# Physics of $\mathbf{B}_{\mathbf{c}}$-mesons 

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## Contents

1. Introduction ..... 1
2. Spectroscopy of $\mathbf{B}_{\mathbf{c}}$-mesons ..... 4
2.1 Mass spectrum of $B_{c}$-mesons; 2.2 Radiative transitions in the $B_{c}$ family; 2.3 Leptonic constant of $B_{c}$-meson
3. Decays of $\mathbf{B}_{\mathbf{c}}$-mesons ..... 14
3.1 Lifetime of $\mathrm{B}_{\mathrm{c}}$-mesons; 3.2 Semileptonic decays of $\mathrm{B}_{\mathrm{c}}$-mesons; 3.3 Hadronic decays of $\mathrm{B}_{\mathrm{c}}$-mesons ..... 254.1 $\mathrm{B}_{\mathrm{c}}$-meson production in $\mathrm{e}^{+} \mathrm{e}^{-}$-annihilation; 4.2 Hadronic production of $\mathrm{B}_{\mathrm{c}}$-mesons; 4.3 $\mathrm{B}_{\mathrm{c}}$-meson production
in $\nu \mathrm{N}$-, ep- and $\gamma \gamma$-collisions
4. Conclusion ..... 32
References ..... 34
Appendices ..... 36
I. Covariant quark model; II. Spectral densities for three-particle functions; III. Sum rule scheme for threepoint correlators


#### Abstract

The mass spectrum for the system ( $\overline{\mathrm{b}} \mathrm{c}$ ) is considered in the framework of potential models for the heavy quarkonium. Spin-dependent splittings, with account taken of the change of a constant representing the effective coulomb interaction between the quarks, and widths of radiative transitions between the ( $\overline{\mathrm{b}}$ ) levels are calculated. In the framework of QCD sum rules, the masses of the lightest vector $B_{c}^{*}$ and pseudoscalar $B_{c}$ states are estimated, the scaling relation for the leptonic constants of heavy quarkonia is derived, and the leptonic constant $f_{\mathrm{B}_{\mathrm{c}}}$ is evaluated. The $B_{c}$ decays are considered in the framework both of the potential models and of the QCD sum rules. The relations, following from the approximate spin symmetry for the heavy quarks in the heavy quarkonium, are analysed for the form factors of the semileptonic weak exclusive decays of $B_{c}$. The $B_{c}$ lifetime is evaluated with account taken of the corrections to the spectator mechanism of the decay, because of the quark binding into the meson. The total and differential cross sections of the $B_{c}$ production in different interactions are calculated. The analytic expressions for the fragmentational production cross sections of $B_{c}$ are derived. The possibility of the


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practical search for $B_{c}$ in current and planned experiments at electron-positron and hadron colliders is analysed.

## 1. Introduction

A complete picture for precise tests of the Standard Model [1] together with a search for effects from new physics requires direct measurement of the three-boson electroweak vertex; searches for Higgs particles [2], the supermultiplets [3] etc., at colliders of super-high energies (LEP200, LHC); as well as a study of CP-violation and a measurement of the fundamental parameters of the electroweak theory (first of all, in the heavy quark sector).

In the next ten years, the main efforts directed to the achievement of this programme will certainly be in the field of heavy quark physics both at the running colliders (LEP and Fermilab) and the B-meson factories (being planned in SLAC, KEK, and at HERA-B). In this case, the extraction of effects related to high values of the energy scale will be essentially determined by the accuracy of the theoretical and empirical knowledge of the mechanisms of the quark interactions at less than high energy and, primarily, about effects caused by QCD dynamics [4]. Therefore, experimental research on processes with heavy $\mathrm{c}-$, b -, and t -quarks has a special importance.

The presence of the small parameter $\Lambda_{\mathrm{QCD}} / m_{\mathrm{Q}}$, where $\Lambda_{\mathrm{QCD}}$ is the scale of the quark confinement and $m_{\mathrm{Q}}$ is the heavy quark mass, has allowed one to develop powerful tools for the study of QCD dynamics in heavy quark interactions. Such methods include the phenomenological potential models [5-10], the QCD sum rules [11-13], and effective heavy quark theory (EHQT) [14], which has been successfully applied to the study of hadrons containing a single heavy quark.

Thus, the investigation of processes with heavy quarks allows one to extract and to study nonperturbative QCD effects causing quark hadronisation, by means of the use of the heavy quark as the 'marked' atoms. The successful implementation of such a programme of studies becomes possible because of progress in the experimental technique of the detection and identification of particles (it is mainly related to the invention and the improvement of the vertex detectors, allowing one to observe the heavy quark particles because of its running gap from the primary vertex of the interaction).

Among the heavy quarkonia ( $\mathrm{Q} \overline{\mathrm{Q}}^{\prime}$ ), the ( $\overline{\mathrm{b}} \mathrm{c}$ ) system with open charm and beauty takes a particular place. In contrast to the hidden charm (c $\bar{c}$ ) and beauty ( $\mathrm{b} \overline{\mathrm{b}}$ ) families which have been studied in detail experimentally [15] and quite accurately described theoretically $[13,16,17]$, the heavy quarkonium ( $\overline{\mathrm{b}} \mathrm{c}$ ) - the family of $\mathrm{B}_{\mathrm{c}}$-mesons - has some specific production and decay mechanisms and spectroscopic features. The study of these mechanisms and features allows one to extend and clarify the quantitative understanding of QCD dynamics as well as to progress in the study of the most important parameters of the electroweak theory.

From the spectroscopy viewpoint, the ( $\overline{\mathrm{b}}$ ) is the heavy quarkonium whose spectrum can be quite reliably calculated in the framework of the QCD-motivated nonrelativistic potential models as well as in the QCD sum rules. ( $\overline{\mathrm{b}} \mathrm{c}$ ) is the only system composed of two heavy quarks where the description of its mass spectrum can test the self-consistency of the potential models and the QCD sum rules, whose parameters (the quark masses, for instance) have been fixed from the fitting of the spectroscopic data on the charmonium and bottomonium.

Thus, the study of $\mathrm{B}_{\mathrm{c}}$-family spectroscopy can serve to improve the quantitative characteristics of the quark models and the QCD sum rules, which are intensively applied in other fields of heavy quark physics (for example, when one extracts values of elements in the matrix of mixings of the heavy quark charged weak currents and one estimates contributions interfering with the effects of the CP-invariance violation, in the heavy hadron decays [18]).

Moreover, there is the problem of the precise description of the P-wave level splittings in the charmonium and bottomonium, when the experimental measurement produces an essential deviation from the values expected in some well-acknowledged quark models [19]. The study of the $B_{c}$-meson family can help in a solution of this problem.

In addition, the ( $\overline{\mathrm{b}}$ ) system is interesting because it allows one, in a new way, to use the phenomenological information obtained from the detailed experimental study of the charmonium and bottomonium. So, for example, $(\overline{\mathrm{b}})$ takes an intermediate place between the charmonium and bottomonium with respect to both the system level masses and the values of average distances between the heavy quarks.

As has been clarified, in the region of the average distances in the ( $c \bar{c}$ ) and ( $b \bar{b}$ ) systems, the heavy quark potential possesses simple scaling properties [8, 20, 57], which state that the kinetic energy of the heavy quarks is practically a constant value, independent of the quark flavours and the excitation level in the heavy quarkonium system. Furthermore, this leads to the fact that the heavy quarkonium level density (the distance between the $n L-$ and $n^{\prime} L$-levels) does not depend on the flavours of quarks
composing the heavy quarkonium. This regularity is quite accurately valid empirically for the (c $\bar{c}$ ) and ( $b \bar{b}$ ) systems and it can be used in the framework of the QCD sum rules, where a scaling relation connecting the leptonic constants of the S-wave levels in the different quarkonia [21, 22] is derived.

Further, having no strong and electromagnetic annihilation channels of decays, the excited ( $\overline{\mathrm{b}}$ ) system levels, being below the threshold of the decay into the BD-meson pair, will decay into the lightest basic pseudoscalar state $\mathrm{B}_{\mathrm{c}}^{+}\left(0^{-}\right)$ due to the radiative cascade transitions into the underlying levels. Therefore, the widths of the electromagnetic $(\gamma)$ and hadronic $(\pi \pi, \eta, \ldots)$ radiative decays of the given excitation into the other levels will compose its total width. As a result, the total widths of the excited levels in the ( $\overline{\mathrm{b}} \mathrm{c}$ ) system turn out to be two orders of magnitude less than the total widths of the charmonium and bottomonium excited levels, for which the annihilation channels are essential.

Moreover, maybe, the data on the radiative hadronic decays in the ( $\overline{\mathrm{b}}$ c)-family provide the possibility of solving some problems in the theory of hadronic transitions in heavy quarkonia (for example, the problem of the anomalous distribution over the $\pi \pi$-pair invariant mass in the decay of $\Upsilon^{\prime \prime} \rightarrow \Upsilon \pi \pi$ [23-28]).

Thus, on the one hand, the methods applied in heavy quark physics are able quite reliably to point out the spectroscopic characteristics of the ( $\overline{\mathrm{b}}$ ) system for one to make a purposefully directed experimental search of the given heavy quarkonium. On the other hand, the measurement of the spectroscopic data in the $\mathrm{B}_{\mathrm{c}}$-family would allow one to improve these methods for the extraction of the fundamental parameters of the Standard Model from both $\mathrm{B}_{\mathrm{c}}$-meson physics and the other fields of heavy quark physics.

Like the other mesons with open flavour, the basic state of the $\mathrm{B}_{\mathrm{c}}$-meson family, the pseudoscalar meson $\mathrm{B}_{\mathrm{c}}^{+}\left(0^{-}\right)$, is a long-living particle, decaying due to the weak interaction and having a lifetime comparable with the lifetimes of $B-$ and D-mesons, so this feature essentially distinguishes $B_{c}$ from the heavy quarkonia $\eta_{c}$ and $\eta_{b}$. Therefore, the study of $\mathrm{B}_{\mathrm{c}}$-meson decays is the rich field of heavy quark physics, where one extracts important information about both the QCD dynamics and the weak interactions.

The spectroscopic $\mathrm{B}_{\mathrm{c}}$-meson characteristics such as the leptonic constant, determining the width of the wave package of the ( $\bar{b} c$ ) system in the basic state, essentially determine the description of the $B_{c}$ decay modes, in which some specific features and effects are observed. First of all, the presence of the valent heavy quark-spectator leads to a large probability for the $B_{c}$ decay modes with the heavy mesons in the final state, i.e. in the decays $B_{c} \rightarrow \psi\left(\eta_{c}\right)$ and $B_{c} \rightarrow B_{s}^{(*)}[29-36]$. The large $\psi$-particle yield is interesting, in addition, in that the $\psi$-particle has a perfect experimental signature in the leptonic decay mode.

Furthermore, in the consideration of the semileptonic $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \psi\left(\eta_{\mathrm{c}}\right) l^{+} v$ decays, the nonrelativistic heavy quark motion inside the quarkonia leads to a major effect caused by large Coulomb-like $\alpha_{\mathrm{s}} / v$-corrections, which notably change the calculation results for these decays in the framework of the QCD sum rules [31]. It is only when these corrections are taken into account that the results of the QCD sum rules and the potential quark models become consistent.

Recently, the semileptonic transitions of the heavy quarks $\mathrm{Q} \rightarrow \mathrm{Q}^{\prime} l v$ in the framework of the effective heavy quark theory (EHQT) for hadrons with a single heavy quark
$(\mathrm{Q} \overline{\mathrm{q}}, \mathrm{Qqq})$, have been taken into consideration to determine the universal regularities [14], which permit, for example, the model independent extraction of the KobayashiMaskawa matrix element value $\left|V_{\mathrm{bc}}\right|$. This universality in the limit of $\Lambda_{\mathrm{QCD}} / m_{\mathrm{Q}} \rightarrow 0$ is caused by the heavy quark flavour-independence of the light quark motion in the gluon field of the static source (the heavy quark), so that the wave functions of such hadrons are universal.

In the case of the heavy quarkonium with two heavy quarks, the distances between the quarks depend on the values and ratios of its masses, i.e. the wave functions of the heavy quarkonia are not universal and depend on the quark flavours. However, in this case, one can neglect the low value of the spin-dependent splitting in the heavy quarkonium and suppose the wave functions of the $n L_{J^{-}}$ quarkonia to be $J$-independent. This fact finds expression in an approximate spin-symmetry for the heavy quarks, so it puts some relations on the form factors of the weak semileptonic exclusive decays of $B_{c}$ [37]. Such relations for the form factors are unique and characteristic of the $\mathrm{B}_{\mathrm{c}}$-meson and reflect the high degree of understanding of heavy quark decay dynamics, which nevertheless need direct experimental verification.

Considering the $B_{c}$ decays with the spectator $b$-quark, one has particularly to note the essential role of the effects caused by the fact that the c-quark is not in the free state, but in the bound one. The decrease of the phase space for the c-quark decay within the heavy quarkonium makes the probability of the decay to be $40 \%$ less than the probability in the $\mathrm{D}-$ and $\mathrm{D}_{\mathrm{s}}$-meson decays [34]. The annihilation channel of the weak $\mathrm{B}_{\mathrm{c}}$-meson decay [52], allowing one to determine the value of the quark wave function at the origin $|\Psi(0)|^{2}$, acquires an important meaning.

As in the case of the ( $\overline{\mathrm{b}}$ ) system spectroscopy, the heavy quark theory is able to make basic predictions on the mechanisms of the $\mathrm{B}_{\mathrm{c}}$-meson decays, whose measurement would allow one essentially to develop methods for description and also to use these methods for the precise investigation of the Standard Model as well as possible deviations from predictions of the latter.

In the case of $\mathrm{B}_{\mathrm{c}}$-meson production, a low value of the $\Lambda_{\mathrm{QCD}} / m_{\mathrm{Q}}$ ratio and, hence, the low value of the quark gluon coupling $\alpha_{\mathrm{s}} \sim 1 / \ln \left(m_{\mathrm{Q}} / \Lambda_{\mathrm{QCD}}\right) \ll 1$ allow one to take into consideration the pair production of the $b \bar{b}$ and $c \bar{c}$ quarks, from which the $B_{c}$-meson is formed, in the framework of perturbative QCD theory, and also, in a way, to factorise contributions caused by the perturbative production of heavy quarks and overcoming nonperturbative binding of the latter into the heavy quarkonium.

So, calculation of the cross sections of the S-wave $B_{c}$ state production in the Z-boson peak is enough to compute the matrix elements for the joint production of the $b \bar{b}$ and $\mathrm{c} \overline{\mathrm{c}}$ pairs in the colour-singlet state of the $(\overline{\mathrm{b}})$ pair with the fixed total spin of quarks ( $S=0,1$ ), when the quarks, being bound into the meson, move with one and the same velocity, equal to the meson velocity. After that, one has to multiply these matrix elements by the nonperturbative factor whose value is determined by the spectroscopic characteristics of the bound state (the quark masses and the leptonic constant, related to the probability of the observation of quarks with zero distance between them in the bound state) [38-47].

The last notion is caused by the fact that the characteristic virtualities of heavy quarks inside the heavy
quarkonium are much less than its masses, since the heavy quarks inside the bound states are moving nonrelativistically, otherwise the quark virtualities in its production are of the order of its masses. Therefore, considering the $B_{c}$ production, one can assume that, inside the meson, the $\overline{\mathrm{b}}-$ and c-quarks are close to the mass shell and practically at rest with respect to each other. Thus, after the extraction of the nonperturbative factor, the analysis of the $B_{c}$ heavy quarkonium production is determined by consideration of the matrix elements, calculated in the perturbation theory of QCD.

Note first of all that the necessity of the two-pair production of heavy quarks in the electromagnetic and strong processes for the $\mathrm{B}_{\mathrm{c}}$ yield leads to the fact that the leading order of perturbative QCD has an additional factor of the suppression $\sim \alpha_{s}^{2}$ with respect to the leading order of the perturbation theory for the production of the singleflavour heavy quarks; for example, the b $\bar{b}$ pair (see Fig. 7, 9), so $\sigma\left(\mathrm{B}_{\mathrm{c}}\right) / \sigma(\mathrm{b} \overline{\mathrm{b}}) \sim \alpha_{\mathrm{s}}^{2}|\Psi(0)|^{2} / m_{\mathrm{c}}^{3}$. This causes the low yield of the $\mathrm{B}_{\mathrm{c}}$-mesons with respect to the $B$-meson production.

The analysis of the leading approximation in the perturbative QCD for the $\mathrm{B}_{\mathrm{c}}$-meson production allows one to derive a number of analytical expressions for the $B_{c}$ production cross sections [38, 39], where one has especially to stress the expressions for the functions of the fragmentation of the heavy quark into the heavy quarkonium in the scaling limit $M^{2} / s \rightarrow 0$, so these functions are determined by the values of $\alpha_{s}$, the quark masses, and the leptonic constant of the meson [42-44]. Thus, fragmentational $B_{c}$ production can be reliably described by analytic expressions, and this opens new possibilities in the study of QCD dynamics, essential in the complete picture of heavy quark physics.

As one can show, fragmentational $B_{c}$ production certainly dominates in Z-boson decays [44], so that it can be straightforwardly studied at the LEP facilities. Moreover, one can analytically study notable spin effects in the fragmentation into the vector $\mathrm{B}_{\mathrm{c}}^{*}$-meson [48], decaying electromagnetically: $\mathrm{B}_{\mathrm{c}}^{*} \rightarrow \mathrm{~B}_{\mathrm{c}} \gamma$.

In hadronic $B_{c}$ production, patron processes at the energies comparable with the $B_{c}$ mass dominate, so that processes having the character of the fragmentational and also recombinational type [38, 46] (see Fig. 9) are essential.

Furthermore, the numerical estimates of the $\mathrm{B}_{\mathrm{c}}$-meson yield at the LEP and Tevatron colliders show that the fraction of $\mathrm{B}_{\mathrm{c}}$-mesons in the production of beauty hadrons is of the order of $10^{-3}$ [38-47, 49]. This leads to the fact that, at the current experimental facilities, a quite large number of $\mathrm{B}_{\mathrm{c}}-$ mesons are being produced.

Thus, one can point out the expected number of $\mathrm{B}_{\mathrm{c}}{ }^{-}$ mesons being produced at different colliders, and the differential $\mathrm{B}_{\mathrm{c}}$-characteristics whose experimental study would significantly clarify the picture of the QCD interactions of heavy quarks.

A solution of the problem of the experimental discovery and study of the $B_{c}$-mesons is determined, first, by the theoretical description of the features of the $\mathrm{B}_{\mathrm{c}}$-meson family (the spectroscopy, the production and decay mechanisms), and so the present review is devoted to this purpose. Second, this programme is determined by the experimental methodology at the current detectors, so that the latter would allow one to observe the events with $B_{c}$ production and decays, predicted by the theory.

As for the second part of the problem, at present, as mentioned, colossal progress being made, related to the use of the electronic vertex detectors that operate rapidly and allow one to isolate processes with long living particles (B, $\left.B_{c}, D\right)$ from the production processes (the technique of distinguishing the primary and secondary vertices), and also accurately to reconstruct the decay vertices of the particles in space [50].

The presence of distinct signatures in $\mathrm{B}_{\mathrm{c}}-$ meson decays and the practical possibility of registering these decay modes have led to a real chance of discovering the $\mathrm{B}_{\mathrm{c}}{ }^{-}$ meson at the LEP and Fermilab detectors [51], as well as to the sharp rise of theoretical interest in the $(\bar{b} c)$ system. The latter is reflected in the achievement of a large number of important results in the consideration of the heavy quark interaction mechanisms in the example of $\mathrm{B}_{\mathrm{c}}$-mesons. So, the present paper is devoted to the review of these results.

## 2. Spectroscopy of $\mathbf{B}_{\mathbf{c}}$-mesons

Some preliminary estimates of the bound state masses of the ( $\overline{\mathrm{b}} \mathrm{c}$ ) system have been made in Refs. [5, 6], devoted to the description of the properties of the charmonium ( $\overline{\mathrm{c}} \mathrm{c}$ ) and bottomonium ( $\overline{\mathrm{b} b}$ ), as well as in Ref. [52]. Recently in Refs [53, 35], the revised analysis of the $B_{c}$ spectroscopy has been performed in the framework of the potential approach and QCD sum rules.

In the present section we consider the $(\bar{b} c)$ spectroscopy with account taken of the change of the effective Coulomb interaction constant, defining spin-dependent splittings of the quarkonium levels. We calculate the widths of radiative transitions between the levels and analyse the leptonic constant $f_{\mathrm{B}_{\mathrm{c}}}$ in the framework of the QCD sum rules in the scheme, allowing one to derive the scaling relation for the leptonic constants of the heavy quarkonia.

### 2.1 Mass spectrum of $\mathbf{B}_{\mathbf{c}}$-mesons

The $\mathrm{B}_{\mathrm{c}}$-meson is the heavy ( $\overline{\mathrm{b}}$ ) quarkonium with open charm and beauty. It occupies an intermediate place in the mass spectrum of the heavy quarkonia between the ( $\overline{\mathrm{c}}$ ) charmonium and the ( $\overline{\mathrm{b}}$ ) bottomonium. The approaches made to the study of the charmonium and bottomonium can be used to describe the properties of the $\mathrm{B}_{\mathrm{c}}$-meson, and an experimental observation of $B_{c}$ could serve as a test of these approaches and could be used for the detailed quantitative study of the mechanisms of heavy quark production, hadronisation, and decays.

In the following, we obtain results on $\mathrm{B}_{\mathrm{c}}$ meson spectroscopy. We will show that below the threshold of hadronic decay of the ( $\overline{\mathrm{b}}$ ) system into the BD meson pair, there are 16 narrow bound states, cascadingly decaying into the lightest pseudoscalar $\mathrm{B}_{\mathrm{c}}^{+}\left(0^{-}\right)$state with mass $m\left(0^{-}\right) \approx 6.25 \mathrm{GeV}$.
2.1.1 Potential. The mass spectra of the charmonium and the bottomonium have been studied in detail experimentally [15] and are properly described in the framework of phenomenological potential models of nonrelativistic heavy quarks $[5-8,10]$. To describe the mass spectrum of the ( $\overline{\mathrm{b}} \mathrm{c}$ ) system, one would prefer to use the potentials whose parameters do not depend on the flavours of the heavy quarks, composing a heavy quarkonium, i.e. one would use the potentials which rather accurately describe the mass spectra of ( $\overline{\mathrm{c}} \mathrm{c}$ ) as well as ( $\overline{\mathrm{b}} \mathrm{b}$ ), with one and the same set of
potential parameters. The use of such potentials allows one to avoid an interpolation of the potential parameters from the values fixed by the experimental data on the ( $\overline{\mathrm{c}}$ ) and $(\overline{\mathrm{b}}$ ) systems, to the values in the intermediate region of the ( $\bar{b} \mathrm{c})$ system.

As has been shown in Ref. [20], with an accuracy up to an additive shift, the potentials, independent of heavy quark flavours $[5-8,10]$, coincide with each other in the region of the average distances between heavy quarks in the ( $\overline{\mathrm{c}}$ ) and ( $\overline{\mathrm{b}} \mathrm{b}$ ) systems, so
$0.1 \mathrm{fm}<r<1 \mathrm{fm}$,
although those potentials have different asymptotic behaviour in the regions of very small $(r \rightarrow 0)$ and very large $(r \rightarrow \infty)$ distances.

In the Cornell model [5], in accordance with the asymptotic freedom in QCD, the potential has a Cou-lomb-like behaviour at small distances, and the term confining the quarks rises linearly at large distances:

$$
\begin{equation*}
V_{\mathrm{C}}(r)=-\frac{4}{3} \frac{\alpha_{\mathrm{s}}}{r}+\frac{r}{a^{2}}+c_{0} \tag{2}
\end{equation*}
$$

so that

$$
\begin{align*}
& \alpha_{\mathrm{s}}=0.36, \quad a=2.34 \mathrm{GeV}^{-1} \\
& m_{\mathrm{c}}=1.84 \mathrm{GeV}, \quad c_{0}=-0.25 \mathrm{GeV} \tag{3}
\end{align*}
$$

The Richardson potential [7] and its modifications in Refs $[10,54]$ also correspond to the behaviour expected in the framework of QCD, so

$$
\begin{align*}
V_{\mathrm{R}}(r)=- & \int \frac{\mathrm{d}^{3} q}{(2 \pi)^{3}} \exp (\mathrm{i} \boldsymbol{r} \cdot \boldsymbol{q})\left[\frac{4}{3} \frac{48 \pi^{2}}{11 N_{c}-2 n_{f}} \frac{1}{q^{2} \ln \left(1+q^{2} / \Lambda^{2}\right)}\right] \\
=- & \int \frac{\mathrm{d}^{3} q}{(2 \pi)^{3}} \exp (\mathrm{i} \cdot \boldsymbol{r} \cdot \boldsymbol{q})\left(\frac{4}{3} \frac{48 \pi^{2}}{27}\right) \\
& \times\left[\frac{1}{q^{2} \ln \left(1+q^{2} / \Lambda^{2}\right)}-\frac{\Lambda^{2}}{q^{4}}\right]+\frac{8 \pi}{27} \Lambda^{2} r \tag{4}
\end{align*}
$$

with

$$
\begin{equation*}
\Lambda=0.398 \mathrm{GeV} \tag{5}
\end{equation*}
$$

In the region of the average distances between heavy quarks (1), the QCD-motivated potentials allow approximations in the forms of the power (Martin) or logarithmic potentials.

The Martin potential has the form [8]

$$
\begin{equation*}
V_{\mathrm{M}}(r)=-c_{\mathrm{M}}+d_{\mathrm{M}}\left(\Lambda_{\mathrm{M}} r\right)^{k} \tag{6}
\end{equation*}
$$

so that

$$
\begin{align*}
& \Lambda_{\mathrm{M}}=1 \mathrm{GeV}, \quad k=0.1 \\
& m_{b}=5.174 \mathrm{GeV}, \quad m_{c}=1.8 \mathrm{GeV} \\
& c_{\mathrm{M}}=8.064 \mathrm{GeV}, \quad d_{\mathrm{M}}=6.869 \mathrm{GeV} \tag{7}
\end{align*}
$$

The logarithmic potential is equal to [9]

$$
\begin{equation*}
V_{\mathrm{L}}(r)=c_{\mathrm{L}}+d_{\mathrm{L}} \ln \left(\Lambda_{\mathrm{L}} r\right) \tag{8}
\end{equation*}
$$

so that

$$
\begin{align*}
& \Lambda_{\mathrm{L}}=1 \mathrm{GeV} \\
& m_{b}=4.906 \mathrm{GeV}, \quad m_{c}=1.5 \mathrm{GeV} \\
& c_{\mathrm{L}}=-0.6635 \mathrm{GeV}, \quad d_{\mathrm{L}}=0.733 \mathrm{GeV} \tag{9}
\end{align*}
$$

The approximations of the nonrelativistic potential of heavy quarks in the region of distances (1) in the form of the power (6) and logarithmic (8) laws, allow one to study its scaling properties.

In accordance with the virial theorem, the average kinetic energy of the quarks in the bound state is determined by the following expression:

$$
\begin{equation*}
\langle T\rangle=\frac{1}{2}\left\langle\frac{r \mathrm{~d} V}{\mathrm{~d} r}\right\rangle . \tag{10}
\end{equation*}
$$

Then, the logarithmic potential allows one to conclude that for the quarkonium states one gets

$$
\begin{equation*}
\left\langle T_{\mathrm{L}}\right\rangle=\text { const } \tag{11}
\end{equation*}
$$

independently of the flavours of the heavy quarks composing the heavy quarkonium,

$$
\frac{d_{\mathrm{L}}}{2}=\text { const } \approx 0.367 \mathrm{GeV}
$$

In the Martin potential, the virial theorem (10) allows one to obtain the expression

$$
\begin{equation*}
\left\langle T_{\mathrm{M}}\right\rangle=\frac{k}{2+k}\left(c_{\mathrm{M}}+E\right) \tag{12}
\end{equation*}
$$

where $E$ is the binding energy of the quarks in the heavy quarkonium.

Phenomenologically, one has $|E| \ll c_{\mathrm{M}}$ [for example, $E(1 \mathrm{~S}, \mathrm{c} \overline{\mathrm{c}}) \approx-0.5 \mathrm{GeV}$ ], so that, neglecting the binding energy of the heavy quarks inside the heavy quarkonium, one can conclude that the average kinetic energy of the heavy quarks is a constant value, independent of the quark flavours and the number of the radial or orbital excitation. The accuracy of such an approximation for $\langle T\rangle$ is about $10 \%$, i.e. $|\Delta T / T| \approx 30-40 \mathrm{MeV}$.

From the Feynman - Hellmann theorem for the system with reduced mass $\mu$, one has

$$
\begin{equation*}
\frac{\mathrm{d} E}{\mathrm{~d} \mu}=-\frac{\langle T\rangle}{\mu} \tag{13}
\end{equation*}
$$

and, in accordance with condition (11), it follows that the difference of the energies for the radial excitations of the heavy quarkonium levels does not depend on the reduced mass of the $\mathrm{QQ}^{\prime}$ system

$$
\begin{equation*}
E(\bar{n}, \mu)-E(n, \mu)=E\left(\bar{n}, \mu^{\prime}\right)-E\left(n, \mu^{\prime}\right) \tag{14}
\end{equation*}
$$

Thus, in the approximation of both the low value for the binding energy of quarks and the zero value for the spin-

Table 1. The mass difference (in MeV ) for the two lightest vector states of different heavy systems, $\Delta M=M(2 \mathrm{~S})-M(1 \mathrm{~S})$.

| System | $\Upsilon$ | $\psi$ | $\mathrm{B}_{\mathrm{c}}$ | $\phi$ |
| :--- | :--- | :--- | :--- | :--- |
| $\Delta M$ | 563 | 588 | 585 | 660 |

dependent splittings of the levels, the heavy quarkonium state density does not depend on the heavy quark flavours:

$$
\begin{equation*}
\frac{\mathrm{d} n}{\mathrm{~d} M_{n}}=\text { const } \tag{15}
\end{equation*}
$$

The given statement has also been derived in Ref. [21] by means of Bohr-Sommerfeld quantisation of the S-wave states for the heavy quarkonium system with Martin potential [8].

Relations (14) and (15) are phenomenologically confirmed for the vector S-levels of the $b \bar{b}, ~ c \bar{c}, ~ s \bar{s}$ systems [15] (see Table 1). Thus, the structure of the nonsplit S-levels of the ( $\overline{\mathrm{b}}$ ) system must repeat not only qualitatively, but quantitatively the structure of the S-levels for the $\overline{\mathrm{b}} \mathrm{b}$ and $\overline{\mathrm{c}} \mathrm{c}$ systems, with an accuracy up to the overall additive shift of masses.

Moreover, in the framework of the QCD sum rules, the universality of the heavy quark nonrelativistic potential [the lack of dependence on the flavours and the scaling properties (11), (14), (15)] allows one to obtain the scaling relation for the leptonic constants of the S-wave quarkonia with mass $M$ [21],

$$
\begin{equation*}
\frac{f^{2}}{M}=\mathrm{const} \tag{16}
\end{equation*}
$$

independently of the heavy quark flavours in the regime when

$$
\left|m_{\mathrm{Q}}-m_{\mathrm{Q}^{\prime}}\right| \text { limited }, \quad \frac{\Lambda_{\mathrm{QCD}}}{m_{\mathrm{Q}, \mathrm{Q}^{\prime}}} \ll 1
$$

i.e., when one can neglect the heavy quark mass difference.

On the other hand, in the regime when the mass difference is not low, one has

$$
\begin{equation*}
\frac{f^{2}}{M}\left(\frac{M}{4 \mu}\right)^{2}=\mathrm{const} \tag{17}
\end{equation*}
$$

where

$$
\mu=\frac{m_{\mathrm{Q}} m_{\mathrm{Q}^{\prime}}}{m_{\mathrm{Q}}+m_{\mathrm{Q}^{\prime}}}
$$

Consider the mass spectrum of the ( $\overline{\mathrm{b}} \mathrm{c}$ ) system with the Martin potential [8].

Solving the Schrodinger equation with potential (6) and the parameters (7), one finds the $B_{c}$ mass spectrum and the characteristics of the radial wave functions $R(0)$ and $R^{\prime}(0)$, shown in Tables 2 and 3, respectively.

Table 3. The characteristics of the radial wave functions $R_{n S}(0)$ (in $\mathrm{GeV}^{3 / 2}$ ) and $R_{n \mathrm{P}}^{\prime}(0)$ (in $\mathrm{GeV}^{5 / 2}$ ), obtained from the Schrodinger equation.

| $n$ | Martin | $[53]$ |
| :--- | :--- | :--- |
| $R_{1 \mathrm{~S}}(0)$ | 1.31 | 1.28 |
| $R_{2 \mathrm{~S}}(0)$ | 0.97 | 0.99 |
| $R_{2 \mathrm{P}}^{\prime}(0)$ | 0.55 | 0.45 |
| $R_{3 \mathrm{P}}^{\prime}(0)$ | 0.57 | 0.51 |

Table 2. The energy levels of the ( $\overline{\mathrm{b}} \mathrm{c}$ ) system, calculated without taking into account relativistic corrections (in GeV ).

| $n$ | [52] | [55] | [54] | $n$ | [52] | [55] | [54] | $n$ | [52] | [55] | [54] |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 S | 6.301 | 6.315 | 6.344 | 2P | 6.728 | 6.735 | 6.763 | 3D | 7.008 | 7.145 | 7.030 |
| 2S | 6.893 | 7.009 | 6.910 | 3P | 7.122 | - | 7.160 | 4D | 7.308 | - | 7.365 |
| 3S | 7.237 | - | 7.024 | 4P | 7.395 | - | - | 5D | 7.532 | - | - |

Table 4. The average kinetic and orbital energies of the quark motion in the ( $\overline{\mathrm{b}} \mathrm{c}$ ) system (in GeV ).

| $n L$ | 1 S | 2 S | 2 P | 3 P | 3 D |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $\langle T\rangle$ | 0.35 | 0.38 | 0.37 | 0.39 | 0.39 |
| $\Delta V_{l}$ | 0.00 | 0.00 | 0.22 | 0.14 | 0.29 |

The average kinetic energy of the levels lying below the threshold for the decay of the ( $\overline{\mathrm{b}} \mathrm{c}$ ) system into the BD pair is presented in Table 4, wherein one can see that the term added to the radial potential due to the orbital rotation,

$$
\begin{equation*}
\Delta V_{l}=\frac{\boldsymbol{L}^{2}}{2 \mu r^{2}} \tag{18}
\end{equation*}
$$

weakly influences the value of the average kinetic energy, and the binding energy for the levels with $L \neq 0$ is essentially determined by the orbital rotation energy, which is approximately independent of the quark flavours (see Table 5), so that the structure of the nonsplit levels of the ( $\overline{\mathrm{b}}$ c) system with $L \neq 0$ must quantitatively repeat the structure of the charmonium and bottomonium levels, too.

Table 5. The average energy of the orbital motion in the heavy quarkonia, in the model with the Martin potential (in GeV ).

| System | $(\overline{\mathrm{c}})$ | $(\mathrm{b})$ | $(\overline{\mathrm{c}})$ |
| :--- | :---: | :---: | :---: |
| $\Delta V_{l}(2 P)$ | 0.23 | 0.22 | 0.21 |

2.1.2 Spin-dependent splitting of the ( $\overline{\mathbf{b}} \mathbf{c}$ ) quarkonium In accordance with the results of Refs [55, 56], one introduces the additional term to the potential to take into account the spin-orbital and spin-spin interactions, causing the splitting of the $n L$-levels ( $n$ is the principal quantum number, $L$ is the orbital momentum), so it has the form

$$
\begin{align*}
V_{\mathrm{SD}}(\boldsymbol{r})= & \left(\frac{\boldsymbol{L} \cdot \boldsymbol{S}_{\mathrm{c}}}{2 m_{\mathrm{c}}^{2}}+\frac{\boldsymbol{L} \cdot \boldsymbol{S}_{\mathrm{b}}}{2 m_{\mathrm{b}}^{2}}\right)\left(-\frac{\mathrm{d} V(r)}{r \mathrm{~d} r}+\frac{8}{3} \alpha_{\mathrm{s}} \frac{1}{r^{3}}\right) \\
& +\frac{4}{3} \alpha_{\mathrm{s}} \frac{1}{m_{\mathrm{c}} m_{\mathrm{b}}} \frac{\boldsymbol{L} \cdot \boldsymbol{S}}{r^{3}}+\frac{4}{3} \alpha_{\mathrm{s}} \frac{2}{3 m_{\mathrm{c}} m_{\mathrm{b}}} \boldsymbol{S}_{\mathrm{c}} \cdot \boldsymbol{S}_{\mathrm{b}}[4 \pi \delta(\boldsymbol{r})] \\
& +\frac{4}{3} \alpha_{\mathrm{s}} \frac{1}{m_{\mathrm{c}} m_{\mathrm{b}}}\left[3\left(\boldsymbol{S}_{\mathrm{c}} \cdot \boldsymbol{n}\right)\left(\boldsymbol{S}_{\mathrm{b}} \cdot \boldsymbol{n}\right)-\boldsymbol{S}_{\mathrm{c}} \cdot \boldsymbol{S}_{\mathrm{b}}\right] \frac{1}{r^{3}}, \boldsymbol{n}=\frac{\boldsymbol{r}}{\boldsymbol{r}} . \tag{19}
\end{align*}
$$

where $V(r)$ is the phenomenological potential confining the quarks. The first term takes into account the relativistic corrections to the potential $V(r)$; the second, third and fourth terms are the relativistic corrections coming from the account of the one-gluon exchange between the $b$ and $c$ quarks; $\alpha_{\mathrm{s}}$ is the effective constant of the quark-gluon interaction inside the ( $\overline{\mathrm{b}}$ ) system.

The value of the $\alpha_{s}$ parameter can be determined in the following way. The splitting of the S-wave heavy quarkonium $\left(\mathrm{Q}_{1} \overline{\mathrm{Q}}_{2}\right)$ is determined by the expression

$$
\begin{equation*}
\Delta M(n \mathrm{~S})=\frac{8}{9} \alpha_{\mathrm{s}} \frac{1}{m_{1} m_{2}}\left|R_{n \mathrm{~S}}(0)\right|^{2} \tag{20}
\end{equation*}
$$

where $R_{n \mathrm{~S}}(0)$ is the value of the radial wave function of the quarkonium, at the origin. Using the experimental value of the 1 S -state splitting in the $\mathrm{c} \overline{\mathrm{c}}$ system [15]

$$
\begin{equation*}
\Delta M(1 \mathrm{~S}, \mathrm{c} \overline{\mathrm{c}})=117 \pm 2 \mathrm{MeV} \tag{21}
\end{equation*}
$$

and the $R_{1 \mathrm{~S}}(0)$ value calculated in the potential model for the $c \bar{c}$ system, one gets the model-dependent value of the $\alpha_{s}(\psi)$ constant for the effective Coulomb interaction of heavy quarks (in the Martin potential, one has $\left.\alpha_{\mathrm{s}}(\psi)=0.44\right)$.

In Ref. [53] the effective constant value, fixed in the described way, has been applied to the description of not only the c $\bar{c}$ system, but also the $\bar{b} c$ and $\bar{b} b$ quarkonia. In the present paper we take into account the variation of the effective Coulomb interaction constant versus the reduced mass of the system $(\mu)$.

In the one-loop approximation at the momentum scale $p^{2}$, the 'running' coupling constant in QCD is determined by the expression

$$
\begin{equation*}
\alpha_{\mathrm{s}}\left(p^{2}\right)=\frac{4 \pi}{b \ln \left(p^{2} / \Lambda_{\mathrm{QCD}}^{2}\right)} \tag{22}
\end{equation*}
$$

where $b=11-2 n_{\mathrm{f}} / 3$, and $n_{\mathrm{f}}=3$, when one takes into account the contribution by the virtual light quarks, $p^{2}<m_{\mathrm{c}, \mathrm{b}}^{2}$.

In the model with the Martin potential, for the kinetic energy of quarks (ç $)$ inside $\psi$, one has

$$
\begin{equation*}
\left\langle T_{1 \mathrm{~S}}(\mathrm{c} \overline{\mathrm{c}})\right\rangle \approx 0.357 \mathrm{GeV} \tag{23}
\end{equation*}
$$

so that, using the expression for the kinetic energy,

$$
\begin{equation*}
\langle T\rangle=\frac{\left\langle p^{2}\right\rangle}{2 \mu}, \tag{24}
\end{equation*}
$$

one gets

$$
\begin{equation*}
\alpha_{\mathrm{s}}\left(p^{2}\right)=\frac{4 \pi}{b \ln \left(2\langle T\rangle \mu / \Lambda_{\mathrm{QCD}}^{2}\right)} \tag{25}
\end{equation*}
$$

so that $\alpha_{\mathrm{s}}(\psi)=0.44$ at

$$
\begin{equation*}
\Lambda_{\mathrm{QCD}} \approx 164 \mathrm{MeV} \tag{26}
\end{equation*}
$$

As has been noted in the previous section, the value of the kinetic energy of the quark motion depends weakly on the heavy quark flavours, and is practically constant; hence, the change of the effective $\alpha_{\mathrm{s}}$ coupling is basically determined by the variation of the reduced mass of the heavy quarkonium. In accordance with Eqns (25)-(26) and Table 4, for the ( $\overline{\mathrm{b}} \mathrm{c}$ ) system one has

| $n L$ | 1 S | 2 S | 2 P | 3P | 3D |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\alpha_{\mathrm{s}}$ | 0.394 | 0.385 | 0.387 | 0.382 | 0.383 |

Note that the Martin potential leads to the $R_{1 \mathrm{~S}}(0)$ values, which - with an accuracy up to $15 \%-20 \%$ agrees with the experimental values of the leptonic decay constants for the heavy $c \bar{c}$ and $b \bar{b}$ quarkonia. The leptonic constants are determined by the expression

$$
\begin{equation*}
\Gamma\left(\mathrm{Q} \overline{\mathrm{Q}} \rightarrow l^{+} l^{-}\right)=\frac{4 \pi}{3} e_{\mathrm{Q}}^{2} \alpha_{\mathrm{em}}^{2} \frac{f_{\mathrm{Q} \overline{\mathrm{Q}}}^{2}}{M_{\mathrm{Q} \overline{\mathrm{Q}}}} \tag{27}
\end{equation*}
$$

where $e_{Q}$ is the heavy quark charge.
In the nonrelativistic model one has

$$
\begin{equation*}
f_{\mathrm{Q} \overline{\mathrm{Q}}}=\left(\frac{3}{\pi M_{\mathrm{Q} \overline{\mathrm{Q}}}}\right)^{1 / 2} R_{1 \mathrm{~S}}(0) \tag{28}
\end{equation*}
$$

For the effective Coulomb interaction of the heavy quarks in the basic 1 S -state one has

$$
\begin{equation*}
R_{1 \mathrm{~S}}^{\mathrm{C}}(0)=2\left(\frac{4}{3} \mu \alpha_{\mathrm{s}}\right)^{3 / 2} \tag{29}
\end{equation*}
$$

Table 6. The leptonic decay constants of the heavy quarkonia, the values, measured experimentally and obtained in the model with the Martin potential, in the model with the effective Coulomb interaction and from the scaling relation (SR) (in MeV ).

| Model | Exp. [15] | Martin | Coulomb | SR |
| :--- | :--- | :--- | :--- | :--- |
| $f_{\psi}$ | $410 \pm 15$ | $547 \pm 80$ | $426 \pm 60$ | $410 \pm 40$ |
| $f_{\mathrm{B}_{\mathrm{c}}}$ | - | $510 \pm 80$ | $456 \pm 70$ | $460 \pm 60$ |
| $f_{\mathrm{Y}}$ | $715 \pm 15$ | $660 \pm 90$ | $772 \pm 120$ | $715 \pm 70$ |

One can see from Table 6 that, taking into account the variation of the effective $\alpha_{\mathrm{s}}$ constant versus the reduced mass of the heavy quarkonium [see Eqn (25)], the Coulomb wave functions give the values of the leptonic constants for the heavy 1S-quarkonia, so that in the framework of the accuracy of the potential models, those values agree with the experimental values and the values obtained by the solution of the Schrodinger equation with the given potential.

Consideration of the variation of the effective Coulomb interaction constant becomes especially important for the $\Upsilon$-particles, for which $\alpha_{s}(\Upsilon) \approx 0.33$ instead of the fixed value $\alpha_{s}=0.44$.

Thus, calculating the splitting of the ( $\overline{\mathrm{b}}$ ) levels, we take into account the $\alpha_{\mathrm{s}}$ dependence on the reduced mass of the heavy quarkonium.

As one can see from Eqn (19), in contrast to the $L S$ coupling in the ( $\overline{\mathrm{c}} \mathrm{c}$ ) and (bb) systems, there is $j j$-coupling in the heavy quarkonium, where the heavy quarks have different masses $\left[\right.$ here, $\boldsymbol{L} \boldsymbol{S}_{\mathrm{c}}$ is diagonalised at the given $\boldsymbol{J}_{\mathrm{c}}$ momentum, $\left(\boldsymbol{J}_{\mathrm{c}}=\boldsymbol{L}+\boldsymbol{S}_{\mathrm{c}}, \boldsymbol{J}=\boldsymbol{J}_{\mathrm{c}}+\boldsymbol{S}_{\mathrm{b}}\right), \boldsymbol{J}$ is the total spin of the system]. We use the following spectroscopic notation for the split levels of the $(\overline{\mathrm{b}})$ system $-n^{2 j_{\mathrm{c}}} L_{J}$.

One can easily show that, independently of the total spin $J$ projection, one has

$$
\begin{align*}
\left.\left.\right|^{2 L+1} L_{L+1}\right\rangle= & |J=L+1, S=1\rangle \\
\left.\left.\right|^{2 L-1} L_{L-1}\right\rangle= & |J=L-1, S=1\rangle, \\
\left.\left.\right|^{2 L+1} L_{L}\right\rangle= & \sqrt{\frac{L}{2 L+1}}|J=L, S=1\rangle \\
& +\sqrt{\frac{L+1}{2 L+1}}|J=L, S=0\rangle, \\
\left|\left.\right|^{2 L-1} L_{L}\right\rangle= & \sqrt{\frac{L+1}{2 L+1}}|J=L, S=1\rangle \\
& \quad-\sqrt{\frac{L}{2 L+1}}|J=L, S=0\rangle, \tag{30}
\end{align*}
$$

where $|J, S\rangle$ are the state vectors with the given values of the total quark spin $S=S_{\mathrm{c}}+\boldsymbol{S}_{\mathrm{b}}$, so that the potential terms of the order of $1 / m_{\mathrm{c}} m_{\mathrm{b}}, 1 / m_{\mathrm{b}}^{2}$ lead, generally speaking, to the mixing of levels with the different $J_{\mathrm{c}}$ values at the given $J$ values. The tensor forces [the last term in Eqn (19)] are equal to zero at $L=0$ or $S=0$.

To calculate values of the level shifts appearing because of the spin - spin and spin-orbital interactions, one has to take the averaged expression (19) over the wave functions of the corresponding states.

The averaging over the angle variables can be performed in the following standard way. Let us represent the matrix element of the unit vector $\boldsymbol{n}=\boldsymbol{r} / r$ pair in the form

$$
\begin{equation*}
\langle L, m| n^{p} n^{q}\left|L, m^{\prime}\right\rangle=a\left(L^{p} L^{q}+L^{q} L^{p}\right)_{m m^{\prime}}+b \delta^{p q} \delta_{m m^{\prime}}, \tag{31}
\end{equation*}
$$

where $L$ are the orbital momentum matrices in the corresponding irreducible representation.

From the conditions of the normalisation of the unit vector, $\left\langle n^{p} n^{q}\right\rangle \delta^{p q}=1$, the orthogonality of the radius-vector to the orbital momentum, $n^{p} L^{p}=0$, the commutation relations for the angular momentum, $\left[L^{p} ; L^{q}\right]=\mathrm{i} \varepsilon^{p l l} L_{l}$, one finds the values of constants $a$ and b in Eqn (31):

$$
\begin{align*}
& a=-\frac{1}{4 \boldsymbol{L}^{2}-3},  \tag{32}\\
& b=\frac{2 \boldsymbol{L}^{2}-1}{4 \boldsymbol{L}^{2}-3} . \tag{33}
\end{align*}
$$

Note further that from the condition for the quark spins $S_{\mathrm{Q}}^{p} S_{\mathrm{Q}}^{q}+S_{\mathrm{Q}}^{q} S_{\mathrm{Q}}^{p}=\delta^{p q} / 2$ it follows, that

$$
\begin{equation*}
3\left(n^{p} n^{q}-\frac{1}{3} \delta^{p q}\right) S_{\mathrm{c}}^{p} S_{\mathrm{b}}^{q}=\frac{3}{2}\left(n^{p} n^{q}-\frac{1}{3} \delta^{p q}\right) S^{p} S^{q} \tag{34}
\end{equation*}
$$

Thus, (see also Ref. [57])

$$
\begin{align*}
\left\langle 6\left(n^{p} n^{q}-\frac{1}{3} \delta^{p q}\right)\right. & \left.S_{\mathrm{c}}^{p} S_{\mathrm{b}}^{q}\right\rangle=-\frac{1}{4 \boldsymbol{L}^{2}-3} \\
& \times\left[6(\boldsymbol{L} \cdot \boldsymbol{S})^{2}+3(\boldsymbol{L} \cdot \boldsymbol{S})-2 \boldsymbol{L}^{2} \boldsymbol{S}^{2}\right] \tag{35}
\end{align*}
$$

Using Eqns (30) and (35), for the level shifts, calculated in the perturbation theory at $S=1$, one gets the following formulae:

$$
\begin{align*}
& \Delta E_{n^{1} \mathrm{~S}_{0}}=-\alpha_{\mathrm{s}} \frac{2}{3 m_{\mathrm{c}} m_{\mathrm{b}}}\left|R_{n S}(0)\right|^{2},  \tag{36}\\
& \Delta E_{n^{1} \mathrm{~S}_{1}}= \alpha_{\mathrm{s}} \frac{2}{9 m_{\mathrm{c}} m_{\mathrm{b}}}\left|R_{n S}(0)\right|^{2},  \tag{37}\\
& \Delta E_{n^{3} \mathrm{P}_{2}}=\alpha_{\mathrm{s}} \frac{6}{5 m_{\mathrm{c}} m_{\mathrm{b}}}\left\langle\frac{1}{r^{3}}\right\rangle \\
&+\frac{1}{4}\left(\frac{1}{m_{\mathrm{c}}^{2}}+\frac{1}{m_{\mathrm{b}}^{2}}\right)\left\langle-\frac{\mathrm{d} V(r)}{r \mathrm{~d} r}+\frac{8}{3} \alpha_{\mathrm{s}} \frac{1}{r^{3}}\right\rangle,  \tag{38}\\
& \Delta E_{n^{1} \mathrm{P}_{0}}=-\alpha_{\mathrm{s}} \frac{4}{m_{\mathrm{c}} m_{\mathrm{b}}}\left\langle\frac{1}{r^{3}}\right\rangle \\
&-\frac{1}{2}\left(\frac{1}{m_{\mathrm{c}}^{2}}+\frac{1}{m_{\mathrm{b}}^{2}}\right)\left\langle-\frac{\mathrm{d} V(r)}{r \mathrm{~d} r}+\frac{8}{3} \alpha_{\mathrm{s}} \frac{1}{r^{3}}\right\rangle,  \tag{39}\\
& \Delta E_{n^{5} \mathrm{D}_{3}=}=\alpha_{\mathrm{s}} \frac{52}{21 m_{\mathrm{c}} m_{\mathrm{b}}}\left\langle\frac{1}{r^{3}}\right\rangle \\
&+\frac{1}{2}\left(\frac{1}{m_{\mathrm{c}}^{2}}+\frac{1}{m_{\mathrm{b}}^{2}}\right)\left\langle-\frac{\mathrm{d} V(r)}{r \mathrm{~d} r}+\frac{8}{3} \alpha_{\mathrm{s}} \frac{1}{r^{3}}\right\rangle,  \tag{40}\\
& \Delta E_{n^{3} \mathrm{D}_{1}=}=-\alpha_{\mathrm{s}} \frac{92}{21 m_{\mathrm{c}} m_{\mathrm{b}}}\left\langle\frac{1}{r^{3}}\right\rangle \\
&-\frac{3}{4}\left(\frac{1}{m_{\mathrm{c}}^{2}}+\frac{1}{m_{\mathrm{b}}^{2}}\right)\left\langle-\frac{\mathrm{d} V(r)}{r \mathrm{~d} r}+\frac{8}{3} \alpha_{\mathrm{s}} \frac{1}{r^{3}}\right\rangle, \tag{41}
\end{align*}
$$

where $R_{n S}(0)$ are the radial wave functions at $L=0$, and $\langle\ldots\rangle$ denote the average values calculated under the wave functions $R_{n L}(r)$.

The mixing matrix elements have the forms

$$
\begin{align*}
& \left\langle{ }^{3} \mathrm{P}_{1}\right| \Delta E\left|{ }^{3} \mathrm{P}_{1}\right\rangle=-\alpha_{\mathrm{s}} \frac{2}{9 m_{\mathrm{c}} m_{\mathrm{b}}}\left\langle\frac{1}{r^{3}}\right\rangle \\
& +\left(\frac{1}{4 m_{\mathrm{c}}^{2}}-\frac{5}{12 m_{\mathrm{b}}^{2}}\right)\left\langle-\frac{\mathrm{d} V(r)}{r \mathrm{~d} r}+\frac{8}{3} \alpha_{\mathrm{s}} \frac{1}{r^{3}}\right\rangle,  \tag{42}\\
& \left\langle{ }^{1} \mathrm{P}_{1}\right| \Delta E\left|{ }^{1} \mathrm{P}_{1}\right\rangle=-\alpha_{\mathrm{s}} \frac{4}{9 m_{\mathrm{c}} m_{\mathrm{b}}}\left\langle\frac{1}{r^{3}}\right\rangle \\
& +\left(-\frac{1}{2 m_{\mathrm{c}}^{2}}+\frac{1}{6 m_{\mathrm{b}}^{2}}\right)\left\langle-\frac{\mathrm{d} V(r)}{r \mathrm{~d} r}+\frac{8}{3} \alpha_{\mathrm{s}} \frac{1}{r^{3}}\right\rangle,  \tag{43}\\
& \left\langle{ }^{3} \mathrm{P}_{1}\right| \Delta E\left|{ }^{1} \mathrm{P}_{1}\right\rangle=-\alpha_{\mathrm{s}} \frac{2 \sqrt{2}}{9 m_{\mathrm{c}} m_{\mathrm{b}}}\left\langle\frac{1}{r^{3}}\right\rangle \\
& -\frac{\sqrt{2}}{6 m_{\mathrm{b}}^{2}}\left\langle-\frac{\mathrm{d} V(r)}{r \mathrm{~d} r}+\frac{8}{3} \alpha_{\mathrm{s}} \frac{1}{r^{3}}\right\rangle,  \tag{44}\\
& \left.\left.\left\langle{ }^{5} \mathrm{D}_{2}\right| \Delta E\right|^{5} \mathrm{D}_{2}\right\rangle=-\alpha_{\mathrm{s}} \frac{4}{15 m_{\mathrm{c}} m_{\mathrm{b}}}\left\langle\frac{1}{r^{3}}\right\rangle \\
& +\left(\frac{1}{2 m_{\mathrm{c}}^{2}}-\frac{1}{5 m_{\mathrm{b}}^{2}}\right)\left\langle-\frac{\mathrm{d} V(r)}{r \mathrm{~d} r}+\frac{8}{3} \alpha_{\mathrm{s}} \frac{1}{r^{3}}\right\rangle,  \tag{45}\\
& \left\langle{ }^{3} \mathrm{D}_{2}\right| \Delta E\left|{ }^{3} \mathrm{D}_{2}\right\rangle=-\alpha_{\mathrm{s}} \frac{8}{15 m_{\mathrm{c}} m_{\mathrm{b}}}\left\langle\frac{1}{r^{3}}\right\rangle \\
& +\left(-\frac{3}{4 m_{\mathrm{c}}^{2}}+\frac{9}{20 m_{\mathrm{b}}^{2}}\right)\left\langle-\frac{\mathrm{d} V(r)}{r \mathrm{~d} r}+\frac{8}{3} \alpha_{\mathrm{s}} \frac{1}{r^{3}}\right\rangle \text {, }  \tag{46}\\
& \left\langle{ }^{5} \mathrm{D}_{2}\right| \Delta E\left|{ }^{3} \mathrm{D}_{2}\right\rangle=-\alpha_{\mathrm{s}} \frac{2 \sqrt{6}}{15 m_{\mathrm{c}} m_{\mathrm{b}}}\left\langle\frac{1}{r^{3}}\right\rangle \\
& -\frac{\sqrt{6}}{10} \frac{m_{\mathrm{b}}^{2}}{2}\left\langle-\frac{\mathrm{d} V(r)}{r \mathrm{~d} r}+\frac{8}{3} \alpha_{\mathrm{s}} \frac{1}{r^{3}}\right\rangle . \tag{47}
\end{align*}
$$

As one can see from Eqn (37), the S-level splitting is essentially determined by the $\left|R_{n S}(0)\right|$ value, which can be related to the leptonic decay constants of the S -states $\left(0^{-}\right.$, $1^{-}$). Section 2.3 is devoted to the calculation of these constants in different ways. We only note here that, with enough accuracy, the predictions of different potential models on the $\left|R_{1 \mathrm{~S}}(0)\right|$ value are in agreement with each other as well as with predictions from other approaches.

For the 2 P level, the mixing matrices of the states with the total quark spin $S=1$ and $S=0$ have the forms

$$
\begin{align*}
& \left|2 \mathrm{P}, 1^{\prime+}\right\rangle=0.294|S=1\rangle+0.956|S=0\rangle,  \tag{48}\\
& \left|2 \mathrm{P}, 1^{+}\right\rangle=0.956|S=1\rangle-0.294|S=0\rangle, \tag{49}
\end{align*}
$$

so that in the $1^{+}$state the probability of the total quark spin value $S=1$ is equal to

$$
\begin{equation*}
w_{1}(2 \mathrm{P})=0.913 \tag{50}
\end{equation*}
$$

For the 3P-level one has

$$
\begin{align*}
& \left|3 \mathrm{P}, 1^{\prime+}\right\rangle=0.371|S=1\rangle+0.929|S=0\rangle,  \tag{51}\\
& \left|3 \mathrm{P}, 1^{+}\right\rangle=0.929|S=1\rangle-0.371|S=0\rangle, \tag{52}
\end{align*}
$$

so that

$$
\begin{equation*}
w_{1}(3 \mathrm{P})=0.863 \tag{53}
\end{equation*}
$$



Figure 1. The mass spectrum of the $\mathrm{B}_{\mathrm{c}}$-meson states with account taken of splittings.

Table 7. The masses (in GeV ) of the bound ( $\overline{\mathrm{b}}$ ) states below the threshold of the decay into the (BD) meson pair ( $*$ is the present paper).

| State | $*$ | $[53]$ | $[54]$ | State | $*$ | $[53]$ | $[54]$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $1{ }^{1} \mathrm{~S}_{0}$ | 6.253 | 6.264 | 6.314 | $3^{1} \mathrm{P}_{0}$ | 7.088 | 7.108 | 7.134 |
| $1^{1} \mathrm{~S}_{0}$ | 6.317 | 6.337 | 6.355 | $3 \mathrm{P} 1^{+}$ | 7.113 | 7.135 | 7.159 |
| $2^{1} \mathrm{~S}_{0}$ | 6.867 | 6.856 | 6.889 | $3 \mathrm{P} 1^{\prime+}$ | 7.124 | 7.142 | - |
| $2^{1} \mathrm{~S}_{0}$ | 6.902 | 6.899 | 6.917 | $3^{3} \mathrm{P}_{2}$ | 7.134 | 7.153 | 7.166 |
| $2{ }^{1} \mathrm{P}_{0}$ | 6.683 | 6.700 | 6.728 | $3 \mathrm{D} 2^{-}$ | 7.001 | 7.009 | - |
| $2 \mathrm{P} 1^{+}$ | 6.717 | 6.730 | 6.760 | $3^{5} \mathrm{D}_{3}$ | 7.007 | 7.005 | - |
| $2 \mathrm{P} 1^{\prime+}$ | 6.729 | 6.736 | - | $3^{3} \mathrm{D}_{1}$ | 7.008 | 7.012 | - |
| $2^{3} \mathrm{P}_{2}$ | 6.743 | 6.747 | 6.773 | $3 \mathrm{D} 2^{\prime-}$ | 7.016 | 7.012 | - |

Table 8. The masses (in GeV ) of the lightest pseudoscalar $B_{c}$ and vector $\mathrm{B}_{\mathrm{c}}^{*}$ states in different models $(*$ is the present paper).

| State | $*$ | $[55]$ | $[54]$ | $[58]$ | $[6]$ | $[59]$ | $[21,65]$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $0^{-}$ | 6.253 | 6.249 | 6.314 | 6.293 | 6.270 | 6.243 | 6.246 |
| $1^{-}$ | 6.317 | 6.339 | 6.354 | 6.346 | 6.340 | 6.320 | 6.319 |
| State | $[53]$ | $[60]$ | $[61]$ | $[62]$ | $[63]$ | $[64]$ | $[35]$ |
| $0^{-}$ | 6.264 | 6.320 | 6.256 | 6.276 | 6.286 | - | 6.255 |
| $1^{-}$ | 6.337 | 6.370 | 6.329 | 6.365 | 6.328 | 6.320 | 6.330 |

For the 3D-level one gets

$$
\begin{align*}
& \left|3 \mathrm{D}, 2^{\prime-}\right\rangle=-0.566|S=1\rangle+0.825|S=0\rangle  \tag{54}\\
& \left|3 \mathrm{D}, 2^{-}\right\rangle=0.825|S=1\rangle+0.566|S=0\rangle \tag{55}
\end{align*}
$$

so that

$$
\begin{equation*}
w_{2}(3 \mathrm{D})=0.680 \tag{56}
\end{equation*}
$$

The $B_{c}$ mass spectrum, with account having been taken of the calculated splittings, is shown in Fig. 1 and Table 7.

The masses of the $B_{c}$ mesons have also been calculated in Ref. [66]. As one can see from Tables 2 and 8, the place of the 1 S -level in the ( $\overline{\mathrm{b}}$ ) system $[m(1 \mathrm{~S}) \approx 6.3 \mathrm{GeV}]$ is predicted by the potential models with the rather high
accuracy $\delta m(1 \mathrm{~S}) \approx 30 \mathrm{MeV}$, and the 1 S-level splitting into the vector and pseudoscalar states is about $m\left(1^{-}\right)-m\left(0^{-}\right) \approx 70 \mathrm{MeV}$.
2.1.3 $B_{c}$ meson masses from QCD sum rules. Potential model estimates for the masses of the lightest ( $\overline{\mathrm{b}} \mathrm{c}$ ) states are in agreement with the results of the calculations for the vector and pseudoscalar ( $\overline{\mathrm{b}} \mathrm{c}$ ) states in the framework of the QCD sum rules $[35,36,67]$, where the accuracy of the calculation is lower than the accuracy of the potential models, because the results essentially depend on both the modelling of the nonresonant hadronic part of the current correlator (the continuum threshold) and the parameter of the sum rule scheme (the moment number for the spectral density of the current correlator or the Borel transformation parameter),

$$
\begin{equation*}
m^{\mathrm{SR}}\left(0^{-}\right) \approx m^{\mathrm{SR}}\left(1^{-}\right) \approx 6.3-6.5 \mathrm{GeV} \tag{57}
\end{equation*}
$$

As has been shown in Ref. [11], for the lightest vector quarkonium, the following QCD sum rules apply

$$
\begin{align*}
\frac{f_{\mathrm{V}}^{2} M_{\mathrm{V}}^{2}}{m_{\mathrm{V}}^{2}-q^{2}}= & \frac{1}{\pi} \int_{s_{\mathrm{i}}}^{s_{\mathrm{h}}} \frac{\mathrm{~d} s}{s-q^{2}} \operatorname{Im} \Pi_{\mathrm{V}}^{\mathrm{QCD}(\text { pert })}(s) \\
& +\Pi_{\mathrm{V}}^{\mathrm{QCD}(\text { nonpert })}\left(q^{2}\right) \tag{58}
\end{align*}
$$

where $f_{\mathrm{V}}$ is the leptonic constant of the vector $(\overline{\mathrm{b}} \mathrm{c})$ state with the mass $M_{\mathrm{V}}$,

$$
\begin{align*}
& \mathrm{i} f_{\mathrm{V}} M_{\mathrm{V}} \varepsilon_{\mu}^{\lambda} \exp (\mathrm{i} p x)=\langle 0| J_{\mu}(x)|V(p, \lambda)\rangle  \tag{59}\\
& J_{\mu}(x)=\bar{c}(x) \gamma_{\mu} b(x) \tag{60}
\end{align*}
$$

where $\lambda, p$ are the $\mathrm{B}_{\mathrm{c}}^{*}$ polarisation and momentum, respectively, and

$$
\begin{align*}
& \begin{array}{l}
\begin{array}{l}
\mathrm{d}^{4} x \exp (\mathrm{i} q x)\langle 0| T J_{\mu}(x) J_{v}(0)|0\rangle \\
\\
\quad=\left(-g_{\mu \nu}+\frac{q_{\mu} q_{\nu}}{q^{2}}\right) \Pi_{\mathrm{V}}^{\mathrm{QCD}}+q_{\mu} q_{\nu} \Pi_{\mathrm{S}}^{\mathrm{QCD}}
\end{array} \\
\Pi_{\mathrm{V}}^{\mathrm{QCD}}\left(q^{2}\right)=\Pi_{\mathrm{V}}^{\mathrm{QCD}(\text { pert })}(s)+\Pi_{\mathrm{V}}^{\mathrm{QCD}(\text { nonpert })}\left(q^{2}\right)
\end{array} \\
& \Pi_{\mathrm{V}}^{\mathrm{QCD}(\text { nonpert })}\left(q^{2}\right)=\sum C_{i}\left(q^{2}\right) O^{i} \tag{61}
\end{align*}
$$

where $O^{i}$ are the vacuum expectation values of the composite operators such as $\langle m \bar{\psi} \psi\rangle,\left\langle\alpha_{\mathrm{s}} G_{\mu \nu}^{2}\right\rangle$, etc. The Wilson coefficients are calculable in the perturbation theory of QCD. $s_{\mathrm{i}}=\left(m_{\mathrm{c}}+m_{\mathrm{b}}\right)^{2}$ is the kinematical threshold of the perturbative contribution, $M_{\mathrm{V}}^{2}>s_{\mathrm{i}}, s_{\mathrm{th}}$ is the threshold of the nonresonant hadronic contribution, which is considered to be equal to the perturbative contribution at $s>s_{\text {th }}$.

Considering the respective correlators, one can write down the sum rules, analogous to Eqn (58), for the scalar and pseudoscalar states.

One believes that the sum rule (58) must rather accurately be valid at $q^{2}<0$. For the $n$th derivative of Eqn (58) at $q^{2}=0$ one gets

$$
\begin{align*}
f_{\mathrm{V}}^{2}\left(M_{\mathrm{V}}^{2}\right)^{-n} & =\frac{1}{\pi} \int_{s_{\mathrm{i}}}^{s_{\mathrm{th}}} \frac{\mathrm{~d} s}{s^{n+1}} \operatorname{Im} \Pi_{\mathrm{V}}^{\mathrm{QCD}(\text { pert })}(s) \\
& +\frac{(-1)^{n}}{n!} \frac{\mathrm{d}^{n}}{\mathrm{~d}\left(q^{2}\right)^{n}} \Pi_{\mathrm{V}}^{\mathrm{QCD}(\text { nonpert })}\left(q^{2}\right) \tag{64}
\end{align*}
$$

so, considering the ratio of the $n$th derivative to the $(n+1)$ th one, one can obtain the value of the vector $\mathrm{B}_{\mathrm{c}}^{*}$ meson mass. The calculated result depends on the $n$ value in the sum rules (64), because of account having been taken both of the finite number of terms in the perturbation theory expansion and of the restricted set of composite operators.

The analogous procedure can be performed in the sum rule scheme with the Borel transform, leading to the dependence of the results on the transformation parameter.

As one can see from Eqn (64), the result obtained in the framework of the QCD sum rules depends on the choice of the values for the hadronic continuum threshold energy and the current masses of quarks. Then, this dependence causes large errors in the estimates of the masses for the lightest pseudoscalar, vector, and scalar ( $\overline{\mathrm{b}} \mathrm{c}$ ) states.

Thus, the QCD sum rules give estimates of the quark binding energy in the quarkonium, and the estimates are in agreement with the results of the potential models, but sum rules involve a considerable parametric uncertainty.

### 2.2 Radiative transitions in the $B_{c}$ family

The $B_{c}$ mesons have no annihilation channels for the decays due to QCD and electromagnetic interactions. Therefore, the mesons, lying below the threshold for the production of $B$ and $D$ mesons, will, in a cascade, decay into the $0^{-}(1 \mathrm{~S})$ state by emission of $\gamma$ quanta and $\pi$ mesons.

Theoretical estimates of the transitions between the levels with the emission of the $\pi$ mesons have uncertainties, and the electromagnetic transitions are quite accurately calculable.
2.2.1 Electromagnetic transitions. The formulae for the radiative E1-transitions have the form [17, 68]

$$
\begin{align*}
\Gamma\left(\bar{n} \mathrm{P}_{J} \rightarrow\right. & \left.n^{1} \mathrm{~S}_{1}+\gamma\right)=\frac{4}{9} \alpha_{\mathrm{em}} Q_{\mathrm{eff}}^{2} \omega^{3} I^{2}(\bar{n} \mathrm{P} ; n \mathrm{~S}) w_{J}(\bar{n} \mathrm{P}) \\
\Gamma\left(\bar{n} \mathrm{P}_{J} \rightarrow\right. & \left.n^{1} \mathrm{~S}_{0}+\gamma\right)=\frac{4}{9} \alpha_{\mathrm{em}} Q_{\mathrm{eff}}^{2} \omega^{3} \\
& \times I^{2}(\bar{n} \mathrm{P} ; n \mathrm{~S})\left[1-w_{J}(\bar{n} \mathrm{P})\right] \\
\Gamma\left(n^{1} \mathrm{~S}_{1} \rightarrow\right. & \left.\bar{n} \mathrm{P}_{J}+\gamma\right)=\frac{4}{27} \alpha_{\mathrm{em}} Q_{\mathrm{eff}}^{2} \omega^{3} \\
& \times I^{2}(n \mathrm{~S} ; \bar{n} \mathrm{P})(2 J+1) w_{J}(\bar{n} \mathrm{P}) \\
\Gamma\left(n^{1} \mathrm{~S}_{0} \rightarrow\right. & \left.\bar{n} \mathrm{P}_{J}+\gamma\right)=\frac{4}{9} \alpha_{\mathrm{em}} Q_{\mathrm{eff}}^{2} \omega^{3} \\
& \times I^{2}(n \mathrm{~S} ; \bar{n} \mathrm{P})(2 J+1)\left[1-w_{J}(\bar{n} \mathrm{P})\right] \\
\Gamma\left(\bar{n} \mathrm{P}_{J} \rightarrow\right. & \left.n \mathrm{D}_{J^{\prime}}+\gamma\right)=\frac{4}{27} \alpha_{\mathrm{em}} Q_{\mathrm{eff}}^{2} \omega^{3} \\
& \times I^{2}(n \mathrm{D} ; \bar{n} \mathrm{P})\left(2 J^{\prime}+1\right) w_{J}(\bar{n} \mathrm{P}) w_{J^{\prime}}(n \mathrm{D}) S_{J^{\prime}} \\
\Gamma\left(n \mathrm{D}{ }_{J} \rightarrow\right. & \left.\bar{n} \mathrm{P}_{J^{\prime}}+\gamma\right)=\frac{4}{27} \alpha_{\mathrm{em}} Q_{\mathrm{eff}}^{2} \omega^{3} \\
& \times I^{2}(n \mathrm{D} ; \bar{n} \mathrm{P})\left(2 J^{\prime}+1\right) w_{J^{\prime}}(\bar{n} \mathrm{P}) w_{J}(n \mathrm{D}) S_{J^{\prime} J} \tag{65}
\end{align*}
$$

where $\omega$ is the photon energy, $\alpha_{\mathrm{em}}$ is the electromagnetic fine structure constant.

In Eqn (65) one uses

$$
\begin{equation*}
Q_{\mathrm{eff}}=\frac{m_{\mathrm{c}} Q_{\overline{\mathrm{b}}}-m_{\mathrm{b}} Q_{\mathrm{c}}}{m_{\mathrm{c}}+m_{\mathrm{b}}} \tag{66}
\end{equation*}
$$

where $Q_{\mathrm{c}, \mathrm{b}}$ are the electric charges of the quarks. For the $\mathrm{B}_{\mathrm{c}}$ meson with the parameters from the Martin potential, one gets $Q_{\text {eff }}=0.41$.
$w_{J}(n L)$ is the probability that the spin $S=1$ in the $n L$ state, so that $w_{0}(n \mathrm{P})=w_{2}(n \mathrm{P})=1, w_{1}(n \mathrm{D})=w_{3}(n \mathrm{D})=1$, and the $w_{1}(n \mathrm{P}), w_{2}(n \mathrm{D})$ values have been presented in the previous section [see Eqns (50), (53), (56)].

The statistical factor $S_{J J^{\prime}}$ takes values [68]

| $J$ | $J^{\prime}$ | $S_{J J^{\prime}}$ |
| :---: | :---: | :---: |
| 0 | 1 | 2 |
| 1 | 1 | $1 / 2$ |
| 1 | 2 | $9 / 10$ |
| 2 | 1 | $1 / 50$ |
| 2 | 2 | $9 / 50$ |
| 2 | 3 | $18 / 25$. |

The $I\left(\bar{n} L ; n L^{\prime}\right)$ value is expressed through the radial wave functions,

$$
\begin{equation*}
I\left(\bar{n} L ; n L^{\prime}\right)=\left|\int R_{\bar{n} L}(r) R_{n L^{\prime}}(r) r^{3} \mathrm{~d} r\right| . \tag{67}
\end{equation*}
$$

In the model with the Martin potential, for the set of transitions one obtains (in $\mathrm{GeV}^{-1}$ ) [52]

$$
\begin{align*}
& I(1 \mathrm{~S}, 2 \mathrm{P})=1.568, \quad I(1 \mathrm{~S}, 3 \mathrm{P})=0.255 \\
& I(2 \mathrm{~S}, 2 \mathrm{P})=2.019, \quad I(2 \mathrm{~S}, 3 \mathrm{P})=2.704 \\
& I(3 \mathrm{D}, 2 \mathrm{P})=2.536, \quad \mathrm{I}(3 \mathrm{D}, 3 \mathrm{P})=2.416 \tag{68}
\end{align*}
$$

For the dipole magnetic transitions one has [5, 17, 68]

$$
\begin{equation*}
\Gamma\left(\bar{n}^{1} S_{\mathrm{i}} \rightarrow n^{1} S_{\mathrm{f}}+\gamma\right)=\frac{16}{3} \mu_{\mathrm{eff}}^{2} \omega^{3}\left(2 S_{\mathrm{f}}+1\right) A_{\mathrm{if}}^{2} \tag{69}
\end{equation*}
$$

Table 9. The energies (in MeV ) and widths (in keV ) of the electromagnetic E1-transitions in the (bc) family ( $*$ is the present paper).

| Transition | $\omega$ | $\Gamma[*]$ | $\Gamma[53]$ |
| :--- | :--- | ---: | ---: |
| $2 \mathrm{P}_{2} \rightarrow 1 \mathrm{~S}_{1}+\gamma$ | 426 | 102.9 | 112.6 |
| $2 \mathrm{P}_{0} \rightarrow 1 \mathrm{~S}_{1}+\gamma$ | 366 | 65.3 | 79.2 |
| $2 \mathrm{P} 1^{\prime+} \rightarrow 1 \mathrm{~S}_{1}+\gamma$ | 412 | 8.1 | 0.1 |
| $2 \mathrm{P}^{+} \rightarrow 1 \mathrm{~S}_{1}+\gamma$ | 400 | 77.8 | 99.5 |
| $2 \mathrm{P} 1^{\prime+} \rightarrow 1 \mathrm{~S}_{0}+\gamma$ | 476 | 131.1 | 56.4 |
| $2 \mathrm{P}^{+} \rightarrow 1 \mathrm{~S}_{0}+\gamma$ | 464 | 11.6 | 0.0 |
| $3 \mathrm{P}_{2} \rightarrow 1 \mathrm{~S}_{1}+\gamma$ | 817 | 19.2 | 25.8 |
| $3 \mathrm{P}_{0} \rightarrow 1 \mathrm{~S}_{1}+\gamma$ | 771 | 16.1 | 21.9 |
| $3 \mathrm{P} 1^{\prime+} \rightarrow 1 \mathrm{~S}_{1}+\gamma$ | 807 | 2.5 | 2.1 |
| $3 \mathrm{P} 1^{+} \rightarrow 1 \mathrm{~S}_{1}+\gamma$ | 796 | 15.3 | 22.1 |
| $3 \mathrm{P} 1^{\prime+} \rightarrow 1 \mathrm{~S}_{0}+\gamma$ | 871 | 20.1 | - |
| $3 \mathrm{P} 1^{+} \rightarrow 1 \mathrm{~S}_{0}+\gamma$ | 860 | 3.1 | - |
| $3 \mathrm{P}_{2} \rightarrow 2 \mathrm{~S}_{1}+\gamma$ | 232 | 49.4 | 73.8 |
| $3 \mathrm{P}_{0} \rightarrow 2 \mathrm{~S}_{1}+\gamma$ | 186 | 25.5 | 41.2 |
| $3 \mathrm{P} 1^{\prime+} \rightarrow 2 \mathrm{~S}_{1}+\gamma$ | 222 | 5.9 | 5.4 |
| $3 \mathrm{P} 1^{+} \rightarrow 2 \mathrm{~S}_{1}+\gamma$ | 211 | 32.1 | 54.3 |
| $3 \mathrm{P} 1^{\prime+} \rightarrow 2 \mathrm{~S}_{0}+\gamma$ | 257 | 58.0 | - |
| $3 \mathrm{P} 1^{+} \rightarrow 2 \mathrm{~S}_{0}+\gamma$ | 246 | 8.1 | - |
| $2 \mathrm{~S}_{1} \rightarrow 2 \mathrm{P}_{2}+\gamma$ | 159 | 14.8 | 17.7 |
| $2 \mathrm{~S}_{1} \rightarrow 2 \mathrm{P}_{0}+\gamma$ | 219 | 7.7 | 7.8 |
| $2 \mathrm{~S}_{1} \rightarrow 2 \mathrm{P} 1^{\prime+}+\gamma$ | 173 | 1.0 | 0.0 |
| $2 \mathrm{~S}_{1} \rightarrow 2 \mathrm{P} 1^{+}+\gamma$ | 185 | 12.8 | 14.5 |
| $2 \mathrm{~S}_{0} \rightarrow 2 \mathrm{P} 1^{\prime+}+\gamma$ | 138 | 15.9 | 5.2 |
| $2 \mathrm{~S}_{0} \rightarrow 2 \mathrm{P} 1^{+}+\gamma$ | 150 | 1.9 | 0.0 |
|  |  |  |  |

where

$$
\begin{align*}
& A_{\text {if }}=\int R_{\bar{n} S}(r) R_{n S}(r) j_{0}\left(\frac{\omega r}{2}\right) r^{2} \mathrm{~d} r, \\
& \mu_{\mathrm{eff}}=\frac{1}{2} \frac{\sqrt{\alpha_{\mathrm{em}}}}{2 m_{\mathrm{c}} m_{\mathrm{b}}}\left(Q_{\mathrm{c}} m_{\mathrm{b}}-Q_{\overline{\mathrm{b}}} m_{\mathrm{c}}\right) . \tag{70}
\end{align*}
$$

Note, in contrast to the $\psi$ - and $\Upsilon$-particles, the total width of the $B_{c}^{*}$ meson is equal to the width of its radiative decay into the $\mathrm{B}_{\mathrm{c}}\left(0^{-}\right)$state.

The electromagnetic widths, calculated in accordance with Eqns (65) and (69), and the frequencies of the emitted photons are presented in Tables 9-11.

Note that the E0-transitions with the conversion of a virtual $\gamma$-quantum into a lepton pair can take place. Moreover, due to the tensor forces, the states with $J>0$ and $S=1$ can, in addition to the $L$-wave, have an admixture of $|L \pm 2|$-waves, giving a quadrupole moment to the corresponding states and causing the E2-transitions. However, these transitions are suppressed by the additional factor $\alpha_{\mathrm{em}}$ in the first case, and by the small value of amplitude, determining, say, the probability of the admixture appearance of the D -wave in the $1^{-}(n S)$-state.

Thus, the registration of the cascade electromagnetic transitions in the ( $\overline{\mathrm{b}}$ ) family can be used for the observa-

Table 10. The energies (in MeV ) and widths (in keV ) of the electromagnetic E1-transitions in the (bc) family ( $*$ is the present paper).

| Transition | $\omega$ | $\Gamma[*]$ | $\Gamma[53]$ |
| :---: | :---: | :---: | :---: |
| $3 \mathrm{P}_{2} \rightarrow 3 \mathrm{D}_{1}+\gamma$ | 126 | 0.1 | 0.2 |
| $3 \mathrm{P}_{2} \rightarrow 3 \mathrm{D} 2^{\prime-}+\gamma$ | 118 | 0.5 | - |
| $3 \mathrm{P}_{2} \rightarrow 3 \mathrm{D} 2^{-}+\gamma$ | 133 | 1.5 | 3.2 |
| $3 \mathrm{P}_{2} \rightarrow 3 \mathrm{D}_{3}+\gamma$ | 127 | 10.9 | 17.8 |
| $3 \mathrm{P}_{0} \rightarrow 3 \mathrm{D}_{1}+\gamma$ | 80 | 3.2 | 6.9 |
| $3 \mathrm{P}_{1}^{\prime+} \rightarrow 3 \mathrm{D}_{1}+\gamma$ | 116 | 0.3 | 0.4 |
| $3 \mathrm{P} 1^{+} \rightarrow 3 \mathrm{D}_{1}+\gamma$ | 105 | 1.6 | 0.3 |
| $3 \mathrm{P} 1^{\prime+} \rightarrow 3 \mathrm{D} 2^{\prime-}+\gamma$ | 108 | 3.5 | - |
| $3 \mathrm{P} 1^{+} \rightarrow 3 \mathrm{D} 2^{-}+\gamma$ | 112 | 3.9 | 9.8 |
| $3 \mathrm{P} 1^{\prime+} \rightarrow 3 \mathrm{D} 2^{-}+\gamma$ | 123 | 2.5 | 11.5 |
| $3 \mathrm{P} 1^{+} \rightarrow 3 \mathrm{D} 2^{\prime-}+\gamma$ | 97 | 1.2 | - |
| $3 \mathrm{D}_{3} \rightarrow 2 \mathrm{P}_{2}+\gamma$ | 264 | 76.9 | 98.7 |
| $3 \mathrm{D}_{1} \rightarrow 2 \mathrm{P}_{0}+\gamma$ | 325 | 79.7 | 88.6 |
| $3 \mathrm{D}_{1} \rightarrow 2 \mathrm{P}^{\prime \prime+}+\gamma$ | 279 | 3.3 | 0.0 |
| $3 \mathrm{D}_{1} \rightarrow 2 \mathrm{P}^{+}+\gamma$ | 291 | 39.2 | 49.3 |
| $3 \mathrm{D}_{1} \rightarrow 2 \mathrm{P}_{2}+\gamma$ | 265 | 2.2 | 2.7 |
| $3 \mathrm{D} 2^{\prime-} \rightarrow 2 \mathrm{P}_{2}+\gamma$ | 273 | 6.8 | - |
| $3 \mathrm{D} 2^{\prime-} \rightarrow 2 \mathrm{P}_{2}+\gamma$ | 258 | 12.2 | 24.7 |
| $3 \mathrm{D} 2^{\prime-} \rightarrow 2 \mathrm{P} 1^{\prime+}+\gamma$ | 287 | 46.0 | 92.5 |
| $3 \mathrm{D} 2^{\prime-} \rightarrow 2 \mathrm{P} 1^{+}+\gamma$ | 301 | 25.0 | - |
| $3 \mathrm{D} 2^{-} \rightarrow 2 \mathrm{P} 1^{\prime+}+\gamma$ | 272 | 18.4 | 0.1 |
| $3 \mathrm{D} 2^{-} \rightarrow 2 \mathrm{P} 1^{+}+\gamma$ | 284 | 44.6 | 88.8 |

Table 11. The energies (in MeV ) and widths (in keV ) of the electromagnetic M1-transitions in the ( $\overline{\mathrm{b}}$ ) family ( $*$ is the present paper).

| Transition | $\omega$ | $\Gamma[*]$ | $\Gamma[53]$ |
| :--- | ---: | :--- | :--- |
| $2 \mathrm{~S}_{1} \rightarrow 1 \mathrm{~S}_{0}+\gamma$ | 649 | 0.098 | 0.123 |
| $2 \mathrm{~S}_{0} \rightarrow 1 \mathrm{~S}_{1}+\gamma$ | 550 | 0.096 | 0.093 |
| $1 \mathrm{~S}_{1} \rightarrow 1 \mathrm{~S}_{0}+\gamma$ | 64 | 0.060 | 0.135 |
| $2 \mathrm{~S}_{1} \rightarrow 2 \mathrm{~S}_{0}+\gamma$ | 35 | 0.010 | 0.029 |

tion of the higher ( $\overline{\mathrm{b}} \mathrm{c}$ ) excitations, having no annihilation channels for the decays.
2.2.2 Hadmonic transitions. In the framework of QCD the consideration of the hadronic transitions between the states of the heavy quarkonium family is based on the multipole expansion for the gluon emission by the heavy nonrelativistic quarks [23], with subsequent hadronisation of gluons, independently of the heavy quark motion.

In the leading approximation over the velocity of the heavy quark motion, the action corresponding to the heavy quark coupling to the external gluon field,

$$
\begin{equation*}
S_{\mathrm{int}}=-g \int \mathrm{~d}^{4} x A_{\mu}^{a}(x) j_{a}^{\mu}(x) \tag{71}
\end{equation*}
$$

can be expressed in the form

$$
\begin{equation*}
S_{\mathrm{int}}=g \int \mathrm{~d} t r^{k} E_{k}^{a}(t, \boldsymbol{x}) \frac{\lambda_{a}^{i j}}{2} \Psi_{n}(\boldsymbol{r}) \Psi_{f}^{j i}(\boldsymbol{r}) K\left(s_{n}, f\right) \mathrm{d}^{3} \boldsymbol{r} \tag{72}
\end{equation*}
$$

where $\Psi_{n}(\boldsymbol{r})$ is the wave function of the quarkonium emitting a gluon, $\Psi_{f}^{i j}(\boldsymbol{r})$ is the wave function of the colouroctet state of the quarkonium, $K\left(s_{n}, f\right)$ corresponds to the spin factor (in the leading approximation, the heavy quark spin is decoupled from the interaction with the gluons).

Then the matrix element for the E1-E1 transition of the quarkonium $n L_{J} \rightarrow n^{\prime} L_{J^{\prime}}^{\prime}+\mathrm{gg}$ can be written in the form

$$
\begin{align*}
M\left(n L_{J}\right. & \left.\rightarrow n^{\prime} L_{J^{\prime}}^{\prime}+\mathrm{gg}\right)=4 \pi \alpha_{\mathrm{s}} E_{k}^{a} E_{m}^{b} \\
& \times \int \mathrm{d}^{3} r \mathrm{~d}^{3} r^{\prime} r_{k} r_{m}^{\prime} G_{s_{n^{\prime}}, s_{n}}^{a b}\left(r, r^{\prime}\right) \Psi_{n L_{J}}(r) \Psi_{n^{\prime} L_{\prime^{\prime}}^{\prime}}\left(r^{\prime}\right), \tag{73}
\end{align*}
$$

where $G_{s_{\bar{n}}, s_{n}}^{a b}\left(r, r^{\prime}\right)$ corresponds to the propagator of the colour-octet state of the heavy quarkonium

$$
\begin{equation*}
G=\frac{1}{\varepsilon-H_{\mathrm{Q} \overline{\mathrm{Q}}}^{\mathrm{c}}} \tag{74}
\end{equation*}
$$

where $H_{\mathrm{Q} \overline{\mathrm{Q}}}^{\mathrm{c}}$ is the Hamiltonian of the coloured state.
One can see from Eqn (73) that the determination of the transition matrix element depends on both the wave function of the quarkonium and the Hamiltonian $H_{\mathrm{Q} Q}^{\mathrm{c}}$. Thus, the theoretical consideration of the hadronic transitions in the quarkonium family is model dependent.

In a number of papers [24], the potential approach has been developed for the calculation of the values such as in Eqn (53). In papers [25] it is shown that nonperturbative conversion of the gluons into the $\pi$ meson pair allows one to give a consideration in the framework of the low-energy theorems in QCD, so that this consideration agrees with the studies performed in the framework of PCAC and soft pion techniques [26].

However, as follows from Eqn (73) and the WignerEckart theorem, the differential width for the E1-E1 transition allows a representation in the form [24]

$$
\begin{align*}
& \frac{\mathrm{d} \Gamma}{\mathrm{~d} m^{2}}\left(n L_{J} \rightarrow n^{\prime} L_{J^{\prime}}^{\prime}+\mathrm{h}\right)=\left(2 J^{\prime}+1\right) \\
& \times \sum_{k=0}^{2}\left\{\begin{array}{ccc}
k & L & L^{\prime} \\
s & J^{\prime} & J
\end{array}\right\}^{2} A_{k}\left(L, L^{\prime}\right) \tag{75}
\end{align*}
$$

where $m^{2}$ is the invariant mass of the light hadron system $\mathrm{h} ;\{\ldots\}$ are $6 j$-symbols; $A_{k}\left(L, L^{\prime}\right)$ is the contribution by the irreducible tensor of the rank equal to $k=0,1,2 ; s$ is the total quark spin inside the quarkonium.

In the limit of soft pions, one has $A_{1}\left(L, L^{\prime}\right)=0$.

From Eqns (73) and (75) it follows that, with an accuracy up to the difference in the phase spaces, the widths of the hadronic transitions in the ( $\mathrm{Q} \overline{\mathrm{Q}}$ ) and $\left(\mathrm{Q} \overline{\mathrm{Q}}^{\prime}\right)$ quarkonia are related to the following expression [23, 24]:

$$
\begin{equation*}
\frac{\Gamma\left(\mathrm{Q} \overline{\mathrm{Q}}^{\prime}\right)}{\Gamma(\mathrm{Q} \overline{\mathrm{Q}})}=\frac{\left\langle r^{2}\left(\mathrm{Q} \overline{\mathrm{Q}}^{\prime}\right)\right\rangle^{2}}{\left\langle r^{2}(\mathrm{Q} \overline{\mathrm{Q}})\right\rangle^{2}} \tag{76}
\end{equation*}
$$

Then the experimental data on the transitions $\psi^{\prime} \rightarrow \mathrm{J} / \psi+\pi \pi, \quad \mathrm{Y}^{\prime} \rightarrow \mathrm{\Upsilon}+\pi \pi, \quad \psi(3770) \rightarrow \mathrm{J} / \psi+\pi \pi[27]$ allow one to extract the values of $A_{k}\left(L, L^{\prime}\right)$ for the transitions $2 \mathrm{~S} \rightarrow 1 \mathrm{~S}+\pi \pi$ and $3 \mathrm{D} \rightarrow 1 \mathrm{~S}+\pi \pi$ [53].

The invariant mass spectrum of the $\pi$ meson pair has the universal form [25, 26]

$$
\begin{equation*}
\frac{1}{\Gamma} \frac{\mathrm{~d} \Gamma}{\mathrm{~d} m}=B \frac{\left|\boldsymbol{k}_{\pi \pi}\right|}{M^{2}}\left(2 x^{2}-1\right) \sqrt{x^{2}-1} \tag{77}
\end{equation*}
$$

where $x=m / 2 m_{\pi},\left|\boldsymbol{k}_{\pi \pi}\right|$ is the $\pi \pi$ pair momentum.
Estimates for the widths of the hadronic transitions in the ( $\overline{\mathrm{b}} \mathrm{c}$ ) family have been made in Ref. [53]. The hadronic transition widths, which have values comparable to the electromagnetic transition width values, are presented in Table 12.

Table 12. The widths (in keV ) of the radiative hadronic transitions in the ( $\overline{\mathrm{b}} \mathrm{c}$ ) family.

| Transition | $\Gamma[53]$ | Transition | $\Gamma[53]$ |
| :--- | :--- | :--- | :--- |
| $2 \mathrm{~S}_{0} \rightarrow 1 \mathrm{~S}_{0}+\pi \pi$ | 50 | $3 \mathrm{D}_{2} \rightarrow 1 \mathrm{~S}_{1}+\pi \pi$ | 32 |
| $2 \mathrm{~S}_{1} \rightarrow 1 \mathrm{~S}_{1}+\pi \pi$ | 50 | $3 \mathrm{D}_{3} \rightarrow 1 \mathrm{~S}_{1}+\pi \pi$ | 31 |
| $3 \mathrm{D}_{1} \rightarrow 1 \mathrm{~S}_{1}+\pi \pi$ | 31 | $3 \mathrm{D}_{2} \rightarrow 1 \mathrm{~S}_{0}+\pi \pi$ | 32 |

The transitions in the ( $\overline{\mathrm{b}}$ ) family with the emission of $\eta$ mesons are suppressed by the low value of the phase space.

Thus, registration of the hadronic transitions in the ( $\overline{\mathrm{b}}$ ) family with the emission of $\pi$ meson pairs can be used to observe the higher 2S- and 3D-excitations of the basic state.

### 2.3 Leptonic constant of $B_{c}$ meson

As we have seen in Section 2.1, the value of the leptonic constant of the $B_{c}$ meson determines the splitting of the basic 1S-state of the ( $\overline{\mathrm{b}} \mathrm{c}$ ) system. Moreover, the higher excitations in the ( $\overline{\mathrm{b}} \mathrm{c}$ ) system transform, in a cascade, into the lightest $0^{-}$state of $B_{c}$, whose widths of decays are essentially determined by the value of $f_{\mathrm{B}_{\mathrm{c}}}$, too.

In the quark models [69-71] used to calculate the weak decay widths of mesons, the leptonic constant, as the parameter, determines the quark wave packet inside the meson (generally, the wave function is chosen in the oscillator form), therefore, the practical problem of the extraction of the value for the weak charged current mixing matrix element $\left|V_{\mathrm{bc}}\right|$ from the data on the weak $\mathrm{B}_{\mathrm{c}}$ decays can be solved only at the known value of $f_{\mathrm{B}_{\mathrm{c}}}$.

Thus, the leptonic constant $f_{\mathrm{B}_{\mathrm{c}}}$ is the most important quantity, characterising the bound state of the ( $\overline{\mathrm{b}} \mathrm{c}$ ) system. In the present section we calculate the value of $f_{\mathrm{B}_{\mathrm{c}}}$ in different ways.

To describe the bound states of the quarks, the use of nonperturbative approaches is required. The bound states of the heavy quarks allow one to consider simplifications, connected both to large values of the quark masses $\Lambda_{\mathrm{QCD}} / m_{\mathrm{Q}} \ll 1$ and to the nonrelativistic quark motion $v \rightarrow 0$. Therefore the value of $f_{\mathrm{B}_{\mathrm{c}}}$ can be quite reliably

Table 13. The leptonic $B_{c}$ meson constant (in MeV ), calculated in the different potential models (the accuracy $\sim 15 \%$ ).

| Model | Martin | Coulomb | $[6]$ | $[53]$ | $[72]$ | $[73,74]$ | $[75]$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $f_{\mathrm{B}_{\mathrm{c}}}$ | 510 | 460 | 570 | 495 | 410 | 600 | 500 |

determined in the framework of the potential models and the QCD sum rules [11].
2.3.1 $f_{\mathrm{B}_{\mathrm{C}}}$ from potential models. In the framework of the nonrelativistic potential models, the leptonic constants of the pseudoscalar and vector mesons [see Eqns (59) and (60)]

$$
\begin{align*}
& \langle 0| \bar{c}(x) \gamma_{\mu} b(x)\left|\mathrm{B}_{\mathrm{c}}^{*}(p, \varepsilon)\right\rangle=\mathrm{i} f_{\mathrm{V}} M_{\mathrm{V}} \varepsilon_{\mu} \exp (\mathrm{i} p x),  \tag{78}\\
& \langle 0| \bar{c}(x) \gamma_{5} \gamma_{\mu} b(x)\left|\mathrm{B}_{\mathrm{c}}(p)\right\rangle=\mathrm{i} f_{\mathrm{P}} p_{\mu} \exp (\mathrm{i} p x), \tag{79}
\end{align*}
$$

are determined by expression (28)

$$
\begin{equation*}
f_{\mathrm{V}}=f_{\mathrm{P}}=\left(\frac{3}{\pi M_{\mathrm{B}_{\mathrm{c}}(1 \mathrm{~S})}}\right)^{1 / 2} R_{1 \mathrm{~S}}(0) \tag{80}
\end{equation*}
$$

where $R_{1 \mathrm{~S}}(0)$ is the radial wave function of the 1 S -state of the ( $\overline{\mathrm{b}}$ ) system, at the origin. The wave function is calculated by solving the Schrodinger equation with different potentials $[5-8,10,54]$ in the quasipotential approach [72] or by solving the Bethe-Salpeter equation with instant potential and in the expansion up to the second order in quark velocity $v / c[73,74]$.

The values of the leptonic $\mathrm{B}_{\mathrm{c}}$ meson constant, calculated in different potential models and effective Coulomb potential with the 'running' constant $\alpha_{\mathrm{s}}$, determined in Section 2.1, are presented in Table 13.

Thus, within the accuracy of this approach, the potential quark models give $f_{\mathrm{B}_{\mathrm{c}}}$ values which are in a good agreement with each other, so that

$$
\begin{equation*}
f_{\mathrm{B}_{\mathrm{c}}}^{\mathrm{pot}}=500 \pm 80 \mathrm{MeV} . \tag{81}
\end{equation*}
$$

2.3.2 $f_{\mathbf{B}_{\mathrm{c}}}$ from QCD sum rules. In the framework of the QCD sum rules [11], expressions (58)-(64) have been derived for the vector states. The expressions have been considered at $q^{2}<0$ in the schemes of the spectral density moments (64) or with the application of the Borel transform [11].

As one can see from Eqns (58)-(64), the result of the QCD sum rule calculations is determined not only by physical parameters such as the quark and meson masses, but also by the unphysical parameters of the sum rule scheme such as the value of the spectral density moment or the Borel transformation parameter.

In the QCD sum rules, this unphysical dependence of the $f_{\mathrm{B}_{\mathrm{c}}}$ value is due to the calculation being performed with a finite number of terms in the expansion of the QCD perturbation theory for the Wilson coefficients of the unit and composite operators. In the calculations, the set of composite operators is also restricted.

Thus, the ambiguity in the choice of the hadronic continuum threshold and the parameter of the sum rule scheme essentially reduces the reliability of the QCD sum rule predictions for the leptonic constants of the vector and pseudoscalar $B_{c}$ states.

Moreover, the nonrelativistic quark motion inside the heavy quarkonium $v \rightarrow 0$ leads to the $\alpha_{s} / v$-corrections to

Table 14. The leptonic $B_{c}$ constant (in MeV ), calculated in the QCD sum rules ( SR is the scaling relation).

| Model | $[76]$ | $[35]$ | $[36]$ | $[67]$ | $[77]$ | $[78]$ | $[79]$ | SR [21] |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $f_{\mathrm{B}_{\mathrm{c}}}$ | 375 | 400 | 360 | 300 | 160 | 300 | 450 | 460 |

the perturbative part of the quark current correlators becoming the most important, where $\alpha_{\mathrm{s}}$ is the effective Coulomb coupling constant in the heavy quarkonium.

As is noted in Refs $[11,21,76]$, the Coulomb $\alpha_{\mathrm{s}} / v-$ corrections can be summed up and represented in the form of the factor corresponding to the Coulomb wave function of the heavy quarks, so that

$$
\begin{equation*}
F(v)=\frac{4 \pi \alpha_{\mathrm{s}}}{3 v}\left[1-\exp \left(-\frac{4 \pi \alpha_{\mathrm{s}}}{3 v}\right)\right]^{-1} \tag{82}
\end{equation*}
$$

where $2 v$ is the relative velocity of the heavy quarks inside the quarkonium. The expansion of the factor (82) in the first order over $\alpha_{s} / v$

$$
\begin{equation*}
F(v) \approx 1+\frac{2 \pi \alpha_{\mathrm{s}}}{3 v} \tag{83}
\end{equation*}
$$

gives the expression obtained in the first order of the QCD perturbation theory [11].

Note that the $\alpha_{\mathrm{s}}$ parameter in Eqn (82) should be on the scale of the characteristic quark virtualities in the quarkonium (see Section 2.1), and not on the scale of the quark or quarkonium masses, as is sometimes done, thereby decreasing the value of factor (82).

The choice of the $\alpha_{s}$ parameter essentially determines the spread of the sum rule predictions for the $f_{\mathrm{B}_{\mathrm{c}}}$ value (see Table 14)

$$
\begin{equation*}
f_{\mathrm{B}_{\mathrm{c}}}^{\mathrm{SR}}=160-570 \mathrm{MeV} \tag{84}
\end{equation*}
$$

As one can see from Eqn (84), the ambiguity in the choice of the QCD sum rule parameters leads to the essential deviations in the results from the $f_{\mathrm{B}_{\mathrm{c}}}$ estimates (81) in the potential models.

However, as has been noted in Section 2.1, the large value of the heavy quark masses $\Lambda_{\mathrm{QCD}} / m_{\mathrm{Q}} \ll 1$, the nonrelativistic heavy quark motion inside the heavy quarkonium $v \rightarrow 0$, and the universal scaling properties of the potential in the heavy quarkonium, when the kinetic energy of the quarks and the quarkonium state density do not depend on the heavy quark flavours [see Eqns (10)(15)], allow one to state the scaling relation (17) for the leptonic constants of the S-wave quarkonia

$$
\frac{f^{2}}{M}\left(\frac{M}{4 \mu}\right)^{2}=\mathrm{const}
$$

Indeed, at $\Lambda_{\mathrm{QCD}} / m_{\mathrm{Q}} \ll 1$ one can neglect the quark gluon condensate contribution, which is of the order of magnitude $O\left(1 / m_{\mathrm{b}} m_{\mathrm{c}}\right)$ (the contribution to the $\psi$ and $\Upsilon$ leptonic constants is less than $15 \%$ ).

At $v \rightarrow 0$ one has to take into account the Coulomb-like $\alpha_{s} / v$-corrections in the form of factor (82), so that the imaginary part of the correlators for the vector and axial quark currents has the form

$$
\begin{equation*}
\operatorname{Im} \Pi_{\mathrm{V}}\left(q^{2}\right) \approx \operatorname{Im} \Pi_{\mathrm{P}}\left(q^{2}\right)=\frac{\alpha_{\mathrm{s}}}{2} q^{2}\left(\frac{4 \mu}{M}\right)^{2} \tag{85}
\end{equation*}
$$

where

$$
v^{2}=1-\frac{4 m_{\mathrm{b}} m_{\mathrm{c}}}{q^{2}-\left(m_{\mathrm{b}}-m_{\mathrm{c}}\right)^{2}}, \quad v \rightarrow 0 .
$$

Moreover, condition (15) can be used in the specific QCD sum rule scheme, so that this scheme excludes the dependence of the results on the parameters such as the value of the spectral density moment or the Borel parameter.

Indeed, for example, the resonance contribution to the hadronic part of the vector current correlator, which is of the form

$$
\begin{equation*}
\Pi_{\mathrm{V}}^{(\mathrm{res})}\left(q^{2}\right)=\int \frac{\mathrm{d} s}{s-q^{2}} \sum_{n} f_{\mathrm{V} n}^{2} M_{\mathrm{V} n}^{2} \delta\left(s-M_{\mathrm{V} n}^{2}\right) \tag{86}
\end{equation*}
$$

can be rewritten as

$$
\begin{equation*}
\Pi_{\mathrm{V}}^{(\mathrm{res})}\left(q^{2}\right)=\int \frac{\mathrm{d} s}{s-q^{2}} s f_{\mathrm{V} n(s)}^{2} \frac{\mathrm{~d} n(s)}{\mathrm{d} s} \frac{\mathrm{~d}}{\mathrm{~d} n} \sum_{k} \theta(n-k) \tag{87}
\end{equation*}
$$

where $n(s)$ is the number of the vector S -state versus the mass, so that

$$
\begin{equation*}
n\left(m_{k}^{2}\right)=k \tag{88}
\end{equation*}
$$

Taking the average value for the derivative of the steplike function, one gets
$\Pi_{\mathrm{V}}^{(\mathrm{res})}\left(q^{2}\right)=\left\langle\frac{\mathrm{d}}{\mathrm{d} n} \sum_{k} \theta(n-k)\right\rangle \int \frac{\mathrm{d} s}{s-q^{2}} s f_{\mathrm{V} n(s)}^{2} \frac{\mathrm{~d} n(s)}{\mathrm{d} s}$,
and, supposing

$$
\begin{equation*}
\left\langle\frac{\mathrm{d}}{\mathrm{~d} n} \sum_{k} \theta(n-k)\right\rangle \approx 1 \tag{90}
\end{equation*}
$$

one can, on average, write down

$$
\begin{equation*}
\operatorname{Im}\left\langle\Pi^{\text {(hadr) }}\left(q^{2}\right)\right\rangle=\operatorname{Im} \Pi^{\mathrm{QCD}}\left(q^{2}\right) \tag{91}
\end{equation*}
$$

so, taking into account the Coulomb factor and neglecting power corrections over $1 / m_{\mathrm{Q}}$, at the physical points $s_{n}=M_{n}^{2}$ one obtains

$$
\begin{equation*}
\frac{f_{n}^{2}}{M_{n}}\left(\frac{M}{4 \mu}\right)^{2}=\frac{\alpha_{\mathrm{s}}}{\pi} \frac{\mathrm{~d} M_{n}}{\mathrm{~d} n}, \tag{92}
\end{equation*}
$$

where one has supposed that

$$
\begin{align*}
& m_{\mathrm{b}}+m_{\mathrm{c}} \approx M_{\mathrm{B}_{\mathrm{c}}}  \tag{93}\\
& f_{\mathrm{V} n} \approx f_{\mathrm{P} n}=f_{n} \tag{94}
\end{align*}
$$

Further, as has been shown in Section 2.1, in the heavy quarkonium the value of $\mathrm{d} n / \mathrm{d} M_{n}$ does not depend on the quark masses [see Eqn (15)], and, with an accuracy up to logarithmic corrections, $\alpha_{\mathrm{s}}$ is a constant value (the last fact is also apparent in the flavour independence of the Coulomb part of the potential in the Cornell model). Therefore, one can draw the conclusion that, in the leading approximation, the right-hand side of Eqn (92) is a constant value, and there is a scaling relation (17) [21]. This relation is valid in the resonant region, where one can neglect the contribution by the hadronic continuum.

Note, scaling relation (17) is in a good agreement with the experimental data on the leptonic decay constants of the $\psi$ - and $\Upsilon$-particles (see Table 6), for which one has $4 \mu / M=1$ [21].

The value of the constant on the right-hand side of Eqn (17) is in agreement with the estimate when we suppose

$$
\begin{equation*}
\left\langle\frac{\mathrm{d} M_{\Upsilon}}{\mathrm{d} n}\right\rangle \approx \frac{1}{2}\left[\left(M_{\Upsilon^{\prime}}-M_{\Upsilon}\right)+\left(M_{\Upsilon^{\prime \prime}}-M_{\Upsilon^{\prime}}\right)\right], \tag{95}
\end{equation*}
$$

and $\alpha_{\mathrm{s}}=0.36$, as in the Cornell model.
Further, in the limiting case of B- and D-mesons, when the heavy quark mass is much greater than the light quark mass $m_{\mathrm{Q}} \gg m_{\mathrm{q}}$, one has

$$
\begin{align*}
& \mu \approx m_{\mathrm{q}} \\
& f^{2} M=\frac{16 \alpha_{\mathrm{s}}}{\pi} \frac{\mathrm{~d} M}{\mathrm{~d} n} \mu^{2} \tag{96}
\end{align*}
$$

Then it is evident that at one and the same value of $\mu$ one gets

$$
\begin{equation*}
f^{2} M=\text { const } \tag{97}
\end{equation*}
$$

Scaling law (97) is very well known in EHQT [14] for mesons with a single heavy quark $(\mathrm{Q} \overline{\mathrm{q}})$ and follows, for example, from the identity of the B- and D-meson wave functions within the limit when an infinitely heavy quark can be considered as a static source of gluon field [then Eqn (97) follows from Eqn (80)].

In our derivation of Eqns (96) and (97) we have neglected power corrections over the inverse heavy quark mass. Moreover, we have taken the masses of the light constituent quark to be

$$
\begin{equation*}
m_{\mathrm{q}} \approx 330 \mathrm{MeV} \tag{98}
\end{equation*}
$$

so that this quark has to be considered as nonrelativistic $v \rightarrow 0$, and the following conditions apply:

$$
\begin{align*}
& m_{\mathrm{Q}}+m_{\mathrm{q}} \approx M_{(\mathrm{Q} \overline{\mathrm{q}})}^{(*)}, \quad m_{\mathrm{q}} \ll m_{\mathrm{Q}}  \tag{99}\\
& f_{\mathrm{V}} \approx f_{\mathrm{P}}=f \tag{100}
\end{align*}
$$

In agreement with Eqns (96) and (98), one finds the estimates $\dagger$

$$
\begin{align*}
& f_{\mathrm{B}^{(*)}}=120 \pm 20 \mathrm{MeV},  \tag{101}\\
& f_{\left.\mathrm{D}^{*}\right)}=220 \pm 30 \mathrm{MeV}, \tag{102}
\end{align*}
$$

which are in an agreement with the estimates in the other schemes of the QCD sum rules [11, 12].

Thus, in the limits of $4 \mu / M=1$ and $\mu / M \ll 1$, scaling relation (17) is consistent.

The $f_{\mathrm{B}_{\mathrm{c}}}$ estimate from Eqn (17) contains an uncertainty connected with the choice of the ratio for the $b$ - and $c$-quark masses, so that (see Table 14)

$$
\begin{equation*}
f_{\mathrm{B}_{\mathrm{c}}}=460 \pm 60 \mathrm{MeV} . \tag{103}
\end{equation*}
$$

In Ref. [76] the sum rule scheme with the double Borel transform was used. Thus, it allows one to study effects related to the power corrections from the gluon condensate, corrections due to nonzero quark velocity and nonzero binding energy of the quarks in the quarkonium.
$\dagger$ In Ref. [21] the dependence of the S-wave state density $\mathrm{d} n / \mathrm{d} M_{n}$ on the reduced mass of the system with the Martin potential has been found by the Bohr - Sommerfeld quantisation, so that at the step from ( $\overline{\mathrm{b}} \mathrm{b}$ ) to ( $\overline{\mathrm{b}} \mathrm{q}$ ), the density changes less than about $15 \%$.

Indeed, for the set of narrow pseudoscalar states, one has the sum rules

$$
\begin{align*}
\sum_{k=1}^{\infty} & \frac{M_{k}^{4} f_{\mathrm{P} k}^{2}}{\left(m_{\mathrm{b}}+m_{\mathrm{c}}^{2}\right)^{2}\left(M_{k}^{2}-q^{2}\right)} \\
& =\frac{1}{\pi} \int \frac{\mathrm{~d} s}{s-q^{2}} \operatorname{Im} \Pi_{\mathrm{P}}(s)+C_{\mathrm{G}}\left(q^{2}\right)\left\langle\frac{\alpha_{\mathrm{s}}}{\pi} G^{2}\right\rangle \tag{104}
\end{align*}
$$

where

$$
\begin{align*}
& C_{\mathrm{G}}\left(q^{2}\right)=\frac{1}{192 m_{\mathrm{b}} m_{\mathrm{c}}} \frac{q^{2}}{\bar{q}^{2}} \\
& \quad \times\left[\frac{3\left(3 v^{2}+1\right)\left(1-v^{2}\right)^{2}}{2 v^{5}} \ln \left(\frac{v+1}{v-1}\right)-\frac{9 v^{4}+4 v^{2}+3}{v^{4}}\right], \tag{105}
\end{align*}
$$

$$
\begin{equation*}
\bar{q}^{2}=q^{2}-\left(m_{\mathrm{b}}-m_{\mathrm{c}}\right)^{2}, \quad v^{2}=1-\frac{4 m_{\mathrm{b}} m_{\mathrm{c}}}{\bar{q}^{2}} . \tag{106}
\end{equation*}
$$

Applying the Borel operator $L_{\tau}\left(-q^{2}\right)$ to Eqn (104), one gets

$$
\begin{align*}
& \sum_{k=1}^{\infty} \frac{M_{k}^{4} f_{\mathrm{P} k}^{2}}{\left(m_{\mathrm{b}}+m_{\mathrm{c}}\right)^{2}} \exp \left(-M_{k}^{2} \tau\right) \\
& =\frac{1}{\pi} \int \mathrm{~d} s \operatorname{Im} \Pi_{\mathrm{P}}(s) \exp (-s \tau)+C_{\mathrm{G}}^{\prime}(\tau)\left\langle\frac{\alpha_{\mathrm{s}}}{\pi} G^{2}\right\rangle \tag{107}
\end{align*}
$$

where

$$
\begin{align*}
& L_{\tau}(x)=\lim _{n, x \rightarrow \infty} \frac{x^{n+1}}{n!}\left(-\frac{\mathrm{d}}{\mathrm{~d} x}\right)^{n}, \quad \frac{n}{x}=\tau,  \tag{108}\\
& C_{\mathrm{G}}^{\prime}(\tau)=L_{\tau}\left(-q^{2}\right) C_{\mathrm{G}}\left(q^{2}\right) \tag{109}
\end{align*}
$$

For the exponential on the left-hand side of Eqn (107), one uses the Euler-MacLaurin formula

$$
\begin{align*}
\sum_{k=1}^{\infty} & \frac{M_{k}^{4} f_{P k}^{2}}{\left(m_{\mathrm{b}}+m_{\mathrm{c}}\right)^{2}} \exp \left(-M_{k}^{2} \tau\right) \\
= & \int_{m_{n}}^{\infty} \mathrm{d} M_{k} \frac{\mathrm{~d} k}{\mathrm{~d} M_{k}} M_{k}^{4} f_{\mathrm{P} k}^{2} \exp \left(-M_{k}^{2} \tau\right) \\
& \quad+\sum_{k=0}^{n-1} M_{k}^{4} f_{\mathrm{P} k}^{2} \exp \left(-M_{k}^{2} \tau\right)+\ldots \tag{110}
\end{align*}
$$

Making the second Borel transform $L_{M_{k}^{2}}(\tau)$ on Eqn (107) with account of Eqn (110), one finds the expression for the leptonic constants of the pseudoscalar ( $\overline{\mathrm{b}} \mathrm{c}$ ) states, so that

$$
\begin{align*}
& f_{\mathrm{P} k}^{2}=\frac{2\left(m_{\mathrm{b}}+m_{\mathrm{c}}\right)^{2}}{M_{k}^{3}} \frac{\mathrm{~d} M_{k}}{\mathrm{~d} k} \\
& \times\left\{\frac{1}{\pi} \operatorname{Im} \Pi_{\mathrm{P}}\left(M_{k}^{2}\right)+C_{\mathrm{G}}^{\prime \prime}\left(M_{k}^{2}\right)\left\langle\frac{\alpha_{\mathrm{s}}}{\pi} G^{2}\right\rangle\right\} \tag{111}
\end{align*}
$$

where we have used the following property of the Borel operator:

$$
\begin{equation*}
L_{\tau}(x) x^{n} \exp (-b x) \rightarrow \delta_{+}^{(n)}(\tau-b) \tag{112}
\end{equation*}
$$

The explicit form for the spectral density and Wilson coefficients can be found in Ref. [76]. Expression (111) is in agreement with the above derivation of scaling relation (17).

The numerical effect from the above corrections is considered to be not large (the power corrections are of
the order of $10 \%$ ), and the uncertainty, connected with the choice of quark masses, dominates in the error in the determination of the $f_{\mathrm{B}_{\mathrm{c}}}$ value [see Eqn (103)].

Thus, we have shown that, in the framework of the QCD sum rules, the most reliable estimate of the $f_{\mathrm{B}_{\mathrm{c}}}$ value (103) comes from the use of the scaling relation (17) for the leptonic decay constants of the quarkonia, and this relation agrees very well with the results of the potential models.

## 3. Decays of $\mathbf{B}_{\mathbf{c}}$-mesons

### 3.1 Lifetime of $\mathbf{B}_{\mathbf{c}}$-mesons

The processes of $\mathrm{B}_{\mathrm{c}}$-meson decay can be subdivided into three classes (Fig. 2): (a) the $\overline{\mathrm{b}}$-quark decay with the spectator c-quark, (b) the c-quark decay with the spectator $\overline{\mathrm{b}}$-quark, (c) the annihilation channel $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow l^{+} v_{l}(\mathrm{c} \overline{\mathrm{s}}, u \overline{\mathrm{~s}})$, $l=\mathrm{e}, \mu, \tau$.

The total width is summed from three partial widths

$$
\begin{equation*}
\Gamma\left(\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{X}\right)=\Gamma(\mathrm{b} \rightarrow \mathrm{X})+\Gamma(\mathrm{c} \rightarrow \mathrm{X})+\Gamma(\mathrm{ann}) \tag{113}
\end{equation*}
$$

The simplest estimates with no account for quark binding inside the $\mathrm{B}_{\mathrm{c}}$-meson and in the framework of the spectator mechanism of the decay for the first and second cases, lead to the expressions

$$
\begin{align*}
& \Gamma(\mathrm{b} \rightarrow \mathrm{X})=\frac{G_{\mathrm{F}}^{2}\left|V_{\mathrm{bc}}\right|^{2} m_{\mathrm{b}}^{5}}{192 \pi^{3}} \times 9 \\
& \Gamma(\mathrm{c} \rightarrow \mathrm{X})=\frac{G_{\mathrm{F}}^{2}\left|V_{\mathrm{cs}}\right|^{2} m_{\mathrm{c}}^{5}}{192 \pi^{3}} \times 5 \tag{114}
\end{align*}
$$

So that $m_{\mathrm{b}}$ and $m_{\mathrm{c}}$ are chosen to represent correctly the spectator parts of the total widths for the B-and D-mesons.

The width of the annihilation channel equals

$$
\begin{equation*}
\Gamma(\mathrm{ann})=\sum_{i} \frac{G_{\mathrm{F}}^{2}}{8 \pi}\left|V_{\mathrm{bc}}\right|^{2} f_{\mathrm{B}_{\mathrm{c}}}^{2} M_{\mathrm{bc}} m_{i}^{2}\left(1-\frac{m_{i}^{2}}{m_{\mathrm{B}_{\mathrm{c}}}^{2}}\right)^{2} C_{i} \tag{115}
\end{equation*}
$$

where $C_{i}=1$ for the $\tau v_{\tau}$ channel and $C_{i}=3\left|V_{\mathrm{cs}}\right|^{2}$ for the $\overline{\mathrm{cs}}$ channel, and $m_{i}$ is the mass of the heaviest fermion ( $\tau$ or c).

Note that in the case of nonleptonic decays, consideration of the strong interaction results in a multiplicative factor of enhancement to formulae (114)-(115) (see Section 3.2).

The mentioned widths, calculated with the use of the known values of parameters $m_{\mathrm{q}},\left|V_{\mathrm{bc}}\right|=0.046,\left|V_{\mathrm{cs}}\right|=0.96$, etc., are presented in Table 15.


Figure 2. Diagrams of the $\mathrm{B}_{\mathrm{c}}$-meson decays: (a) the c -spectator decay; (b) the b-spectator decay; (c) the annihilation.

Table 15. The widths (in $10^{-6} \mathrm{eV}$ ) of the inclusive decays of $b$ - and c-quarks in free and bound states (in the $\mathrm{B}_{\mathrm{c}}$-meson) and the branching ratios ( BR in \%) of inclusive $\mathrm{B}_{\mathrm{c}}$ decays.

| Decay mode | Free quarks | $\mathrm{B}^{+}$ | BR | Decay mode | Free quarks | $\mathrm{B}_{\mathrm{c}}^{+}$ | BR |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{b} \rightarrow \overline{\mathrm{c}}+\mathrm{e}^{+} \mathrm{v}_{\mathrm{e}}$ | 62 | 62 | 4.7 | $\mathrm{c} \rightarrow \mathrm{s}+\mathrm{e}^{+}+\mathrm{v}_{\mathrm{e}}$ | 124 | 74 | 5.6 |
| $\overline{\mathrm{b}} \rightarrow \overline{\mathrm{c}}+\mu^{+} v_{\mu}$ | 62 | 62 | 4.7 | $\mathrm{c} \rightarrow \mathrm{s}+\mu^{+}+v_{\mu}$ | 124 | 74 | 5.6 |
| $\overline{\mathrm{b}} \rightarrow \overline{\mathrm{c}}+\tau^{+} \nu_{\tau}$ | 14 | 14 | 1.0 | $\mathrm{c} \rightarrow \mathrm{s}+\mathrm{u}+\overline{\mathrm{d}}$ | 675 | 405 | 30.5 |
| $\overline{\mathrm{b}} \rightarrow \overline{\mathrm{c}}+\overline{\mathrm{d}}+\mathrm{u}$ | 248 | 248 | 18.7 | $\mathrm{c} \rightarrow \mathrm{s}+\mathrm{u}+\overline{\mathrm{s}}$ | 33 | 20 | 1.5 |
| $\overline{\mathrm{b}} \rightarrow \overline{\mathrm{c}}+\overline{\mathrm{s}}+\mathrm{u}$ | 13 | 13 | 1.0 | $\mathrm{c} \rightarrow \mathrm{d}+\mathrm{e}^{+} \nu$ | 7 | 4 | 0.3 |
| $\overline{\mathrm{b}} \rightarrow \overline{\mathrm{c}}+\overline{\mathrm{s}}+\mathrm{c}$ | 87 | 87 | 6.5 | $\mathrm{c} \rightarrow \mathrm{d}+\mu^{+}+v_{\mu}$ | 7 | 4 | 0.3 |
| $\overline{\mathrm{b}} \rightarrow \overline{\mathrm{c}}+\overline{\mathrm{d}}+\mathrm{c}$ | 5 | 5 | 0.4 | $\mathrm{c} \rightarrow \mathrm{d}+\mathrm{u}+\overline{\mathrm{d}}$ | 39 | 23 | 1.7 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \tau^{+}+\nu_{\tau}$ | - | 63 | 4.7 | $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{c}+\overline{\mathrm{s}}$ | - | 162 | 12.2 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{c}+\overline{\mathrm{d}}$ | - | 8 | 0.6 | $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{all}$ | - | 1328 | 100 |



Figure 3. The Dalitz diagrams for the semileptonic decays: (1) $\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{B}_{\mathrm{s}}^{*} \operatorname{lv}$, (2) $\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{B}_{\mathrm{s}} \mathrm{lv}$, (3) $\mathrm{D} \rightarrow \mathrm{K}^{*} l \mathrm{l}$, (4) $\mathrm{D} \rightarrow \mathrm{Klv}$, (5) $\mathrm{c} \rightarrow \mathrm{slv}$ $\left(m_{\mathrm{c}}=1.7 \mathrm{GeV}, \quad m_{\mathrm{s}}=0.55 \mathrm{GeV}\right), \quad(6) \quad \mathrm{c} \rightarrow \mathrm{slv} \quad\left(m_{\mathrm{c}}=1.5 \mathrm{GeV}\right.$, $m_{\mathrm{s}}=0.15 \mathrm{GeV}$ ); $E$ is the lepton energy, $q^{2}$ is the square of the lepton pair mass.

Thus, a rough estimate of the lifetime leads to $\tau_{\mathrm{B}_{\mathrm{c}}} \approx(2-5) \times 10^{-13} \mathrm{~s}$. So, the fraction of the c-quark decay is approximately $50 \%$, the b-quark one is $45 \%$, and the annihilation channel is $5 \%$. However, these estimates do not take into account the quite strong binding of the quarks inside the $\mathrm{B}_{\mathrm{c}}$-meson: corresponding corrections to the estimates can reach about $40 \%$.

Let us consider this effect in the semileptonic modes of decay with the spectator $\bar{b}$-quark. The final state of such decays generally contains the $\mathrm{B}_{\mathrm{s}}^{(*)}$-mesons with the smaller phase space of the lepton pair.

The effect of the phase space decrease is shown in Fig. 3, where the kinematic borders of the Dalitz plot for the $B_{\mathrm{c}}^{+} \rightarrow \mathrm{B}_{\mathrm{s}} \mathrm{e}^{+} v$ decay are compared with the borders for the $\mathrm{c}-$ quark and calculated at different values of the c-quark mass. As one can see from Fig. 3, the end-point of the leptonic spectrum is approximately one and the same in the different decays

$$
\begin{equation*}
E^{\max }=\frac{M_{\mathrm{B}_{\mathrm{c}}}^{2}-M_{\mathrm{B}_{\mathrm{s}}}^{2}}{2 M_{\mathrm{B}_{\mathrm{c}}}} . \tag{116}
\end{equation*}
$$



Figure 4. Dalitz diagrams for the semileptonic decays: (1) $\mathrm{B}_{\mathrm{c}} \rightarrow \psi / v$, (2) $\mathrm{B}_{\mathrm{c}} \rightarrow \eta_{\mathrm{c}} l v$, (3) $\mathrm{B} \rightarrow \mathrm{Dlv}$, (4) $\mathrm{B} \rightarrow \mathrm{D}^{*} l v$, (5) $\mathrm{b} \rightarrow \mathrm{clv}$; $E$ is the lepton energy, $q^{2}$ is the square of the lepton pair mass.

However, the maximum values of the leptonic pair masses $q_{\text {max }}^{2}$ are different.

One can easily show that the spectator model better describes the semileptonic decay $\mathrm{D} \rightarrow \mathrm{K}$. In the case of the $\mathrm{B}_{\mathrm{c}}$-meson decay, the admissible kinematical region is strongly reduced. With account taken of the phase space reduction in the spectator model, one can get [34]

$$
\begin{equation*}
\Gamma\left(\mathrm{B}_{\mathrm{c}}^{+} \rightarrow X_{\mathrm{b}} \mathrm{e}^{+} v\right) \approx 0.71 \Gamma\left(\mathrm{D}^{+} \rightarrow \mathrm{X}_{\mathrm{s}} e^{+} v\right) \tag{117}
\end{equation*}
$$

The effect of the phase space reduction does not notably appear in the case of decays with the spectator c-quark. For such decays, as one can see from Fig. 4, the spectator model well describes the B -meson decays as well as the $\mathrm{B}_{\mathrm{c}}$-meson decays, and one can believe that

$$
\begin{equation*}
\Gamma\left(\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{X}_{\mathrm{c}} \mathrm{e}^{+} \mathrm{v}\right) \approx \Gamma\left(\mathrm{B}^{+} \rightarrow \mathrm{X}_{\mathrm{c}} \mathrm{e}^{+} \mathrm{v}\right) . \tag{118}
\end{equation*}
$$

Another possible manner of estimation is related to the summation of the exclusive decays into the channels $\mathrm{B}_{\mathrm{s}} \mathrm{e}^{+} v$ and $\mathrm{B}_{\mathrm{s}}^{*} \mathrm{e}^{+} v$. In agreement with the same kinematical arguments, their sum is the main fraction of the semileptonic decays [82]. If one neglects the decaying quark momentum inside the $\mathrm{B}_{\mathrm{c}}$-meson, the admissible region of
masses in the inclusive semileptonic decay $\mathrm{Q} \rightarrow \mathrm{Q}^{\prime} \mathrm{ev}$ is varied within the limits

$$
\begin{equation*}
\left(m_{\mathrm{q}^{\prime}}+m_{\mathrm{sp}}\right)^{2}<M_{\mathrm{x}}^{2}<m_{\mathrm{q}^{\prime}}^{2}+m_{\mathrm{sp}}^{2}+m_{\mathrm{sp}} \frac{m_{\mathrm{q}^{\prime}}^{2}}{m_{\mathrm{q}}} \tag{119}
\end{equation*}
$$

From approximate formula (119) with the use of the constituent quark masses, one can see that the admissible $M_{\mathrm{x}}$ region in the decay $\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{X}_{\mathrm{c}}$ is varied in the limit of 200 MeV and, hence, the final state is saturated by the lowest states. For the considered case ( $m_{\mathrm{q}}=m_{\mathrm{c}}=1.7 \mathrm{GeV}$, $m_{\mathrm{q}^{\prime}}=m_{\mathrm{s}}=0.55 \mathrm{GeV}$ and $m_{\mathrm{sp}}=m_{\mathrm{b}}=5.1 \mathrm{GeV}$ ), this region has widths equal to 340 MeV , that is less than the expected difference of masses between the basic state and the first orbital excitation of the $\overline{\mathrm{b}}$ s system.

Thus, one can consider that
$\Gamma\left(\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{X}_{\mathrm{b}} \mathrm{e}^{+} \mathrm{v}\right) \approx \Gamma\left(\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{B}_{\mathrm{s}}+\mathrm{ev}\right)+\Gamma\left(\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{B}_{\mathrm{s}}^{*}+\mathrm{ev}\right)$.

The results of different quark models for the semileptonic $B_{c}$ decays (see Section 3.2) lead to the following sum of the widths of decays into $B_{s}$ and $B_{s}^{*}$ :

$$
\begin{align*}
& \Gamma\left(\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{~B}_{\mathrm{s}}+\mathrm{ev}\right)+\Gamma\left(\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{~B}_{\mathrm{s}}^{*}+\mathrm{ev}\right) \\
& \quad \approx(60 \pm 7) \times 10^{-15} \mathrm{GeV} \approx 0.5 \Gamma\left(\mathrm{D}^{+} \rightarrow \mathrm{X}_{\mathrm{s}} \mathrm{e}^{+} v\right) \tag{121}
\end{align*}
$$

Accounting for the current theoretical uncertainties, one can calculate

$$
\begin{equation*}
\Gamma\left(\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{X}_{\mathrm{b}}+\mathrm{e}^{+} v\right)=(0.6 \pm 0.2) \Gamma\left(\mathrm{D}^{+} \rightarrow \mathrm{X}_{\mathrm{s}} \mathrm{e}^{+} v\right) \tag{122}
\end{equation*}
$$

For the c-spectator decays, the calculations in quark models and QCD sum rules show that the semileptonic decays are saturated by the transitions into the lowest $\eta_{\mathrm{c}^{-}}$ and $J / \psi$-states, i.e.

$$
\begin{align*}
& \Gamma\left(\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{X}_{\mathrm{c}} \mathrm{e}^{+} \mathrm{v}\right) \\
& \quad \approx \Gamma\left[\mathrm{B}_{\mathrm{c}}^{+} \rightarrow\left(\eta_{\mathrm{c}}+\mathrm{J} / \psi\right) \mathrm{e}^{+} \mathrm{v}\right] \approx \Gamma\left(\mathrm{B}^{+} \rightarrow \mathrm{X}_{\mathrm{c}} \mathrm{e}^{+} v\right) \tag{123}
\end{align*}
$$

The probabilities of inclusive decays are presented in Table 15 with these factors taken into account. The widths of the hadronic inclusive decays, which are discussed in detail in Section 3.3, are also shown.

The compact sizes of $\mathrm{B}_{\mathrm{c}}$-mesons lead to the large value of the weak decay constant ( $f_{\mathrm{B}_{\mathrm{c}}} \approx 500 \mathrm{MeV}$ ), which enforces the role of the annihilation channel into the massive fermions $c, \tau$. The decays of $\mathrm{B}_{\mathrm{c}}$-mesons into the light fermions are suppressed because they are forbidden by the spirality. Although the use of the effective masses for the $u$-and d-quarks instead of the current masses can increase the width of the annihilation channel into $u \bar{d}$, the latter will yet be much less than the width into the heavy massive fermions. In agreement with Eqn (115), conservative estimates of the annihila-tion decay probabilities are presented in Table 15.

Thus, the consideration of three types of processes for the $\mathrm{B}_{\mathrm{c}}$-meson decay leads to the lifetime estimate

$$
\tau_{\mathrm{B}_{\mathrm{c}}} \approx 5 \times 10^{-13} \mathrm{~s}
$$

with the following approximate sharing of branching fractions: $37 \%, 45 \%$, and $18 \%$, corresponding to the cspectator mechanism, the $\bar{b}$-spectator mechanism, and annihilation, respectively.

The uncertainty in the estimation of the $\mathrm{B}_{\mathrm{c}}$-meson lifetime is generally related to the choice of quark masses. The mass of b-quark $m_{\mathrm{b}}=4.9 \mathrm{GeV}$ is chosen so that one can describe the B-meson lifetime in the framework of the spectator mechanism. Note that the differences of the lifetimes for the charged and neutral B-mesons are insignificant and, hence, the given choice of the mass is sufficiently unambiguous. For the D-mesons, this is not the case, since the lifetimes of $D^{+}$- and $D^{0}{ }^{-}$mesons differ by a factor of two.

Nevertheless, there is a more reliable way to obtain the c-quark mass, and this is the consideration of the semileptonic decays of D-mesons. Indeed, the value $m_{\mathrm{c}}=1.5 \mathrm{GeV}$ in the spectator mechanism well describes the decays $\mathrm{D}^{+} \rightarrow \overline{\mathrm{K}}^{0} \mathrm{e}^{+} v$ and $\mathrm{D}^{0} \rightarrow \overline{\mathrm{~K}}^{-} \mathrm{e}^{+} v$, whose widths are approximately equal to each other. Note, at any other reasonable choice of $m_{\mathrm{c}}$ (from the total widths, say), the error in the $\mathrm{B}_{\mathrm{c}}$-meson lifetime will not be large, since the summed branching ratio of the $\mathrm{B}_{\mathrm{c}}$-meson decays due to the c-quark decays is about $40 \%$.

### 3.2 Semileptonic decays of $\mathbf{B}_{\mathbf{c}}$-mesons

3.2.1 Quark models. The semileptonic decays of $\mathrm{B}_{\mathrm{c}}$-mesons are considered in Refs [30, 32, 34] in the framework of quark models. A detailed study of the $\mathrm{B}_{\mathrm{c}}$-meson decays in the quark model of the WSB relativistic oscillator [71] was first made in Ref. [32] and further in Ref. [34], where the ISGW quark model [70] was also used. The covariant description approach, proposed earlier for the composed quarkonium model, is developed in Ref. [30].

Consider the amplitude of the $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{M}_{\mathrm{X}} \mathrm{e}^{+} v_{\mathrm{e}}$ transition with the weak decay of quark 1 into quark 2 (Fig. 5)

$$
\begin{equation*}
A=\frac{G_{\mathrm{F}}}{\sqrt{2}} V_{12} l_{\mu} H^{\mu} \tag{124}
\end{equation*}
$$

where $G_{\mathrm{F}}$ is the Fermi constant, $V_{12}$ is an element of the Kobayashi-Maskawa matrix. The lepton current $l_{\mu}$ is determined by the expression

$$
\begin{equation*}
l_{\mu}=\bar{e}\left(q_{1}\right) \gamma_{\mu}\left(1-\gamma_{5}\right) v\left(q_{2}\right) \tag{125}
\end{equation*}
$$

where $q_{1}$ and $q_{2}$ are the lepton and neutrino momenta, respectively, $\left(q_{1}+q_{2}\right)^{2}=t$.


Figure 5. Diagram of the semileptonic decay of the $\mathrm{B}_{\mathrm{c}}$-meson.

The $H_{\mu}$ quantity in Eqn (124) is the matrix element of the hadronic current $J_{\mu}$

$$
\begin{equation*}
J_{\mu}=V_{\mu}-A_{\mu}=\bar{Q}_{1} \gamma_{\mu}\left(1-\gamma_{5}\right) Q_{2} \tag{126}
\end{equation*}
$$

The matrix element for the $\mathrm{B}_{\mathrm{c}}$-meson decay into the pseudoscalar state P can be written down in the form

$$
\left\langle\mathrm{B}_{\mathrm{c}}(p)\right| A_{\mu}|\mathrm{P}(k)\rangle=F_{+}(t)(p+k)_{\mu}+F_{-}(t)(p-k)_{\mu},(127)
$$

and for the transition into the vector meson V with mass $M_{\mathrm{V}}$ and polarisation $\lambda$, one has

$$
\begin{align*}
& \left\langle\mathrm{B}_{\mathrm{c}}(p)\right| J_{\mu}|\mathrm{V}(k, \lambda)\rangle \\
& \quad=-\left(M+M_{\mathrm{V}}\right) A_{1}(t) \varepsilon_{\mu}^{(\lambda)}+\frac{A_{2}(t)}{M+M_{\mathrm{V}}}\left(\varepsilon^{(\lambda)} p\right)(p+k)_{\mu} \\
& \quad+\frac{A_{3}(t)}{M+M_{\mathrm{V}}}\left(\varepsilon^{(\lambda)} p\right)(p-k)_{\mu}+\mathrm{i} \frac{2 V(t)}{M+M_{\mathrm{V}}} \varepsilon_{\mu v \alpha \beta} \varepsilon_{v}^{(\lambda)} p^{\alpha} k^{\beta} . \tag{128}
\end{align*}
$$

Relations (127) and (128) define the form factors of the $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{M}_{\mathrm{X}} \mathrm{e}^{+} v_{\mathrm{e}}$ transitions, so, for the massless leptons, $F_{-}$ and $A_{3}$ do not contribute to matrix element (124).

In the covariant model of the quarkonium (see Appendix I), one can easily find

$$
\begin{align*}
& F_{+}(t)=\frac{1}{2}\left(m_{1}+m_{2}\right) \sqrt{\frac{M_{\mathrm{P}}}{M}} \frac{1}{m_{2}} \xi_{\mathrm{P}}(t)  \tag{129}\\
& F_{-}(t)=-\frac{1}{2}\left(m_{1}-m_{2}+2 m_{\mathrm{sp}}\right) \sqrt{\frac{M_{\mathrm{P}}}{M}} \frac{1}{m_{2}} \xi_{\mathrm{P}}(t) \tag{130}
\end{align*}
$$

Here $m_{\text {sp }}$ is the mass of the spectator quark (see Fig. 5), and the function $\xi_{\mathrm{X}}(t)$ has the form

$$
\begin{gather*}
\xi_{\mathrm{X}}(t)=\left(\frac{2 \omega \omega_{\mathrm{X}}}{\omega^{2}+\omega_{\mathrm{X}}^{2}}\right)^{3 / 2} \exp \left\{-\frac{m_{\mathrm{sp}}^{2}}{\omega^{2}+\omega_{\mathrm{X}}^{2}} \frac{t_{\mathrm{max}}-t}{M M_{\mathrm{X}}}\right. \\
\left.\times\left[1+\frac{\omega^{2}}{\omega_{\mathrm{X}}^{2}}\left(1-\frac{t_{\mathrm{max}}-t}{4 M M_{\mathrm{X}}}\right)\right]\right\} \tag{131}
\end{gather*}
$$

where $M_{\mathrm{X}}$ is the recoil meson mass, $\omega_{\mathrm{X}}$ is the wave function parameter (I.6)-(I.8) for the recoil meson, and

$$
\begin{equation*}
t_{\max }=\left(M-M_{\mathrm{X}}\right)^{2} \tag{132}
\end{equation*}
$$

is the maximal square of the lepton pair mass.
For the vector state one has $M_{\mathrm{V}}=M_{\mathrm{X}}$, and we obtain

$$
\begin{gather*}
V(t)=\frac{1}{2}\left(M+M_{\mathrm{V}}\right) \sqrt{\frac{M_{\mathrm{V}}}{M}} \frac{1}{m_{2}} \xi_{\mathrm{V}}(t),  \tag{133}\\
A_{1}(t)=\frac{1}{2} \frac{M^{2}+M_{\mathrm{V}}^{2}-t+2 M\left(m_{2}-m_{\mathrm{sp}}\right)}{M+M_{\mathrm{V}}} \sqrt{\frac{M_{\mathrm{V}}}{M}} \frac{1}{m_{2}} \xi_{\mathrm{V}}(t), \tag{134}
\end{gather*}
$$

$$
\begin{align*}
& A_{2}(t)=\frac{1}{2}\left(M+M_{\mathrm{V}}\right)\left(1-\frac{2 m_{\mathrm{sp}}}{M}\right) \sqrt{\frac{M_{\mathrm{V}}}{M}} \frac{1}{m_{2}} \xi_{\mathrm{V}}(t)  \tag{135}\\
& A_{3}(t)=-\frac{1}{2}\left(M+M_{\mathrm{V}}\right)\left(1+\frac{2 m_{\mathrm{sp}}}{M}\right) \sqrt{\frac{M_{\mathrm{V}}}{M}} \frac{1}{m_{2}} \xi_{\mathrm{V}}(t) \tag{136}
\end{align*}
$$

It is interesting to note that the exponential form of the dependence of the form factor on $t$ (131) can be quite accurately represented, in the admissible kinematical region, by the form corresponding to the model of meson dominance

$$
\begin{equation*}
\xi_{k}(t)=\xi_{k}(0) \frac{1}{1-t / m_{k}^{2}} \tag{137}
\end{equation*}
$$

where $m_{k}$ are presented in Table 16.
One can see from Eqns (130)-(136), that the form factors [excepting $A_{1}(t)$ ] are also representable in form (137), and the degeneration takes place

$$
\begin{equation*}
m_{\mathrm{V}}=m_{A_{2}}=m_{A_{3}} \approx m_{+}, \tag{138}
\end{equation*}
$$

if $\omega_{\mathrm{P}} \approx \omega_{\mathrm{V}}, M_{\mathrm{P}} \approx M_{\mathrm{V}}$.

Table 16. The $m_{k}$ parameters (in GeV ) for the $\xi_{k}(t)$ representation in Eqn (137).

| Mode | $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \psi \mathrm{e}^{+} v_{\mathrm{e}}$ | $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \eta_{\mathrm{c}} \mathrm{e}^{+} v_{\mathrm{e}}$ | $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}_{\mathrm{s}} \mathrm{e}^{+} v_{\mathrm{e}}$ | $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}_{\mathrm{s}}^{*} \mathrm{e}^{+} v_{\mathrm{e}}$ |
| :--- | :--- | :--- | :--- | :--- |
| $m_{k}$ | 6.3 | 6.45 | 1.9 | 1.95 |

As for the $A_{1}(t)$ form factor, it can be represented in the form

$$
\begin{equation*}
A_{1}(t)=\varphi(t) \frac{1}{1-t / m_{A_{1}}^{2}}=a_{1}+\frac{A_{1}^{\prime}(0)}{1-t / m_{A_{1}}^{2}} \tag{139}
\end{equation*}
$$

where

$$
\begin{align*}
& m_{A_{1}}=m_{\mathrm{V}}  \tag{140}\\
& \begin{aligned}
A_{1}^{\prime}(0)= & \frac{1}{2} \frac{M^{2}+M_{\mathrm{V}}^{2}-m_{A_{1}}^{2}+2 M\left(m_{2}-m_{\mathrm{sp}}\right)}{M+M_{\mathrm{V}}} \\
& \times \sqrt{\frac{M_{\mathrm{V}}}{M}} \frac{1}{m_{2}} \xi_{\mathrm{V}}(0), \\
a_{1}= & A_{1}(0)-A_{1}^{\prime}(0) .
\end{aligned}
\end{align*}
$$

The values of the transition form factors at zero mass of the lepton pair are shown in Table 17. The numerical calculations in [30] have been performed for the mass values $m_{\mathrm{b}}=4.9 \mathrm{GeV}, m_{\mathrm{c}}=1.6 \mathrm{GeV}, m_{\mathrm{s}}=0.5-0.55 \mathrm{GeV}$.(143)
The element of the Kobayashi-Maskawa matrix has been taken equal to $V_{b c}=0.046$.

Table 17. The form factors of the semileptonic $B_{c}$ decays.

| Mode | $F_{+}(0)$ | $A_{1}(0)$ | $A_{1}^{\prime}(0)$ | $A_{2}(0)$ | $V(0)$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{e}^{+} v_{\mathrm{e}}$ | - | 0.73 | 0.14 | 0.67 | 1.31 |
| $\mathrm{~B}_{\mathrm{c}}^{+} \rightarrow \eta_{\mathrm{c}} \mathrm{e}^{+} v_{\mathrm{e}}$ | 0.89 | - | - | - | - |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}_{\mathrm{s}} \mathrm{e}^{+} v_{\mathrm{e}}$ | 0.61 | - | - | - | - |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}_{\mathrm{s}}^{*} \mathrm{e}^{+} v_{\mathrm{e}}$ | - | 0.52 | - | -2.79 | 5.03 |

The $f_{\mathrm{B}_{\mathrm{c}}}$ constant in Ref. [30] has been varied in the limits

$$
\begin{equation*}
f_{\mathrm{B}_{\mathrm{c}}}=360-570 \mathrm{MeV} \tag{144}
\end{equation*}
$$

where the upper limit corresponds to the values obtained in the nonrelativistic potential model [34,52], in the parton model [75], and in the QCD sum rules [36, 76]. The lower limit corresponds to the value obtained in the Borel sum rules of QCD [36, 76]. Note that for the $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \psi \mathrm{e}^{+} v_{\mathrm{e}}$ decay, the result weakly depends on the $f_{\mathrm{B}_{\mathrm{c}}}$ choice (3\%).

It has also been supposed that

$$
\begin{equation*}
f_{\eta_{\mathrm{c}}}=f_{\psi} \tag{145}
\end{equation*}
$$

and the values

$$
\begin{align*}
& f_{\mathrm{B}_{\mathrm{s}}}=100-110 \mathrm{MeV}  \tag{146}\\
& f_{\mathrm{B}_{\mathrm{s}}^{*}}=160-180 \mathrm{MeV} \tag{147}
\end{align*}
$$

have been varied, which does not contradict the estimates made in the QCD sum rules [12].

Note that for the semileptonic $\mathrm{B}_{\mathrm{c}}$-meson decays $B_{c}^{+} \rightarrow M_{X} e^{+} v_{e}$, where $M_{X}$ is the recoil meson, the explicit covariance of the model allows one to take into account corrections to the velocity of the $\mathrm{M}_{\mathrm{X}}$ meson. As for
corrections due to the quark motion inside the meson, they are taken into account by means of the difference between the constituent and current masses of the quark.

In the ISGW model for the meson state vector, the following nonrelativistic expression is used:

$$
\begin{align*}
\left|X\left(\boldsymbol{p}_{\mathrm{X}} ; s_{\mathrm{x}}\right)\right\rangle & =\sqrt{2 m_{\mathrm{x}}} \int \mathrm{~d}^{3} p \sum C_{m_{L} m_{S}}^{s_{\mathrm{X}} L S} \phi_{\mathrm{X}}(\boldsymbol{p})_{L m_{L}} \chi_{\mathrm{S} \overline{\mathrm{~s}}}^{S m_{S}} \\
& \times\left|\mathrm{q}\left(\frac{m_{\mathrm{q}}}{m_{x}} \boldsymbol{p}_{\mathrm{X}}+\boldsymbol{p}, s\right) \overline{\mathrm{q}}\left(\frac{m_{\overline{\mathrm{q}}}}{m_{x}} \boldsymbol{p}_{\mathrm{X}}-\boldsymbol{p}, \bar{s}\right)\right\rangle, \tag{148}
\end{align*}
$$

where $\chi_{\bar{s} \bar{s}}^{S m_{s}}$ is the spin wave function of the quark antiquark pair in the state with the total spin $S$ and the spin projection $m_{S}, C_{m_{L} m_{S}}^{s_{L} L S}$ is the coupling between the orbital momentum $L$ and the total spin $S$ of the system with the total momentum $s_{x}, \phi_{\mathrm{X}}(\boldsymbol{p})_{L m_{L}}$ is the corresponding nonrelativistic wave function, $\boldsymbol{p}_{\mathrm{X}}$ is the meson momentum, $\boldsymbol{p}$ is the relative momentum of quarks. In the considered model, the meson mass is equal to the sum of quark masses only in the approximation of infinitely narrow wave packets.

As the probe functions, the nonrelativistic oscillator wave functions have been chosen:

$$
\begin{aligned}
& \Psi^{1 \mathrm{~S}}=\frac{\beta_{\mathrm{S}}^{3 / 2}}{\pi^{3 / 4}} \exp \left(-\frac{\beta_{\mathrm{S}}^{2} r^{2}}{2}\right) \\
& \Psi^{1 \mathrm{P}}=-\frac{\beta_{\mathrm{P}}^{5 / 2}}{\pi^{3 / 4}} r \exp \left(-\frac{\beta_{\mathrm{P}}^{2} r^{2}}{2}\right) \\
& \Psi^{2 \mathrm{~S}}=\left(\frac{2}{3}\right)^{1 / 2} \frac{\beta_{\mathrm{S}}^{7 / 2}}{\pi^{3 / 4}}\left(r^{2}-\frac{3}{2} \beta_{\mathrm{S}}^{-2}\right) \exp \left(-\frac{\beta_{\mathrm{S}}^{2} r^{2}}{2}\right) .
\end{aligned}
$$

The $\beta$ parameters have been determined by the variational principle and the Cornell potential [5].

In the WSB model, the mesons are considered as a relativistic bound state of a quark $\mathrm{q}_{1}$ and an antiquark $\overline{\mathrm{q}}_{2}$ in the system of infinitely large momentum [71]:

$$
\begin{aligned}
& \left|P, m, j, j_{z}\right\rangle=\sqrt{2}(2 \pi)^{3 / 2} \sum_{s_{1}, s_{2}} \int \mathrm{~d}^{3} p_{1} \mathrm{~d}^{3} p_{2} \delta^{3}\left(\boldsymbol{P}-\boldsymbol{p}_{1}-\boldsymbol{p}_{2}\right) \\
& \times L_{m}^{j, j_{z}}\left(\boldsymbol{p}_{1 \mathrm{t}}, x, s_{1}, s_{2}\right) a_{1}^{s_{1}^{+}}\left(\boldsymbol{p}_{1}\right) b_{2}^{s_{2}^{+}}\left(\boldsymbol{p}_{2}\right)|0\rangle
\end{aligned}
$$

where $P_{\mu}=\left(P_{0}, 0,0, P\right), \quad$ and $\quad$ at $P \rightarrow \infty, \quad x=p_{1 z} / p$ corresponds to the momentum fraction carried out by the nonspectator quark, $p_{1 t}$ is the transverse momentum.

For the orbital part of the wave function, the solution of the relativistic oscillator is used:

$$
\begin{align*}
L_{m}\left(\boldsymbol{p}_{\mathrm{t}}, x\right) & =N_{m} \sqrt{x(1-x)} \exp \left(-\frac{\boldsymbol{p}_{\mathrm{t}}^{2}}{2 \omega^{2}}\right) \\
& \times \exp \left[-\frac{m^{2}}{2 \omega^{2}}\left(x-\frac{1}{2}-\frac{m_{\mathrm{q}_{1}}^{2}-m_{\mathrm{q}_{2}}^{2}}{2 m^{2}}\right)^{2}\right] \tag{149}
\end{align*}
$$

In both models, the calculation of hadronic matrix elements $\left\langle\mathrm{B}_{\mathrm{c}}(p)\right| J_{\mu}|\mathrm{X}(k)\rangle$ corresponds to the calculation of the matrix elements of the quark currents between the quark states and the overlapping corresponding wave functions.

In the potential models, the bound state of two particles is described by the wave function whose argument is the relative momentum of the particle motion relative to the centre of mass of the meson system. However, in the case of
decays with large recoil momenta, one cannot choose a system where both mesons (the initial one and the decay product) would be at rest, so that one has a kinematical uncertainty in the form factor values.

For instance, in the ISGW model the form factor dependence on the invariant mass of lepton pair $t$ is determined by the function
$\xi_{\mathrm{IGSW}}(t)=\left(\frac{2 \beta \beta_{1}}{\beta^{2}+\beta_{1}^{2}}\right)^{3 / 2} \exp \left(-\frac{m_{\mathrm{sp}}^{2}}{2 \tilde{M}_{\mathrm{i}} \tilde{M}_{\mathrm{f}}} \frac{t_{\mathrm{max}}-t}{k^{2}\left(\beta_{\mathrm{i}}^{2}+\beta_{\mathrm{f}}^{2}\right)}\right)$,
where $\beta_{\mathrm{i}}$ and $\beta_{\mathrm{f}}$ are the parameters of wave functions for the initial and final mesons, $m_{\text {sp }}$ is the spectator quark mass, and $\tilde{M}_{\mathrm{i}}$ and $\tilde{M}_{\mathrm{f}}$ are the model parameters (the masses of the initial and final 'mock'-mesons) [70].

The $k$ parameter in Eqn (150) is introduced synthetically for the correct description of the electromagnetic form factor of $\pi$-meson ( $k=0.7$ ). So, the authors of Ref. [70] related this factor with possible relativistic corrections at large recoil momenta.

Recently in Ref. [83], a model for the description of the heavy quarkonium decays has been offered. In this model, the required behaviour of form factors (at $k=0.7$ ) is automatic with no introduction of additional parameters. In contrast to the above approaches (the covariant quark model and ISGW model), the nonrelativistic approximation is performed for the hadronic matrix element as a whole, but it is not performed separately for the wave functions of just the initial and final states. At small recoil momenta, this formalism practically repeats the ISGW model, but at large momenta there are some differences in the structure of the spin part of the wave function and the argument of the wave function of the final meson. So, the latter change is the most important and leads to the difference in the form factor dependence on $t$ [84].

The transition form factors in the ISGW model depend on $\beta_{\mathrm{B}_{\mathrm{c}}}$ and $\beta_{\mathrm{B}_{\mathrm{s}}}$. For its values, $\beta_{\mathrm{B}_{\mathrm{c}}}=0.82$ and $\beta_{\mathrm{B}_{\mathrm{s}}}=0.51$ are obtained from the variational principle. Since the considered model is the nonrelativistic approximation, the form factors are the most accurately predicted at $q^{2}=q_{\text {max }}^{2}=$ $\left(M_{\mathrm{B}_{\mathrm{c}}}-M_{\mathrm{X}}\right)^{2}$ (at the maximal value of the lepton pair invariant mass).

One can calculate the form factors in the region of low $q^{2}$ values in two different ways: by the use of the exponential dependence on $q^{2}$ as in ISGW or in the pole model of meson dominance. The results for the decay widths, calculated in these ways, are presented in Table 18. The additional parameter in the ISGW model is $k=1$ [see Eqn (150)].

The results obtained in [83] are also presented in the same table. In the constituent quark model, the exponential dependence of the form factors can be represented in the pole form. As one can see from Table 18, in the ISGW model for the decays, where the c-quark is the spectator, the exponential dependence and the pole model give different results.

In the WSB model, the form factor values at $q^{2}=0$ are predicted in terms of the $\omega$ parameter [see Eqn (149)], which corresponds to the average transverse momentum of quarks inside the meson. In Ref. [34] the $\omega$ values were equal to the average $p_{\mathrm{t}}^{2}$ values, estimated in the ISGW model $\left(\omega_{\mathrm{X}} \approx \beta_{\mathrm{X}}\right)$. Note that the $\omega$ parameter is external for the WSB model.

The results of these approaches are presented in Table 18.

Table 18. The partial widths (in $10^{-6} \mathrm{eV}$ ) of the semileptonic $\mathrm{B}_{\mathrm{c}}$ decays (ISGW1 and ISGW2 are the results of the ISGW model with the exponential dependence of the form factors and the pole model, respectively).

| Mode | ISGW1 <br> $[34]$ | ISGW2 <br> $[34]$ | WSB [34] | $[30]$ | $[83]$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |  |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \psi \mathrm{e}^{+} v_{\mathrm{e}}$ | 38.5 | 53.1 | 21.8 | 37.3 | 34.4 |
| $\mathrm{~B}_{\mathrm{c}}^{+} \rightarrow \eta_{\mathrm{c}} \mathrm{e}^{+} v_{\mathrm{e}}$ | 10.6 | 16.1 | 16.5 | 20.4 | 14.2 |
| $\mathrm{~B}_{\mathrm{c}}^{+} \rightarrow \mathrm{D}^{0} \mathrm{e}^{+} v_{\mathrm{e}}$ | 0.033 | 0.12 | 0.002 | - | 0.094 |
| $\mathrm{~B}_{\mathrm{c}}^{+} \rightarrow \mathrm{D}^{0} \mathrm{e}^{+} v_{\mathrm{e}}$ | 0.13 | 0.32 | 0.011 | - | 0.268 |
| $\mathrm{~B}_{\mathrm{c}}^{+} \rightarrow \psi(2 \mathrm{~S}) \mathrm{e}^{+} v_{\mathrm{e}}$ | - | - | - | - | 1.45 |
| $\mathrm{~B}_{\mathrm{c}}^{+} \rightarrow \eta_{\mathrm{c}}^{\prime} \mathrm{e}^{+} v_{\mathrm{e}}$ | - | - | - | - | 0.727 |
| $\mathrm{~B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}_{\mathrm{s}} \mathrm{e}^{+} v_{\mathrm{e}}$ | 16.4 | 17.9 | 11.1 | $16 \pm 4$ | 26.6 |
| $\mathrm{~B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}_{\mathrm{s}}^{*} \mathrm{e}^{+} v_{\mathrm{e}}$ | 40.9 | 46.3 | 43.7 | $41 \pm 6$ | 44.0 |
| $\mathrm{~B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}_{\mathrm{d}} \mathrm{e}^{+} v_{\mathrm{e}}$ | 1.0 | 1.1 | 0.5 | - | 2.30 |
| $\mathrm{~B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}_{\mathrm{d}}^{*} \mathrm{e}^{+} v_{\mathrm{c}}$ | 2.5 | 3.0 | 2.9 | - | 3.32 |
|  |  |  |  |  |  |

Note that the relative yield of the pseudoscalar states with respect to the vector states is much greater in Ref. [83], where $\Gamma^{*} / \Gamma \approx 2$ in comparison with $\Gamma^{*} / \Gamma \approx 3-4$ in the ISGW model. This leads to the fact that, for example, the exclusive decay modes $B_{c}^{+} \rightarrow \psi\left(\eta_{c}\right) \mathrm{e}^{+} v_{\mathrm{e}}$ practically saturate the $b \rightarrow c e v$ transition. This feature is analogous to the consideration of the $B \rightarrow D^{(*)} e v$ decay, which also saturates free b-quark decay. The decays into the excited states and many-particle modes are suppressed.

As one can see, these three models for decays with the spectator b-quark, give the close values

$$
\begin{aligned}
\Gamma\left(\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{~B}_{\mathrm{s}}+\mathrm{e}+\mathrm{v}\right)+\Gamma\left(\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{~B}_{\mathrm{s}}^{*}\right. & +\mathrm{e}+\mathrm{v}) \\
& =(60 \pm 7) \times 10^{-6} \mathrm{eV}
\end{aligned}
$$

Note also that in the case of the heavy quarkonium $B_{c}$, the application of the nonrelativistic wave function instead of the wave function of the relativistic oscillator in the meson of the WSB model seems to be more acceptable. This circumstance and uncertainty in $\omega$ perhaps explains why the WSB model gives an underestimated value for the width of the $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{J} / \psi+\mathrm{e}+v$ decay.
3.2.2 $\mathbf{B}_{\mathbf{c}}^{+} \rightarrow \mathbf{J} / \Psi\left(\boldsymbol{\eta}_{\mathbf{c}}\right) \mathbf{e}^{+} \boldsymbol{v}$ decay in $\mathbf{Q C D}$ sum rules. The most suitable for the registration modes of the $B_{c}$ decays are the semileptonic or hadronic transitions with the $J / \Psi-$ particle in the final state. But in the QCD sum rules (SR) $[30,36,35]$ and in the quark models, one found different results both for the widths of the corresponding decays and for the form factors of the transitions; although, in the framework of the separate approach the calculations performed in different ways coincided with each other. Recently in Ref. [31], we have shown that the existing discrepancy can be cancelled by the taking into account of the higher QCD corrections in SR.

The widths of the semileptonic $\mathrm{B}_{\mathrm{c}}$ decays are defined, in general, by the form factors $F_{+}, V, A_{1}$, and $A_{2}$ [see Eqns (127) and (128)]. Following the notation of Ref. [31], the form factors (127) and (128) are redefined as follows:

$$
\begin{aligned}
& f_{+}=F_{+}, \quad F_{0}^{\mathrm{A}}=\left(M_{\mathrm{B}_{\mathrm{c}}}+M_{\mathrm{V}}\right) A_{1} \\
& F_{+}^{\mathrm{A}}=-\frac{A_{2}}{M_{\mathrm{B}_{\mathrm{c}}}+M_{\mathrm{V}}}, \quad F_{\mathrm{V}}=\frac{V}{M_{\mathrm{B}_{\mathrm{c}}}+M_{\mathrm{V}}}
\end{aligned}
$$

For the calculation of these form factors in the QCD SR, let us consider the three-point functions

$$
\begin{align*}
& \Pi_{\mu}\left(p_{1}, p_{2}, q^{2}\right)=\mathrm{i}^{2} \int \mathrm{~d} x \mathrm{~d} y \exp \mathrm{i}\left(p_{2} x-p_{1} y\right) \\
& \times\langle 0| T\left\{\bar{c}(x) \gamma_{5} c(x), V_{\mu}(0), \overline{\mathrm{b}}(y) \gamma_{5} c(y)\right\}|0\rangle  \tag{151}\\
& \Pi_{\mu \nu}^{\mathrm{V}, \mathrm{~A}}\left(p_{1}, p_{2}, q^{2}\right)=\mathrm{i}^{2} \int \mathrm{~d} x \mathrm{~d} y \exp \mathrm{i}\left(p_{2} x-p_{1} y\right) \\
& \times\langle 0| T\left\{\bar{c}(x) \gamma_{\nu} c(x), J_{\mu}^{\mathrm{V}, \mathrm{~A}}(0), \overline{\mathrm{b}}(y) \gamma_{5} c(y)\right\}|0\rangle \tag{152}
\end{align*}
$$

We introduce the Lorentz structures in the correlators:

$$
\begin{align*}
\Pi_{\mu}= & \Pi_{+}\left(p_{1}+p_{2}\right)_{\mu}+\Pi_{-} q_{\mu}  \tag{153}\\
\Pi_{\mu \nu}^{\mathrm{V}}= & \mathrm{i} \Pi_{V} \varepsilon_{\mu \nu \alpha} p_{2}^{\alpha} p_{1}^{\beta}  \tag{154}\\
\Pi_{\mu \nu}^{\mathrm{A}}= & \mathrm{i} \Pi_{0}^{\mathrm{A}} g_{\mu \nu}+\Pi_{1}^{\mathrm{A}} p_{2}^{\mu} p_{1}^{v}+\Pi_{2}^{\mathrm{A}} p_{1}^{\mu} p_{1}^{v} \\
& \quad+\Pi_{3}^{\mathrm{A}} p_{2}^{\mu} p_{2}^{v}+\Pi_{4}^{\mathrm{A}} p_{1}^{\mu} p_{2}^{v} \tag{155}
\end{align*}
$$

The form factors $f_{+}, F_{\mathrm{V}}, F_{0}^{\mathrm{A}}$ and $F_{+}^{\mathrm{A}}$ are determined by the amplitudes $\Pi_{+}, \Pi_{\mathrm{V}}, \Pi_{0}^{\mathrm{A}}$ and $\Pi_{+}^{\mathrm{A}}=\left(\Pi_{1}+\Pi_{2}\right) / 2$, respectively. For the amplitudes, one can write down the double dispersion relation
$\Pi_{i}\left(p_{1}^{2}, p_{2}^{2}, q^{2}\right)=-\frac{1}{(2 \pi)^{2}} \int \frac{\rho_{i}\left(s_{1}, s_{2}, Q^{2}\right)}{\left(s_{1}-p_{1}^{2}\right)\left(s_{2}-p_{2}^{2}\right)} \mathrm{d} s_{1} \mathrm{~d} s_{2}$,
where $Q^{2}=-q^{2}>0$.
The integration region in Eqn (156) is determined by the condition

$$
\begin{equation*}
-1<\frac{2 s_{1} s_{2}+\left(s_{1}+s_{2}-q^{2}\right)\left(m_{\mathrm{b}}^{2}-m_{\mathrm{c}}^{2}-s_{1}\right)}{\lambda^{1 / 2}\left(s_{1}, s_{2}, q^{2}\right) \lambda^{1 / 2}\left(m_{\mathrm{c}}^{2}, s_{1}, m_{\mathrm{b}}^{2}\right)}<1 \tag{157}
\end{equation*}
$$

where $\lambda\left(x_{1}, x_{2}, x_{3}\right)=\left(x_{1}+x_{2}-x_{3}\right)^{2}-4 x_{1} x_{2}$.
In accordance with the general ideology of the QCD sum rules [11], the right-hand (theoretical) side of Eqn (156) can be calculated at large euclidian $p_{1}^{2}$ and $p_{2}^{2}$ values by the use of the operator product expansion (OPE). The perturbative parts of the corresponding spectral densities (the unit operator in OPE) of the one-loop approximation are presented in Appendix II. Since we are considering systems composed of heavy quarks, one can neglect the power corrections [36].

Consider the physical part of SR. As has been already mentioned in the consideration of the axial constant of $B_{c}$, there are two approaches. In the first one, one assumes that the physical part includes the contribution of the lowest mesons and the continuum that is approximated by the perturbative part of the spectral function from some threshold values $s_{0}^{1}$ and $s_{0}^{2}[35,36]$. The contribution of the higher excitations and the continuum is suppressed because of the Borel transformations over two variables $-p_{1}^{2}$ and $-p_{2}^{2}$. The numerical results obtained the same way as in Refs $[35,36]$ are presented below.

In the second way, one saturates the spectral density by an infinite number of narrow resonances [30], so that

$$
\begin{gather*}
\rho_{+}\left(s_{1}, s_{2}, Q^{2}\right)=(2 \pi)^{2} \sum_{i, j=1}^{\infty} f_{\mathrm{B}_{\mathrm{c}}}^{i} \frac{M_{\mathrm{B}_{\mathrm{c}}}^{i 2}}{m_{\mathrm{b}}+m_{\mathrm{c}}} f_{\eta_{\mathrm{c}}}^{j} \frac{M_{\mathrm{\eta}_{\mathrm{c}}}^{j 2}}{2 m_{\mathrm{c}}} f_{+}^{i j}\left(Q^{2}\right) \\
\times \delta\left(s_{1}-M_{\mathrm{B}_{\mathrm{c}}}^{i 2}\right) \delta\left(s_{2}-M_{\eta_{\mathrm{c}}}^{j 2}\right) \tag{158}
\end{gather*}
$$

$$
\begin{align*}
\rho_{\mathrm{V}}\left(s_{1}, s_{2}, Q^{2}\right)= & 2(2 \pi)^{2} \sum_{i, j=1}^{\infty} f_{\mathrm{B}_{\mathrm{c}}}^{i} \frac{M_{\mathrm{B}_{\mathrm{c}}}^{i 2}}{m_{\mathrm{b}}+m_{\mathrm{c}}} \frac{M_{\psi}^{j 2}}{g_{\psi}} F_{\mathrm{V}}^{i j}\left(Q^{2}\right) \\
& \times \delta\left(s_{1}-M_{\mathrm{B}_{\mathrm{c}}}^{i 2}\right) \delta\left(s_{2}-M_{\psi}^{j 2}\right),  \tag{159}\\
\rho_{0,+}^{\mathrm{A}}\left(s_{1}, s_{2}, Q^{2}\right)= & (2 \pi)^{2} \sum_{i, j=1}^{\infty} f_{\mathrm{B}_{\mathrm{c}}}^{i} \frac{M_{\mathrm{B}_{\mathrm{c}}}^{i 2}}{m_{\mathrm{b}}+m_{\mathrm{c}}} \frac{M_{\psi}^{j 2}}{g_{\psi}} F_{0,+}^{i j}\left(Q^{2}\right) \\
& \times \delta\left(s_{1}-M_{\mathrm{B}_{\mathrm{c}}}^{i 2}\right) \delta\left(s_{2}-M_{n_{\mathrm{c}}}^{j 2}\right) . \tag{160}
\end{align*}
$$

Substituting the expressions for the spectral densities (158)-(160) in the dispersion relations for correlators (156) on the one hand, and their perturbative values on the other, one gets the corresponding sum rules.

Applying the procedure described in Appendix III for both sums over the resonances, one obtains for the form factors under consideration:

$$
\begin{align*}
f_{+}^{k l}\left(Q^{2}\right)= & \frac{8 m_{\mathrm{c}}\left(m_{\mathrm{b}}+m_{\mathrm{c}}\right)}{M_{\mathrm{B}_{\mathrm{c}}}^{k} M_{\eta_{\mathrm{c}}}^{l} f_{\mathrm{B}_{\mathrm{c}}}^{k} f_{\eta_{\mathrm{c}}}^{l}} \frac{\mathrm{~d} M_{\mathrm{B}_{\mathrm{c}}}^{k}}{\mathrm{~d} k} \frac{\mathrm{~d} M_{{\eta_{\mathrm{c}}}^{l}}^{\mathrm{d} l}}{} \\
\times & \times \frac{1}{(2 \pi)^{2}} \rho_{+}\left(M_{\mathrm{B}_{\mathrm{c}}}^{k 2}, M_{\mathrm{n}_{\mathrm{c}}}^{l 2}, Q^{2}\right)  \tag{161}\\
F_{\mathrm{V}}^{k l}\left(Q^{2}\right)= & \frac{2\left(m_{\mathrm{b}}+m_{\mathrm{c}}\right) g_{\psi}^{l}}{M_{\mathrm{B}_{\mathrm{c}}}^{k} M_{\psi}^{l} f_{\mathrm{B}_{\mathrm{c}}}^{k}} \frac{\mathrm{~d} M_{\mathrm{B}_{\mathrm{c}}}^{k}}{\mathrm{~d} k} \frac{\mathrm{~d} M_{\psi}^{l}}{\mathrm{~d} l} \\
& \times \frac{1}{(2 \pi)^{2}} \rho_{\mathrm{V}}\left(M_{\mathrm{B}_{\mathrm{c}}}^{k 2}, M_{\psi}^{l 2}, Q^{2}\right)  \tag{162}\\
F_{0,+}^{k l}\left(Q^{2}\right)= & \frac{4\left(m_{\mathrm{b}}+m_{\mathrm{c}}\right) g_{\psi}^{l}}{M_{\mathrm{B}_{\mathrm{c}}}^{k} M_{\psi}^{l} f_{\mathrm{B}_{\mathrm{c}}}^{k}} \frac{\mathrm{~d} M_{\mathrm{B}_{\mathrm{c}}}^{k}}{\mathrm{~d} k} \frac{\mathrm{~d} M_{\psi}^{l}}{\mathrm{~d} l} \\
& \times \frac{1}{(2 \pi)^{2}} \rho_{0,+}\left(M_{\mathrm{B}_{\mathrm{c}}}^{k 2}, M_{\psi}^{l 2}, Q^{2}\right) \tag{163}
\end{align*}
$$

Choosing the $k$ and $l$ values, one can extract the transitions between the given resonances. At $k=l=1$, one gets the required form factors for the $B_{c}^{+} \rightarrow \mathrm{J} / \psi\left(\eta_{\mathrm{c}}\right) \mathrm{e}^{+} v$ decays.

Thus, we use the phenomenological parameters $\mathrm{d} M_{k} / \mathrm{d} k$ instead of the additional parameters such as the continuum thresholds. As has been mentioned, the former is, in a sense, the density of the quarkonium states with the given quantum numbers. One can quite accurately calculate these factors. The masses of the radial excitations of $\psi$ are known experimentally [15], and for the $B_{c}$ and $\eta_{c}$ systems composed of heavy quarks, one can use the predictions of the potential models [5-10, 52, 57-66].

The $\mathrm{d} M_{k} / \mathrm{d} k$ values at $k=1$ for the systems under consideration, are presented in Table 19.

Let us choose the following values of the parameters: $f_{\mathrm{B}_{\mathrm{c}}}=360 \mathrm{MeV}, f_{\eta_{\mathrm{c}}}=330 \mathrm{MeV}[36,76], m_{\mathrm{b}}=4.6 \pm 0.1 \mathrm{GeV}$, $m_{\mathrm{c}}=1.4 \pm 0.05 \mathrm{GeV}, \quad g_{\mathrm{J} / \psi}=8.1 \quad[$ from the data on $\left.\Gamma\left(\mathrm{J} / \psi \rightarrow \mathrm{e}^{+} \mathrm{e}^{-}\right)\right]$. For the axial constant, we choose 360 MeV [30] instead of 460 MeV , to compare the form factor values with the results of Ref. [36]. The $\mathrm{B}_{\mathrm{c}}$-meson mass will be varied from 6.245 to 6.284 GeV (the data of the different potential models). Note that with this choice of parameters we do not depart from the integration region (157). In Ref. [36] $M_{\mathrm{B}_{\mathrm{c}}}=6.35 \mathrm{GeV}$ was used.

Table 19. The derivaties $\mathrm{d} M_{k} / \mathrm{d} k$ (in GeV ) for the lowest states at $k=1$.

| Quarkonium | $\mathrm{B}_{\mathrm{c}}$ | $\mathrm{J} / \psi$ | $\eta_{c}$ |
| :--- | :--- | :--- | :--- |
| $\mathrm{~d} M_{k} / \mathrm{d} k$ | 0.75 | 0.75 | 0.76 |

The values of form factors obtained in Refs [30, 36, 35] at $Q^{2}=0$ are shown in Table 20. The deviation from the central values in Table 20 corresponds to the variation of the quark and $\mathrm{B}_{\mathrm{c}}$-meson masses within the limits mentioned above (for Ref. [30]). As in the case of the potential models, the SR predictions agree with each other.
Table 20. The form factors of the $B_{c} \rightarrow J / \psi\left(\eta_{c}\right) e v$ transitions at $Q^{2}=0$.

| $f_{+}(0)$ | $F_{\mathrm{V}}(0) / \mathrm{GeV}^{-1}$ | $F_{+}^{\mathrm{A}}(0) / \mathrm{GeV}^{-1}$ | $F_{0}^{\mathrm{A}}(0) / \mathrm{GeV}$ | Ref. |
| :--- | :--- | :--- | :--- | :--- |
| $0.23 \pm 0.01$ | $0.035 \pm 0.03$ | $-0.024 \pm 0.002$ | $2.0 \pm 0.2$ | $[30]$ |
| $0.20 \pm 0.02$ | $0.04 \pm 0.01$ | $-0.03 \pm 0.01$ | $2.5 \pm 0.3$ | $[36]$ |
| $0.55 \pm 0.1$ | $0.048 \pm 0.007$ | $-0.030 \pm 0.003$ | $3.0 \pm 0.5$ | $[35]$ |

In Ref. [30], the form factors have the following pole behaviour:

$$
\begin{equation*}
F_{i}\left(Q^{2}\right)=\frac{F_{i}(0)}{1+Q^{2} / m_{\text {pole }}^{2}} \phi_{i}\left(Q^{2}\right) \tag{164}
\end{equation*}
$$

where $m_{\text {pole }}=6.3-6.4 \mathrm{GeV}$, and $\phi_{i}\left(Q^{2}\right)=1+a_{i} Q^{2}$. The representation of $f_{+}, F_{\mathrm{V}}, F_{0}^{\mathrm{A}}$ and $F_{+}^{\mathrm{A}}$ by the form (161)(163) gives the following $a_{i}$ values, which are quite low and equal to $-0.025,-0.007,-0.012$, and -0.02 , respectively. The behaviour considered above hardly differs from the ordinary pole behaviour [30], where $a_{i}=0$. The results for the transition widths are presented in Table 21.

Table 21. The widths (in $10^{-6} \mathrm{eV}$ ) of the semileptonic $\mathrm{B}_{\mathrm{c}}$ decays in the QCD sum rules with no account of $\alpha_{s} / v$-corrections

| Mode | $[30]$ | $[36]$ | $[35]$ |
| :--- | :--- | :--- | :--- |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{J} / \psi \mathrm{e}^{+} v$ | 4.6 | 7 | 10.5 |
| $\mathrm{~B}_{\mathrm{c}}^{+} \rightarrow \eta_{\mathrm{c}} \mathrm{e}^{+} v$ | 1.4 | 1 | 9 |

As one can see from Tables 20 and 21, the results of the Borel SR are, in general, in good agreement with the results of the considered approach within the accuracy of the model. The widths obtained in Ref. [35] are greater than in Refs [30, 36], since in Ref. [35] the $q^{2}$-dependence of the transition form factors strongly differs from the behaviour expected in the meson dominance model.

The deviation from the quark models is related, from our point of view, to that in the calculations of the transition form factors in the QCD sum rules; one has to account for $\alpha_{\mathrm{s}} / v$-corrections, where $v$ is the relative velocity of the quarks inside the meson. For the heavy quarkonia, where the velocity of the quark motion is small, such corrections, corresponding to Coulomb-like interactions (Fig. 6), can play an essential role [31].


Figure 6. The Coulomb corrections in the semileptonic $B_{c}$-meson decay.

Indeed, the spectral densities $\rho_{i}\left(s_{1}, s_{2}, Q^{2}\right)$, determining the $B_{c}$ decay form factors, are calculated near the threshold $s_{1}=M_{\mathrm{B}_{\mathrm{c}}}^{2}, \quad s_{2}=M_{\eta_{\mathrm{c}}}^{2}, \psi$. When the recoil meson momentum is small, the calculation of the ladder diagrams in the formalism of the nonrelativistic quantum mechanics (see Ref. [17], Fig. 5) leads to the finite renormalisation of $\rho$, so that

$$
\begin{equation*}
\bar{\rho}_{i}\left(s_{1}, s_{2}, Q_{\max }^{2}\right)=C \rho_{i}\left(s_{1}, s_{2}, Q_{\max }^{2}\right), \tag{165}
\end{equation*}
$$

where the factor $C$ has the form

$$
\begin{equation*}
C=\left|\frac{\Psi_{\mathrm{B}_{\mathrm{c}}}^{\mathrm{C}}(0) \Psi_{\eta_{c}, \psi}^{\mathrm{C}}(0)}{\Psi_{\mathrm{B}_{\mathrm{c}}}^{\mathrm{fre}}(0) \Psi_{\eta_{\mathrm{c}}, \psi}^{\mathrm{free}}(0)}\right|, \tag{166}
\end{equation*}
$$

and $\Psi^{\mathrm{C}, \text { free }}(0)$ are the Coulomb and free wave functions of quarks, so that

$$
\begin{equation*}
\left|\frac{\Psi^{\mathrm{C}}(0)}{\Psi^{\mathrm{free}}(0)}\right|^{2}=\frac{4 \pi \alpha_{\mathrm{s}}}{3 v}\left[1-\exp \left(-\frac{4 \pi \alpha_{\mathrm{s}}}{3 v}\right)\right]^{-1} \tag{167}
\end{equation*}
$$

For the two-point quark correlators, determining the decay constants $f$ of the heavy quarkonia $\psi, \Upsilon, \mathrm{B}_{\mathrm{c}}$, the consideration of factor (166) led to the essential enhancement of $f$, so that one observes agreement with the experimental data on $f_{\psi}$ and $f_{\mathrm{Y}}$. Note that the expansion in Eqn (167) over $\alpha_{\mathrm{s}} / v \rightarrow 0$ leads exactly to the dominant term appearing in account of the one-loop $\alpha_{s}$-corrections to the two-point correlator of currents. Moreover, these corrections have been taken into account in the evaluation of $f$ for the three-point correlators, but one did not take into account the loop corrections in the determination of the three-point spectral densities.

For the sake of consistency, one should either ignore the $\alpha_{s}$-corrections in the evaluation of $f$ as well as in the determination of $\rho$, or one should take into account these corrections in both cases. As one can see, for example, from Eqn (161), one can write down

$$
\begin{align*}
f_{+}^{k l}\left(Q^{2}\right) & =\frac{8 m_{\mathrm{c}}\left(m_{\mathrm{b}}+m_{\mathrm{c}}\right) C}{M_{\mathrm{B}_{\mathrm{c}}} M_{\eta_{\mathrm{c}}} f_{\mathrm{B}_{\mathrm{c}}}^{(0)} f_{\eta_{\mathrm{c}}}^{(0)} C_{\mathrm{B}_{\mathrm{c}}}^{1 / 2} C_{\eta_{\mathrm{c}}}^{1 / 2}} \\
& \times \frac{\mathrm{d} M_{\mathrm{B}_{\mathrm{c}}}^{k}}{\mathrm{~d} k} \frac{\mathrm{~d} M_{\eta_{\mathrm{c}}}^{l}}{\mathrm{~d} l} \frac{1}{(2 \pi)^{2}} \rho_{+}^{(0)}\left(M_{\mathrm{B}_{\mathrm{c}}}^{k 2}, M_{\eta_{\mathrm{c}}}^{l 2}, Q^{2}\right), \tag{168}
\end{align*}
$$

where the $f^{(0)}$ and $\rho^{(0)}$ values are calculated with no account for the $\alpha_{\mathrm{s}}$-corrections, and the factors $C$ appear because of Coulomb-like corrections and are defined in Eqns (166), (167). It is evident that

$$
\begin{equation*}
\frac{C}{C_{\mathrm{B}_{\mathrm{c}}}^{1 / 2} C_{\mathrm{\eta}_{\mathrm{c}}}^{1 / 2}}=1 \tag{169}
\end{equation*}
$$

Thus, in the determination of the transition form factors, we can use the 'bare' $f$ and $\rho$ quantities, calculated in the zero approximation over $\alpha_{s}$ [85], instead of that, say, done in Ref. [36], where $\rho^{(0)}$ was used without the $C$ factor and the $f$ constants with the $\alpha_{s}$-corrections taken into account, (i.e. with the factors $C_{B_{c}}$ and $C_{\eta_{\mathrm{c}}, \psi}$ ) were instantaneously used.

As a result, one gets the following values for the form factors $f_{+}$and $F_{0}^{\mathrm{A}}[31]$ :

$$
f_{+}(0)=0.85 \pm 0.15, \quad F_{0}^{\mathrm{A}}=6.5 \pm 1 \mathrm{GeV}
$$

For the corresponding widths, one has found [31]

$$
\begin{aligned}
& \Gamma\left(\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \psi \mathrm{e}^{+} v\right) \approx 44 \times 10^{-6} \mathrm{eV} \\
& \Gamma\left(\mathrm{~B}_{\mathrm{c}}^{+} \rightarrow \eta_{\mathrm{c}} \mathrm{e}^{+} v\right) \approx 15 \times 10^{-6} \mathrm{eV}
\end{aligned}
$$

Note that we have neglected the contributions of the form factors $F_{\mathrm{V}}$ and $F_{+}^{\mathrm{A}}$ in the decay $\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{J} / \psi \mathrm{ev}$. This can result in an overestimation of the value of the widths of up to $10 \%-20 \%$. One can make agreement between the obtained values of the widths and the results of the quark models (see Table 18) within the limits of the theoretical uncertainties of the methods used.

Comparing the results of the QCD SR and the quark models, one can accept as a central value of the $B_{c} \rightarrow J / \psi e v$ decay width (with an accuracy of about $40 \%$ )

$$
\Gamma\left(\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{~J} / \psi \mathrm{ev}\right) \approx 40 \times 10^{-6} \mathrm{eV}
$$

which corresponds to a branching fraction equal to $3 \%$. Then the relative probability of a three-lepton yield in $B_{c}$ decays, when two of them reconstruct $J / \psi$, is

$$
\operatorname{BR}\left(\mathrm{B}_{\mathrm{c}}^{+} \rightarrow\left(1^{+} 1^{-}\right)_{\mathrm{J} / \psi} 1^{\prime+} v\right) \approx 8 \times 10^{-3}
$$

where $l, l^{\prime}$ denotes e or $\mu$.
3.2.3 Approximate spin symmetry. In the bound state, the heavy quark virtualities are much less than their masses, i.e. the following kinematic expansion for the quark momentum $p_{\mathrm{Q}}$ is accessible

$$
\begin{equation*}
p_{\mathrm{Q}}^{\mu}=m_{\mathrm{Q}} v^{\mu}+k^{\mu} \tag{170}
\end{equation*}
$$

so that

$$
\begin{equation*}
v k \approx 0, \quad\left|k^{2}\right| \ll m_{\mathrm{Q}}^{2} \tag{171}
\end{equation*}
$$

Then, in the system where $v=(1, \mathbf{0})$, the heavy quark Hamiltonian in a gluon field of an external source has the form

$$
\begin{equation*}
H=m_{\mathrm{Q}}+V(\boldsymbol{r})+\frac{\boldsymbol{k}^{2}}{2 m_{\mathrm{Q}}}+g \frac{\boldsymbol{\sigma} \cdot \boldsymbol{B}}{2 m_{\mathrm{Q}}}+\mathrm{O}\left(\frac{1}{m_{\mathrm{Q}}^{2}}\right) \tag{172}
\end{equation*}
$$

so that in the limit $\Lambda_{\mathrm{QCD}} \ll m_{\mathrm{Q}}$, the spin-flavour symmetry EHQT [14] occurs for hadrons with a single heavy quark.

For the heavy quarkonium, one has purely phenomenologically that the kinetic energy is practically independent of their flavours; however, the value of the potential energy term $V(\boldsymbol{r})$ is determined by the average distance between the heavy quarks. This distance depends on the quark masses, i.e. the flavours. Therefore, there is no flavour symmetry of the wave functions in the heavy quarkonium. However, the magnetic field of the heavy quark is determined by its motion velocity (as well as magnetic moment). The quark motion is nonrelativistic in the heavy quarkonium, so that

$$
\begin{equation*}
\boldsymbol{B} \sim \mathrm{O}(\boldsymbol{v}) \sim \mathrm{O}\left(\frac{1}{m_{\mathrm{Q}}}\right) \tag{173}
\end{equation*}
$$

From Eqns (172) and (173) it follows that the spindependent potential in the heavy quarkonium appears in the second order over the inverse heavy quark masses (see Section 2)

$$
\begin{equation*}
V_{\mathrm{SD}} \sim \mathrm{O}\left(\frac{1}{m_{\mathrm{Q}}^{2}}\right) \tag{174}
\end{equation*}
$$

Thus, in the leading approximation for the heavy quarkonium, one can neglect the spin-dependent forces in comparison with the kinetic energy and the nonrelativistic potential. This means that in this approximation the quark spin is decoupled from interaction with the gluons of low virtualities, therefore the masses of the $n L_{J}$-quarkonium states are degenerated over $J$, and these states have identical wave functions.

Thus, there is an approximate spin symmetry for the heavy quarks in the heavy quarkonium.

Further, let us consider the matrix element

$$
\begin{equation*}
M=\left\langle n^{S} L_{J}\left(\mathrm{Q}^{\prime}\right)\right| \Gamma|h\rangle, \tag{175}
\end{equation*}
$$

where $\Gamma$ is the operator of quark currents, $h$ is a state. Then the spin symmetry means that the action of spin operators of the heavy quark is factorised, and the matrix element $\bar{M}$, obtained by the action of the quark $Q$ spin (or by the antiquark $\overline{\bar{Q}^{\prime}}$ spin)

$$
\begin{equation*}
S_{\mu}^{\mathrm{Q}}=\frac{1}{4} \varepsilon_{\mu v \alpha \beta} v_{\mathrm{Q}}^{v} \sigma^{\alpha \beta}, \quad \sigma^{\alpha \beta}=\frac{\mathrm{i}}{2}\left[\gamma^{\alpha} ; \gamma^{\beta}\right] \tag{176}
\end{equation*}
$$

is related to the matrix element $M$ by the equation

$$
\begin{equation*}
\bar{M}=\left\langle n^{S} L_{J}\left(\mathrm{Q}^{\prime}\right)\right| S_{\mu}^{\mathrm{Q}} \Gamma|h\rangle=\sum C_{S S^{\prime}}^{J J^{\prime}}\left\langle n^{S^{\prime}} L_{J^{\prime}}\left(\mathrm{Q} \overline{\mathrm{Q}}^{\prime}\right)\right| \Gamma|h\rangle, \tag{177}
\end{equation*}
$$

where $\bar{M}$ is the sum of the matrix elements with $J^{\prime}$, and $C_{S S}^{J J^{\prime}}$, are defined by the rules for the spin operator action.

For the semileptonic $B_{c}^{+} \rightarrow \eta_{c}(\psi) 1^{+} v$ decays, the spin symmetry is valid at the point of zero recoil of $\eta_{c}(\psi)$. Indeed, in this case the spectator c-quark and the $\overline{\mathrm{c}}$-quark, produced in the weak decay of the $\bar{b}$-quark, are practically at rest with respect to each other so, binding into the state, they interact with the low virtualities characteristic of the heavy quarkonium. At a nonzero velocity of the $\overline{\mathrm{c}}$-quark, it must exchange with the c-quark by a momentum comparable to its mass in order to make the bound state, where their velocities are close. Thus, at the nonzero meson recoil, the gluons with high virtualities can shift the heavy quark spin, and the spin symmetry does not appear.

At the zero recoil of the charmonium $v_{\mathrm{B}_{\mathrm{c}}}=v_{\eta_{\mathrm{c}}(\psi)}$, in the covariant amplitude of the weak current, the nonzero contributions are given by $A_{1}(t), F_{ \pm}(t)$ at $t=t_{\text {max }}$, and the heavy quark spin symmetry means that

$$
\begin{align*}
& \left(M_{\mathrm{B}_{\mathrm{c}}}+M_{\eta_{\mathrm{c}}}\right) F_{+}+\left(M_{\mathrm{B}_{\mathrm{c}}}-M_{\eta_{\mathrm{c}}}\right) F_{-}=\left(M_{\mathrm{B}_{\mathrm{c}}}+M_{\psi}\right) A_{1}, \\
& t=t_{\mathrm{max}}, \quad M_{\eta_{\mathrm{c}}}=M_{\psi} . \tag{178}
\end{align*}
$$

Thus, in the approximation of zero spin-dependent splitting of the heavy quarkonium, one derives the specific relation for the form factors of the semileptonic exclusive $B_{c}$ decays into the charmonium.

Note now that the covariant model considered above gives the semileptonic form factor values for the $B_{c}$ decay into the charmonium, so that these quantities satisfy the symmetry relation (178). In contrast to the decays of the heavy hadrons with a single heavy quark, where the form factor normalisation at zero recoil is fixed due to the flavour symmetry, the normalisation of form factors for the weak semileptonic transitions between the heavy quarkonia is determined by the overlapping of their wave functions, which depend on the quarkonium model.

For the oscillator wave functions in the considered potential model, we get

$$
\begin{equation*}
\left(M_{\mathrm{B}_{\mathrm{c}}}+M_{\psi}\right) A_{1}\left(t_{\max }\right)=\sqrt{2 M_{\mathrm{B}_{\mathrm{c}}} \times 2 M_{\psi}} \xi\left(t_{\max }\right) \tag{179}
\end{equation*}
$$

where

$$
\begin{equation*}
\xi\left(t_{\max }\right)=\left(\frac{2 \omega_{\mathrm{B}_{\mathrm{c}}} \omega_{\psi}}{\omega_{\mathrm{B}_{\mathrm{c}}}^{2}+\omega_{\psi}^{2}}\right)^{3 / 2} \tag{180}
\end{equation*}
$$

In Ref. [37] the factor $\xi\left(t_{\max }\right)$ was determined in the quarkonium model with the Coulomb potential, which is quite a rough approximation.

Note further that in the semileptonic $\mathrm{B}_{\mathrm{c}}$ decay, the lepton pair kinematically has, on average, large invariant masses $m\left(l^{+} v\right) \approx 1.9 \mathrm{GeV}$, thus the $A_{1}$ form factor contribution dominates, so that in accordance with the meson dominance of the $t$-dependence of the form factors, relation (178), giving $A_{1}\left(t_{\text {max }}\right)$, determines, in a sense, the matrix element of the semileptonic $B_{c}^{+} \rightarrow \psi\left(\eta_{c}\right) 1^{+} v$ decay. This feature can be used for the determination of the $\mathrm{B}_{\mathrm{c}}{ }^{-}$ meson mass from the $\psi l^{+}$mass spectrum as well as the element $\left|V_{\mathrm{bc}}\right|$ of the Kobayashi-Maskawa matrix.

### 3.3 Hadronic decays of $\mathbf{B}_{\mathbf{c}}$-mesons

Although the semileptonic $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{J} / \psi \mu^{+}\left(\mathrm{e}^{+}\right) \nu_{\mu}\left(v_{\mathrm{e}}\right)$ decays can serve as a good trigger for the $B_{c}$ registration, the complete $B_{c}$ reconstruction needs large statistics because of the neutrino presence in the decay products. The direct measurement of the $\mathrm{B}_{\mathrm{c}}$-meson mass is possible only in the hadronic exclusive decays. The preliminary estimates of some nonleptonic decay widths with the $\mathrm{J} / \psi$-particle in the final state were made in Refs [29, 33, 81] in the framework of the potential models.

The hadronic decays were considered in detail in Refs [32, 34, 83]. In Ref. [34] the transition form factors were calculated with the use of the WSB and ISGW models, mentioned above. In the calculation of the decay widths, the reduction of the phase space for the c -spectator decays was taken into account (see Section 3.1), in contrast to some other calculations [33, 81]. In the following analysis of the hadronic decays of the $\mathrm{B}_{\mathrm{c}}$-meson, we will follow the results of the latter paper.

The effective four-fermion Hamiltonian for the nonleptonic decays of the c- and b-quarks has the form [86]

$$
\begin{aligned}
& \mathcal{H}_{\mathrm{eff}}^{\mathrm{c}}=\frac{G}{2 \sqrt{2}} V_{\mathrm{uq}_{1}} V_{\mathrm{cq}_{1}}^{*}\left[C_{+}^{\mathrm{c}}(\mu) O_{+}^{\mathrm{c}}+C_{-}^{\mathrm{c}}(\mu) O_{-}^{\mathrm{c}}\right]+\text { h.c. }, \\
& \mathcal{H}_{\mathrm{eff}}^{\mathrm{b}}=\frac{G}{2 \sqrt{2}} V_{\mathrm{q}_{1} \mathrm{~b}} V_{\mathrm{q}_{2} \mathrm{q}_{3}}^{*}\left[C_{+}^{\mathrm{b}}(\mu) O_{+}^{\mathrm{b}}+C_{-}^{\mathrm{b}}(\mu) O_{-}^{\mathrm{b}}\right]+\text { h.c. },(182)
\end{aligned}
$$

where
$O_{ \pm}^{\mathrm{c}}=\left[\bar{q}_{1 \alpha} \gamma_{v}\left(1-\gamma_{5}\right) c_{\beta}\right]\left[\bar{u}_{\gamma} \gamma^{\nu}\left(1-\gamma_{5}\right) q_{2 \delta}\right]\left(\delta_{\alpha \beta} \delta_{\gamma \delta} \pm \delta_{\alpha \delta} \delta_{\gamma \beta}\right)$,
$O_{ \pm}^{\mathrm{b}}=\left[\bar{q}_{1 \alpha} \gamma_{v}\left(1-\gamma_{5}\right) b_{\beta}\right]\left[\bar{q}_{3 \gamma} \gamma^{\nu}\left(1-\gamma_{5}\right) q_{2 \delta}\right]\left(\delta_{\alpha \beta} \delta_{\gamma \delta} \pm \delta_{\alpha \delta} \delta_{\gamma \beta}\right)$.
The factors $C_{ \pm}^{\mathrm{c}, \mathrm{b}}(\mu)$ account for the strong corrections to the corresponding four-fermion operators because of hard gluons [34, 86].

The transition amplitudes should not depend on the subtraction point $\mu$ if one consistently calculates them in the perturbation theory, i.e. one constructs the corresponding functions of the initial and final hadronic states in the perturbation theory, in accordance with the operators.

The problem is complicated when one deals with the factorisation approximation used for the calculation of the matrix elements. In this approximation, one assumes that the current is proportional to a single stable or quasistable hadronic field, and one calculates its matrix element between the vacuum and the corresponding asymptotic hadronic state; this procedure gives a value proportional to the decay constant of the hadron. After that, the amplitude of the weak decay is factorised and is completely determined by the hadronic matrix element of another current that can be calculated by the use of a model, as

Table 22. The widths (in $10^{-6} \mathrm{eV}$ ) of two-particle hadronic $\overline{\mathrm{b}}$-spectator decays ( $M_{\mathrm{B}_{\mathrm{c}}}=6.27 \mathrm{GeV}, M_{\mathrm{B}_{\mathrm{s}}}=5.39 \mathrm{GeV}, M_{\mathrm{B}_{\mathrm{s}}^{*}}=5.45 \mathrm{GeV}$ ).

| Decay mode | WSB | $\begin{aligned} & a_{1}=1.23 \\ & a_{2}=0.33 \end{aligned}$ | ISGW | $\begin{aligned} & a_{1}=1.23 \\ & a_{2}=0.33 \end{aligned}$ | [83] | $\begin{aligned} & a_{1}=1.12 \\ & a_{2}=-0.26 \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}_{\mathrm{s}}+\pi^{+}$ | $a_{1}^{2} 31.1$ | 47.8 | $a_{1}^{2} 44.0$ | 67.7 | $a_{1}^{2} 58.4$ | 73.3 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}_{\mathrm{s}}+\mathrm{\rho}^{+}$ | $\begin{array}{ll}a_{1}^{2} & 12.5\end{array}$ | 19.2 | $a_{1}^{2} 20.2$ | 3.1 | $a_{1}^{2} 44.8$ | 56.1 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}_{\mathrm{s}}^{*}+\pi^{+}$ | $a_{1}^{2} 25.6$ | 39.4 | $a_{1}^{2} 34.7$ | 53.4 | $a_{1}^{2} 51.6$ | 64.7 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{Bs}^{*}+\mathrm{\rho}^{+}$ | $a_{1}^{2} 115.6$ | 177.8 | $a_{1}^{2} 152.1$ | 234 | $a_{1}^{2} 150$ | 188 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}^{+}+\overline{\mathrm{K}}^{0}$ | $a_{2}^{2} 28.2$ | 3.1 | $a_{2}^{2} 61.4$ | 6.7 | $a_{2}^{2} 96.5$ | 4.25 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}^{+}+\overline{\mathrm{K}}^{* 0}$ | $\begin{array}{ll}a_{2}^{2} & 10.0\end{array}$ | 1.1 | $a_{2}^{2} 24.1$ | 2.6 | $a_{2}^{2} 68.2$ | 3.01 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}^{*+}+\overline{\mathrm{K}}^{0}$ | $a_{2}^{2} 31.0$ | 3.4 | $a_{2}^{2} 28.3$ | 3.1 | $a_{2}^{2} 73.3$ | 3.23 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}^{*+}+\overline{\mathrm{K}}^{* 0}$ | $a_{2}^{2} 147.1$ | 16 | $\begin{array}{lll}a_{2}^{2} & 163.8\end{array}$ | 18 | $\begin{array}{ll}a_{2}^{2} & 141\end{array}$ | 6.23 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}^{0}+\pi^{+}$ | $a_{1}^{2} \quad 0.97$ | 1.49 | $a_{1}^{2} 1.89$ | 2.9 | $a_{1}^{2} 3.30$ | 4.14 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}^{0}+\mathrm{\rho}^{+}$ | $a_{1}^{2} 0.94$ | 1.45 | $a_{1}^{2} 2.14$ | 3.3 | $a_{1}^{2} 5.97$ | 7.48 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}^{* 0}+\pi^{+}$ | $a_{1}^{2} 1.58$ | 2.42 | $a_{1}^{2} 1.28$ | 2.0 | $a_{1}^{2} 2.90$ | 3.64 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}^{* 0}+\mathrm{\rho}^{+}$ | $a_{1}^{2} 8.82$ | 13.6 | $a_{1}^{2} 8.86$ | 12 | $a_{1}^{2} 11.9$ | 15.0 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}^{+}+\pi^{0}$ | $a_{2}^{2} 0.48$ | 0.05 | $a_{2}^{2} 0.95$ | 0.1 | $a_{2}^{2} 1.65$ | 0.074 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}^{+}+\rho^{0}$ | $a_{2}^{2} 0.47$ | 0.05 | $a_{2}^{2} 1.07$ | 0.12 | $a_{2}^{2} 2.98$ | 0.132 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}^{+}+\omega$ | $a_{2}^{2} 0.38$ | 0.04 | $a_{2}^{2} 0.87$ | 0.009 | - | - |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}^{*+}+\pi^{0}$ | $a_{2}^{2} 0.79$ | 0.09 | $a_{2}^{2} 0.64$ | 0.07 | $a_{2}^{2} 1.45$ | 0.064 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}^{*+}+\rho^{0}$ | $a_{2}^{2} 4.41$ | 0.48 | $a_{2}^{2} 4.43$ | 0.48 | $a_{2}^{2} 5.96$ | 0.263 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}^{*+}+\omega$ | $a_{2}^{2} 3.60$ | 0.39 | $a_{2}^{2} 3.53$ | 0.38 | - | - |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}_{\mathrm{s}}+\mathrm{K}^{+}$ | $a_{1}^{2} 2.18$ | 3.35 | $a_{1}^{2} 3.28$ | 5 | $a_{1}^{2} 4.2$ | 5.27 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{B}_{\mathrm{s}}^{*}+\mathrm{K}^{+}$ | $a_{1}^{2} 1.71$ | 2.6 | $a_{1}^{2} 2.52$ | 3.9 | $a_{1}^{2} 2.96$ | 3.72 |
| $\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{B}^{0}+\mathrm{K}^{+}$ | - | - | - | - | $a_{1}^{2} 0.255$ | 0.32 |
| $\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{B}^{0}+\mathrm{K}^{*+}$ | - | - | - | - | $a_{1}^{2} 0.180$ | 0.226 |
| $\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{B}^{* 0}+\mathrm{K}^{+}$ | - | - | - | - | $a_{1}^{2} 0.195$ | 0.244 |
| $\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{B}^{* 0}+\mathrm{K}^{*+}$ | - | - | - | - | $\begin{array}{lll}a_{1}^{2} & 0.374\end{array}$ | 0.47 |

in the case of the semileptonic decays. In this approximation, the interaction in the final state is neglected.

Note that the exact factorisation takes place in the leading order of the $1 / N_{c}$ expansion [87]. In this approximation, one has to be careful in the choice of the subtraction point, since the matrix elements depend on $\mu$. (The dependence of coefficients for the four-fermion operators of the effective Hamiltonian on the subtraction point is not compensated by the functions of the initial and final states.) The most suitable choice is $\mu \approx m_{\mathrm{c}}$, since the radius of the $\mathrm{B}_{\mathrm{c}}$-meson is determined by the mass of the c quark, and the transferred momenta in the decays are about $m_{\mathrm{c}}$ [34].

The anomalous dimensions of the $O_{+}^{\mathrm{c}}$ and $O_{-}^{\mathrm{c}}$ operators at $\mu=m_{\mathrm{c}}$ have the form

$$
\begin{equation*}
\gamma_{ \pm}=-\frac{\alpha_{\mathrm{s}}}{2 \pi} \frac{3}{N_{\mathrm{c}}}\left(1 \mp N_{\mathrm{c}}\right) . \tag{183}
\end{equation*}
$$

In the leading logarithm approximation at $\mu>m_{\mathrm{c}}$, one has [88]

$$
\begin{align*}
& C_{+}^{\mathrm{c}}(\mu)=\left[\frac{\alpha_{\mathrm{s}}\left(M_{\mathrm{W}}^{2}\right.}{\alpha_{\mathrm{s}}\left(m_{\mathrm{b}}^{2}\right)}\right]^{6 / 23}\left[\frac{\alpha_{\mathrm{s}}\left(m_{\mathrm{b}}^{2}\right)}{\alpha_{\mathrm{s}}\left(\mu^{2}\right)}\right]^{6 / 25}, \\
& C_{-}^{\mathrm{c}}(\mu)=\left[C_{+}^{\mathrm{c}}(\mu)\right]^{-2} \tag{184}
\end{align*}
$$

with $\alpha_{\mathrm{s}}\left(m_{\mathrm{c}}^{2}\right)=0.27, \alpha_{\mathrm{s}}\left(m_{\mathrm{b}}^{2}\right)=0.19, \alpha_{\mathrm{s}}\left(M_{\mathrm{W}}^{2}\right)=0.11$, one has the values $C_{+}^{\mathrm{c}}\left(m_{\mathrm{c}}\right)=0.80$ and $C_{-}^{\mathrm{c}}\left(m_{\mathrm{c}}\right)=1.57$.

When $\mu>m_{\mathrm{b}}$, the anomalous dimensions of the $C_{ \pm}^{\mathrm{b}}$ operators are determined by Eqn (183), but when $m_{\mathrm{c}}<\mu<m_{\mathrm{b}}$ one finds

$$
\begin{equation*}
\gamma_{ \pm}=-\frac{\alpha_{\mathrm{s}}}{2 \pi}\left[3 \frac{N_{\mathrm{c}}^{2}-1}{4 N_{\mathrm{c}}}+\frac{3}{2 N_{\mathrm{c}}}\left(1 \mp N_{\mathrm{c}}\right)\right] \tag{185}
\end{equation*}
$$

$$
\begin{align*}
& C_{+}^{\mathrm{b}}(\mu)=\left[\frac{\alpha_{\mathrm{s}}\left(M_{\mathrm{W}}^{2}\right.}{\alpha_{\mathrm{s}}\left(m_{\mathrm{b}}^{2}\right)}\right]^{6 / 23}\left[\frac{\alpha_{\mathrm{s}}\left(m_{\mathrm{b}}^{2}\right)}{\alpha_{\mathrm{s}}\left(\mu^{2}\right)}\right]^{-3 / 25},  \tag{186}\\
& C_{-}^{\mathrm{b}}(\mu)=\left[\frac{\alpha_{\mathrm{s}}\left(M_{\mathrm{W}}^{2}\right)}{\alpha_{\mathrm{s}}\left(m_{\mathrm{b}}^{2}\right)}\right]^{-12 / 23}\left[\frac{\alpha_{\mathrm{s}}\left(m_{\mathrm{b}}^{2}\right)}{\alpha_{\mathrm{s}}\left(\mu^{2}\right)}\right]^{-12 / 25} . \tag{187}
\end{align*}
$$

The numerical values are $C_{+}^{\mathrm{b}}\left(m_{\mathrm{c}}\right)=0.90$ and $C_{-}^{\mathrm{b}}\left(m_{\mathrm{c}}\right)=1.57$.

For the nonleptonic inclusive spectator decays of the $\mathrm{B}_{\mathrm{c}}$-meson, the enhancement factor due to the 'dressing' of the four-fermion operators by hard gluons is equal to

$$
\begin{equation*}
3\left[C_{+}^{2} \frac{N_{\mathrm{c}}+1}{2 N_{\mathrm{c}}}+C_{-}^{2} \frac{N_{\mathrm{c}}-1}{2 N_{\mathrm{c}}}\right], \tag{188}
\end{equation*}
$$

where 3 is the colour factor. For the annihilation decays, the corresponding factor equals

$$
\begin{equation*}
3\left[C_{+} \frac{N_{\mathrm{c}}+1}{2 N_{\mathrm{c}}}+C_{-} \frac{N_{\mathrm{c}}-1}{2 N_{\mathrm{c}}}\right]^{2} . \tag{189}
\end{equation*}
$$

The widths of the annihilation and inclusive spectator decays are presented in Table 15. As has been mentioned, the quark masses have the following values: $m_{\mathrm{c}}=1.5 \mathrm{GeV}$, $m_{\mathrm{b}}=4.9 \mathrm{GeV}$ and $m_{s}=0.15 \mathrm{GeV}$, i.e. one makes a choice that provides a good description both of the semileptonic decays of B- and D-mesons and of the total B-meson width. The enhancement factors (188) and (189) are calculated in the large $N_{\mathrm{c}}$ limit (this approximation gives a good description of B- and D-meson decays [71, 89]). For the b-spectator decays, one accounts for the phase space reduction, in contrast to calculations in Ref. [32].

The results of calculations for the widths of exclusive decays (here we consider the two-particle states) were performed in the models of WSB, ISGW, and Ref. [83],

Table 23. The widths (in $10^{-6} \mathrm{eV}$ ) of two-particle hadronic c-spectator decays.

| Decay mode | ISGW | $a_{1}=1.18$ | [83] | $a_{1}=1.26$ |
| :---: | :---: | :---: | :---: | :---: |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \eta_{\mathrm{c}}+\pi^{+}$ | $a_{1}^{2} 1.71$ | 2.63 | $a_{1}^{2} 2.07$ | 3.29 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \eta_{\mathrm{c}}+\rho^{+}$ | $a_{1}^{2} 4.04$ | 6.2 | $a_{1}^{2} 5.48$ | 8.70 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{J} / \psi+\pi^{+}$ | $a_{1}^{2} 1.79$ | 2.75 | $a_{1}^{2} 1.97$ | 3.14 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{J} / \psi+\mathrm{\rho}^{+}$ | $a_{1}^{2} 5.07$ | 7.8 | $a_{1}^{2} 5.95$ | 9.45 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \eta_{\mathrm{c}}+\mathrm{K}^{+}$ | $\begin{array}{lll}a_{1}^{2} & 0.127\end{array}$ | 0.195 | $a_{1}^{2} 0.161$ | 0.256 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \eta_{\mathrm{c}}+\mathrm{K}^{*+}$ | $\begin{array}{lll}a_{1}^{2} & 0.203\end{array}$ | 0.31 | $a_{1}^{2} 0.286$ | 0.453 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{J} / \psi+\mathrm{K}^{+}$ | $a_{1}^{2} 0.130$ | 0.2 | $\begin{array}{lll}a_{1}^{2} & 0.152\end{array}$ | 0.242 |
| $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{J} / \psi+\mathrm{K}^{*+}$ | $a_{1}^{2} 0.263$ | 0.4 | $\begin{array}{lll}a_{1}^{2} & 0.324\end{array}$ | 0.514 |
| $\mathrm{B}_{\mathrm{c}} \rightarrow \psi(2 \mathrm{~S})+\pi^{+}$ | - | - | $a_{1}^{2} 0.251$ | 0.398 |
| $\mathrm{B}_{\mathrm{c}} \rightarrow \psi(2 S)+\mathrm{\rho}^{+}$ | - | - | $a_{1}^{2} 0.71$ | 1.13 |
| $\mathrm{B}_{\mathrm{c}} \rightarrow \psi(2 \mathrm{~S})+\mathrm{K}^{+}$ | - | - | $\begin{array}{ll}a_{1}^{2} & 0.018\end{array}$ | 0.029 |
| $\mathrm{B}_{\mathrm{c}} \rightarrow \psi(2 \mathrm{~S})+\mathrm{K}^{*+}$ | - | - | $a_{1}^{2} 0.038$ | 0.060 |

and are presented in Tables 22 and 23. The Cabibbo nonsuppressed widths of $b$-spectator decays are shown in Table 22.

The $a_{1}$ and $a_{2}$ coefficients, accounting for the renormalisation of the four-fermion operators, are defined in the following way:

$$
\begin{align*}
& a_{1}=C_{+} \frac{N_{\mathrm{c}}+1}{2 N_{\mathrm{c}}}+C_{-} \frac{N_{\mathrm{c}}-1}{2 N_{\mathrm{c}}}  \tag{190}\\
& a_{2}=C_{+} \frac{N_{\mathrm{c}}+1}{2 N_{\mathrm{c}}}-C_{-} \frac{N_{\mathrm{c}}-1}{2 N_{\mathrm{c}}} \tag{191}
\end{align*}
$$

In the limit $N_{\mathrm{c}} \rightarrow \infty$ one has

$$
\begin{align*}
& a_{1} \approx 0.5\left(C_{+}+C_{-}\right)  \tag{192a}\\
& a_{2} \approx 0.5\left(C_{+}-C_{-}\right) \tag{192b}
\end{align*}
$$

The $a_{1}$ and $a_{2}$ values used in Refs [34, 83], differ from each other because of the different choices for the quark masses and the $\Lambda_{\mathrm{QCD}}$ parameter.

Note that for decays with B-mesons in the final state, the contribution of the annihilation and 'penguin' diagrams is suppressed as $\mathrm{O}\left(\sin ^{10} \theta_{\mathrm{c}}\right)$. As one can see from Table 22, the WSB results agree with the ISGW model, and the sum of the two-particle decay widths is equal to the total inclusive width of the $b$-spectator decay (see Table 15).

The widths in the model of Ref. [83] are slightly greater. The reason for the deviation from the results of two other models might be the fact that for the $b$-spectator decays there are $\mathrm{B}-$ and $\mathrm{B}_{\mathrm{s}}$-mesons in the final state, so that these mesons are relativistic systems because of the presence of the light quark and, hence, the nonrelativistic approximation would work poorly.

Among the c-spectator decays, widths (Table 23) are given for the decays, where the WSB and ISGW models result in close values and one can neglect the contributions of the annihilation and 'penguin' diagrams. As one can see from Table 23, the data of the ISGW model agree well with the results of the model in Ref. [83].

The total inclusive nonleptonic width of the $\mathrm{B}_{\mathrm{c}}$-meson decay with the $J / \psi$-particle in the final state can be obtained from the corresponding width of the semileptonic decay:

$$
\begin{equation*}
\Gamma\left(\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{~J} / \psi \mathrm{X}_{\mathrm{u}(\overline{\mathrm{~d}}(\overline{\mathrm{~s}}}\right)=3 a_{1}^{2} \Gamma\left(\mathrm{~B}_{\mathrm{c}} \rightarrow \mathrm{~J} / \psi \mathrm{ev}\right)\left|V_{\mathrm{ud}(\mathrm{~s})}\right|^{2} \tag{193}
\end{equation*}
$$

In the limit of large $N_{\mathrm{c}}$, one has $3 a_{1}^{2}=4.6\left(a_{1}=1.18\right)$ and

$$
\Gamma\left(\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{~J} / \psi \mathrm{X}_{\mathrm{ud}(\overline{\mathrm{~s}})}\right) \approx 190 \times 10^{-6} \mathrm{eV}
$$

The branching ratio of the $B_{c}$ decay with the $J / \psi$-particle in the final state is

$$
\mathrm{BR}\left(\mathrm{~B}_{\mathrm{c}} \rightarrow \mathrm{~J} / \psi+\mathrm{X}\right) \approx 0.2
$$

The WSB and ISGW models give close results for the two-particle $\mathrm{B}_{\mathrm{c}}$-meson decays with the B -mesons in the final state. Unfortunately, it is complex to detect the $\mathrm{B}_{\mathrm{c}}$-mesons in such decay modes, since one has to reconstruct the $B$ mesons from the products of their weak decays. The $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{J} / \psi \pi^{+}$decay is more suitable for the detection of the $B_{c}$-meson and the measurement of its mass [34]. Its branching ratio is

$$
\mathrm{BR}\left(\mathrm{~B}_{\mathrm{c}}^{+} \rightarrow \mathrm{J} / \psi \pi^{+}\right) \approx 2 \times 10^{-3}
$$

The $\mathrm{B}_{\mathrm{c}}$-meson decays in which CP -violation can be observed $-B_{c}^{ \pm} \rightarrow(c \bar{c}) D^{ \pm}, B_{c} \rightarrow D \rho(\pi)$ and $B_{c} \rightarrow D^{0} D_{s}-$ are of a special interest.

Approximate estimates for the decay branching fractions and the asymmetry parameter of CP-violation were obtained in Ref. [80]. The corresponding results are presented in Table 24. The asymmetry parameter $A$ is defined in the following way:

$$
\begin{equation*}
A=\frac{\Gamma\left(\mathrm{B}_{\mathrm{c}}^{-} \rightarrow \overline{\mathrm{X}}\right)-\Gamma\left(\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{X}\right)}{\Gamma\left(\mathrm{B}_{\mathrm{c}}^{-} \rightarrow \overline{\mathrm{X}}\right)+\Gamma\left(\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{X}\right)} . \tag{194}
\end{equation*}
$$

Table 24. The branching ratios (BR) and asymmetries $A$ for the CPviolating $\mathrm{B}_{\mathrm{c}}$-decays

| X | $\mathrm{BR}\left(\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \mathrm{X}\right)$ | $A$ |
| :--- | :--- | ---: |
| $\eta_{\mathrm{c}} \mathrm{D}^{*+}$ | $1.0 \times 10^{-4}$ | $1.5 \times 10^{-2}$ |
| $\eta_{\mathrm{c}} \mathrm{D}^{+}$ | $1.2 \times 10^{-4}$ | $-0.3 \times 10^{-2}$ |
| $\mathrm{~J} / \psi \mathrm{D}^{+}$ | $0.5 \times 10^{-4}$ | $0.6 \times 10^{-2}$ |
| $\mathrm{D}^{0} \rho^{+}$ | $2.8 \times 10^{-5}$ | $1.9 \times 10^{-3}$ |
| $\mathrm{D}^{+} \rho^{0}$ | $1.6 \times 10^{-5}$ | $3.0 \times 10^{-3}$ |
| $\mathrm{D}^{* 0} \pi^{+}$ | $3.3 \times 10^{-5}$ | $1.3 \times 10^{-3}$ |
| $\mathrm{D}^{*+} \pi^{0}$ | $1.8 \times 10^{-5}$ | $2.0 \times 10^{-3}$ |
| $\mathrm{D}^{0} \pi^{+}$ | $1.6 \times 10^{-6}$ | $-8.9 \times 10^{-3}$ |
| $\mathrm{D}^{+} \pi^{0}$ | $0.4 \times 10^{-6}$ | $-13.8 \times 10^{-3}$ |

A large value of the asymmetry is expected in the $B_{c} \rightarrow D_{s}^{*} D^{0}$ decays with the $D^{0}$-meson, decaying into the CP -invariance eigenstate. However, the branching ratio of such an event is too low:

$$
\mathrm{BR}\left(\mathrm{~B}_{\mathrm{c}}^{+} \rightarrow \mathrm{D}_{\mathrm{s}}^{*+} \mathrm{D}^{0}\right) \approx 10^{-6}
$$

The identification of the $D_{s}^{*+}$-meson is also complicated. As one can see from Table 24, the best mode for the observation of CP -violation in the $\mathrm{B}_{\mathrm{c}}$-meson decay is $B_{c}^{ \pm} \rightarrow(c \bar{c}) D^{ \pm}$. However, even at the expected statistics of the $B_{c}$-meson yield in future colliders (about $10^{9}-10^{11}$ events), it is difficult to observe such events because of the branching fractions of the ( $\mathrm{c} \overline{\mathrm{c}}$ )-states and D-meson decays.

It is difficult to estimate the decay widths, but it is worth mentioning $\mathrm{B}_{\mathrm{c}}$-meson decay modes such as $\mathrm{B}_{\mathrm{c}} \rightarrow 3 \mathrm{DX}$ or $B_{c} \rightarrow D_{s} \phi$ and $B_{c} \rightarrow \bar{D} K$. The $B_{c} \rightarrow \psi(3 S) D$ decay can be of a great interest when $\psi(3 S)$ decays into a D-meson pair. However, it is probable that this decay width, like the width of the decay into three $D$-mesons, is small because of the smallness of the phase space. The $\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{D}_{\mathrm{s}} \phi$ decay width can be roughly estimated to be of the order of $2 \%$ [52], but it will be very difficult to observe the $\mathrm{B}_{\mathrm{c}}$-meson in such mode because of the complex reconstruction of the $\mathrm{D}_{\mathrm{s}}$-meson.

## 4. Production of $B_{c}$-mesons

The electromagnetic and hadronic production of $\mathrm{B}_{\mathrm{c}}$ as the particle with mixed flavour requires the joint production of the heavy quarks $\bar{b}$ and $c$. This explains the low value of the $B_{c}$ production cross section in comparison with the production cross sections of particles from the $\psi$ and $\Upsilon$ families. On the other hand, the absence of $\mathrm{B}_{\mathrm{c}}$ decay channels into light hadrons because of the strong interactions implies that all bound ( $\overline{\mathrm{b}}$ ) states are basically transformed into the lowest state with a probability close to unity because of radiative transitions (see Section 2).

From the theoretical point of view, the production of small $\mathrm{B}_{\mathrm{c}}$-mesons takes place with virtualities of the order of the sum of the heavy quark mass. This fact assures the applicability of perturbation theory to the processes of $B_{c}$ production. The nonperturbative part, associated with the $B_{c}$ wave function, is quite reliably calculated in this case.

## 4.1 $\mathbf{B}_{\mathbf{c}}$-meson production in $\mathrm{e}^{+} \mathrm{e}^{-}$-annihilation

The simplest example of $B_{c}$ production in $\mathrm{e}^{+} \mathrm{e}^{-}$-annihilation (in the region of a Z peak) is described by the diagrams in Fig. 7.


Figure 7. Diagrams of single $\mathrm{B}_{\mathrm{c}}$-meson production in $\mathrm{e}^{+} \mathrm{e}^{-}$-annihilation.

The matrix element of the ( $\overline{\mathrm{b}} \mathrm{c}$ ) quarkonium production is transformed from the corresponding matrix element $T\left(p_{\overline{\mathrm{b}}}, p_{\mathrm{c}}\right)$ for four-heavy-quark production by integration over the relative momentum of the $\bar{b}$ - and c-quarks, weighted by the quarkonium wave function:

$$
\begin{align*}
T_{j}=\int & \frac{\mathrm{d}^{3} \boldsymbol{q}}{(2 \pi)^{3}} \Psi(\boldsymbol{q}) T_{\alpha \beta}^{a b}\left(p_{\overline{\mathrm{b}}}, p_{\mathrm{c}}\right)\left(-\hat{p}_{\overline{\mathrm{b}}}+m_{\mathrm{b}}\right)^{\alpha^{\prime} \alpha} \\
& \times\left(-\hat{p}_{\mathrm{c}}+m_{\mathrm{c}}\right)^{\beta^{\prime} \beta} \Gamma_{j}^{\alpha^{\prime} \beta^{\prime}} \frac{\sqrt{2 M}}{\sqrt{2 m_{\mathrm{b}} \times 2 m_{\mathrm{c}}}} \frac{\delta^{a b}}{\sqrt{3}}, \tag{195}
\end{align*}
$$

where $M$ is the meson mass and

$$
\begin{equation*}
\Gamma^{\alpha \beta}=\frac{1}{\sqrt{2}} \gamma_{5}^{\alpha \beta}, \quad \Gamma_{\lambda}^{\alpha \beta}=\frac{1}{\sqrt{2}} \gamma_{\mu}^{\alpha \beta} \varepsilon_{\lambda}^{\mu} \tag{196}
\end{equation*}
$$



Figure 8. The functions of $\bar{b}$-quark fragmentation into $\mathrm{B}_{\mathrm{c}}-$ and $\mathrm{B}_{\mathrm{c}}^{*}$ mesons.
for the pseudoscalar state and the vector state $\left(B_{c}, B_{c}^{*}\right)$, respectively. The quark momenta are determined by the relations

$$
\begin{align*}
& p_{\overline{\mathrm{b}}}=\frac{m_{\mathrm{b}}}{M} p+q, \quad p_{\mathrm{c}}=\frac{m_{\mathrm{c}}}{M} p-q,  \tag{197}\\
& p q=0 . \tag{198}
\end{align*}
$$

For the heavy quarkonium one has $|\boldsymbol{q}| \ll m_{\mathrm{b}}, m_{\mathrm{c}}$, and Eqn (195) can be simplified by the substitution of $T_{\alpha \beta}^{a b}\left(p_{\bar{b}}, p_{\mathrm{c}}\right)$ by its value at $\boldsymbol{q}=\mathbf{0}$. Then

$$
\begin{equation*}
\int \frac{\mathrm{d}^{3} \boldsymbol{q}}{(2 \pi)^{3}} \Psi(\boldsymbol{q})=\Psi(x)_{x=0} \tag{199}
\end{equation*}
$$

In Refs [41-44] the total cross sections of the $\mathrm{B}_{\mathrm{c}}$ - and $\mathrm{B}_{\mathrm{c}}^{*}$-mesons and their distributions over the variable $z=2 E_{B_{\mathrm{c}}} / \sqrt{s}$ have been obtained. Fig. 8 shows the result of the precise numerical calculation with the technique of spiral amplitudes and Monte Carlo integration over the phase space. One can see that this distribution is rather sharp and is maximum at $z_{\max }=M /\left(M+m_{\mathrm{c}}\right) \approx 0.8$. At this value of $z_{\max }$ the $\mathrm{B}_{\mathrm{c}}$-meson and $\overline{\mathrm{c}}$-quark have zero relative velocity.

If one recalls that in our approximation the $\mathrm{c}-\mathrm{and} \overline{\mathrm{b}}-$ quarks inside the $\mathrm{B}_{\mathrm{c}}$-meson have no relative motion, then one clearly finds that the maximum in the distribution corresponds to the configuration in which all quarks move as a whole with one and the same velocity. In this case the minimum virtualities of the initial $\bar{b}$-quark $p^{2}=\left(m_{\mathrm{b}}+2 m_{\mathrm{c}}\right)^{2}$ and the gluon $k^{2} \approx 4 m_{\mathrm{c}}^{2}$ are achieved. At any other $z$ values, these virtualities increase.

Note that these speculations are correct only for the last two diagrams in Fig. 7, in which one can neglect the contribution of the first and second diagrams, suppressed up to two orders of magnitude with respect to the former. In the asymptotic limit $s \rightarrow \infty$, in which one can neglect terms of the order of $M^{2} / s$ and higher powers of this ratio, choosing the special gauge condition (the axial gauge with the four- vector $n=(1,0,0,-1)$ along the direction of motion of the b-quark), one can show that the contribution of only the last diagram in Fig. 7 survives. In this case the
expression for $\sigma^{-1} \mathrm{~d} \sigma / \mathrm{d} z$ acquires the sense of the fragmentation function of the $\overline{\mathrm{b}}$-quark into the $\mathrm{B}_{\mathrm{c}}$-meson, if one chooses the b-quark production cross section $\sigma$ as the normalisation factor at the same energy.

The function of the $\bar{b} \rightarrow B_{c}$ fragmentation, where $B_{c}$ is the pseudoscalar state, has the following form:

$$
\begin{align*}
D(z)_{\bar{b} \rightarrow \mathrm{~B}_{\mathrm{c}}}= & \frac{8 \alpha_{\mathrm{s}}^{2}|\Psi(0)|^{2}}{81 m_{\mathrm{c}}^{3}} \frac{r z(1-z)^{2}}{[1-(1-r) z]^{6}} \\
& \times\left[6-18(1-2 r) z+\left(21-74 r+68 r^{2}\right) z^{2}\right. \\
& -2(1-r)\left(6-19 r+18 r^{2}\right) z^{3} \\
& \left.+3(1-r)^{2}\left(1-2 r+2 r^{2}\right) z^{4}\right] \tag{200}
\end{align*}
$$

And for fragmentation into the vector state one has

$$
\begin{align*}
D(z)_{\bar{b} \rightarrow \mathrm{~B}_{\mathrm{c}}^{*}}= & \frac{8 \alpha_{\mathrm{s}}^{2}|\Psi(0)|^{2}}{27 m_{\mathrm{c}}^{3}} \frac{r z(1-z)^{2}}{[1-(1-r) z]^{6}} \\
& \times\left[2-2(3-2 r) z+3\left(3-2 r+4 r^{2}\right) z^{2}\right. \\
& -2(1-r)\left(4-r+2 r^{2}\right) z^{3} \\
& \left.+(1-r)^{2}\left(3-2 r+2 r^{2}\right) z^{4}\right] \tag{201}
\end{align*}
$$

where $r=m_{\mathrm{c}} /\left(m_{\mathrm{b}}+m_{\mathrm{c}}\right)$.
As one can see from Fig. 8, $D_{\bar{b} \rightarrow \mathrm{~B}_{\mathrm{c}}}(z)$ and $D_{\bar{b} \rightarrow \mathrm{~B}_{\mathrm{c}}^{*}}(z)$ are in a good agreement with the results of precise calculations. The $\bar{b} \rightarrow B_{c}^{*}$ process has a slightly sharper distribution in comparison with the $\bar{b} \rightarrow B_{c}$ one. At $\alpha_{s}=0.22$, $|\Psi(0)|^{2}=f_{\mathrm{B}_{\mathrm{c}}}^{2} M_{\mathrm{B}_{\mathrm{c}}} / 12, \quad f_{\mathrm{B}_{\mathrm{c}}}=560 \mathrm{MeV}$ and $m_{\mathrm{c}}=1.5 \mathrm{GeV}$, the corresponding integral probabilities are equal to $3.8 \times 10^{-4}$ for $\bar{b} \rightarrow B_{c}$ and $5.4 \times 10^{-4}$ for $\bar{b} \rightarrow B_{c}^{*}$. The probabilities of fragmentation of the c-quark into $B_{c}$ are suppressed by two orders of magnitude with respect to the values given above.

As a fraction of the $\mathrm{b} \overline{\mathrm{b}}$ production, the total number of produced $\mathrm{B}_{\mathrm{c}}\left(\overline{\mathrm{B}}_{\mathrm{c}}\right)$-mesons, with the $\mathrm{B}_{\mathrm{c}}^{*}\left(\overline{\mathrm{~B}}_{\mathrm{c}}^{*}\right)$ states and the first radial excitations taken into account, is (at $\alpha_{\mathrm{s}}=0.22$ )
$R_{\mathrm{B}_{\mathrm{c}}}=\frac{\sigma\left(\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow \mathrm{B}_{\mathrm{c}}^{+}+\mathrm{x}\right)+\sigma\left(\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow \mathrm{B}_{\mathrm{c}}^{-}+\mathrm{x}\right)}{\sigma\left(\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow \mathrm{b} \overline{\mathrm{b}}\right)}=2 \times 10^{-3}$.

Owing to the quark-hadron duality, there is an independent way of estimating this ratio. To reach this goal, one has to compare the obtained cross section for the production of the bound $\overline{\mathrm{b}} \mathrm{c}$ state with the cross section for the production of the colour-singlet ( $\overline{\mathrm{b}}$ c) pair in the process $\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow \mathrm{b} \overline{\mathrm{b}} c \overline{\mathrm{c}}$ with the low values of invariant mass $M_{\overline{\mathrm{b}}}$ :

$$
\begin{equation*}
\int_{m_{0}^{2}}^{M_{\mathrm{th}}^{2}} \frac{\mathrm{~d} \sigma\left(\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow \mathrm{b} \overline{\mathrm{~b}} \mathrm{c} \overline{)_{\overline{\mathrm{b}}-\text { singl }}}\right.}{\mathrm{d} M_{\overline{\mathrm{bc}}}^{2}} \mathrm{~d} M_{\overline{\mathrm{bc}}}^{2}, \tag{203}
\end{equation*}
$$

where $m_{0}=m_{\mathrm{b}}+m_{\mathrm{c}} \leqslant M_{\overline{\mathrm{bc}}} \leqslant M_{\mathrm{B}}+M_{\mathrm{D}}+\Delta M=M_{\mathrm{th}}$ and $\Delta M \approx 0.5-1 \mathrm{GeV}$. Supposing $m_{0}=6.1 \mathrm{GeV}$ and $M_{\mathrm{th}}=8$ GeV as the threshold value, one gets a $\overline{\mathrm{b}} \mathrm{c}$ system production cross section of the order of 7 pb . On the other hand, the sum of the cross sections for the production of $\mathrm{B}_{\mathrm{c}}$ and its first excitations equals 9.3 pb , as is seen from Table 25.

Comparison of these two independent estimates indicates, on the one hand, good agreement. On the other hand, it means that the contribution of the higher excitations is

Table 25. The cross sections (in pb ) for the production of the S-wave states of $\mathrm{B}_{\mathrm{c}}$-mesons in the Z-boson peak at $\alpha_{\mathrm{s}}=0.22$.

| State | $1^{1} \mathrm{~S}_{0}$ | $1^{1} \mathrm{~S}_{1}$ | $2^{1} \mathrm{~S}_{0}$ | $2^{1} \mathrm{~S}_{1}$ |
| :--- | :--- | :--- | :--- | :--- |
| $\sigma$ | 3.14 | 4.37 | 0.805 | 1.078 |

not large, and the total cross section is saturated by the Swave levels.

Recent direct calculations of the cross sections for P level production [91] confirm this conclusion. According to the estimates of this paper, the sum over the cross sections for the production of P -wave levels is less than $10 \%$ of the sum of the S-wave level contributions.

In Ref. [48] the functions of the fragmentation of the heavy quark into the heavy polarised vector quarkonium have been studied and for the longitudinally polarised quarkonium, one has found the expression

$$
\begin{align*}
D(z)_{\underset{\mathrm{b}}{\mathrm{~L}}}^{\mathrm{L}} \mathrm{~B}_{\mathrm{c}}^{*}= & \frac{8 \alpha_{\mathrm{s}}^{2}|\Psi(0)|^{2}}{81 m_{\mathrm{c}}^{3}} \frac{r z(1-z)^{2}}{[1-(1-r) z]^{6}} \\
& \times\left[2-2(3-2 r) z+\left(9-10 r+16 r^{2}\right) z^{2}\right. \\
& -2(1-r)\left(4-5 r+6 r^{2}\right) z^{3} \\
& \left.+(1-r)^{2}\left(3-6 r+6 r^{2}\right) z^{4}\right] \tag{204}
\end{align*}
$$

which does not depend on the polarisation of the fragmenting quark. At $r=\frac{1}{2}$, expression (204) coincides with the result obtained for the heavy quarkonium with the hidden flavour ( $\Upsilon, \psi$ ) [48].

Fragmentation function (204) agrees with the analysis of the fragmentation of the heavy quark into the heavy meson ( $\mathrm{Q} \overline{\mathrm{q}}$ ), where, in the limit of an infinitely heavy quark, EHQT leads to equal probability of production of the vector quarkonium with an arbitrarily orientated spin, i.e. to the absence of spin alignment and to a ratio of the vector and pseudoscalar state yields equal to $V / P=3$ [94].

For the heavy quarkonium, the relative yield of the vector and pseudoscalar mesons is close to unity, and the spin alignment of the vector state has a significant value. For the $\mathrm{B}_{\mathrm{c}}^{*}$-meson, this can be observed in the angular distribution of the $B_{c}^{*} \rightarrow B_{c} \gamma$ decay, which composes the total $\mathrm{B}_{\mathrm{c}}^{*}$ width. This distribution has the form

$$
\begin{equation*}
\frac{\mathrm{d} \Gamma}{\mathrm{~d} \cos \theta} \sim 1-\frac{3 \xi-2}{2-\xi} \cos ^{2} \theta \tag{205}
\end{equation*}
$$

where $\theta$ is the angle between the photon and the $\mathrm{B}_{\mathrm{c}}^{*}$ polarisation axis in the system in which $\mathrm{B}_{\mathrm{c}}^{*}$ is at rest, and the asymmetry parameter $\xi$ determines the relative yield of the transversely polarised $\mathrm{B}_{\mathrm{c}}^{*}$ state

$$
\begin{equation*}
\xi=\frac{T}{L+T} \tag{206}
\end{equation*}
$$

For the integral asymmetry at the small mass of the generated quark entering the meson, $r \ll 1$, one has

$$
\begin{equation*}
\xi=\frac{2}{3}+\frac{5}{16} r+\mathrm{O}\left(r^{2}\right) \tag{207}
\end{equation*}
$$

The anisotropy in the $\mathrm{B}_{\mathrm{c}}^{*} \rightarrow \mathrm{~B}_{\mathrm{c}} \gamma$ decay is numerically equal to $6 \%$.

In Ref. [48] the vector quarkonium spin alignment was studied as a function of the transverse momentum with respect to the fragmentation axis. Quite bulky analytic
expressions have been derived for the fragmentation functions $D_{\overline{\mathrm{b}} \rightarrow \mathrm{B}_{\mathrm{c}}^{\mathrm{s}}}^{\mathrm{L}, \mathrm{T}}\left(p_{\mathrm{t}}\right)$, that linearly tend to zero at $p_{\mathrm{t}} \rightarrow 0$ and decrease as $1 / p_{\mathrm{t}}^{3}$ at $p_{\mathrm{t}} \rightarrow \infty$. It is interesting that the average transverse momentum in the fragmentation into the longitudinally polarised vector $\mathrm{B}_{\mathrm{c}}$-quarkonium is twice as large as the average transverse momentum in the fragmentation into the transversely polarised $\mathrm{B}_{\mathrm{c}}^{*}$-meson, $\left\langle p_{\mathrm{t}}\right\rangle \approx 7 \mathrm{GeV}$.

The event with a $\mathrm{B}_{\mathrm{c}}$-meson has a characteristic signature. The hadron jet from the b-quark must be produced in the direction opposite to the $\mathrm{B}_{\mathrm{c}}$ motion. The $\mathrm{B}_{\mathrm{c}}$-meson must be accompanied by a $\overline{\mathrm{D}}$-meson with an average ratio of the momenta $\left\langle z_{D}\right\rangle /\left\langle z_{\mathrm{B}_{\mathrm{c}}}\right\rangle \approx 0.3$ and an average angle between the momenta of about $20^{\circ}$ [44].

The production of a single $\mathrm{B}_{\mathrm{c}}$-meson in $\mathrm{e}^{+} \mathrm{e}^{-}$-annihilation has also been considered in Refs [41, 52].

In Ref. [39] the exclusive production of $\mathrm{B}_{\mathrm{c}}^{(\times)+} \mathrm{B}_{\mathrm{c}}^{(*)-}$ pairs in $\mathrm{e}^{+} \mathrm{e}^{-}$-annihilation has been calculated at low energies, where one can neglect the Z-boson contribution. The total cross sections of the vector and pseudoscalar states have the form

$$
\begin{align*}
& \sigma\left[\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow\left(\mathrm{Q}_{1} \overline{\mathrm{Q}}_{2}\right)_{\mathrm{P}}\left(\overline{\mathrm{Q}}_{1} \mathrm{Q}_{2}\right)_{\mathrm{P}}\right] \\
& =\frac{\pi^{3} \alpha_{\mathrm{s}}^{2}\left(4 m_{2}^{2}\right) \alpha_{\mathrm{em}}^{2}}{3^{7} \times 4 m_{2}^{6}} f_{\mathrm{P}}^{4}\left(1-v^{2}\right)^{3} v^{3} \\
& \times \frac{m_{1}^{2}}{M^{2}}\left\{3 e_{1}\left[2 \frac{m_{2}}{m_{1}}-\left(1-v^{2}\right)\right]\right. \\
& \left.-3 e_{2}\left[2-\left(1-v^{2}\right) \frac{m_{2}}{m_{1}}\right] \frac{m_{2}^{3} \alpha_{\mathrm{s}}\left(4 m_{1}^{2}\right)}{m_{1}^{3} \alpha_{\mathrm{s}}\left(4 m_{2}^{2}\right)}\right\}^{2},  \tag{208}\\
& \sigma\left[\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow\left(\mathrm{Q}_{1} \overline{\mathrm{Q}}_{2}\right)_{\mathrm{P}}\left(\overline{\mathrm{Q}}_{1} \mathrm{Q}_{2}\right)_{\mathrm{V}}\right] \\
& =\frac{\pi^{3} \alpha_{\mathrm{s}}^{2}\left(4 m_{2}^{2}\right) \alpha_{\mathrm{em}}^{2}}{3^{7} \times 2 m_{2}^{6}} f_{\mathrm{P}}^{2} f_{\mathrm{V}}^{2}\left(1-v^{2}\right)^{4} v^{3} \\
& \times\left[3 e_{1}-3 e_{2} \frac{m_{2}^{3} \alpha_{s}\left(4 m_{1}^{2}\right)}{m_{1}^{3} \alpha_{\mathrm{s}}\left(4 m_{2}^{2}\right)}\right]^{2},  \tag{209}\\
& \sigma\left[\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow\left(\mathrm{Q}_{1} \overline{\mathrm{Q}}_{2}\right)_{\mathrm{V}}\left(\overline{\mathrm{Q}}_{1} \mathrm{Q}_{2}\right)_{\mathrm{V}}\right] \\
& =\frac{\pi^{3} \alpha_{\mathrm{s}}^{2}\left(4 m_{2}^{2}\right) \alpha_{\mathrm{em}}^{2}}{3^{7} \times 2 m_{2}^{6}} f_{\mathrm{V}}^{4}\left(1-v^{2}\right)^{3} v^{3} \\
& \times\left[3 e_{1}-3 e_{2} \frac{m_{2}^{3} \alpha_{\mathrm{s}}\left(4 m_{1}^{2}\right)}{m_{1}^{3} \alpha_{\mathrm{s}}\left(4 m_{2}^{2}\right)}\right]^{2}\left[3\left(1-v^{2}\right)\right. \\
& \left.+\left(1+v^{2}\right)(1-a)^{2}+\frac{a^{2}}{2}\left(1-v^{2}\right)\left(1-3 v^{2}\right)\right], \tag{210}
\end{align*}
$$

where $v=\sqrt{1-4 M^{2} / s}, M=m_{1}+m_{2}$, and

$$
\begin{equation*}
a=\frac{m_{1}}{M}\left[1-\frac{e_{2} m_{2}^{4} \alpha_{\mathrm{s}}\left(4 m_{1}^{2}\right)}{e_{1} m_{1}^{4} \alpha_{\mathrm{s}}\left(4 m_{2}^{2}\right)}\right]\left[1-\frac{e_{2} m_{2}^{3} \alpha_{\mathrm{s}}\left(4 m_{1}^{2}\right)}{e_{1} m_{1}^{3} \alpha_{\mathrm{s}}\left(4 m_{2}^{2}\right)}\right]^{-1} \tag{211}
\end{equation*}
$$

The relative yield of the $\mathrm{B}_{\mathrm{c}}$-meson pairs $R=\sigma\left(\mathrm{B}_{\mathrm{c}}^{+} \mathrm{B}_{\mathrm{c}}^{-}\right) / \sigma(\mathrm{b} \overline{\mathrm{b}})$ reaches its maximum at the energy $\sqrt{s}=14 \mathrm{GeV}$, where $R \approx 10^{-4}$. This ratio rapidly decreases with increasing energy, where single- $\mathrm{B}_{\mathrm{c}}$ production becomes dominant.

As one can see, the study of $\mathrm{B}_{\mathrm{c}}$-meson production in $\mathrm{e}^{+} \mathrm{e}^{-}$-annihilation allows one to make analytical studies of heavy quark dynamics.

Thus, in the $\mathrm{Z}^{0}$-boson pole, where the b-quark production cross section is large, one has to expect of the order of 2 events with $B_{c}$ production per thousand $b \bar{b}$ pairs. It is expected that, in the experiments at the LEP accelerator, about $2 \times 10^{7} \mathrm{Z}^{0}$-bosons will be detected. This means that the total number of $\mathrm{B}_{\mathrm{c}}\left(\overline{\mathrm{B}}_{\mathrm{c}}\right)$ events has to be of the order of $10^{4}$. However, the real number of reconstructed events will be less, if one takes into account the particular modes of the decay.

### 4.2 Hadronic production of $B_{c}$-mesons

As has been mentioned, the process of $\mathrm{B}_{\mathrm{c}}$-meson production in $\mathrm{e}^{+} \mathrm{e}^{-}$-annihilation at high energies can be reformulated as the process of the $\bar{b} \rightarrow B_{c}\left(B_{c}^{*}\right)$ fragmentation, appearing with a probability about $10^{-3}$.

The hadronic $B_{c}$ production turns out to be more complex. First, in hadronic production the region of low partonic energies dominates, so that the asymptotic regime with the cross section factorisation

$$
\begin{equation*}
\frac{\mathrm{d} \sigma}{\mathrm{~d} z} \sim \sigma_{\mathrm{b} \overline{\mathrm{~b}}} D_{\overline{\mathrm{b}} \rightarrow \mathrm{~B}_{\mathrm{c}}}(z) \tag{212}
\end{equation*}
$$

is not yet achieved. Second, in the hadron interactions a new type of diagram appears which we shall label a recombinational diagram, for which the factorisation does not take place.

The contribution of such diagrams, dominating at low masses of the $B_{c}^{-} \bar{b} c$ system, decreases with the growth in this mass; however, it remains significant even at large masses and large transverse momenta. The contribution of these type of diagrams to the $\mathrm{B}_{\mathrm{c}}^{(*)}$ production was first calculated


Figure 9. Diagrams of single $\mathrm{B}_{\mathrm{c}}$-meson production in gluon and quark subprocesses.
for the exclusive $\mathrm{B}_{\mathrm{c}}^{(*)}$ pair production in the quark antiquark annihilation at low energies [38].

A typical set of QCD diagrams of the fourth order in $\alpha_{s}$ is shown on Fig. 9. Here, as in the case of the $B_{c}$ production in $\mathrm{e}^{+} \mathrm{e}^{-}$-annihilation, the matrix element of the ( $\overline{\mathrm{b}} \mathrm{c}$ )quarkonium production is obtained from the corresponding matrix element of four-quark production by integration over the relative momentum of the $c-$ and $\bar{b}$-quarks, weighted by the quarkonium wave function.

At high energies, where the $B_{c}$ production cross sections permit meson observation, the gluon-gluon contribution to the production dominates.

The energetic spectra of the $\mathrm{B}_{\mathrm{c}^{-}}$and $\mathrm{B}_{\mathrm{c}}^{*}$-mesons in the centre-of-mass system (c.m.s.) for two colliding gluons are shown in Fig. 10 at different values of the total energy $\sqrt{s}=20,40$, and 100 GeV .

The $\sigma\left(\mathrm{gg} \rightarrow \mathrm{B}_{\mathrm{c}}\left(\mathrm{B}_{\mathrm{c}}^{*}\right) \overline{\mathrm{c}} \mathrm{b}\right)$ values are presented in Fig. 11 at several energies of the interacting gluons for $m_{\mathrm{b}}=5.1 \mathrm{GeV}$, $m_{\mathrm{c}}=1.5 \mathrm{GeV}$, and $\alpha_{\mathrm{s}}=0.2$. The ratio of the cross sections $\sigma_{\mathrm{B}_{\mathrm{c}}^{*}} / \sigma_{\mathrm{B}_{\mathrm{c}}}$ is about 3 at the energies 20,40 , and 100 GeV , and is about 2 at 1 TeV . In $\mathrm{e}^{+} \mathrm{e}^{-}$-annihilation, where the $\overline{\mathrm{b}} \rightarrow \mathrm{B}_{\mathrm{c}}$ fragmentation dominates, this ratio is $\sigma_{\mathrm{B}_{\mathrm{c}}^{*}} / \sigma_{\mathrm{B}_{\mathrm{c}}} \approx 1.3$.


Figure 10. The differential $\mathrm{d} \sigma / \mathrm{d} z$ cross sections for the single production of the $\mathrm{B}_{\mathrm{c}}$-mesons (a) and $\mathrm{B}_{\mathrm{c}}^{*}$-mesons (b) in gluon annihilation at different values of the total energy $\sqrt{s}$.


Figure 11. The total cross sections for the single production of $\mathrm{B}_{\mathrm{c}}$ mesons (empty triangles) and $\mathrm{B}_{\mathrm{c}}^{*}$-mesons (solid triangles) in the gluon annihilation in comparison with the production cross section (multiplied by the factor $2 \times 10^{-3}$ ) of the $\overline{\mathrm{b}}$-quark pairs (solid line).

The variation of the $\sigma_{\mathrm{B}_{\mathrm{c}}^{*}} / \sigma_{\mathrm{B}_{\mathrm{c}}}$ ratio is the consequence of the change in the production mechanism. The fragmentational component gives a low contribution in comparison with the contribution of the recombination diagrams. This can be noted from Fig. 10, where the differential cross sections for the $\mathrm{B}_{\mathrm{c}}-$ and $\mathrm{B}_{\mathrm{c}}^{*}$-meson production, calculated by Monte Carlo integration of the exact expression for the matrix element squared, are presented in comparison with the cross section, calculated by the fragmentation formulae (200) and (201).

The total cross section of the $\mathrm{B}_{\mathrm{c}}\left(\mathrm{B}_{\mathrm{c}}^{*}\right)$-mesons is obtained from the partonic one $\sigma_{i j}(\hat{s})$ by convolution with the functions of the parton distributions in the initial hadrons:

$$
\begin{align*}
\sigma_{\mathrm{tot}}(s) & =\int_{4\left(m_{\mathrm{b}}+m_{\mathrm{c}}\right)^{2}}^{s} \frac{\mathrm{~d} \hat{s}}{s} \int_{-1+\hat{s} / s}^{1-\hat{s} / s} \frac{\mathrm{~d} x}{x^{*}} \sum_{i j} f_{a}^{i}\left(x_{1}\right) f_{b}^{j}\left(x_{2}\right) \hat{\sigma}_{i j}(\hat{s}) \\
x^{*} & =\left(x^{2}+\frac{4 \hat{s}}{s}\right)^{1 / 2} \tag{213}
\end{align*}
$$

The cross sections, calculated with account taken of the known parameterisations for $f_{a, b}^{i, j}(x)$ [92], are presented in Table 26.

The energy 40 GeV is close to the c.m.s. energy for carrying out fixed-target experiments at the HERA accelerator. At $\sqrt{s}=1.8 \mathrm{TeV}$ we present the cross section of the $\mathrm{B}_{\mathrm{c}}$ production in $\mathrm{p} \overline{\mathrm{p}}$-collisions at Tevatron, and, finally, the energy $\sqrt{s}=16 \mathrm{TeV}$ corresponds to the conditions of the pp -experiment at the future LHC collider. The energetic

Table 26. The cross sections (in nb) of hadronic production of the $\mathrm{B}_{\mathrm{c}}\left(\mathrm{B}_{\mathrm{c}}^{*}\right)$-mesons (the standard deviation in the last digit is shown in brackets).

| $n^{2 S+1} L_{j}$ | $1^{1} \mathrm{~S}_{0}$ | $1^{3} \mathrm{~S}_{1}$ | $2^{1} \mathrm{~S}_{0}$ | $2^{3} \mathrm{~S}_{1}$ |
| :--- | :---: | :---: | :--- | :---: |
| $\sigma_{\text {tot }} 10^{5}(40 \mathrm{GeV})$ | $1.63(2)$ | $9.5(2)$ | $0.13(1)$ | $0.75(2)$ |
| $\sigma_{\text {tot }} 10^{3}(100 \mathrm{GeV})$ | $7.8(2)$ | $36(1)$ | $1.1(2)$ | $5.2(2)$ |
| $\sigma_{\text {tot }}(1.8 \mathrm{TeV})$ | $13.3(8)$ | $53(3)$ | $2.7(2)$ | $10.4(5)$ |
| $\sigma_{\text {tot }} 10^{-2}(16 \mathrm{TeV})$ | $1.96(8)$ | $7.6(2)$ | $0.43(2)$ | $1.66(8)$ |



Figure 12. The total cross sections (in nb) for the single production of $B_{c}$-mesons (empty circles) in $p \bar{p}$-interactions at different energies and the cross sections (in $\mu \mathrm{b}$ ) for beauty particle production (solid circles).
dependence of the cross section, summed over the particle and antiparticle $\overline{\mathrm{B}}_{\mathrm{c}}$ production, is shown in Fig. 12.

From the values presented in Table 26, it follows that at $\sqrt{s}=40 \mathrm{GeV}$ the summed cross section $\sigma_{\text {sum }}$ for the meson production is about $10^{-4}$ of the total cross section $\sigma_{\text {tot }}$ of the $b \bar{b}$ production, so this makes the study of $B_{c}$ practically impossible in this experiment. One should note that in this case we cannot restrict ourselves by the $\mathrm{gg} \rightarrow \mathrm{B}_{\mathrm{c}} \mathrm{c} \mathrm{b}$ contribution and we have taken into account the contribution of the $q \bar{q} \rightarrow B_{c} \bar{c} b$ process.

The experiments at Tevatron and LHC, where $\sigma_{\mathrm{sum}} / \sigma_{\mathrm{b} \overline{\mathrm{b}}}$ is about $10^{-2}$, will provide a real possibility for observing hadronic $\mathrm{B}_{\mathrm{c}}$ production. Therefore, at the energies of these two facilities, we present the most interesting distributions of the cross sections for the production of the $1^{1} \mathrm{~S}_{0^{-}}$and $1^{3} \mathrm{~S}_{1}$-states (note that, as our calculations show, the cross section at the energies under consideration is completely determined by the gluon-gluon interaction, since the quark-quark contribution is suppressed by two orders of magnitude, $10^{-2}$ ).

The distributions for the $1{ }^{1} \mathrm{~S}_{0}$ pseudoscalar and $1^{3} \mathrm{~S}_{1}$ vector mesons are shown on Figs 13 and 14 at the energy of the interacting hadrons, 1.8 TeV .

The distributions $\mathrm{d} \sigma / \mathrm{d} x$ (see Fig. 14b) show that we are dealing with the central $\mathrm{B}_{\mathrm{c}}$ production, where the complete cross section is collected in the interval from -0.3 to 0.3 . The average transverse momentum of $B_{c}$ is about 6 GeV , and from the distribution over the angle between the directions of the $\mathrm{B}_{\mathrm{c}}$ and $\overline{\mathrm{c}}$-quark motions, one can conclude that the $\overline{\mathrm{c}}-$ quark generally moves in a direction close to that of $\mathrm{B}_{\mathrm{c}}$ [46].

One should note that these diagrams of the QCD perturbation theory are of the fourth order in $\alpha_{s}$. This results in the strong dependence of the cross section on the particular $\alpha_{\mathrm{s}}$ choice. The latter must be determined by the typical virtuality in the production process. The analysis shows that this virtuality is large in the contributions, decreasing faster that $1 / \hat{s}$. In the remaining contributions, including the fragmentational one, it is not large and is about $4 m_{\mathrm{c}} m_{\mathrm{b}}$. For this reason the $\alpha_{\mathrm{s}}=0.2$ value, chosen as the strong coupling constant, is the most reasonable at this scale. The use of the running coupling constant $\alpha_{s}(\hat{s})$, for example, leads to a decrease of the


Figure 13. The differential $\mathrm{d} \sigma / \mathrm{d} p_{\mathrm{t}}$ cross sections for the single produc-tion of $\mathrm{B}_{\mathrm{c}^{-}}$and $\mathrm{B}_{\mathrm{c}}^{*}$-mesons in $\mathrm{p} \overline{\mathrm{p}}$ interactions at the energy 1.8 TeV .



Figure 14. The differential cross sections for the single production of $\mathrm{B}_{\mathrm{c}}{ }^{-}$and $\mathrm{B}_{\mathrm{c}}^{*}$-mesons in $p \overline{\mathrm{p}}$-interactions at the energy 1.8 TeV : (a) $\mathrm{d} \sigma / \mathrm{d} y$, where $y$ is the particle rapidity, (b) $\mathrm{d} \sigma / \mathrm{d} x$, where $x=2 E / \sqrt{s}$.
$\mathrm{B}_{\mathrm{c}}\left(\mathrm{B}_{\mathrm{c}}^{*}\right)$ production cross section by a factor of about 7 . Pessimistic estimates of $B_{c}\left(B_{c}^{*}\right)$ production are presented in Ref. [93], where, as one can see, the $\alpha_{\mathrm{s}}(\hat{s})$ value was used.

At low energies of hadron collisions, the quarkantiquark annihilation with $B_{c}$ production dominates with respect to the gluonic one, since, in this case, the latter has a much lower luminosity, which decreases also with the growth of the total energy of the partonic subprocess. At low energies of quark - antiquark annihilation, exclusive $\mathrm{B}_{\mathrm{c}}^{+} \mathrm{B}_{\mathrm{c}}^{-}$pair production can be significant.

The total cross sections for the production of vector and pseudoscalar $\mathrm{B}_{\mathrm{c}}$-mesons due to the quark-antiquark annihilation have the form

$$
\begin{gather*}
\sigma\left(1^{-}, 1^{-}\right)=\frac{\alpha_{\mathrm{s}}^{4} \pi^{3}}{8 \times 3^{8}} \frac{f_{\mathrm{V}}^{4}}{\mu^{6}} \lambda^{3} \sqrt{1-\lambda}\left(1.3+1.4 \lambda+0.3 \lambda^{2}\right),  \tag{214}\\
\sigma\left(1^{-}, 0^{-}\right)=\frac{\alpha_{\mathrm{s}}^{4} \pi^{3}}{16 \times 3^{8}} \frac{f_{\mathrm{V}}^{2} f_{\mathrm{P}}^{2}}{\mu^{6}} \frac{\left(m_{\mathrm{b}}-m_{\mathrm{c}}\right)^{2}}{M^{2}} \\
\quad \times \lambda^{3} \sqrt{1-\lambda}(1+2 \lambda),  \tag{215}\\
\sigma\left(0^{-}, 0^{-}\right)=\frac{\alpha_{\mathrm{s}}^{4} \pi^{3}}{16 \times 3^{8}} \frac{f_{\mathrm{P}}^{4}}{\mu^{6}} \lambda^{3} \sqrt{1-\lambda}(1-\lambda)^{2}, \tag{216}
\end{gather*}
$$

from which one can see that the vector state production dominates. In Eqns (214)-(216) we have introduced the notation

$$
\lambda=\frac{4 M^{2}}{s}, \quad \mu=\frac{m_{\mathrm{b}} m_{\mathrm{c}}}{m_{\mathrm{b}}+m_{\mathrm{c}}} .
$$

The numerical estimates of the total cross sections for the $B_{c}$ production in $p \bar{p}$ interactions are presented in Table 27.

Table 27. The total cross sections (in $10^{-14} \mathrm{~b}$ ) for the pair production of $\mathrm{B}_{\mathrm{c}}$-mesons due to the quark -antiquark annihilation in $\mathrm{p} \overline{\mathrm{p}}(\mathrm{pp})$ interactions at low energies.

| $\sqrt{s} / \mathrm{GeV}$ | $\sigma\left(1^{-}, 1^{-}\right)$ | $\sigma\left(1^{-}, 0^{-}\right)$ | $\sigma\left(0^{-}, 0^{-}\right)$ |
| :--- | :---: | :--- | :--- |
| 30 | $0.9(0.08)$ | $0.24(0.022)$ | $0.006(0.0004)$ |
| 40 | $5.8(0.94)$ | $1.6(0.25)$ | $0.054(0.007)$ |
| 50 | $15.8(3.5)$ | $4.3(0.95)$ | $0.18(0.034)$ |

Summing up the analysis of hadronic production, one can draw the following conclusions:
-The mechanism of hadronic $\mathrm{B}_{\mathrm{c}}\left(\mathrm{B}_{\mathrm{c}}^{*}\right)$ production strongly differs from the production in $\mathrm{e}^{+} \mathrm{e}^{-}$-annihilation;
-the relative fraction of the fragmentation contribution is low even in the region of large transverse momenta; - the vector state production is enforced with respect to $\mathrm{e}^{+} \mathrm{e}^{-}$-annihilation.

Thus, the hadronic $B_{c}$ production requires an analysis of the large number of diagrams and its detailed study raises the possibility of investigating the effects of the heavy quark dynamics in the higher orders of QCD perturbation theory. The $B_{c}$ yield at the real physical facilities is quite high, but the registration of the $B_{c}$ events is essentially determined by the detector acceptance (cut-off over the transverse momenta of particles, characteristics of the vertex detector, and so on).

## 4.3 $\mathrm{B}_{\mathbf{c}}$-meson production in $\mathbf{v N}$-, ep- and $\gamma \boldsymbol{\gamma}$-collisions

In the previous sections we considered $\mathrm{B}_{\mathrm{c}}$-meson production in processes where one has the maximal current statistics for the production of hadrons with heavy quarks,
i.e. at the Fermilab and the LEP colliders. In the present section we consider estimates for $\mathrm{B}_{\mathrm{c}}$-meson production in processes of deep inelastic scattering of neutrinos and electrons by nucleons and in $\gamma \gamma$-interactions at future facilities.
4.3.1 $\mathbf{B}_{\mathbf{c}}$-meson production in $\mathbf{v N}$-interactions. The diagrams of the neutrino production of $\mathrm{B}_{\mathrm{c}}$-mesons on quarks and gluons are show in Fig. 15. Note that for $B_{c}$ production in the neutrino collisions with gluons, the suppression of the partonic subprocess cross section by the factor $\left|V_{\mathrm{bc}}\right|^{2}$ in comparison with the partonic subprocess of $\mathrm{B}_{\mathrm{c}}$ neutrino production on light quarks is compensated by the higher luminosity of the gluonic subprocess in comparison with the quark one. Thus, both mechanisms of $B_{c}$ production in neutrino-nucleon scattering give comparable contributions, and $\sigma\left(v \mathrm{~N} \rightarrow \mathrm{~B}_{\mathrm{c}} \mathrm{X}\right) \approx 10^{-43} \mathrm{~cm}^{2}$ at the neutrino energy $E_{v} \approx 500-1000 \mathrm{GeV}$ in the laboratory system.


Figure 15. Diagrams of $\mathrm{B}_{\mathrm{c}}$-meson production in processes of the neutrino scattering on gluons (a) and quarks (b).

After integration over the valent parton d distribution, c-quark production in the $\mathrm{W}^{*+} \mathrm{d} \rightarrow \mathrm{c}$ process, suppressed as $\sin ^{2} \theta_{\mathrm{c}}$, has a value comparable with c -quark production in the $\mathrm{W}^{*+} \mathrm{s} \rightarrow \mathrm{c}$ process, since the strange quark 'sea' is suppressed with respect to both the valent quark distribution and the 'sea' of the lighter d-quark.

The estimates of the $\mathrm{B}_{\mathrm{c}}$-meson production cross sections, calculated on the basis of the diagrams in Fig. 15, agree with the estimates obtained in the model of vector meson dominance (Fig. 16) and in the model of the soft gluon emission of the ( $\overline{\mathrm{b}} \mathrm{c}$ ) pair that, in the colour-singlet state and with the low invariant mass $M(\overline{\mathrm{~b}})<M_{\mathrm{B}}+M_{\mathrm{D}}$, transforms, in accordance with the quark - hadron duality, into the ( $\overline{\mathrm{b}} \mathrm{c}$ ) bound state, which radiatively decays into the basic $1 \mathrm{~S}_{0}$-state, in a cascade, with a probability of 1 .

As a result, one can reliably state that the total cross section for $\mathrm{B}_{\mathrm{c}}$-meson production in vN -collisions is of the order of $10^{-6}$ from the total cross section of the $v \mathrm{~N}$ scattering, so that, at a characteristic statistic of about $10^{6}$ events in neutrino experiments, one can expect only a few events with $\mathrm{B}_{\mathrm{c}}$-meson production.


Figure 16. Diagrams of $\mathrm{B}_{\mathrm{c}}^{*}$-meson production in the model of vector meson dominance.
4.3.2 Production of $\mathbf{B}_{\mathbf{c}}$-mesons in ep-scattering. In contrast to $v \mathrm{~N}$-scattering, in ep-collisions in addition to processes of weak charged current exchange, the main contribution to the $\mathrm{B}_{\mathrm{c}}$-meson production will give processes with virtual $\gamma-$ quanta exchange (Fig. 17).


Figure 17. Diagrams of $\mathrm{B}_{\mathrm{c}}$-meson production in parton process of $\gamma^{*} \mathrm{~g}^{-}$ scattering.

The exact calculation of the diagrams in Fig. 17 has not yet been performed. However, one can imagine that the estimate of the Monte Carlo simulation system for hadron production HERWIG [49] is quite reliable, since the HERWIG parameters have been chosen to give correct values for total hadronic cross sections of the production of charmed and beauty particles, being in agreement with the experimental values. Moreover, the HERWIG estimates of the $B_{c}$ production cross sections in $\mathrm{e}^{+} \mathrm{e}^{-}$and hadronic interactions agree with the values obtained in the exact calculation of the diagrams in the QCD perturbation theory.

Thus, in accordance with the estimates in the HER WIG system, one can expect about $10^{3}$ events per year with the $B_{c}$ production at the HERA facilities. This $B_{c}$ yield is comparably close to that of LEP. However, the extraction of $B_{c}$ events at HERA is complicated by the presence of a hadronic background, which is significantly lower at LEP.
4.3.3 Photonic production of $\mathbf{B}_{\mathbf{c}}$-mesons. Future $\gamma \boldsymbol{\gamma}$-colliders with the high luminosity ( $\sim 10^{34} \mathrm{~cm}^{-2} \mathrm{~s}^{-1}$ ) have been intensively discussed. In this section we calculate the cross section of single $\mathrm{B}_{\mathrm{c}}$ production at energies $\sqrt{s}$ about 30 GeV in accordance with the diagrams shown in Fig. 18. The calculation technique coincides with that described in the section on the hadronic production of $\mathrm{B}_{\mathrm{c}}$.


Figure 18. The types of diagrams in the photonic production of $\mathrm{B}_{\mathrm{c}^{-}}$ mesons.

The total cross sections of $B_{c}$ and $B_{c}^{*}$ production are presented in Table 28, in which $\alpha_{\mathrm{s}} \approx 0.2$. One can see that near the threshold the pseudoscalar state production is suppressed in comparison with the production of the vector one, so at $\sqrt{s}=15 \mathrm{GeV}$ one has $\sigma_{\mathrm{B}_{\mathrm{c}}^{*}} / \sigma_{\mathrm{B}_{\mathrm{c}}} \approx 55$. Such behaviour of the $\sigma_{\mathrm{B}_{\mathrm{c}}^{*}} / \sigma_{\mathrm{B}_{\mathrm{c}}}$ ratio has been noted in Ref. [6], where the strong suppression of pseudoscalar meson pair production with respect to the vector one occurs in quark - antiquark annihilation. At high energies of the initial photons this ratio decreases and becomes $\sigma_{\mathrm{B}_{\mathrm{c}}^{*}} / \sigma_{\mathrm{B}_{\mathrm{c}}} \approx 4$. The inclusive cross sections $\sigma_{\mathrm{B}_{\mathrm{c}}}$ and $\sigma_{\mathrm{B}_{\mathrm{c}}^{*}}$ have

Table 28. The cross section (in pb ) of the photonic production of $B_{c}\left(B_{c}^{*}\right)$.

| $\sqrt{s} / \mathrm{GeV}$ | 15 | 20 | 40 | 100 |
| :--- | :--- | :--- | :--- | :--- |
|  | $5.1 \times 10^{-3}$ | $3.8 \times 10^{-2}$ | $6.7 \times 10^{-2}$ | $2.5 \times 10^{-2}$ |
| $\sigma_{\mathrm{B}_{\mathrm{c}}}$ | $2.8 \times 10^{-1}$ | $6.0 \times 10^{-1}$ | $4.0 \times 10^{-1}$ | $1.1 \times 10^{-1}$ |
| $\sigma_{\mathrm{B}_{\mathrm{c}}^{*}}$ |  |  |  |  |

their maximum at $\sqrt{s}=20-30 \mathrm{GeV}$. As $s$ increases they fall like the total cross section for heavy quark production $\sigma_{\mathrm{b} \overline{\mathrm{b}}}$.

The distributions $\sigma^{-1} \mathrm{~d} \sigma / \mathrm{d} z$ over the variable $z=2|\boldsymbol{p}| / \sqrt{s}$, where $\boldsymbol{p}$ is the meson momentum, are shown on Figs 19 and 20 for the $\mathrm{B}_{\mathrm{c}^{-}}$and $\mathrm{B}_{\mathrm{c}}^{*}$-mesons. It follows from these figures that the scaling in these distributions is broken: as energy increases a shift to low $z$ values takes place. Note that an analogous picture has been observed in the gluonic production of $\mathrm{B}_{\mathrm{c}}$-mesons.

Note that detailed consideration shows that in the matrix element of the $\gamma \gamma \rightarrow b \bar{b} c \bar{c}$ process and hence in the $\gamma \gamma \rightarrow \mathrm{B}_{\mathrm{c}} \overline{\mathrm{b}} \mathrm{c}$ matrix element one can distinguish three groups


Figure 19. The cross section distributions, normalised to the unity as functions of $z$ for $\mathrm{B}_{\mathrm{c}}$-meson production at different energies.


Figure 20. The cross section distributions, normalised to unity as a function of $z$ for $\mathrm{B}_{\mathrm{c}}^{*}$-meson production at different energies.
of contributions which are separately gauge invariant under both the gluon field transformation and the photon one.

The first group of contributions is composed of the diagrams in which quark production is independent (we will label these diagrams as recombination diagrams), the second group consist of diagrams where the cé pair is produced from the b-quark line (we will call these diagrams the b-quark fragmentation diagrams, their contribution will be denoted as $\sigma^{b-f r a g}$ ), the third group contains diagrams with $\mathrm{b} \overline{\mathrm{b}}$ pair production from the c-quark line, so that they are c-fragmentation diagrams with the corresponding contributions denoted as $\sigma^{\text {c-frag }}$.

In Refs [43, 91, 93] the assumption was made that the b-fragmentation contribution has to dominate at large values of the $B_{c}$ transverse momentum, independently of the type of process. So the following approximate equation has to be valid:

$$
\begin{equation*}
\frac{\mathrm{d} \sigma_{\mathrm{B}_{\mathrm{c}}}^{\mathrm{q} \text { frag }}}{\mathrm{d} p_{\mathrm{t}}} \approx \int_{2 p_{\mathrm{t}} / \sqrt{s}}^{1} \frac{\mathrm{~d} \sigma_{\mathrm{q} \bar{q}}}{\mathrm{~d} k_{\mathrm{t}}}\left(\frac{p_{\mathrm{t}}}{z}\right) \frac{D_{\mathrm{q} \rightarrow \mathrm{~B}_{\mathrm{c}}}(z)}{z} \mathrm{~d} z, \tag{217}
\end{equation*}
$$

where $\mathrm{d} \sigma_{\mathrm{q} \bar{q}} / \mathrm{d} k_{\mathrm{t}}$ is the differential cross section for the production of the fragmenting $q$-quark in the Born approximation, $k_{\mathrm{t}}$ is its transverse momentum, and $D_{\mathrm{q} \rightarrow \mathrm{B}_{\mathrm{c}}}(z)$ is the function of the $\mathrm{q} \rightarrow \mathrm{B}_{\mathrm{c}}+\mathrm{X}$ fragmentation.

Recall that in the $\mathrm{e}^{+} \mathrm{e}^{-}$-annihilation the b -quark fragmentation dominates and the c-quark fragmentation contribu-tion is suppressed by two orders of magnitude. In the $\gamma \gamma$-interactions, the c -quark fragmentation contribution is enlarged due to the quark charge ratio $\left(Q_{c} / Q_{b}\right)^{4}=16$ and therefore we cannot neglect it (as one does in $\mathrm{e}^{+} \mathrm{e}^{-}-$ annihilation). Note further that the c-quark fragmentation contribution and the b-quark fragmentation one are related to each other by simple permutation of the quark masses and charges ( $m_{\mathrm{c}} \leftrightarrow m_{\mathrm{b}}$ and $Q_{\mathrm{c}} \leftrightarrow Q_{\mathrm{b}}$ ) (217).

The distributions $\mathrm{d} \sigma^{\mathrm{tot}} / \mathrm{d} p_{\mathrm{t}}, \quad \mathrm{d} \sigma^{\mathrm{c} \text {-frag }} / \mathrm{d} p_{\mathrm{t}} \quad$ and $\mathrm{d} \sigma^{\mathrm{b}-\mathrm{frag}} / \mathrm{d} p_{\mathrm{t}}$ at 100 GeV for $\mathrm{B}_{\mathrm{c}}-$ and $\mathrm{B}_{\mathrm{c}}^{*}-$ meson production


Figure 21. The $\mathrm{d} \sigma^{\text {tot }} / \mathrm{d} p_{\mathrm{t}}, \mathrm{d} \sigma^{\mathrm{c}-\text { frag }} / \mathrm{d} p_{\mathrm{t}}$, and $\mathrm{d} \sigma^{\mathrm{b}-\text { frag }} / \mathrm{d} p_{\mathrm{t}}$ distributions as functions of the transverse momentum for the invariant contributions to the cross section of $\mathrm{B}_{\mathrm{c}}$-meson production at 100 GeV . The curves 1 and 2 correspond to the prediction of the fragmentational mechanism (217) for the b-quark (1) and c-quark (2).


Figure 22. The $\mathrm{d} \sigma^{\text {tot }} / \mathrm{d} p_{\mathrm{t}}, \mathrm{d} \sigma^{\mathrm{c}-\mathrm{frag}} / \mathrm{d} p_{\mathrm{t}}$, and $\mathrm{d} \sigma^{\mathrm{b}-\mathrm{frag}} / \mathrm{d} p_{\mathrm{t}}$ distributions as functions of the transverse momentum for the invariant contributions to the cross section of $\mathrm{B}_{\mathrm{c}}^{*}$-meson production at 100 GeV . The curves 1 and 2 correspond to the prediction of the fragmentational mechanism (217) for the b-quark (1) and c-quark (2).
are shown in Figs 21 and 22. The distributions predicted in accordance with Eqn (217) for the b-fragmentation (curve 1) and c-fragmentation (curve 2) are also shown.

One can see that, as in the hadronic production, the contribution of the recombinational type diagram is significant at any reasonable values of the transverse momentum of the $\mathrm{B}_{\mathrm{c}}$-meson and cannot be neglected when one calculates the cross sections even at large transverse momenta. One can see from the figure that for the b -fragmentation contribution in terms of $p_{\mathrm{t}}$ greater than about 30 GeV , the fragmentational mechanism gives correct predictions.

Thus, at its maximum at the energy $20-30 \mathrm{GeV}$, the total cross section, including the $\mathrm{B}_{\mathrm{c}}^{*}$ and corresponding antiparticle production, is about 1 pb . This corresponds to $10^{5} \mathrm{~B}_{\mathrm{c}}$, produced at a $\gamma \gamma$-collider with luminosity of $10^{34} \mathrm{~cm}^{-2} \mathrm{~s}^{-1}$. At large energies, the cross section falls like that of the $\mathrm{b} \overline{\mathrm{b}}$-pair production. The $\mathrm{B}_{\mathrm{c}}$ production mechanism is close to that of the gluon-gluon interactions, and is also not reduced to the simple b-quark fragmentation.

## 5. Conclusion

The discovery and study of the family of the ( $\overline{\mathrm{b}}$ ) heavy quarkonium with open charm and beauty will allow one significantly to specify the notion of the dynamics of heavy quark interactions and the parameters of the Standard Model of elementary particles (such values as the b - and c quark masses, the coupling of the $\mathrm{b}-$ and $\mathrm{c}-\mathrm{quarks}-\left|V_{\mathrm{bc}}\right|$, etc.). The present review is aimed at the creation of a theoretical basis for object-oriented experimental research and study of the ( $\overline{\mathrm{b}}$ ) heavy quarkonium family.

Summarising the problems considered, one can note the following.

We have shown that below the threshold at which the $(\overline{\mathrm{b}})$ system decays into the BD meson pair, there are 16 narrow states of the $B_{c}$ meson family, whose masses can be
reliably calculated in the framework of the nonrelativistic potential models of the heavy quarkonia.

The flavour independence of the QCD-motivated potentials in the region of average distances between the quarks in the ( $\bar{b} b$ ), ( $\bar{c} c$ ) and ( $\bar{b} c$ ) systems and their scaling properties allow one to find the regularity of the spectra for the levels that are not split by the spin-dependent forces: in the leading approximation the state density of the system does not depend on the heavy quark flavours, i.e. the distances between the $n L$-levels of the heavy quarkonium do not depend on the heavy quark flavours.

We have described the spin-dependent splittings of the $(\bar{b} c)$ system levels, i.e. the splittings appearing in the second order in the inverse heavy quark masses, $V_{\mathrm{SD}} \approx \mathrm{O}\left(1 / m_{\mathrm{b}} m_{\mathrm{c}}\right)$, with account taken of the variation of the effective Coulomb coupling constant of the quarks (the interaction is due to relativistic corrections, coming from the one-gluon exchange).

The approaches developed to describe emission by the heavy quarks have been applied to the description of the radiative transitions in the ( $\overline{\mathrm{b}} \mathrm{c}$ ) family whose states have no electromagnetic or gluonic channels of annihilation. The last fact means that, due to the cascade processes with the emission of photons and pion pairs, the higher excitations decay into the lightest pseudoscalar $B_{c}$ meson, decaying in the weak way. Therefore, the excited states of the ( $\overline{\mathrm{b}} \mathrm{c}$ ) system have widths significantly less (by two orders of magnitude) than those in the charmonium (c $\bar{c}$ ) and bottomonium ( $\mathrm{b} \overline{\mathrm{b}}$ ) systems.

The value of the leptonic decay constant $f_{\mathrm{B}_{\mathrm{c}}}$ can be measured in the annihilation channels of decay, for example $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \tau^{+} v_{\tau}$. It can be most reliably estimated from the scaling relation for the leptonic constants of the heavy quarkonia, due to the relation obtained in the framework of the QCD sum rules in the specific scheme. In the other schemes of the QCD sum rules, it is necessary to do an interpolation of the scheme parameters (the hadronic continuum threshold and the number of the spectral density moment or the Borel parameter) into the region of the ( $\overline{\mathrm{b}} \mathrm{c}$ ) system, so this procedure leads to significant uncertainties.

The $f_{\mathrm{B}_{\mathrm{c}}}$ estimate from the scaling relation agrees with the results of the potential models, whose accuracy for the leptonic constants is notably lower. The value of $f_{\mathrm{B}_{\mathrm{c}}}$ essentially determines the decay widths and the production cross sections of the $B_{c}$ mesons.

The theoretical study of semileptonic $B_{c}$ decays shows that the results of the potential quark models agree with the predictions of the QCD sum rules, if one accounts for the Coulomb-like $\alpha_{\mathrm{s}} / v$-corrections. In this case, the approximate spin symmetry in the sector of heavy quarks allows one to derive the relations for the form factors of semileptonic $B_{c}$ decays at the rest point of the recoil meson.
$\mathrm{B}_{\mathrm{c}}$-meson production allows in some cases a description on the level of analytic expressions, such as the universal functions of the heavy quark fragmentation into the heavy quarkonium. The fragmentational mechanism dominates in the $B_{c}$ production in the $\mathrm{e}^{+} \mathrm{e}^{-}$-annihilation at high energies (at the $Z$-boson peak) and can be studied at the LEP facilities.

The hadronic production of $B_{c}$ is basically determined by the processes of the $\bar{b}$ - and c-quark recombination, since the partonic subprocesses have the highest luminosity in the region of low invariant masses of the resultant system
( $b \bar{b} c \bar{c}$ ). The $B_{c}-$ meson yield with respect to the production of the beauty hadrons is of the order $\dagger$ of $10^{-3}$.

Modes of $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \psi \mathrm{X}$ decays with the characteristic signature of the $J / \psi$-particle have the quite large probability

$$
\mathrm{BR}\left(\mathrm{~B}_{\mathrm{c}}^{+} \rightarrow \psi \mathrm{X}\right) \approx 0.2
$$

Therefore, the search for the $B_{c}-$ particle can start from the separation of events containing the $J / \psi$-particle, whose production vertex is beyond the primary intersection point.

The selected set of events will, of course, contain a background from decays of ordinary heavy - light B-mesons ( $\overline{\mathrm{b}} \mathrm{u}, \overline{\mathrm{b}} \mathrm{d}, \overline{\mathrm{b}} \mathrm{s}$ ), since the probability of the $\mathrm{B} \rightarrow \mathrm{J} / \psi \mathrm{KX}$ decay is about $1 \%$, and the heavy-light $\mathrm{B}-\mathrm{meson}$ yield is three orders of magnitude greater than that of the $B_{c}$. The separation from background requires a cut-off from below the effective mass of the $J / \psi X$ system, where $X$ denotes those charged particles whose tracks originate from the $J / \psi$ vertex.

The most preferable channel for $B_{c}$ extraction is that of the $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \psi 1^{+} v_{1}$ decay, since $\mathrm{B}_{\mathrm{c}}$ is the only heavy particle with the three-lepton vertex of the decay $\psi 1^{+} \rightarrow 1^{\prime+} 1^{\prime-} 1^{+}$. The probability of this channel is

$$
\mathrm{BR}\left(\mathrm{~B}_{\mathrm{c}}^{+} \rightarrow \psi \mathrm{l}^{+} v_{1}\right) \approx 8 \%, \quad \mathrm{l}=\mathrm{e}, \mu, \tau
$$

In a quite large statistical sample $\ddagger$, events with the decay $\mathrm{B}_{\mathrm{c}}^{+} \rightarrow \psi l^{+} v$ can raise the possibility of the determination of the $B_{c}$ mass value under the $\psi 1$ mass spectrum or the missed transverse momentum of the neutrino with respect to the direction of the $B_{c}$ motion (see Fig. 23). The necessary condition for the such measurement is a quite high separation of charged hadrons and leptons.

A straightforward measurement of the $B_{c}$ mass can be made in the mode of the $B_{c}^{+} \rightarrow J / \psi \pi^{+}$decay, having the branching ratio

$$
\mathrm{BR}\left(\mathrm{~B}_{\mathrm{c}}^{+} \rightarrow \psi \pi^{+}\right) \approx 0.2 \%
$$

The mode of the $B_{c}^{ \pm} \rightarrow \mathrm{J} / \psi \pi^{ \pm} \pi^{ \pm} \pi^{\mp}$ decay, where three $\pi$ mesons can compose the $a_{1}$-meson, is also of interest. This mode must have a significantly greater probability than the $\mathrm{B}_{\mathrm{c}} \rightarrow \mathrm{J} / \psi \pi$ decay.

Since $\mathrm{B}_{\mathrm{c}}$ production in colliding $\mathrm{e}^{+} \mathrm{e}^{-}$beams has, as mentioned, fragmentational character, in general (see Fig. 7) it must be accompanied by a D-meson presence in the jet where the $B_{c}$ candidate is being observed. Such a signature of the event would provide a large advantage to the search for $\mathrm{B}_{\mathrm{c}}$-mesons at $\mathrm{e}^{+} \mathrm{e}^{-}$colliders compared with that at hadron colliders, where the recombinational mechanism dominates in $\mathrm{B}_{\mathrm{c}}$-meson production at energies accessible in the immediate future (see Fig. 9).

However, one must take into account the possibility that the probability of the b -quark fragmentation production of the free c $\bar{c}$-quark pair is one order of magnitude greater than the probability of the fragmentation into the $\mathrm{B}_{\mathrm{c}}$-meson and the single free c-quark. This means that with account for the branching ratios for the $B-$ and $B_{c}$-decays into $J / \psi X$, events with $B_{c}$ decay and a single D-meson will appear only
$\dagger$ In the present review we do not consider in detail $\mathrm{B}_{\mathrm{c}}$ production in the neutrino-nucleon interactions, where one can expect only a few events with $B_{c}$ production per year, since the coupling constant of the b- and c-quarks is low [52], so that these processes have no practical significance for the experimental search for $B_{c}$.
$\ddagger$ The CDF facility with the vertex detector at the Tevatron FNAL has, in this sense, a preferable position.


Figure 23. The distribution over the invariant masses of the $\psi l$ (a) and $\psi l v_{\text {mis }}$ (b) systems in the $B_{c}^{+} \rightarrow \psi l^{+} v_{1}$ decay, where $v_{\text {mis }}$ is a neutrino with momentum equal to the missing transverse momentum with respect to the direction of the $\mathrm{B}_{\mathrm{c}}$-meson motion.
twice as often as the decay of the heavy-light B-meson into $\mathrm{J} / \psi \mathrm{X}$, with the instantaneous production of two D mesons in the same jet. It is not clear whether one can quite effectively separate these two processes in the present vertex detectors, i.e. whether one cannot lose the vertex of the second D-meson.

It is evident that progress in the experimental study of the $\mathrm{B}_{\mathrm{c}}$-meson and the general physics of heavy quarks will be mainly related to the development of the vertex detectors, so that the latter would give the possibility of the reliable observation of several heavy quarks instantaneously (to search the cascade decays, for example). However, since at the present yields of LEP and Fermilab several dozen $\mathrm{B}_{\mathrm{c}}$-meson production events may be observed, one would think that the practical detection of $B_{c}$ will be achieved in the near future.

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## Appendices

## I. Covariant quark model

Consider the general statements of the covariant description of the composed quarkonium model.

By definition, the energy fraction carried out by quark $i$ in the $\left(\mathrm{Q}^{\prime}\right)$ meson is its constituent mass $m_{i}$, so that

$$
\begin{equation*}
M=m+m^{\prime}, \tag{I.1}
\end{equation*}
$$

where $M$ is the meson mass, and $m$ and $m^{\prime}$ are fixed values. For the four-momenta, one has

$$
\begin{align*}
k & =\frac{m}{M} P+q \\
k^{\prime} & =\frac{m^{\prime}}{M} P-q \tag{I.2}
\end{align*}
$$

where $P$ is the meson momentum, and $q$ is the relative momentum of quarks inside the meson.

For the quark propagator, one has

$$
\begin{equation*}
S(k)=\left(k_{\mu} \gamma^{\mu}+m\right) D(k) \tag{I.3}
\end{equation*}
$$

The constituent quark has, in fact, fixed energy, so that in the $D(k)$ function, only the imaginary part contributes. In the meson rest frame, one has

$$
\begin{equation*}
\operatorname{Im} D(k)=\frac{\pi}{m} \delta\left(\left|k_{0}\right|-m\right) \tag{I.4}
\end{equation*}
$$

Eqn (I.4) with account for Eqn (I.2) can be rewritten in the covariant form

$$
\begin{equation*}
\operatorname{Im} D(k)=\frac{\pi M}{m} \delta(P q) \tag{I.5}
\end{equation*}
$$

The quark - meson vertex can be represented as

$$
\begin{equation*}
L_{\mathrm{q} \bar{q} \mathrm{M}}=\bar{v}(k) \Gamma v^{\prime}\left(k^{\prime}\right) D^{-1}(k) D^{-1}\left(k^{\prime}\right) \chi(P ; q), \tag{I.6}
\end{equation*}
$$

where $v$ and $v^{\prime}$ are the quark spinors, the $D(k)$ function is defined in Eqn (I.3), and $\Gamma$ is the spinor matrix determining the quantum numbers of the meson.

The nonrelativistic description of the meson means that the form factor is determined by the expression

$$
\begin{equation*}
\chi(P ; q)=2 \pi \delta(P q) \phi\left(q^{2}\right) \tag{I.7}
\end{equation*}
$$

In the following, we suppose

$$
\begin{equation*}
\phi\left(q^{2}\right)=N \exp \frac{q^{2}}{\omega^{2}} \tag{I.8}
\end{equation*}
$$

The choice (I.8) reflects the typical form of the S-wave functions of the charmonium and bottomonium, and it allows one to perform the analytic calculation of the semileptonic decay widths for $\mathrm{B}_{\mathrm{c}}$-meson.

Let us define the decay constants $f$ for the pseudoscalar and vector mesons,

$$
\begin{align*}
& \langle 0| J_{5 \mu}(x)|\mathrm{P}(q)\rangle=\mathrm{i} f_{\mathrm{P}} q_{\mu} \exp (\mathrm{i} q x)  \tag{I.9}\\
& \langle 0| J_{\mu}(x)|\mathrm{V}(q, \lambda)\rangle=\mathrm{i} f_{\mathrm{V}} M_{\mathrm{V}} \varepsilon_{\mu}^{(\lambda)} \exp (\mathrm{i} q x) \tag{I.10}
\end{align*}
$$

where $\lambda$ is the vector meson polarisation, and the quark currents are

$$
\begin{align*}
& J_{5 \mu}(x)=\bar{Q}(x) \gamma_{5} \gamma_{\mu} Q^{\prime}(x),  \tag{I.11}\\
& J_{\mu}(x)=\bar{Q}(x) \gamma_{\mu} Q^{\prime}(x) . \tag{I.12}
\end{align*}
$$

In the nonrelativistic potential model, one has

$$
\begin{equation*}
f_{\mathrm{P}} \approx f_{\mathrm{V}}=f \tag{I.13}
\end{equation*}
$$

so that

$$
\begin{equation*}
f=2 \sqrt{\frac{3}{M}} \Psi(0) \tag{I.14}
\end{equation*}
$$

where $\Psi(0)$ is the quarkonium wave function at the origin.
The oscillator function, resulting in Eqn (I.8), has the form

$$
\begin{equation*}
\Psi(\boldsymbol{r})=\left(\frac{\omega^{2}}{2 \pi}\right)^{3 / 4} \exp \left(-\frac{r^{2} \omega^{2}}{4}\right) \tag{I.15}
\end{equation*}
$$

Condition (I.14) means that the normalisation constant $N$ in Eqn (I.8) is

$$
\begin{equation*}
N=\frac{M}{m m^{\prime}} \frac{\sqrt{6}}{f} \tag{I.16}
\end{equation*}
$$

Thus, for the quark - meson form factor, one finds

$$
\begin{equation*}
\chi(P ; q)=2 \pi \delta(P q) \frac{M}{m m^{\prime}} \frac{\sqrt{6}}{f} \exp \frac{q^{2}}{\omega^{2}}, \tag{I.17}
\end{equation*}
$$

where $\omega$ is determined by Eqns (I.14) and (I.15), so that the only free parameter of the model is the constant $f$. For the $\psi$-particle, $f_{\psi}$ can, for example, be related to the width of the $\psi \rightarrow \mathrm{e}^{+} \mathrm{e}^{-}$decay

$$
\begin{equation*}
\Gamma\left(\psi \rightarrow \mathrm{e}^{+} \mathrm{e}^{-}\right)=\frac{4 \pi}{3} \alpha_{\mathrm{em}}^{2} e_{\mathrm{c}}^{2} \frac{f_{\psi}^{2}}{M_{\psi}} \tag{I.18}
\end{equation*}
$$

where $e_{\mathrm{c}}=2 / 3$ is the electric charge on the c-quark. From Eqn (I.18), the experimental value of the leptonic width [15] gives

$$
\begin{equation*}
f_{\psi}=410 \pm 15 \mathrm{MeV} \tag{I.19}
\end{equation*}
$$

The values of the $f_{\mathrm{B}_{\mathrm{c}}}$ and $f_{\mathrm{B}_{\mathrm{S}}}$ constants are determined theoretically in the framework of the QCD sum rules and in the potential models.

Note that the stated model of the composed quarkonium gives, for instance, the exact formula of the nonrelativistic M 1-transition for the electromagnetic decay of the vector state into the pseudoscalar state, $\mathrm{V} \rightarrow \mathrm{P} \gamma$

$$
\begin{equation*}
\Gamma(\mathrm{V} \rightarrow \mathrm{P} \gamma)=\frac{16}{3} \mu^{2} \omega_{\gamma}^{3} \tag{I.20}
\end{equation*}
$$

where $\omega_{\gamma}$ is the energy of the $\gamma$-quantum, and the magnetic moment $\mu$ is

$$
\begin{equation*}
\mu=\frac{1}{2} \sqrt{\alpha_{\mathrm{em}}}\left(\frac{e}{2 m}+\frac{e^{\prime}}{2 m^{\prime}}\right) \tag{I.21}
\end{equation*}
$$

where $e$ and $e^{\prime}$ are the electric charges on quarks in units of the electron charge.

## II. Spectral densities for three-particle functions

The spectral densities for the three-particle functions are determined in the following way [36]:

$$
\begin{align*}
\rho_{+}\left(s_{1}, s_{2}, Q^{2}\right)= & \frac{3}{2 k^{3 / 2}}\left\{k \frac{\Delta_{1}+\Delta_{2}}{2}\right. \\
& -k\left[m_{3}\left(m_{3}-m_{1}\right)+m_{3}\left(m_{3}-m_{2}\right)\right] \\
& -\left[2\left(s_{1} \Delta_{2}+s_{2} \Delta_{1}\right)-u\left(\Delta_{1}+\Delta_{2}\right)\right] \\
& \left.\times\left(m_{3}^{2}-\frac{u}{2}+m_{1} m_{2}-m_{2} m_{3}-m_{1} m_{3}\right)\right\} \tag{II.1}
\end{align*}
$$

$$
\begin{align*}
\rho_{\mathrm{V}}\left(s_{1}, s_{2}, Q^{2}\right)= & \frac{3}{k^{3 / 2}}\left[\left(2 s_{1} \Delta_{2}-u \Delta_{1}\right)\left(m_{3}-m_{2}\right)\right. \\
& \left.+\left(2 s_{2} \Delta_{1}-u \Delta_{2}\right)\left(m_{3}-m_{1}\right)+m_{3} k\right],  \tag{II.2}\\
\rho_{0}^{\mathrm{A}}\left(s_{1}, s_{2}, Q^{2}\right)= & \frac{3}{k^{1 / 2}}\left\{\left(m_{1}-m_{2}\right)\right. \\
& \times\left[m_{3}^{2}+\frac{1}{k}\left(s_{1} \Delta_{2}^{2}+s_{2} \Delta_{1}^{2}-u \Delta_{1} \Delta_{2}\right)\right] \\
& -m_{2}\left(m_{3}^{2}-\frac{\Delta_{1}}{2}\right)-m_{1}\left(m_{3}^{2}-\frac{\Delta_{2}}{2}\right) \\
& \left.+m_{3}\left(m_{3}^{2}-\frac{\Delta_{1}+\Delta_{2}-u}{2}+m_{1} m_{2}\right)\right\} \tag{II.3}
\end{align*}
$$

$$
\rho_{+}^{\mathrm{A}}\left(s_{1}, s_{2}, Q^{2}\right)=\frac{3}{k^{3 / 2}}\left\{m_{1}\left(2 s_{2} \Delta_{1}-u \Delta_{2}+4 \Delta_{1} \Delta_{2}+2 \Delta_{2}^{2}\right)\right.
$$

$$
+m_{1} m_{3}^{2}\left(4 s_{2}-2 u\right)+m_{2}\left(2 s_{1} \Delta_{2}-u \Delta_{1}\right)
$$

$$
-m_{3}\left[2\left(3 s_{2} \Delta_{1}+s_{1} \Delta_{2}\right)-u\left(3 \Delta_{2}+\Delta_{1}\right)+k\right.
$$

$$
\left.+4 \Delta_{2} \Delta_{1}+2 \Delta_{2}^{2}+m_{3}^{2}\left(4 s_{2}-2 u\right)\right]
$$

$$
+\frac{6}{k}\left(m_{1}-m_{3}\right)\left[4 s_{1} s_{2} \Delta_{1} \Delta_{2}\right.
$$

$$
-u\left(2 s_{2} \Delta_{1} \Delta_{2}+s_{1} \Delta_{2}^{2}+s_{2} \Delta_{1}^{2}\right)
$$

$$
\begin{equation*}
\left.\left.+2 s_{2}\left(s_{1} \Delta_{2}^{2}+s_{2} \Delta_{1}^{2}\right)\right]\right\} \tag{II.4}
\end{equation*}
$$

where

$$
\begin{aligned}
& k=\left(s_{1}+s_{2}+Q^{2}\right)^{2}-4 s_{1} s_{2}, \quad u=s_{1}+s_{2}+Q^{2}, \\
& \Delta_{1}=s_{1}-m_{1}^{2}+m_{3}^{2}, \quad \Delta_{2}=s_{2}-m_{2}^{2}+m_{3}^{2} .
\end{aligned}
$$

In the $\mathrm{B}_{\mathrm{c}} \rightarrow \eta_{\mathrm{c}}(\mathrm{J} / \psi) \mathrm{ev}$ decays, one has $m_{1}=m_{\mathrm{b}}$ and $m_{2}=m_{3}=m_{\mathrm{c}}$ for the masses.

## III. QCD sum rule scheme for three-point correlators

Let us consider the sum rules for the $f_{+}\left(Q^{2}\right)$ form factor

$$
\begin{align*}
\sum_{i, j=1}^{\infty} f_{\mathrm{B}_{\mathrm{c}}}^{i} & \frac{M_{\mathrm{B}_{\mathrm{c}}}^{i 2}}{m_{\mathrm{b}}+m_{\mathrm{c}}} f_{\eta_{\mathrm{c}}}^{j} \frac{M_{{\mathrm{n}_{\mathrm{c}}}^{j 2}}^{2 m_{\mathrm{c}}} f_{+}^{i j}\left(Q^{2}\right)}{} \\
& \times \frac{1}{\left(M_{\mathrm{B}_{\mathrm{c}}}^{i 2}-p_{1}^{2}\right)\left(M_{\eta_{\mathrm{c}}}^{j 2}-p_{2}^{2}\right)} \\
= & \frac{1}{(2 \pi)^{2}} \int \mathrm{~d} s_{1} \mathrm{~d} s_{2} \frac{\rho_{+}\left(s_{1}, s_{2}, Q^{2}\right)}{\left(s_{1}-p_{1}^{2}\right)\left(s_{2}-p_{2}^{2}\right)} \tag{III.1}
\end{align*}
$$

Applying the Borel operators $\hat{L}_{\tau_{1}}\left(-p_{1}^{2}\right)$ and $\hat{L}_{\tau_{2}}\left(-p_{2}^{2}\right)$, defined in section 2, to Eqn (III.1), one derives the following sum rules

$$
\begin{align*}
& \sum_{i, j=1}^{\infty} f_{\mathrm{B}_{\mathrm{c}}}^{i} M_{\mathrm{B}_{\mathrm{c}}}^{i 2} f_{\mathrm{\eta}_{\mathrm{c}}}^{j} M_{{\eta_{\mathrm{c}}}^{j 2}}^{f_{+}^{i j}\left(Q^{2}\right) \exp \left(-M_{\mathrm{B}_{\mathrm{c}}}^{i 2} \tau_{1}-M_{\left.{\eta_{\mathrm{c}}}^{j 2} \tau_{2}\right)}\right.} \begin{array}{l}
=\frac{2\left(m_{\mathrm{b}}+m_{\mathrm{c}}\right) m_{\mathrm{c}}}{(2 \pi)^{2}} \int \mathrm{~d} s_{1} \mathrm{~d} s_{2} \rho_{+}\left(s_{1}, s_{2}, Q^{2}\right) \\
\quad \times \exp \left(-s_{1} \tau_{1}-s_{2} \tau_{2}\right)
\end{array} .
\end{align*}
$$

Introduce the notation

$$
\begin{equation*}
S_{i}=\sum_{j=1}^{\infty} f_{\eta_{\mathrm{c}}}^{j} M_{\eta_{\mathrm{c}}}^{j 2} f_{+}^{i j}\left(Q^{2}\right) \exp \left(-M_{\eta_{\mathrm{c}}}^{j 2} \tau_{2}\right) \tag{III.3}
\end{equation*}
$$

and transform the left-hand side of Eqn (III.2) with the use of the Euler-MacLaurin formula [90]

$$
\begin{align*}
& \sum_{i=1}^{\infty} f_{\mathrm{B}_{\mathrm{c}}}^{i} M_{\mathrm{B}_{\mathrm{c}}}^{i 2} S_{i} \exp \left(-M_{\mathrm{B}_{\mathrm{c}}}^{i 2} \tau_{1}\right) \\
& \quad=\int_{M_{\mathrm{B}_{\mathrm{c}}}^{k}}^{\infty} \mathrm{d} M_{\mathrm{B}_{\mathrm{c}}}^{n} \frac{\mathrm{~d} n}{\mathrm{~d} M_{\mathrm{B}_{\mathrm{c}}}^{n}} f_{\mathrm{B}_{\mathrm{c}}}^{n} M_{\mathrm{B}_{\mathrm{c}}}^{n 2} S_{n} \exp \left(-M_{\mathrm{B}_{\mathrm{c}}}^{n 2} \tau_{1}\right) \\
& \quad  \tag{III.4}\\
& \quad+\sum_{n=0}^{n=k-1} f_{\mathrm{B}_{\mathrm{c}}}^{n} M_{\mathrm{B}_{\mathrm{c}}}^{n 2} S_{n} \exp \left(-M_{\mathrm{B}_{\mathrm{c}}}^{n 2} \tau_{1}\right)+\ldots
\end{align*}
$$

Applying $\hat{L}_{\tau^{\prime}}\left(M_{\mathrm{B}_{\mathrm{c}}}^{k 2}\right)$ to Eqn (III.2) and accounting for Eqn (III.4), one gets

$$
\begin{gather*}
\sum_{j=1}^{\infty} f_{\mathfrak{\eta}_{\mathrm{c}}}^{j} M_{\eta_{\mathrm{c}}}^{j 2} f_{+}^{k j}\left(Q^{2}\right) \exp \left(-M_{\mathrm{n}_{\mathrm{c}}}^{j 2} \tau_{2}\right)=\frac{2 m_{\mathrm{c}}\left(m_{\mathrm{b}}+m_{\mathrm{c}}\right)}{(2 \pi)^{2}} \frac{\mathrm{~d} M_{\mathrm{B}_{\mathrm{c}}}^{k}}{\mathrm{~d} k} \\
\quad \times \frac{2}{M_{\mathrm{B}_{\mathrm{c}}}^{k} f_{\mathrm{B}_{\mathrm{c}}}^{k}} \int \mathrm{~d} s_{2} \rho\left(M_{\mathrm{B}_{\mathrm{c}}}^{k 2}, s_{2}, Q^{2}\right) \exp \left(-s_{2} \tau_{2}\right) . \tag{III.5}
\end{gather*}
$$

Using the analogous procedure for the sum of the $\eta_{\mathrm{c}}^{i}$ resonances, one obtains

$$
\begin{align*}
f_{+}^{k l}\left(Q^{2}\right)= & \frac{8 m_{\mathrm{c}}\left(m_{\mathrm{b}}+m_{\mathrm{c}}\right)}{M_{\mathrm{B}_{\mathrm{c}}}^{k} M_{\eta_{\mathrm{c}}}^{l} f_{\mathrm{B}_{\mathrm{c}}}^{k} f_{\eta_{\mathrm{c}}}^{l}} \frac{\mathrm{~d} M_{\mathrm{B}_{\mathrm{c}}}^{k}}{\mathrm{~d} k} \frac{\mathrm{~d} M_{\mathrm{n}_{\mathrm{c}}}^{l}}{\mathrm{~d} l} \\
& \times \frac{1}{(2 \pi)^{2}} \rho_{+}\left(M_{\mathrm{B}_{\mathrm{c}}}^{k 2}, M_{\eta_{\mathrm{c}}}^{l 2}, Q^{2}\right) . \tag{III.6}
\end{align*}
$$

Here we have used the property of the Borel operator

$$
\hat{L}_{\tau}(x)\left[x^{n} \exp (-b x)\right] \rightarrow \delta_{+}^{(n)}(\tau-b)
$$

It is not difficult to generalise this procedure to the remaining form factors.


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