}\,\frac{\partial^2}{\partial \xi^2}A(F_0)\,, \tag{4.8.9a}$$

$$\frac{\partial \zeta_2}{\partial \xi} = \frac{\partial^2 F_2}{\partial \xi^2} - \frac{\kappa_0}{6}\frac{\partial^4 F_1}{\partial \xi^4} + \frac{v_0^2}{\kappa_0}\frac{\partial^2 F_1}{\partial y^2} + 2\frac{\partial}{\partial \xi}\left(\frac{\partial F_0}{\partial \xi}\frac{\partial F_1}{\partial \xi}\right) + \frac{\partial^2 F_1}{\partial \xi\,\partial \tau} + C(F_0) - \frac{\partial}{\partial \tau}A(F_0) - \frac{\partial}{\partial \xi}\left[A(F_0)\frac{\partial F_0}{\partial \xi}\right]\,, \tag{4.8.9b}$$

with

$$B(F_0) = -\frac{\kappa_0^2}{24}\frac{\partial^5 F_0}{\partial \xi^5} + \frac{\kappa_0}{2}\left(\frac{\partial^2 F_0}{\partial \xi^2}\right)^2 - \frac{\kappa_0}{2}\frac{\partial F_0}{\partial \xi}\frac{\partial^3 F_0}{\partial \xi^3} + \kappa_0\frac{\partial}{\partial \xi}\left(\frac{\partial F_0}{\partial \xi}\frac{\partial^2 F_0}{\partial \xi^2}\right) - \frac{\kappa_0}{2}\frac{\partial^3 F_0}{\partial \xi^2\,\partial \tau} + v_0^2\left[\frac{1}{2\kappa_0}\left(\frac{\partial F_0}{\partial y}\right)^2 + \left(\frac{1}{2} - \mathrm{We}\right)\frac{\partial^3 F_0}{\partial \xi\,\partial y^2}\right]\,, \tag{4.8.10a}$$

$$C(F_0) = \frac{\kappa_0^2}{120}\frac{\partial^6 F_0}{\partial \xi^6} - \frac{\kappa_0}{6}\frac{\partial}{\partial \xi}\left(\frac{\partial F_0}{\partial \xi}\frac{\partial^3 F_0}{\partial \xi^3}\right) + v_0^2\left[\frac{\partial}{\partial y}\left(\frac{\partial F_0}{\partial \xi}\frac{\partial F_0}{\partial y}\right) - \frac{1}{6}\frac{\partial^4 F_0}{\partial \xi^2\,\partial y^2}\right]\,. \tag{4.8.10b}$$

Finally, from the two equations (4.8.9a) and (4.8.9b) for F_2 and ζ_2, we can eliminate the function ζ_2; we then obtain the following inhomogeneous (but linear) equation for the function $F_1(\xi,\tau,y)$:

$$\frac{\partial}{\partial \xi}\left[2\frac{\partial F_1}{\partial \tau} + 3\frac{\partial F_0}{\partial \xi}\frac{\partial F_1}{\partial \xi} + \kappa_0\left(\frac{1}{3} - \mathrm{We}\right)\frac{\partial^3 F_1}{\partial \xi^3}\right] + \frac{v_0^2}{\kappa_0}\frac{\partial^2 F_1}{\partial y^2} = \frac{\partial}{\partial \xi}\left(A(F_0)\frac{\partial F_0}{\partial \xi}\right) + \left(\frac{\partial}{\partial \tau} + \kappa_0\,\mathrm{We}\,\frac{\partial^2}{\partial \xi^2}\right)A(F_0) - \frac{\partial[B(F_0)]}{\partial \xi} - C(F_0)\,. \tag{4.8.11}$$

Thus the KP equation (4.8.5) for F_0 and the linear inhomogeneous equation (4.8.11) for F_1 describe the *second-order KP approximation*. We can also calculate ζ_0 [from Eqn (4.8.6)] and ζ_1 [from Eqn (4.8.7) with Eqns (4.8.8) and (4.8.11)].

It is now well established that the KP equation is the lowest-order nontrivial consequence of the perturbation approximation for the Q1DGB equations describing weakly dispersive waves.

It is also important to note that the KP equation admits solitary wave solutions. Indeed, if we write the KP equation (4.8.5) for F_0 as an equation for the function $\zeta_0 = \partial F_0/\partial \xi$ when $\mathrm{We} = 0$,

$$\frac{\partial}{\partial \xi}\left(\frac{\partial \zeta_0}{\partial \tau} + \frac{3}{2}\zeta_0\frac{\partial \zeta_0}{\partial \xi} + \frac{\kappa_0}{6}\frac{\partial^3 \zeta_0}{\partial \xi^3}\right) + \frac{v_0^2}{2\kappa_0}\frac{\partial^2 \zeta_0}{\partial y^2} = 0\,, \tag{4.8.12}$$

then we can seek the following solution

$$\zeta_0 = \zeta_0(\theta)\,, \qquad \theta = \xi - \alpha\tau + \beta y\,. \tag{4.8.13}$$

In this case we obtain the *KP soliton* solution of Eqn (4.8.12) in the following form

$$\zeta_0 = \mathrm{sech}^2\left[\xi - \left(1 + \frac{v_0^4}{3}\right)\tau + v_0^2 y\right]\,, \tag{4.8.14}$$

$$\kappa_0 = \frac{3}{4}\,, \qquad \beta = v_0^2\,, \qquad \alpha = 1 + \frac{\beta}{3}v_0^2\,, \tag{4.8.15}$$

where v_0^2 is a parameter describing a (small) inclination of the wave relative to the main direction of propagation. In the absence of the y direction (when $v_0^2 = 0$ in the one-dimensional case), the solution (4.8.14) reduces to the KdV soliton solution.

Introducing this above solution (4.8.14) for $\partial F_0/\partial \xi = \zeta_0$ into Eqn (4.8.11) for F_1, we can find the second-order term $F_1(\theta)$ and we can introduce the notion of a '*dressed KP soliton*', that is a KP soliton with second-order corrections.

However, this dressed KP soliton solution may also involve the appearance of secular terms (as in the KdV theory). Elimination of these secularities, in addition to ξ and τ, requires — in our above reductive perturbation method — introduction of the following new slow variables: $X = \varepsilon(x - t)$, $T = \varepsilon^2 t, \ldots$. Actually, we are not certain if it is necessary to introduce also a new slow transverse variable $\eta = \varepsilon y$. Naturally, in this case Eqn (4.8.11) for F_1 changes and in the transformed equation for $F_1(\xi,\tau,y,X,T,\ldots)$ we have some new terms containing derivatives with respect to $X, T, \ldots$. As consequence, we can assume the following soliton solution:

$$F_1(\theta,X,T,\ldots) = A\,\mathrm{sech}^2\left[B(\theta + C)\right]\,, \tag{4.8.16}$$

with $A = A(X,T,\ldots)$, $B = B(X,T,\ldots)$ and $C = C(X,T,\ldots)$.

We can now use the added freedom to eliminate the secular-producing terms. As the secular-free conditions, we obtain a set of equations for the 'modulating' functions $A(X,T,\ldots)$, $B(X,T,\ldots)$ and $C(X,T,\ldots)$. In the KdV theory these secular-producing terms were eliminated by Sugimoto and Kakutani (1977) [56] (see, also, the book by Jeffrey and Kawahara [57]).

4.9 Some features of wave solutions of the Kadomtsev–Petviashvili equation

The canonical form of the KP equation is

$$\frac{\partial}{\partial \xi}\left[\frac{\partial f}{\partial \theta} + 6f\frac{\partial f}{\partial \xi} + \frac{\partial^3 f}{\partial \xi^3}\right] + 3\frac{\partial^2 f}{\partial \eta^2} = 0\,, \tag{4.9.1}$$

where

$$\zeta = \frac{3}{2}\varepsilon f + O(\varepsilon^2)\,.$$

The N-soliton wave solution of the above KP equation was derived by Satsuma (1976) [67] and its structure was elucidated by Miles [22] (see, also, the useful review paper by Freeman [52]).

One interesting use of the computer is to plot known analytic wave solutions: when the solutions are compli-

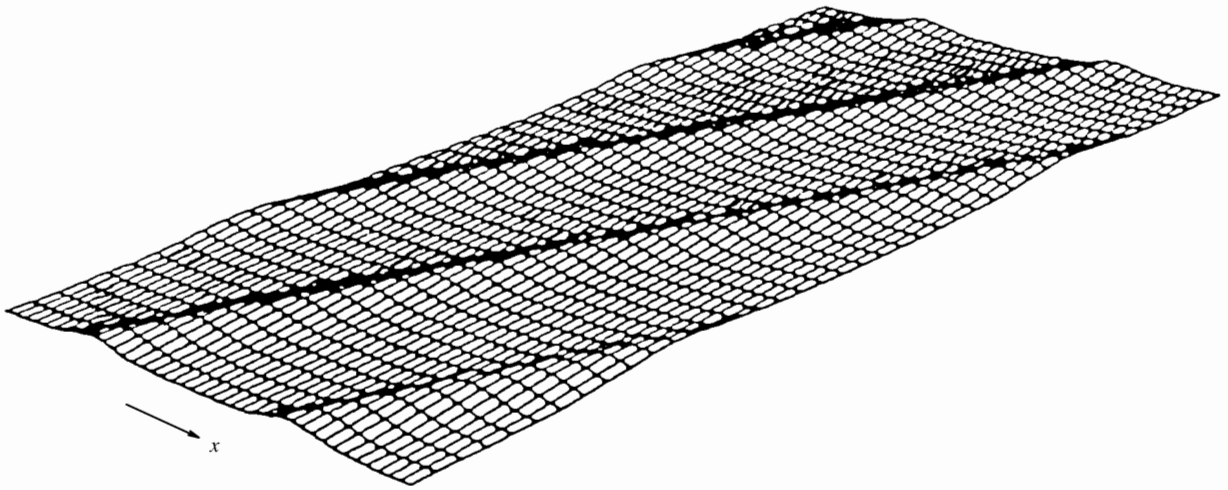

Figure 5. Cnoidal wave solution of the KP equation.

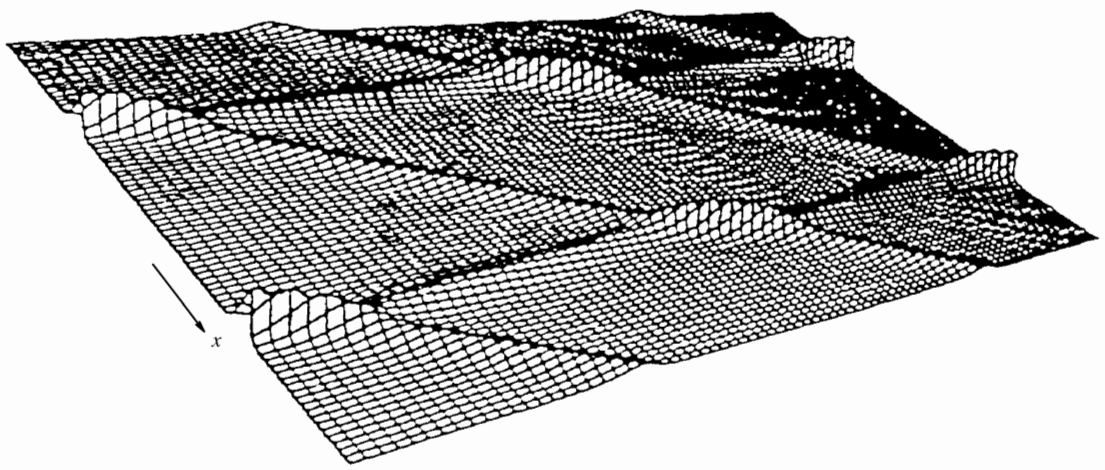

Figure 6. Two-soliton wave solution of the KP equation.

cated, a diagram is often worth pages of analysis! A typical *cnoidal KP wave*, a solution of the KP equation, is shown in Fig. 5; Fig. 6 shows a two-soliton solution and Fig. 7 gives a *three-soliton* wave solution of the KP equation. In Fig. 7 there are three plane-wave solitons with an interaction region where several short sections of waves appear. There is a phase shift in each plane soliton caused by the interaction. This diagram corresponds to Fig. 6a in Johnson's review [48], and the reader is referred to this paper for further details.

According to Segur and Finkel [68] (Figs 5 and 6 corresponds to Figs 1 and 2 in their paper), the KP equation (4.9.1) is Galilean-invariant, so that spatially periodic solutions may be normalised by imposing the condition

$$\lim_{L\to\infty}\left[\frac{1}{L}\int_0^L \zeta(x+x^0, y, t)\right] = 0\,, \tag{4.9.2}$$

which represents normalisation of a free surface $z=\zeta(x, y, t)$ and which implies that h_0 (we suppose that the bottom is the plane $z=-h_0$) is the mean depth of the liquid, and that $z=0$ ($\zeta=0$) in the absence of any motion! Naturally, the KP equation admits waves that travel along any direction in (x,y) plane, but we can expect them to model water waves accurately only if they propagate primarily in the x direction! In contrast to the KdV equation, the contribution of transverse (perpendicular) dynamics seems to be modest in the KP equation (4.9.1). However, the additional term $3\partial^2 f/\partial\eta^2$ in Eqn (4.9.1), absent from the KdV equation, opens the door to a wealth of physical effects.

The reader is referred to the recent book by Infeld and Rowlands [44] for deeper analysis of this KP equation.

In the review paper by Freeman [52], the reader can find also a very interesting exposition of the soliton interaction in two dimensions based on an examination of the structure of the two- and three-soliton solutions of the KP equation. As in the case of the KdV equation, the general technique for solution of the KP equation is the inverse scattering transform (IST, see Section 7), which can be used to construct a general multisoliton solution. For example, the interaction of two plane waves in the far field, *after a long time*, when $\tau=\varepsilon t=O(1)$ [52, 69], can be considered, and it is sufficient to discuss the far-field development of such waves. In this situation we assume that the disturbances are localised near some line $\boldsymbol{n}\cdot\boldsymbol{r}-t=\text{const}$.

Such waves need only be considered in the *neighbourhood of their interaction zone*, since far from this zone the

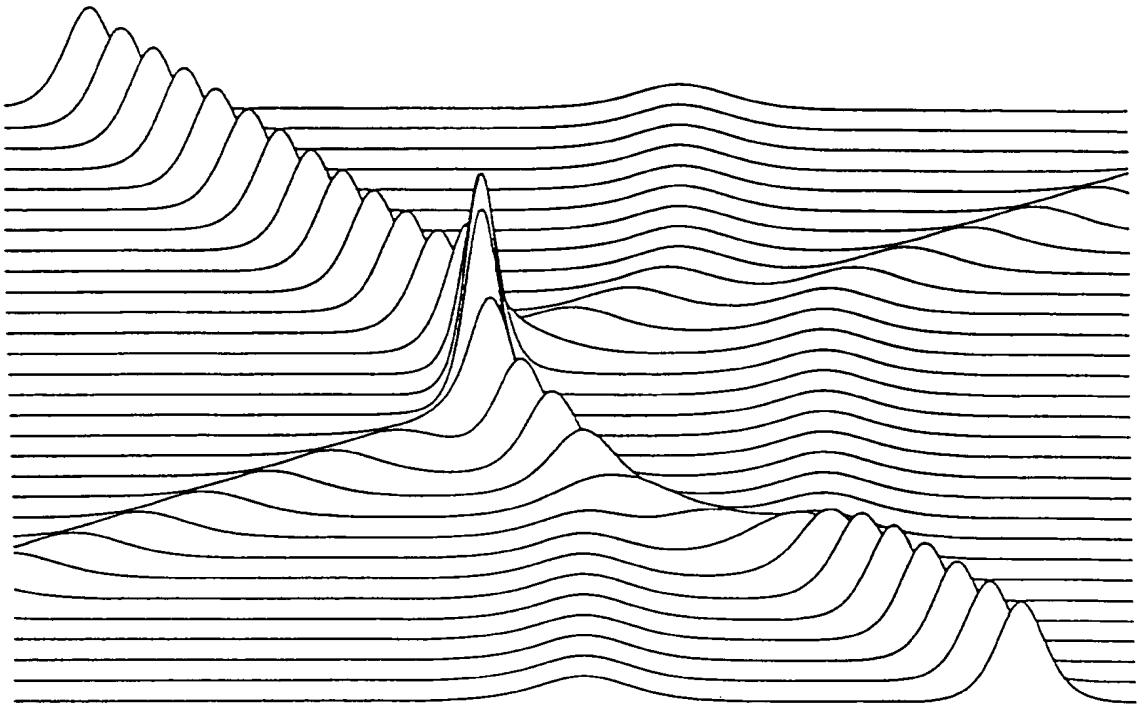

Figure 7. Three-soliton wave solution of the KP equation.

waves are uninfluenced by the presence of each other. This interaction was called 'weak' by Miles [69]. If we introduce two plane-wave coordinates ξ_1 and ξ_2, where $\xi_i = \boldsymbol{n}_i \cdot \boldsymbol{r} - t$, $\boldsymbol{r} = (x, y)$, $\boldsymbol{n}_i = (\cos\theta_i, \sin\theta_i)$, and $i = 1, 2$, we obtain [from the classical three-dimensional classical problem, assuming that the amplitude parameter ε and the dispersive (long waves) parameter δ are small, but $\delta^2/\varepsilon = 1$] the KdV equation if $1 - \cos(\theta_1 - \theta_2) = O(1)$. But the above expansion technique (for the derivation of the KdV equation) obviously fails when

$$1 - \cos(\theta_1 - \theta_2) = O(\varepsilon) \ . \tag{4.9.3}$$

In this case the waves are *almost aligned* and $\theta_1 - \theta_2 = O(\varepsilon^{1/2})$. The interaction is then *no longer weak* and, following Miles [69], is referred to as *strong*. The two phases now differ only by the order ε and it is convenient to introduce, as coordinates, ξ_1 and the normal to ξ_1 with an appropriate scaling of η. Hence we write

$$\eta = \varepsilon^{1/2}(y\cos\theta_1 - x\sin\theta_2) \ , \qquad \xi = \xi_1 \ . \tag{4.9.4}$$

For this 'strong' case we can derive the following KP equation:

$$\frac{\partial}{\partial\xi}\left(2\frac{\partial\zeta_0}{\partial\tau} + 3\zeta_0\frac{\partial\zeta_0}{\partial\xi} + \frac{1}{3}\frac{\partial^3\zeta_0}{\partial\xi^3}\right) + \frac{\partial^2\zeta_0}{\partial\eta^2} = 0 \ , \tag{4.9.5}$$

$$\zeta = \varepsilon\zeta_0 + \ldots$$

The linear dispersion relationship for the above KP equation can be written as $2\omega k = k^4/3 + m^2$, where the phase func-tion is $\beta = k\xi + m\eta - \omega\tau$. A convenient parametrisation of this relation is obtained by putting $k = \sqrt{6}(1+n)$, $m = 6(n^2 - l^2)$, $\omega = 4\sqrt{6}(l^3 + n^3)$. A single skewed soliton solution of this KP equation thus becomes:

$$\zeta_0 = 2(l+n)^2\,\mathrm{sech}^2\left[\frac{1}{2}\beta(l,n)\right] \ , \tag{4.9.6}$$

which corresponds to a KdV soliton

$$\zeta_0 = 8l^2\,\mathrm{sech}^2\left[\sqrt{6}\,l(\xi - 4l^2\tau)\right] \ ,$$

when $l = n$ and the wave propagates in the ξ direction.

The wave amplitude is now $2(l+n)^2$ and, in general, the wave is characterised by the two parameters l and n.

Finally, the two-soliton solution of the KP equation is

$$\zeta_0 = \frac{4}{3}\frac{\partial^2}{\partial\xi^2}\log f \ , \tag{4.9.7}$$

where

$$f = 1 + \exp\beta_1 + \exp\beta_2 + N_2\exp(\beta_1 + \beta_2) \ , \tag{4.9.8}$$

$N_2 = (l_1 - l_2)(n_1 - n_2)/(l_1 + n_2)(l_2 + n_1)$, and $\beta_i = \beta(l_i, n_i)$, where $i = 1, 2$, are the phases of the two solitons.

5. Nonlinear Schrodinger and Schrodinger–Poisson equations

5.1 Nonlinear Schrodinger equation

A truly linear system has a dispersion relationship which is independent of the amplitude. However, let us assume that the growth of a harmonic wave in a *weakly nonlinear* system can be represented by a dispersion relationship which is amplitude-dependent! Such a situation actually occurs in the nonlinear theory of water waves and we can suppose that

$$\omega = \omega\big(k;\,|a|^2\big) \ . \tag{5.1.1}$$

A Taylor expansion around some suitable wave number k_0 and frequency ω_0 gives:

$$\omega - \omega_0 = \left(\frac{\partial\omega}{\partial k}\right)_0 (k - k_0) + \frac{1}{2}\left(\frac{\partial^2\omega}{\partial k^2}\right)_0 (k - k_0)^2 + \left(\frac{\partial\omega}{\partial|a|^2}\right)_0 |a|^2 + \ldots \ . \tag{5.1.2}$$

Eqn (5.1.2) is the Fourier-space equivalent of an operator equation which, when operating on the amplitude a, yields:

$$\mathrm{i}\left[\frac{\partial}{\partial t}+\left(\frac{\partial\omega}{\partial k}\right)_0\frac{\partial}{\partial x}\right]a+\frac{1}{2}\left(\frac{\partial^2\omega}{\partial k^2}\right)_0\frac{\partial^2 a}{\partial x^2}-\left(\frac{\partial\omega}{\partial|a|^2}\right)_0 a|a|^2=0\ , \tag{5.1.3}$$

where higher terms are neglected [32].

Eqn (5.1.3) is the nonlinear Schrodinger equation and the name 'nonlinear Schrodinger' (NLS) has been coined precisely because its structure is that of the Schrodinger equation of quantum mechanics with $|a|^2$ as the potential, although in most situations it is unrelated to the real quantum Schrodinger equation other than in name. In fact, it plays a significant role in the theory of the propagation of the envelopes of wave trains in many stable dispersive physical system in which no dissipation occurs. The above rather heuristic derivation of the NLS equation shows how the effect of the nonlinear term can be crudely modelled by thinking of the system as having an amplitude-dependent dispersion relationship.

This quick derivation method tells us how the NLS equation arises but, unfortunately, for a specific set of model equations it does not give us the values of the coefficients in the final NLS equation, in particular the $(\partial\omega/\partial|a|^2)_0$ term. As we shall see below, the sign of this term is rather important. At this point it is preferable to introduce a more formal mathematical method which can be applied in general to a large range of nonlinear equations when we want to know the development of a *slowly varying envelope modulating a fast carrier wave*.

This latter property means that many wavelengths of the carrier wave are contained in just one wavelength of the envelope. Consequently,

$$\alpha=\frac{\lambda_c}{\lambda_e}\ll 1\ , \tag{5.1.4}$$

where λ_c and λ_e are typical wavelengths of the carrier wave and of the envelope, respectively. Since x and t are normal space and time variables, for a two-dimensional carrier wave we can define a set of 'slow' space and time variables:

$$X_n=\alpha^n x\ ,\qquad T_n=\alpha^n t\ . \tag{5.1.5}$$

These variables describe the motion of the envelope and from now on they will be considered as independent variables (MS or 'two time' method). In searching for the method to find the evolution equation of the envelope of oscillations for a given nonlinear equation, it is better to proceed by example than by a general approach.

5.1.1 Let us choose the KdV equation as the initial nonlinear equation and let us derive the associated NLS equation. Therefore, for the function $u(t,x)$, we have the following KdV equation:

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\beta\frac{\partial}{\partial x}u^2+\frac{\partial^3 u}{\partial x^3}=0\ , \tag{5.1.6}$$

with the (linear) dispersion relationship: $\omega=k-k^3$.

Expanding u as

$$u=\alpha u_1+\alpha^2u_2+\alpha^3u_3+\dots\ , \tag{5.1.7}$$

$$u_n=u_n(t,x;X_1,T_1,T_2,\dots),\qquad n=1,2,3,\dots, \tag{5.1.8}$$

$$\frac{\partial}{\partial t}\to\frac{\partial}{\partial t}+\alpha\frac{\partial}{\partial T_1}+\alpha^2\frac{\partial}{\partial T_2}+\dots,\quad \frac{\partial}{\partial x}\to\frac{\partial}{\partial x}+\alpha\frac{\partial}{\partial X_1}+\dots\ , \tag{5.1.9}$$

we find, as expected, at $O(\alpha)$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial^3}{\partial x^3}\right)u_1=0\ , \tag{5.1.10}$$

$$u_1=A(X_1,T_1,T_2,\dots)E+\text{c.c.}\ ,$$

$$E=\exp\mathrm{i}\theta,\qquad \theta=kx-\omega t\ , \tag{5.1.11}$$

where c.c. denotes the complex conjugate.

At $O(\alpha^2)$, we find:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial^3}{\partial x^3}\right)u_2=-\left[\frac{\partial A}{\partial T_1}+(1-3k^2)\frac{\partial A}{\partial X_1}\right]E+\text{c.c.}$$
$$-2\mathrm{i}k\beta A^2E^2+2\mathrm{i}k\beta A^{*2}E^{-2}\ , \tag{5.1.12}$$

and the E term is secular, so that we take

$$X^*=X_1-(1-3k^2)T_1\to A=A(X^*,X_2,T_2,\dots)\ . \tag{5.1.13}$$

Integrating Eqn (5.1.12) to find u_2, we obtain

$$u_2=\frac{\beta}{3k^2}(A^2E^2+A^{*2}E^{-2})+B(X^*,T_1)\ , \tag{5.1.14}$$

where $B(X^*,T_1)$ is an integration constant for the fast scales x and t, but can be made a function of the slow scales.

At $O(\alpha^3)$, we now find

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial^3}{\partial x^3}\right)u_3=-\left[3\frac{\partial^3}{\partial x\,\partial X_1^2}+\frac{\partial}{\partial T_2}\right]\left[AE+A^*E^{-1}\right]$$
$$-\left[3\frac{\partial^3}{\partial x^2\,\partial X_1}+\frac{\partial}{\partial T_1}+\frac{\partial}{\partial X_1}\right]\left[\frac{\beta}{3k^2}(A^2E^2+A^{*2}E^{-2})\right.$$
$$\left.+B(X^*,T_1)\right]-\beta\frac{\partial}{\partial X_1}(A^2E^2+A^{*2}E^{-2}+2|A|^2)$$
$$-2\beta\frac{\partial}{\partial x}\left[ABE+\frac{\beta}{3k^2}(A|A|^2E+A^3E^3)+\text{c.c.}\right]\ . \tag{5.1.15}$$

There are two types of secular terms in Eqn (5.1.15). The first are functions of the slow scales only, which will give rise to terms in u_3, which are explicit functions of x and y. Removal of these gives:

$$\left[\frac{\partial}{\partial X_1}+\frac{\partial}{\partial T_1}\right]B+2\beta\frac{\partial}{\partial X_1}|A|^2=0\ . \tag{5.1.16}$$

Next, removal of the E secular terms gives:

$$3\mathrm{i}k\frac{\partial^2 A}{\partial X^{*2}}+\frac{\partial A}{\partial T_2}-\mathrm{i}k\frac{4\beta^2}{3k^2}A|A|^2+\mathrm{i}k\frac{2\beta^2}{3k^2}A|A|^2=0\ . \tag{5.1.17}$$

Finally, from Eqns (5.1.16) and (5.1.17), we obtain the following NLS equation for A:

$$-\mathrm{i}\frac{\partial A}{\partial\tau}+\frac{\partial^2 A}{\partial\xi^2}+\frac{2\beta^2}{3k^2}A|A|^2=0\ , \tag{5.1.18}$$

where $\tau=kT_2$ and $\xi=X^*/\sqrt{3}$.

Therefore, the time scale on which the envelope NLS equation operates is quite long, since one unit of time on the τ scale is $1/\alpha^2$ units of real time (far-field equation).

5.1.2 The amplitude $A(\tau,\xi)$ is a complex function and it therefore contains information about the phase of the wave. Eqn (5.1.18) may be expressed in terms of real functions by assuming that

$$A=a\exp\left(\mathrm{i}\int W\,\mathrm{d}\xi\right)\ , \tag{5.1.19}$$

where $a = a(\tau, \xi)$ and $W = W(\tau, \xi)$. Separating the real and imaginary parts, we get:

$$\frac{\partial}{\partial\tau} a^2 - 2\frac{\partial}{\partial\xi} Wa^2 = 0 , \tag{5.1.20a}$$

$$\frac{\partial W}{\partial\tau} - \frac{\partial}{\partial\xi}\left(\frac{1}{a}\frac{\partial^2 a}{\partial\xi^2} - W^2 + \frac{2\beta^2}{3k^2} a^2\right) = 0 . \tag{5.1.20b}$$

These equations are in the form of conservation laws

$$\frac{\partial P}{\partial\tau} + \frac{\partial Q}{\partial\xi} = 0 .$$

They were derived for deep-water waves by Chu and Mei (1970) [70], and by Whitham (1967) [71] without the term $(1/a)\partial^2 a/\partial\xi^2$. The connection between Eqns (5.1.20) and (5.1.18) was pointed by Davey (1972) [72].

5.1.3 It is also necessary to stress that, in general, when the long-wave parameter δ is not small but fixed when $\varepsilon \to 0$, the NLS equation describes the amplitude of a harmonic wave profile as a function of slow space and time variables. The wavelength of the carrier wave is taken to be $O(1)$ as $\varepsilon \to 0$, and this corresponds to δ being fixed in going to the limit. The basic wave is therefore sought in the form

$$\zeta \sim A(\xi, \tau) \exp \mathrm{i}p + \text{c.c.}, \quad \varepsilon \to 0 , \tag{5.1.21}$$

$$p = x - c_\mathrm{p} t , \quad \xi = \varepsilon(x - c_\mathrm{g} t) , \quad \tau = \varepsilon^2 t . \tag{5.1.22}$$

The expansion for ζ (and ϕ) is so constructed that it is periodic (to all orders) in p. Hence, higher-order terms must contain higher harmonics generated by the nonlinear coupling. The carrier wave moves at the phase velocity (c_p) and the amplitude modulation travels at the corresponding group velocity (c_g), although the specific forms of c_p and c_g are not assumed *a priori*! From the pivotal two-dimensional water wave problem with the flat-bottom condition, when the surface tension is ignored, the leading and next-order approximations give

$$c_\mathrm{p}^2 = \frac{\tanh\delta}{\delta} , \quad c_\mathrm{g} = \frac{1}{2} c_\mathrm{p}\left(1 + \frac{2\delta}{\sinh 2\delta}\right) , \tag{5.1.23}$$

respectively. The very next order, which incorporates the cubic nonlinearity, yields the following general NLS (GNLS) equation:

$$-2\mathrm{i}c_\mathrm{p}\frac{\partial A}{\partial\tau} + q\frac{\partial^2 A}{\partial\xi^2} + rA|A|^2 = 0 , \tag{5.1.24}$$

where q and r are involved functions of δ. The coefficient is $r(\delta) = 0$ for $\delta = \delta^0 = 1.363$, and it is well known that the Stokes wave is then unstable if $\delta \geqslant \delta^0$ [73]. This suggests that the nature of the Benjami–Feir instability could be examined via a suitable generalisation of the NLS equation (5.1.24) valid near $r = 0$ [74]. We must stress that the coefficient q is always positive, whereas r changes its sign from positive to negative at $\delta = \delta^0 = 1.363$, as δ increases.

The NLS equation (5.1.24) was first derived (for a finite depth) by Hasimoto and Ono (1972) in the following form [36]

$$\frac{1}{\mathrm{i}}\frac{\partial A}{\partial\tau} = \mu\frac{\partial^2 A}{\partial\xi^2} + \nu A|A|^2 , \tag{5.1.25}$$

where

$$\mu = -\frac{g}{8k_0\sigma\omega_0}\left\{\left[\sigma - k_0h_0(1-\sigma^2)\right]^2 + 4k_0^2h_0^2\sigma^2(1-\sigma^2)\right\} , \tag{5.1.26a}$$

$$\nu = -\frac{k_0^4}{2\omega_0}\left\{\frac{1}{c_\mathrm{g}^2 - gh_0}\left[4c_0^2 + 4(1-\sigma^2)c_0c_\mathrm{g} + gh_0(1-\sigma^2)^2\right] + \frac{1}{2\sigma^2}(9 - 10\sigma^2 + 9\sigma^4)\right\} , \tag{5.1.26b}$$

$$\sigma = \tanh k_0h_0 , \quad c_0 = \frac{\omega_0}{k_0} = \left(\frac{g\sigma}{k_0}\right)^{1/2} . \tag{5.1.26c}$$

It can be seen from Eqn (5.1.26a) that μ is always negative, whereas n changes its sign from negative to positive at $k_0h_0 = 1.363$, as k_0h_0 decreases.

It is known that the NLS equation (5.1.25) has the following solution representing a *nonlinear plane wave*

$$A = A_0 \exp\left[\mathrm{i}(\alpha\tau - \kappa\xi)\right] , \tag{5.1.27}$$

where $A_0 = \text{const}$ and $\alpha = -\mu\kappa^2 + \nu|A_0|^2$.

5.1.4 In the limit $k_0h_0 \to 0$ with k_0 of the order of unity, the coefficients m and n in Eqn (5.1.25) become, respectively:

$$\mu \to \mu_s = -\frac{1}{2} c_0^{1/2} k_0 h_0^2 , \tag{5.1.28a}$$

$$\nu \to \nu_\mathrm{s} = \frac{9}{4} c_0^{-1/2} k_0 h_0^{-2} , \tag{5.1.28b}$$

where $c_0 = (gh_0)^{1/2}$. In this case the nonlinear plane wave [for $\varepsilon < (k_0h_0)^3 \ll 1$] assumes the following form:

$$\zeta = \varepsilon a\cos\theta_\mathrm{s} - \frac{3}{4}\frac{\varepsilon^2 a^2}{h_0(k_0h_0)^2}(1 - \cos 2\theta_\mathrm{s}) , \tag{5.1.29}$$

where $\theta_\mathrm{s} = k_0x - (\omega_0 - \varepsilon^2\alpha_\mathrm{s})t$ and $\alpha_\mathrm{s} = \nu_\mathrm{s} g^2a^2/4\omega_0^2$.

On the other hand, as is well known, the shallow-water waves are governed by the KdV equation:

$$\frac{\partial\zeta}{\partial t} + c_0\frac{\partial\zeta}{\partial x} + \frac{3c_0\zeta}{2h_0}\frac{\partial\zeta}{\partial x} + \frac{c_0h_0^2}{6}\frac{\partial^3\zeta}{\partial x^3} = 0 , \tag{5.1.30}$$

which has the steady periodic solution called a cnoidal wave

$$\zeta = \varepsilon a\left\{\zeta_\infty + \frac{2}{m^2}\,\mathrm{Dn}^2\left[\left(\frac{\varepsilon a}{6m^2}\right)^{1/2}(x - c_\mathrm{g}t)\,|\,m\right]\right\} , \tag{5.1.31}$$

$$c_\mathrm{g} = c_0\left\{1 + \frac{3\varepsilon a}{2h_0}\left[\zeta_\infty + \frac{2}{3}\left(\frac{2}{m^2} - 1\right)\right]\right\} , \tag{5.1.32}$$

and the mean depth, say ζ^*, is given by $\zeta^* = \varepsilon a(\zeta_\infty + 2E/m^2K)$, where K, E and m are respectively, the first and the second kinds of the complete elliptic integral and its modulus.

As in Hasimoto and Ono [36], putting

$$k_0^2 = \frac{3}{2}\frac{\varepsilon a\pi^2}{m^2h_0^3K^2} , \quad \zeta^* = -\frac{3}{4}\frac{\varepsilon^2a^2}{h_0^3k_0^2} ,$$

and expanding solution (5.1.31) for small value of m, we obtain solution (5.1.29). Thus we find that the nonlinear plane wave solution corresponds to a weak cnoidal wave in the shallow-water limit.

Finally, we may conclude that a weak cnoidal wave is modulation-stable against small disturbances because $\nu_\mathrm{s}\mu_\mathrm{s} < 0$ (see Section 8).

5.2 Soliton solution of the nonlinear Schrodinger equation

The sign of β in the NLS equation (5.1.18) is important as it determines whether the isospectral operator (in this case the corresponding eigenvalues are independent of time!) of the NLS equation is self- or skew-adjoint.

For $\beta > 0$ the operator is skew-adjoint, giving rise to imaginary eigenvalues. In this case, solitons originate from a discrete spectrum which in turn arises from negative (bound) energy states (these negative energy states are associated with negative imaginary eigenvalues). One could consider the condition $\beta < 0$ as representing the case when focusing or bunching of the wave envelope occurs.

We can seek a solution of the canonical NLS equation (5.1.18) in the form

$$A = a_m \operatorname{sech}\left[\left(\frac{\beta}{2}\right)^{1/2} a_m(\xi - \xi_0 - U\tau)\right] \times \exp\left[-\mathrm{i}r(\xi - V\tau - \varphi)\right], \quad \beta > 0 , \tag{5.2.1}$$

where $r = U/2$ and $a_m = 2U\left[(U/8) - V\right]/\beta$.

The *envelope soliton* is characterised by the free parameters a_m and U and by the phases φ and ξ_0.

One of the most important characteristics of the NLS equation is that it can be solved exactly for initial conditions that decay sufficiently rapidly as $|\xi| \to \infty$. This was done by Zakharov and Shabat (1972) [35] using what was then the newly discovered IST (Gardner et al. [49]).They showed that any initial wave packet eventually evolves into a number of 'envelope solitons' and a dispersive tail. The bulk of the energy is contained in the solitons, which have solitary-wave-solution shapes and propagate with a permanent profile once produced. Solitons also survive interactions with other solitons or wave packets. Since the NLS equation describes the envelope of long waves in shallow water with a carrier frequency, the theory predicts the existence of packets of long waves in shallow water with soliton properties. The existence of these envelope soliton properties would hardly have been expected on the basis of experience with linear wave systems in which wave components are uncoupled and highly dispersive.

We note that the NLS equation yields a rich variety of nonlinear wave structures, namely solitons, rarefaction solitons, several kinds of periodic nonlinear waves, and a pair of shocks. Indeed, as a result of this overabundance, scientists are not sure that all these solutions correspond to physical waves!

Naturally, we can look for a simple solution of the NLS equation (5.1.18):

$$A(\tau, \xi) = a(\xi) \exp(\mathrm{i}\gamma^2\tau) , \tag{5.2.2}$$

where $a(\xi)$ is a real function, and γ is a constant (to be determined) representing a frequency correction to the individual waves.

In this case, it follows from the NLS equation (5.1.18) that $a(\xi)$ can be described by the equation

$$\frac{\mathrm{d}^2 a}{\mathrm{d}\xi^2} + \gamma^2 a + \beta a^3 = 0 ,$$

and this equation admits solutions in terms of the Dn function, which is the Jacobi elliptic function of the second kind:

$$a(\xi) = a^0 \operatorname{Dn}\left[a^0(\xi - \xi^0) | m\right] , \tag{5.2.3}$$

where m is the modulus of the Dn function with the properties

$$a^0 = \gamma\left(\frac{2}{2 - m^2}\right)^{1/2} , \quad 0 \leqslant m \leqslant 1 . \tag{5.2.4}$$

In the limit $m \to 0$, we have $\operatorname{Dn}[\xi|0] \to 1$ and in the limit $m \to 1$, we find that $\operatorname{Dn}[\xi|1] \to \operatorname{sech} \xi$.

The soliton envelope and the periodic envelope for the NLS equation are represented schematically in Fig. 8.

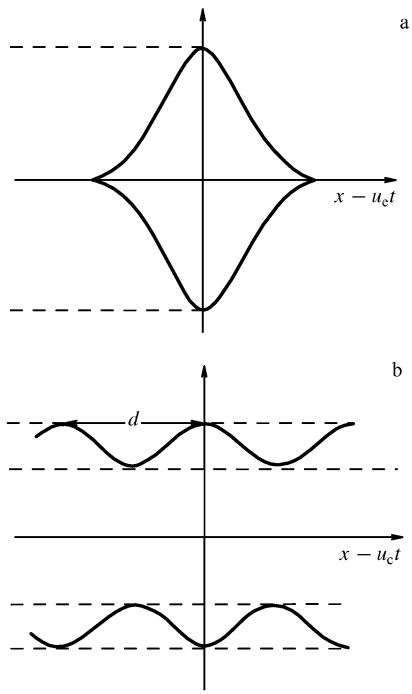

Figure 8. Soliton (a) and periodic (b) envelopes for the NLS equation.

More precisely, the carrier travels at a velocity u_c and the envelope travels at a velocity u_e; the pulse amplitude is

$$A_0 = \left[\frac{u_e}{2\beta}(u_e - 2u_c)\right]^{1/2} .$$

5.3 Asymptotic derivation of nonlinear Schrodinger–Poisson equations

We shall now return to the GKP equation (4.7.3) with the relationships (4.7.4). Hence, in this section, our initial system of equations for the two functions ζ_0 and F_0 is

$$2\chi_0^2 \frac{\partial \zeta_0}{\partial \tau} + \frac{1}{3}\frac{\partial \zeta_0}{\partial \xi} + \chi_0 \frac{\partial \zeta_0}{\partial \sigma} + 3\chi_0\zeta_0 \frac{\partial \zeta_0}{\partial \xi} + 3\chi_0^2\zeta_0 \frac{\partial \zeta_0}{\partial \sigma} + \frac{1}{3}\left(\frac{\partial^3 \zeta_0}{\partial \xi^3} + 3\chi_0 \frac{\partial^3 \zeta_0}{\partial \xi^2 \partial \sigma} + 3\chi_0^2 \frac{\partial^3 \zeta_0}{\partial \sigma^2 \partial \xi} + \chi_0^3 \frac{\partial^3 \zeta_0}{\partial \sigma^3}\right) + \chi_0^2 \frac{\partial^2 F_0}{\partial \eta^2} = 0 , \tag{5.3.1}$$

$$\zeta_0 = \frac{\partial F_0}{\partial \xi} + \chi_0 \frac{\partial F_0}{\partial \sigma} . \tag{5.3.2}$$

In these equations the small parameter (in the long-wave limit) is

$$\chi_0 = \frac{1}{\kappa_0} \equiv \frac{\varepsilon}{\delta^2} \ll 1 . \tag{5.3.3}$$

First, we expand the unknown functions ζ_0 and F_0 in terms of χ_0

$$\zeta_0 = h_0 + \chi_0 h_1 + \chi_0^2 h_2 + \dots , \tag{5.3.4a}$$

$$F_0 = f_0 + \chi_0 f_1 + \chi_0^2 f_2 + \dots \tag{5.3.4b}$$

If we successively equate like terms in χ_0^0, χ_0^1 and χ_0^2, we obtain the following equations for h_0, f_0, h_1, f_1, h_2 and f_2

$$h_0 = \frac{\partial f_0}{\partial \xi} , \tag{5.3.5a}$$

$$h_1 = \frac{\partial f_1}{\partial \xi} + \frac{\partial f_0}{\partial \sigma} , \tag{5.3.5b}$$

$$h_2 = \frac{\partial f_2}{\partial \xi} + \frac{\partial f_1}{\partial \sigma} \tag{5.3.5c}$$

and

$$\left(\frac{\partial^2}{\partial \xi^2} + 1\right) \frac{\partial h_0}{\partial \xi} = 0 , \tag{5.3.6a}$$

$$\left(\frac{\partial^2}{\partial \xi^2} + 1\right) \frac{\partial h_1}{\partial \xi} = -3\left(\frac{\partial h_0}{\partial \sigma} + 3h_0 \frac{\partial h_0}{\partial \xi} + \frac{\partial^3 h_0}{\partial \xi^2 \partial \sigma}\right) , \tag{5.3.6b}$$

$$\left(\frac{\partial^2}{\partial \xi^2} + 1\right) \frac{\partial h_2}{\partial \xi} = -3\left[2\frac{\partial h_0}{\partial \tau} + 3h_0 \frac{\partial h_0}{\partial \sigma} + \frac{\partial^2 f_0}{\partial \eta^2} + 3\frac{\partial}{\partial \xi} h_0 h_1 + \frac{\partial h_1}{\partial \sigma} + \frac{\partial^2}{\partial \xi \partial \sigma}\left(\frac{\partial h_1}{\partial \xi} + \frac{\partial h_0}{\partial \sigma}\right)\right] . \tag{5.3.6c}$$

Now, from Eqns (5.3.6a) and (5.3.5a), we determine h_0 and f_0 in the following form:

$$h_0 = A_{01} E + A_{01}^* E^{-1} , \tag{5.3.7a}$$

$$f_0 = B_{00} + B_{01} E + B_{01}^* E^{-1} , \tag{5.3.7b}$$

where $E = \exp \mathrm{i}\xi$ and $E^{-1} = \exp(-\mathrm{i}\xi)$; the asterisk denotes a complex conjugate. In the above set of relationships (5.3.7), we have

$$B_{01} = -\mathrm{i}A_{01} , \quad B_{01}^* = \mathrm{i}A_{01}^* . \tag{5.3.8}$$

Next, we can determine the function h_1 from Eqn (5.3.5b):

$$h_1 = A_{10} + A_{11} E + A_{11}^* E^{-1} + \frac{\partial f_1}{\partial \xi} , \tag{5.3.9a}$$

where

$$A_{10} = \frac{\partial B_{00}}{\partial \sigma} , \quad A_{11} = \frac{\partial B_{01}}{\partial \sigma} , \quad A_{11}^* = \frac{\partial B_{01}^*}{\partial \sigma} . \tag{5.3.10}$$

Now, if we take into account the expressions for h_0, (5.3.7a), and also for h_1, (5.3.9a), we obtain the following equation for the function f_1 from Eqn (5.3.6b):

$$\left(\frac{\partial^2}{\partial \xi^2} + 1\right) \frac{\partial^2 f_1}{\partial \xi^2} = -\frac{9}{2}(A_{01}^2 E^2 + A_{01}^{*2} E^{-2}) ,$$

and the expression of f_1 is then

$$f_1 = B_{10} + B_{12} E^2 + B_{12}^* E^{-2} , \tag{5.3.9b}$$

where

$$B_{12} = -\mathrm{i}\,\frac{3}{4} A_{01}^2 , \quad B_{12}^* = \mathrm{i}\,\frac{3}{4} A_{01}^{*2} . \tag{5.3.11}$$

The following expression for h_1 follows from expressions (5.3.9a) and (5.3.9b)

$$h_1 = A_{10} + A_{11} E + A_{11}^* E^{-1} + A_{12} E^2 + A_{12}^* E^{-2} , \tag{5.3.9′a}$$

$$A_{12} = 2\mathrm{i}B_{12} , \quad A_{12}^* = -2\mathrm{i}B_{12}^* . \tag{5.3.12}$$

More precisely, in the above relationships (5.3.7)–(5.3.12) the coefficients A_{01}, A_{01}^*, B_{00}, B_{01}, B_{01}^*, A_{10}, A_{11}, A_{11}^*, B_{10}, B_{12}, B_{12}^*, A_{12}, A_{12}^* are all functions of σ, τ and η.

Let us now consider Eqns (5.3.5c) and (5.3.6c). In fact, Eqn (5.3.6c) defines h_2 and we can write this equation in the following form

$$\left(\frac{\partial^2}{\partial \xi^2} + 1\right) \frac{\partial h_2}{\partial \xi} = -3\left(L_0 + L_1 E + L_2 E^2 + 9\mathrm{i}A_{01} \frac{\partial A_{12}}{\partial \sigma} E^3 + \text{c.c.}\right) , \tag{5.3.13}$$

where

$$L_0 = \frac{\partial A_{10}}{\partial \sigma} + 3\left(A_{01} \frac{\partial A_{01}^*}{\partial \sigma} + A_{01}^* \frac{\partial A_{01}}{\partial \sigma}\right) + \frac{\partial^2 B_{00}}{\partial \eta^2} , \tag{5.3.14a}$$

$$L_1 = 2\frac{\partial A_{01}}{\partial \tau} + \mathrm{i}\,\frac{\partial^2 A_{01}}{\partial \sigma^2} + 3\mathrm{i}(A_{10} A_{01} + A_{01}^* A_{12}) + \frac{\partial^2 B_{01}}{\partial \eta^2} , \tag{5.3.14b}$$

$$L_2 = \frac{3}{2}\frac{\partial}{\partial \sigma} A_{01}^2 + 6\mathrm{i}A_{11} A_{01} + 3A_{01} \frac{\partial A_{01}}{\partial \sigma} - 4\frac{\partial A_{12}}{\partial \sigma} \equiv 0 , \tag{5.3.14c}$$

if we utilise the above expressions (5.3.7a), (5.3.7b) and (5.3.9′a) for h_0, f_0 and h_1.

We see that the L_0 and L_1 terms in (5.3.13) are *secular*. However, the E^3 term is not secular (the E^2 term is zero), because it does not resonate with the homogeneous solution. Therefore, in order to ensure that perturbation theory is valid for long times, we must have

$$L_0 = 0 , \quad L_1 = 0 . \tag{5.3.15}$$

Now, from Eqns (5.3.5c) and (5.3.9b), we can determine also the function h_2

$$h_2 = A_{20} + A_{22} E^2 + A_{22}^* E^{-2} + \frac{\partial f_2}{\partial \xi} , \tag{5.3.16}$$

$$A_{20} = \frac{\partial B_{10}}{\partial \sigma} , \quad A_{22} = \frac{\partial B_{12}}{\partial \sigma} , \quad A_{22}^* = \frac{\partial B_{12}^*}{\partial \sigma} . \tag{5.3.17}$$

It follows from expressions (5.3.15) and (5.3.16), that the left-hand side of Eqn (5.3.13) obeys the following relationship:

$$\left(\frac{\partial^2}{\partial \xi^2} + 1\right) \frac{\partial h_2}{\partial \xi} = \frac{\partial}{\partial \xi}\left(\frac{\partial^2}{\partial \xi^2} + 1\right) \frac{\partial f_2}{\partial \xi} - 6\mathrm{i}(A_{22} E^2 - A_{22}^* E^{-2}) , \tag{5.3.18}$$

and we conclude that, to obtain f_2, it is necessary to solve the following equation

$$\frac{\partial}{\partial \xi}\left(\frac{\partial^2}{\partial \xi^2} + 1\right) \frac{\partial f_2}{\partial \xi} = -3\mathrm{i}\left(-2\mathrm{i}A_{22} E^2 + 9\mathrm{i}A_{01} \frac{\partial A_{12}}{\partial \sigma} E^3 + \text{c.c.}\right) . \tag{5.3.19}$$

But, at this stage the functions A_{01}, B_{00}, B_{01}, A_{10}, A_{11} and $A_{12} = 2\mathrm{i}B_{12}$ obey the following four relationships:

$$B_{01} = -\mathrm{i}A_{01} , \quad A_{10} = \frac{\partial B_{00}}{\partial \sigma} , \quad A_{11} = \frac{\partial B_{01}}{\partial \sigma} , \quad B_{12} = -\mathrm{i}\,\frac{3}{4} A_{01}^2 \tag{5.3.20}$$

and two equations [from expressions (5.3.15), (5.3.14a), (5.3.14b)], since $A_{01}A_{01}^* \equiv |A_{01}|^2$:

$$\frac{\partial A_{10}}{\partial \sigma} + 3\,\frac{\partial |A_{01}|^2}{\partial \sigma} + \frac{\partial^2 B_{00}}{\partial \eta^2} = 0\,, \tag{5.3.21a}$$

$$2\,\frac{\partial A_{01}}{\partial \tau} + \mathrm{i}\,\frac{\partial^2 A_{01}}{\partial \sigma^2} + 3\mathrm{i}(A_{10}A_{01} + A_{01}^* A_{12}) + \frac{\partial^2 B_{01}}{\partial \eta^2} = 0\,. \tag{5.3.21b}$$

From relationships (5.3.20) and (5.3.21), we can eliminate the functions A_{10}, A_{11}, A_{12} and B_{01}. We then obtain the following system of two equations for A_{01} and B_{00}:

$$\frac{\partial^2 B_{00}}{\partial \sigma^2} + \frac{\partial^2 B_{00}}{\partial \eta^2} + 3\,\frac{\partial |A_{01}|^2}{\partial \sigma} = 0\,, \tag{5.3.22}$$

$$2\mathrm{i}\,\frac{\partial A_{01}}{\partial \tau} - \frac{\partial^2 A_{01}}{\partial \sigma^2} - \frac{9}{2}\,A_{01}|A_{01}|^2 + \frac{\partial^2 A_{01}}{\partial \eta^2} = 3A_{01}\,\frac{\partial B_{00}}{\partial \sigma}\,. \tag{5.3.23}$$

These limiting equations (5.3.22) and (5.3.23) represent the *nonlinear Schrodinger–Poisson* (*NLS–P*) *system of two equations* (valid in the long-wave limit) obtained first by Davey and Stewartson (1974) [39] and also by Freeman and Davey (1975) [33].

Hence, modulation of the amplitude A_{01} of a travelling wave packet of small amplitude (propagating in quasi-one direction on water of finite depth) may be described by the nonlinear Schrodinger equation (5.3.23) coupled to the Poisson equation (5.3.22) for the middle part of the flow velocity potential B_{00}.

The NLS–P system of two equations (5.3.22), (5.3.23) was derived by Davey and Stewartson [39] in the long-wave limit ($\varepsilon \to 0$ and then $\delta \to 0$), but without any formal justification. According to Freeman and Davey [33], the above two evolutionary NLS–P equations (5.3.22), (5.3.23) are derived by a double expansion procedure assuming that an expansion in terms of δ can be used first, followed by an expansion in χ_0 ($\chi_0 = 1/\kappa_0 \equiv \varepsilon/\delta^2 \ll 1$). This procedure would seem to imply that the parameters δ and χ_0 are quite independent of each other. A close examination of the method indicates however that the results still remain true even if χ_0 is dependent on δ. At first sight, the retention of terms of order $\chi_0\delta^2$ in the course of derivation of the GKP equation (4.7.3), neglecting terms of order δ^4 in the expansion of ϕ in powers of δ, would suggest that some restriction on the magnitude of χ_0 relative to δ is implied. However, it should be realised that the terms of order δ^4 neglected in the expansion of ϕ are just those terms which vanish to the first order in χ_0 because the value of c_{p} is chosen in accordance with the linearised theory to achieve exactly that. A similar observation applies to certain terms of order $\delta^4\chi_0$, because of the choice of c_{g}.

We can confidently assert therefore that the double limit, in which we have first $\delta \to 0$ and then $\chi_0 \to 0$, as described in this present section, is valid and correct. Since a more formal procedure with first $\chi_0 \to 0$ and then $\delta \to 0$ yields the same result, the double limit $\delta, \chi_0 \to 0$ must be valid and uniform for Eqns (5.3.22), (5.3.23) as the appropriate evolutionary equations: *the double limit* $\delta, \chi_0 \to 0$ *is uniform since the sequence in which the limits are taken is immaterial*!

Eqns (5.3.22), (5.3.23) suffice to determine A_{01} and B_{00}, given appropriate boundary conditions. On physical grounds a 'reasonable' boundary condition is that, at any fixed time τ, the wave decays completely at a distance sufficiently far from its centre, so that

$$|A_{01}| \to 0\,, \qquad \frac{\partial B_{00}}{\partial \sigma} \to 0\,, \qquad \frac{\partial B_{00}}{\partial \eta} \to 0\,, \qquad \sigma^2 + \eta^2 \to \infty\,. \tag{5.3.24}$$

Furthermore, if we suppose that at time $t = 0$ a travelling wave is formed and the elevation of the free surface is raised to $z = \varepsilon\zeta$ (in dimensionless form), where $\zeta = \zeta_0(\chi_0 x, \eta)\exp \mathrm{i}x + \text{c.c.}$, then the appropriate initial condition on A_{01} is that

$$A_{01}(\sigma, \eta, 0) = \zeta_0(\chi_0 x, \eta)\,, \qquad \sigma_{\tau=0} \equiv \chi_0 x\,. \tag{5.3.25}$$

Thus, at this stage, we can use the following two asymptotic expansions for the functions ζ_0 and F_0:

$$\zeta_0 = A_{01}E + A_{01}^* E^{-1} + \chi_0\left(\frac{\partial B_{00}}{\partial \sigma} - \mathrm{i}\,\frac{\partial A_{01}}{\partial \sigma}\,E + \mathrm{i}\,\frac{\partial A_{01}^*}{\partial \sigma}\,E^{-1} + \frac{3}{2}\,A_{01}^2 E^2 + \frac{3}{2}\,A_{01}^{*2}E^{-2}\right) + O(\chi_0^2)\,, \tag{5.3.26}$$

$$F_0 = B_{00} - \mathrm{i}A_{01}E + \mathrm{i}A_{01}^* E^{-1} + O(\chi_0)\,, \tag{5.3.27}$$

where $E = \exp \mathrm{i}\xi$.

5.4 Consistent asymptotic expansions

In principle, we can extend the asymptotic expansions (5.3.26) and (5.3.27) up to the term of χ_0^3 for ζ_0, and up to the term of the order of χ_0^2 for F_0. But this makes it necessary to solve, first, Eqn (5.3.19) for f_2. Surprisingly, the expression for f_2 is then of the following form:

$$f_2 = B_{20} + \frac{3}{8}\,\frac{\partial A_{01}^2}{\partial \sigma}\,E^2 + \frac{3}{8}\,\frac{\partial A_{01}^{*2}}{\partial \sigma}\,E^{-2} - \mathrm{i}\frac{9}{16}\,A_{01}^3 E^3 + \mathrm{i}\,\frac{9}{16}\,A_{01}^{*3}E^{-3}\,, \tag{5.4.1}$$

and from expression (5.3.16) we obtain for h_2 (the terms with E^2 and E^{-2} cancel out!):

$$h_2 = \frac{\partial B_{10}}{\partial \sigma} + \frac{27}{16}\,A_{01}^3 E^3 + \frac{27}{16}\,A_{01}^{*3}E^{-3}\,. \tag{5.4.2}$$

Therefore, if we wish to extend expansions (5.3.26) and (5.3.27), we have to determine the function $B_{10}(\sigma, \tau, \eta)$. We then must consider the equation for the function h_3 and determine the structure of the right-hand side of this equation. From the initial equation (5.3.1) we deduce easily the following equation for h_3

$$\left(\frac{\partial^2}{\partial \xi^2} + 1\right)\frac{\partial h_3}{\partial \xi} = -3\left(2\,\frac{\partial h_1}{\partial \tau} + \frac{\partial h_2}{\partial \sigma} + 3h_0\,\frac{\partial h_1}{\partial \sigma} + 3h_1\,\frac{\partial h_0}{\partial \sigma} + 3h_0\,\frac{\partial h_2}{\partial \xi} + 3h_1\,\frac{\partial h_1}{\partial \xi} + 3h_2\,\frac{\partial h_0}{\partial \xi} + \frac{\partial^3 h_2}{\partial \xi^2\,\partial \sigma} + \frac{\partial^3 h_1}{\partial \xi\,\partial \sigma^2} + \frac{\partial^3 h_0}{\partial \sigma^3} + \frac{\partial^2 f_1}{\partial \eta^2}\right)\,. \tag{5.4.3}$$

Now, if we take into account expressions (5.3.7a), (5.3.9a) and (5.4.2) for h_0, h_1 and h_2, respectively, we obtain the following equation for h_3:

$$\left(\frac{\partial^2}{\partial \xi^2} + 1\right)\frac{\partial h_3}{\partial \xi} = -3(N_0 + N_1 E + N_2 E^2 + N_3 E^3 + N_4 E^4 + \text{c.c.})\,, \tag{5.4.4}$$

where

$$N_0 = 2\,\frac{\partial A_{10}}{\partial\tau} + \frac{\partial A_{20}}{\partial\sigma} + 3A_{01}\,\frac{\partial A_{11}^*}{\partial\sigma} + 3A_{01}^*\,\frac{\partial A_{11}}{\partial\sigma}$$

$$+3A_{11}^*\,\frac{\partial A_{01}}{\partial\sigma} + 3A_{11}\,\frac{\partial A_{01}^*}{\partial\sigma} + \frac{\partial^2 B_{10}}{\partial\eta^2}\,. \qquad (5.4.5)$$

At this stage we shall not write down the corresponding expressions for the terms proportional to E, E^2, E^3 and E^4, since we only intend to derive the equation for B_{10}. For Eqn (5.4.4) the first compatibility condition is $N_0 = 0$. Then, Eqn (5.4.5), and the relationships $A_{10} = \partial B_{00}/\partial\sigma$, $A_{11} = \partial B_{01}/\partial\sigma$, $A_{11}^* = \partial B_{01}^*/\partial\sigma$, $B_{01} = -\mathrm{i}A_{01}$, $B_{01}^* = \mathrm{i}A_{01}^*$ and $A_{20} = \partial B_{10}/\partial\sigma$, yield the following Poisson equation for B_{10}

$$\frac{\partial^2 B_{10}}{\partial\sigma^2} + \frac{\partial^2 B_{10}}{\partial\eta^2} = -2\,\frac{\partial^2 B_{00}}{\partial\tau\,\partial\sigma} + 3\mathrm{i}\left(A_{01}^*\,\frac{\partial^2 A_{01}}{\partial\sigma^2} - A_{01}\,\frac{\partial^2 A_{01}^*}{\partial\sigma^2}\right)\,. \qquad (5.4.6)$$

Finally, we obtain the following consistent asymptotic expansions for ζ_0 and F_0:

$$\zeta_0 = A_{01}E + A_{01}^*E^{-1} + \chi_0\left(\frac{\partial B_{00}}{\partial\sigma} - \mathrm{i}\,\frac{\partial A_{01}}{\partial\sigma}\,E\right.$$
$$\left.+\mathrm{i}\,\frac{\partial A_{01}^*}{\partial\sigma}\,E^{-1} + \frac{3}{2}\,A_{01}^2E^2 + \frac{3}{2}\,A_{01}^{*2}E^{-2}\right)$$
$$+\chi_0^2\left(\frac{\partial B_{10}}{\partial\sigma} + \frac{27}{16}\,A_{01}^3E^3 + \frac{27}{16}\,A_{01}^{*3}E^{-3}\right) + O(\chi_0^3)\,, \qquad (5.4.7)$$

$$F_0 = B_{00} - \mathrm{i}A_{01}E + \mathrm{i}A_{01}^*E^{-1}$$
$$+\chi_0\left(B_{10} - \mathrm{i}\,\frac{3}{4}\,A_{01}^2E^2 + \mathrm{i}\,\frac{3}{4}\,A_{01}^{*2}E^{-2}\right) + O(\chi_0^2)\,. \qquad (5.4.8)$$

The relevant equations for the functions $A_{01}(\tau,\sigma,\eta)$, $B_{00}(\tau,\sigma,\eta)$ and $B_{10}(\tau,\sigma,\eta)$ are Eqns (5.3.22), (5.3.23) and (5.4.6).

In principle, we can also extend the above expansions (5.4.7) and (5.4.8) for ζ_0 and F_0, if we consider the corresponding equations for h_4, h_5, ... and f_4, f_5, ... in the expansions (5.3.4a), (5.3.4b). But it is then necessary also to introduce, in addition to the slow variables τ, σ and η, several new slow variables, for example, $\tau_1 = \chi_0\tau$, $\sigma_1 = \chi_0\sigma$, ...

Indeed, if we make explicit the term proportional to E on the right-hand side of Eqn (5.4.4)

$$N_1 = 2\,\frac{\partial A_{11}}{\partial\tau} + 3\mathrm{i}(A_{20}A_{01} + A_{10}A_{11} + A_{12}A_{11}^*)$$
$$+3\,\frac{\partial}{\partial\sigma}(A_{10}A_{01} + A_{12}A_{01}^*) + \mathrm{i}\,\frac{\partial^2 A_{11}}{\partial\sigma^2} + \frac{1}{3}\,\frac{\partial^3 A_{01}}{\partial\sigma^3}\,, \qquad (5.4.9)$$

we obtain a new equation from the second compatibility relation for Eqn (5.4.4) with $N_1 = 0$, namely

$$\frac{4}{3}\,\frac{\partial^3 A_{01}}{\partial\sigma^3} - 2\mathrm{i}\,\frac{\partial^2 A_{01}}{\partial\sigma\,\partial\tau} + 3\,\frac{\partial}{\partial\sigma}\left[A_{01}\left(\frac{\partial B_{00}}{\partial\sigma} + \frac{3}{2}\,|A_{01}|^2\right)\right]$$
$$+3\mathrm{i}\left(\frac{\partial B_{00}}{\partial\sigma}\,\frac{\partial B_{10}}{\partial\sigma} - \mathrm{i}\,\frac{\partial B_{00}}{\partial\sigma}\,\frac{\partial A_{01}}{\partial\sigma} + \mathrm{i}\,\frac{3}{2}\,\frac{\partial A_{01}^*}{\partial\sigma}\,A_{01}^2\right) = 0\,. \qquad (5.4.10)$$

The above equation is complementary for the unknown functions A_{01}, B_{00} and B_{10}, which satisfy already three equations (5.3.22), (5.3.23) and (5.4.6)!

It is not at all evident that Eqn (5.4.10) is an *identity*. Therefore, seemingly the problem of A_{01}, B_{00} and B_{10} is *overdetermined*! To remedy this difficulty, we can assume that our unknown functions A_{01}, B_{00} and B_{10} are also dependent of the slow variables: τ_1, σ_1, ... — clearly more research is needed in this direction!

5.5 Cnoidal wave and soliton solutions

5.5.1 If we introduce new variables (p, T, Y) via the transformation $\xi = p + (1/6\chi_0)T$, $\sigma = \chi_0[p + (1/2\chi_0)T]$, $\eta = \chi_0^{1/2}Y$, $\tau = \chi_0 T$, then from the GKP equation (4.7.3) we can describe the function

$$\zeta_0\left[\chi_0 T,\, p + \frac{1}{6\chi_0}\,T,\; \chi_0\left(p + \frac{1}{2\chi_0}\,T\right),\; \chi_0^{1/2}Y\right] \equiv h(p, T, Y)\,,$$

by the following canonical KP equation

$$2\,\frac{\partial h}{\partial T} + 3h\,\frac{\partial h}{\partial p} + \frac{1}{3\chi_0}\,\frac{\partial^3 h}{\partial p^3} + \frac{\partial^2 f}{\partial Y^2} = 0\,, \qquad (5.5.1)$$

where $h = \partial f/\partial p$ and χ_0 is of order of unity. The above equation admits transverse cnoidal wave solutions in which both h and f are functions of $\theta = lp + mY - cT$ only.

In this case the relevant equation for $h = h(\theta)$ is

$$l^3h''' + 3\chi_0(3l^2h - 2cl + m^2)h' = 0\,,$$

and this equation has solutions of the form

$$h(\theta) = a + b\,\mathrm{Cn}^2[\theta|v]\,, \qquad (5.5.2)$$

where Cn represents a Jacobian elliptic function and a, b and v are constants. These constants must satisfy the algebraic relationships:

$$3l^2a - 2lc + m^2 = \frac{4l^3}{3\chi_0}(1 - 2v)\,, \qquad b = \frac{4lv}{3\chi_0}\,, \qquad (5.5.3)$$

for given values of l, m, a. Then, in the limit when $\chi_0 \to 0$, if the amplitude b is to remain of order of unity, it is clear that v and $(1/c)$ must both be of order of χ_0; note also that c will be negative. In this limit, therefore, cnoidal waves become harmonic and the above solution, together with F_0 as given by $\zeta_0 = \partial F_0/\partial\xi$, may be identified with the corresponding solution of the NLS–P equations (5.3.22) and (5.3.23).

For example, if we write $l = 1 + k\chi_0\cos\theta$, $m = k\chi_0^{1/2}\sin\theta$, and require that the solution (5.5.2) has zero mean and unity amplitude, then this solution with $a = 0$ and $b = 1$ implies that

$$\zeta_0 = \exp\left\{\mathrm{i}\left[\xi + k(\sigma\cos\theta + \eta\sin\theta)\right.\right.$$
$$\left.\left.+\left(\frac{9}{4}\cos 2\theta + \frac{k^2}{2}\cos 2\theta\right)T\right]\right\} + \text{c.c.} + O(\chi_0)\,, \qquad (5.5.4)$$

and the leading term on the right-hand side is a solution of the NLS–P equations (5.3.22) and (5.3.23) for A_{01}; more precisely, we have

$$A_{01} = \exp\left\{\mathrm{i}\left[k(\sigma\cos\theta + \eta\sin\theta)\right.\right.$$
$$\left.\left.+\left(\frac{9}{4}\cos 2\theta + \frac{k^2}{2}\cos 2\theta\right)T\right]\right\}\,. \qquad (5.5.5)$$

5.5.2 Rather remarkably, Anker and Freeman [75] have used the two-dimensional IST of Zakharov and Shabat [35] to show that the NLS–P equations (5.3.22), (5.3.23) are integrable. These NLS–P equations, together with the KP equation, are one of the few physically relevant two-dimensional equations known to be solvable by the IST.

First, we can rewrite the NLS–P system of two equations (5.3.22), (5.3.23) in the following new form for P and $\mathcal{Q}$, if we introduce the relationships

$$A_{01} = C^0 P \exp(\mathrm{i}p\tau)\,, \qquad Q = Q^0 + \mathcal{Q}\,,$$
$$\frac{\partial B_{00}}{\partial \sigma} = Q - 3|A_{01}|^2\,, \tag{5.5.6}$$

where P, C^0 and Q^0 are constants,

$$2\mathrm{i}\,\frac{\partial P}{\partial \tau} - \frac{\partial^2 P}{\partial \sigma^2} + \frac{\partial^2 P}{\partial \eta^2} = -\frac{9}{2}\,C^{02} P\left(|P|^2 - 1\right) + 3P\mathcal{Q}\,, \tag{5.5.7a}$$

$$\frac{\partial^2 \mathcal{Q}}{\partial \sigma^2} + \frac{\partial^2 \mathcal{Q}}{\partial \eta^2} = 3C^{02}\,\frac{\partial^2 |P|^2}{\partial \eta^2}\,, \tag{5.5.7b}$$

choosing $C^{02} = 2(Q^0 + 2p)/3$.

In the one-dimensional problem, independent of η, the system of equations (5.5.7) gives the NLS equation. Linearisation of the above equations in the one-dimensional case shows that it is stable (see Section 8) in the sense of Hasimoto and Ono [36]. It has also been shown by Zakharov and Shabat (1972) [35] that in this case there exist soliton-type solutions of the form:

$$P = \frac{(\lambda + \mathrm{i}\nu)^2 + \exp\left[2\nu(\sigma - \sigma^0 + \lambda\tau)\right]}{1 + \exp\left[2\nu(\sigma - \sigma^0 + \lambda\tau)\right]}\,, \tag{5.5.8}$$

with $\nu = (1 - \lambda^2)^{1/2}$, and multisoliton solutions can also be constructed.

In a later (1974) paper, Zakharov and Shabat [76] describe a general method for constructing equations which are solvable by the IST and which have soliton and multisoliton solutions. The one-dimensional NLS equation is shown to belong to this class of equations. According to Anker and Freeman [75], the NLS–P (Freeman–Davey) system of equations (5.5.7) can be constructed by the method of Zakharov and Shabat (Z–S) [76] and hence soliton and multisoliton solutions are available. The pseudo-two-dimensional solutions thus obtained describe the interaction of solitons of the form given by solution (5.5.8), which are *skewed* with respect to each other.

The first task is to set up operators in the Z–S theory [76] corresponding to the two-dimensional case. Such operators describe a set of linear partial differential equations {see Eqns (2.23)–(2.25) in the paper by Anker and Freeman [75]} which underlie the nonlinear set of partial differential equations (5.5.7). Simple exponential solutions of these linear equations then give the solitons corresponding to those of solution (5.5.8) for the two-dimensional case and sums of these solutions describe the multisoliton interactions.

Examination of the N-soliton solution enables the phase shift and centre shift of the individual solitons to

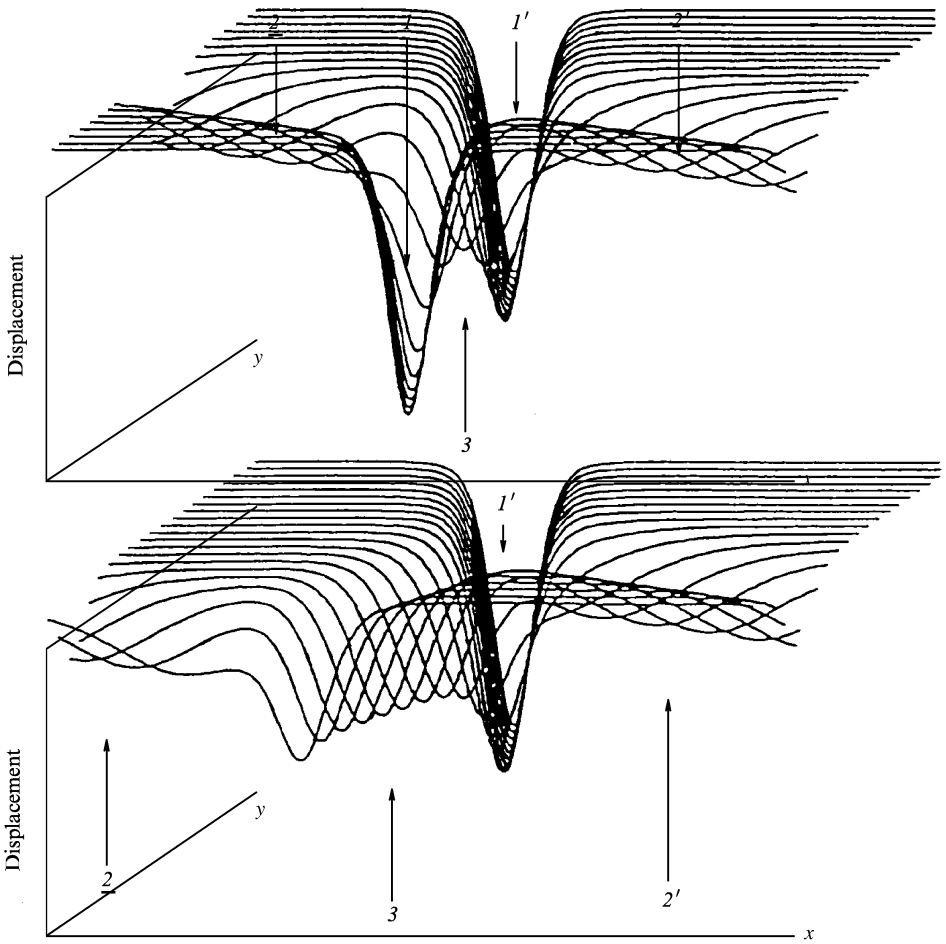

Figure 9. Amplitude variation in a two-soliton interaction for two values of phase shift: *1* and *2* are the incident solitons; *1′* and *2′* are the solitons after the interaction; *3* is the interaction soliton (in the second case, the incident soliton *1* is not drawn).

be calculated. The centre shift describes the displacement of the wave envelope and the phase shift describes the change in the phase of the modulated wave. In special cases, the phase shift becomes infinite and a limiting solution is obtained in which the resonance condition determines a third soliton. The results of two numerical computations are shown in Fig. 9 for two values of the phase shift (from Anker and Freeman [75]).

5.6 Generalised nonlinear Schrodinger–Poisson equations and their matching to the Kadomtsev–Petviashvili equation

First, it is important to note that, in a more general case, it is possible to derive asymptotically a coupled system of two evolution equations for a packet of water waves when $\varepsilon \to 0$ and $\Delta = v_0\varepsilon$, *but* δ *and* v_0 *are fixed*. These represent the GNLS–P system of two equations deduced from the classical dimensionless problem described by Eqns (2.3.3), (2.3.5)–(2.3.7). Below, we shall present the main features of this derivation and we shall consider also the matching between the KP and these GNLS–P equations.

For the derivation of the GNLS–P equations the reader can consult the papers by Benney and Roskes [38], Davey and Stewartson [39], Djordjevic and Redekopp [40], Ablowitz and Segur [41], and also the book by Mei ([43], pp 607–618). The most general analysis was reported in Ref. [40]: it includes the effects of surface tension and arbitrary depth and yields the GNLS–P system of two coupled evolution equations.

Without surface tension (when $\mathrm{We} = 0$) and with the dimensionless variables, our initial problem is described by the Laplace equation (2.3.3), in water, together with the free-surface conditions (2.3.6) and (2.3.7) and the flat-bottom boundary condition (2.3.5).

Only a brief outline of the perturbation analysis need be given here.

5.6.1 The wavelength of the carrier wave is taken to be $O(1)$ as $\varepsilon \to 0$, and this corresponds to δ being fixed in the limiting process:

$$\varepsilon \to 0\,, \quad \Delta = v_0\varepsilon\,, \quad \text{with} \quad \delta \text{ and } v_0 \text{ fixed}\,. \tag{5.6.1}$$

Indeed, as shown by the earlier work of Benney and Roskes (1969) [38] and Davey and Stewartson (1974) [39], it is convenient to introduce the following multiple slow scales:

$$q = \varepsilon(x - c_{\mathrm{g}}t)\,, \quad y^* = \varepsilon\frac{\delta}{\Delta}\,y = \frac{\delta}{v_0}\,y\,, \quad t^* = \varepsilon^2 t\,. \tag{5.6.2}$$

The carrier wave moves at the phase velocity c_{p} and the amplitude modulation moves at the corresponding group velocity c_{g}, although the specific forms of c_{p} and c_{g} are not assumed *a priori*.

The wavetrain is so constructed that it is periodic (to all orders in ε) in

$$p^* = x - c_{\mathrm{p}}t\,, \tag{5.6.3}$$

with the fundamental periodicity $E = \exp ip^*$ and the amplitude modulation described by the scaled coordinates (5.6.2). Therefore, higher-order terms (in the series expansions in ε, given below) must contain higher harmonics generated by the nonlinear coupling. Now, if we assume that the solution of our problem — described by Eqns (2.3.3), (2.3.5)–(2.3.7) and expressions (5.6.1)–(5.6.3) — is given by the following asymptotic expansions:

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \ldots\,, \qquad \zeta = \zeta_0 + \varepsilon\zeta_1 + \varepsilon^2\zeta_2 + \ldots\,, \tag{5.6.4}$$

we are faced with the following set of problems described by equations for the functions φ_n and ζ_n, $n = 0, 1, 2\ldots$:

$$\frac{\partial^2\phi_n}{\partial z^2} + \delta^2\frac{\partial^2\phi_n}{\partial p^{*2}} = -F_n\,, \qquad \left.\frac{\partial\phi_n}{\partial z}\right|_{z=-1} = 0\,, \tag{5.6.5a}$$

$$\left.\frac{\partial\phi_n}{\partial z}\right|_{z=0} + c_{\mathrm{p}}\delta^2\frac{\partial\zeta_n}{\partial p^*} = G_n\Big|_{z=0}\,, \tag{5.6.5b}$$

$$\zeta_n = c_{\mathrm{p}}\left.\frac{\partial\phi_n}{\partial p^*}\right|_{z=0} + H_n\Big|_{z=0}\,, \tag{5.6.5c}$$

$$(n = 0, 1, 2, \ldots)\,,$$

where

$$F_0 = 0\,, \quad G_0 = 0\,, \quad H_0 = 0\,; \tag{5.6.6a}$$

$$F_1 = 2\delta^2\frac{\partial^2\phi_0}{\partial p^*\,\partial q}\,,$$

$$G_1 = -c_{\mathrm{g}}\delta^2\frac{\partial\zeta_0}{\partial q} + \delta^2\frac{\partial\phi_0}{\partial p^*}\frac{\partial\zeta_0}{\partial p^*} - \zeta_0\frac{\partial^2\phi_0}{\partial z^2}\,,$$

$$H_1 = c_{\mathrm{g}}\frac{\partial\phi_0}{\partial q} + c_{\mathrm{p}}\zeta_0\frac{\partial^2\phi_0}{\partial p^*\,\partial z} + c_{\mathrm{p}}\frac{\partial\zeta_0}{\partial p^*}\frac{\partial\phi_0}{\partial z}$$
$$-\frac{1}{2}\left(\frac{\partial\phi_0}{\partial p^*}\right)^2 - \frac{1}{2\delta^2}\left(\frac{\partial\phi_0}{\partial z}\right)^2\,; \tag{5.6.6b}$$

$$F_2 = 2\delta^2\frac{\partial^2\phi_1}{\partial p^*\,\partial q} + \delta^2\left(\frac{\partial^2\phi_0}{\partial q^2} + \frac{\partial^2\phi_0}{\partial y^{*2}}\right)\,,$$

$$G_2 = -c_{\mathrm{g}}\delta^2\frac{\partial\zeta_1}{\partial q} + \delta^2\frac{\partial\phi_1}{\partial p^*}\frac{\partial\zeta_0}{\partial p^*} + \delta^2\frac{\partial\phi_0}{\partial p^*}\frac{\partial\zeta_1}{\partial p^*} - \zeta_0\frac{\partial^2\phi_1}{\partial z^2}$$
$$-\zeta_1\frac{\partial^2\phi_0}{\partial z^2} + \delta^2\frac{\partial\zeta_0}{\partial t^*} + \delta^2\frac{\partial\phi_0}{\partial q}\frac{\partial\zeta_0}{\partial p^*} + \delta^2\frac{\partial\phi_0}{\partial p^*}\frac{\partial\zeta_0}{\partial q}$$
$$+\delta^2\zeta_0\frac{\partial\zeta_0}{\partial p^*}\frac{\partial^2\phi_0}{\partial p^*\,\partial z} + \delta^2\left(\frac{\partial\zeta_0}{\partial p^*}\right)^2\frac{\partial\phi_0}{\partial z} - \frac{1}{2}\zeta_0^2\frac{\partial^3\phi_0}{\partial z^3}\,,$$

$$H_2 = c_{\mathrm{g}}\frac{\partial\phi_1}{\partial q} + c_{\mathrm{p}}\zeta_0\frac{\partial^2\phi_1}{\partial p^*\,\partial z} + c_{\mathrm{p}}\zeta_1\frac{\partial^2\phi_0}{\partial p^*\,\partial z} + c_{\mathrm{p}}\frac{\partial\zeta_0}{\partial p^*}\frac{\partial\phi_1}{\partial z}$$
$$+c_{\mathrm{p}}\frac{\partial\zeta_1}{\partial p^*}\frac{\partial\phi_0}{\partial z} - \frac{\partial\phi_0}{\partial p^*}\frac{\partial\phi_1}{\partial p^*} - \frac{1}{\delta^2}\frac{\partial\phi_0}{\partial z}\frac{\partial\phi_1}{\partial z} - \frac{\partial\phi_0}{\partial t^*}$$
$$+c_{\mathrm{g}}\zeta_0\frac{\partial^2\phi_0}{\partial q\,\partial z} + c_{\mathrm{g}}\frac{\partial\zeta_0}{\partial q}\frac{\partial\phi_0}{\partial z} - \frac{\partial\phi_0}{\partial p^*}\frac{\partial\phi_0}{\partial q} + c_{\mathrm{p}}\zeta_0^2\frac{\partial^3\phi_0}{\partial z^2\,\partial p^*}$$
$$+c_{\mathrm{p}}\zeta_0\frac{\partial\zeta_0}{\partial p^*}\frac{\partial^2\phi_0}{\partial z^2} - \zeta_0\frac{\partial\phi_0}{\partial p^*}\frac{\partial^2\phi_0}{\partial z\,\partial p^*} - \frac{\partial\phi_0}{\partial p^*}\frac{\partial\phi_0}{\partial z}\frac{\partial\zeta_0}{\partial p^*}$$
$$-\frac{1}{\delta^2}\zeta_0\frac{\partial\phi_0}{\partial z}\frac{\partial^2\phi_0}{\partial z^2}\,. \tag{5.6.6c}$$

First, if we combine, for $n=0$, Eqns (5.6.5a), (5.6.5b) with (5.6.5c), we obtain for ϕ_0 the following homogeneous problem

$$\frac{\partial^2\phi_0}{\partial z^2}+\delta^2\frac{\partial^2\phi_0}{\partial p^{*2}}=0\,,\qquad \left.\frac{\partial\phi_0}{\partial z}\right|_{z=-1}=0\,, \tag{5.6.7a}$$

$$\left.\frac{\partial\phi_0}{\partial z}\right|_{z=0}+c_{\mathrm{p}}^2\delta^2\left.\frac{\partial^2\phi_0}{\partial p^{*2}}\right|_{z=0}=0\,, \tag{5.6.7b}$$

and we can easily find the following results for ϕ_0 and ζ_0:

$$\phi_0=\phi_{00}(q,y^*,t^*)+F_{00}(z)\left[A(q,y^*,t^*)E+A^*E^{-1}\right]\,, \tag{5.6.8a}$$

$$\zeta_0=c_{\mathrm{p}}\left.\frac{\partial\phi_0}{\partial p^*}\right|_{z=0}=\mathrm{i}c_{\mathrm{p}}\left[A(q,y^*,t^*)E-A^*E^{-1}\right]\,, \tag{5.6.8b}$$

with

$$F_{00}(z)=\frac{\cosh[\delta(z+1)]}{\cosh\delta}\,,\qquad E^{-1}=\exp(-\mathrm{i}p^*)\,, \tag{5.6.9}$$

and c_{p} is calculated from the dispersion relationship in the linear theory,

$$c_{\mathrm{p}}=\frac{\omega(\delta)}{\delta}\,,\qquad \omega(\delta)=(\delta\sigma)^{1/2}\,,\qquad \sigma\equiv\tanh\delta\,. \tag{5.6.10}$$

Next, for ϕ_1 we have an inhomogeneous problem

$$\frac{\partial^2\phi_1}{\partial z^2}+\delta^2\frac{\partial^2\phi_1}{\partial p^{*2}}=-2\mathrm{i}\delta^2F_{00}(z)\left[\frac{\partial A}{\partial q}E-\frac{\partial A^*}{\partial q}E^{-1}\right]\,, \tag{5.6.11a}$$

$$\left.\frac{\partial\phi_1}{\partial z}\right|_{z=-1}=0\,, \tag{5.6.11b}$$

$$\left.\frac{\partial\phi_1}{\partial z}\right|_{z=0}+c_{\mathrm{p}}^2\delta^2\left.\frac{\partial^2\phi_1}{\partial p^{*2}}\right|_{z=0}=N(\phi_0)_{z=0}\,, \tag{5.6.11c}$$

with

$$N(\phi_0)=-2\mathrm{i}\delta^2c_{\mathrm{p}}c_{\mathrm{g}}\left(\frac{\partial A}{\partial q}E-\frac{\partial A^*}{\partial q}E^{-1}\right)+3\mathrm{i}\delta^2c_{\mathrm{p}}(\sigma^2-1)(A^2E^2-A^{*2}E^{-2})\,. \tag{5.6.12}$$

The solution of the problem, described by the system of equations (5.6.11) in combination with expressions (5.6.12), is:

$$\begin{aligned}\phi_1&=\phi_{10}(q,y^*,t^*)+F_{00}(z)\left[B(q,y^*,t^*)E+B^*E^{-1}\right]\\&\quad-2\mathrm{i}\delta^2F_{10}(z)\left(\frac{\partial A}{\partial q}E-\frac{\partial A^*}{\partial q}E^{-1}\right)\\&\quad+F_{11}(z)(A^2E^2-A^{*2}E^{-2})\,,\end{aligned} \tag{5.6.13a}$$

where

$$F_{10}(z)=\frac{1}{2\delta}\left\{\frac{(z+1)\sinh[\delta(z+1)]}{\cosh\delta}-\frac{\sigma\cosh[\delta(z+1)]}{\cosh\delta}\right\}\,, \tag{5.6.14a}$$

$$F_{11}(z)=\frac{3\mathrm{i}}{4\sigma^2}(\sigma^2+1)(\sigma^2-1)\frac{\cosh[2\delta(z+1)]}{\cosh 2\delta}\,, \tag{5.6.14b}$$

and for ζ_1 we find:

$$\begin{aligned}\zeta_1&=c_{\mathrm{p}}\frac{\partial\phi_{00}}{\partial q}-(1-\sigma^2)|A|^2+\mathrm{i}c_{\mathrm{p}}\left[BE-B^*E^{-1}\right]\\&\quad+c_{\mathrm{g}}\left(\frac{\partial A}{\partial q}E+\frac{\partial A^*}{\partial q}E^{-1}\right)\\&\quad+\frac{1}{2\sigma^2}(\sigma^2-3)(A^2E^2+A^{*2}E^{-2})\,.\end{aligned} \tag{5.6.13b}$$

The above results (5.6.13a), (5.6.13b) for ϕ_1 and ζ_1 are obtained when we assume that:

$$c_{\mathrm{g}}=c_{\mathrm{p}}\frac{\sigma+\delta(1-\sigma^2)}{2\sigma}=\frac{\mathrm{d}\omega(\delta)}{\mathrm{d}\delta} \tag{5.6.15}$$

according to the linear theory.

We can then obtain ϕ_2 and ζ_2, once the solutions for the mean flow (ϕ_0, ζ_0) and second harmonic (ϕ_1, ζ_1) [described by expressions (5.6.8) and (5.6.13)] have been found: the evaluation of F_2, G_2 and H_2 from the set of expressions (5.6.6c) is a straightforward, but tedious task!

For example, we can find the following solution for ϕ_2:

$$\begin{aligned}\phi_2&=-\frac{\delta^2}{2}(z+1)^2\left(\frac{\partial^2\phi_{00}}{\partial q^2}+\frac{\partial^2\phi_{00}}{\partial y^{*2}}\right)+\phi_{20}(q,y^*,t^*)\\&\quad+F_{00}(z)\left[C(q,y^*,t^*)E+C^*E^{-1}\right]\\&\quad+\delta^2F_{10}(z)\left[\left(2\delta\sigma\frac{\partial^2A}{\partial q^2}-\frac{\partial^2A}{\partial y^{*2}}-2\mathrm{i}\frac{\partial B}{\partial q}\right)E+\text{c.c.}\right]\\&\quad-\frac{\delta^2}{2}F_{00}(z)\left[(z+1)^2-1\right]\left(\frac{\partial^2A}{\partial q^2}E+\frac{\partial^2A^*}{\partial q^2}E^{-1}\right)\\&\quad+\text{higher harmonic terms}\,.\end{aligned} \tag{5.6.16}$$

Then, imposing the boundary condition at $z=0$, described by expression (5.6.7b) and assuming that $n=2$, we find from expression (5.6.6c) for G_2 that the *leading-order mean flow* or *long-wave component*, $\phi_{00}(q,y^*,t^*)$, is described by the equation

$$(1-c_{\mathrm{g}}^2)\frac{\partial^2\phi_{00}}{\partial q^2}+\frac{\partial^2\phi_{00}}{\partial y^{*2}}=-\left[2c_{\mathrm{p}}+c_{\mathrm{g}}(1-\sigma^2)\right]\frac{\partial}{\partial q}|A|^2\,. \tag{5.6.17}$$

The above equation shows that the long-wave component ϕ_{00} is generated by the self-interaction of the short-wave component [characterised by the amplitude function $A(q,\,y^*,\,t^*)$].

Finally, comparing the first-harmonic terms in the boundary condition (5.6.5b) (at $z=0$) for $n=2$ with the corresponding expression (5.6.6c) for G_2, the expression (5.6.5c) for ζ_2, when $n=2$, the expression (5.6.6c) for H_2, we find that the derived two equations are compatible only if the amplitude function $A(q,y^*,t^*)$ satisfies the following evolutionary (Schrodinger) equation:

$$\begin{aligned}2\mathrm{i}c_{\mathrm{p}}\frac{\partial A}{\partial t^*}-\left[c_{\mathrm{g}}^2-(1-\sigma^2)(1-\sigma\delta)\right]\frac{\partial^2A}{\partial q^2}+c_{\mathrm{p}}c_{\mathrm{g}}\frac{\partial^2A}{\partial y^{*2}}&=\left[2c_{\mathrm{p}}+c_{\mathrm{g}}(1-\sigma^2)\right]A\frac{\partial\phi_{00}}{\partial q}\\&\quad+\left(\frac{9}{2\sigma^2}-6+\frac{13}{2}\sigma^2-\sigma^4\right)A|A|^2\,.\end{aligned} \tag{5.6.18}$$

Eqns (5.6.17) and (5.6.18) taken together describe the evolution of a travelling wave, to the first order in ε and with δ fixed.

For the capillary–gravity water waves (when We $\neq 0$) it is also possible to derive an analogous GNLS -P system of two coupled equations

$$\mathrm{i}\frac{\partial A}{\partial t^*}+\lambda\frac{\partial^2A}{\partial q^2}+\mu\frac{\partial^2A}{\partial y^{*2}}=\chi_1A\frac{\partial\phi_{00}}{\partial q}+\chi A|A|^2\,, \tag{5.6.19a}$$

$$\alpha\frac{\partial^2\phi_{00}}{\partial q^2}+\frac{\partial^2\phi_{00}}{\partial y^{*2}}=-\beta\frac{\partial}{\partial q}|A|^2\,, \tag{5.6.19b}$$

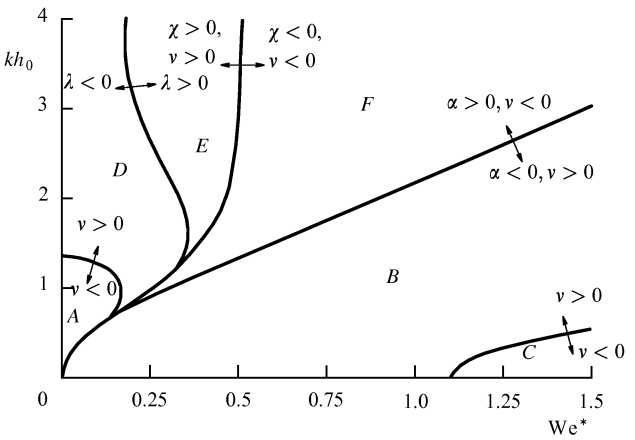

Figure 10. Dependences of λ, χ, α and ν on kh_0 and We^*. Curves indicate where the various coefficients change sign.

where λ, μ, χ, χ_1, α and β are known real constants. The expressions for the various constants are given by Ablowitz and Segur (see [41], p. 697), but note misprint in their Eqn (2.24d) for the coefficient χ.

In particular, we note that (in terms of dimensional quantities) $\alpha = 1 - (c_g^2/gh_0)$, that μ, χ_1 and β are non-negative and bounded, and that α, λ, χ and $\nu = \chi - (\beta/\alpha)\chi_1$ change sign as shown in Fig. 10. In Fig. 10, the two axes represent the dimensionless wavenumber kh_0 and the surface tension parameter [according to expressions (2.3.9) and (2.3.20)]: $We^* = \delta^2 We$. Each line denotes a simple zero of the designated coefficient except for the lines bounding region F, which denote singularities of ν and χ.

These singularities arise when $We^* = \sigma^2/(3-\sigma^2)$ and $\sigma = \tanh kh_0$, and this is the condition for the second-harmonic resonance, at which our perturbation expansion breaks down. Cases where $\alpha = 0$ are also singular.

5.6.2 We shall now consider the GNLS-P system of two coupled equations, (5.6.17) and (5.6.18), for the functions $A(q, y^*, t^*)$ and $\phi_{00}(q, y^*, t^*)$ in the shallow water limit when $\delta \to 0$. In this case, first, we find the following limiting values from expressions (5.6.10) and (5.6.15) for c_p and c_g:

$$c_p = 1 - \frac{1}{6}\delta^2 + \ldots , \quad c_g = 1 - \frac{1}{2}\delta^2 + \ldots , \quad \delta \to 0 . \tag{5.6.20}$$

Then, in place of Eqn (5.6.17), we obtain the following Poisson equation for the limiting value of ϕ_{00}:

$$\left(\delta^2 \frac{\partial^2 \phi_{00}}{\partial q^2} + \frac{\partial^2 \phi_{00}}{\partial y^{*2}}\right)\phi_{00}^0 = -3\,\frac{\partial}{\partial q}\,|A^0|^2 , \tag{5.6.21}$$

since $\sigma = \delta - (1/3)\delta^2 + \ldots$ when $\delta \to 0$; $\phi_{00}^0 = \lim_{\delta\to 0}\phi_{00}$ and $A^0 = \lim_{\delta\to 0} A$.

Next, when $\delta \to 0$, in place of the Schrodinger equation (5.6.18), we find

$$2i\delta^2 \frac{\partial A^0}{\partial t^*} - \delta^4 \frac{\partial^2 A^0}{\partial q^2} + \delta^2 \frac{\partial^2 A^0}{\partial y^{*2}} = 3\delta^2 A^0 \frac{\partial \phi_{00}^0}{\partial q} + \frac{9}{2} A^0 |A^0|^2 . \tag{5.6.22}$$

It is now necessary to compare the slow variables $q = \varepsilon(x - c_g t)$, $y^* = \varepsilon y \delta/\Delta = y\delta/v_0$ and $t^* = \varepsilon^2 t$, described by expression (5.6.2), with the variables (σ, η, τ) in the NLS–P (Freeman–Davey) equations (5.3.22) and (5.3.23). The variables (σ, η, τ) are defined, according to expression (4.7.3), by $\sigma = \chi_0(x - c_g t)$, $\tau = \chi_0^2 \delta^2 t$ and $\eta = y/v_0$, where $\chi_0 = 1/\kappa_0 \equiv \varepsilon/\delta^2$. This comparison yields the following relationships:

$$q = \delta^2 \sigma , \quad y^* = \delta\eta , \quad t^* = \delta^2 \tau , \tag{5.6.23}$$

since $\Delta = v_0 \varepsilon$ and $\chi_0 \equiv \varepsilon/\delta^2$.

Expression (5.6.23), together with Eqns (5.6.21) and (5.6.22) for ϕ_{00}^0 and A^0, yield the NLS–P system of two equations (5.3.22), (5.3.23), but for $A^0(\sigma, \eta, \tau)$ and $\phi_{00}^0(\sigma, \eta, \tau)$:

$$\left(\frac{\partial^2 \phi_{00}}{\partial \sigma^2} + \frac{\partial^2 \phi_{00}}{\partial \eta^2}\right)\phi_{00}^0 = -3\,\frac{\partial}{\partial \sigma}\,|A^0|^2 , \tag{5.6.24a}$$

$$2i\,\frac{\partial A^0}{\partial \tau} - \frac{\partial^2 A^0}{\partial \sigma^2} + \frac{\partial^2 A^0}{\partial \eta^2} = 3A^0 \frac{\partial \phi_{00}^0}{\partial \sigma} + \frac{9}{2} A^0 |A^0|^2 . \tag{5.6.24b}$$

Therefore, it is clear that Eqns (5.6.24) and (5.3.22), (5.3.23), match, i.e., $B_{00} = \phi_{00}^0$ and $A_{01} = A^0$.

Thus, the long-wave limit of the GNLS–P equations (5.6.17), (5.6.18) matches precisely the short-wave limit of the KP equation (derived in Sections 4.5 and 4.6). It confirms a measure of agreement between the GNLS–P equations for long waves ($\delta \to 0$) and the KP equation for short waves ($\kappa_0 \to \infty$). This can be stated more formally:

$$\lim_{\delta\to 0}[\text{GNLS–P}] = \lim_{\kappa_0\to\infty}[\text{KP}] , \tag{5.6.25}$$

and since matching occurs, the coefficients in the GNLS–P equations, when $\delta(= kh_0) \to 0$, can be checked against those deduced from the KP equation, when $\kappa_0 \to \infty$ ($\chi_0 = 1/\kappa_0 \equiv \varepsilon/\delta^2$).†

6. Influence of an uneven bottom

6.1 Quasi-one-dimensional Boussinesq equation for a variable depth

If we want to take into account the influence of an uneven bottom, we have to consider the dimensionless classical problem (3.2.1) but, in place of (3.2.1b), in this case we can apply the uneven-bottom condition:

$$\phi_z = \alpha\left[\delta^2 \phi_x G_x + \Delta^2 \phi_y G_y\right] \quad \text{on} \quad z = -1 + \alpha G(x^*, y^*) , \tag{6.1.1}$$

according to expansion (3.2.11), where $\alpha = g_0/h_0$, $\beta = \lambda_0/l_0$, $\gamma = \mu_0/m_0$ and $x^* = \beta x$, $y^* = \gamma y$.

Here, we assume that $We = 0$ and we consider, subject to the uneven-bottom condition (6.1.1), the dimensionless Laplace equation

$$\phi_{zz} + \delta^2 \phi_{xx} + \Delta^2 \phi_{yy} = 0 ,$$
$$-1 + \alpha G(x^*, y^*) \leqslant z \leqslant \varepsilon\zeta(x, y, t) , \tag{6.1.2}$$

and the following two free-surface dimensionless conditions:

$$\phi_z = \delta^2 \zeta_t + \varepsilon(\delta^2 \phi_x \zeta_x + \Delta^2 \phi_y \zeta_y) , \tag{6.1.3}$$

$$\phi_t + \frac{1}{2}\left(\varepsilon\phi_x^2 + \frac{\varepsilon\Delta^2}{\delta^2}\,\phi_y^2 + \frac{\varepsilon}{\delta^2}\,\phi_z^2\right) + \zeta = 0 , \tag{6.1.4}$$
$$\text{on} \quad z = \varepsilon\zeta(x, y, t) .$$

†For the KdV and NLS equations, in the one-dimensional case, and for more details of the matching procedure, see Ref. [77] and Ref. [78], p. 25.

As in Section 3.2, from the Laplace equation (6.1.2) and the uneven-bottom condition (6.1.1), we can find, in place of expansion (3.2.8), the following asymptotic expansion for the velocity potential ϕ

$$\phi = F(x,\,y,\,t) - \frac{\varepsilon\kappa_0}{2}(z+h)^2(F)_{xx} - \beta\varepsilon\kappa_0(z+h)(F)_x\,\frac{\partial h}{\partial x^*}$$
$$+\varepsilon^2\left[\frac{\kappa_0^2}{24}(z+h)^4(F)_{xx\,xx} - \frac{v_0^2}{2}(z+h)^2(F)_{yy}\right]$$
$$+\beta\varepsilon^2\frac{\kappa_0^2}{2}(z+h)^3(F)_{xx\,x}\,\frac{\partial h}{\partial x^*} - \gamma\varepsilon^2 v_0^2(z+h)(F)_y\,\frac{\partial h}{\partial y^*}$$
$$+O(\varepsilon^3;\,\beta^2\varepsilon^2)\,, \qquad (6.1.5)$$

when we assume again that $\delta^2 = \kappa_0\varepsilon$ and $\Delta = v_0\varepsilon$ in the limit $\varepsilon \to 0$.

In expansion (6.1.5), we have $h(x^*, y^*;\,\alpha) \equiv 1 - \alpha G(x^*, y^*)$ and we can specify that expansion (6.1.5) is valid when $\gamma = O(1)$, but it is necessary to postulate that $\beta \gg \varepsilon$. If $\beta = O(\varepsilon)$, then the fifth term in expansion (6.1.5), proportional to $\beta\varepsilon^2$, is of order of $O(\varepsilon^3)$ and we do not take this term into account in this case! In the paper by Liu, Yoon and Kirby [79] this last case is considered correctly and these authors have conjectured a form of the 'modified' KP equation for a variable depth (in Section 6.3 we shall consider the influence of a variable depth on the KP equation).

From expansion (6.1.5) we can easily obtain the values of the derivatives ϕ_s, with $s = (t, x, y)$, and ϕ_z, on $z = \varepsilon\,\zeta(s)$:

$$\phi_t = (F)_t - \frac{\varepsilon\kappa_0}{2}\,h^2(F)_{xx\,t} - \beta\varepsilon\kappa_0 h\,\frac{\partial h}{\partial x^*}(F)_{xt} + \ldots\,, \qquad (6.1.6a)$$

$$\phi_x = (F)_x - \frac{\varepsilon\kappa_0}{2}\,h^2(F)_{xx\,x} - 2\beta\varepsilon\kappa_0 h\,\frac{\partial h}{\partial x^*}(F)_{xx} + \ldots\,, \qquad (6.1.6b)$$

$$\phi_y = (F)_y - \frac{\varepsilon\kappa_0}{2}\,h^2(F)_{xx\,y} - \beta\varepsilon\kappa_0 h\,\frac{\partial h}{\partial x^*}(F)_{xy}$$
$$-\gamma\varepsilon\kappa_0 h\,\frac{\partial h}{\partial y^*}(F)_{xx} + \ldots\,, \qquad (6.1.6c)$$

$$\phi_z = -\varepsilon\kappa_0\Bigg[h(F)_{xx} + \beta\,\frac{\partial h}{\partial x^*}(F)_x + \varepsilon\,\zeta(F)_{xx}$$
$$-\frac{\varepsilon\kappa_0}{6}\,h^3(F)_{xx\,xx} + \frac{\varepsilon v_0^2}{\kappa_0}\,h(F)_{yy} - \frac{3}{2}\,\beta\varepsilon\kappa_0 h^2\,\frac{\partial h}{\partial x^*}(F)_{xxx}$$
$$+\frac{\gamma\varepsilon v_0^2}{\kappa_0}\,\frac{\partial h}{\partial y^*}(F)_y\Bigg] + \ldots \qquad (6.1.6d)$$

Now, from the free-surface boundary condition (6.1.3), we find, according to expressions (6.1.6), the following approximate equation

$$\zeta_t + h(F)_{xx}$$
$$+\varepsilon\left[\zeta_x(F)_x + \zeta(F)_{xx} - \frac{\kappa_0}{6}\,h^3(F)_{xx\,xx} + \frac{v_0^2}{\kappa_0}\,h(F)_{yy}\right]$$
$$= -\beta\,\frac{\partial h}{\partial x^*}(F)_x - \frac{3}{2}\,\beta\varepsilon\kappa_0 h^2\,\frac{\partial h}{\partial x^*}(F)_{xx\,x} + \frac{\gamma\varepsilon v_0^2}{\kappa_0}\,\frac{\partial h}{\partial y^*}(F)_y\,, \qquad (6.1.7)$$

with an error of $O(\varepsilon^2)$.

Next, the second free-boundary condition (6.1.4), gives, still according to expressions (6.1.6), a second approximate equation

$$(F)_t + \zeta + \varepsilon\left[\frac{1}{2}(F)_x^2 - \frac{\kappa_0}{2}\,h^2(F)_{xx\,t}\right] = \beta\varepsilon\kappa_0 h\,\frac{\partial h}{\partial x^*}(F)_{xt}\,, \qquad (6.1.8)$$

again with an error of $O(\varepsilon^2)$.

The two equations, (6.1.7) and (6.1.8), are our *quasi-one-dimensional Boussinesq equations for a variable, uneven bottom of the form* $z = -h(x^*, y^*;\,\alpha)$, with $x^* = \beta x$ and $y^* = \gamma y$.

If $h \equiv 1$, we obtain again, from Eqns (6.1.7) and (6.1.8), the classical Q1DB system of two equations for F and ζ, similar to Eqns (3.2.20)–(3.2.23).

If $h \neq 1$, we can also write down the above Boussinesq equations (6.1.7), (6.1.8) for an uneven bottom:

$$\zeta_t + \left[(h+\varepsilon\,\zeta)u\right]_x + \frac{\varepsilon v_0^2}{\kappa_0}(hv)_y - \frac{\varepsilon\kappa_0}{6}(h^3 u_{xx})_x$$
$$= \beta\varepsilon\kappa_0 h^2\,\frac{\partial h}{\partial x^*}\,u_{xx}\,, \qquad (6.1.9)$$

$$u_t + \varepsilon u u_x + \zeta_x - \frac{\varepsilon\kappa_0}{2}\,h^2 u_{xx\,t} = 2\beta\varepsilon\kappa_0 h\,\frac{\partial h}{\partial x^*}\,u_{xt}\,, \qquad (6.1.10)$$

where $u = (F)_x$, $v = (F)_y$, $u_y = v_x$.

Again, from Eqns (6.1.7) and (6.1.8), we can eliminate the function ζ and derive a *single Boussinesq equation for F*. Indeed, the following expression for ζ can be obtained from Eqn (6.1.8):

$$\zeta = -(F)_t - \frac{\varepsilon}{2}(F)_x^2 + \frac{\varepsilon\kappa_0}{2}\left[h^2(F)_{xt}\right]_x\,,$$

and if we take into account the above relationship in Eqn (6.1.7), we find for $F(x,\,y,\,t)$ a *single approximate Boussinesq equation for an uneven bottom*:

$$(F)_{tt} - \left[h(F)_x\right]_x - \frac{\varepsilon v_0^2}{\kappa_0}\left[h(F)_y\right]_y + \varepsilon\left[(F)_x^2 + \frac{1}{2}(F)_t^2\right]_t$$
$$-\frac{\varepsilon\kappa_0}{2}\left[h^2(F)_{xt}\right]_{xt} + \frac{\varepsilon\kappa_0}{6}\left[h^3(F)_{xx\,x}\right]_x$$
$$+\beta\varepsilon\kappa_0 h^2\,\frac{\partial h}{\partial x^*}(F)_{xxx} = 0 \qquad (6.1.11)$$

with an error of $O(\varepsilon^2)$ when $\beta \gg \varepsilon$.

Naturally, if $\beta = O(\varepsilon)$, then in place of Eqn (6.1.11) it is necessary to write the following reduced Boussinesq equation, again with an error of $O(\varepsilon^2)$ and with $\gamma = O(1)$:

$$(F)_{tt} - h(F)_{xx} - \varepsilon\gamma^2\,\frac{v_0^2}{\kappa_0}\,h(F)_{y^*y^*}$$
$$+\varepsilon\Bigg\{\left[(F)_x^2 + \frac{1}{2}(F)_t^2\right]_t - \gamma^2\,\frac{v_0^2}{\kappa_0}\,\frac{\partial h}{\partial y^*}(F)_{y^*}$$
$$-\frac{\kappa_0}{2}\,h^2\left[(F)_{tt} - \frac{1}{3}\,h(F)_{xx}\right]_{xx}\Bigg\} = 0\,, \qquad (6.1.12)$$

for $F(x, t, y^*;\,h)$, where $h = h(\varepsilon x, y^*;\,\alpha)$ and $y^* = \gamma y$.

6.2 Korteweg–de Vries equation for variable depth

We return now to the Boussinesq equations (6.1.9) and (6.1.10) for an uneven bottom. Here, for one-dimensional water waves, when

$$\beta = \varepsilon\,, \qquad \gamma = 0\,, \qquad h = h(\varepsilon x)\,, \qquad (6.2.1)$$

we obtain, as the Boussinesq equations for variable depth, the following two dimensionless equations for $u(t, x)$ and $\zeta(t, x)$:

$$\zeta_t + \left[(h+\varepsilon\zeta)u\right]_x - \frac{\varepsilon\kappa_0}{6}\left[h^3 u_{xx}\right]_x = 0\,,$$
$$u_t + \varepsilon u u_x + \zeta_x - \frac{\varepsilon\kappa_0}{2}\,h^2 u_{xxt} = 0\,, \tag{6.2.2}$$

with an error of $O(\varepsilon^2)$. But an alternate form of the Boussinesq equations (6.2.2), for an uneven bottom $z=-h(\varepsilon x)$, can be derived if we introduce the depth-averaged horizontal velocity $U(x,t)$ defined by

$$U = \frac{1}{h+\varepsilon\zeta}\int_{-h}^{\varepsilon\zeta}\phi_x\,\mathrm{d}z$$
$$= \frac{1}{h+\varepsilon\zeta}\int_{-h}^{\varepsilon\zeta}\left[(F)_x - \frac{\varepsilon\kappa_0}{2}(z+h)^2 h^2 (F)_{xxx} + \ldots\right]\mathrm{d}z\,,$$

in accordance with expression (6.1.6b). Hence, we find:

$$U = u - \frac{\varepsilon\kappa_0}{6}\,h^2 u_{xx} + O(\varepsilon^2)\,, \qquad u = (F)_x\,, \tag{6.2.3}$$

which can be inverted to give the following approximate expression for u:

$$u = U + \frac{\varepsilon\kappa_0}{6}\,h^2 U_{xx} + O(\varepsilon^2)\,. \tag{6.2.4}$$

After this relationship (6.2.4) for u is substituted into the system of equations (6.2.2), it follows directly that

$$\zeta_t + \left[(h+\varepsilon\zeta)U\right]_x = 0\,,$$
$$U_t + \varepsilon U U_x + \zeta_x - \frac{\varepsilon\kappa_0}{3}\,h^2 U_{xxt} = 0\,, \tag{6.2.5}$$

where $h = h(\varepsilon x)$ and with an error of $O(\varepsilon^2)$.

Let the new variables be

$$x^* = \varepsilon x\,, \qquad \xi = \frac{1}{\varepsilon}\int^{x^*}\frac{\mathrm{d}x^*}{[h(x^*)]^{1/2}} - t\,, \tag{6.2.6}$$

where ξ is the coordinate moving at the local linear velocity.

The changes

$$\frac{\partial}{\partial t} = -\frac{\partial}{\partial\xi}\,, \qquad \frac{\partial}{\partial x} = \varepsilon\frac{\partial}{\partial x^*} + \frac{1}{[h(x^*)]^{1/2}}\frac{\partial}{\partial\xi} \tag{6.2.7}$$

make the system of equations (6.2.5)

$$-\frac{\partial\zeta}{\partial\xi} + \varepsilon u\frac{\mathrm{d}h}{\mathrm{d}x^*} + \varepsilon h\frac{\partial u}{\partial x^*} + \varepsilon u h^{-1/2}\frac{\partial\zeta}{\partial\xi} + \varepsilon\zeta h^{-1/2}\frac{\partial u}{\partial\xi}$$
$$+ h^{-1/2}\frac{\partial u}{\partial\xi} = 0\,, \tag{6.2.8a}$$

$$-h^{1/2}\frac{\partial u}{\partial\xi} + \varepsilon u\frac{\partial u}{\partial\xi} + \varepsilon h^{1/2}\frac{\partial\zeta}{\partial x^*} + \frac{\partial\zeta}{\partial\xi} + \frac{\varepsilon\kappa_0}{3}\,h^{3/2}\frac{\partial^3 u}{\partial\xi^3} = 0 \tag{6.2.8b}$$

with an error of $O(\varepsilon^2)$. Adding the two above equations and using the leading approximation, when $\varepsilon\to 0$ and $u\sim h^{-1/2}\zeta$, we get, to the leading order:

$$h^{1/2}\frac{\partial\zeta}{\partial x^*} + \frac{1}{2}\,\zeta h^{-1/2}\frac{\mathrm{d}h}{\mathrm{d}x^*} + 3h^{-1}\zeta\frac{\partial\zeta}{\partial\xi} + \frac{\kappa_0}{3}\,h\,\frac{\partial^3\zeta}{\partial\xi^3} = 0\,. \tag{6.2.9}$$

This *extended KdV equation* was first deduced by Kakutani (1971) [80] and may be expressed in several forms. For example, we can apply the following Ono transformation [47]

$$\zeta = -\frac{2}{3}\,\kappa_0 h^2 Z\,, \qquad T = \frac{\kappa_0}{6}\int_{x^{*0}}^{x^*} h^{1/2}\,\mathrm{d}x^*\,, \tag{6.2.10}$$

where the exponents of h are chosen to remove most of the variable coefficient. In this case we obtain for the function $Z(T,\xi)$, in place of Eqn (6.2.9), the following reduced KdV equation for an uneven bottom:

$$\frac{\partial Z}{\partial T} - 6Z\frac{\partial Z}{\partial\xi} + \frac{\partial^3 Z}{\partial\xi^3} + \lambda(T)Z = 0\,, \tag{6.2.11}$$

where the coefficient

$$\lambda(T) = \frac{27}{2\kappa_0}\,h^{-3/2}\frac{\mathrm{d}h}{\mathrm{d}x^*}\,, \qquad \mathrm{d}T = \frac{\kappa_0}{6}\left[h(x^*)\right]^{1/2}\mathrm{d}x^* \tag{6.2.12}$$

represents the *effect of variable depth*.

6.2.1 Let us consider two invariants of Eqn (6.2.11). Integrating Eqn (6.2.11) with respect to ξ from $-\infty$ to $+\infty$, we obtain

$$\frac{\partial}{\partial T}\left(\int_{-\infty}^{+\infty} Z\,\mathrm{d}\xi\right) + \left[-3Z^2 + \frac{\partial^2 Z}{\partial\xi^2}\right]_{-\infty}^{+\infty} + \lambda(T)\int_{-\infty}^{+\infty} Z\,\mathrm{d}\xi = 0\,.$$

Now, if Z and its derivatives are assumed to vanish at infinities, then

$$\int_{-\infty}^{+\infty} Z\,\mathrm{d}\xi\left[\exp\int_0^T\lambda(T)\,\mathrm{d}T\right] = \text{const} = J\,. \tag{6.2.13}$$

Now, multiplying Eqn (6.2.11) by Z and integrating with respect to ξ, we get

$$\frac{1}{2}\frac{\partial}{\partial T}\left(\int_{-\infty}^{+\infty} Z^2\,\mathrm{d}\xi\right) + \left[-2Z^3 + Z\frac{\partial^2 Z}{\partial\xi^2} - \frac{1}{2}\frac{\partial(Z^2)}{\partial\xi}\right]_{-\infty}^{+\infty}$$
$$+\frac{1}{2}\lambda(T)\int_{-\infty}^{+\infty} Z^2\,\mathrm{d}\xi = 0\,,$$

which may be also integrated with respect to T to give

$$\exp\left[2\int_0^T\lambda(T)\,\mathrm{d}T\right]\left(\int_{-\infty}^{+\infty} Z^2\,\mathrm{d}\xi\right) = \text{const} = H\,. \tag{6.2.14}$$

Relationships (6.2.13) and (6.2.14) are *two invariants of the extended KdV equation* (6.2.11) *for an uneven bottom*.

But from the expression (6.2.12) describing the coefficient $\lambda(T)$ it follows that

$$\exp\int_0^T\lambda(T)\,\mathrm{d}T = h^{9/4}\,, \qquad \mathrm{d}T = \frac{\kappa_0}{6}\,h^{1/2}\,\mathrm{d}x^*\,,$$

and hence our two invariants are

$$J = h^{9/4}\int_{-\infty}^{+\infty} Z\,\mathrm{d}\sigma\,, \qquad H = h^{9/2}\int_{-\infty}^{+\infty} Z^2\,\mathrm{d}\sigma\,. \tag{6.2.15}$$

6.2.2 We note that the approximate extended KdV equation (6.2.11), valid for wave propagation to the right, cannot account for reflection during transmission which, however, can be predicted by the more complete Boussinesq equations of Peregrine [46]. In particular, Peregrine notes that weak reflection should be describable by the linearised Airy equations but for variable depth, which can be handled analytically by the method of characteristics.

6.2.3 The KdV equation (6.2.9) with variable coefficients is often also rewritten [27, 48, 78] in the following form

$$\frac{\partial Z}{\partial T} - 6Z\frac{\partial Z}{\partial\xi} + \frac{\partial^3 Z}{\partial\xi^3} = \frac{9}{4}\frac{Z}{h}\frac{\mathrm{d}h}{\mathrm{d}T}\,. \tag{6.2.16}$$

This 'perturbed' KdV equation predicts soliton fission that occurs as a solitary wave moves into the shelf region [40]. This equation (6.2.16) has also been used as the basis for a

discussion of the effects of a perturbation on the KdV equation, when $(1/h)\,\mathrm{d}h/\mathrm{d}T$ is small. This can be accomplished either by direct methods [81] or, more satisfactory, via the inverse scattering methods [82]. In particular the phenomenon of the shelf that appears behind a solitary wave is now well understood [51].

6.3 Kadomtsev–Petviashvili equation for an uneven bottom

We shall use Eqns (6.1.7) and (6.1.8) to derive the KP equation for an uneven bottom. When $v_0 = 0$, we obtain the classical KdV equation for an uneven bottom in several forms (see, for instance, the book of Mei [43], pp 560 to 561). For the derivation of the '*extended KP*' equation for an *uneven bottom* it is necessary to consider the Boussinesq equations (6.1.7) and (6.1.8) with $\beta = \varepsilon$.

These equations then have the following variables

$$t, x, y, \quad \text{and also} \quad x^* = \varepsilon x\,, \quad y^* = \gamma y\,, \tag{6.3.1}$$

and we can write, with an error of $O(\varepsilon^2)$, the following two equations for F and ζ

$$\zeta_t + h(F)_{xx} + \varepsilon\left[\zeta_x (F)_x + \zeta (F)_{xx} - \frac{\kappa_0}{6}\, h^3 (F)_{xxxx} + \frac{v_0^2}{\kappa_0}\, h(F)_{yy} + \frac{\partial h}{\partial x^*}(F)_x\right] = \gamma\varepsilon\, \frac{v_0^2}{\kappa_0}\, \frac{\partial h}{\partial y^*}(F)_y\,, \tag{6.3.2a}$$

$$(F)_t + \zeta + \varepsilon\left[\frac{1}{2}(F)_x^2 - \frac{\kappa_0}{2}\, h^2 (F)_{xxt}\right] = 0\,. \tag{6.3.2b}$$

By analogy with the one-dimensional case (see Section 6.2), we shall now introduce the following new variables

$$\tau = \varepsilon t\,, \qquad \xi = \int_0^x h^{-1/2}(\varepsilon x, y^*;\, \alpha)\,\mathrm{d}x - t\,. \tag{6.3.3}$$

We now have the following fomulae for the derivatives:

$$\frac{\partial}{\partial x} = \varepsilon\, \frac{\partial}{\partial x^*} + h^{-1/2}\, \frac{\partial}{\partial \xi}\,, \qquad \frac{\partial}{\partial t} = -\frac{\partial}{\partial \xi} + \varepsilon\, \frac{\partial}{\partial \tau}\,,$$

$$\frac{\partial}{\partial y} = \gamma\, \frac{\partial}{\partial y^*} + G\, \frac{\partial}{\partial \xi}\,,$$

with $h = h(x^*, y^*;\, \alpha)$ and

$$G(x^*, y^*) = \frac{\partial}{\partial y^*} \int_0^x h^{-1/2}(\varepsilon x, y^*;\, \alpha)\,\mathrm{d}x\,. \tag{6.3.4}$$

If we assume that

$$F = F_0(\tau, \xi, x^*, y^*) + \varepsilon F_1 + \ldots\,,$$
$$\zeta = \zeta_0(\tau, \xi, x^*, y^*) + \varepsilon\, \zeta_1 + \ldots \tag{6.3.5}$$

then, equating terms of order ε^0 and ε^1, we obtain the following equations for the functions F_0, ζ_0 and F_1, ζ_1:

$$\zeta_0 = \frac{\partial F_0}{\partial \xi}\,, \tag{6.3.6a}$$

$$-\frac{\partial \zeta_1}{\partial \xi} + \frac{\partial^2 F_1}{\partial \xi^2} = \frac{\partial \zeta_0}{\partial \tau} + 2h^{1/2}\, \frac{\partial^2 F_0}{\partial \xi\, \partial x^*} + \frac{1}{2}\, h^{-1/2}\, \frac{\partial h}{\partial x^*}\, \frac{\partial F_0}{\partial \xi} + h^{-1}\left(\zeta_0\, \frac{\partial^2 F_0}{\partial \xi^2} + \frac{\partial F_0}{\partial \xi}\, \frac{\partial \zeta_0}{\partial \xi}\right) - \frac{\kappa_0}{6}\, h\, \frac{\partial^4 F_0}{\partial \xi^4} + \frac{v_0^2}{\kappa_0}\left[\gamma^2\, \frac{\partial}{\partial y^*}\left(h\, \frac{\partial F_0}{\partial y^*}\right) + \gamma\, \frac{\partial}{\partial y^*}\left(hG\, \frac{\partial F_0}{\partial \xi}\right) + \gamma h G\, \frac{\partial^2 F_0}{\partial \xi\, \partial y^*} + hG^2\, \frac{\partial^2 F_0}{\partial \xi^2}\right]\,, \tag{6.3.6b}$$

$$-\frac{\partial F_1}{\partial \xi} + \zeta_1 = -\frac{1}{2}\, h^{-1}\left(\frac{\partial F_0}{\partial \xi}\right)^2 - \frac{\kappa_0}{2}\, h\, \frac{\partial^3 F_0}{\partial \xi^3}\,. \tag{6.3.6c}$$

As expected, the first equation (6.3.6a) is insufficient to determine both functions ζ_0 and F_0, and it is necessary to go to the second order in ε (terms ε^1) to obtain a consistency condition to do this.

Differentiating Eqn (6.3.6c) with respect to ξ and subtracting from Eqn (6.3.6b), we get the following equation for the leading term of the elevation ζ_0 of the free surface:

$$\frac{\partial \zeta_0}{\partial \tau} + 2h^{1/2}\, \frac{\partial \zeta_0}{\partial x^*} + \frac{1}{2}\, h^{-1/2}\zeta_0\, \frac{\partial h}{\partial x^*} + 3h^{-1}\zeta_0\, \frac{\partial \zeta_0}{\partial \xi} + \frac{\kappa_0}{3}\, h\, \frac{\partial^3 \zeta_0}{\partial \xi^3} + \frac{v_0^2}{\kappa_0}\, h\left(G^2\, \frac{\partial \zeta_0}{\partial \xi} + \gamma G\, \frac{\partial \zeta_0}{\partial y^*}\right) + \frac{v_0^2}{\kappa_0}\, \gamma\, \frac{\partial}{\partial y^*}\left(hG\zeta_0 + \gamma h\, \frac{\partial F_0}{\partial y^*}\right) = 0\,, \tag{6.3.7}$$

with

$$F_0 = \int_\infty^\xi \zeta_0\, \mathrm{d}\xi\,.$$

From the above equation, when ζ_0 is independent of the slow time τ, we get the equation derived by Xue-Nong Chen {[55], Eqn (22)}.

Naturally, our extended KP equation (6.3.7), in ζ_0, for an uneven bottom can be also derived directly from the single Boussinesq equation (6.1.12). If the topography is even ($h = 1$), Eqn (6.3.7) is reduced to the classical KP equation and if $v_0 = 0$, this equation is reduced to a variable-coefficient KdV equation which is the same (when $\partial\zeta_0/\partial\tau = 0$) as that obtained by Johnson [48, 81] (see Section 6.2).

To get a more concise form of Eqn (6.3.7), we take $\zeta_0 = h^{-1/4} H(x^*, y^*, \xi)$ corresponding to $\partial\zeta_0/\partial\tau = 0$, so that Eqn (6.3.7) becomes

$$\frac{\partial H}{\partial x^*} + \frac{3}{2}\, h^{-7/4} H\, \frac{\partial H}{\partial \xi} + \frac{\kappa_0}{6}\, h^{1/2}\, \frac{\partial^3 H}{\partial \xi^3} + \frac{v_0^2}{2\kappa_0}\left[h^{1/2} G^2\, \frac{\partial H}{\partial \xi} + \gamma h^{3/4} G\, \frac{\partial}{\partial y^*}(h^{-1/4} H) + \gamma h^{-1/4}\, \frac{\partial}{\partial y^*}\left(\gamma h\, \frac{\partial F_0}{\partial y^*} + h^{3/4} GH\right)\right. = 0\,, \tag{6.3.8}$$

with

$$F_0 = h^{-1/4} \int_\infty^\xi H\, \mathrm{d}\xi\,.$$

Again, when $v_0 = 0$, we can rederive the classical KdV equation for an uneven bottom in several forms and for this see the book by Mei [43], pp 560–561.

7. Some aspects of the solitary wave-soliton phenomenon

7.1 John Scott Russell's discovery

The history (story!) of the solitary wave (SW) begins with the observation by J Scott Russell of 'the great wave of translation' (first observed on the Edinburgh to Glasgow canal in 1834). Russell reported his discovery to the British Association for the Advancement of Science in 1844 as

follows (the discovery of the SW excited strongly his scientific and poetic imagination):

"... I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped — not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a round, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon..."

In fact, he knew that the velocity was proportional to its height and proposed after much experimental work the law:

$$c^2 = g(h+a) \ , \tag{7.1.1}$$

where g, h and a are the acceleration due to gravity, the undisturbed depth, and the maximum height of the wave, as measured from the undisturbed level, respectively. The SW is therefore a gravity wave. He knew about the interaction of solitary waves, but did not appear to have noticed their soliton quality, a property I will discuss shortly in Sections 7.4 and 7.5. He also knew how to create them! But, unfortunately, at first, Russell's idea faced great hostility and scepticism from the leading lights in the scientific community of his day. Both Airy and Stokes questioned whether a wave which travelled without change in shape could be totally above the water and cited the diminution of amplitude as an indication that the wave was inherently nonpermanent. Russell had suggested (correctly!) that this failure was due to friction. From expression (7.1.1) we note that higher waves travel faster.

Hidden away in Russell's "Report on waves" (1844) (see Ref. [83], plate XLVII) is the diagram reproduced in Fig. 11 (this figure is Fig. 1.5 in the book of Drazin and Johnson [31]) together with the associated description.

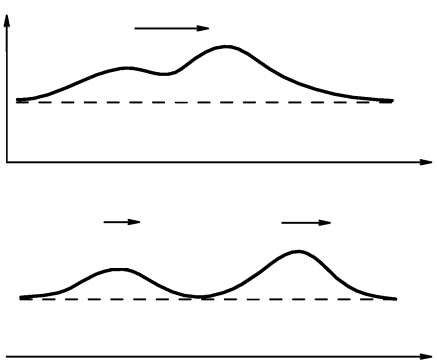

Figure 11. A sketch of J Scott Russell's 'compound wave'. This figure "... represents the genesis by a large low column of fluid of a compound or double wave of the first order, which immediately breaks down by spontaneous analysis into two, the greater moving faster and altogether leaving the smaller" (see Ref. [83], p. 384).

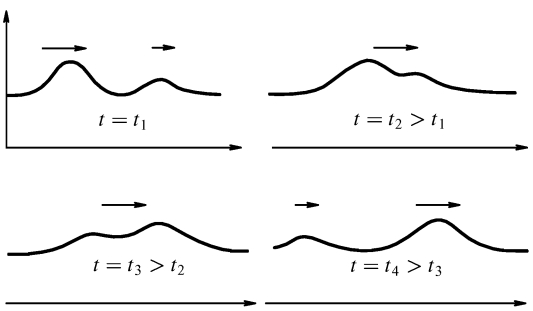

Figure 12. The taller wave catches up, interacts with and then passes the shorter one. The taller one, therefore, appears to overtake the shorter one and continue on its way intact and undistorted as an SW.

One interpretation of this Russell's result (with a little hindsight!) is that an arbitrary initial profile (which is not an exact SW!) will evolve into two (or more!) waves which then move apart progressively, approaching the form of single SWs as $t \to \infty$, since an SW is defined for $(-\infty, +\infty)$.

This alone is rather surprising, but another remarkable property can also be observed. If we start with an initial profile like that given in Fig. 11, but with the taller wave somewhat to the left of the shorter, then the evolution is as depicted in Fig. 12.

The experimental work of Russell, on the SWs, summarised in Ref. [83], led immediately to the theoretical work of Airy [8] and Stokes [3], which underlie almost all subsequent theoretical work on water waves except, surprisingly, that on the SWs!

This was first described much later by Boussinesq [16 to 19], but has been brought to prominence in recent years by the development of soliton solutions initially for the KdV equation, which describes the SWs.

7.2 Boussinesq and Rayleigh solitary wave solution

It was not until the 1870s that Russell's work was finally vindicated and its scientific importance can be measured by the eminence of the men who did the job (according to Newell's book [27], p. 3).

The conflict between Russell's observations and Airy's shallow-water theory [see the system of equations (3.1.12)] was resolved independently by Boussinesq and Rayleigh. Boussinesq [16] and Rayleigh [10] found the hyperbolic secant squared solution for the free surface elevation.

To put Russell's formula (7.1.1) on a firmer footing, both Boussinesq and Rayleigh assumed that an SW has a length scale much greater than the depth of water. They deduced, from the equation of motion for an inviscid incompressible liquid, Russell's formula for c.

In fact, they also showed that the profile $z = \zeta(x,t)$ is given by

$$\zeta = a\,\mathrm{sech}^2\left[\beta(x-ct)\right] \ , \tag{7.2.1}$$

where $\beta^2 = 3a/4h^2(h+a)$ for any $a > 0$, although the sech^2 profile is strictly only correct if $a/h \ll 1$! But, these authors did not, however, write down a simple equation (of the KdV type!), for $\zeta(x,t)$ which admits formula (7.2.1) as a solution.

Boussinesq derived his solution from the equation

$$\frac{\partial^2\zeta}{\partial t^2} = C_0^2\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{3}{2h_0}\frac{\partial^2\zeta^2}{\partial x^2} + \frac{1}{3}h_0^2\frac{\partial^4\zeta}{\partial x^4}\right) \tag{7.2.2}$$

approximating the water wave equation that now bears his name. In this approximate equation, the motion can be still bidirectional, but the basic idea of the balance between nonlinearity and dispersion is present!

Boussinesq also showed that any local section of a unidirectional solution of Eqn (7.2.2) moves at the approximate velocity:

$$C = C_0\left(1 + \frac{3}{4h_0}\,\zeta + \frac{1}{6\zeta}\,h_0^2\,\frac{\partial^2\zeta}{\partial x^2}\right)\,, \tag{7.2.3}$$

where C_0 is the velocity of infinitesimal long waves and the second and third terms within the parentheses in expression (7.2.3) represent nonlinearity and dispersion, respectively. He infers from expression (7.2.3) that an initial elevation of water for which al^2 is significantly in excess of the value determined by $U = 3al^2/4h^3 = 1$ would tend to disintegrate into two or more solitary waves (plus, in most cases, a residual wave train) and that an initial depression would tend to decay into an oscillatory wave train, all in conformity with Russell's observations. We shall assume that l is the characteristic length for the SWs. Rayleigh obtained

$$l^2 = \frac{4h^2(h+a)}{3a}\,,$$

which reduces to $U = 1$ for $a/h \ll 1$. Rayleigh's [10] derivation of the equivalents of expressions (7.1.1) and (7.2.1) is reproduced by Lamb ([4], Section 252). It is more direct but less illuminating than that of Boussinesq.

7.3 Korteweg – de Vries and Kadomtsev – Petviashvili solitary waves

Unfortunately, both Boussinesq and Rayleigh did not write down a simple equation for $\zeta(x,t)$ which admits formula (5.3.2) as a solution! This final step (in the first period of the history of the solitary waves) was completed by Korteweg and de Vries (1895) [24]. These authors, who 'apparently!' did not know the work of Boussinesq and Rayleigh and who were still trying to answer the objections of Airy and Stokes, wrote down the unidirectional equation (KdV equation) for $\zeta(x,t)$ which now bears their names. In fact, the Boussinesq equation (7.2.2) reduces to the KdV equation (4.2.4) by factoring the operator $C_0^2\,\partial^2/\partial x^2 - \partial^2/\partial t^2$, invoking the prior assumption of unilateral propagation, and integrating with respect to x.

Indeed, Korteweg and de Vries [24] derived a somewhat more general equation, in which they allowed for any uniform translation of the reference frame and incorporated surface tension:

$$\frac{\partial\zeta}{\partial t} = \frac{3}{2}\left(\frac{g}{h_0}\right)^{1/2}\left(\zeta\,\frac{\partial\zeta}{\partial\chi} + \frac{2}{3}\,\alpha\,\frac{\partial\zeta}{\partial\chi} + \frac{1}{3}\,\sigma\,\frac{\partial^3\zeta}{\partial\chi^3}\right)\,, \tag{7.3.1}$$

where α is a small but arbitrary constant, which is closely related to the exact velocity of uniform motion imparted to the liquid, and where

$$\sigma = \frac{1}{3}\,h_0^3 - \frac{h_0 T}{\rho_0 g} \tag{7.3.2}$$

depends on the surface tension T of a liquid of constant density ρ_0.

They then obtained a family of periodic solutions of the form $\zeta = \zeta(x - Ct)$, which they called cnoidal waves (see Section 3.3 for details). Boussinesq [18, 19] also discussed periodic solutions of Eqn (7.2.2), but did not obtain explicit integrals. This family of cnoidal waves comprises the Boussinesq solitary wave described by expression (7.2.1) in the limit of an infinite period. More precisely, an SW can be claimed to possess a wavelength λ, not in the usual sense of a spatial period, but in the sense of the distance within which the surface elevation does not fall below (say) 3% of its maximum value. In this sense we obtain

$$\frac{a_0\lambda^2}{h_0^3} \approx 16\,, \tag{7.3.3}$$

and in Fig. 2 (Section 3.3) an SW is plotted (lower curve) for this value of the wavelength (Ref. [61], pp 465 – 466).

We have seen that the KdV equation is indeed valid in an appropriate region of the (x,t)-space for small-amplitude waves (see, for instance, Section 4.1). However, we are left with one final connection to make: that between the KdV equation and the sech^2 profile! To demonstrate this, according to Ref. [31], let us return to the equation derived by Korteweg and de Vries themselves, which is Eqn (7.3.1). This has the advantage that it is written in terms of physical variables and can therefore more readily be related to the work of Russell, Boussinesq and Rayleigh as expressed by relationship (7.1.1) and solution (7.2.1).

If the solution of Eqn (7.3.4) is stationary in the frame χ [χ is a coordinate chosen to be moving (almost) with the wave], then $\zeta = \zeta(\chi)$ and

$$\zeta\,\frac{\partial\zeta}{\partial\chi} + \frac{2}{3}\,\alpha\,\frac{\partial\zeta}{\partial\chi} + \frac{1}{3}\,\sigma\,\frac{\partial^3\zeta}{\partial\chi^3} = 0\,. \tag{7.3.4}$$

If we assume that $\zeta \to 0$ as $|\chi| \to \infty$ (as is the SW case), then Eqn (7.3.4) can be integrated twice to yield

$$2\alpha\zeta^2 + \zeta^3 + \sigma\left(\frac{\partial\zeta}{\partial\chi}\right)^2 = 0\,, \tag{7.3.5}$$

the second integration introducing the integration factor $\partial\zeta/\partial\chi$.

The last equation may be integrated once again, but it is more easily verified by direct substitution which shows that $\zeta(\chi) = a\,\mathrm{sech}^2\,\beta\chi$ is a solution, provided $a = 4\sigma\beta^2$ and $\alpha = -2\sigma\beta^2$.

The coordinate χ is defined by Korteweg and de Vries [24] as

$$\chi = x - \left(\frac{g}{h_0}\right)^{1/2}\left(1 - \frac{\alpha}{h_0}\right)t\,,$$

and so the SW solution becomes:

$$\zeta(x,t) = a\,\mathrm{sech}^2\left\{\frac{1}{2}\left(\frac{a}{\sigma}\right)^{1/2}\left[x - \left(\frac{g}{h_0}\right)^{1/2}\left(1 + \frac{a}{2h_0}\right)t\right]\right\}\,. \tag{7.3.6}$$

This agrees with expressions (7.1.1) and (7.2.1) if we neglect surface tension (so that $\sigma = h_0^3/3$) and assume that $a/h_0 \ll 1$, for then we have

$$C \sim \left(\frac{g}{h_0}\right)^{1/2}\left(1 + \frac{a}{2h_0}\right), \quad \beta \sim \frac{1}{2}\left(\frac{3a}{h_0^3}\right)^{1/2}\,. \tag{7.3.7}$$

Thus Russell's SW is a solution of the KdV equation.

In conclusion, we find that [31]: (a) from the SW solution (7.3.6), we see that the *velocity* of the SW relative to $(g/h_0)^{1/2}$ (the velocity of infinitesimal waves) is proportional to the amplitude a of the SW; (b) the *width* of the SW (defined as the distance between the points of height $a/2$, say!) is inversely proportional to $a^{1/2}$. In other words: *the taller SWs travel faster and are narrower.*

Finally, it is very interesting to compare the appearance of the amplitude a in solution (7.3.6) with the way $\varepsilon = a/h_0$ appears in the scaled variables that were used in our asymptotic derivation of the KdV equation [see, for example, Section 4.1, formulae (4.1.3)].

7.3.1 The two-dimensional generalisation of the KdV SW is performed in accordance with the KP equation [see Eqn (4.6.8)]:

$$\frac{\partial}{\partial x}\left(\frac{1}{c_0}\frac{\partial\zeta}{\partial t}+\frac{\partial\zeta}{\partial x}+\frac{3}{2h_0}\zeta\frac{\partial\zeta}{\partial x}+\frac{h_0^2}{6}\frac{\partial^3\zeta}{\partial x^3}\right)+\frac{1}{2}\frac{\partial^2\zeta}{\partial y^2}=0\ , \quad (7.3.8)$$

with $c_0 = (g/h)9^{1/2}$, and it is clear that this KP equation has also solitary wave solutions that are skewed versions of those given by Boussinesq, by Rayleigh, and by Korteweg and de Vries. Written in the same notation, they become

$$\zeta(x,\,y,\,t) = a\,\mathrm{sech}^2\left\{\frac{1}{2}\left(\frac{3a}{h_0^3}\right)^{1/2}\times\left[x+my-\left(1+\frac{a}{2h_0}+\frac{m^2}{2}\right)\left(\frac{g}{h_0}\right)^{1/2}t\right]\right\}\ , \quad (7.3.9)$$

when $\sigma = h_0^3/3$ and m is a parameter describing the (small) inclination of the KP SW relative to the main direction of propagation.

In conclusion, we should note that the SW solution of the KdV equation remained a curiosity in the literature until Zabusky and Kruskal (1965) [25] showed by their numerical studies that, as Russell had intimated, that SWs were of more ubiquitous nature!

7.4 Zabusky and Kruskal (1965) numerical investigations

Fig. 1 (see Section 1.1), taken from the famous 1965 paper of Zabusky and Kruskal (ZK) [25] announcing the *soliton* (see Section 7.5), shows the results of the ZK numerical experiment in which they used a centred difference, mass and (almost) energy conserving scheme, to solve the KdV equation for $u(x,t)$

$$\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\delta^2\frac{\partial^3 u}{\partial x^3}=0\ . \quad (7.4.1)$$

They used periodic boundary conditions and their initial profile was sinusoidal:

$$u(x,0)=\cos\pi x\ , \qquad 0\leqslant x<2\ , \quad (7.4.2)$$

and u, $\partial u/\partial x$, $\partial^2 u/\partial x^2$ are periodic in the interval [0, 2] for all t; they chose $\delta = 0.022$.

Initially the negative slope steepens, then the third derivative term induces fine-structure wiggles of wavelength d near and to the left of the maximum of u. In time the wiggles separate, forming a train of pulses travelling to the right, with the largest on the right, each pulse seeming to take on a life and identity of its own (!) and having a velocity proportional to its amplitude. These pulses each may be approximately described by the sech^2 SW solution, although strictly this is a solution valid for an isolated pulse on an infinite line. Because of the periodic boundary conditions, the solitary pulses eventually reappear on the left boundary and, owing to their higher velocity, the larger pulses overtake the smaller ones.

At this point, ZK noticed a remarkable phenomenon. Whereas two pulses behaved in almost a nonlinear way during the interaction, they afterwards reappeared with the larger one in front, each bearing precisely its former identity (height, width, and velocity).

The only evidence of a collision at all was a phase shift whereby the larger one appeared to be ahead of the position it would have been had it travelled alone and the smaller one appeared behind. When the two pulses were almost equal, the interaction seemed to take place by an exchange of identities in which the forward and smaller soliton became taller and narrower when it felt the leading edge of the larger one which then, in turn, took on the identity of the smaller one.

When the two pulses had very different amplitudes, the larger one rode over the smaller one in an adiabatic fashion. For amplitude differences in the in-between range, the interaction was more complicated. In a later analysis of the interaction, Lax (1968) [84] verified these observations rigorously.

The fact that the SWs emerge from a collision with exactly the same shape is surprising since it might be thought that the strong nonlinearity during the collision process would break up the pulses. This property is important because it shows that energy can be propagated in localised stable 'packets' without being dispersed. This behaviour is not a property of the KdV equation alone!

We note also that, after a very long time, the initial profile — or something very close to it — reappears, a phenomenon requiring the topology of the torus for its explanation; this is an example of *recurrence*.

This persistence of the wave induced ZK to coin the name 'soliton', to emphasise the particle-like character of these waves which seem to retain their identities in a collision. The discovery has led, in turn, to an intense study over the last thirty years!

7.5 From solitary wave to soliton

Although the term soliton was originally applied only to the SWs of the KdV equation, several nonlinear wave (NLW) equations are now known to exhibit similar effects (for example, the NLS equation derived in Section 5), and the term is often used in a wider context without formal definition. In fact, a soliton is an SW solution of an NLW equation (or a soliton equation!) which asymptotically preserves its shape and velocity upon collision with other SWs.

It can be proved that arbitrary initial motion (for example, that predicted by the KdV equation) breaks up, ultimately, into an 'ensemble of solitons'. Indeed, the significance of the name soliton for the SWs of the KdV equation is that by the use of the IST it can be shown that solitons appear for a wide range of initial conditions. For example, Fig. 1 (see Section 1.1) demonstrates formation of eight more-or-less distinct solitons, whose crests lie close to a straight line and have a period of 2.

In summary, the initial hump eventually disintegrates into N solitons, each of which corresponds to a discrete eigenvalue of the initial 'potential well' (in the IST). By a more elaborate analysis of the Gel'fand–Levitan–Marchenko (GLM) integral equation (see Section 7.6), an oscillatory tail can be shown to follow a train of solitons. However, the lag increases with time, so that the solitons are eventually alone at the front. This disintegration of an initial pulse into a train of solitons is also called *fission*.

7.5.1 Indeed, it was found numerically and confirmed experimentally that a soliton travelling from one constant depth to another constant but smaller depth, disintegrates into several solitons of varying sizes, trailed by an oscillatory tail. One the other hand, it was shown (numerically, but also analytically!) that over an uneven bottom the SWs exhibit peculiar behaviour such as damping, growing, or splitting, depending upon the local slope of the bottom (disintegration or degeneration of a soliton in shallow water with an uneven bottom). It has proved also that if the depth decreases to form a shelf, then for particular new depths an SW breaks up into a finite number of solitons asymptotically far along the shelf. It has been shown that if the depth changes from h^* to h^{**}, then only solitons are formed provided [48]:

$$\frac{h^{**}}{h^*} = \left[\frac{n(n+1)}{2}\right]^{-4/9}, \tag{7.5.1}$$

where n (integer) is the number of solitons far enough along the shelf.

For increasing depth (a wave moving into deeper water) an oscillatory asymptotic solution can be derived and such a solution describes an SW (soliton) degenerating into a cnoidal wave! More precisely, the fission of solitons was first discovered and studied by Masden and Mei [50] who used a numerical method.

7.5.2 Perhaps the most striking discovery resulting from the computations relating to (numerically) strong solitons is that the '*highest*' *soliton is not the most energetic*. If a soliton is a *localised entity* which may keep its identity after an interaction (almost as if the principle of superposition were valid), it may be regarded also as a *local confinement of the energy of the wave field* and when two solitons collide, each may come away with the same character as it had before the collision. When a soliton meets an 'anti-soliton', both may be annihilated. We note that, in fact, a soliton is a specific solution for waves of permanent form, although such a solution is not in general a soliton.

The phenomenon of the interaction of two solitons is shown (schematically) again in Fig. 13.

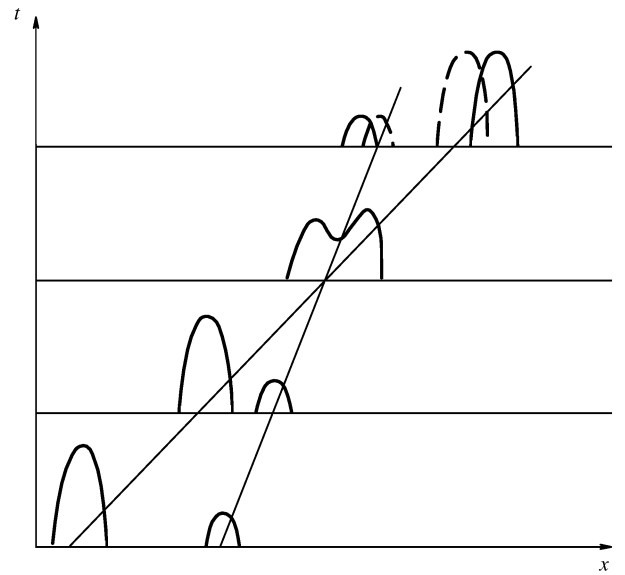

Figure 13. A sketch depicting again the interaction of two solitons.

These special 'soliton solutions' of the NLW equation are likely to be important in many ways. Gardner, Greene, Kruskal and Miura (1967) [49] developed an ingenious series of steps to tie the KdV equation to an inverse scattering problem, i.e. to determination of the scattering potential from the spectral functions, which can be done with the aid of the famous GLM integral equation (see Section 7.6).

7.5.3 Unfortunately, it is not easy to give a comprehensive and rigorous definition of a soliton! However, following Drazin and Johnson [31], we shall associate the term with: any solution of the NLW equation (or system of equations) which (a) represents a *wave of permanent form*, (b) is *localised*, so that it decays or approaches a *constant at infinity*, (c) can *interact* strongly with other solitons and *retain its identity*.

Naturally, there are more formal definitions, some of which concern discrete eigenvalues of a scattering problem. In the context of the KdV equation, it is usual to refer to the single-soliton solution as the SW, but when more than one of them appear in a solution they are called solitons.

Another way of expressing this is to say a soliton becomes a SW when it is infinitely far from any other soliton. We must mention also the fact that, for equations other than the KdV (or KP) equation, the SW solution may not be a sech^2 function! Furthermore, some NLW equations (or systems of equations) have SWs but not solitons, whereas others (like the KdV equation) have SWs *which are solitons*.

7.5.4 Now, I want to describe some properties related to 'soliton dynamics' and I shall follow a very interesting book [27].

At first, one tends to think of the soliton equation as a nonlinear evolution equation, a prescription which describes how a given function of a space-like variable x evolves with respect to a time-like variable t. This is certainly the point of view one takes when one applies the IST, in which the evolution equation is clearly considered to be a Cauchy initial boundary-value problem. However, as the various miracles of soliton equation unfold, it becomes clearer that this equation is best thought of as a local relationship between a function (or functions) of an infinite number of independent variables and its various derivatives with respect to the independent variables, a relationship which is special because of some underlying algebraic structure. Because the equation is local, there is no need to think of any one variable as space-like and therefore particularly distinguished.

The soliton equation is magic purely for algebraic reasons which have to do with the structure of the equation as a very special relationship between a function and its various derivatives. No global properties are required to give it its special significance.

The soliton itself is a dramatic new concept in nonlinear science. Here at last, on the classical level, is the entity that the field theorists had been postulating for years, a local travelling wave pulse, a lump-like coherent structure, the solution of a field equation with remarkable stability and particle-like properties. It is intrinsically nonlinear and owes its existence to the balance between two forces, one of which is linear and acts to disperse the pulse, the other is nonlinear and acts to focus it.

Whereas the NLS equation (considered in Section 5) was the first-born among soliton equations (see, Zakharov and Shabat [35]), it was the celebrated KdV equation which fathered the soliton. It, too, is universal and is also ubiquitous and, just as in the case of the NLS equation, one can give the recipe for the circumstances under which it applies (it describes the evolution of shallow water waves). Both the KdV and NLS equations arise as asymptotic solvability conditions: such a condition on the leading order approximation to the solution of a more complicated set of equations ensures that the later iterates of the approximation remain uniformly bounded. It is very interesting to note that many of the equations, derived as asymptotic solvability conditions under very general and widely applicable premises, are also soliton equations!

One of the key properties of a soliton equation is that it has an infinite number of conservation laws and associated symmetries.

What do we mean by a soliton equation ? A true soliton, a solution to an equation with very special qualities, is much more than a solitary wave. The SW solutions of soliton equations have additional properties, however. One property is that two such SWs pass through each other without any loss of identity: after a nonlinear interaction, two pulses will emerge again, with the larger one in front, and each will regain its former identity precisely. There will be no radiation, no other mode created by the scattering process — the only interaction memory will be a phase shift. Whereas this interaction property is remarkable and indeed is often used as the test of soliton equations, it is not, by itself, sufficient. There are equations which admit solutions that are a nonlinear superposition of two SWs, but which do not have all the properties enjoyed by the soliton equations.

A soliton equation, when it admits SW solutions, must admit a solution which is a nonlinear superposition of N SWs for arbitrary N. To date, all known soliton equations have Hamiltonian structures and an infinite number of independent motion constants in involution. There is also a canonical transformation which converts a soliton equation into an infinite sequence of separate equations for the action–angle variables, each member of which can be trivially integrated. In this way, one can, in principle, solve the Cauchy initial-value problem.

It turns out that some of the action variables are the soliton parameters and this is the reason that a soliton's identity (namely the parameters giving its shape, velocity, amplitude, internal frequency etc.) is preserved under collision.

Among many of the special properties of the soliton, there are two which are very interesting:

(a) the first of these is the *Hirota property* and is due to Hirota, who discovered a very useful and important method for calculating multisoliton solutions (see Section 7.6);

(b) the existence of these rational solutions (of Hirota) is equivalent to another property enjoyed by soliton equations, the *Painleve property*.†

The discovery of the soliton, initiated as it was by the computer, has ironically shown that the modern tendency to reach for a computer to solve all problems is premature to say the least. The full power of such techniques as the inverse scattering theory has yet to be realised. The ingenuity of workers in this field leads to the speculation that at least for nondissipative systems there are many more useful applications yet to be discovered.

The main stumbling block to such advances is at present the absence of a standard technique for constructing the associated eigenvalue problem or the lack even of a criterion for its existence (according to Freeman [52], p. 35).

Finally, the soliton solutions of the KdV equation (for example!) have received much recent exposure in meetings and publications. An elementary introduction is Drazin's and Johnson's (1983) book [31]; the more substantial texts and review papers on 'soliton dynamics' are Refs [22, 27, 31, 32, 44, 52, 53, 62, 85–103].

†The Painleve property in the language of the Hirota f function (see Section 7.6) seems to demand that the function $f(x,t)$ has no movable critical point! This observation is significant and has potential consequences not only in the context of evolution equations, but also for other exactly solvable models.

7.6 Soliton 'mathematics'

7.6.1 Schrodinger equation and conservation laws. If V satisfies the 'modified KdV' equation

$$Q(V) \equiv \frac{\partial V}{\partial \tau} + \alpha V^2 \frac{\partial V}{\partial \xi} + \beta \frac{\partial^3 V}{\partial \xi^3} = 0 ,$$

then the function U given by

$$U = V^2 + \frac{\partial V}{\partial \xi} , \tag{7.6.1}$$

satisfies the classical KdV equation

$$P(U) \equiv \frac{\partial U}{\partial \tau} + \alpha U \frac{\partial U}{\partial \xi} + \beta \frac{\partial^3 U}{\partial \xi^3} = 0 , \qquad \alpha + 6\beta = 0 .$$

In fact, using relationship (7.6.1), after some manipulation we can show that

$$P(U) = \left(\frac{\partial}{\partial \xi} + 2V\right) Q(V) + (\alpha + 6\beta) \frac{\partial V}{\partial \xi} \frac{\partial^2 V}{\partial \xi^2} . \tag{7.6.2}$$

Naturally, the inverse statement: if $P(U) = 0$, then $Q(V) = 0$, is not valid.

If U is given, relationship (7.6.1) becomes the Riccati equation for V, so that the usual transformation to linearise the Riccati equation

$$V = \frac{1}{W} \frac{\partial W}{\partial \xi} , \tag{7.6.3}$$

leads to the equation

$$\frac{\partial^2 W}{\partial \xi^2} - UW = 0 , \tag{7.6.4}$$

and here U is a solution of the classical KdV equation $P(U) = 0$.

If we take note of the Galilean invariance of the KdV equation $P(U) = 0$, which is invariant under the transformation

$$U \Rightarrow U - \lambda , \qquad \xi \Rightarrow \xi + \alpha\lambda\tau , \qquad \tau \Rightarrow \tau , \tag{7.6.5}$$

we can at once generalise Eqn (7.6.4) to

$$-\frac{\partial^2 W}{\partial \xi^2} + UW = \lambda W . \tag{7.6.6}$$

This is just the *Schrodinger equation for the function* $W(\xi, \tau; \lambda)$ *with a potential* and $P(U) = 0$.

Nevertheless, it is a result which is essentially different from the Schrodinger equation in quantum mechanics, because of the fact that U is the solution of the classical

KdV equation $P(U) = 0$, so that it changes with time τ. Hence, in Eqn (7.6.6), the time τ must be considered as a parameter. In other words, it is necessary that at each instant of time Eqn (7.6.6) is valid for $U(\xi, \tau)$ at that time!

The eigenvalue λ would thus also seem to be time-dependent, but surprisingly all *the eigenvalues λ are time-independent*, provided only that U decreases sufficiently rapidly at infinity with respect to ξ, or that it satisfies a periodic boundary condition.

More precisely, if we use the original form of the KdV equation:

$$\frac{\partial \zeta}{\partial t} = \frac{3}{2}\left(\frac{g}{h_0}\right)^{1/2}\left(\zeta\,\frac{\partial \zeta}{\partial x} + \frac{2}{3}\,\alpha\,\frac{\partial \zeta}{\partial x} + \frac{1}{3}\,\sigma\,\frac{\partial^3 \zeta}{\partial x^3}\right),$$

the transformation

$$t' = \frac{1}{2}\left(\frac{g}{h_0}\right)^{1/2} t, \quad x' = \frac{x}{\sigma^{1/2}}, \tag{7.6.7a}$$

$$u = -\frac{1}{2}\,\zeta - \frac{1}{3}\,\alpha \tag{7.6.7b}$$

gives

$$P(u) = \frac{\partial u}{\partial t} - 6u\,\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{7.6.8}$$

where we have dropped the primes. For the reduced KdV equation (7.6.8) the condition $\alpha + 6\beta = 0$ is verified!

Next, let us consider the question of *conservation laws* of the KdV equation (7.6.8), namely equations of the form:

$$\frac{\partial N}{\partial t} + \frac{\partial J}{\partial x} = 0, \tag{7.6.9}$$

where N is a '*density*' and J is the associated '*flux*'.

Ten such conserved quantities for the KdV equation (7.6.8) were found [88]. But these conserved quantities have none of the immediate physical significance that we associate with the continuity equation.

The first few are:

$$N_1 = u,$$

$$J_1 = -3u^2 + \frac{\partial^2 u}{\partial x^2} \quad \text{(KdV equation)},$$

$$N_2 = u^2, \quad J_2 = -4u^3 + 2u\,\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial u}{\partial x}\right)^2,$$

$$N_3 = u^3 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2,$$

$$J_3 = -\frac{9}{4}\,u^4 + 3u^2\,\frac{\partial^2 u}{\partial x^2} - 6u\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial u}{\partial x}\,\frac{\partial^3 u}{\partial x^3} - \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^2.$$

It was conjectured that the KdV equation had an infinite number of such conservation laws, and this was later proved by Kruskal and Miura, and simultaneously by Gardner (see the survey paper of Miura [88]). To generate a whole sequence of constants of motion for the KdV equation, in fact an infinite sequence, we can first introduce a function w defined by

$$u = w + e\,\frac{\partial w}{\partial x} + e^2 w^2, \tag{7.6.10}$$

where e is an arbitrary constant. Substitution of this into the KdV equation (7.6.8) shows that w must satisfy the following conservation law

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}\left(3w^2 + 2e^2 w^3 - \frac{\partial^2 w}{\partial x^2}\right) = 0 \tag{7.6.11}$$

for all e. Integration over all x, assuming that w and its derivatives vanish at $x \to \pm\infty$, gives $\langle w \rangle = 0$, which is a constant of motion.

One may formally solve Eqn (7.6.10) by expanding in e to give:

$$w = w_0 + e w_1 + e^2 w_2 + \ldots = u - e\,\frac{\partial u}{\partial x} - e^2\left(u^2 - \frac{\partial^2 u}{\partial x^2}\right) + \ldots.$$

The important point is that *u is independent* of e!

Therefore, the condition $\mathrm{d}\langle w \rangle / \mathrm{d}t = 0$ leads to an infinite set of conditions:

$$\frac{\mathrm{d}\langle w_n \rangle}{\mathrm{d}t} = 0. \tag{7.6.12}$$

Thus, we have generated an infinite set of constants of motion for the KdV equation (7.6.8), which are the integrated values of w_n. One can prove, however, that only the coefficients of even powers of e lead to nontrivial constants!

The fact that for the KdV equation one has an infinite set of constants of motion makes one suspect that the KdV equation is equivalent to an infinite-order integrable Hamiltonian system, in which case relatively simple analytic solutions such as solitons exist! In fact, a *necessary* (*but not sufficient*) *condition for that the KdV equation* (7.6.8) *to have N soliton-type solutions is just the existence of an infinite set of constants of motion*!

In any case, the above observation concerning the 'Schrodinger equation' and the possible importance of the fact that the KdV equation has an infinite number of conservation laws, apparently advanced the solution of the KdV equation very little!

However, the presence of a Schrodinger-like equation puts a new face on the problem and a new perspective from which to attack the problem. This has led to collaboration between some ingenious researchers, each adding his individual insight, to discover the beautiful and highly original inverse scattering transform (IST) method.

The general solution of $u(x,t)$ of the KdV equation

$$P(u) = \frac{\partial u}{\partial t} - 6u\,\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0,$$

is obtained by

(a) considering both scattering and bound solutions of $W(x, 0)$,

(b) finding its time evolution,

(c) obtaining then the inverse solution of the GLM integral equation.

This general scheme is called the IST, which is a sophisticated generalisation of the Fourier transform for solving a linear equation.

7.6.2 Inverse scattering transform (IST). If we return to the Schrodinger equation (7.6.6), then on the basis this equation we may write: $U = \lambda + \left[\partial^2 W / \partial \xi^2\right] / W$. Substituting this value of U into the KdV equation, assuming that $P(U) = 0$, and integrating resulting equation over all $x = \xi$, we obtain the following relationship:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t}\int W^2\,\mathrm{d}x = 0 \tag{7.6.13}$$

if we assume that W and its derivatives approach zero as $x \to +\infty$.

By hypothesis, we associate with relationship (7.6.13) N solutions $W_n(x,t)$ which are bounded and such that the integral in this relationship exists and is finite. Hence, if the potential $u(x,t)$ in the (linear) Schrodinger (LS) equation

$$\frac{\partial^2 W}{\partial x^2} - (u-\lambda)W = 0 \tag{7.6.14}$$

satisfies the KdV equation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 , \tag{7.6.15}$$

then *the eigenvalues λ_n are constants.*

More precisely, by virtue of the assumption that $u(x,t)$ decays rapidly as $x \to \pm\infty$ for all t, the LS equation (7.6.14) admits a *finite* number of eigenstates of negative energy $\lambda_n = -k_n^2$, $n = 1,2,3,\ldots,N$ and also a *continuous spectrum* of positive energy $\lambda = k^2$.

The discrete eigenvalues $\lambda_n = -k_n^2$, which are the values λ leading to eigenfunction solutions, vanish at infinity and are square integrable. If the eigenfunction corresponding to k_n is normalised,

$$\int_{-\infty}^{+\infty} W_n^2 \,\mathrm{d}x = 1 , \tag{7.6.16}$$

then the coefficient c_n is defined by the asymptotic behaviour of W_n:

$$W_n \sim c_n(t)\exp(-k_n t) , \quad x \to \infty . \tag{7.6.17}$$

For the continuous spectrum, the wave function W is a linear combination of $\exp(+\mathrm{i}kx)$. Since $u(x,t)$ vanishes as $x \to \pm\infty$, we have to impose the conditions:

$$W_n \sim \exp(-\mathrm{i}kx) + b(k,t)\exp(\mathrm{i}kx) , \quad x \to +\infty , \tag{7.6.18a}$$

$$W_n \sim a(k,t)\exp(-\mathrm{i}kx) , \quad x \to -\infty . \tag{7.6.18b}$$

Physically, the term on the right corresponds to steady emission of plane waves propagating into the potential from infinity, to an amount $b(k,t)$, called the *reflection coefficient*, being reflected from the potential, and to an amount $a(k,t)$, called the *transmission coefficient*, being transmitted through the potential. In particular, $|a|^2 + |b|^2 = 1$.

Now, we have the following *theorem* (Miura [88]):

if $u(x,t)$ vanishes sufficiently rapidly as $x \to \pm\infty$, then

$$\begin{aligned} c_n(t) &= c_n(0)\exp(4k_n^3 t) , \\ b(k,t) &= b(k,0)\exp(8\mathrm{i}k^3 t) , \\ a(k,t) &= a(k,0) , \end{aligned} \tag{7.6.19}$$

where $c_n(0)$, $b(k,0)$ and $a(k,0)$ are determined from the initial data relating to the KdV equation (7.6.15) *for $u(x,t)$.*

The literature treating the inverse scattering problem (ISP) is extensive (see, for example, Gel'fand and Levitan [104]) and in fact the solution of the ISP is reduced to the problem of solving a linear integral equation, which is the Gel'fand–Levitan–Marchenko (GLM) integral equation:

$$K(x,y) + B(x+y) + \int_x^{+\infty} B(y+z)K(x,z)\,\mathrm{d}z = 0 . \tag{7.6.20}$$

Finally, we wish to solve the initial-value KdV problem for $u(x,t)$:

$$\begin{aligned} &\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 , \\ &-\infty < x < +\infty , \quad t > 0 , \quad u(x,0) = u^0(x) . \end{aligned}$$

First, we solve the eigenvalue problem:

$$\frac{\partial^2 W}{\partial x^2} - \left[u^0(x) - \lambda\right]W = 0 , \tag{7.6.21}$$

from which K_n, $c_n(0)$ and $b(k,0)$ are determined.

Then the set of expressions (7.6.19) yields the time-dependent quantities $c_n(t)$ and $b(k,t)$ and these determine $B(x+y)$ in Eqn (7.6.20) explicitly as:

$$\begin{aligned} B(x+y) = {} & \frac{1}{2\pi}\int_{-\infty}^{+\infty} b(k,0)\exp\left[\mathrm{i}k(8\mathrm{i}k^2 t + x + y)\right]\mathrm{d}k \\ & + \sum_{n=1}^{N}\left[c_n(0)\right]^2 \exp\left[8k_n^3 t - k_n(x+y)\right] , \end{aligned} \tag{7.6.22}$$

so that the GLM equation (7.6.20) is defined.

If we can solve this GLM integral equation, then the solution of the initial-value problem for the KdV equation is simply:

$$u(x,t) = -2\frac{\mathrm{d}}{\mathrm{d}x}\left[K(x,x;t)\right] , \tag{7.6.23}$$

where t in K is treated as a parameter.

Naturally, one might argue that we have merely replaced one difficult problem (nonlinear!) with another one! However, two major simplifications have been achieved:

(a) the equation involved, (7.6.21), and the GLM equation are *linear*;

(b) the time t enters the problem only *parametrically*.

Unfortunately, it is in general not possible to solve the basic GLM integral equation (7.6.20) analytically except of course for the reflection-free potentials $u^0(x)$ such that $b(k,0) = 0$. In general, the long-term solution is in the form of N solitons travelling at different velocities to the right and noise-like behaviour (*oscillatory state*) travelling to the left. The major mathematical difficulty arises from the integral contribution in $B(x+y)$.

Fig. 14 shows the solution of the KdV equation with both solitons and an oscillatory state (for $N = 2$), in addition to two solitons propagating to the right, we have a dispersing oscillatory state propagating to the left (because of the negative group velocity of the linear waves).

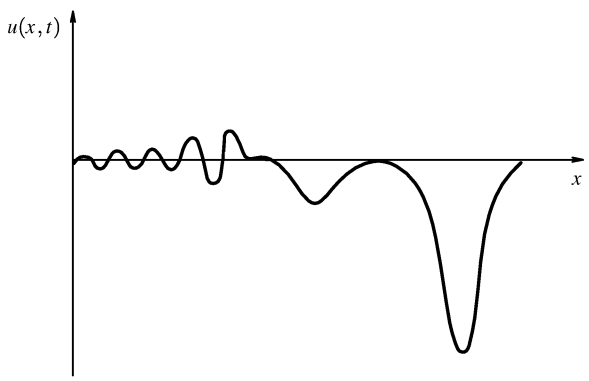

Figure 14. Solution of the KdV equation with two solitons and an oscillatory state.

7.6.3 Backlund transformation. In a particular case, the Backlund transformation [BT — sometimes called, also, auto-BT (A-BT)] can be used to transform a *zero soliton solution*, $u = 0$, of the KdV equation

$$\frac{\partial f}{\partial t} - 6\left(\frac{\partial f}{\partial x}\right)^2 + \frac{\partial^3 f}{\partial x^3} = 0 \ , \tag{7.6.24}$$

to the (non-zero) solution

$$w = -k_0 \tanh \xi_0 \ , \qquad \xi_0 = k_0(x - 4k_0 t) \ , \tag{7.6.25}$$

which is simply related to the *one-soliton solution* of the KdV equation (7.6.24). Next, the solution (7.6.25) itself can be transformed to the solution:

$$v = \frac{k_1^2 - k_0^2}{q - v} \ , \tag{7.6.26a}$$

with

$$q = -k_1 \coth \xi_1 \ , \qquad \xi_1 = k_1\left[x - 4k_1^2 t\right] \ , \tag{7.6.26b}$$

which corresponds to the *two-soliton solution* of (7.6.24).

This process can be continued to give solutions with an increasing number of solitons. At each stage, though, one has to solve two equations:

$$\frac{\partial w}{\partial x} = -\frac{\partial u}{\partial x} - k^2 + (w - u)^2 \ , \tag{7.6.27a}$$

$$\frac{\partial w}{\partial t} = -\frac{\partial u}{\partial t} + 4\left[k^4 + k^2\frac{\partial u}{\partial x} - k^2(w-u)^2 + \frac{\partial u}{\partial x}(w-u)^2 + \frac{\partial^2 u}{\partial x^2}(w-u)\right] \tag{7.6.27b}$$

with the *integrability condition* on w: $\partial^2 w/\partial x\,\partial t = \partial^2 w/\partial t\,\partial x$.

Unfortunately, the derivation of these 'initial' equations (7.6.27), such that both w and u satisfy the KdV equation (7.6.24), is not at all straightforward!

On the other hand, once the BT has been discovered, one has relatively simple way of generating a hierarchy of solutions! In our particular case, we start with the two equations (7.6.27), for u and w, where k is an arbitrary constant. The integrability condition on w demonstrates, after little algebra, that both w and u satisfy the KdV equation (7.6.24), for all k and in this case we have the A-BT. Differentiation of this KdV equation (7.6.24) with respect to x shows that $F = -2\,\partial f/\partial x$ satisfies the KdV equation

$$\frac{\partial F}{\partial t} + 6F\frac{\partial F}{\partial x} + \frac{\partial^3 F}{\partial x^3} = 0 \ . \tag{7.6.28}$$

We now choose u to satisfy the KdV equation (7.6.24): one solution is simply $u = 0$, in which case Eqns (7.6.27a) and (7.6.27b) reduce to

$$\frac{\partial w}{\partial x} = -k_0^2 + w^2 \ , \tag{7.6.29a}$$

$$\frac{\partial w}{\partial t} = 4(k_0^4 - k_0^2 w^2) \ , \tag{7.6.29b}$$

where k_0 is the corresponding value of k.

These equations are readily solved to give (7.6.25). We now take this to be the value of u and substitute it on the right-hand side of (7.6.27a), with a different value of k, for example k_1 (permutability!), to obtain the next equation in the hierarchy:

$$\frac{\partial v}{\partial x} = k_0 \operatorname{sech}^2 \xi_0 - k_1^2 + (v + k_0 \tanh \xi_0)^2 \ . \tag{7.6.30}$$

Next, if we introduce a function q, defined so that solution (7.6.26a) is true, and use the above equations, we find that

$$\frac{\partial q}{\partial x} = -k_1^2 + q^2 \ , \tag{7.6.31}$$

which is identical in form to Eqn (7.6.29a). This last equation, for q, has a solution (physically admissible!) of the form (7.6.26b), which is admissible as solution (7.6.26a), since it gives a bounded solution for v.

More details and other results can be found in Ref. [105]. Here, we shall mention only three important uses of the BT, namely:

(a) algebraic (as above) construction of solutions by application of the theorem of 'permutability' (due to Bianchi [106]);

(b) derivation of an associated ISP, since $u = v^2 + \partial v/\partial x$ essentially corresponds to half the BT relating to the solution of the KdV equation and the modified KdV equation;

(c) generation of conservation laws, by virtue of expression (7.6.10).

Finally, we hasten to point out that, for the KdV equation, much hindsight has been used in deriving the above results, but for some other equations for which the BT, IST and conservation laws have not been found, derivation of these results is not straightforward! For a more profound exposition, we refer the reader to a book by Dodd et al. [32].

7.6.4 Hirota method. We shall now consider the KdV equation in the form:

$$\frac{\partial^3 u}{\partial x^3} + 12u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0 \ , \tag{7.6.32}$$

which, with $u = \partial w/\partial x$, reduces to

$$\frac{\partial^3 w}{\partial x^3} + 6\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial w}{\partial t} = 0 \ . \tag{7.6.33}$$

The transformation

$$w = \frac{\partial}{\partial x}\log f \tag{7.6.34}$$

reduces the KdV equation to a *homogeneous* equation for $f(x,t)$ [107]:

$$f\frac{\partial^4 f}{\partial x^4} - 4\frac{\partial f}{\partial x}\frac{\partial^3 f}{\partial x^3} + 3\left(\frac{\partial^2 f}{\partial x^2}\right)^2 + f\frac{\partial^2 f}{\partial x\,\partial t} - \frac{\partial f}{\partial x}\frac{\partial f}{\partial t} = 0 \ . \tag{7.6.35}$$

The transformation (7.6.34) is known as the Cole–Hopf transformation [108, 109]. In trying to solve Eqn (7.6.35), we first take note of the fact that the classical single-soliton wave solution, for the KdV equation (7.6.32), is

$$u = \frac{a^2}{2}\operatorname{sech}^2\frac{\theta}{2} \ , \qquad \theta = ax - a^3 t + b \tag{7.6.36}$$

and is obtained by taking

$$f = 1 + \exp\theta_1 \ , \tag{7.6.37}$$

where $\theta_i = a_i x - a_i^3 t + b_i$, $i = 1, 2, 3, \ldots$. This encourages us to look for a solution of Eqn (7.6.35) in the form

$$f = 1 + \sum_{n=1}^{N} \varepsilon^n f^{(n)} \ , \tag{7.6.38}$$

with ε a convenient expansion parameter.

For $O(\varepsilon)$, we can easily obtain the exact solution in the form of the single exponential function, as in expression (7.6.37). However, since in this case we have a linear homogeneous equation for $f^{(1)}$, we can introduce as many exponentials as we like, although here we shall restrict ourselves to *two*:

$$f^{(1)} = \exp\theta_1 + \exp\theta_2 \ . \tag{7.6.39}$$

Next, this exact solution, for $f^{(1)}$, can be substituted on the right-hand side of the $O(\varepsilon^2)$ equation for $f^{(2)}$, to give:

$$\frac{\partial^4 f^{(2)}}{\partial x^4} + \frac{\partial^2 f^{(2)}}{\partial x\,\partial t} = 3a_1a_2(a_1-a_2)^2\exp(\theta_1+\theta_2) \ . \tag{7.6.40}$$

which integrates to

$$f^{(2)} = \left(\frac{a_1-a_2}{a_1+a_2}\right)^2 \exp(\theta_1+\theta_2) \ . \tag{7.6.41}$$

Surprisingly, the equation, for $f^{(3)}$, of the order $O(\varepsilon^3)$ is simply:

$$\frac{\partial^4 f^{(3)}}{\partial x^4} + \frac{\partial^2 f^{(3)}}{\partial x\,\partial t} = 0 \ ,$$

since the right-hand side is zero!

Naturally, we can take as solution $f^{(3)} = 0$ and we can easily see that with this trivial solution for $f^{(3)}$, all the subsequent functions are $f^{(n)} = 0$ for $n \geqslant 3$. This *self-truncation* of the series for f is absolutely crucial to obtaining the exact solution of the KdV equation (7.6.32).

The factors of ε can be absorbed into the phase of each θ and we have the *exact two-soliton solution*:

$$u = \frac{\partial^2}{\partial x^2}\log f(x,t) \ ,$$

$$f = 1 + \exp\theta_1 + \exp\theta_2 + A\exp(\theta_1+\theta_2) \ , \tag{7.6.42}$$

$$A = \left(\frac{a_1-a_2}{a_1+a_2}\right)^2 \ .$$

The same process will work if we take a three-parameter solution, but the algebra becomes rather forbidding! The KdV equation was solved in this way for N solitons by Hirota [107].

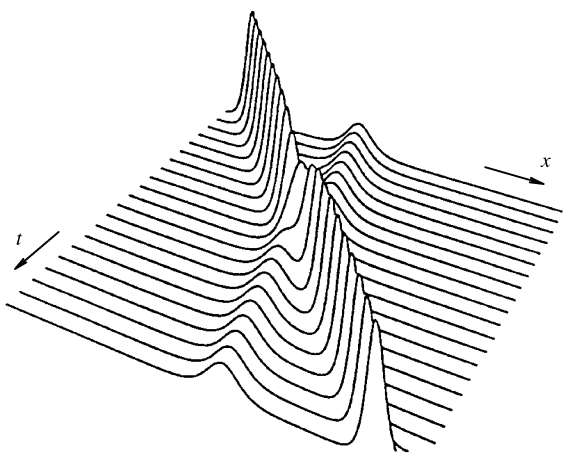

Figure 15. Collision of two KdV solitary waves.

Formula (7.6.42) can be analysed for the case when two solitons are far apart. The result is that the larger soliton $a_1 > a_2 > 0$ is shifted forward and the smaller $a_2 < a_1$ shifted backward relative to the motion that would have taken place if no interaction had occurred.

The trajectories of the maxima of the solitons in Fig. 15 make this result plain. Whether we think of the soliton collision as a process in which the solitons pass through one another or whether they exchange identities is only a matter of interpretation. Finally, it is interesting to note that Hirota noted that the terms in Eqn (7.6.35) were very like the Leibnitz formulae for the derivatives of products. Except for signs, Eqn (7.6.35) looks somewhat like:

$$\frac{\partial^2 f^{(2)}}{\partial x\,\partial t} + \frac{\partial^4 f^{(2)}}{\partial x^4} \ .$$

Hirota invented a new operator D_x, defined for ordered pairs of functions $g(x)$, $f(x)$ as follows:

$$\mathrm{D}_x\, g\cdot f = f\frac{\partial g}{\partial x} - g\frac{\partial f}{\partial x} \ ,$$

but this definition can be extended to functions $g(x,t)$, $f(x,t)$. For example,

$$\mathrm{D}_x\mathrm{D}_t f\cdot f = 2\left(f\frac{\partial^2 f}{\partial x\,\partial t} - \frac{\partial f}{\partial x}\frac{\partial f}{\partial t}\right) \ ,$$

$$\mathrm{D}_x^4 f\cdot f = 2\left[f\frac{\partial^4 f}{\partial x^4} - 4\frac{\partial f}{\partial x}\frac{\partial^3 f}{\partial x^3} + 3\left(\frac{\partial^2 f}{\partial x^2}\right)^2\right] \ ,$$

and in this notation, the KdV equation (7.6.35) becomes very compact:

$$\left[\mathrm{D}_x\mathrm{D}_t + \mathrm{D}_x^4\right] f\cdot f = 0 \ . \tag{7.6.43}$$

The multisoliton solutions are also easier to obtain. We can see this if we look at how the operators D_x act on exponential functions. It is easy to show that

$$\mathrm{D}_x^m \exp kx \cdot \exp lx = (k-l)^m \exp(k+l)x \ .$$

8. Well-posed problem: existence, uniqueness, stability results

8.1 Existence and uniqueness

If we consider the two-dimensional classical problem for the physical velocity potential $\phi(x,z,t)$, then this boundary-value problem is extremely difficult (even if $T = 0$) mostly because of the *dynamic* boundary condition,

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\zeta = 0 \ , \qquad z = \zeta(x,t) \tag{8.1.1}$$

which is *nonlinear* and one that is imposed at an *unknown boundary* $z = \zeta(x,y,t)$.

Some idea of the difficulty of this classical problem may be obtained by asking what is known about it! Naturally, we can introduce a new vertical independent variable in place of z. For example, in the case of *deep-water waves*, if

$$\eta = z - \zeta(\xi,\tau) \ , \qquad x = \xi \ , \qquad t = \tau \tag{8.1.2}$$

are the new variables, where $0 < \eta < -\infty$, we can find the derivatives:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \frac{\partial \zeta}{\partial \xi}\frac{\partial}{\partial \eta} \ , \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \frac{\partial \zeta}{\partial \tau}\frac{\partial}{\partial \eta} \ , \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial \eta} \ ,$$

and also

$$\phi_{xx} = \frac{\partial^2\phi}{\partial\xi^2} - \frac{\partial^2\zeta}{\partial\xi^2}\frac{\partial\phi}{\partial\eta} - 2\frac{\partial\zeta}{\partial\xi}\frac{\partial^2\phi}{\partial\xi\,\partial\eta} + \left(\frac{\partial\zeta}{\partial\xi}\right)^2\frac{\partial^2\phi}{\partial\eta^2}\,,$$

$$\phi_{zz} = \frac{\partial^2\phi}{\partial\eta^2}\,.$$

As consequence, we obtain the following two-dimensional *canonical classical problem for the deep-water waves*:

$$\frac{\partial^2\phi}{\partial\xi^2} + \left[1 + \left(\frac{\partial\zeta}{\partial\xi}\right)^2\right]\frac{\partial^2\phi}{\partial\eta^2} - 2\frac{\partial\zeta}{\partial\xi}\frac{\partial^2\phi}{\partial\xi\,\partial\eta} - \frac{\partial^2\zeta}{\partial\xi^2}\frac{\partial\phi}{\partial\eta} = 0\,,$$

$$0 < \eta < -\infty\,, \quad (8.1.3a)$$

$$\frac{\partial\zeta}{\partial\tau} + \frac{\partial\zeta}{\partial\xi}\frac{\partial\phi}{\partial\xi} - \left[1 + \left(\frac{\partial\zeta}{\partial\xi}\right)^2\right]\frac{\partial\phi}{\partial\eta} = 0\,, \qquad \eta = 0\,, \qquad (8.1.3b)$$

$$\frac{\partial\phi}{\partial\tau} + \frac{1}{2}\left(\frac{\partial\phi}{\partial\xi}\right)^2 - \frac{1}{2}\left[1 + \left(\frac{\partial\zeta}{\partial\xi}\right)^2\right]\left(\frac{\partial\phi}{\partial\eta}\right)^2 + g\zeta = 0\,,$$

$$\eta = 0\,. \quad (8.1.3c)$$

For deep water, we have also the following behavioural condition:

$$\frac{\partial\phi}{\partial\eta} \to 0\,, \qquad \eta \to -\infty\,. \qquad (8.1.3d)$$

The strongly nonlinear water-wave problem (8.1.3) with the two functions ϕ and ζ is *terribly difficult* and it is clear that a mathematical theory for these deep-water waves problem is practically impossible to construct! Numerical integration of this (8.1.3) problem is not easy either!

The simplest nontrivial statement that a mathematician can make about a physical problem is that it has a solution. According to Shinbrot's book [58], p. 87, historically the first main results proving the existence of a solution were as follows.

In 1925, Levi-Civita [12] proved that in *water of infinite depth* ($h_0 = \infty$), there is a *periodic wave* that progresses without change of shape. This means that ϕ does not depend on t and x separately, but only on a combination $x - ct$ for some constant velocity c. Naturally, ζ also depends only on $x - ct$, while ϕ and ζ are both *periodic functions* of $x - ct$.

Shortly after Levi-Civita proved his result, Struik [13] showed that it could be generalised to the case of a flat *horizontal bottom* ($h_0 = \text{const}$). Again, Struik proved the *existence of a periodic wave progressing without change of shape*.

In 1954, Friedrichs and Hyers [59] proved, again for $h_0 = \text{const}$, the existence of another type of wave, once more progressing without change of shape at a constant velocity (*solitary wave*). This can be looked on as a periodic wave, '*a la Struik*', but with an *infinite wavelength*.

If the bottom is periodic and has only one maximum and one minimum per period, Gerber [60] proved that there is *steady flow* in which the free surface has the same properties as the bottom. In addition, the troughs of the free surface lie directly over the troughs of the bottom, and the crests lie over the crests. In the same 1955 paper [60], Gerber proved also that over a *monotonic bottom*, there is a flow with a *monotonic free surface*. Again this can be looked upon as flow over a periodic bottom with an infinite period. The results of Gerber has been generalised by Krasovskii [110].

It should be noted that all these examples represent essentially *steady two-dimensional flows* — the last two are steady to begin with and the first three become steady in a coordinate system moving at the velocity c.

Concerning the *three-dimensional problem*, we note the papers by Lavrent'ev [111, 112], in which use was made of the theory of quasi-conformal mappings of three-dimensional domains. Ter-Krikorov [113–115] proved the existence of *periodic waves which degenerate into a solitary wave* and also the existence of a solitary wave on the surface of a liquid with vorticity.

The paper of Ovsyannikov [116] given the existence theorem for the *Cauchy–Poisson problem* about waves on a water surface (unsteady incompressible motion of a liquid with a free surface) as a result of an initial disturbance. Concerning the works of Soviet scientists, the reader can consult a review book edited by the Academy of Sciences of the USSR in 1970 (in Russian, see a survey paper by Moiseev [117], p. 55).

More recent results, again for the two-dimensional potential problem, were published by Showalter [118]. In his paper the existence–uniqueness–stability results are obtained from the corresponding results for the abstract Cauchy problem of an evolution equation in a Hilbert space. The existence theory for irrotational water waves is discussed in a paper by Keady and Norbury [119].

The justification of the '*shallow-water*' model equations [the 'Airy equations' (3.1.12)] was provided by Ovsyannikov [120]. A rigorous mathematical justification of the *validity of the shallow-water equations* for a two-dimensional channel with analytical data was given by Kano and Nishida [121]. For the three-dimensional case with *a priori* assumptions about the free surface, the justification was put forward by Berger (1976) [122].

The existence of *travelling-wave solutions of the KdV model equation* (for the KdV and KP equations, see Section 4) was analysed and proved by Showalter (1988) [118]. According to Showatter, the appropriate *initial–boundary problem for the KdV equation is well-posed*. A *global existence* theorem for the solution of the KdV equation for a general channel was established by Shen (1983) [123] as a consequence of the existence results due to Kato [124, 125].

Kato [126] considered a mathematical problem arising in the theory of solitary water waves in the presence of *surface tension*. For an extended survey of nonlinear waves under external forces (nonlinearly resonant surface waves), the reader is directed to a review paper [127]. In more recent work [128, and also 129] there are rigorous results concerning the Boussinesq equations (derived in Section 3) and also the KdV limiting equation for water waves, including a 'rigorous' derivation of these equations and estimates of the differences between solutions. The results are better justified for the KdV equation, due to the well-posed nature of the initial-value problem, while the results for the single Boussinesq equation are less satisfactory.†

More precisely, Kano and Nishida [128] worked within a class of analytic functions and used an abstract form of the Cauchy–Kovalevskaya theorem to prove the existence and to obtain further estimates. Craig (1985) [129] used a different functional framework and posed the problem of

† For this initial-value problem for the KdV equation, see Ref. [130], p. 508.

existence in a *Sobolev* space, obtaining a *long-term* existence theorem for the water wave problem in the long-wave scaling regime. Additionally, the latter setting is natural for the water-wave problem, as it does not display a small-scale linear instability, in contrast to the Kelvin–Helmholtz instability problem. Both these papers provide a mathematical justification for the Boussinesq and KdV equations as approximations to the classical water-wave problem. But, the results for the Boussinesq equation are different from those for the KdV equation because of the differences in the well-posedness of the initial-value problems for these two equations. The statement of justification for the Boussinesq equation is roughly that the Boussinesq operator gives rise to an error of $O(\varepsilon^2)$, in an appropriate function space, when applied to a solution of the water-wave problem.

In recent work Craig et al. [131], present a rigorous analysis of the use of modulation theory in the problem of water waves in a two-dimensional channel, and justify the approximation of the solution by a wave packet modulated according to the NLS equation (see, Section 5 for an asymptotic derivation of the NLS equation in the *long-wave limit*). The results in Ref. [131] include a rigorous derivation of the NLS equation and also an estimate within a class of Sobolev spaces, which shows that the modulation approximation satisfies the classical water-wave problem to the leading order in the scaling parameter. The results are not as well founded as those in the two preceding cited references for the KdV equation, but they are justified better than the results in these two references for the Boussinesq scaling regime.

The justification of the KdV scaling limit in both Refs [129] and [128] is substantially stronger than the results on the NLS or Boussinesq equations, basically proving that solutions to the classical water-wave problem converge strongly to solutions of the KdV equation as $\varepsilon \to 0$. In this case, Craig [129] reached a somewhat stronger conclusion, stemming from a difference in functional analytic setting between Refs [129] and [128]. The first poses the problem in Lagrangian coordinates, which, according to Nalimov (1974) [132] and Yosihara (1982) [133], is well-posed in an appropriate Sobolev space.

More precisely, Craig showed [129] that in the long-wave scaling regime the existence time of $O(\varepsilon^{-3/2})$ is obtained. Since the KdV time scaling reveals variations in the solution only over time intervals of $O(\varepsilon^{-3/2})$, solutions to the water-wave problem (with the initial data satisfying a unidirectional condition) are shown to converge strongly to solutions of the KdV in an interval of length $O(1)$ measured in KdV time! Kano and Nishida [128] formulated the problem in terms of a time-dependent conformal mapping on a reference domain, for the initial data in a space of analytic functions, and they appeal to a generalised version of the Cauchy–Kovalevskaya theorem to solve the initial-value problem. This gives an existence time of $O(1/\varepsilon)$ only!

Finally, we should note that the Lagrangian formulation of the water-wave problem, is a very convenient setting for rigorous estimates of the asymptotic procedure. In order to work with Lagrangian variables, coordinates of the free surface are taken in the form $\zeta(x,t) = [x + X(x,t), Y(x,t)]$. Then motion is considered for which (X, Y) are bounded localised perturbations of the free surface $(x, 0)$ for the liquid at rest. To describe the motion of the surface, we take the point $(x + X, Y)$ to be the coordinate of a Lagrange particle on the free surface. Writing $\boldsymbol{X} = (X, Y)$, then

$$\frac{\partial \boldsymbol{X}}{\partial t} = \boldsymbol{u}(t, x + X, Y) \,, \tag{8.1.4}$$

and eliminating the pressure term from the Euler equation, we can write *the equations of motion of the free surface* in the following form:

$$\left(1 + \frac{\partial X}{\partial x}\right)\frac{\partial^2 X}{\partial t^2} + \frac{\partial Y}{\partial x}\left(g + \frac{\partial^2 Y}{\partial t^2}\right) = 0 \,, \tag{8.1.5}$$

$$\frac{\partial Y}{\partial t} = \mathrm{K}(\boldsymbol{X})\,\frac{\partial X}{\partial t} \,, \tag{8.1.6}$$

where the operator K is the *Hilbert transform* for the variable domain occupied by the liquid. More precisely, the liquid velocity components $\boldsymbol{u} = (u, v)$ satisfy the Cauchy–Riemann conditions for an analytic function $f(z)$ with $f = u - \mathrm{i}v$ and $z = x + \mathrm{i}y$ and, combined with the bottom boundary condition $v(x, -h) = 0$, the results lead to a *singular* operator on the top surface. The boundary values of v are obtained from the boundary values of suitably behaved $\boldsymbol{u}$: $v = \mathrm{K}(\boldsymbol{X})u$.

The classical solutions of Eqn (8.1.5), (8.1.6) describing the free surface and the liquid velocity on that surface were studied by Craig [129] and Yosihara [133]. Together with the bottom-boundary condition, this enabled them to compute the liquid velocity throughout the liquid at each fixed time by solving the Laplace equation. The operator $\mathrm{K}(\boldsymbol{X})$ is computed in Ref. [133]. An important observation made by Coifman and Meyer [134] is that $\mathrm{K}(\boldsymbol{X})$ depends analytically upon $\boldsymbol{X}$ within a neighbourhood of the origin in the space of Lipschitz continuous functions. We thus expand

$$\mathrm{K}(\boldsymbol{X}) = \sum_{n=0}^{\infty} \mathrm{K}_n(\boldsymbol{X}) \,, \tag{8.1.7}$$

where $\mathrm{K}_n(\boldsymbol{X})$ is an operator homogeneous of degree n in $\boldsymbol{X}$, which is a concatenation of powers of $\boldsymbol{X}(x)$ and its derivatives with explicit Fourier multipliers.†

8.2 Stability – instability

8.2.1 Boussinesq showed that the solutions of Eqn (7.2.2) are characterised by the invariants

$$Q = \int_{-\infty}^{+\infty} \zeta \,\mathrm{d}x \,, \quad E = \int_{-\infty}^{+\infty} \zeta^2 \,\mathrm{d}x \,, \tag{8.2.1a}$$

$$M = \int_{-\infty}^{+\infty} \left(\zeta_x^2 - \frac{3}{h_0^3}\,\zeta^3\right) \mathrm{d}x \,, \tag{8.2.1b}$$

provided that ζ vanishes sufficiently rapidly as $x \to \pm\infty$. Q and E evidently represent the *volume* and *energy* of a wave. Boussinesq designated M as the moment of instability and demonstrated that solitary waves represent the unique solution of the *variational problem*: $\delta(M) = 0$, with E fixed (Boussinesq omitted the implicit constraint that Q be fixed, but this has no effect on the end result, for which the corresponding Lagrange multiplier vanishes).

Boussinesq also showed that the amplitude and volume of a solitary wave of prescribed energy are given by

$$a_0 = \frac{3}{4h_0}\,E^{2/3} \,, \quad Q = 2h_0 E^{1/3} \tag{8.2.2}$$

and he remarked that (see Ref. [22], p. 15):

†A simple derivation of these coefficients is presented in Appendix 1 of the paper [131].

"When a wave propagates along a canal of which the depth h_0 is slowly decreasing from one point to the next ... the bottom of the canal must continuously reflect a small part of the movement, in a manner such that the volume and energy of the wave must divide between the direct wave and this reflected wave, the latter being of an increasing length and of a height which is at once proportional both to this volume and to its height, ceasing to remain very small. The direct wave thus will conserve, approximately, all the energy of the wave, and, as it retains effectively the form of a solitary wave, ... its height a_0 and ... its volume Q ... will be obtained at any particular instant by means of Eqn (8.2.2), where E will be invariable: one sees that the wave will become higher, shorter, and consequently less stable, until finally it lacks a base and breaks. The opposite would occur if the depth were increasing".

These Boussinesq predictions appear to have been overlooked in most of the current literature (although not by Keulegan and Patterson [135]).

8.2.2 We shall now consider the NLS equation (5.1.25):

$$\mathrm{i}^{-1}\frac{\partial A}{\partial \tau} = \mu\frac{\partial^2 A}{\partial \xi^2} + \nu A|A|^2 \ . \tag{8.2.3}$$

As is seen from expression (5.1.26a), μ is always negative, whereas ν [according to expression (5.1.26b)] changes its sign from negative to positive across $k_0h_0 = 1.363$, as k_0h_0 decreases. It is known that the NLS equation (8.2.3) has the following solution representing a *nonlinear plane wave* $A = A_0\exp\left[\mathrm{i}(\alpha\tau - \kappa\xi)\right]$, where $A_0 = \text{const}$ and $\alpha = -\mu\kappa^2 + \nu|A_0|^2$. Let us now consider the meaning of this solution in terms of the original physical variables. In particular, if we set $\kappa = 0$ and $A_0 = ag/2\mathrm{i}\omega_0$, a being a real constant, and if for $\omega_0(k_0)$ we assume the classical dispersion relationship $\omega_0^2 = gk_0\sigma$, then the perturbed free surface takes the following form:

$$\zeta = \varepsilon a\cos\theta + \frac{1}{4}\frac{\varepsilon^2 a^2}{k_0\sigma}(\gamma - \lambda\cos 2\theta) \ , \tag{8.2.4}$$

where $\theta = k_0x - (\omega_0 - \varepsilon^2\alpha_0)t$ and $\alpha_0 = \nu g^2a^2/4\omega_0^2$.

In expression (8.2.4) we also have: $\gamma = [1/(c_g^2 - gh_0)][2\omega_0k_0c_g + (1-\sigma^2)gh_0k_0^2]$ and $\lambda = (\sigma^2 - 3)k_0^2/2\sigma^2$. This is simply the *Stokes wavetrain in the second-order approximation*. Here, $\omega = \omega_0 - \varepsilon^2\alpha_0$ is the nonlinear dispersion relationship for a Stokes wave, including the effect of an induced horizontal current. Moreover, the dispersion term in Eqn (8.2.3) is unimportant in this solution because $\kappa = 0$.

In addition to the plane-wave solution described above, Eqn (8.2.3) has another type of solution in terms of the Jacobian elliptic functions, exhibiting a dynamical balance between nonlinear and dispersion effects, which we shall call the *equilibrium solution*:

$$A = B(\xi)\exp \mathrm{i}\alpha\tau \ , \tag{8.2.5}$$

where α is constant and β is real. If $\mu\nu > 0$,

$$B(\xi) = B^0\,\mathrm{Dn}\left[B^0\left(\frac{\nu/2}{\mu}\right)^{1/2}\xi \mid m\right] \tag{8.2.6}$$

with the modulus m and $m^2 = 2 - 2\alpha/\nu(B^0)^2$.

We shall show that time evolution of the unstable modes may be regarded as a special case of the general modulation described by Eqn (8.2.3). In order to reproduce a Stokes wave, let us set $\alpha = \alpha_0$, $\kappa = 0$ and $A_0 = ag/2\mathrm{i}\omega_0$ in expression (5.1.27). Then we can consider a disturbed Stokes wave given by

$$A = [A_0 + \lambda A']\exp\left[\mathrm{i}(\alpha_0\tau + \lambda\theta')\right] \ , \tag{8.2.7}$$

where A' and θ' are assumed to be real functions representing a disturbance and λ is a small parameter. Substituting the above expression into the NLS equation (8.2.3) for $A(\tau,\xi)$ and linearising it with respect to λ, we obtain

$$\frac{\partial A'}{\partial \tau} + \mu|A_0|\frac{\partial^2\theta'}{\partial\xi^2} = 0 \ , \tag{8.2.8a}$$

$$\frac{\partial \theta'}{\partial \tau} - 2\nu|A_0|^2A' - \mu\frac{\partial^2 A'}{\partial\xi^2} = 0 \ . \tag{8.2.8b}$$

Since these equations form a set of linear differential equations with constant coefficients, we can assume a solution of the form:

$$(A',\theta') = (A_0',\theta_0')\exp\left[\mathrm{i}(k'\xi - \omega'\tau)\right] + \text{c.c.} \ , \tag{8.2.9}$$

where A_0' and θ_0' are constants. From the condition that Eqns (8.2.8a) and (8.2.8b) have a nontrivial solution, we obtain the following dispersion relationship:

$$\omega'^2 = \mu^2k'^2\left(k'^2 - \frac{2\nu}{\mu}|A_0'|^2\right) \ , \tag{8.2.10}$$

which shows that, *if* $\mu\nu < 0$, then ω' is *always real for arbitrary* values of k' so that the Stokes wave given by expression (5.1.27) is *neutrally stable*. On the other hand, if $\mu\nu > 0$, ω' becomes imaginary for

$$k' < 2\left(\frac{\nu}{\mu}\right)^{1/2}|A_0'| \ . \tag{8.2.11}$$

Hence, the disturbance will grow exponentially. In this sense, the Stokes wave given by expression (5.1.27) is unstable against the above modulational disturbance and the maximum growth rate, say $d_{\max}$, is given by $d_{\max} = |\nu A_0'^2|$ for $k' = (\nu/\mu)^{1/2}|A_0'|$.

According to Hasimoto and Ono (1972) [36], if we return to the original NLS equation (8.2.3), we can investigate further time evolution of such unstable modes even to the stage when the linear theory ceases to be valid. For example, when $m = 1$, the equilibrium solution (8.2.6) degenerates into a solitary modulational wave propagating at the group velocity. This wave is

$$B(\xi) = \left(\frac{2\alpha}{\nu}\right)^{1/2}\mathrm{sech}\left[\left(\frac{\alpha}{\mu}\right)^{1/2}\xi\right] \ , \tag{8.2.12}$$

and its width is $(\mu/\alpha\varepsilon^2)^{1/2}$. When $\alpha = \alpha_0$, this width agrees with the wavelength of the unstable mode with the maximum growth rate. This fact leads us to a *conjecture* that *modulation of the Stokes wave eventually deforms it into the solitary wave described by* expression (8.2.12).

8.3.3 The instability of deep water waves was also established in Ref [71, 73, 136–141]: *the Stokes waves in deep water are definitely unstable*!

For a liquid of *finite depth*, the coupling with the induced mean flow becomes significant and it has a stabilising effect. In this case the Whitham modulation equations of the wavetrains are *elliptic* or *hyperbolic* for $kh_0 > 1.36$ or $kh_0 < 1.36$, respectively.

In the former case, the modulation process is unstable and, there is a remarkable agreement between the Benjamin and Whitham theoretical predictions.

The discovery of the instability of the weakly nonlinear Stokes waves has led to questioning of the evolution of the unstable nonlinear wavetrains. As a consequence, special attention has been given to the derivation of evolution equations valid for *long times*! Indeed, many nonlinear instability problems of diverse nature can also be described by the NLS equation.† Finally, when several dominant (linear) wave modes are present, their *mutual interaction* is significant. This is especially so when some of these modes *resonate*. The simplest and most important case is the *three-wave resonance*. But not all systems exhibit a three-wave resonance and, ironically, one of the first searches for such a resonance, among inviscid surface gravity waves, yielded negative results and Phillips 'bravely' (see, Craik's book [42], p. 73) continued his analysis to third order in amplitude, to determine the *cubic* interaction coefficients of resonant *quartets*! For a deep discussion of the three-wave resonance, cubic three- and four-wave interactions, strong interactions, local instabilities, and transition to turbulence see the book by Craik [42], Chapters 5, 7 and 8.

9. Conclusion

As pointed out by Craik ([42], p. 288): "... The *key* to understanding nonlinear wave motion and transition to turbulence is not any one of solitons, bifurcation theory, catastrophe theory, strange attractors, period-doubling cascades, et cetera. Fashionable, and fascinating, theoretical bandwagons add momentum to scientific progress but can also carry the unwary up blind alleys. The richness of fluid mechanics is such that many new surprises and insights still await discovery".

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†For a discussion see Ref. [98], pp 242–252. Concerning the *bifurcation of large-amplitude waves*, see also Ref. [98], pp 252–255.

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