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Collective effects of plasma particles in bremsstrahlung

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Contents

Abstract. In the classical limit, bremsstrahlung corresponds to scattering of virtual fields of the colliding electron and ion on the incident electron. This statement is no longer correct in the case when the collective effects in bremsstrahlung are taken into account. These effects can change drastically the cross-sections of bremsstrahlung because of the important role of scattering of virtual waves on the Debye shielding shells of colliding particles. In particular, the scattering on ions (their Debye shielding clouds) is important (while in the absence of collective effects the scattering on a single ion is negligible). In this article it is shown how the present theory of fluctuations in a plasma, with nonlinear fluctuations taken into account, leads to the determination of cross-sections of bremsstrahlung, which take into account all collective effects that are important for this process.

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1. Introduction

It is often wrongly believed that to describe the kinetics of evolution of a physical system consisting of many particles, for example a plasma, it is sufficient to know the cross-sections of the physical processes of individual particle interactions. In reality, the ensemble of interacting particles can change the cross-sections drastically and these effects are usually called the collective effects (the term collective effects is also used in plasma physics to define the development of instabilities of collective plasma motions, but in what follows we will completely ignore the effects due to instabilities and use the word collective only to describe the collective change of cross-sections). The example of a plasma, the simplest case of an ensemble of almost free particles, with collective changes in cross-sections brings this to light more clearly. The fact that the cross-sections of the processes in a plasma are indeed very different from those for individual particles should be well known to physicists working in such fields as plasma diagnostics, plasma theory, or plasma astrophysics. Nevertheless, even in the textbooks on plasmas, the terminology used can lead to misunderstandings and probably shows that there exists not only an improper use of terminology but also that a clear physical understanding of the processes in plasmas could be lost.

The most representative effect of this kind is the change of cross-sections of wave scattering for wavelengths larger than the screening length, when electrons and ions appear to change their places. In vacuum (for individual particles), the scattering on ions is m_e^2/m_i^2 times less than the scattering on electrons, while in a plasma, in the case when the collective effects dominate, the scattering on electrons is approximately by the same factor smaller than the scattering on ions.

At first glance, this fact has a simple explanation. An electromagnetic wave displaces mainly the electrons, and both electrons and ions take part in the screening of electrons and ions (the electrons are screened by a decrease in electron density and an increase in ion density in their vicinity and the ions are screened by an increase in electron density and a decrease in ion density in their vicinity). For wavelengths larger than the screening length (which is the Debye length) the 'central' electron, moving with a velocity greater than the thermal ion velocity, and its screening electron cloud are together almost neutral (since an electron moving with a velocity larger than the mean ion thermal velocity is screened mainly by electrons) and the incident wave almost does not create an alternating dipole moment to excite the scattered wave. For ions, only the screening electrons are displaced by the incident wave and the total charge of the screening electrons is nearly equal in absolute value and opposite in sign to the charge of the ion. In the case when the ion is singly ionised, it will scatter almost at the same rate as an electron in a vacuum.

This explanation, in fact, assumes that all plasma particles are screened by other particles, which turn out to be the same plasma particles. Thus the explanation is not at all trivial. One should prove that this explanation is indeed correct, i.e. one should prove that any plasma particle can serve as a 'centre' of scattering and simultaneously take part in screening of the fields of other particles. Such a proof can be obtained only by a theory of fluctuations in a plasma in which the particle motion is decomposed into an average part (in which the particle appears as a scattering 'centre') and a fluctuating part (in which the particle is screening the fields of other particles). This picture appears to be a correct and true picture describing the processes which indeed occur.

It is interesting to note that it was shown previously that such a concept is correct for particle collisions and wave scattering. Here it will be shown that this concept also applies to the emission of waves in particle collisions, i.e. to bremsstrahlung. This then indicates that this concept could be a general one for all changes of cross-sections due to collective effects.

At this point it should be stressed that the word 'screening' used before means dynamical screening (the exact description is given below), i.e. for a sufficiently low particle velocity it is close to static screening, while for large particle velocities the polarisation charges are stripped off from particles and the cross-sections can approach values corresponding to those when the collective effects are not taken into account. The critical velocity is the mean particle thermal velocity.

One important point should be made clear before starting the discussion on collective effects in bremsstrahlung. This concerns the explanation of the physical processes leading to the changes in cross-sections. It is best to illustrate this point for the process of wave scattering on ions. Although the scattering is produced by an electron screening shell, the changes in energy and momentum of the waves taking part in the scattering process apply only to ions. This statement follows from expressions obtained from the fluctuation theory and particularly from equations describing the changes of ion distributions due to scattering. These equations show that the change of the total ion energy and total ion momentum is equal in absolute value and opposite in sign to the total change in energy and momentum of the incident and scattered waves. This is not often mentioned in the literature and is probably the reason for the misunderstanding when such a scattering is attributed to the scattering on electron fluctuations.

The fluctuations indeed produce the polarisation shell and, when one does not take into account the recoil effect in scattering and assumes that the ion distribution is fixed, one can have an impression that the scattering is produced by electron fluctuations. But the electrons play only the role of an intermediate chain, and by ambipolar field the energy and momentum are transferred to ions. Note that, when one neglects the Doppler shifts of the incident and the scattered waves, the difference in their frequencies is zero and the field acting on ions is almost a static field.

In connection with this statement it is necessary to mention the problem considered by I E Tamm [1] concerning the emission of a particle moving in a medium with a velocity greater than the velocity of light in that medium. The question considered was: is the Vavilov–Cherenkov emission produced by the polarisation shell of the particle moving in the medium or by the particle itself? The answer is that the emission is produced by the particle itself because the energy and momentum of radiation are derived from the particle energy and momentum.

The simplest way to see it is to use the quantum description of Vavilov–Cherenkov radiation given for the first time by V L Ginzburg [2]. From Ginzburg's description it is obvious that the energy and momentum emitted are lost by the particle itself. For scattering, the same arguments work but instead of the energy and momentum of the wave the differences of the energies and momenta of the initial and scattered waves are considered. We looked for this in the literature because, up to the present time, in the literature on scattering the term collective effects has been used to imply scattering on electrons, yet one does not mention that the scattering is mainly produced by ions. Nevertheless even in the very early papers on the scattering of waves in plasmas [3–5, 14], scattering on ions had already been considered.

The definite statement that the scattering is indeed occurring on ions and a simple physical interpretation of this effect were first given in Ref. [6], and then a simple method, independent of the fluctuation, for calculation of the additional amplitude in scattering due to particle polarisation shells was proposed in Refs [7, 8]. In the monograph [9] a simple method, also independent of the fluctuation approach, was proposed for calculating additional matrix elements in bremsstrahlung due to the particle polarisation shells.

The present review serves not only as a review of recent results concerning the collective effects in bremsstrahlung but also contains an original proof of how the additional collective matrix elements mentioned earlier can be derived from the theory of fluctuations. Proof will be provided that the concept of dynamically screened particles is appropriate for bremsstrahlung processes. It should be mentioned that the processes of bremsstrahlung were considered previously in the general theory of fluctuations in a plasma. But as in the case of scattering processes, it was not shown that the total effect can be expressed through the probability of bremsstrahlung, containing a square of the sum of matrix elements of the process on individual particles and collective processes connected with the screening shell. This proof is given in the present article. With such a proof available it becomes very easy to analyse the bremsstrahlung processes and particularly new ones which result from collective effects (for example, an increase of bremsstrahlung radiation intensity in ion–ion collisions).

2. Qualitative description of collective effects in bremsstrahlung

2.1 Why is the concept of bremsstrahlung of 'bare' particles wrong for any plasma density?

The term 'bare' indicates that the particle is isolated in vacuum. One could think that if the particle is in a plasma and the plasma density tends to zero, each particle can be regarded as 'bare' and the emission of waves by it will be the same as in the case of a 'bare' particle. But this is not correct.

Indeed, the collective effects should be most pronounced for wavelengths much larger than the screening length. This length is inversely proportional to $\sqrt{n}$, where n is the plasma density. The lower the plasma density, the larger the wavelengths for which the collective effects are important.

The statement that if the bremsstrahlung is considered to be that of 'bare' particles, the correct value of real bremsstrahlung cannot be obtained for any low plasma density becomes obvious when one takes into account that in a plasma only those electrostatic waves can exist that have wavelengths larger than the Debye screening length. This is the case for Langmuir electrostatic waves or ion–sound waves. For these, the collective effects are always important.

The wavelength of electromagnetic waves exceeds the Debye length for

$$\omega \gg \omega_{\mathrm{pe}} \frac{c}{v_{T\mathrm{e}}} ,$$

where ω_{pe} is the electron plasma frequency, c is the light velocity, and $v_{T\mathrm{e}}$ is the mean electron thermal velocity. The frequency range from ω_{pe} to $\omega_{\mathrm{pe}} c / v_{T\mathrm{e}}$ is quite large in many real conditions.

2.2 Why does the bremsstrahlung of longitudinal waves dominate for nonrelativistic particles?

Often, when speaking about bremsstrahlung, one means the bremsstrahlung of electromagnetic waves. But all waves which are modes of the system can be emitted during particle collisions. Among them are also longitudinal electrostatic waves. When the particles have nonrelativistic velocities, an expression for the intensity of bremsstrahlung of longitudinal waves cannot contain the velocity of light, since the latter enters neither in the disturbance of particle motion by the wave and the work performed by the emitted wave on the particle, nor in the dispersion characteristics of the electrostatic waves.

For electromagnetic wave bremsstrahlung by nonrelativistic particles, the velocity of light also does not enter in the disturbances of particle motion by the wave (when the intensity of the wave is not so high as to make the particle oscillation velocity in the wave relativistic) or in the work performed by the emitted wave on the particle disturbances.

Nevertheless it is well known that the factor c^3 appears in the denominator in the standard formula for bremsstrahlung of electromagnetic waves by an electron with the momentum $\boldsymbol{p}$:

$$Q_{\boldsymbol{p},\omega} = \frac{16 e_\alpha^4 e_\beta^2 n_\beta}{3 m_\alpha^2 v_\alpha c^3} \ln \frac{\rho_{\max}}{\rho_{\min}} , \qquad (1)$$

where $Q_{\boldsymbol{p},\omega}$ is the power of emission in the interval $\mathrm{d}\omega$, e_α is the charge of the incident particle (electron), m_α is the mass of the incident particle, e_β is the charge of the heavy particles (ions), n_β is the density of the heavy particles (ions), v_α is the velocity of the incident particle (determined by its momentum $\boldsymbol{p}$), and $\rho_{\max}/\rho_{\min}$ is the ratio of the maximum impact parameter to the minimum impact parameter. Thus we conclude that the factor c^3 derives only from the dispersion relation for electromagnetic waves, $\omega = kc$.

From dimensional arguments it is obvious that for electrostatic waves the factor $1/v_{\mathrm{ph}}^3$ for electrostatic waves should be used instead of the factor $1/c^3$ for electromagnetic waves, where $v_{\mathrm{ph}} = \omega/k$ is the phase velocity of the electrostatic waves. The characteristic property of longitudinal waves in a plasma is that they can have a rather low phase velocity. For example, the phase velocities of Langmuir waves can be as low as the mean electron thermal velocity and the phase velocities of ion–sound waves can be as low as the ion thermal velocity.

This estimate directly shows that the bremsstrahlung of electrostatic waves by nonrelativistic particles is much greater than the bremsstrahlung of electromagnetic waves. Therefore for nonrelativistic particles it is possible to restrict the consideration to the case of bremsstrahlung of electrostatic waves (if, of course, one is not specially interested in the bremsstrahlung of electromagnetic waves). The radiation losses of nonrelativistic particles will be determined mainly by the bremsstrahlung of electrostatic waves, because the contribution of bremsstrahlung of electromagnetic waves in the total particle energy losses due to radiation will be small. This point is seldom taken into account in the applications of bremsstrahlung in laboratory experiments and in the interpretation of emission from cosmic sources of radiation.

In the literature there is no detailed description of the problem of bremsstrahlung of longitudinal waves. Therefore we should start with the simplest estimates.

2.3 The bremsstrahlung of longitudinal waves by 'bare' particles

For longitudinal waves, it is impossible to neglect collective effects in bremsstrahlung since their wavelengths are always larger than the screening length, but for comparison with the results of collective bremsstrahlung it is necessary to know the expression for bremsstrahlung of 'bare', nonscreened particles. I consider here the case when the two colliding particles (electron and ion) are not screened and the emission is due only to the deviation of the electron from its straight trajectory by the field of the ion at rest, i.e. exactly the kind of process which is usually in one's mind when one thinks of bremsstrahlung of electromagnetic waves by 'bare' particles.

We can find the emitted power by calculating the work of the emitted wave performed on the current produced by a change of trajectory of the incident electron averaged over all ion positions $\boldsymbol{r}_\beta$, which appear along the path of the electron with an initial momentum $\boldsymbol{p}$:

$$Q_p = \int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}_\beta\,(\boldsymbol{j}\cdot\boldsymbol{E})\,n_\beta\ . \tag{2}$$

The current density $\boldsymbol{j}$ can be expressed through the change in charge density $\delta\rho$ by the use of the continuity equation and the fact that the field is longitudinal. Then the electric field strength $\boldsymbol{E}$ can be expressed through $\delta\rho$ by means of the Poisson equation:

$$Q_p = 4\pi\mathrm{i}(2\pi)^3 \int \mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega\,\mathrm{d}\omega'\,\frac{\omega' n_\beta}{k^2\varepsilon_{\boldsymbol{k},\omega}} \times \exp(\mathrm{i}\omega t - \mathrm{i}\omega' t) \int \mathrm{d}\boldsymbol{r}_\beta\,\delta\rho_{\boldsymbol{k},\omega}\,\delta\rho_{-\boldsymbol{k},-\omega'}\ . \tag{3}$$

The disturbance in the electron motion can be found from the equation for electron motion:

$$m_\alpha \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = e_\alpha \boldsymbol{E}^\beta = -\frac{\mathrm{i}e_\alpha e_\beta}{2\pi^2}\int \mathrm{d}\boldsymbol{q}\,\frac{\boldsymbol{q}}{q^2}\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r} - \mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_\beta)\ . \tag{4}$$

For a distant collision, in the first approximation the electron proceeds along a straight line with a constant velocity $\boldsymbol{r} = vt$, and in the next approximation it is weakly disturbed:

$$\delta\boldsymbol{r} = \frac{\mathrm{i}e_\alpha e_\beta}{2\pi^2 m_\alpha}\int \mathrm{d}\boldsymbol{q}\,\frac{\boldsymbol{q}}{q^2(\boldsymbol{q}\cdot v)^2}\exp(\mathrm{i}\boldsymbol{q}\cdot vt - \mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_\beta)\ . \tag{5}$$

The Fourier components of the charge density for an arbitrary moving charge are described by the equation

$$\rho_{\boldsymbol{k},\omega} = \frac{e_\alpha}{(2\pi)^4}\int \mathrm{d}t\,\exp\left[-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}(t) + \mathrm{i}\omega t\right]\ , \tag{6}$$

and the change in charge density due to $\delta\boldsymbol{r}$ is given by

$$\delta\rho_{\boldsymbol{k},\omega} = -\frac{\mathrm{i}e_\alpha}{(2\pi)^4}\int \mathrm{d}t\,(\boldsymbol{k}\cdot\delta\boldsymbol{r})\exp(\mathrm{i}\omega - \mathrm{i}\boldsymbol{k}\cdot v)\ . \tag{7}$$

From this expression, and after substituting Eqn (5), we get

$$\delta\rho_{\boldsymbol{k},\omega} = \frac{e_\alpha^2 e_\beta}{\pi(2\pi)^4 m_\alpha} \times \int \mathrm{d}\boldsymbol{q}\,\frac{(\boldsymbol{k}\cdot\boldsymbol{q})\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_\beta)}{q^2(\omega - \boldsymbol{k}\cdot v)^2}\,\delta\left[\omega - (\boldsymbol{k}-\boldsymbol{q})\cdot v\right]\ . \tag{8}$$

This result [Eqn (8)] allows us to calculate the expression for the product of charge densities averaged over the ion positions:

$$\int \mathrm{d}\boldsymbol{r}_\beta\,\delta\rho_{\boldsymbol{k},\omega}\,\delta\rho_{-\boldsymbol{k},-\omega'} = \frac{e_\alpha^4 e_\beta^2}{\pi^2(2\pi)^5 m_\alpha^2}\,\delta(\omega-\omega') \times \int \mathrm{d}\boldsymbol{q}\,\frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2}{q^4(\omega-\boldsymbol{k}\cdot v)^4}\,\delta\left[\omega - (\boldsymbol{k}-\boldsymbol{q})\cdot v\right]\ . \tag{9}$$

The expression for the emitted power [Eqn (3)] will contain only the imaginary part of the inverse dielectric permittivity:

$$\mathrm{Im}\,\frac{1}{\varepsilon_{\boldsymbol{k},\omega}} = -\frac{\mathrm{i}\pi\omega}{|\omega|}\,\delta(\mathrm{Re}\,\varepsilon_{\boldsymbol{k},\omega}) = -\frac{\mathrm{i}\pi\omega}{|\omega|}\,\frac{\delta(\omega-\omega_k)+\delta(\omega+\omega_k)}{\left(\partial\varepsilon_{\boldsymbol{k},\omega}/\partial\omega\right)_{\omega=\omega_k}}\ . \tag{10}$$

Finally we find an expression for the power of the bremsstrahlung of longitudinal waves:

$$Q_p = \frac{2e_\alpha^4 e_\beta^2}{\pi^2 m_\alpha^2} \times \int \mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{q}\,\frac{\omega_k(\boldsymbol{k}\cdot\boldsymbol{q})^2 n_\beta}{k^2 q^4(\omega_k - \boldsymbol{k}\cdot v)^4}\,\frac{\delta[\omega_k - (\boldsymbol{k}-\boldsymbol{q})\cdot v]}{(\partial\varepsilon_{\boldsymbol{k},\omega}/\partial\omega)_{\omega=\omega_k}}\ . \tag{11}$$

The power emitted in Langmuir waves is obtained by integrating this expression over the angles of vectors $\boldsymbol{k}$ and $\boldsymbol{q}$, and over q, on the assumption that $\omega \gg kv_{Te}$, and with account taken that for these conditions $\partial\varepsilon/\partial\omega \approx 2/\omega_{\mathrm{pe}}$:

$$Q_p = \int \mathrm{d}k\,Q_{p,k}\ , \tag{12}$$

where

$$Q_{p,k} = \frac{8e_\alpha^4 e_\beta^2 k^2 n_\beta}{3m_\alpha^2\omega_{\mathrm{pe}}^2 v}\ln\frac{\rho_{\max}}{\rho_{\min}}\ . \tag{13}$$

In expression (13) we used $q \approx 1/\rho$ and $q_{\max} \approx 1/\rho_{\min}$, and $q_{\min} \approx 1/\rho_{\max}$. We will see that q, or more exactly $\hbar q$, is the momentum transferred in the processes of bremsstrahlung.

If we can compare formula (1) with formula (13) we see that the intensity of bremsstrahlung increases with an increase in the value of the wavevector, i.e. it increases as the phase velocity ω_{pe}/k decreases. One can conclude also that the intensity of bremsstrahlung of longitudinal waves is much larger than that of electromagnetic waves. Note that when considering the intensity emitted in a unit frequency interval, one should divide Eqn (13) by $\partial\omega_k/\partial k$, which gives the light velocity in the case of electromagnetic waves. Together with the factor k^2/ω_k^2 we get in the denominator the factor c^3 for electromagnetic waves. An additional factor of 1/2 in Eqn (13), as compared with Eqn (1), appears there because of the difference in the polarisation of longitudinal and electromagnetic waves — instead of the average value of $(1+\cos^2\theta)/2$, equal to 2/3, we have the average value of $\cos^2\theta$, equal to 1/3.

This calculation is of interest not only as an illustration of the statement that the emission of longitudinal waves is the dominant process for nonrelativistic particles, but also as an illustration of the error committed by neglecting the collective screening effects and considering the particles as 'bare' particles. We will also use these calculations to introduce the probabilities of bremsstrahlung; changes in the latter, as a result of collective effects, will be the main subject of the discussion that follows. Also, collective effects for longitudinal waves are the most spectacular.

2.4 The probabilities of wave bremsstrahlung

Let us suppose that in the process of bremsstrahlung particle β loses the momentum $\boldsymbol{q}$ (its initial momentum is $\boldsymbol{p}'$ and its final momentum is $\boldsymbol{p}'-\boldsymbol{q}$) and particle α loses the momentum $\boldsymbol{k}-\boldsymbol{q}$ (its initial momentum is $\boldsymbol{p}$ and its final momentum is $\boldsymbol{p}-\boldsymbol{k}+\boldsymbol{q}$, and the momentum $\boldsymbol{k}$ is taken up by the emitted wave).

The energy conservation law in the elementary process of bremsstrahlung is

$$\varepsilon_p^\alpha + \varepsilon_{p'}^\beta = \varepsilon_{p-k+q}^\alpha + \varepsilon_{p'-q}^\beta + \omega_k \;, \tag{14}$$

where $\hbar = 1$, and we use for the energy of particle α the relativistic expression $\varepsilon_p^\alpha = \sqrt{m_\alpha^2 c^4 + c^2 p^2}$.

In the case when the transferred momentum and the momentum taken up by the wave are small compared with the momenta of the particles, the conservation law (14) can be written in a simpler form:

$$\omega_k - \boldsymbol{q}\cdot v' - (\boldsymbol{k} - \boldsymbol{q})\cdot v = 0 \;. \tag{15}$$

It was exactly this conservation law for $v' = 0$ which appeared under the sign of the δ-function in expression (11).

It is possible to define the probability of bremsstrahlung of any eigenmode in a plasma including the longitudinal and electromagnetic waves. We will not introduce in the definition of the probability any superscript specifying the type of mode emitted, bearing in mind that it can be any mode. We will find a relation for the probability of bremsstrahlung by a particle with momentum $\boldsymbol{p}$, colliding with a particle with momentum $\boldsymbol{p}'$, normalised to the unit phase volume of emitted waves $\mathrm{d}\boldsymbol{k}/(2\pi)^3$ and the unit volume of the transferred momenta $\mathrm{d}\boldsymbol{q}/(2\pi)^3$. The expression for the power emitted in bremsstrahlung can serve for the definition of the probability $w_{p,p'}^{\alpha,\beta}(\boldsymbol{k},\boldsymbol{q})$:

$$Q_p = \int \frac{\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{p}'\,\mathrm{d}\boldsymbol{q}}{(2\pi)^9}\,\omega_k\, w_{p,p'}^{\alpha,\beta}(\boldsymbol{k},\,\boldsymbol{q})\,\Phi_{p'}^\beta \;, \tag{16}$$

where $\Phi_{p'}^\beta$ is the distribution function of particles β normalised on the unit phase volume of momenta $\boldsymbol{p}'$ and is given by

$$\int \frac{\mathrm{d}\boldsymbol{p}'}{(2\pi)^3}\,\Phi_{p'}^\beta = n_\beta \;. \tag{17}$$

From expressions (11) and (16), we can find the probability of bremsstrahlung of longitudinal waves by a ‘bare’ particle α, which can be expressed in terms of the matrix element M:

$$w_{p,p'}^{\alpha,\beta}(\boldsymbol{k},\,\boldsymbol{q}) = \frac{16\pi e_\alpha^2 e_\beta^2 (2\pi)^3}{(\partial\varepsilon_k/\partial\omega)_{\omega=\omega_k}} \times |M|^2\,\delta\big[\omega_k - \boldsymbol{q}\cdot v' - (\boldsymbol{k}-\boldsymbol{q})\cdot v\big] \;, \tag{18}$$

where $M = M_{\mathrm{noncoll}}^\alpha$ and

$$M_{\mathrm{noncoll}}^\alpha = \frac{e_\alpha}{m_\alpha q}\,\frac{1}{(\omega_k - \boldsymbol{k}\cdot v)^2}\,\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{kq} \;. \tag{19}$$

The subscript ‘noncoll’ indicates that collective effects have not been taken into account.

As mentioned earlier, the approximation of a ‘bare’ particle is not applicable for the description of bremsstrahlung of longitudinal electrostatic waves for which the collective effects are very important. Note that these effects cannot be taken into account simply by replacing the field of a ‘bare’ ion by the field of an ion screened by the plasma. If this were possible, it would have been sufficient to put in the denominator of the matrix element [Eqn (19)] for the value of the dielectric permittivity obtained from the Poisson equation. Thus it will be sufficient to put $\varepsilon_{q,\boldsymbol{q}\cdot v'}$ in the denominator (the effect of screening corresponds to the Debye screening only for an ion at rest; for a moving ion ω corresponds to $\boldsymbol{q}\cdot v'$):

$$M_{\mathrm{noncoll}}^\alpha = \frac{e_\alpha}{m_\alpha q}\,\frac{1}{(\omega_k - \boldsymbol{k}\cdot v)^2}\,\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{kq}\,\frac{1}{\varepsilon_{q,\boldsymbol{q}\cdot v'}} \;. \tag{20}$$

Expression (20) is also incorrect, since it does not take into account the emission due to displacements of the shielding cloud produced by the colliding particles — this is called transition bremsstrahlung (see Ref. [9]).

I will use the relation between the coefficient of absorption of waves due to the process inverse to bremsstrahlung and the probability of bremsstrahlung to find the probabilities from the theory of fluctuations in a plasma. From the balance relations of the direct and inverse processes of stimulated emission and stimulated absorption we obtain the decrement of attenuation:

$$\gamma_k = \frac{1}{2}\int \frac{\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{p}'}{(2\pi)^9}\, w_{p,p'}^{\alpha,\beta}(\boldsymbol{k},\boldsymbol{q}) \times \left[(\boldsymbol{k}-\boldsymbol{q})\cdot\frac{\partial\Phi_p^\alpha}{\partial\boldsymbol{p}}\,\Phi_{p'}^\beta + \boldsymbol{q}\cdot\frac{\partial\Phi_{p'}^\beta}{\partial\boldsymbol{p}'}\,\Phi_p^\alpha\right] \;. \tag{21}$$

Expression (21) is valid both for the case when the collective effects are not taken into account and for the case when they are taken into account, since the value of the probability was not specified in this expression. I will show that the theory of fluctuations in a plasma leads to expression (21) and will compare the expression found for the probability from the theory of fluctuations with expressions (18) and (19), in which the collective effects were not taken into account.

2.5 Qualitative differences in the bremsstrahlung of ‘dressed’ and ‘bare’ particles. The effective charge responsible for the generation of bremsstrahlung

Any charge in a plasma is screened both by electrons and by ions (a positive charge is screened by a surplus of electrons and a shortfall of ions, and a negative charge is screened by a surplus of ions and a shortfall of electrons). If the ion charge is Z, the screening charge of electrons will be (in units of electron charge e)

$$Z_{\mathrm{eff}} = Z\,\frac{1/d_{\mathrm{e}}^2}{1/d_{\mathrm{e}}^2 + 1/d_{\mathrm{i}}^2} \;, \tag{22}$$

where d_{e} and d_{i} are the electron Debye radius and the ion Debye radius, respectively.

For wavelengths much larger than the Debye radius (all electrostatic waves satisfy this relation in the absence of magnetic fields) the screening charge can be considered approximately as a point charge and it is possible to consider its displacement as a whole under the action of the incident electron. Since the charge given by Eqn (22) is also an electron charge, it will be disturbed under the action of the field of the electron in a manner similar to that when the incident electron was disturbed by the field of a stationary ion in the previous discussion.

The simplest way to see this is to go over to another frame of reference in which the incident electron is at rest. Then the physical picture of wave emission during the collision will be exactly the same as in the previous description, with the only difference that the value of the emitting electron charge is now determined by Eqn (22) and can be larger than the value of a single electron charge (if for the ion $Z > 1$). The intensity of bremsstrahlung can then be even larger than that determined above. In short, one must never neglect this emission since it can even exceed the usual bremsstrahlung emission.

A simple estimate of this effect can be made by adding a factor $Z_{\rm eff}^2$ to the formula for bremsstrahlung [Eqn (18)]. But this again leads to a wrong result. The reason is that the two mechanisms of bremsstrahlung interfere with each other and partly suppress each other. The latter is clear from the following arguments.

Let us consider a fast electron with a velocity much higher than the average thermal velocity. Then, at the high frequencies emitted by the electron, a collision with electrons in the shielding shell corresponds to a collision of free electrons. It is known that, in the dipole approximation, no bremsstrahlung is emitted as a result of such collisions, i.e., interference should quench the amplitudes of both bremsstrahlung emission mechanisms. Under certain conditions, such quenching can be partial. In addition one should take into account that the shielding clouds are present around both colliding particles and it is necessary to take into account the emission of the shielding cloud of the incident particle.

These arguments show why the emission in electron–electron collisions can be changed substantially by collective effects. Such changes can lead to large variations in the intensity only for nonequilibrium particle distributions; for equilibrium distributions the collective effects can change the intensity not more than by a factor of the order of unity.

The new qualitative effect introduced by collective effects is bremsstrahlung in collisions of two heavy particles (ions, for example). This emission is negligibly small in the absence of collective effects. The collective effects in ion–ion collisions correspond to the emissions of screening shells during these collisions and are determined by $Z_{\rm eff}$ when the interference effects are absent. The collective effects also change the emission in electron–electron collisions.

To write down the expression for an additional matrix element in bremsstrahlung probability, describing the emission of polarisation shells of both colliding particles, one can use the approach described in Ref. [9]. Namely, one can, in a first approximation, neglect the changes in trajectories of colliding particles and consider the oscillations of the polarisation shells of each colliding particle under the action of the other colliding particle.

This perturbation for nonrelativistic particles is determined by the Poisson equation with a nonlinear charge density $\rho^{\rm N}$ taken into account:

$$\rho^{\rm N}_{k,\omega} = \int \mathrm{d}\boldsymbol{q}\,\mathrm{d}\nu\,\rho^{\rm N,2}_{q,\nu;k-q,\omega-\nu} E_{q,\nu} E_{k-q,\omega-\nu}\,, \tag{23}$$

where $\boldsymbol{E}_{k,\omega} = (\boldsymbol{k}/k)E_{k,\omega}$.

Substitution in Eqn (23) of the fields of uniformly moving colliding charges makes it possible to find, by the same method as before, the additional power of the bremsstrahlung radiation due to the screening shells (see Ref. [9]). But the correct result is obtained not by summing the intensities but by summing the matrix elements.

From the expression for intensity determined from Eqn (23), one can find the square of the matrix element describing the emission of polarisation shells and from that get its absolute value; however, the sign of the matrix element is determined from a separate argument. Therefore to find the correct result, which takes into account the interference effects, one needs to calculate the intensity of emission, taking into account the changes in the trajectories of colliding particles and the oscillations of their polarisation shells. This gives two additional matrix elements, which should be added to expression (20): M^{β}, which is due to a change in the path of a charge β (neglected previously, because an ion is assumed to be very heavy; we now have to take this into account in order to determine the collective effects in bremsstrahlung, for example, those occurring in electron–electron or ion–ion collisions), and $M^{\alpha,\beta}$, which is due to both polarisation shells of the colliding particles:

$$M^{\beta} = \frac{e_\beta}{m_\beta|\boldsymbol{k}-\boldsymbol{q}|}\,\frac{1}{(\omega_k - \boldsymbol{k}\cdot\boldsymbol{v}')^2}\,\frac{\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{q})}{k|\boldsymbol{k}-\boldsymbol{q}|}\,\frac{1}{\varepsilon_{k-q,(k-q)\cdot v}}\,, \tag{24}$$

and

$$M^{\alpha,\beta} = -\frac{8\pi\rho^{\rm N,2}_{q,q\cdot v';k-q,(k-q)\cdot v}}{kq|\boldsymbol{k}-\boldsymbol{q}|}\,\frac{1}{\varepsilon_{q,q\cdot v'}\,\varepsilon_{k-q,(k-q)\cdot v}}\,. \tag{25}$$

Both colliding particles are equivalent, as can be seen by changing the notations of the momentum of the virtual quanta (transferred momentum) $\boldsymbol{q} \leftrightarrow \boldsymbol{k}-\boldsymbol{q}$.

All the given relations can be considered to be preliminary expressions intutively obtained from physical arguments, since their exact proof can be obtained only from the theory of fluctuations. It is important to know that the theory of fluctuations proves that these relations are correct and that each plasma particle in its average motion appears as a colliding particle, and in fluctuation motion is able to screen the fields of other colliding particles. In short, the simple picture of collisions of dynamically screening particles and emission during these collisions corresponds to the true picture of plasma emission due to particle collisions. The aim of the following description is to prove this statement.

3. Nonlinear interaction of electrostatic waves with fluctuations of plasma particles and fields

3.1 Fluctuations of particles and fields in a plasma

We will start by dividing the distribution function of a plasma particle α into its regular average component Φ_p^α and its fluctuating component δf_p^α. When the particles are neutral (do not have charges), their fluctuations in a plasma will be described by fluctuations of independent particles.

Let us denote by $\delta f_p^{\alpha(0)}$ the corresponding part of the fluctuating component of a distribution function. These fluctuations play the role of 'zero' fluctuations, serving as a background on which the processes of particle collisions and the processes of emission of waves during the particle collision develop. By taking into account, in the first approximation, the particle charges, one obtains the known collision integrals in a plasma (see Ref. [10]). Obviously the above-mentioned 'zero' fluctuations should satisfy the equation:

$$\frac{\partial}{\partial t}\,\delta f_p^{\alpha(0)} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}}\,\delta f_p^{\alpha(0)} = 0\,. \tag{26}$$

In the first approximation, the regular component Φ_p^α satisfies the same equation, since it is supposed to describe the homogeneous and stationary state. The fluctuating component describes very inhomogeneous and nonstationary perturbations and therefore each term in equation (26) is large but the two terms compensate each other such that equation (26) is satisfied. The 'zero' fluctuations in fact imply that the average value of the square of particle fluctuations in a given volume is equal to the average number of particles in this volume. Mathematically this relation can be written as a relation for the average value of the product of two fluctuating components of the distribution function (see Ref. [10]):

$$\left\langle \delta f^{\alpha(0)}_{p,k,\omega}\, \delta f^{\beta(0)}_{p',k',\omega'} \right\rangle$$

$$= \Phi^{\alpha}_{p}\, \delta_{\alpha,\beta}\, \delta(\boldsymbol{p}-\boldsymbol{p}')\, \delta(\boldsymbol{k}+\boldsymbol{k}')\, \delta(\omega+\omega')\, \delta(\omega-\boldsymbol{k}\cdot v)\ . \quad (27)$$

These 'zero' particle fluctuations lead, according to the Poisson equation, to 'zero' fluctuations of electric fields. Let us denote these fields by $\boldsymbol{E}^{(0)}$. From expression (27) it is not difficult to obtain a formula for averaging the fields of 'zero' fluctuations:

$$\left\langle E^{(0)}_{i,k,\omega}\, E^{(0)}_{j,k',\omega'} \right\rangle = \frac{k_i k_j}{k^2}\, \left|E^{(0)}\right|^2_{k,\omega}\, \delta(\boldsymbol{k}+\boldsymbol{k}')\, \delta(\omega+\omega')\ , \quad (28)$$

where

$$\left|E^{(0)}\right|^2_{k,\omega} = \sum_{\alpha} \frac{16\pi^2 e_\alpha^2}{k^2 |\varepsilon_{k,\omega}|^2} \int \mathrm{d}\boldsymbol{p}'\, \delta(\omega-\boldsymbol{k}\cdot v')\, \Phi^{\alpha}_{p'}\ . \quad (29)$$

These relations help in determining the properties of 'zero' fluctuations for arbitrary average particle distributions, as well as the fields of these 'zero' fluctuations, which are necessary for the description used in this paper. The interactions with these fluctuations are nonlinear processes. This includes the bremsstrahlung processes. So far, not much attention has been paid to this aspect of the description of bremsstrahlung. From this point of view, even the process of linear wave absorption should be regarded as a nonlinear process of interaction of the propagating wave with the 'zero' fluctuations described above.

In what follows I will be interested in the linear processes in the field of a propagating wave. Let us denote its field by E^{σ}, where σ is a superscript denoting the type of mode propagating in a plasma (for example, the electrostatic mode or elec-tromagnetic mode). The frequency of the propagating wave is denoted by $\omega_{\boldsymbol{k}}$. The frequency and the wavevector of the 'zero' fluctuations is denoted by subscript zero — ω_0 and $\boldsymbol{k}_0$.

The nonlinear interactions should create the fields at frequencies which generally do not correspond to the frequencies of the propagating waves: $\omega_{\boldsymbol{k}} \pm \omega_0$ and $\boldsymbol{k} \pm \boldsymbol{k}_0$. The fields of these frequencies are called virtual fields and are denoted by $\boldsymbol{E}^{\mathrm{v}}$. Thus the total field can be written as

$$\boldsymbol{E} = \boldsymbol{E}^{\sigma} + \boldsymbol{E}^{(0)} + \boldsymbol{E}^{\mathrm{v}}\ . \quad (30)$$

I shall develop a pertubation theory of the total field described by expression (30) and, as the zeroth approximation, I shall use $\Phi^{\alpha}_{p} + \delta f^{\alpha(i)}_{p}$. An approximation of the ith order in this field is denoted by $\delta f^{\alpha(i)}_{p}$, which will consist of a part related to the initial distribution Φ^{α}_{p} (and denoted by $\delta f^{\alpha(\mathrm{R},i)}_{p}$) and a part related to the initial distribution $\delta f^{\alpha(0)}_{p}$ (and denoted by $\delta f^{\alpha(0,i)}_{p}$). Thus

$$\delta f^{\alpha(i)}_{p} = \delta f^{\alpha(\mathrm{R},i)}_{p} + \delta f^{\alpha(0,i)}_{p}\ .$$

The perturbation theory equations have the form:

$$\frac{\partial}{\partial t}\, \delta f^{\alpha(1)}_{p} + v\cdot \frac{\partial}{\partial \boldsymbol{r}}\, \delta f^{\alpha(1)}_{p} = -e_\alpha \boldsymbol{E}\cdot \frac{\partial}{\partial \boldsymbol{p}}\, \Phi^{\alpha}_{p}\ , \quad (31)$$

$$\frac{\partial}{\partial t}\, \delta f^{\alpha(\mathrm{R},i)}_{p} + v\cdot \frac{\partial}{\partial \boldsymbol{r}}\, \delta f^{\alpha(\mathrm{R},i)}_{p}$$

$$= -e_\alpha \boldsymbol{E}\cdot \frac{\partial}{\partial \boldsymbol{p}}\, \delta f^{\alpha(\mathrm{R},i-1)}_{p} + \left\langle e_\alpha \boldsymbol{E}\cdot \frac{\partial}{\partial \boldsymbol{p}}\, \delta f^{\alpha(\mathrm{R},i-1)}_{p} \right\rangle\ , \quad (32)$$

$$\frac{\partial}{\partial t}\, \delta f^{\alpha(0,i)}_{p} + v\cdot \frac{\partial}{\partial \boldsymbol{r}}\, \delta f^{\alpha(0,i)}_{p}$$

$$= -e_\alpha \boldsymbol{E}\cdot \frac{\partial}{\partial \boldsymbol{p}}\, \delta f^{\alpha(0,i-1)}_{p} + \left\langle e_\alpha \boldsymbol{E}\cdot \frac{\partial}{\partial \boldsymbol{p}}\, \delta f^{\alpha(0,i-1)}_{p} \right\rangle\ . \quad (33)$$

Here $\delta f^{\alpha(0,0)}_{p} = \delta f^{\alpha(0)}_{p}$.

In practice, in Eqn (33) it is sufficient to consider $i = 1, 2$. This is due to the fact that we will be interested in the contributions of the lower order in nonlinearities, i.e. in cubic nonlinearities in the total field and therefore only in the contributions quadratic in 'zero' fluctuations. The first part of the contribution described by expression (32) is unrelated to fluctuations of the plasma particles. These fluctuations represent natural inhomogeneities of the refractive index for the waves. The scattering by the inhomogeneities gives rise to additional radiation.

3.2 Nonlinear interactions not produced directly by plasma particle fluctuations

In this section I will consider the nonlinear effects created by that part of the fields of 'zero' fluctuations which are caused by the regular component of the particle distribution function Φ^{α}_{p}. In this case the expansion in the total field corresponds to a standard procedure of deriving the nonlinear equations (see Ref. [11]). I shall use the notations of this standard nonlinear theory:

$$E_1 = E_{k_1,\omega_1}\ , \qquad E_2 = E_{k_2,\omega_2}\ , \qquad E_3 = E_{k_3,\omega_3}\ ,$$

$$k = \{\boldsymbol{k},\, \omega\}\ , \qquad \mathrm{d}k = \mathrm{d}\boldsymbol{k}\, \mathrm{d}\omega\ ,$$

$$\mathrm{d}_{1,2} = \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2\, \mathrm{d}\omega_1\, \mathrm{d}\omega_2\, \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\, \delta(\omega-\omega_1-\omega_2)\ ,$$

$$\mathrm{d}_{1,2,3} = \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2\, \mathrm{d}\boldsymbol{k}_3\, \mathrm{d}\omega_1\, \mathrm{d}\omega_2\, \mathrm{d}\omega_3$$

$$\times\, \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\, \delta(\omega-\omega_1-\omega_2-\omega_3)\ ,$$

$$\rho^{\mathrm{N},2}_{1,2} = \rho^{\mathrm{N},2}_{k_1,\omega_1;k_2,\omega_2}\ , \qquad \rho^{\mathrm{N},3}_{1,2,3} = \rho^{\mathrm{N},3}_{k_1,\omega_1;k_2,\omega_2;k_3,\omega_3}\ .$$

Here $\rho^{\mathrm{N},2}_{1,2}$ and $\rho^{\mathrm{N},3}_{1,2,3}$ are the standard nonlinear plasma responses expressed through Φ^{α}_{p}, and describe the quadratic and cubic nonlinear charge densities, respectively .

With the given notations, the standard nonlinear equation for fields in a plasma, which takes into account the cubic and quadratic nonlinearities, has the form:

$$\mathrm{i}k\varepsilon_k E_k = 4\pi\rho^{\mathrm{N}}_{k} = 4\pi \int \mathrm{d}_{1,2}\, \rho^{\mathrm{N},2}_{1,2} \left(E_1E_2 - \langle E_1E_2\rangle\right)$$

$$+\, 4\pi \int \mathrm{d}_{1,2,3}\, \rho^{\mathrm{N},3}_{1,2,3} \left(E_1E_2E_3 - E_1\langle E_2E_3\rangle - \langle E_1E_2E_3\rangle\right). \quad (34)$$

Without losing generality, one can make the nonlinear quadratic response coefficient symmetric over the subscripts 1 and 2 and the nonlinear cubic response coefficient symmetric over the subscripts 2 and 3. For completeness of the representation, I will write down the known plasma nonlinear responses, which can also be directly derived from the system of equations (32):

$$\rho^{\mathrm{N},2}_{1,2} = -\sum_{\alpha}\frac{e_\alpha^3}{2k_1k_2}\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\left(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+\mathrm{i}0\right)^{-1}$$
$$\times\left[\left(\boldsymbol{k}_1\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\left(\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v}+\mathrm{i}0\right)^{-1}\left(\boldsymbol{k}_2\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\right.$$
$$\left.+\left(\boldsymbol{k}_2\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\left(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v}+\mathrm{i}0\right)^{-1}\left(\boldsymbol{k}_1\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\right]\Phi^\alpha_{\boldsymbol{p}}\ , \quad (35)$$

and

$$\rho^{\mathrm{N},3}_{1,2,3} = \sum_{\alpha}\frac{\mathrm{i}e_\alpha^4}{2k_1k_2k_3}\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\left(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+\mathrm{i}0\right)^{-1}$$
$$\times\left(\boldsymbol{k}_1\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\left[(\omega-\omega_1)-(\boldsymbol{k}-\boldsymbol{k}_1)\cdot\boldsymbol{v}+\mathrm{i}0\right]^{-1}$$
$$\times\left[\left(\boldsymbol{k}_2\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\left(\omega_3-\boldsymbol{k}_3\cdot\boldsymbol{v}+\mathrm{i}0\right)^{-1}\left(\boldsymbol{k}_3\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\right.$$
$$\left.+\left(\boldsymbol{k}_3\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\left(\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v}+\mathrm{i}0\right)^{-1}\left(\boldsymbol{k}_2\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\right]\Phi^\alpha_{\boldsymbol{p}}\ . \quad (36)$$

In the general nonlinear equation (34), the total field is present. We are interested in deriving from this equation the linear equation for the wave field E^σ. Hence we can average the nonlinear equation over the fluctuations, leaving on the right-hand side only the terms linear in the wave field. The other two fields in the cubic nonlinear term should then be the fluctuating fields, described by Eqns (28) and (29), because the virtual fields correspond to higher nonlinearities (and, as can be seen from the expressions given below, they are expressed through the quadratic combinations of fields).

Then it is easy to find that, after linearising the last term of Eqn (34) with respect to the wave field and after averaging the result over the fluctuations, the nonzero contribution will be made by the term in which the wave field enters as E_2. In the case of a quadratic nonlinearity, none of these two fields can be the wave field, since the second field should then be virtual. By definition, a virtual field consists of wave and fluctuation fields. The result is then a quadratic function of the wave field, but we are interested in the effects which are linear in the wave field. Nor can the second field be the fluc-tuating field, because after averaging over the fluctuations the result will be zero. Thus only the virtual field and the field of fluctuations can appear in the quadratic nonlinearity.

We then obtain (taking into account the mentioned symmetry properties of the nonlinear responses):

$$\mathrm{i}k\varepsilon_k E_k = 8\pi\int \mathrm{d}_{1,1'}\,\rho^{\mathrm{N},2}_{1,1'}\,E^{(0)}_1\,E^{\mathrm{v}}_{1'}$$
$$+\,8\pi\int \mathrm{d}_{1,2,3}\,\rho^{\mathrm{N},3}_{1,2,3}\,E^{(0)}_1\,E^\sigma_2\,E^{(0)}_3\ . \quad (37)$$

At this stage, an important point should be made. The description of the virtual fields given here leads to a deviation from the standard nonlinear theory. Indeed, in the case considered here the virtual field can be determined not only from the same nonlinear equations (34), where the responses are determined by the regular component of the particle distribution, but also through the fluctuating component of the particle distribution $\delta f^{\alpha(0)}_{\boldsymbol{p}}$. The first part of the virtual field determined by the regular component of the distribution function is denoted by $E^{\mathrm{v}(1)}_k$, while the second part related to the fluctuating component of particle distribution is denoted by $E^{\mathrm{v}(2)}_k$. We will postpone the consideration of the second part of the virtual field to the next subsection . We will see that the additional contribution to the standard nonlinear theory (with which we are dealing in this subsection) will appear and this will give additional contributions to the nonlinear dielectric permittivity. Among these will be the contribution due to the second part of the virtual fields.

Here I will write down the result which corresponds to the standard nonlinear theory. To find the virtual field we use the nonlinear equation (34):

$$\mathrm{i}k'\varepsilon_{k'}E^{\mathrm{v}(1)}_{k'} = 8\pi\int \mathrm{d}_{2,3}\,\rho^{\mathrm{N},2}_{2,3}\,E^\sigma_2\,E^{(0)}_3\ . \quad (38)$$

After substituting this expression into equation (37) we find:

$$\mathrm{i}k\varepsilon_k E^\sigma_k = 8\pi\int \mathrm{d}_{1,2,3,}\,\rho^{\mathrm{eff}}_{1,2,3}\,E^{(0)}_1\,E^\sigma_2\,E^{(0)}_3\ , \quad (39)$$

where

$$\rho^{\mathrm{eff}}_{1,2,3} = \rho^{\mathrm{N},3}_{1,2,3} + \frac{8\pi}{\mathrm{i}|\boldsymbol{k}_2+\boldsymbol{k}_3|\varepsilon_{2+3}}\,\rho^{\mathrm{N},2}_{1,2+3}\,\rho^{\mathrm{N},2}_{2,3}\ . \quad (40)$$

This result agrees with the standard nonlinear theory [11].

After averaging over fluctuations we get the equation for the wave field,

$$\left(\varepsilon_k+\varepsilon^{\mathrm{N},1}_k\right)E^\sigma_k = 0\ , \quad (41)$$

which contains an additional nonlinear permittivity denoted by $\varepsilon^{\mathrm{N},1}_k$, where

$$\varepsilon^{\mathrm{N},1}_k = \frac{8\pi}{\mathrm{i}k}\int \mathrm{d}k_1\,\rho^{\mathrm{eff}}_{k,k_1,-k_1}\,\left|E^{(0)}\right|^2_{k_1}\ . \quad (42)$$

I have denoted this nonlinear contribution to the nonlinear permittivity by the superscript 1 because there appear other contributions to the nonlinear permittivity which in fact cause the present derivation to differ from the standard nonlinear theory. In a general case, the total nonlinear permittivity contains several contributions denoted by $\varepsilon^{\mathrm{N},i}_k$, and the total $\varepsilon^{\mathrm{N}}_k$ enters in the dispersion wave equation, where

$$\varepsilon^{\mathrm{N}}_k = \sum_i \varepsilon^{\mathrm{N},i}_k\ . \quad (43)$$

3.3 Fluctuations of virtual fields

Let us consider now the contribution of the virtual field created by particle fluctuations and determine first the expression for $E^{\mathrm{v}(2)}_k$. The physical reason for the appearance of this contribution is the presence in 'zero' approximation of strong inhomogeneities and time variations. The standard nonlinear theory does not take into account these effects since it assumes that the initial state is stationary and homogeneous (or that the initial state is slowly evolving in time and is slightly inhomogeneous).

The sharp spatial changes in the particle distribution in the 'zero' fluctuations can be comparable with or less than the wavelength of the wave we are interested in, and the frequency related to short time variations of the 'zero' fluctuations can be of the order of or larger than the frequency of that wave. Under these conditions, new effects of emission and absorption of waves are important, which are known as transition emission and transition absorption [9]. They will thus contribute to the absorption of the wave we are interested in. The 'zero' fluctuations imply the appearance of these processes.

To find the expression for the additional virtual field we use the first approximation of the perturbation theory for the random component of particle distribution driven by the 'zero' particle fluctuations, i.e. we use the first equation of the system (33) describing the fluctuating part of the particle distribution in which we substitute the wave field for $\boldsymbol{E}$.

We denote the corresponding contribution by $\delta f^{\alpha(0,1,\sigma)}_{\boldsymbol{p},k}$, emphasising with superscript σ that this contribution is due to the wave field. This contribution will be the only additional contribution to the nonlinear equation (34). We find:

$$\delta f^{\alpha(0,1,\sigma)}_{\boldsymbol{p},k} = \frac{e_\alpha}{\mathrm{i}}(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0)^{-1} \times \int \mathrm{d}k_1\, E^\sigma_{k_1} \frac{1}{k_1}\left(\boldsymbol{k}_1\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\delta f^{\alpha(0)}_{\boldsymbol{p},k-k_1}\,. \tag{44}$$

This gives

$$E^{\mathrm{v}(2)}_k = \sum_\alpha \frac{4\pi e_\alpha}{\mathrm{i}k\varepsilon_k}\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\,\delta f^{\alpha(0,1)}_{\boldsymbol{p},k} = -\sum_\alpha \frac{4\pi e^2_\alpha}{k\varepsilon_k}(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0)^{-1} \times \int\frac{\mathrm{d}k_2\,\mathrm{d}\boldsymbol{p}}{(2\pi)^3}E^\sigma_{k_2}\frac{1}{k_2}\left(\boldsymbol{k}_2\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\delta f^{\alpha(0)}_{\boldsymbol{p},k-k_2}\,. \tag{45}$$

After substituting expression (45) into the quadratic nonlinear term in equation (34) we find an additional contribution to the nonlinear plasma permittivity:

$$\varepsilon^{\mathrm{N},2}_k = -\frac{8\pi}{\mathrm{i}kE^\sigma_k}\int \mathrm{d}k_1\,\rho^{\mathrm{N},2}_{k_1,k-k_1}\left\langle E^{(0)}_{k_1}E^{\mathrm{v}(2)}_{k-k_1}\right\rangle = \sum_\alpha\frac{32\pi^2 e^2_\alpha}{\mathrm{i}kE^\sigma_k}\int\frac{\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\rho^{\mathrm{N},2}_{k_1,k-k_1}E^\sigma_{k_2} \times \frac{1}{|\boldsymbol{k}-\boldsymbol{k}_1|\varepsilon_{k-k_1}}\left[\omega-\omega_1-(\boldsymbol{k}-\boldsymbol{k}_1)\cdot\boldsymbol{v}+\mathrm{i}0\right]^{-1} \times \frac{1}{k_2}\left(\boldsymbol{k}_2\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\left\langle E^{(0)}_{k_1}\,\delta f^{\alpha(0)}_{\boldsymbol{p},k-k_1-k_2}\right\rangle\,. \tag{46}$$

The procedure of averaging can be performed by the use of formula (27) and the Poisson equation. Namely we use the expression

$$\left\langle E^{(0)}_k\,\delta f^{\alpha(0)}_{\boldsymbol{p},k'}\right\rangle = \frac{e_\alpha}{2\pi^2\mathrm{i}k\varepsilon_k}\,\Phi^\alpha_{\boldsymbol{p}}\,\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\,\delta(k+k')\,. \tag{47}$$

Finally we find

$$\varepsilon^{\mathrm{N},2}_k = -\sum_\alpha\frac{16e^3_\alpha}{k^2}\int\frac{\mathrm{d}k_1\,\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\rho^{\mathrm{N},2}_{k_1,k-k_1} \times \frac{1}{|\boldsymbol{k}-\boldsymbol{k}_1|k_1\varepsilon_{k-k_1}\varepsilon_{k_1}}\left[\omega-\omega_1-(\boldsymbol{k}-\boldsymbol{k}_1)\cdot\boldsymbol{v}+\mathrm{i}0\right]^{-1} \times \left(\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\Phi^\alpha_{\boldsymbol{p}}\,\delta(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v})\,, \tag{48}$$

which ends the calculation of the additional contribution of the fluctuations in the virtual fields. However, these virtual fields (and this applies to both components) can contribute to that additional nonlinear interaction which is the direct result of the particle fluctuations.

3.4 Contribution of virtual fields to the interactions due to particle fluctuations

In the expression for the fluctuating part of the particle distribution function due to the 'zero' fluctuations we considered above the wave field as a field which disturbs the particle motion [see Eqn (44)]. If such a field is a virtual field, then the cubic terms appear and we should take them into account, restricting ourselves to the cubic nonlinearities. In this case both virtual fields contribute.

We denote the corresponding contribution to the fluctuating part of the particle distribution function by $\delta f^{\alpha(0,1,\mathrm{v})}_{\boldsymbol{p},k}$ (in contrast to $\delta f^{\alpha(0,1,\sigma)}_{\boldsymbol{p},k}$ already defined). The expression for $\delta f^{\alpha(0,1,\mathrm{v})}_{\boldsymbol{p},k}$ can be found by substituting $E^{\mathrm{v}}_{k_1}$ for $E^\sigma_{k_1}$:

$$\delta f^{\alpha(0,1,\mathrm{v})}_{\boldsymbol{p},k} = \frac{e_\alpha}{\mathrm{i}}(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+\mathrm{i}0)^{-1} \times \int\mathrm{d}k_1\,E^{\mathrm{v}}_{k_1}\frac{1}{k_1}\left(\boldsymbol{k}_1\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\delta f^{\alpha(0)}_{\boldsymbol{p},k-k_1}\,. \tag{49}$$

A substitution of the field $E^{\mathrm{v}(1)}_k$ in this expression gives:

$$\rho^{(0,1,\mathrm{v}(1))}_k = \sum_\alpha e_\alpha\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\delta f^{\alpha(0,1,\mathrm{v})}_{\boldsymbol{p},k} = -\sum_\alpha 8\pi e^2_\alpha\int\frac{\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}\boldsymbol{p}}{(2\pi)^3}(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+\mathrm{i}0)^{-1} \times \rho^{\mathrm{N},2}_{k_2,k_1-k_2}\frac{E^\sigma_{k_2}}{k^2_1\varepsilon_{k_1}}\left(\boldsymbol{k}_1\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\left\langle E^{(0)}_{k_1-k_2}\delta f^{(0)}_{\boldsymbol{p},k-k_1}\right\rangle\,. \tag{50}$$

This gives the following contribution to the dielectric permittivity (denoted as $\varepsilon^{\mathrm{N},3}_k$):

$$\varepsilon^{\mathrm{N},3}_k = -\frac{4\pi}{\mathrm{i}kE^\sigma_k}\rho^{[0,1,\mathrm{v}(1)]}_k = -\sum_\alpha\frac{16e^3_\alpha}{k}\int\frac{\mathrm{d}k_1\,\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\rho^{\mathrm{N},2}_{k,k_1-k} \times \frac{1}{k^2_1\varepsilon_{k_1}}\frac{1}{|\boldsymbol{k}-\boldsymbol{k}_1|\varepsilon_{k_1-k}}(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+\mathrm{i}0)^{-1} \times \left(\boldsymbol{k}_1\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\delta\left[\omega-\omega_1-(\boldsymbol{k}-\boldsymbol{k}_1)\cdot\boldsymbol{v}\right]\Phi^\alpha_{\boldsymbol{p}}\,. \tag{51}$$

If the other virtual field $E_k^{\mathrm{v}(2)}$ is substituted in expression (49) we get

$$\delta f_{\boldsymbol{p}',k}^{[0,1,\mathrm{v}(2)]} = -\sum_{\beta} \frac{4\pi e_{\beta}^2 e_{\alpha}}{\mathrm{i}} \left(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0\right)^{-1} \times \int \mathrm{d}k_1\,\mathrm{d}k_2\, \frac{E_{k_2}^{\sigma}}{k_1^2 k_2 \varepsilon_{k_1}} \left(\boldsymbol{k}_1\cdot\frac{\partial}{\partial \boldsymbol{p}'}\right)\left(\omega_1 - \boldsymbol{k}_1\cdot\boldsymbol{v} + \mathrm{i}0\right)^{-1} \times \left(\boldsymbol{k}_2\cdot\frac{\partial}{\partial \boldsymbol{p}}\right)\left\langle \delta f_{\boldsymbol{p},k_1-k_2}^{\beta(0)}\, \delta f_{\boldsymbol{p}',k-k_1}^{\alpha(0)}\right\rangle . \tag{52}$$

After averaging over the fluctuations, we get another contribution to the nonlinear permittivity

$$\varepsilon_k^{\mathrm{N},4} = -\sum_{\alpha} \frac{4\pi e_{\alpha}}{\mathrm{i}k E_k^{\sigma}} \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\, \delta f_{\boldsymbol{p},k}^{[0,1,\mathrm{v}(2)]} = -\sum_{\alpha} \frac{2e_{\alpha}^4}{\pi m_{\alpha}^2} \int \frac{\mathrm{d}k_1\,\mathrm{d}\boldsymbol{p}}{(2\pi)^3} \frac{(\boldsymbol{k}\cdot\boldsymbol{k}_1)^2}{k^2 k_1^2 \varepsilon_k} \left(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0\right)^{-4} \times \delta\left[\omega - \omega_1 - (\boldsymbol{k} - \boldsymbol{k}_1)\cdot\boldsymbol{v}\right] \Phi_{\boldsymbol{p}}^{\alpha} . \tag{53}$$

This concludes the consideration of the contribution of virtual fields in the frame of approximations used in this paper.

3.5 Direct interactions with particle fluctuations

Let us now calculate the contributions of the fluctuations associated with direct perturbations of the propagating wave by the field. They are described according to the second equation of the system of equations (33) by:

$$\delta f_{\boldsymbol{p},k}^{\alpha(0,2)} = \frac{e_{\alpha}}{\mathrm{i}} \left(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0\right)^{-1} \int \mathrm{d}k_1 \frac{1}{k_1}\left(\boldsymbol{k}_1\cdot\frac{\partial}{\partial \boldsymbol{p}}\right) \times \left(E_{k_1}\, \delta f_{\boldsymbol{p},k-k_1}^{\alpha(0,1)} - \left\langle E_{k_1}\, \delta f_{\boldsymbol{p},k-k_1}^{\alpha(0,1)}\right\rangle\right) . \tag{54}$$

It is necessary to substitute in this expression the solution for $\delta f_{\boldsymbol{p},k}^{\alpha(0,1)}$, which for the variables we are considering here has the form:

$$\delta f_{\boldsymbol{p},k-k_1}^{\alpha(0,1)} = \frac{e_{\alpha}}{\mathrm{i}} \left[\omega - \omega_1 - (\boldsymbol{k} - \boldsymbol{k}_1)\cdot\boldsymbol{v} + \mathrm{i}0\right]^{-1} \times \int \mathrm{d}k_2\, \frac{1}{k_2}\left(\boldsymbol{k}_2\cdot\frac{\partial}{\partial \boldsymbol{p}}\right) \times \left(E_{k_2}\, \delta f_{\boldsymbol{p},k-k_1-k_2}^{\alpha(0)} - \left\langle E_{k_2}\, \delta f_{\boldsymbol{p},k-k_1-k_2}^{\alpha(0)}\right\rangle\right) . \tag{55}$$

After substitution there remains in expression (54) only one field which should be taken as a wave field.

Thus we obtain

$$\varepsilon_k^{\mathrm{N},5} = -\sum_{\alpha} \frac{4\pi e_{\alpha}}{\mathrm{i}k E_k^{\sigma}} \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\, \delta f_{\boldsymbol{p},k}^{\alpha(0,2)} = -\sum_{\alpha} \frac{2e_{\alpha}^2}{\pi} \int \frac{\mathrm{d}k_1\,\mathrm{d}\boldsymbol{p}}{(2\pi)^3} \frac{1}{k^2 k_1^2 \varepsilon_{k_1}} \left(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}0\right)^{-1} \times \left(\boldsymbol{k}_1\cdot\frac{\partial}{\partial \boldsymbol{p}}\right)\left[\omega - \omega_1 - (\boldsymbol{k} - \boldsymbol{k}_1)\cdot\boldsymbol{v} + \mathrm{i}0\right]^{-1} \times \left(\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{p}}\right) \delta(\omega_1 - \boldsymbol{k}_1\cdot\boldsymbol{v})\, \Phi_{\boldsymbol{p}}^{\alpha} . \tag{56}$$

This concludes the calculation of all the contributions to the nonlinear permittivity, which govern the collective processes in bremsstrahlung and, as it proves, the collective effects in scattering.

The expressions obtained from the fluctuation theory can be used to demonstrate rigorously that the above clear picture of bremsstrahlung generated by dynamically shielded particles corresponds exactly to the emission of radiation from a plasma due to natural statistical fluctuations.

4. Collective effects in bremsstrahlung of longitudinal waves

4.1 Derivation of bremsstrahlung probabilities from the expressions for nonlinear permittivities for waves interacting with particle and field fluctuations

The nonlinear permittivities derived in the previous section contain complete information on the propagation and absorption of waves in a plasma (more exactly, they contain the information on the linear effects in the wave amplitudes). These effects also describe the inverse bremsstrahlung effects producing wave absorption. We can separate them from other effects if we take into account that the elementary process of particle collisions accompanied by wave emission should satisfy certain conservation laws.

We recall that, in the process under consideration, the particles exchange momentum during collisions. If particle β had a momentum $\boldsymbol{p}'$ before the collision and it has a momentum $\boldsymbol{p}' - \boldsymbol{q}$ after the collision (i.e. it loses the momentum $\boldsymbol{q}$), then particle α should change its momentum by $\boldsymbol{k} - \boldsymbol{q}$, where $\boldsymbol{k}$ is the momentum of the emitted wave during the collision. If the initial momentum of the particle α was $\boldsymbol{p}$ then its final momentum will be $\boldsymbol{p} - \boldsymbol{k} + \boldsymbol{q}$.

The energy conservation law for bremsstrahlung then reads as

$$\varepsilon_{\boldsymbol{p}}^{\alpha} + \varepsilon_{\boldsymbol{p}'}^{\beta} = \varepsilon_{\boldsymbol{p}-\boldsymbol{k}+\boldsymbol{q}}^{\alpha} + \varepsilon_{\boldsymbol{p}'-\boldsymbol{q}}^{\beta} + \omega_{\boldsymbol{k}} , \tag{57}$$

and can be used to separate from the general expressions for nonlinear permittivities only those terms which contain under the sign of the δ-function the expressions corresponding to the energy conservation in the elementary process of bremsstrahlung.

This procedure can be used to determine the probabilities of bremsstrahlung of longitudinal waves, since we have already determined the longitudinal dielectric permittivities above.

A similar procedure can be used to find the probabilities of bremsstrahlung of electromagnetic waves. For this purpose one needs to know the transverse dielectric permittivity.

The procedure described above can be easily generalised to obtain the nonlinear transverse permittivity. The general important features of collective effects in bremsstrahlung can be illustrated by considering the bremsstrahlung of longitudinal waves, bearing in mind that the nonrelativistic particle emits the longitudinal waves with a probability much larger than the probabilities of emission of electromagnetic waves.

We start with the expression $\varepsilon_k^{\mathrm{N},1}$ [see Eqn (39)] and in particular with the contribution of the cubic nonlinear charge density $\rho^{\mathrm{N},3}$ [see Eqn (40)]. In expression $\varepsilon_k^{\mathrm{N},1}$ we substitute the relation (29) for the correlation function of fluctuating fields and the term with ρ^{eff}. We get

$$\varepsilon_k^{\mathrm{N},1} = \varepsilon_k^{\mathrm{N}(3),1} + \varepsilon_k^{\mathrm{N}(2),1}\,, \tag{58}$$

where the superscripts N(3) and N(2) are used to denote contributions of the first term of Eqn (40) containing $\rho^{\mathrm{N},3}$ and of the second term of Eqn (40) containing $\rho^{\mathrm{N},2}$, respectively. We find:

$$\varepsilon_k^{\mathrm{N}(3),1} = \sum_\alpha \frac{2(4\pi)^3 e_\alpha^2}{\mathrm{i}k} \times \int \mathrm{d}\boldsymbol{p}'\,\mathrm{d}k_1\,\frac{\rho^{\mathrm{N},3}_{k_1,k,-k_1}}{k_1^2|\varepsilon_{k_1}|^2}\,\delta(\omega_1 - \boldsymbol{k}_1\cdot\boldsymbol{v}')\,\Phi^\alpha_{\boldsymbol{p}'}\,. \tag{59}$$

For $\rho^{\mathrm{N},3}$ we can use expression (36) assuming that for all the waves the conditions for a Cherenkov resonance are not fulfilled, but the conditions for scattering can be fulfilled (we can speak about scattering only tentatively because the scattering in bremsstrahlung occurs only for virtual fields but not for real waves; nevertheless, refer to the next section with regard to the scattering of propagating waves). Then, by putting $k_1 = -\{\boldsymbol{q},\,\omega_1\}$, we find:

$$\mathrm{Im}\left(\frac{1}{\mathrm{i}}\,\rho^{\mathrm{N},3}_{k_1,k,-k_1}\right) = \sum_\alpha \frac{e_\alpha^4\pi}{2q^2km_\alpha^2}\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\,\frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v})^4} \times \delta\big[\omega-\omega_1-(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}\big]\left[(\boldsymbol{k}-\boldsymbol{q})\cdot\frac{\partial\Phi^\alpha_{\boldsymbol{p}}}{\partial\boldsymbol{p}}\right]\,. \tag{60}$$

This expression can be used to find a contribution to the absorption of waves due to this part of the interaction with the fluctuating fields (this contribution will correspond only to a part of the general expression; nevertheless it is useful to find exactly this part and compare it with the expression for wave damping containing the probability of bremsstrahlung). This contribution will be denoted by $\gamma_{\boldsymbol{k}}^{\mathrm{N},1}$. Note that the total damping rate will be a sum of several contributions,

$$\gamma_{\boldsymbol{k}}^{\mathrm{N}} = \sum_i \gamma_{\boldsymbol{k}}^{\mathrm{N},i}\,, \tag{61}$$

and all other terms in this sum will be determined below. By the definition

$$\gamma_{\boldsymbol{k}}^{\mathrm{N},1} = -\frac{\mathrm{Im}\,\varepsilon_k^{\mathrm{N}(3),1}}{(\partial\varepsilon_k/\partial\omega)_{\omega=\omega_k}} = \sum_{\alpha,\beta}\frac{8\pi^4 e_\alpha^4 e_\beta^2}{m_\alpha^2(\partial\varepsilon_k/\partial\omega)_{\omega=\omega_k}} \times \int\frac{\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{p}'}{(2\pi)^6}\,\frac{\delta\big[\omega_{\boldsymbol{k}}-\boldsymbol{q}\cdot\boldsymbol{v}'-(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}\big]}{q^2|\varepsilon_{q,\boldsymbol{q}\cdot\boldsymbol{v}'}|^2(\omega_{\boldsymbol{k}}-\boldsymbol{k}\cdot\boldsymbol{v})^4} \times\left(\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{kq}\right)^2\left[(\boldsymbol{k}-\boldsymbol{q})\cdot\frac{\partial\Phi^\alpha_{\boldsymbol{p}}}{\partial\boldsymbol{p}}\right]\Phi^\beta_{\boldsymbol{p}'}\,. \tag{62}$$

If we assume that this contribution corresponds to the first term in expression (21), we can calculate the corresponding contributions to the bremsstrahlung probability. [We shall later obtain the remaining terms in expression (21).] It is interesting that the probability 'term' found in this way (which generally need not be expressible in terms of the square of the modulus of a matrix element) is nevertheless described by the probability:

$$w^{\alpha,\beta}_{\boldsymbol{p},\boldsymbol{p}'}(\boldsymbol{k},\boldsymbol{q}) = \frac{16\pi\,e_\alpha^4e_\beta^2(2\pi)^3}{m_\alpha^2(\partial\varepsilon_k/\partial\omega)_{\omega=\omega_k}} \times\frac{\delta[\omega_{\boldsymbol{k}}-\boldsymbol{q}\cdot\boldsymbol{v}'-(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}]}{q^2|\varepsilon_{q,\boldsymbol{q}\cdot\boldsymbol{v}'}|^2(\omega_{\boldsymbol{k}}-\boldsymbol{k}\cdot\boldsymbol{v})^4}\left(\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{kq}\right)^2\,. \tag{63}$$

This expression is the same as that given by formulas (18) and (20) , which take into account the screening of the fields of particle β but ignore completely the oscillations of the screening shell. Let us note that expression (20) was 'derived' from expression (19) by the use of a qualitative physical argument and that its exact proof was not given. Eqn (63) provides this proof.

It is important that expression (63) contains only a part of the total effect. Apart from the absence in it of the contribution of the emission due to the oscillations of the screening shell of particle β, all effects related to the screening shell of the particle α are also absent. But even the 'correctness' of the part described by Eqn (63) is not proved, since we obtained only the first term in the expression for the damping decrement (21) which is proportional to the derivative of the distribution function of the particle α, but did not find the second term, which contains the derivative of the distribution function of particles β.

To prove the existence of this term in the general expressions for nonlinear permittivity, let us consider expression (53) for $\varepsilon_k^{\mathrm{N},4}$. Let us write the imaginary part of $1/\varepsilon_{k_1}$, with $k_1 = \{\boldsymbol{q},\omega_1\}$:

$$\mathrm{Im}\,\frac{1}{\varepsilon_{k_1}} = -\mathrm{Im}\,\frac{\varepsilon_{k_1}}{|\varepsilon_{k_1}|^2} = \sum_\beta\frac{4\pi^2e_\beta^2}{q^2}\int\frac{\mathrm{d}\boldsymbol{p}'}{(2\pi)^3}\,\frac{\delta(\omega_1-\boldsymbol{q}\cdot\boldsymbol{v}')}{|\varepsilon_{q,\boldsymbol{q}\cdot\boldsymbol{v}'}|^2}\left(\boldsymbol{q}\cdot\frac{\partial\Phi^\beta_{\boldsymbol{p}'}}{\partial\boldsymbol{p}'}\right)\,. \tag{64}$$

We then find the following contribution to wave damping after substituting expression (64) into expression (53):

$$\gamma_{\boldsymbol{k}}^{\mathrm{N},2} = -\frac{\mathrm{Im}\,\varepsilon_k^{\mathrm{N},4}}{(\partial\varepsilon_k/\partial\omega)_{\omega=\omega_k}} = \sum_{\alpha,\beta}\frac{8\pi e_\alpha^4e_\beta^2}{m_\alpha^2(\partial\varepsilon_k/\partial\omega)_{\omega=\omega_k}}\int\frac{\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{p}'\,\mathrm{d}\boldsymbol{q}}{(2\pi)^6} \times\frac{\delta[\omega_{\boldsymbol{k}}-\boldsymbol{q}\cdot\boldsymbol{v}'-(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}]}{q^2|\varepsilon_{q,\boldsymbol{q}\cdot\boldsymbol{v}'}|^2(\omega_{\boldsymbol{k}}-\boldsymbol{k}\cdot\boldsymbol{v})^4}\left(\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{kq}\right)^2\left(\boldsymbol{q}\cdot\frac{\partial\Phi^\beta_{\boldsymbol{p}'}}{\partial\boldsymbol{p}'}\right)\Phi^\alpha_{\boldsymbol{p}}\,. \tag{65}$$

We then find the same expression (63) for the probability after comparing Eqn (65) with the second term in the expression (21). Thus we have proved that this probability does in fact enter in the total expression for the damping [Eqn (21)].

Let us consider the other terms in the nonlinear permittivity, which were not taken into account up to now. We return to expression $\varepsilon^{\mathrm{N}(2),1}$ which describes the contributions in $\varepsilon^{\mathrm{N},1}$ due to quadratic nonlinear charge densities. It is not difficult to find by integrating by parts the general expressions [Eqn (35)] for quadratic nonlinear charge densities that the following relation is valid in the absence of Cherenkov resonance:

$$\rho^{\mathrm{N},2}_{k,-k_1} = \frac{|\boldsymbol{k}-\boldsymbol{k}_1|}{k}\,\rho^{\mathrm{N},2}_{k_1,k-k_1}\,. \tag{66}$$

We can write the last relation in the form:

$$\rho^{\mathrm{N},2}_{k,-k_1} = \frac{|\boldsymbol{k}-\boldsymbol{k}_1|}{k}\,\rho^{\mathrm{N},2*}_{k_1,k-k_1} + 2\mathrm{i}\,\frac{|\boldsymbol{k}-\boldsymbol{k}_1|}{k}\,\mathrm{Im}\,\rho^{\mathrm{N},2}_{k_1,k-k_1}\,. \tag{67}$$

We then take into account only that part of the expression $\varepsilon^{N(2),1}$ which corresponds to the first term in Eqn (67) and denote its contribution to the damping of the waves by $\gamma_{\boldsymbol{k}}^{N,3}$. Then

$$\gamma_{\boldsymbol{k}}^{N,3} = \frac{(8\pi)^2}{k^2(\partial\varepsilon_k/\partial\omega)_{\omega=\omega_k}} \times \int \mathrm{d}k_1\, |E^{(0)}|^2_{k_1}\, |\rho^{N,2}_{k_1,k-k_1}|^2 \,\mathrm{Im}\,\frac{1}{\varepsilon_{k-k_1}}\,. \tag{68}$$

The imaginary part of the linear dielectric permittivity, which enters in expression (68), can be written in the form:

$$\mathrm{Im}\,\frac{1}{\varepsilon_{k-k_1}} = \sum_\alpha \frac{4\pi^2 e_\alpha^2}{(\boldsymbol{k}-\boldsymbol{q})^2} \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3} \times \frac{\delta[\omega-\omega_1-(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}]}{|\varepsilon_{\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}|^2}\left[(\boldsymbol{k}-\boldsymbol{q})\cdot\frac{\partial\Phi^\alpha_{\boldsymbol{p}}}{\partial\boldsymbol{p}}\right]. \tag{69}$$

Finally expression (68) will have the form:

$$\gamma_{\boldsymbol{k}}^{N,3} = \sum_{\alpha,\beta} \frac{(8\pi)^2 e_\alpha^2 e_\beta^2}{(\partial\varepsilon_k/\partial\omega)_{\omega=\omega_k}} \int \frac{\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{p}'\,\mathrm{d}\boldsymbol{q}}{(2\pi)^6} \times \frac{|\rho^{N,2}_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}';\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}|^2}{k^2q^2|\varepsilon_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}'}|^2|\varepsilon_{\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}|^2} \times \delta[\omega_k-\boldsymbol{q}\cdot\boldsymbol{v}'-(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}]\left[\frac{(\boldsymbol{k}-\boldsymbol{q})}{|\boldsymbol{k}-\boldsymbol{q}|^2}\cdot\frac{\partial\Phi^\alpha_{\boldsymbol{p}}}{\partial\boldsymbol{p}}\right]\Phi^\beta_{\boldsymbol{p}'}\,. \tag{70}$$

The last expression is symmetric for the substitutions $\alpha \leftrightarrow \beta$, $\boldsymbol{q} \leftrightarrow \boldsymbol{k}-\boldsymbol{q}$, and $\boldsymbol{p} \leftrightarrow \boldsymbol{p}'$. Therefore given concrete values for α and β, it is necessary in expression (70) to sum over α and β to take into account the term in which the values for α and β are equal to those we are considering, and the term in which the value for α corresponds to the value for β we are considering, and vice versa. These two terms resulting from the aforementioned symmetry give the two terms in expression (21) with the following value of the probability:

$$w^{\alpha,\beta}_{\boldsymbol{p},\boldsymbol{p}'}(\boldsymbol{k},\boldsymbol{q}) = \frac{16\pi e_\alpha^2 e_\beta^2 (2\pi)^3(8\pi)^2}{(\partial\varepsilon_k/\partial\omega)_{\omega=\omega_k}} \times \frac{|\rho^{N,2}_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}';\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}|^2}{k^2q^2|\varepsilon_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}'}|^2\,|\varepsilon_{\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}|^2} \times \frac{1}{|\boldsymbol{k}-\boldsymbol{q}|^2}\,\delta[\omega_k-\boldsymbol{q}\cdot\boldsymbol{v}'-(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}]\,. \tag{71}$$

It is easily seen that this formula can be written in the form prescribed by expression (18) with the matrix element given by Eqn (24). Thus expression (71) describes the bremsstrahlung caused by disturbances of the polarisation charges of colliding particles and expression (18) is thus proved.

There takes place a natural interference of the two mechanisms of bremsstrahlung considered here, and the total intensity of bremsstrahlung is not equal to the sum of the intensity of the usual bremsstrahlung (due to changes in trajectories of colliding particles) and the bremsstrahlung due to perturbations of the polarisation shells of the colliding particles. One should sum the matrix elements but not the intensities. This means that the total intensity of bremsstrahlung should contain the square of the sum of matrix elements [Eqns (20), (24), and (25)].

It should be mentioned that, in deriving expression (63) from expressions (62) and (65), one should take into account the term with $\alpha = \beta$ in the sum over α. In this term the substitution $\boldsymbol{k}-\boldsymbol{q}$ for $\boldsymbol{q}$ in the argument of the δ-function, which describes the energy conservation law in bremsstrahlung, is converted to the form corresponding to relation (18) with a probability containing the matrix element (25). Thus we obtained, in probability terms, the terms with squares of all of the three matrix elements.

Let us prove that the probability is indeed described by expression (18), which contains the square of the matrix element M, where

$$M = M^\alpha + M^\beta + M^{\alpha,\beta}\,, \tag{72}$$

and M^α corresponds to expression (20). Let us obtain all the interference terms. First let us consider the contribution of the neglected second term of expression (67) and denote its contribution to the damping of waves by $\gamma_{\boldsymbol{k}}^{N,4}$. We find

$$\gamma_{\boldsymbol{k}}^{N,4} = \frac{2(8\pi)^2}{k(\partial\varepsilon_k/\partial\omega)_{\omega=\omega_k}} \times \int \mathrm{d}k_1\,|E^{(0)}|^2_{k_1}\,\mathrm{Im}\,\frac{\rho^{N,2}_{k,-k_1}}{|\boldsymbol{k}-\boldsymbol{k}_1|}\,\mathrm{Re}\,\frac{\rho^{N,2}_{k_1,k-k_1}}{\varepsilon_{k-k_1}}\,. \tag{73}$$

We then get from definition (35) for the quadratic nonlinear charge density:

$$\mathrm{Im}\,\rho^{N,2}_{k,-k_1} = \sum_\alpha \frac{e_\alpha^3\pi}{2m_\alpha}\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\,\frac{\boldsymbol{k}\cdot\boldsymbol{k}_1}{kk_1}\,\frac{1}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v})^2} \times \delta[\omega-\omega_1-(\boldsymbol{k}-\boldsymbol{k}_1)\cdot\boldsymbol{v}]\left[(\boldsymbol{k}-\boldsymbol{k}_1)\cdot\frac{\partial\Phi^\alpha_{\boldsymbol{p}}}{\partial\boldsymbol{p}}\right]. \tag{74}$$

By substituting this expression into Eqn (73), we find

$$\gamma_{\boldsymbol{k}}^{N,4} = \sum_{\alpha,\beta}\frac{8\pi e_\alpha^2 e_\beta^2(2\pi)^3}{(\partial\varepsilon_k/\partial\omega)_{\omega=\omega_k}}\int\frac{\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{p}'\,\mathrm{d}\boldsymbol{q}}{(2\pi)^9} \times \delta[\omega_k-\boldsymbol{q}\cdot\boldsymbol{v}'-(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}]\,\Phi^\beta_{\boldsymbol{p}'}\left[(\boldsymbol{k}-\boldsymbol{q})\cdot\frac{\partial\Phi^\alpha_{\boldsymbol{p}}}{\partial\boldsymbol{p}}\right] \times 2\,\mathrm{Re}\left[-\frac{8\pi\rho^{N,2}_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}';\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}}{kq|\boldsymbol{k}-\boldsymbol{q}|\,\varepsilon_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}'}\,\varepsilon_{\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}}\right] \times \frac{e_\alpha\boldsymbol{k}\cdot\boldsymbol{q}}{m_\alpha q^2k\,\varepsilon_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}'}\,(\omega_{\boldsymbol{k}}-\boldsymbol{k}\cdot\boldsymbol{v})^2}\,. \tag{75}$$

It is not difficult to find that expression (75) contains a part of the interference terms of the square of the total matrix element M:

$$|M|^2 = |M^\alpha|^2 + |M^\beta|^2 + |M^{\alpha,\beta}|^2 + 2\,\mathrm{Re}\{M^\alpha M^{\alpha,\beta}\} + 2\,\mathrm{Re}\{M^\beta M^{\alpha,\beta}\} + 2\,\mathrm{Re}\{M^\alpha M^\beta\}\,. \tag{76}$$

The first three terms in Eqn (76), corresponding to the squares of matrix elements, have already been obtained above. Expression (75) contains a part of the two terms in Eqn (76), namely, a part of $2\,\mathrm{Re}\{M^{\alpha}M^{\alpha,\beta}\}$ and a part of $2\,\mathrm{Re}\{M^{\beta}M^{\alpha,\beta}\}$.

Indeed, for those terms in the sum over α and β in relationship (75) for which the values of α and β are identical with the α and β of interest to us, relationship (75) does contain $2\,\mathrm{Re}\{M^{\beta}M^{\alpha,\beta}\}$, but then this relationship contributes only the first term in expression (21) [the second term with $\boldsymbol{q}\cdot(\partial\Phi^{\beta}_{p'}/\partial\boldsymbol{p}')\,\Phi^{\alpha}_{p}$ is absent]. For the same terms in the sum over α and β in relationship (75), for which the values of α and β are identical with the α and β of interest to us, relationship (75) again contains $2\,\mathrm{Re}\{M^{\beta}M^{\alpha,\beta}\}$, but it then contributes only the second term in expression (21) [the first term with $(\boldsymbol{k}-\boldsymbol{q})\cdot(\partial\Phi^{\alpha}_{p}/\partial\boldsymbol{p})\,\Phi^{\beta}_{p'}$ is absent]. These two missing terms are contained in $\varepsilon^{\mathrm{N},3}_{k}$ [see expression (51)] and partly in $\varepsilon^{\mathrm{N},2}_{k}$ [see expression (48)].

The nonlinear permittivities mentioned above can be written in the following form (integrating by parts over the particle momentum):

$$\varepsilon^{\mathrm{N},3}_{k}=\sum_{\alpha}\frac{16e^{3}_{\alpha}}{m_{\alpha}}\int\frac{\mathrm{d}k_{1}\,\mathrm{d}\boldsymbol{p}}{(2\pi)^{3}}\,\frac{\boldsymbol{k}\cdot\boldsymbol{k}_{1}}{kk^{2}_{1}|\boldsymbol{k}-\boldsymbol{k}_{1}|\,\varepsilon_{k_{1}}\varepsilon_{k_{1}-k}}\times\frac{\rho^{\mathrm{N},2}_{k,k_{1}-k}}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v})^{2}}\,\delta\big[\omega-\omega_{1}-(\boldsymbol{k}-\boldsymbol{k}_{1})\cdot\boldsymbol{v}\big]\,\Phi^{\alpha}_{p}\,,\qquad(77)$$

$$\varepsilon^{\mathrm{N},2}_{k}=\sum_{\beta}\frac{16e^{3}_{\beta}}{m_{\beta}}\int\frac{\mathrm{d}k_{1}\,\mathrm{d}\boldsymbol{p}'}{(2\pi)^{3}}\,\frac{\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{k}_{1})}{k_{1}k^{2}|\boldsymbol{k}-\boldsymbol{k}_{1}|\,\varepsilon_{k-k_{1}}\varepsilon_{k_{1}}}\times\frac{\rho^{\mathrm{N},2}_{k_{1},k-k_{1}}}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v}')^{2}}\,\delta(\omega_{1}-\boldsymbol{k}_{1}\cdot\boldsymbol{v}')\,\Phi^{\beta}_{p'}\,.\qquad(78)$$

Let us substitute in expression (77) the first term from the relation

$$\rho^{\mathrm{N},2}_{k,k_{1}-k}=\frac{k_{1}}{k}\,\rho^{\mathrm{N},2*}_{k_{1},k-k_{1}}+2\mathrm{i}\,\frac{k_{1}}{k}\,\mathrm{Im}\,\rho^{\mathrm{N},2}_{k_{1},k-k_{1}}\,,\qquad(79)$$

and let us denote its contribution to the damping due to $\mathrm{Im}\,(1/\varepsilon_{k_{1}})$ by $\gamma^{\mathrm{N},5}_{k}$. We will consider separately the term with $\mathrm{Im}\,\rho^{\mathrm{N},2}_{k_{1},k-k_{1}}$ [note that its double contribution to Eqn (79) plus its negative value from $\rho^{\mathrm{N},2*}$ give its value without the coefficient 2]. In expression (78) we take into account only the imaginary part of $1/\varepsilon_{k-k_{1}}$, leaving the term with $\mathrm{Im}\,\rho^{\mathrm{N},2}_{k_{1},k-k_{1}}$ for subsequent consideration. Let us denote by $\gamma^{\mathrm{N},6}_{k}$ the corresponding contribution to the wave damping.

We then obtain the two contributions in the form:

$$\gamma^{\mathrm{N},5}_{k}=-\frac{\mathrm{Im}\,\varepsilon^{\mathrm{N},3}_{k}}{(\partial\varepsilon_{k}/\partial\omega)_{\omega=\omega_{k}}}=\sum_{\alpha,\beta}\frac{8\pi e^{2}_{\alpha}e^{2}_{\beta}(2\pi)^{3}}{(\partial\varepsilon_{k}/\partial\omega)_{\omega=\omega_{k}}}\times\int\frac{\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{p}'\,\mathrm{d}\boldsymbol{q}}{(2\pi)^{9}}\,\delta\big[\omega_{k}-\boldsymbol{q}\cdot\boldsymbol{v}'-(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}\big]\times\Phi^{\alpha}_{p}\left(\boldsymbol{q}\cdot\frac{\partial\Phi^{\beta}_{p'}}{\partial\boldsymbol{p}'}\right)\left[-\frac{8\pi\,\rho^{\mathrm{N},2*}_{q,\boldsymbol{q}\cdot\boldsymbol{v}';k-q,(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}}{kq\,\varepsilon^{*}_{q,\boldsymbol{q}\cdot\boldsymbol{v}'}|\boldsymbol{k}-\boldsymbol{q}|\varepsilon^{*}_{k-q,(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}}\right]\times\frac{e_{\alpha}\boldsymbol{k}\cdot\boldsymbol{q}}{kq^{2}m_{\alpha}\varepsilon_{q,\boldsymbol{q}\cdot\boldsymbol{v}'}(\omega_{k}-\boldsymbol{k}\cdot\boldsymbol{v})^{2}}\,,\qquad(80)$$

$$\gamma^{\mathrm{N},6}_{k}=-\frac{\mathrm{Im}\,\varepsilon^{\mathrm{N},2}_{k}}{(\partial\varepsilon_{k}/\partial\omega)_{\omega=\omega_{k}}}=\sum_{\alpha,\beta}\frac{8\pi e^{2}_{\alpha}e^{2}_{\beta}(2\pi)^{3}}{(\partial\varepsilon_{k}/\partial\omega)_{\omega=\omega_{k}}}\times\int\frac{\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{p}'\,\mathrm{d}\boldsymbol{q}}{(2\pi)^{9}}\,\delta\big[\omega_{k}-\boldsymbol{q}\cdot\boldsymbol{v}'-(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}\big]\times\Phi^{\beta}_{p'}\left[(\boldsymbol{k}-\boldsymbol{q})\cdot\frac{\partial\Phi^{\alpha}_{p}}{\partial\boldsymbol{p}}\right]\left[-\frac{8\pi\,\rho^{\mathrm{N},2}_{q,\boldsymbol{q}\cdot\boldsymbol{v}';k-q,(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}}{kq\,\varepsilon_{q,\boldsymbol{q}\cdot\boldsymbol{v}'}|\boldsymbol{k}-\boldsymbol{q}|\varepsilon_{k-q,(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}}\right]\times\frac{e_{\beta}\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{q})}{k(\boldsymbol{k}-\boldsymbol{q})^{2}m_{\beta}\varepsilon^{*}_{k-q,(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}(\omega_{k}-\boldsymbol{k}\cdot\boldsymbol{v}')^{2}}\,.\qquad(81)$$

From the last expression it is easy to find that expression (80) contains the product of $M^{\alpha,\beta*}$ by M^{α}. By substituting β for α (extracting the corresponding terms from the sum over α and β) and by the substitution $\boldsymbol{q}\leftrightarrow\boldsymbol{k}-\boldsymbol{q}$ it is easy to show that expression (80) contains the product of $M^{\alpha,\beta*}$ by M^{β}. It is easy to show also that expression (81) contains the product of $M^{\alpha,\beta}$ by $M^{\beta*}$ and by substituting β for α (extracting the corresponding terms from the sum over α and β) and by the substitution $\boldsymbol{q}\leftrightarrow\boldsymbol{k}-\boldsymbol{q}$ it is easy to show that Eqn (81) contains the product of $M^{\alpha,\beta}$ by $M^{\alpha*}$.

Thus we have proved the existence in wave absorption of all the interference terms corresponding to the contribution of $2\mathrm{Re}\,M^{\alpha,\beta}(M^{\alpha}+M^{\beta})$ in the square of the total matrix element. What is left now is to prove the existence of the interference terms corresponding to the contribution of $2\mathrm{Re}\,M^{\alpha}M^{\beta}$ in the square of the total matrix element. This contribution comes from the terms in $\varepsilon^{\mathrm{N},3}_{k}$ and $\varepsilon^{\mathrm{N},2}$ containing $\mathrm{Im}\,\rho^{\mathrm{N},2}_{k,k_{1}-k}$ not used up to now.

By using the relation,

$$\mathrm{Im}\,\rho^{\mathrm{N},2}_{k,k_{1}-k}=-\sum_{\beta}\frac{e^{3}_{\beta}\pi}{2k|\boldsymbol{k}-\boldsymbol{k}_{1}|}\int\frac{\mathrm{d}\boldsymbol{p}'}{(2\pi)^{3}}\,\frac{\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{k}_{1})}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v}')^{2}}\times\delta(\omega_{1}-\boldsymbol{k}_{1}\cdot\boldsymbol{v}')\left(\boldsymbol{k}\cdot\frac{\partial\Phi^{\beta}_{p'}}{\partial\boldsymbol{p}'}\right)-\sum_{\alpha}\frac{e^{3}_{\alpha}\pi}{2k|\boldsymbol{k}-\boldsymbol{k}_{1}|}\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^{3}}\,\frac{\boldsymbol{k}\cdot\boldsymbol{k}_{1}}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v})^{2}}\times\delta\big[\omega-\omega_{1}-(\boldsymbol{k}-\boldsymbol{k}_{1})\cdot\boldsymbol{v}\big]\left[(\boldsymbol{k}-\boldsymbol{k}_{1})\cdot\frac{\partial\Phi^{\alpha}_{p}}{\partial\boldsymbol{p}}\right],\qquad(82)$$

we will take into account the first and second terms of Eqn (82) in Eqns (77) and (78), respectively, to consider only a δ-function which describes the energy conservation in bremsstrahlung. Then with the same substitution of β for α as described above we find that their imaginary parts give a contribution to damping that is exactly equal to the contribution $2\mathrm{Re}\,M^{\alpha}M^{\beta}$ in the square of the matrix element. Denoting this contribution as $\gamma^{\mathrm{N},7}_{k}$, we find:

$$\gamma^{\mathrm{N},7}_{k}=\sum_{\alpha,\beta}\frac{8\pi e^{2}_{\alpha}e^{2}_{\beta}(2\pi)^{3}}{(\partial\varepsilon_{k}/\partial\omega)_{\omega=\omega_{k}}}\int\frac{\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{p}'\,\mathrm{d}\boldsymbol{q}}{(2\pi)^{9}}\times\delta\big[\omega_{k}-\boldsymbol{q}\cdot\boldsymbol{v}'-(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}\big]\,\Phi^{\alpha}_{p}\left(\boldsymbol{q}\cdot\frac{\partial\Phi^{\beta}_{p'}}{\partial\boldsymbol{p}'}\right)\times2\mathrm{Re}\left[\frac{e_{\alpha}\boldsymbol{k}\cdot\boldsymbol{q}}{kq^{2}m_{\alpha}\,\varepsilon_{q,\boldsymbol{q}\cdot\boldsymbol{v}'}(\omega_{k}-\boldsymbol{k}\cdot\boldsymbol{v})^{2}}\right]\times\frac{e_{\beta}\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{q})}{k(\boldsymbol{k}-\boldsymbol{q})^{2}m_{\beta}\,\varepsilon_{k-q,(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}(\omega_{k}-\boldsymbol{k}\cdot\boldsymbol{v}')^{2}}\,.\qquad(83)$$

This completes the construction of the general theory of bremsstrahlung with all the collective effects taken into account. Using the approach of fluctuations in a plasma in its simplest form, we performed the calculations which illustrate the major points and therefore we used the simplest example of the bremsstrahlung of longitudinal waves. All results obtained here are easily transformed to the case of waves with arbitrary polarisation.

4.2 Analysis of matrix elements of the bremsstrahlung of longitudinal waves

The simplest case is the case most similar to the vacuum case, when the collective effects are unimportant and which generally is not valid for longitudinal waves. But this case is useful for comparison with the case in which the collective effects are taken into account.

To consider this case, let us neglect the matrix element $M^{\alpha,\beta}$, and the Doppler corrections $\boldsymbol{k}\cdot\boldsymbol{v}$, as compared with the frequency $\omega_{\boldsymbol{k}}$, in the matrix elements M^{α} and M^{β}. We will also suppose that the wave momentum is small compared with the transferred momentum. Then we have

$$M^{\alpha}+M^{\beta}\approx\frac{1}{q\omega_{\boldsymbol{k}}^{2}}\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{kq}\left(\frac{e_{\alpha}}{m_{\alpha}}-\frac{e_{\beta}}{m_{\beta}}\right), \tag{84}$$

which leads to the known result, i.e. the absence of emission for $e_{\alpha}/m_{\alpha}-e_{\beta}/m_{\beta}=0$.

It is clear that this result cannot hold if the collective effects are taken into account in bremsstrahlung; first, because the mentioned sum is not zero if one takes into account the screening of the fields of the colliding particles and, second, because the matrix element $M^{\alpha,\beta}$ for collisions of the same particles is not zero. To prove this statement let us write the matrix elements M^{α} and M^{β}, neglecting the Doppler corrections to the frequencies emitted:

$$M^{\alpha}\approx\frac{e_{\alpha}}{m_{\alpha}\omega_{\boldsymbol{k}}^{2}q\,\varepsilon_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}'}}\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{kq}, \tag{85}$$

$$M^{\beta}\approx\frac{e_{\beta}}{m_{\beta}\omega_{\boldsymbol{k}}^{2}|\boldsymbol{k}-\boldsymbol{q}|\,\varepsilon_{\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}}\frac{\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{q})}{k|\boldsymbol{k}-\boldsymbol{q}|}. \tag{86}$$

Even for $k\ll q$ the zero value of the sum of the matrix elements occurs only for particle velocities much less than the average thermal velocity when the Debye approximation can be used for the screening. But the matrix element due to oscillations of the polarisation charges does not tend to zero even for collisions of very heavy particles when the matrix elements M^{α} and M^{β} are small due to the large values of particle masses.

Indeed, let us find the approximate expressions for the matrix element $M^{\alpha,\beta}$ assuming that: (1) the main contribution to nonlinear charge densities comes from plasma electrons (this is a good approximation for high-frequency waves such as Langmuir waves); and (2) it is possible to neglect the Doppler corrections with respect to the frequency of the emitted wave. Then in the general expression for the nonlinear response (the electron charge is $-e$),

$$\begin{aligned}\rho^{\mathrm{N},2}_{\boldsymbol{k}_1,\boldsymbol{k}-\boldsymbol{k}_1}&=\frac{e^{3}}{2k_{1}|\boldsymbol{k}-\boldsymbol{k}_{1}|}\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^{3}}\frac{1}{(\omega_{\boldsymbol{k}}-\boldsymbol{k}\cdot\boldsymbol{v})}\\&\times\left\{\left(\boldsymbol{k}_{1}\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\left[\omega-\omega_{1}-(\boldsymbol{k}-\boldsymbol{k}_{1})\cdot\boldsymbol{v}+\mathrm{i}0\right]^{-1}\right.\\&\times\left[(\boldsymbol{k}-\boldsymbol{k}_{1})\cdot\frac{\partial}{\partial\boldsymbol{p}}\right]+\left[(\boldsymbol{k}-\boldsymbol{k}_{1})\cdot\frac{\partial}{\partial\boldsymbol{p}}\right]\\&\times\left.\left(\omega_{1}-\boldsymbol{k}_{1}\cdot\boldsymbol{v}+\mathrm{i}0\right)^{-1}\left(\boldsymbol{k}_{1}\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\right\}\Phi^{\mathrm{e}}_{\boldsymbol{p}},\end{aligned} \tag{87}$$

it is possible to use integration by parts for the first derivative, with respect to the momentum, and then neglect the Doppler corrections compared with the frequency and obtain

$$\begin{aligned}\rho^{\mathrm{N},2}_{\boldsymbol{k}_1,\boldsymbol{k}-\boldsymbol{k}_1}&\approx-\frac{e\,\boldsymbol{k}\cdot\boldsymbol{k}_{1}|\boldsymbol{k}-\boldsymbol{k}_{1}|}{8\pi k_{1}m_{\mathrm{e}}\omega_{\boldsymbol{k}}^{2}}\left(\varepsilon^{\mathrm{e}}_{\boldsymbol{k}-\boldsymbol{k}_1}-1\right)\\&\quad-\frac{e\,\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{k}_{1})k_{1}}{8\pi|\boldsymbol{k}-\boldsymbol{k}_{1}|m_{\mathrm{e}}\omega_{\boldsymbol{k}}^{2}}\left(\varepsilon^{\mathrm{e}}_{\boldsymbol{k}_1}-1\right),\end{aligned} \tag{88}$$

where ε^{e} is the electron part of dielectric permittivity (i.e. that part which is obtained from the general expression for the permittivity in the limit when the masses of all particles except electrons tend to infinity). Note that the total ε contains the contributions of all plasma particles. Using the result (88) we can write the matrix element $M^{\alpha,\beta}$ in a rather simple form:

$$\begin{aligned}M^{\alpha,\beta}&\approx\frac{e}{m_{\mathrm{e}}\omega_{\boldsymbol{k}}^{2}q\,\varepsilon_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}'}}\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{kq}\frac{\varepsilon^{\mathrm{e}}_{\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}-1}{\varepsilon_{\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}}\\&\quad+\frac{e}{m_{\mathrm{e}}\omega_{\boldsymbol{k}}^{2}|\boldsymbol{k}-\boldsymbol{q}|\,\varepsilon_{\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}}\frac{\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{q})}{k|\boldsymbol{k}-\boldsymbol{q}|}\frac{\varepsilon^{\mathrm{e}}_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}'}-1}{\varepsilon_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}'}}.\end{aligned} \tag{89}$$

The first term in expression (89) differs from M^{α} [see expression (85)] by a factor which contains the ratio of the electron part of the permittivity minus unity (i.e. the electron polarisability) to the total permittivity (both permittivities apply to the wavenumber and the frequency governing the change in the momentum and energy of particle β). The second term in expression (89) differs from M^{β} [see expression (86)] by a factor which again contains the ratio of the electron part of the permittivity minus unity (i.e. the electron polarisability) to the total permittivity (both apply to the wavenumber and frequency which govern the change in the momentum and energy of particle α).

This means in particular that the bremsstrahlung due to the disturbances of the polarisation charge can be of the same order of magnitude as the usual bremsstrahlung. Moreover, in the sum of the matrix elements, a substantial cancellation of particular contributions can occur. This can be seen from the fact that for electrons $e_{\alpha}=-e$, $m_{\alpha}=m_{\mathrm{e}}$. For collisions of heavy particles the main contribution is made by $M^{\alpha,\beta}$.

4.3 Bremsstrahlung of longitudinal waves in electron – electron collisions

Let us show that the collective effects change the cross-sections of bremsstrahlung in electron – electron collisions substantially. The total matrix element for bremsstrahlung in electron – electron collisions can be found easily from the expressions already obtained:

$$\begin{aligned}M&=-\frac{e}{m_{\mathrm{e}}\omega_{\boldsymbol{k}}^{2}\,\varepsilon_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}'}}\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{kq}\frac{\varepsilon^{\mathrm{i}}_{\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}}{\varepsilon_{\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}}\\&\quad-\frac{e}{m_{\mathrm{e}}\omega_{\boldsymbol{k}}^{2}\,\varepsilon_{\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}}\frac{\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{q})}{k|\boldsymbol{k}-\boldsymbol{q}|}\frac{\varepsilon^{\mathrm{i}}_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}'}}{\varepsilon_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}'}},\end{aligned} \tag{90}$$

where ε^{i} is the ion dielectric permittivity

$$\begin{aligned}\varepsilon^{\mathrm{i}}_{\boldsymbol{k},\omega}&=1+\sum_{\mathrm{i}}\frac{4\pi e_{\mathrm{i}}^{2}}{k^{2}}\\&\times\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^{3}}\left(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+\mathrm{i}0\right)^{-1}\left(\boldsymbol{k}\cdot\frac{\partial}{\partial\boldsymbol{p}}\right)\Phi^{\mathrm{i}}_{\boldsymbol{p}},\end{aligned} \tag{91}$$

and the summation in Eqn (91) is performed over all types of ions i.

When the electron velocities are much larger than the average thermal ion velocity and for

$$q \gg \frac{\omega_{\rm pi}}{v} \tag{92}$$

($\omega_{\rm pi} = \sqrt{4\pi e_{\rm i}^2 n_{\rm i}/m_{\rm i}}$ is the ion plasma frequency and v is the absolute value of the electron velocity), we can put $\varepsilon^{\rm i} = 0$ in Eqn (90). Then:

$$M \approx -e\frac{\boldsymbol{k}\cdot\boldsymbol{q}/q^2 + \boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{q})/(\boldsymbol{k}-\boldsymbol{q})^2}{k m_{\rm e}\omega_{\boldsymbol{k}}^2\, \varepsilon^{\rm e}_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}'}\, \varepsilon^{\rm e}_{\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}}\,. \tag{93}$$

Let us consider the limit $k \ll q$; then the energy conservation law in an elementary process of bremsstrahlung gives

$$q > \frac{\omega_{\rm pe}}{u}\,, \tag{94}$$

where $u = |v - v'|$ is the relative velocity of the two colliding electrons so that for u of the order of v, the validity of Eqn (92) is confirmed by relation (94).

It should be noted that in the case considered here the velocity of the particles relative to the plasma plays a substantial role. This is the reason why in relation (92) the value of the electron velocity is present.

The neglect of the Doppler corrections implies that $k \ll \omega_{\rm pe}/v$, which for u of the order of v — in combination with equality (94) — gives $k \ll q$.

In the limit (when one neglects completely the collective effects) the matrix element is equal to

$$M_{\rm noncoll} = -\frac{e}{k m_{\rm e}\omega_{\boldsymbol{k}}^2}\left[\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{q^2} + \frac{\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{q})}{(\boldsymbol{k}-\boldsymbol{q})^2}\right]. \tag{95}$$

We will write the power emitted in collisions of two electrons in accordance with Eqns (16) and (18) in the form ($k = |\boldsymbol{k}|$):

$$Q_{\boldsymbol{p}}^{\rm e,e} = \int\frac{{\rm d}k\,{\rm d}\boldsymbol{p}'}{(2\pi)^3}\,\omega_{\rm pe}\,Q^{\rm e,e}_{\boldsymbol{p},\boldsymbol{p}'}(k)\,\Phi^{\rm e}_{\boldsymbol{p}'}\,, \tag{96}$$

where, for the case of Langmuir waves,

$$Q^{\rm e,e}_{\boldsymbol{p},\boldsymbol{p}'}(k) = 8\pi e^4\int\frac{{\rm d}\boldsymbol{q}\,{\rm d}\boldsymbol{\Omega}_{\boldsymbol{k}}}{(2\pi)^3}\,\omega_{\rm pe}^2 k^2\,|M|^2\,\delta(\omega_{\rm pe} - \boldsymbol{q}\cdot\boldsymbol{u})\,, \tag{97}$$

and ${\rm d}\boldsymbol{\Omega}_{\boldsymbol{k}}$ is the solid angle differential of the vector $\boldsymbol{k}$.

It is important that $M_{\rm noncoll}$ decreases rapidly with an increase of the transferred momentum q, and that the main contribution to the integral is made by the momenta close to their minimum possible value of $q = \omega_{\rm pe}/u$. For non-collective bremsstrahlung the integration over q and the solid angle ${\rm d}\boldsymbol{\Omega}_{\boldsymbol{k}}$ gives

$$Q^{\rm (e,e)\,noncoll}_{\boldsymbol{p},\boldsymbol{p}'}(k) = \frac{28 e^6 k^4 u}{15 m_{\rm e}^2\omega_{\rm pe}^4}\,. \tag{98}$$

For collective bremsstrahlung, the result depends on whether the velocities of colliding electrons are greater or less than their mean thermal velocity. For velocities much less than the mean thermal velocity, one can use a Debye screening approximation for dielectric permittivity:

$$\varepsilon_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}} \approx 1 + \frac{\omega_{\rm pe}^2}{q^2 v_{T\rm e}^2}\,.$$

If $q \ll \omega_{\rm pe}/v_{T\rm e}$, the matrix element is small, but for $q \gg \omega_{\rm pe}/v_{T\rm e}$, it is practically identical to the noncollective element. Since $q_{\min} = \omega_{\rm pe}/u \gg \omega_{\rm pe}/v_{T\rm e}$, the collective effects are small for collisions of slow electrons.

Even for collisions of a slow electron ($v' \ll v_{T\rm e}$) with a fast one ($v \gg v_{T\rm e}$), collective effects are very important. As $v \gg v'$ the energy conservation law in the process of bremsstrahlung gives $\omega_{\rm pe} = \boldsymbol{q}\cdot\boldsymbol{v}$, and thus for the expression for the dielectric permittivity $\varepsilon_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}}$ it is possible to use an approximate version of this law:

$$\varepsilon_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}} \approx 1 - \frac{\omega_{\rm pe}^2}{(\boldsymbol{q}\cdot\boldsymbol{v})^2} - \frac{3\omega_{\rm pe}^2 q^2 v_{T\rm e}^2}{(\boldsymbol{q}\cdot\boldsymbol{v})^4} \approx \frac{3q^2 v_{T\rm e}^2}{\omega_{\rm pe}^2}\,, \tag{99}$$

whereas for the other expression for dielectric permittivity $\varepsilon_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}'}$ it is possible to use the Debye screening approximation. Then

$$Q^{\rm (e,e)\,coll}_{\boldsymbol{p},\boldsymbol{p}'}(k) = \frac{28 e^6 k^4 v}{135 m_{\rm e}^2\omega_{\rm pe}^4}\,, \tag{100}$$

This value is 9 times smaller than the result obtained from Eqn (98) in the limit $v \gg v'$.

When both electrons are fast (v, $v' \gg v_{T\rm e}$), it is necessary to use the first approximate expression (99) for both dielectric permittivities. Then

$$Q^{\rm (e,e)\,coll}_{\boldsymbol{p},\boldsymbol{p}'}(k) = \frac{28 e^6 k^4}{15 m_{\rm e}^2\omega_{\rm pe}^2}\int\frac{{\rm d}q^2}{q^4}\,F(q^2)\,, \tag{101}$$

where

$$F(q^2) = \left\{\frac{\delta[\omega_{\rm pe} - \boldsymbol{q}\cdot(v'-v)]}{\left([1 - \omega_{\rm pe}^2/(\boldsymbol{q}\cdot v)^2 - 3\omega_{\rm pe}^2 q^2 v_{T\rm e}^2/(\boldsymbol{q}\cdot v)^4\right]^2} \times \frac{1}{\left[1 - \omega_{\rm pe}^2/(\boldsymbol{q}\cdot v')^2 - 3\omega_{\rm pe}^2 q^2 v_{T\rm e}^2/(\boldsymbol{q}\cdot v')^4\right]^2}\right\}_{\rm av} \tag{102}$$

and the curly brackets with a subscript 'av' indicate angular averaging.

To avoid making the representation cumbersome, we will give here only the result for angular averaging and subsequent integration over q^2 for the case $v \gg v'$ (remember that v, $v' \gg v_{T\rm e}$):

$$Q^{\rm (e,e)\,coll}_{\boldsymbol{p},\boldsymbol{p}'}(k) = \frac{28 e^6 k^4 v}{15 m_{\rm e}^2\omega_{\rm pe}^4}\,\frac{v'^4}{9 v_{T\rm e}^4} \times\left(\frac{1}{3}\cos^4\chi + \frac{1}{2}\sin^2\chi\cos^2\chi + \frac{1}{8}\sin^4\chi\right), \tag{103}$$

where χ is the angle between the velocities v' and v. Let us emphasise that $v'^4/v_{T\rm e}^4 \gg 1$, and therefore the collective effects increase substantially the intensity of bremsstrahlung of fast electrons in electron – electron collisions.

Thus we have proved that the collective effects can not only substantially change the numerical coefficients in the intensity of bremsstrahlung, but also change the major qualitative characteristics of bremsstrahlung.

4.4 Bremsstahlung of longitudinal waves in ion – ion collisions

In ion – ion collisions, the matrix elements M^{α} and M^{β} are small owing to the large ion masses, and the emission is determined by the matrix element $M^{\alpha,\beta}$ [see Eqn (89)].

Let us consider first the case when the velocity of ions is much less than the mean thermal electron velocity v, $v' \ll v_{T\rm e}$. Then for the electron part of the dielectric permittivity in the numerator of expression (89) it is possible to use the Debye screening approximation. We find that

$$M^{\alpha,\beta} \approx \frac{ek}{T_e q^2 (\boldsymbol{k}-\boldsymbol{q})^2} \frac{1}{\varepsilon_{q,q\cdot v'}\,\varepsilon_{k-q,(k-q)\cdot v}} . \tag{104}$$

Because $q_{\min} = \omega_{pe}/u \gg \omega_{pe}/v_{Te}$ the dielectric permittivities in expression (104) can be put equal to unity independently of the relation between the ion velocities and the mean ion thermal velocities. Indeed, in the case when the ion velocities are much smaller than their mean thermal velocity, the dielectric permittivity contains the total Debye radius d and $q_{\min} \gg 1/d$. When the ion velocities are much larger than their mean thermal value and (as we assumed) are much smaller than the mean thermal electron velocity, the dielectric permittivity contains the electron Debye radius d_e, but again $q_{\min} \gg 1/d_e$.

Thus for $k \ll q$ the matrix element of bremsstrahlung in ion–ion collisions can be written approximately in the form:

$$M \approx \frac{ek}{T_e q^4} , \tag{105}$$

and

$$Q^{(i,i)\,\mathrm{coll}}_{p,p'}(k) = \frac{4k^4 u^5 e^6 Z_\alpha^2 Z_\beta^2}{9T_e^2 \omega_{pe}^6} . \tag{106}$$

This expression shows that for $T_e \approx T_i$ the collective bremsstrahlung of ions for $v \gg v_{Ti}$ exceeds substantially the bremsstrahlung obtained when the collective effects are not taken into account. In the case when the ion velocity v is of the order of v_{Te}, the bremsstrahlung in ion–ion collisions will be of the same order of magnitude as the bremsstrahlung in electron–electron collisions, which in turn is also much larger than the bremsstrahlung in ion–ion collisions when the collective effects are not taken into account.

Let us finally consider the case of fast ions $v \gg v_{Te}$. Here, for the electron part of dielectric permittivity in the matrix element, one can use an approximate expression $\varepsilon^e_{q,\omega} - 1 \approx -\omega_{pe}^2/\omega^2$. The power of emission then decreases with increasing ion velocities according to the law $1/v^5$. Thus the intensity of emission in ion–ion collisions is maximal for v of the order of v_{Te}.

4.5 Bremsstrahlung of longitudinal waves in electron–ion collisions

This is the main process in the approximation in which we can ignore the collective effects and, in fact, it remains the main process when the collective effects are taken into account. However, the collective effects modify the process of bremsstrahlung emission significantly.

When the collective effects are not taken into account, the matrix element of bremsstrahlung is described by expression (19), which for $k \ll q$ can be written in the form:

$$M_{\mathrm{noncoll}} = -\frac{e\boldsymbol{k}\cdot\boldsymbol{q}}{m_e \omega_{pe}^2 k q^2} . \tag{107}$$

In the case when, from physical arguments of field screening (which, strictly speaking, should be calculated by taking into account the collective effects), one writes in the denominator of the matrix element an additional factor equal to the dielectric permittivity for the frequency which the ion is 'seeing', then one should use expression (20) and:

$$M_{\mathrm{noncoll}} = -\frac{e\boldsymbol{k}\cdot\boldsymbol{q}}{m_e \omega_{pe}^2 k q^2 \varepsilon_{q,q\cdot v'}} . \tag{108}$$

These two expressions can then be compared with the correct one obtained from the formulas given above when the collective effects have been taken into account. In the same approximations with which expressions (107) and (108) were determined we find:

$$M_{\mathrm{coll}} = -\frac{e\boldsymbol{k}\cdot\boldsymbol{q}}{m_e \omega_{pe}^2 k q^2 \varepsilon_{q,q\cdot v'}} \frac{\varepsilon^i_{k-q,(k-q)\cdot v}}{\varepsilon_{k-q,(k-q)\cdot v}} + \frac{e\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{q})}{m_e \omega_{pe}^2 k (\boldsymbol{k}-\boldsymbol{q})^2 \varepsilon_{k-q,(k-q)\cdot v}} \frac{\varepsilon^e_{q,q\cdot v'} - 1}{\varepsilon_{q,q\cdot v'}} . \tag{109}$$

Let us study the limit when the electron velocity is much larger than the average ion thermal velocity and the ion velocity is much smaller than the average electron thermal velocity. We consider the case when $k \ll q$. Then in Eqn (109) we can put $\varepsilon^i = 1$ and the Debye screening approximation can be used for ε^e. If the electron velocity is much less than the mean electron thermal velocity one can use $\varepsilon^e_{q,q\cdot v'} - 1 \approx \omega_{pe}^2/k^2 v_{Te}^2 < v^2/v_{Te}^2 \ll 1$, and thus the second term in the matrix element can be neglected. One can use $\varepsilon = 1$ in all the other expressions for dielectric permittivities in the matrix element. We find then that the matrix element of bremsstrahlung in this approximation coincides with the one in which the collective effects are neglected.

In a more general case, for arbitrary electron velocities (but much larger than the average ion thermal velocity) and for $k \ll q$ we have

$$M_{\mathrm{coll}} \approx -\frac{e\boldsymbol{k}\cdot\boldsymbol{q}}{m_e k q^2 \omega_{pe}^2 \varepsilon^e_{-q,-q\cdot v}} \frac{1 + 1/q^2 d_e^2}{1 + 1/q^2 d^2} , \tag{110}$$

where d is the total Debye radius while d_e is the electron Debye radius and

$$\frac{1}{d^2} = \frac{1}{d_e^2} + \frac{1}{d_i^2} .$$

When the electron velocity is much larger than the average thermal electron velocity, the main contribution to the probability is made by those q values which are much less than the inverse Debye radius, and $\varepsilon_{-q,-q\cdot v}$ can be approximated by $-3q^2 v_{Te}^2/\omega_{pe}^2$. Then the matrix element has the approximate form:

$$M_{\mathrm{coll}} \approx \frac{e\boldsymbol{k}\cdot\boldsymbol{q}\, d^2}{3T_e k q^4 d_e^2} . \tag{111}$$

The intensity of bremsstrahlung will be determined by the relation:

$$Q^{(e,i)\,\mathrm{coll}}_{p,p'}(k) = \frac{4k^2 u^3 e^6 Z_i^2}{27T_e^2 \omega_{pe}^4} . \tag{112}$$

We can compare expressions (112) and (106) [although expression (106) is valid for $u \ll v_{Te}$ and expression (112) is valid for the opposite sign of the inequality]. This comparison shows that in expression (106) there exists a factor k^2u^2/ω_{pe}^2 which is generally small. But at the limit of applicability, this factor can be of the order of one and these formulas can be compared when v is of the order of v_{Te} (remember that for $v \gg v_{Te}$ the intensity of bremsstrahlung in ion–ion collisions decreases rapidly with an increase in ion velocities).

Nevertheless, taking into account all the restrictions we have just mentioned, the comparison made gives a very strange result: the emissions in ion–ion collisions and electron–ion collisions can have the same order of magnitude. Let us recall that without the collective effects the bremsstrahlung in ion – ion collisions is at least m_e^2/m_i^2 times less than the emission in electron – ion collisions.

The other important point, which follows from the above discussion, is that the emission of fast particles is changed substantially by collective effects. This fact can be, at first glance, surprising because the fast particles do not have a substantial polarisation screening charge. Moreover, from previous results it follows that the emission of slow particles can be close to that in which the collective effects are not taken into account. An explanation of this is that an essential role play here oscillations of the screening charge of an ion, the velocities of which are small, and for large electron velocities the emission due to oscillation of the polarisation charge surrounding the ion emission interferes with the emission due to changes in the electron trajectory produced by the ion.

5. Scattering of longitudinal waves in plasma

5.1 Scattering as a resonant bremsstrahlung

The absorption of waves due to inverse bremsstrahlung and the scattering are closely related to each other. Indeed, under the conditions when the frequency of the field (which transfers the momentum and energy from one colliding particle to another colliding particle) is close to the wave eigenfrequency, this wave is emitted additionally to a 'bremsstrahlung photon', i.e. wave scattering takes place. This can by seen also from the conservation law [Eqn (15)] which can be written in the form:

$$(\boldsymbol{q}-\boldsymbol{k})\cdot v-\omega+\omega_{\boldsymbol{k}}=0\,, \tag{113}$$

and

$$\omega=\boldsymbol{q}\cdot v'\,. \tag{114}$$

For $\omega=\omega_{\boldsymbol{q}}$ relation (113) corresponds to the energy conservation law in the process of scattering. In vacuum, such a situation is impossible since the virtual waves never correspond to real waves on the so-called 'mass shell'. But in a medium in which the phase velocities can be small, such a situation is not only possible but often occurs.

The nonlinear dielectric permittivity, obtained in Section 3, also describes the processes of scattering. An important point is that the scattering corresponds to a resonant condition when $\varepsilon_{\boldsymbol{q},\omega}$ is close to zero and when one can use an approximate relation:

$$\operatorname{Im}\frac{1}{\varepsilon_{\boldsymbol{q},\omega}}\approx-\pi\,\frac{\omega}{|\omega|}\,\delta(\varepsilon_{\boldsymbol{q},\omega})\,. \tag{115}$$

Relation (115) shows explicitly that the frequency and the wavevector $\boldsymbol{q}$ of the field should be equal to the eigenfrequency of the wave $\omega_{\boldsymbol{q}}$ (in the particular case we are considering here, it is the plasma wave), which satisfies the dispersion relation for longitudinal waves $\varepsilon_{\boldsymbol{q},\omega_{\boldsymbol{q}}}=0$.

In the previous section we found an expression for dielectric permittivity for a wave of small amplitude, taking into account its nonlinear interaction with fluctuations of particles and fields. The expressions obtained are linear in the amplitudes (intensities) of the propagating waves. In the approximation linear in the amplitudes of the waves, the processes of scattering are described by two terms in the way in which they can be derived from the balance equation. One of these terms describes the generation of the scattered waves and is proportional to the intensity of waves being scattered, and the other one describes the extinction of scattered waves and is proportional to the intensity of the waves created in the scattering process.

In the approach used to calculate the nonlinear permittivity we had taken into account only the effects which are proportional to the amplitude (intensity) of the wave considered. Thus with the help of this permittivity we can find the coefficient of extinction of the wave due to scattering only. But this is sufficient to find the probability of scattering since this coefficient of extinction has a definite relation with the scattering probability. Such an approach makes it possible to find all the collective effects in scattering. From the balance equation we can obtain the damping decrement (the extinction coefficient) of the waves, expressed through the probability of scattering $w_{\boldsymbol{p}}^{\mathrm{sc},\alpha}(\boldsymbol{k},\boldsymbol{k}')$ of a wave with a momentum (wavevector) $\boldsymbol{k}$ on particle α with a momen-tum $\boldsymbol{p}$ generating in a unit time a scattered wave with momentum $\boldsymbol{k}'$, normalised on the unit phase volume $\mathrm{d}\boldsymbol{k}'/(2\pi)^3$:

$$\gamma_{\boldsymbol{k}}^{\mathrm{sc}}=-\frac{\operatorname{Im}\varepsilon_{\boldsymbol{k},\omega}^{\mathrm{N}}}{\left[\partial(\operatorname{Re}\varepsilon_{\boldsymbol{k},\omega})/\partial\omega\right]_{\omega=\omega_{\boldsymbol{k}}}}-\frac{1}{2}\sum_{\alpha}\int\frac{\mathrm{d}\boldsymbol{k}'\,\mathrm{d}\boldsymbol{p}}{(2\pi)^6}\,w_{\boldsymbol{p}}^{\mathrm{sc},\alpha}(\boldsymbol{k},\boldsymbol{k}')\,\Phi_{\boldsymbol{p}}^{\alpha}\,. \tag{116}$$

We define the probability of scattering through the matrix element M^{sc}:

$$w_{\boldsymbol{p}}^{\mathrm{sc},\alpha}(\boldsymbol{k},\boldsymbol{k}')=4e_\alpha^2(2\pi)^3|M^{\mathrm{sc}}|^2\times\frac{\delta\left[\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}_1}-(\boldsymbol{k}-\boldsymbol{k}_1)\cdot v\right]}{(\partial\varepsilon_k/\partial\omega)_{\omega=\omega_{\boldsymbol{k}}}(\partial\varepsilon_{k_1}/\partial\omega_1)_{\omega_1=\omega_{\boldsymbol{k}_1}}}\,. \tag{117}$$

In general the collective effects change the matrix element of scattering substantially, which then consists of the two parts:

$$M^{\mathrm{sc}}=M_{\mathrm{noncoll}}^{\mathrm{sc}}+M_{\mathrm{coll}}^{\mathrm{sc}}\,. \tag{118}$$

Let us now study how these components of the scattering matrix element can be derived from the theory of fluctuations.

5.2 Cross-sections of scattering of longitudinal waves

We start with expression (53) for the nonlinear permittivity $\varepsilon_k^{\mathrm{N},4}$, since from this expression we get the probability of scattering in which the collective effects are not taken into account. In calculating the imaginary part of $\varepsilon_k^{\mathrm{N},4}$, we used the general expression for $\operatorname{Im}(1/\varepsilon_k)$. But it is more appropriate to use relation (115) for resonant conditions [in particular, to use relation (115) in which $\boldsymbol{q}=\boldsymbol{k}$]. Obviously this approximate expression can be obtained from the general one used before, but only when we are interested in the frequency range close to resonance.

Let us denote the corresponding contribution in the damping of waves by $\gamma_{\boldsymbol{k}}^{\mathrm{sc}(1)}$. We find that

$$\gamma_{\boldsymbol{k}}^{\mathrm{sc}(1)}=-\frac{\operatorname{Im}\varepsilon_k^{\mathrm{N},4}}{(\partial\varepsilon_k/\partial\omega)_{\omega=\omega_{\boldsymbol{k}}}}=-\sum_{\alpha}\frac{2e_\alpha^4}{m_\alpha^2}\times\int\frac{\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\,\frac{(\boldsymbol{k}\cdot\boldsymbol{k}_1)^2\Phi_{\boldsymbol{p}}^{\alpha}}{k^2k_1^2(\omega_{\boldsymbol{k}}-\boldsymbol{k}\cdot v)^4}\times\frac{\delta\left[\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}_1}-(\boldsymbol{k}-\boldsymbol{k}_1)\cdot v\right]}{(\partial\varepsilon_k/\partial\omega)_{\omega=\omega_{\boldsymbol{k}}}(\partial\varepsilon_{k_1}/\partial\omega_1)_{\omega_1=\omega_{\boldsymbol{k}_1}}}\,. \tag{119}$$

This expression leads to a matrix element of noncollective scattering:

$$M^{\rm sc}_{\rm noncoll} = \frac{e_\alpha}{m_\alpha}\frac{\boldsymbol{k}\cdot\boldsymbol{k}_1}{kk_1}\frac{1}{(\omega_{\boldsymbol{k}} - \boldsymbol{k}\cdot v)^2}\,. \tag{120}$$

The expression obtained describes the Thomson scattering and differs from the known expression for the Thomson scattering of electromagnetic waves by the polarisation vectors. In particular, expression (120) contains the scalar product of the unit vectors along the propagation of the two longitudinal waves taking part in the scattering process.

Let us consider expression (51) for $\varepsilon^{\rm N,3}_k$. Since the resonance $1/\varepsilon_{k-k_1}$ corresponds to the corrections to Cherenkov resonance we will consider only the resonance related to $1/\varepsilon_{k_1}$. Denoting by $\gamma^{\rm sc(2)}_{\boldsymbol{k}}$ the expression obtained from $\varepsilon^{\rm N,3}_k$ for this resonance, we find:

$$\gamma^{\rm sc(2)}_{\boldsymbol{k}} = -\frac{{\rm Im}\,\varepsilon^{\rm N,3}_k}{(\partial\varepsilon_k/\partial\omega)_{\omega=\omega_k}} = {\rm Re}\sum_\alpha \frac{16\pi e^3_\alpha}{m_\alpha}$$

$$\times\int\frac{{\rm d}\boldsymbol{k}_1\,{\rm d}\boldsymbol{p}}{(2\pi)^3}\frac{(\boldsymbol{k}\cdot\boldsymbol{k}_1)\Phi^\alpha_{\boldsymbol{p}}}{kk_1(\omega_{\boldsymbol{k}} - \boldsymbol{k}\cdot v)^2}\frac{\rho^{\rm N,2}_{k,k_1-k}}{|\boldsymbol{k}-\boldsymbol{k}_1|\varepsilon_{k_1-k}}$$

$$\times\frac{\delta\left[\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}_1} - (\boldsymbol{k}-\boldsymbol{k}_1)\cdot v\right]}{(\partial\varepsilon_k/\partial\omega)_{\omega=\omega_{\boldsymbol{k}}}(\partial\varepsilon_{k_1}/\partial\omega_1)_{\omega_1=\omega_{k_1}}}\,. \tag{121}$$

If we define

$$M^{\rm sc}_{\rm coll} = \frac{8\pi\rho^{\rm N,2}_{k,k_1-k}}{k_1|\boldsymbol{k}-\boldsymbol{k}_1|\varepsilon_{k_1-k}}\,, \tag{122}$$

we find that expression (121) describes part of the square of the total matrix element of scattering $|M^{\rm sc}|^2$ equal to ${\rm Re}\,(M^{\rm sc}_{\rm coll}M^{\rm sc}_{\rm noncoll})$ [which amounts to half of the interference term equal to $2\,{\rm Re}\,(M^{\rm sc}_{\rm coll}M^{\rm sc}_{\rm noncoll})$].

The second half appears from expression (48) for $\varepsilon^{\rm N,2}_k$ on replacing k_1 by $k-k_1$ and integrating by parts with respect to particle momenta and using relation (115). We denote the result obtained from expression (48) in the resonance conditions by $\gamma^{\rm sc(3)}_{\boldsymbol{k}}$ and find that

$$\gamma^{\rm sc(3)}_{\boldsymbol{k}} = \gamma^{\rm sc(2)}_{\boldsymbol{k}}\,. \tag{123}$$

Finally the square of the matrix element of collective scattering is obtained as resonant absorption from $\varepsilon^{\rm N,1}_k$. The resonance effects exist only in the second term of expression (40). We denote the decrement for resonant damping obtained from this expression by $\gamma^{\rm sc(4)}_{\boldsymbol{k}}$ and use relation (115). We find:

$$\gamma^{\rm sc(4)}_{\boldsymbol{k}} = -\frac{{\rm Im}\,\varepsilon^{\rm N,1}_k}{(\partial\varepsilon_k/\partial\omega)_{\omega=\omega_k}}$$

$$= 64\pi^3\int{\rm d}\boldsymbol{k}_1\,\frac{|E^{(0)}|^2_{k-k_1}\,\rho^{\rm N,2}_{k-k_1,k_1}\rho^{\rm N,2}_{k,k_1-k}}{kk_1(\partial\varepsilon_k/\partial\omega)_{\omega=\omega_k}(\partial\varepsilon_{k_1}/\partial\omega_1)_{\omega_1=\omega_{k_1}}}\,. \tag{124}$$

By using expression (29) for the field fluctuations and expression (66) with the substitution of $k-k_1$ for k_1 we indeed confirm that expression (124) describes the effect of scattering determined by $|M^{\rm sc}_{\rm coll}|^2$ in the probability of scattering.

We have thus shown rigorously how the fluctuation theory leads to expressions for the scattering cross-sections, which contain the square of the modulus of the matrix elements and take into account the collective effects. It is important to stress that both the probability of bremsstrahlung emission and the scattering probability are expressed in terms of the relevant squares of the matrix elements, but the collective effects enter the matrix elements additively. The interference effects in the scattering and bremsstrahlung can considerably reduce or increase the cross-sections of the processes.

When one uses an approximate expression for the nonlinear responses and neglects the Doppler corrections one finds that

$$\rho^{\rm N,2}_{k,k_1-k} \approx \frac{e(\boldsymbol{k}\cdot\boldsymbol{k}_1)|\boldsymbol{k}-\boldsymbol{k}_1|}{8\pi k m_{\rm e}\omega^2_{\rm pe}}\left(\varepsilon^{\rm e}_{k_1-k} - 1\right)\,, \tag{125}$$

from which the known expressions for probabilities of scattering of Langmuir waves on electrons and ions can be derived (see Refs [8, 9]). Particularly for electrons, these interference effects lead to an additional small factor in the Thomson cross-section. This factor is equal to

$$\left|\frac{\varepsilon^{\rm i}_{k_1-k}}{\varepsilon_{k_1-k}}\right|^2\,.$$

The electron mass enters in the cross-section of scattering on ions, which is equal to the Thomson cross-section of scattering (on electrons) with the following factor of the order of unity:

$$\left|\frac{\varepsilon^{\rm e}_{k_1-k} - 1}{\varepsilon_{k_1-k}}\right|^2\,.$$

The cross-section of scattering on ions in a plasma is then of the order of the Thomson cross-section of scattering on electrons, and the cross-section of scattering on electrons is much less than the Thomson cross-section of scattering.

6. Collective effects in bremsstrahlung of electromagnetic waves

6.1 Probabilities of bremsstrahlung of waves of arbitrary polarisation by nonrelativistic particles

The role of collective effects in the bremsstrahlung of transverse waves is important for experiments in which the electromagnetic radiation emitted by a plasma is measured. The first attempts to take into account the role of shell electrons in bremsstrahlung of electromagnetic waves in electron – atom collisions were made by L D Landau and Yu B Rumer [12]. But it appears that this process is more complex than they thought; however, at the present time it can be described in a precise manner both for plasma and for other media including nonionised atoms or molecules. In the case of this effect in a plasma, the electrons of the screening shell are free electrons and the description of their role in bremsstrahlung is simple. In the present representation we will use the physical picture already discussed in detail for longitudinal waves and point out the specific features for the case of electromagnetic waves.

As above, an important point is that the probability of bremsstrahlung of electromagnetic waves can be expressed through the square of a matrix element. To find the latter it is not necessary to use the general theory of fluctuations, although in the literature the bremsstrahlung and scattering are derived in this way (see, for example, Ref. [13]). The fluctuation theory is a very cumbersome way of obtaining the results, and, in addition, the result obtained cannot easily be expressed in a form proportional to the square of a matrix element. But this kind of proof can be given in all cases and the expression we will use coincides with that found from fluctuation theory. After that we only need to analyse the matrix element, which is a much simpler way of getting a definite answer and a clearer one from the physical point of view. All the cancellations due to the interference of the two mechanisms of bremsstrahlung appear in the matrix elements. Thus it is necessary to check only once that the fluctuation theory and the simple arguments give the same result, in order to use the expression for matrix elements in future applications. We will not present the fluctuation theory here but just start with general expressions for matrix elements which can be derived from the fluctuation theory and from another simpler approach. For their application to concrete problems we need to have simplified expressions and these will be provided even for an arbitrary wave polarisation valid only for the case of nonrelativistic particles.

In the general case the new aspects of bremsstrahlung of electromagnetic waves, as compared to the case already discussed, can be reduced to the following three:

(1) The polarisation vectors of the electromagnetic waves in the absence of an external magnetic field are transverse.

(2) In the force acting on plasma particles, one should take into account the Lorentz force.

(3) If the particles are relativistic, the virtual transverse fields should be taken into account (for longitudinal waves the process through the transverse virtual wave should also be added, but it plays a less important role and can be neglected completely in the case of nonrelativistic particles).

The general expression for the probability of bremsstrahlung for a wave with an arbitrary polarisation can be given by the following expression [compare with Eqn (18)]:

$$w^{\alpha,\beta}_{p,p'}(\boldsymbol{k},\boldsymbol{q}) = \frac{16\pi e_\alpha^2 e_\beta^2 (2\pi)^3}{\left(\partial\varepsilon_k^\sigma \omega^2/\partial\omega\right)_{\omega=\omega_k}} |\boldsymbol{M}\cdot\boldsymbol{e}_k^\sigma|^2 \omega^2 \times \delta\left[\omega_k - \boldsymbol{q}\cdot\boldsymbol{v}' - (\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}\right] , \qquad (126)$$

where $\boldsymbol{M}$ is the vector matrix element, $\boldsymbol{e}_k^\sigma$ is the unit polarisation vector of the emitted waves, $\varepsilon_k^\sigma = \varepsilon_{i,j}(k)\, e^\sigma_{i,k} e^\sigma_{j,k}$ and $\varepsilon_{i,j}(k)$ is the tensor of dielectric plasma permittivity.

The expression for the vector matrix element for the case of arbitrary relativistic particle distributions is given in Ref. [9] (their definition differs from that used in the present article by a factor ω_k; in the definition of the probability the factor ω_k^2 is absent).

We will give here approximate expressions for the matrix element vectors valid only if (1) the particle velocities are assumed to be nonrelativistic (both plasma particles and particles colliding with each other and creating the emission) and thus the virtual wave can be considered to be longitudinal, and (2) it is possible to neglect the Doppler corrections in comparison with the frequency of the wave emitted.

The expressions have a form very similar to that obtained previously, and on multiplying them by the unit longitudinal polarisation vectors we can obtain the previous expressions. Thus the given matrix element vectors just extend in a simple manner the previous results for the case of waves of arbitrary polarisation. These expressions are:

$$\boldsymbol{M}^\alpha = \frac{e_\alpha \boldsymbol{q}}{m_\alpha \omega_k^2 q^2\, \varepsilon_{q,\boldsymbol{q}\cdot\boldsymbol{v}'}} , \qquad (127)$$

$$\boldsymbol{M}^\beta = \frac{e_\beta(\boldsymbol{k}-\boldsymbol{q})}{m_\beta \omega_k^2 (\boldsymbol{k}-\boldsymbol{q})^2\, \varepsilon_{k-q,(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}} , \qquad (128)$$

$$\boldsymbol{M}^{\alpha,\beta} \approx \frac{e\boldsymbol{q}}{m_e \omega_k^2 q^2\, \varepsilon_{q,\boldsymbol{q}\cdot\boldsymbol{v}'}} \frac{\varepsilon^e_{k-q,(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}} - 1}{\varepsilon_{k-q,(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}} + \frac{e(\boldsymbol{k}-\boldsymbol{q})}{m_e \omega_k^2 (\boldsymbol{k}-\boldsymbol{q})^2\, \varepsilon_{k-q,(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}} \frac{\varepsilon^e_{q,\boldsymbol{q}\cdot\boldsymbol{v}'} - 1}{\varepsilon_{q,\boldsymbol{q}\cdot\boldsymbol{v}'}} . \qquad (129)$$

Here ε is the longitudinal dielectric permittivity:

$$\varepsilon_k = \varepsilon_{i,j}(k)\, \frac{k_i k_j}{k^2} . \qquad (130)$$

The result given here is new and not previously given in the literature. Its derivation from the general expressions given in Ref. [9] is very cumbersome. I hope that it will be useful in applications. It can be used even for an anisotropic particle distribution when, strictly speaking, one cannot separate the dielectric permittivity into its longitudinal and transverse parts. But the expression which determines the matrix element vectors for nonrelativistic particles contains the dielectric permittivity defined by Eqn (130). The conditions for applicability of Eqns (127)–(129) are $v,\, v' \ll c$ and $\omega_k \gg kv,\, kv'$.

The result is also applicable for a plasma in an external magnetic field when, apart from the relations of applicability already given, there appear further conditions for particles to be nonmagnetised: $kv,\, kv' \gg \omega_{H,\alpha}$ and $qv,\, qv' \gg \omega_{H,\alpha}$, where $\omega_{H,\alpha} = e_\alpha H/m_\alpha c$ is the cyclotron plasma frequency. The last inequality containing the value of the transferred momentum is not very restrictive if the transferred momenta are large, for example, if they are of the order of the minimum impact parameter. But the collective effects, as can be seen from previous considerations, enhance the bremsstrahlung of identical or heavy particles when the effective transferred momenta are determined either by particle velocity or by the Debye radius. Then the stated restriction is important. In any case, the general expressions for matrix element vectors given here can be useful in many different applications.

6.2 Bremsstrahlung of electromagnetic waves in electron–electron and ion–ion collisions

In this and subsequent subsections I will discuss the processes of bremsstrahlung of electromagnetic waves in the absence of an external magnetic field. One important difference between electromagnetic and longitudinal waves lies in the dispersion of waves (their frequency dependence on the wavenumber):

$$\omega_k = \sqrt{\omega_{pe}^2 + c^2 k^2} . \qquad (131)$$

The waves with $\omega_k \approx \omega_{pe}$ for $k \ll \omega_{pe}/c$ are similar to the longitudinal waves and are sometimes called transverse plasmons. We shall therefore consider them separately. The waves with $\omega_k \gg \omega_{pe}$ when $\omega_k \approx kc$ are similar to electromagnetic waves in vacuum. We will study them separately and simply call them electromagnetic waves.

The matrix element vector for electron–electron collisions can be found from expression (90) in a simple manner. By taking into account that the probability contains the square of the matrix element, we find that the sum over the two transverse polarisations contains the square of the vector product of the matrix element vector and the wavevector $\boldsymbol{k}$. Therefore, without loss of generality, the vector matrix element can be represented by the product of its vector product and a unit vector $\boldsymbol{k}/k$.

For electron velocities much larger than the average thermal velocity, instead of Eqn (93) for the electromagnetic waves we find the following expression:

$$\frac{\boldsymbol{k}}{k}\times\boldsymbol{M}\approx -e\frac{\boldsymbol{k}\times\boldsymbol{q}}{k}\,\frac{1/q^2-1/(\boldsymbol{k}-\boldsymbol{q})^2}{m_{\mathrm{e}}\omega_{\boldsymbol{k}}^2\,\varepsilon^{\mathrm{e}}_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}'}\,\varepsilon^{\mathrm{e}}_{\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}}\,. \qquad (132)$$

In the opposite case, when the velocities of electrons are much less than the average thermal electron velocities, as was found previously, the value of the dielectric permittivity can be put equal to unity. In this case for $k\ll q$ we find:

$$\frac{\boldsymbol{k}}{k}\times\boldsymbol{M}\approx -e\frac{(\boldsymbol{k}\times\boldsymbol{q})}{kq^4}\,\frac{2\boldsymbol{k}\cdot\boldsymbol{q}}{km_{\mathrm{e}}\omega_{\boldsymbol{k}}^2}\,. \qquad (133)$$

For transverse plasmons it is useful to introduce the expression for the power emitted in a unit interval of the absolute values of the wavevectors, as it was used for longitudinal plasmons. We also take into account the difference in the dispersion of transverse and longitudinal plasmons:

$$\omega_{\boldsymbol{k}}=\omega_{\boldsymbol{k}}^{\mathrm{tpl}}\approx\omega_{\mathrm{pe}}+\frac{k^2c^2}{2\omega_{\mathrm{pe}}}\,.$$

Instead of Eqn (97) we get:

$$Q^{(\mathrm{e,e})}_{\boldsymbol{p},\boldsymbol{p}'}(k)=8\pi e^4\int\frac{\mathrm{d}\boldsymbol{q}\,\mathrm{d}\Omega_{\boldsymbol{k}}}{(2\pi)^3}\times\left|\frac{\boldsymbol{k}}{k}\times\boldsymbol{M}\right|^2\omega_{\mathrm{pe}}^2k^2\,\delta(\omega_{\boldsymbol{k}}^{\mathrm{tpl}}-\boldsymbol{q}\cdot\boldsymbol{u})\,. \qquad (134)$$

For the transverse plasmons, the result of angular integration of Eqn (134) differs from expression (98) by a numerical factor only:

$$Q^{(\mathrm{e,e})\,\mathrm{tpl}}_{\boldsymbol{p},\boldsymbol{p}'}(k)=\frac{32e^6k^4u}{15m_{\mathrm{e}}^2\omega_{\mathrm{pe}}^4}\,. \qquad (135)$$

For electromagnetic waves it is useful to introduce the intensity normalised on a unit frequency interval:

$$Q^{(\mathrm{e,e})}_{\boldsymbol{p}}=\int\frac{\mathrm{d}\boldsymbol{p}'\,\mathrm{d}\omega}{(2\pi)^3}\,Q^{(\mathrm{e,e})}_{\boldsymbol{p},\boldsymbol{p}'}(\omega)\,\Phi^{\mathrm{e}}_{\boldsymbol{p}'}\,, \qquad (136)$$

where

$$Q^{(\mathrm{e,e})}_{\boldsymbol{p},\boldsymbol{p}'}(\omega)=8\pi\int\frac{\mathrm{d}\boldsymbol{q}\,\mathrm{d}\Omega_{\boldsymbol{k}}}{(2\pi)^3}\times\left|\frac{\boldsymbol{k}}{k}\times\boldsymbol{M}\right|^2\frac{\omega^4e^4}{c^3}\,\delta(\omega-\boldsymbol{q}\cdot\boldsymbol{u})\,. \qquad (137)$$

For slow electrons, expression (137) does not depend on k, i.e. on ω, and becomes

$$Q^{(\mathrm{e,e})}_{\boldsymbol{p},\boldsymbol{p}'}(\omega)=\frac{32e^6u}{15c^5m_{\mathrm{e}}^2}\,. \qquad (138)$$

As can be seen by a comparison of expressions (138) and (135) the spectral density of radiation increases with an increase of k (frequency) and reaches a constant value for $k\gg\omega_{\mathrm{pe}}/c$.

For fast electrons, expressions (99) should be used for dielectric permittivities, but the last approximate expression of the system of Eqns (99) can be used only for $k\ll v_{T\mathrm{e}}\omega_{\mathrm{pe}}/cu$. In this limit, the intensities of emission of longitudinal and transverse plasmons differ only by a numerical factor [32 for transverse plasmons instead of 28 in relation (103) for longitudinal plasmons].

For a large k, the role of collective effects decreases in the range of wavenumber values $v_{T\mathrm{e}}\omega_{\mathrm{pe}}/cu\ll k\ll\omega_{\mathrm{pe}}/c$, and the large factor $v^4/v_{T\mathrm{e}}^4$ finally disappears.

For $k\gg\omega_{\mathrm{pe}}/c$, the permittivities can be assumed to be unity and the collective effects are weak. In this case we can use relationship (138). Thus both for velocities much less than the average thermal velocity and for velocities much larger than the average thermal velocity, expression (138) is valid. As can be seen, it is also valid for any intermediate values of electron velocities.

The total power of emission of a unit plasma volume will differ from Eqn (138) by a factor n_{e}^2 and then for u one could use its value averaged over a thermal electron distribution equal to $2v_{T\mathrm{e}}/\sqrt{\pi}$.

The bremsstrahlung of transverse waves in the ion–ion collisions is, as a rule, much less than that for longitudinal waves. This stems from the fact that, for velocities of ions much less than the main thermal electron velocity, the Debye approximation can be used for the expression for dielectric permittivity and the transverse part of the matrix element vector $M^{\alpha,\beta}$ vanishes. But in the next approximation the matrix element $M^{\alpha,\beta}$ for transverse waves contains a small parameter equal to the ratio of the ion velocity to the mean electron thermal velocity:

$$\varepsilon^{\mathrm{e}}_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}-1\approx\frac{\omega_{\mathrm{pe}}^2}{k^2v_{T\mathrm{e}}^2}\left(1+\sqrt{\frac{\pi}{2}}\,\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{kv_{T\mathrm{e}}}\right)\,.$$

In spite of this, bremsstrahlung resulting from ion–ion collisions is governed totally by the collective effects, since the square of the matrix element contains a small (of the order of $m_{\mathrm{e}}/m_{\mathrm{i}}$) parameter $v^2/v_{T\mathrm{e}}^2$, and the term representing the noncollective process contains the square of this parameter.

6.3 Bremsstrahlung of electromagnetic waves in electron–ion collisions

The bremsstrahlung in electron–ion collisions is still the dominant process of bremsstrahlung when the collective effects are taken into account. This is in spite of the drastic increase of bremsstrahlung in ion–ion collisions and of the essential modification of the bremsstrahlung in electron–electron collisions.

The qualitative change of bremsstrahlung due to collective effects described in previous sections for longitudinal waves also occurs for electromagnetic waves. In the corresponding squares of the matrix elements, vector products appear instead of scalar products. Thus instead of Eqns (108) and (109) we need to use

$$\left(\frac{\boldsymbol{k}}{k}\times\boldsymbol{M}\right)_{\mathrm{noncoll}}=-\frac{e(\boldsymbol{k}\times\boldsymbol{q})}{m_{\mathrm{e}}\omega_{\boldsymbol{k}}^2kq^2\,\varepsilon_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}'}}\,, \qquad (139)$$

$$\left(\frac{\boldsymbol{k}}{k}\times\boldsymbol{M}\right)_{\mathrm{coll}}=-\frac{e(\boldsymbol{k}\times\boldsymbol{q})}{m_{\mathrm{e}}\omega_{\boldsymbol{k}}^2k\,\varepsilon_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}'}\,\varepsilon_{\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}}\times\left[\frac{\varepsilon^{\mathrm{i}}_{\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}}{q^2}+\frac{\varepsilon^{\mathrm{e}}_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}'}-1}{(\boldsymbol{k}-\boldsymbol{q})^2}\right]\,. \qquad (140)$$

For electron velocities much larger than the thermal ion velocity we can substitute 1 for the dielectric permittivity for frequencies corresponding to electron velocities; and for the electron permittivity for ion velocities we can use the Debye screening approximation. Then for $k \ll q$ we find that

$$\left(\frac{\boldsymbol{k}}{k}\times\boldsymbol{M}\right)_{\mathrm{coll}} = -\frac{e(\boldsymbol{k}\times\boldsymbol{q})}{m_{\mathrm{e}}\omega_{\boldsymbol{k}}^2 k q^2\,\varepsilon^{\mathrm{e}}_{\boldsymbol{k}-\boldsymbol{q},(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{v}}} \times \frac{1+\omega_{\mathrm{pe}}^2/q^2 v_{T\mathrm{e}}^2}{\omega_{\mathrm{pe}}^2/q^2 v_{T\mathrm{e}}^2+\varepsilon^{\mathrm{i}}_{\boldsymbol{q},\boldsymbol{q}\cdot\boldsymbol{v}'}}\,. \tag{141}$$

Under these conditions the power of bremsstrahlung will be determined by the relation:

$$Q^{(\mathrm{e},\mathrm{i})}_{\boldsymbol{p},\boldsymbol{p}'}(\omega) = 8\pi\int\frac{\mathrm{d}\boldsymbol{q}\,\mathrm{d}\Omega_{\boldsymbol{k}}}{(2\pi)^3}\left|\frac{\boldsymbol{k}}{k}\times\boldsymbol{M}\right|^2\frac{\omega^4 e^4 Z_{\mathrm{i}}^2}{c^3}\,\delta(\omega-\boldsymbol{q}\cdot\boldsymbol{v})\,, \tag{142}$$

where Z_{i} is the ion charge in units of an electron charge.

An analysis of these expressions makes clear the role of collective effects. We will illustrate them by using the expression averaged over the thermal distributions of electrons and ions. We can perform the integration over the ion distribution by using the following relation, which can be obtained from the fluctuation–dissipation theorem (one can also obtain this relation directly by integrating the corresponding expressions containing the ion dielectric permittivity):

$$\beta\sum_{\mathrm{i}} Z_{\mathrm{i}}^2 n_{\mathrm{i}}\int\frac{\mathrm{d}y_{\mathrm{i}}}{\sqrt{\pi}}\frac{\exp(-y_{\mathrm{i}}^2)}{\left|\omega_{\mathrm{pe}}^2/q^2 v_{T\mathrm{e}}^2+\sum_{\mathrm{i}}\varepsilon^{\mathrm{i}}_{q,y_{\mathrm{i}}}\right|^2} = \frac{\beta\sum_{\mathrm{i}} Z_{\mathrm{i}} n_{\mathrm{i}}}{(1+\omega_{\mathrm{pe}}^2/q^2 v_{T\mathrm{e}}^2)\left[1+(1+\beta)\omega_{\mathrm{pe}}^2/q^2 v_{T\mathrm{e}}^2\right]}\,, \tag{143}$$

where the summation is performed over all types of ions i; the ion dielectric permittivity is supposed to be dependent on q and

$$y_{\mathrm{i}} = \frac{\omega}{q\sqrt{2}v_{T\mathrm{i}}}$$

($v_{T\mathrm{i}}$ is the ion thermal velocity of ions of type i); the parameter β is equal to

$$\beta = \frac{\sum_{\mathrm{i}} Z_{\mathrm{i}}^2 n_{\mathrm{i}}}{\sum_{\mathrm{i}} Z_{\mathrm{i}} n_{\mathrm{i}}} \tag{144}$$

(n_{i} is the density of ions of type i).

The total power of bremsstrahlung emission of a unit plasma volume in electron–ion collisions averaged over the thermal distributions of electrons and ions will then have the form:

$$Q^{(\mathrm{e},\mathrm{i})\,\mathrm{coll}}(\omega) = \frac{32e^6 n_{\mathrm{e}}^2\beta}{3m_{\mathrm{e}}^2 c^3}(1-\omega_{\mathrm{pe}}^2/\omega^2)^{1/2}\int\frac{\mathrm{d}y\,\mathrm{d}q}{\sqrt{\pi}} \times\frac{\exp(-y^2)\,\delta(\omega-qv_{T\mathrm{e}}\sqrt{2}y)}{\left|1+W(y)\omega_{\mathrm{pe}}^2/q^2 v_{T\mathrm{e}}^2\right|^2} \times\frac{1+\omega_{\mathrm{pe}}^2/q^2 v_{T\mathrm{e}}^2}{1+(1+\beta)\omega_{\mathrm{pe}}^2/q^2 v_{T\mathrm{e}}^2}\,, \tag{145}$$

where

$$W(y) = 1 - y\exp(-y^2)\int_0^y \mathrm{d}t\,\exp t^2 + \mathrm{i}\sqrt{\pi}y\exp(-y^2)\,. \tag{146}$$

We give here also the result for the case when the collective effects are neglected but the Debye shielding of the ion field is taken into account, in order to compare it with the correct result given by expression (146):

$$Q^{(\mathrm{e},\mathrm{i})\,\mathrm{noncoll}}(\omega) = \frac{32e^6 n_{\mathrm{e}}^2\beta}{3m_{\mathrm{e}}^2 c^3}(1-\omega_{\mathrm{pe}}^2/\omega^2)^{1/2}\int\frac{\mathrm{d}y\,\mathrm{d}q}{\sqrt{\pi}} \times\frac{\exp(-y^2)\,\delta(\omega-qv_{T\mathrm{e}}\sqrt{2}y)}{\left|1+\omega_{\mathrm{pe}}^2/q^2 v_{T\mathrm{e}}^2\right|^2}\,. \tag{147}$$

By comparing these results one can see that, when the collective effects are ignored, the difference from unity of the factor

$$\frac{1+\omega_{\mathrm{pe}}^2/q^2 v_{T\mathrm{e}}^2}{1+(1+\beta)\omega_{\mathrm{pe}}^2/q^2 v_{T\mathrm{e}}^2}\,,$$

(for which no reason can be given) is neglected and the function $W(y)$, which appears in the denominator under the sign of a square of absolute value, is absent. Both factors change the bremsstrahlung qualitatively . The first one contains a dependence on ion density and its distribution among the species of different ions [see Eqn (144)], and the second one makes the screening of ions negligible for $y \gg 1$, as the function $W(y)$ is very small in this case. This means that the screening of the ion field is then almost zero (the factor inside the absolute value signs is equal to untiy). When the collective effects are not taken into account, the phenomenon of ion 'stripping' is not taken into account. But in accordance with the argument given in the qualitative description of bremsstrahlung in the previous sections, this process of 'stripping' is obviously present.

The analysis shows that the collective effects are most important for frequencies of the order of the plasma frequency. This can be seen from the δ-function in Eqn (157), because for $\omega \gg \omega_{\mathrm{pe}}$ we have $q \gg \omega_{\mathrm{pe}}/v_{T\mathrm{e}}$. In the presence of external magnetic fields, the collective effects in bremsstrahlung can substantially alter the emission for frequencies close to either the lower-hybrid or the upper-hybrid frequency.

7. Conclusion

In this section I will emphasise those points which are new to the field and which have been discussed in this paper.

First, it should be recalled that the theory of fluctuations, scattering of waves, and plasma emission due to fluctuations is a rather widely developed field of research. But up to the present time it has not been shown that all the effects of wave scattering and of bremsstrahlung can be expressed through the probabilities of scattering and bremsstrahlung, which contain the squares of absolute values of the matrix elements of scattering and bremsstrahlung.

It has also not ben clear how the linear wave absorption can be derived from the nonlinear interactions of test waves with fluctuations of plasma particles and fields in a plasma. The existence of such a proof gives an additional insight into the physical processes which are important, for example, transitional absorption, and it also simplifies the investigations of the particular processes.

The proof given is easily generalised for the case of any relativistic effect if one adds the processes occurring through the virtual transverse fields. We have discussed only the case of longitudinal virtual fields but the results obtained in this approximation (before the assumption that the particles are nonrelativistic) were relativistically invariant. Thus if the processes occurring through transverse virtual fields are added to these expressions (which can be given also in a relativistic invariant form), the complete expressions for probabilities can be written for any relativistic particle velocity and any relativistic particle distribution in the plasma.

The general proof given here that all the collective effects can be included as additional contributions to the matrix elements, seems to be very useful for investigations of the particular processes of scattering and bremsstrahlung in more complicated conditions than those used in this review (presence of external fields, inhomogeneities, etc.). This could be of particular importance when one looks for a precise calculation of these effects so as to compare them with experimental results or astrophysical observations (such as the solar opacity problem). In the usual approach it is rather distressing to find that the final result can only be expressed through the square of the absolute value of a rather cumbersome and long expression.

Second, a simple physical picture of the role of collective effects have been given which can be used in some simple estimates. Previously in the literature the physical picture of a new process of transition scattering and transition bremsstrahlung was given [9] and it was stated that the total matrix elements of bremsstrahlung and scattering should also contain the contributions of transition bremsstrahlung and transition scattering. This statement was based on physical arguments. The discussion given here contains an exact proof of these statements obtained with the use of the fluctuation theory.

The new content of this statement is that, except for contributions from transition bremsstrahlung and transition scattering, there exist no other contributions which can follow from fluctuation theory, and those effects are the only ones which describe all collective effects in bremsstrahlung and scattering. Obviously, in other more complicated media, there can also be effects of transition bremsstrahlung and transition scattering and the latter effects have a much wider physical meaning than that in plasmas. But for other media we have not established up to the present time whether there exist additional effects which can also contribute to scattering and bremsstrahlung. For a plasma, such a proof exists and it is possible to demonstrate directly the way in which these processes appear because of fluctuations.

Third, in this review it is shown that the two processes of bremsstrahlung and scattering are deeply related to one another. It is shown that close to the eigenmodes of the system they can be converted from one to the other. It is also shown that the collective effects are large, especially for the frequencies close to the plasma eigenmodes, even when the processes of bremsstrahlung cannot be reduced to the processes of scattering.

Fourth, it is shown for the case of fast particles that the bremsstrahlung can be enhanced by many orders of magnitude for frequencies close to the plasma eigenmodes. This means that an observation of enhanced emission from a plasma at frequencies close to eigenmodes (plasma frequency, for example) does indicate the presence of an instability (as one would guess from a quick study of experimental data). It may also imply the presence in a plasma of a superthermal stable particle distribution. This comment seems to be important in the interpretation of the emission from thermonuclear plasma or solar corona plasma.

Fifth, for a nonrelativistic particle distribution, a new general expression is derived for matrix elements of bremsstrahlung of a wave of any polarisation valid in the presence of external magnetic fields.

Sixth, the collective effects in ion – ion and electron – electron collisions have been thoroughly analysed and it is shown that the collective effects substantially enhance these processes but that in a nonrelativistic plasma they are still much weaker than the bremsstrahlung in electron – ion collisions. It is also shown that the collective effects change quantitatively and qualitatively the bremsstrahlung processes in electron – ion collisions, but only for frequencies not very much different from the plasma frequency.

Seventh, it is shown that the dynamics of changes in particle distributions are largely governed by the processes of longitudinal wave bremsstrahlung, which for nonrelativistic particles are much more effective than the changes in particle distributions due to the bremsstrahlung of electromagnetic waves.

For these reasons (presentation of a new exact proof of familiar expressions, as well as the proof of the possibility of writing them in a compact form useful for applications), references are made only to general reviews in which the original investigations were cited. An exception is made only for some pioneer works important for the description of problems of principal importance.

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