Physics – Uspekhi **38** (11) 1283 – 1288 (1995)

METHODOLOGICAL NOTES

PACS numbers: 47.37.Tq; 97.60.Gb

Paradoxes of superfluid rotation

D A Kirzhnits, S N Yudin

Contents

Abstract. The phenomenon of superfluidity (as well as the related phenomenon of superconductivity) is of quantum nature and as such looks paradoxical from the standpoint of both classical physics and common sense. Suffice it to say that these phenomena imply the ability to flow without exhibiting viscosity or resistance (for this reason, for example, the circulation of a superconducting current within a closed circuit in the absence of external sources would last without damping for a period much in excess of the age of the universe). At the same time, a number of unusual and quite unexpected features appear to be intrinsic to specific events of superfluid physics. Some of them will be considered in the present study.

1. Preliminary data

The following discussion will be confined to slow rotation of a superfluid with an angular velocity below the critical level for the formation of the first vortex filament [1, 2]. Rotation is considered to be steady and to occur at a fixed momentum of the system. The order parameter of the superfluid is assumed to be scalar. This study is specifically concerned with the case of a nonrelativistic superfluid medium whose pressure is small compared with the resting energy density.

Hereinafter, we understand by 'superfluid' electrically neutral media: liquid ^{4}He and neutron fluid. The latter is known to form the core of neutron stars, pulsars (see, for instance, Ref. [3]). However, results of the present study cannot be directly applied to pulsars because their real angular velocity is many orders of magnitude higher than Ω_c and the core contains a broad network of vortex filaments [4]. Nevertheless, we are going to use selected facts of pulsar physics by way of a starting point in the formulation of paradoxes discussed below which actually have more general implication.

This introductory section examines certain universally-known statements concerning the rotation of fluids (including superfluids) in a vessel. The velocities of points of a solid container which rotates at an angular velocity $\boldsymbol{\Omega}$ is defined by the elementary formula

$$v = \boldsymbol{\Omega} \times \boldsymbol{r} \, . \tag{1}$$

The same law of 'solid-body rotation' describes the regime of fluid rotation which ensures the minimum value of $E - \boldsymbol{\Omega} \cdot \boldsymbol{M}$, i.e. energy E at a given momentum $\boldsymbol{M}$. In this situation, the continuity condition for the velocity vector (for a viscous fluid) or of its normal component with respect to the vessel surface (for a superfluid) is fulfilled at the fluid – wall interface.

This law (1) is really satisfied for the rotation of a viscous fluid. However, it corresponds to the non-potential flow $\operatorname{curl} v = 2\boldsymbol{\Omega}$ and cannot be fulfilled for a superfluid because the superfluid velocity v_s must obey the condition

$$\operatorname{curl} v_s = 0 \tag{2}$$

(see Appendix I). In the case of an axially symmetric vessel, considered in this section, the normal component of its velocity is equal to zero, and the superfluid is not brought into rotation together with the vessel:

$$v_s = 0 \, , \tag{3}$$

which directly corresponds to the absence of frictional forces in the superfluid.

However, this is true only of the case of a sufficiently slow rotation $\Omega < \Omega_c$. The thing is that, as was mentioned before, it is the solid-body rotation (1) rather than the resting state (2) of the fluid that corresponds to the energy

D A Kirzhits, S N Yudin Lebedev Physical Institute, Leninskii pr. 53, 117924 Moscow, Russia
Tel. (7-095) 135-75-11
E-mail: kirzhnit@lpi.ac.ru

Received 11 August 1995
Uspekhi Fizicheskikh Nauk **165** (11) 1335 – 1340 (1995)
Translated by Yu V Morozov; edited by H Milligan

minimum, and even partial involvement of the fluid at a sufficiently large Ω would result in energy saving. Such a motion of the fluid does occur at $\Omega \geqslant \Omega_c$ when the thin normal 'core' of a vortex filament forms on the axis of rotation, and the superfluid enters into the state of potential 'irrotational' rotation in the non-simply connected domain thus formed. The number of vortex filaments increases with rising angular velocity whereas the averaged velocity of the induced complex flow approaches Eqn (1) [1, 2].

This picture is in line with the general analogy of rotational and magnetic phenomena of which an earlier example was given by the Larmor theorem [5] implying the possibility to substitute the effect of a weak magnetic field with induction B by transition to the coordinate system which rotates with angular velocity $\Omega = eB/2mc$. Accordingly, $\Omega \Leftrightarrow B$, $M \Leftrightarrow H$ (similar to momentum M, the magnetic field H or the field of external sources is fixed) allows for the comparison of superfluid rotation patterns and the behaviour of a type 2 superconductor in the magnetic field [1]: as $\boldsymbol{H}(\boldsymbol{M})$ grows, the complete expulsion of field $\boldsymbol{B}$ (angular velocity $\boldsymbol{\Omega}$), i.e. the Meissner effect or the absence of superfluid rotation, is replaced by the appearance of one and then many vortex filaments until field $\boldsymbol{B}$ (velocity rotor) completely penetrates the system.

2. Non-axially symmetrical rotation

If the axis of rotation does not coincide with the axis of superfluid axial symmetry, the component of its velocity normal with respect to the vessel surface is not zero, and the superfluid cannot remain at rest. However, the superfluid is carried along by friction against the vessel wall and must execute a potential motion [see Eqn (2)]. This requires that the superfluid rotate relative to the vessel in the direction opposite to that of the vessel's rotation in order that the velocity rotors of these two motions may compensate each other†.

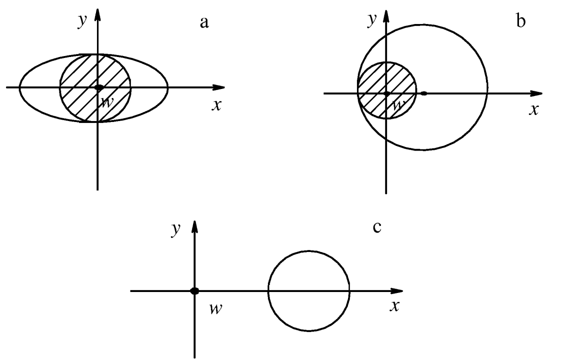

Figure 1.

We shall consider three problems (see Fig. 1): (a) central rotation of a nonequilibrium ellipsoid (or elliptic cylinder), (b) eccentric rotation of a sphere (or circular cylinder) with the axis of rotation inside the body, and (c) a similar sphere or cylinder with the axis of rotation outside the body. These problems arise in the description of a nonspherical atomic nucleus, a cylinder with helium at the North Pole slightly shifted relative to the Earth's axis and involved in its daily revolutions, and a pulsar of a binary system participating in the orbital motion about the system's centre of mass, respectively. In considering these problems, let us assume the boundary between the superfluid and the vessel wall to be a surface of second order:

$$a_1 x^2 + a_2 y^2 + b_1 x + b_2 y = c\,,$$

where (quadratic) dependence on z is moved to the right-hand side. Focusing on the streamline patterns in the established flow, it is possible to confine oneself strictly to the examination of the moment of time when axes x and y coincide with the axes of symmetry (main axes) of the system (see Ref. [6], paragraph 10, where the solution of the first problem is proposed). Axis x is taken to be the axis of rotation.

Since the flow turns out to be planar (it occurs in planes parallel to plane xy) and the fluid is assumed to be incompressible ($\nabla \cdot v_s = 0$), it seems convenient to use the complex potential method [6]. This method detects the analytic function $w = \varphi + \mathrm{i}\psi$ of variable $\zeta = x + \mathrm{i}y$, where φ is the velocity potential (proportional to the condensate phase, see Appendix I), and ψ is the function of the flow which defines streamlines by the equation $\psi = \mathrm{const}$. The condition at the superfluid–vessel interface requires that it should coincide with one of the streamlines in the reference system in which the vessel is at rest (transition to it may be accomplished by subtracting quantity $\boldsymbol{\Omega} \times \boldsymbol{r}$ from the velocity). All these conditions are satisfied by the expression

$$w = \frac{\Omega}{a_1 + a_2}\left[\frac{\mathrm{i}\,(a_1 - a_2)}{2}\,\zeta^2 + (b_2 + \mathrm{i}\,b_1)\,\zeta + \mathrm{const}\right], \quad (4)$$

whose analyticity rules out the possibility of a partially resting superfluid (see the previous footnote).

The streamline patterns in the reference system of a resting vessel are presented in Fig. 2. They appear to correspond to the relative motion of the superfluid which is known to restore potential flow (see above). A similar picture in a laboratory system is shown in Fig. 3. The result is really surprising in the case of eccentric rotation (Figs 3b, 3c): potentiality is achieved by the superfluid flowing as a whole with constant velocity which does not depend on the distance from the axis of rotation:

$$v_x = \frac{\Omega b_2}{2a}\,, \quad v_y = -\frac{\Omega b_1}{2a} \quad (a = a_1 = a_2)\,. \quad (4a)$$

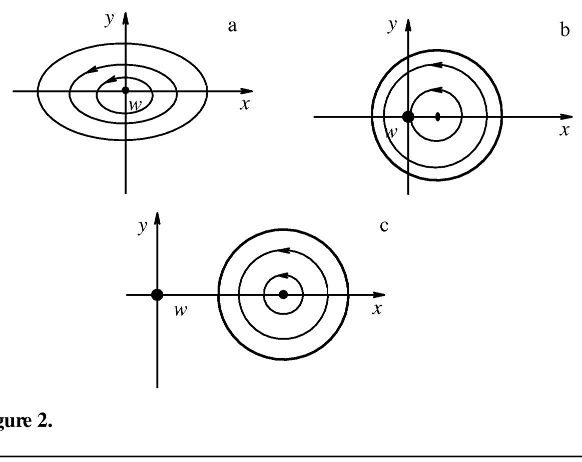

Figure 2.

†Certain authors postulate flow patterns in which a part of superfluid that is not immediately caused to rotate together with the vessel (shown as the shaded area in Fig. 1) remains at rest. This picture is wrong (see below).

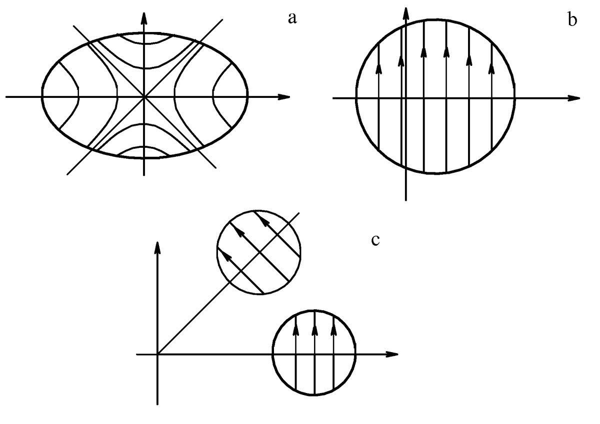

Figure 3.

To conclude this section, it is worthwhile to emphasise certain facts pertaining to the eccentric rotation of a circular cylinder. It is possible to demonstrate that the critical angular velocity for the formation of the first vortex filament [which appears in the centre of the cylinder because only this point of the superfluid remains at rest during rotation in the reference system of the resting vessel (see above)] will not change as compared with Ω_c for the symmetrical rotation (see Refs [1, 2]). On the assumption that the angular velocity of the eccentric rotation of a circular cylinder is $\Omega \gg \Omega_c$ and there exists a vortex massif, the superposition of the averaged velocity caused by the vortex massif and the velocity (4a) leads to the solid-body rotation of the superfluid about the eccentric axis.

3. Frictionless rotation of superfluid

Transference of rotation from a vessel to the enclosed superfluid occurs not only as a result of the direct contact between them but also under the influence of a long-range field caused by the rotation of the vessel (or a part of it). We shall consider this and related problems in application to the simplest axially symmetric case (see Ref. [7]).

Let us start from the well-studied problem of superconductor behaviour in the external magnetic field $\boldsymbol{B}$ induced by the rotational motion of electrons in the electromagnet winding with velocity v_e. This field gives rise to a counterflow of superconductor electrons (they form a charged superfluid) which serves as a screen for the source of the external field and repels this field from the superconductor volume (the Meissner effect). The effect of the magnetic field is quantitatively described by the 'long' gradient of the order parameter $\nabla - \mathrm{i}e\boldsymbol{A}$ in the expression for superconducting current which leads to the London relation $\operatorname{curl} v_s = -e\boldsymbol{B}/m$ or, in the gauge $\nabla\cdot\boldsymbol{A} = 0$, to

$$v_s = -\frac{e}{m}\boldsymbol{A} . \quad (5)$$

Here $\boldsymbol{A}$ is the vector potential, e and m are the charge and the mass of the electron, respectively. Therefore, it can be seen that in contrast to the uncharged superfluid case [see Eqn (3)], the effect of the field that 'lengthens' the order parameter gradient causes the superfluid to rotate. It is in this way that the direct dynamic relationship between electrons of the 'vessel' (i.e. electromagnet winding) and the superfluid is established.

This inference could be of value for pulsar physics. It is well known that the period of a pulsar monotonically grows on scales ranging from hundreds to one thousand years but is from time to time subject to sudden drops followed by relaxation within a few months to one year [3]. Such a huge relaxation time suggests abnormally weak dynamic links between the shell of the pulsar (whose rotation determines the period) and its core (containing the bulk of the mass). This anomaly gives evidence of superfluidity of the neutron matter in the core which results from the Cooper pairing of nucleons by nuclear forces.

It should be emphasised that this explanation requires sufficiently low efficiency of mechanisms other than viscosity for dynamic links between the core and the shell. One of them resembles the mechanism discussed in a previous paragraph which underlies the relationship between the outer and superconducting currents and takes into account magnetoid (velocity-dependent) forces of the general theory of relativity caused by shell rotation. Such gravimagnetic forces are defined by components $g_{0\alpha}$ ($\alpha = 1, 2, 3$) of the metric tensor and lead to the Lense–Thirring effects [5, 8].

In the assessment of the efficacy of this mechanism as applied to a real pulsar, a well-developed network of vortex filaments needs to be taken into consideration (see the beginning of the paper). The remaining part of this study will be confined to a simpler problem which is beyond the scope of pulsar physics, that is whether the superfluid is caused to rotate by gravimagnetic forces (as a superconducting fluid is under the action of a magnetic field) or remains at rest (as an uncharged superfluid like ^{4}He). Paradoxically, both options appear to occur concurrently.

4. Gravimagnetic rotation of superfluid

The problem in question turns out to be conflicting as soon as it is approached. Arguments ensuing from structural considerations with regard to the order parameter gradient (see Section 3) seem to be in favour of the absence of superfluid rotation: 'lengthening' in the general theory of relativity consists in the transition from the ordinary derivative to the covariant one; in application to the scalar order parameter ψ, the latter coincides with the ordinary derivative (see formula A.2). Hence, the superfluid should be regarded as motionless, as is the case with helium, and relation (3) holds true.

On the other hand, there is close similarity between the equations of general relativity (for a weak field) and of electrodynamics, which is apparent on substitution (for transversal field components):

$$e\boldsymbol{A} \Leftrightarrow m\boldsymbol{g} \quad (g_\alpha = g_{0\alpha}) , \quad (6)$$

$$e^2 \to -4m^2 G , \quad (7)$$

where G is the gravitational constant. If applied to Eqn. (5), this substitution leads to the Dewitt relation [9]

$$v_s = -\boldsymbol{g} , \quad (8)$$

whereas its application to the equation $\nabla^2\boldsymbol{A} = 4\pi e\rho v/m$ (ρ being density) yields the hydromagnetic analogue of the London equation of superconductivity theory:

$$(\nabla^2 + \varkappa_s^2)\boldsymbol{g} = \varkappa_n^2 v_n \quad (\varkappa^2 = 16\pi G\rho) , \quad (9)$$

where s and n are indices of the superfluid and normal components of matter, respectively.

Formulas (8) and (9) point to similarity between the gravimagnetic and superfluid cases. In either case a respective field forces the superfluid to rotate, and this rotation gives rise to a secondary field, etc. However, these cases are significantly different in that there is the 'wrong' sign in front of the second term in the brackets in Eqn (9) which is caused by the 'minus' sign in expression (7) (in gravitation, in contrast to electrodynamics, the like charges, i.e. masses, are mutually attractive rather than repulsive). Therefore, in the gravimagnetic case, the superfluid is brought into rotation instead of being involved in the countercurrent, and the secondary field amplifies the primary one rather than weakens it (ideal paramagnetism, see Ref. [10], chapter 9). In the dynamic conditions corresponding to Eqn. (9), the tachyonic excitation spectrum $\omega^2 = k^2 - \varkappa_s^2$ would arise which could result in 'gathering self-rotation' by the system (a rotational analogue of the Jeans instability).

To resolve the contradiction between Eqns (3) and (8), it is necessary to take into consideration the difference between covariant and contravariant components of velocity, one of which can vanish if the other is finite, namely in the case of $g_{0\alpha} \neq 0$. Moreover, one needs to get rid of the excess accuracy in Eqn (9) (the second term in the brackets has an extra order in G). All this can be achieved by the transition to the consistent (in terms of general relativity) description of a spherical body (the Schwarzschild problem) brought into the state of slow rotation, in the linear approximation over Ω. In the spherical coordinates $x^\alpha = (r, \theta, \varphi)$ with the axis parallel to the axis of rotation, only components of vectors with spatial index 3 differ from zero, and physical quantities are independent of the coordinate x^3 as a result of axial symmetry.

Specifically, the normal velocity corresponding to solid-body rotation (1) $x^3 = x_0^3 + \Omega t$ is not zero:

$$v_n^3 = \dot{x}^3 h^{-1/2} = \Omega h^{-1/2} \quad (h = g_{00}) .$$

With regard to the superfluid velocity, the equality (3) contains the covariant component of the 4-velocity vector [5]

$$u_i = (h^{1/2}, 0, 0, v_3) , \qquad u^i = (h^{-1/2}, 0, 0, v^3) ,$$

and may be written in the form

$$v_{s3} = 0 . \tag{3a}$$

This results from the fact that the phase gradient of the order parameter (see Appendix I) is proportional to quantity u_i whereas equality (3a) immediately follows from the axial symmetry of the system. This equality, written as $g_{3i} u^i = 0$, defines the contravariant component of the 4-velocity by the relation which generalises Eqn (8):

$$v_s^3 = -g^3 h^{-1/2} \qquad (g^3 = g_{03}/g_{33}) . \tag{8a}$$

Therefore, the velocity contained in Eqn (8) is in fact the contravariant component of the 4-velocity.

5. Gravimagnetic rotation of superfluid (physical aspects)

After having formally settled the above contradiction by the statement that superfluid behaves like a superconductor in terms of the contravariant component of its velocity and like superfluid ^{4}He in terms of the covariant component, we may turn to examining nontrivial physical aspects of this statement.

The vanishing of the covariant component of velocity [see Eqn (3a)] cannot but lead to the disappearance of at least a part of the dynamic manifestation of superfluid rotation. This specifically refers to the generation of a gravimetric field by rotating matter, the source of the generation being rotation of the normal but not the superfluid component of the matter. This is immediately apparent from the Einstein equation (where p is pressure):

$$R_{03} - \frac{1}{2} g_{03} R = \frac{\varkappa^2}{2} u_0 v_3 - 8\pi G p g_{03} ,$$

in which the source corresponding to the superfluid is zero. By deriving an equation analogous to Eqn (9) from the Einstein equation for R_0^3, it is possible to show (see Appendix I) that there is correspondence between the source $(\varkappa^2/2)\, u_0 v_s^3$ on the right-hand side and the geometric term of the highest order in G on the left-hand side, the latter accurately compensating the former source as a consequence of Eqn (8a). Accordingly, there is no contribution of superfluid rotation to asymptotics of the gravimagnetic field g^3 at large distances from the body which defines the system's momentum (see Ref. [5]):

$$g^3 \to -2GM/r^3 \qquad (r \to \infty) .$$

Thus, the superfluid rotation does not contribute to the momentum of the system.

However, such rotation, i.e. nonzero velocity v_s^3 [see Eqn (8a)], is apparent in the next order in Ω as the appearance of a meniscus on the free surface of the superfluid, if any. This immediately follows from the Bernoulli equation of the potential flow which can be derived from hydrodynamic equations of general relativity and has the standard form in the limit of small velocities, nonrelativistic matter, and weak field:

$$p + \rho \frac{v^2}{2} + \chi = \text{const} , \qquad h = 1 + 2\chi + \ldots ,$$

where $v^2 = g_{\alpha\beta} v^\alpha v^\beta$ (one should distinguish between v_{s3} and $g_{3\alpha} v_s^3$ because these quantities are actually the covariant components of the 4-vector and the 3-vector respectively).

This becomes especially clear after a new reference system is introduced (respective quantities are denoted by a prime) which rotates with respect to the initial one (the Galilean system at infinity) with the angular velocity ω about the same axis. Since $x^{3'} = x^3 + \omega t$ (the remaining coordinates in the two systems coincide), the following transformation law holds for tensor g_{ik}:

$$h' = h - 2\omega g^3 g_{33} + \omega^2 g_{33} ,$$
$$g^{3\prime} = g^3 - \omega , \qquad g'_{33} = g_{33} , \tag{10}$$

and components of the 4-velocity (see Section 4):

$$u'_0 = u_0 - \omega v_3 , \qquad v'_3 = v_3 ;$$
$$u^{0\prime} = u^0 , \qquad v^{3\prime} = v^3 + \omega u^0 . \tag{11}$$

It appears from Eqn (10) that the choice of $\omega = g^3$ leads to $g^{3\prime} = 0$ which means cancellation between the Coriolis acceleration [term $-\omega$ in Eqns (10)] and the acceleration caused by the Lense–Thirring forces (term g^3 in the same equations). In this sense, the scheme thus chosen may be regarded as inertial. This explains why so many authors speak about involvement of the inertial system in the rotation of a massive body in the general theory of relativity.

It is essential that in such a system $v^{3'} = 0$, as follows from the last equality in Eqns (11) and (8). Simultaneously, the component $v'_3 = g_{33} v^{3'}$ also disappears, in accordance with equality $g^{3'} = 0$. Because of this, the rotation of a massive body sets in motion the superfluid which is at rest in the inertial system. This accounts for the absence of any physical manifestation of the superfluid rotation. According to the second equality in Eqns (11), this is true of all reference systems including the initial one†.

Why do not such considerations forbid the appearance of a meniscus? Just for the same reason for which the Larmor theorem holds only in the first order in the angular velocity. The fact is that the property of inertiality of a rotating reference system holds only in the same order: acceleration quadratic in ω [including the centrifugal acceleration, i.e. the last term in the first equality of Eqns (10)] is not compensated by the corresponding gravimagnetic terms and remains nonzero in the new reference system. Meanwhile, the meniscus is the effect of the second order in ω, unlike all other effects being examined.

References

1. Landau L D, Lifshitz E M *Statistical Physics*, Part 2 (Oxford, Pergamon Press, 1980)
2. Khalatnikov I M *Teoriya Sverkhtekuchesti* (The Theory of Superfluidity) (Moscow: Mir, 1971);
Tilley D R, Tilley J *Superfluidity and Superconductivity* (New York: Halsted, 1975)
3. Manchester R N, Taylor J H *Pulsars* (San Francisco, CA: Freeman, 1977)
4. Ginzburg V L, Kirzhnits D A *Zh. Eksp. Teor. Fiz.* **47** 2006 (1964) [*Sov. Phys. JETP* **20** 1346 (1965)]
5. Landau L D, Lifshitz E M *The Classical Theory of Fields* (Oxford: Pergamon Press, 1980)
6. Landau L D, Lifshitz E M *Fluid Mechanics* (Oxford: Pergamon Press, 1987)
7. Andreev A Yu, Kirzhnits D A, Yudin S N *Pis'ma v Zh. Eksp. Teor. Fiz.* **61** 825 (1995) [*JETP Lett.* **61** 846 (1995)]
8. Weinberg S *Gravitation and Cosmology; Principles and Applications of the General Theory of Relativity* (New York: Wiley, 1972)
9. DeWitt B S *Phys. Rev. Lett.* **16** 10922 (1966)
10. V L Ginzburg, D A Kirzhnits (Eds) *Problema Vysokotemperaturnoi Provodimosti* (Problems of High-Temperature Conductivity) (Moscow: Nauka, 1977)
11. Thorne K S, Price R H, Macdonald D A (Eds) *Black Holes: The Membrane Paradigm* (New Haven, CT: Yale University Press, 1986)
12. Grib A A, Mamaev S G, Mostepanenko V M *Kvantovye Effekty v Intensivnykh Vneshnikh Polyakh* (Quantum Effects in Strong External Fields) (Moscow: Atomizdat, 1980)

Appendices

I. Hydrodynamics of superfluid in general relativity

We shall proceed from the general relativity Klein–Gordon equation for the scalar wave function of condensate ψ (see, for instance, Ref. [12]):

$$D^i D_i \psi + F(|\psi|^2)\,\psi = \frac{1}{\sqrt{-g}}\,\partial_i\left(\sqrt{-g}\,g^{ik}\,\partial_k\psi\right) + F(|\psi|^2)\,\psi = 0\,, \qquad (A.1)$$

where D_i denotes the covariant derivative and g is the determinant of the metric tensor. Substitution of $\psi = ve^{i\alpha}$ into Eqn (A.1) leads to the hydrodynamic formulation of quantum mechanics (Modelung representation). The imaginary part of the resultant expression gives the continuity equation

$$D_i j^i = 0\,, \quad j_i = \frac{i}{2}\left(\partial_i\bar{\psi}\psi - \bar{\psi}\,\partial_i\psi\right) = v^2\partial_i\alpha\,, \qquad (A.2)$$

and its real part yields the relation

$$\partial_i\alpha\,\partial^i\alpha = k^2 = F(v^2) + \frac{D^i D_i v}{v}\,. \qquad (A.3)$$

The current vector $j_i = nu_i$ gives expressions for the concentration $u_i u^i = 1$ and the 4-velocity n with due regard for condition u_i and Eqn (A.3):

$$n = kv^2\,, \quad u_i = \frac{\partial_i\,\alpha}{k}\,. \qquad (A.4)$$

It can be seen from Eqns (A.4) that in the general case velocity has no potential; instead it exists for quantity w/nu_i where $k = (tF + D_i D^i v/v)^{1/2}$ is identified as the heat function per particle (w/n) (see Ref. [6]). It is only in the nonrelativistic limit when $k \to m$ that one may have potential flow in the common sense of the term and relations (2), (3).

Differentiation of (A.3) using Eqns (A.4) yields the Euler equation [6]

$$(u^k D_k)\,u_i = \left[\partial_i - (u^k\,\partial_k)\,u_i\right]\ln\frac{w}{n}\,. \qquad (A.5)$$

In the case in question $\partial_0 = 0$ and $u_\alpha = 0$, and the general relativistic Bernoulli equation can be obtained from Eqn (A.5). To this effect, the following general relations should be used:

$$g^{00} = \frac{\gamma^2}{g_{00}}\,, \quad g^{0\alpha} = -\gamma^2 g^{-1}_{\alpha\beta}\,\frac{g_{0\beta}}{g_{00}}\,,$$

$$g^{\alpha\beta} = g^{-1}_{\alpha\beta} + \gamma^2 g^{-1}_{\alpha\mu}\,g_{0\mu}\,g^{-1}_{\beta\nu}\,\frac{g_{0\nu}}{g_{00}}\,;$$

$$u_0 = \frac{\sqrt{g_{00}}}{\gamma}\,, \quad u_\alpha = 0\,;$$

$$u^0 = \frac{\gamma}{\sqrt{g_{00}}}\,, \quad u^\alpha = -\gamma g^{-1}_{\alpha\beta}\,\frac{g_{0\beta}}{\sqrt{g_{00}}}\,,$$

where $\gamma = (1 - v^2)^{1/2}$, $v^2 = -g_{\alpha\beta}\,v^\alpha v^\beta$. The Bernoulli equation has the form

$$\frac{\sqrt{g_{00}}}{\gamma}\,\frac{w}{n} = \text{const} \qquad (A.6)$$

and turns into the standard Bernoulli equation in the case of weak fields, low velocity of motion, and nonrelativistic state equation (see Section 5).

II. Equation for gravimagnetic field

Statistical Einstein equations linear in the angular velocity of rotation of a spherically symmetric body have the form (see Ref. [5], paragraph 95):

†The absence of momentum in a body which is at rest in a rotating inertial system is universally known (see, for instance, Ref. [11]).

$$D_\beta f^{\alpha\beta} + 3\partial_\beta \ln \sqrt{h} f^{\alpha\beta} = -\frac{\varkappa^2}{2\sqrt{h}} v^\alpha , \qquad (A.7)$$

where

$$f_{\alpha\beta} = \partial_\alpha \tilde{g}_\beta - \partial_\beta \tilde{g}_\alpha , \qquad \tilde{g}_\alpha = -\frac{g_{0\alpha}}{g_{00}} .$$

Equation (A.7) is reduced to

$$\left[\beta\left(\partial_r^2 + \frac{4}{r}\partial_r\right) - \frac{1}{4}(\varkappa_n^2 + \varkappa_s^2)\, r\partial_r - \varkappa_n^2\right] g^3 = \varkappa_n^2 \Omega , \qquad (A.8)$$

where

$$\beta = -g_{11}^{-1} = 1 - \frac{1}{2r}\int_0^r \mathrm{d}r\, r^2(\varkappa_n^2 + \varkappa_s^2) .$$

It is clear that the term describing generation of field g of the rotating superfluid is actually absent.

Eqn (A.8) can be solved in the case of a superfluid core of a body (with mass M_s and radius R) and a thin normal shell (with mass M_n) if $n_{s,n}$ are constant. Then, quantity g^3 in the core is also constant. After denoting $\lambda = r_g/R$, $r_g = 2GM$, $\theta = (1-\sigma)/(1-\sigma/4)$, $\sigma = [1-\lambda_n/(1-\lambda_s)]^{1/2}$, the expression for the momentum is found:

$$M = \frac{\theta R^3 \Omega}{2G} , \qquad (A.9)$$

and for the effective angular velocity of the condensate:

$$\frac{\Omega_{\mathrm{ef}}}{\Omega} = \frac{v_s^3}{v_n^3} = g^3 = \theta . \qquad (A.10)$$

At $\lambda_n \ll \lambda_s$,

$$M = \frac{2}{3} M_n R^2 \frac{\Omega}{1-\lambda_s} , \qquad \frac{\Omega_{\mathrm{ef}}}{\Omega} = \frac{2}{3}\frac{\lambda_n}{1-\lambda_s} ,$$

where the denominator has a purely geometric sense, being related to quantity β in Eqn (A.8) which is contained in the Laplacian in curved space.