

FROM THE CURRENT LITERATURE

PACS numbers: 41.20.bt

The approximability problem of the electromagnetic field

B Z Katsenelenbaum

Contents

Abstract. The relationship between the geometric properties of a surface and the properties of the electromagnetic field generated by a current arbitrarily distributed on the surface is discussed. There is a continual cardinality of surfaces for which this field cannot even approximately describe any randomly chosen pattern or any field in the near region. Study of these surfaces is based on the fact that they are zero surfaces of some auxiliary electromagnetic field which obeys the Maxwell equation. The mere proximity of the surface to any surface having these properties gives rise to nontrivial properties in the fields generated by the currents induced on the surface.

1. Introduction

1.1 Introductory comments

We shall review the properties of the electromagnetic fields generated by currents distributed on some special surfaces, i.e. induced on some metallic bodies.

In the case of some simple surfaces these properties of the fields are obvious; it has been found that 'very many' surfaces display them. The essence of these properties is that any complete system of currents on the surface generates an incomplete system of fields, and in particular an incomplete system of patterns. Therefore 'very many' fields cannot be described even approximately by these fields: more precisely, all the fields which fail to satisfy a given condition. Many nontrivial physical corollaries follow from this statement.

In Sections 1 – 4 the two-dimensional scalar fields generated by current distributed on lines will be discussed. In this model all the formulations and the mathematical apparatus are simpler, and the main results can be transferred to three-dimensional vector fields almost automatically (see, especially Section 5.1), with only one exception (Section 5.3). In this review the discussion will be confined to electric currents and to monochromatic fields [$\sim \exp(\mathrm{i}\omega t)$]. Some of the results have recently been published [1, 2].

Let us examine the relationship between the field $u(r, \varphi)$ and the line C on which the currents which generate this field are distributed. The field $u(r, \varphi)$ satisfies the homogeneous wave equation

$$\nabla^2 u + k^2 u = 0 \tag{1.1}$$

($k = \omega/c$), the Sommerfeld radiation condition

$$u(r, \varphi) \to \frac{\exp(-\mathrm{i}\,kr)}{\sqrt{kr}} f(\varphi) \,, \tag{1.2}$$

where $f(\varphi)$ is a pattern, and the condition that the field itself is continuous on C but its normal derivative $\partial u/\partial N$ has a discontinuity equal to the given current $j(s)$ (where s

B Z Katsenelenbaum Institute of Radio Engineering and Electronics, Russian Academy of Sciences, 11 ul. Mokhovaya, 103907 Moscow
Tel. (095) 203 48 36;
Fax (095) 203 84 14

Received 22 September 1993, revised 6 June 1994
Uspekhi Fizicheskikh Nauk **164** (9) 983 – 993 (1994)
Translated by J I Carasso

is the coordinate along C). The $f(\varphi)$ pattern is obtained from $j(s)$ by the integral transformation

$$f(\varphi) = \int_C \mathcal{K}(s, \varphi) j(s)\,\mathrm{d}s\,, \tag{1.3a}$$

$$\mathcal{K}(s, \varphi) = \exp[-\mathrm{i}kr(\theta)\cos(\varphi - \theta)]\,, \tag{1.3b}$$

where $r = r(\theta)$ is the equation of the line C, and $\theta = \theta(s)$. An unimportant factor has been omitted in Eqn (1.3b): in similar circumstances the omission will not be pointed out again in the discussion to follow.

If $f(\varphi)$ is given, Eqn (1.3a) is a known integral equation of the first kind for $j(s)$; by definition it has a solution if the current norm

$$N = \left(\int_C |j(s)|^2\,\mathrm{d}s\right)^{1/2} \tag{1.4}$$

is finite ($N < \infty$). The patterns generated by any type of current in C will be called patterns generated by the line C.

In most of the results reported below $N < \infty$ is not a necessary condition. The current can have singularities which make $|j(s)|^2$ not integrable: we only require the existence of the integral on the left-hand side of Eqn (2.6) (see below), and for this it is sufficient for the current itself to be integrable. The condition is satisfied also for the current near the edge of a half-plane (for any polarisation), and for $j(s) \approx \delta(s - s_0)$, i.e. in the approximation often used in the theory of antenna arrays. We shall use the condition $N < \infty$ in order to simplify the treatment (especially in Section 4).

1.2 Realisability, approximability, amplitude approximability

The following relationships between the patterns generated by the line C and any function $F(\varphi)$ are possible:

(a) Realisability. A current which generates the pattern of $F(\varphi)$ exists. Eqn (1.3a) has a solution if $f(\varphi)$ is replaced by $F(\varphi)$.

(b) Approximability. For any $\delta > 0$ there is a current such that the distance (in root-mean-square metrics) between the pattern generated by that current and $F(\varphi)$ is not greater than δ:

$$\left(\int_0^{2\pi} |F(\varphi) - f(\varphi)|^2\,\mathrm{d}\varphi\right)^{1/2} \leqslant \delta\,. \tag{1.5}$$

(c) Amplitude approximability. There exists a real function $\psi(\varphi)$ (the 'phase') such that the function $F(\varphi)\exp[-\mathrm{i}\psi(\varphi)]$ is approximable

(d) Amplitude nonapproximability. The amplitudes of all the patterns generated by the line are not close to $|F(\varphi)|$.

Of course, any other metrics apart from the quadratic can be used to determine the approximability.

The realisability depends on the analytic properties of the $F(\varphi)$ function [3], and a realisable function can be made unrealisable (and vice-versa) by an infinitesimally small change. The approximability is a coarser function: its presence (or absence) in any function is shared with all the functions close to it.

2. Nonapproximable patterns

2.1 Nonapproximability and zero lines of a wave field

The line C for which nonapproximable functions exist has one specific property: it has a corresponding function $\widehat{F}(\varphi)$ which can be normalised over the range $(0, 2\pi)$ and is orthogonal to all the $f(\varphi)$ patterns generated by the line C. The orthogonality of the functions $f(\varphi)$ and $\widehat{F}(\varphi)$ means that the product of these function $(f, \widehat{F})$, equal (by definition) to the integral

$$(f, \widehat{F}) = \int_0^{2\pi} f(\varphi)\widehat{F}^*(\varphi)\,\mathrm{d}\varphi\,, \tag{2.1}$$

vanishes. The normalisability condition means that the product $(\widehat{F}, \widehat{F})$ is finite; we shall normalise $\widehat{F}(\varphi)$ by using the equation $(\widehat{F}, \widehat{F}) = 1$.

We know [4] that if the functions $f(\varphi)$ satisfy the condition $(f, \widehat{F}) = 0$, a finite distance must exist between them and any function $F(\varphi)$ for which $(F, \widehat{F}) \neq 0$:

$$\left(\int_0^{2\pi} |F(\varphi) - f(\varphi)|^2\,\mathrm{d}\varphi\right)^{1/2} \geqslant |(F, \widehat{F})|\,. \tag{2.2}$$

If (furthermore) we stipulate that both $F(\varphi)$ and all the functions $f(\varphi)$ are normalised to unity, i.e. that $(F, F) = 1$, $(f, f) = 1$, the minimum distance between $F(\varphi)$ and $f(\varphi)$ becomes greater than the right-hand side of condition (2.2) and equal to

$$\varDelta = \left(2 - 2\sqrt{1 - |(F, \widehat{F})|^2}\right)^{1/2}. \tag{2.3}$$

The lines C for which all the resulting patterns have this property will be called special lines and designated by the symbol $\widehat{C}$.

There are $\widehat{C}$ lines corresponding not to a single $\widehat{F}(\varphi)$ function but to several such functions $\widehat{F}_p(\varphi)$ ($p = 1, 2, \ldots, P$). All these functions produce an orthogonal complement in the space of the patterns. They can be orthonormalised, $(\widehat{F}_p, \widehat{F}_q) = \delta_{pq}$. In this case in formula (2.3) for the 'gap width' between $f(\varphi)$ and $F(\varphi)$ the term $|(f, \widehat{F})|^2$ must be replaced by the sum over all the p of the squares of these products, and a corresponding change must be made in form-ula (2.2). The necessary and sufficient condition for the approximability of any function $F(\varphi)$ by the special line $\widehat{C}$ is the equality of all the $(F, \widehat{F}_p)$ products to zero. The existence of one or several orthogonal complement functions shows that the complete system of currents on C gives an incomplete system of patterns.

We shall show that all the special lines are zero lines of some wave field, and that all the zero lines of any wave field are special lines.

We require a proof of the equivalence of the condition that all the patterns created by the line $\widehat{C}$ are orthogonal to some function $\widehat{F}(\varphi)$ to the conditions that there exists a field $\widehat{u}(r, \varphi)$ equal to zero on $\widehat{C}$:

$$\widehat{u}(r, \varphi)\Big|_{\widehat{C}} = 0\,. \tag{2.4}$$

The field $\widehat{u}(r, \varphi)$ should obey the same homogeneous wave equation (1.1) as the field $u(r, \varphi)$, and it should have no singularities over the whole plane. At $r \to \infty$ it contains [as in expression (1.2)], in addition to an outgoing cylindrical wave, an incoming ('non-Sommerfeld') wave which is effectively the source of the field $\widehat{u}(r, \varphi)$, so that its asymptote at $r \to \infty$ becomes

$$\widehat{u}(r, \varphi) \to \frac{\exp(-ikr)}{\sqrt{kr}} \Phi(\varphi) + \frac{\exp(ikr)}{\sqrt{kr}} \widehat{F}^*(\varphi) . \quad (2.5)$$

Here $\widehat{F}(\varphi)$ is a function to which all the patterns generated by currents on the line $\widehat{C}$ found in condition (2.4) are orthogonal. The function $\Phi(\varphi)$ is unimportant in the subsequent calculations; if, as is often the case, $\widehat{u}(r, \varphi)$ is real we have $\Phi(\varphi) = \widehat{F}(\varphi)$.

The proof of the equivalence of the two conditions is elementary. It is a scalar variant of Lorentz's lemma for two fields: the field $u(r, \varphi)$, generated by the current $j(s)$ on $\widehat{C}$, and the field $\widehat{u}(r, \varphi)$, generated by the approaching cylindrical wave of amplitude $F^*(\varphi)$. We shall apply Green's second formula to these fields over the entire surface. The line $\widehat{C}$ must be eliminated by a supplementary contour (which is doubly bound if $\widehat{C}$ is closed). Since both fields obey Eqn (1.1) the formula retains only the integral over this contour, i.e. the integral over $\widehat{C}$, and the integral to infinity. According to (1.2) and (2.5) and to the condition for $u(r, \varphi)$ on $\widehat{C}$ it takes the form

$$\int_{\widehat{C}} \widehat{u}(r, \varphi) j(s) \, ds = \int_0^{2\pi} f(\varphi) \widehat{F}^*(\varphi) \, d\varphi . \quad (2.6)$$

If condition (2.4) is satisfied we have $(f, \widehat{F}) = 0$ for all currents. And conversely, if $(f, \widehat{F}) = 0$ for all currents we find, by applying Eqn (2.6) to any complete system of currents on $\widehat{C}$, that the value of $\widehat{u}(r, \varphi)$ on $\widehat{C}$ is orthogonal, in the usual meaning, to the complete system of functions, i.e. that Eqn (2.4) applies.

The inequality (2.2), i.e. the nonapproximability of 'nearly all' the functions, is an important property of the line $\widehat{C}$ for practical applications. It follows directly from the property $(f, \widehat{F}) = 0$, but a direct proof of this property is not usually possible. This proof of the equivalence means that inequality (2.2) is a consequence of the existence of some field $\widehat{u}(r, \varphi)$ having the property (2.4) on $\widehat{C}$, and often it is possible either to demonstrate the existence of this field (sometimes by a simple plot) or to prove that it cannot exist.

Thus, the approximability problem is related to the analytic properties of the fields, i.e. to the solution of the homogeneous wave equation. The special lines are the zero lines of the fields, and the orthogonal complement functions which characterise the degree of nonapproximability of any function $F(\varphi)$ [i.e. the right-hand side of inequality (2.2) or expression (2.3)] contribute to the asymptotics of these fields. This result is the starting point of the present review. In it, the properties of the fields $\widehat{u}(r, \varphi)$ and of their zero lines $\widehat{C}$, and also the physical consequences of the nonapproximability phenomenon, will be examined.

2.2 Properties of special lines

The function $\widehat{F}(\varphi)$ and the field $\widehat{u}(r, \varphi)$ are unambiguously interrelated. All solutions of the homogeneous wave equation without singularities can be expressed in the form

$$\widehat{u}(r, \varphi) = \sum C_n J_n(kr) \cos n\varphi , \quad (2.7)$$

where only fields which are even in φ are considered in order to simplify the treatment. In this case $\widehat{F}(\varphi)$ can be expressed as a Fourier series with the same coefficients; more precisely

$$\widehat{F}(\varphi) = \sum_n C_n (-i)^n \cos n\varphi . \quad (2.8)$$

Usually a function having a zero line is real. In that case C_n are also real, and $\widehat{F}(\varphi)$ satisfies the condition $\widehat{F}(\varphi + \pi) = \widehat{F}^*(\varphi)$.

The assertion that there are 'very many' special lines follows, essentially, from the existence of 'very many' different $\widehat{u}(r, \varphi)$ fields. In the geometric neighbourhood of any special line we find other special lines. More exactly, if the equation of the line C_1 is $r = r_1(\varphi)$ we can construct, for any $\varepsilon > 0$, a line $\widehat{C}_2$ with an equation $r = r_2(\varphi)$, where $|r_2(\varphi) - r_1(\varphi)| \leqslant \varepsilon$, and the functions $\widehat{F}_1(\varphi)$ and $\widehat{F}_2(\varphi)$ are not brought closer together by shifting or rotating the system of coordinates. Similarly, in any neighbourhood (in mean-square metrics) of any function $\widehat{F}_1(\varphi)$ we always find another function $\widehat{F}_2(\varphi)$. If two special lines with different $\widehat{F}(\varphi)$ functions are known, we can construct a series of $\widehat{C}$ lines continuously depending on the parameter.

Any arc of a special line is a special line. A closed special line produces a resonant contour, i.e. in its inner region we find a nonzero solution of Eqn (1.1), which becomes zero on the contour. As many special lines as required can be drawn through any point on the surface.

2.3 Examples of special lines. 'Forbidden' forms of antennas

The relationship between $\widehat{C}$ and $\widehat{F}(\varphi)$ is of practical interest, the field $\widehat{u}(r, \varphi)$ being only an auxiliary function. However, in examples studied to illustrate the relationship between $\widehat{C}$ and $\widehat{F}(\varphi)$ it is convenient to set the latter field and calculate from it (usually with a computer) $\widehat{C}$ and $\widehat{F}(\varphi)$. It is also relatively easy to find $\widehat{u}(r, \varphi)$ from a given $\widehat{F}(\varphi)$ [for example, by comparing Eqns (2.7) and (2.8)], and hence to find $\widehat{C}$. Constructing $\widehat{u}(r, \varphi)$ and $\widehat{F}(\varphi)$ from a given line $\widehat{C}$ is a more difficult problem (see Section 3.3).

We shall take as our first example

$$\widehat{u}(r, \varphi) = J_n(kr) \sin n\varphi , \qquad n = 1, 2, \ldots . \quad (2.9)$$

The line $\widehat{C}$ is composed of the resonant circles $r = \mu_{nm}/k (m = 1, 2, \ldots)\pi$ and $2n$ rays $\varphi = m\pi/n$ $(m = 0, 1, \ldots, 2n-1)$; μ_{nm} is the mth root of the nth Bessel function. To all these lines corresponds the function

$$\widehat{F}(\varphi) = \sin n\varphi \frac{1}{\sqrt{\pi}} .$$

Patterns produced by a cylindrical mirror antenna as arcs of a resonant circle corresponding to some value of the index n cannot be approximated (with the appropriate polarisation) by any function whose Fourier series includes $\sin n\varphi$ or $\cos n\varphi$.

The fact that a function containing $\sin n\varphi$ cannot be approximated by the patterns of currents on any of the given straight lines can also be interpreted as a consequence of the following obvious fact: the function $\sin n\varphi$ is odd with respect to a straight line, whereas all the patterns are even with respect to the straight line on which their generating currents are distributed. In general the mutual relationship between the line and $\widehat{F}(\varphi)$ can be treated as a generalisation of the symmetry properties of fields generated by currents on a straight line. The only nontrivial consequence is that these simple lines are not the only special lines.

Two straight lines on which $\hat{u}(r, \varphi)$ vanishes [Eqn (2.9)] intersect at an angle which is a rational fraction of π, i.e. it has the form $\alpha = \pi t/n$ ($t = 1, 2, \ldots, n$). On the other hand, if the angle α between the two straight lines does not have this form, the lines do not form a special line, and their currents give rise to a complete system of patterns. Contrasting these two cases has no physical significance, since any number α/π can be approximated by a rational fraction. The apparent paradox is eliminated by noting that for large n the right-hand side of inequality (2.2) is small, so that even for $\alpha = \pi t/n$ the nonapproximability almost vanishes when n is large.

The surface of the antenna does not have to be special. In particular, the walls of horn antennas do not need to intersect at angles of the type of $\alpha = \pi t/n$ with small n. We must avoid divergence angles equal or close to [see Section (4.4)] the angles

$$\alpha = 90^\circ\ (n=2)\,;\ 60^\circ,\ 120^\circ\ (n=3)\,;\ 45^\circ,\ 135^\circ\ (n=4)\ . \tag{2.10}$$

In an elliptical system of coordinates we can define the field $\hat{u}$ in the same elementary form as in Eqn (2.9); in the form of the product of two Mathieu functions. In this case the special lines take the form of arcs of ellipses or hyperbolas. The eccentricity of the ellipses will depend on the frequency or, more strictly, on the parameter kc, where $2c$ is the distance between the foci. For example, an ellipse with an eccentricity of 0.91 becomes a special line for $kc = 3.86$, an ellipse with an eccentricity of 0.63 for $kc = 1.73$, etc. Hyperbolas are conveniently characterised by the angle β between the asymptotes. For example, for $kc = 2.0$ and $kc = 6.3$ the special lines will be hyperbolas for which the angle β is equal (respectively) to 106°, 66°, 124°, 99°... and 144°, 116°, 146°, 118°.... These special lines correspond to orthogonal complement functions equal to Mathieu functions of φ. Elliptical and hyperbolic mirrors with appropriate eccentricities or angles β produce patterns with which only functions orthogonal to these functions can be approximated.

The resonance rectangle is not a special line. The field whose zero lines are the sides of the rectangle $a \times b$ ($\pi^2/a^2 + \pi^2/b^2 = k^2$) is described over the whole surface by the formula

$$\hat{u}(x, y) = \cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right) .$$

This field is produced by the interference of four approaching plane waves, and therefore the function $\hat{F}(\varphi)$ contains a δ-function, i.e. it is not normalisable. Although all the patterns generated by this line satisfy the condition $(f, \hat{F} = 0)$, as can also be shown directly from symmetry considerations, inequality (2.2) does not follow from this argument: there is no nonapproximability. For the same reason two parallel straight lines do not form a special line if the distance between them is greater than one-half of the wavelength, nor is a special line formed by an equilateral resonant triangle (or by several other resonant triangles [5]) on which the field produced by the interference of six plane waves falls to zero. We should also note that for near fields (but not for patterns) these resonant contours form special lines (in a slightly loose application of the term).

2.4 Example of amplitude nonapproximability

The special line corresponding to two straight lines forming an angle $\alpha = \pi t/n$ is the zero line of the denumerable set of fields $\hat{u}(r, \varphi)$ corresponding to different orthogonal complement functions,

$$\hat{F}_p = \sin pn\varphi \frac{1}{\sqrt{\pi}} \qquad (p = 1, 2, \ldots) .$$

The function $F(\varphi)$ is approximable if the system of conditions $(F, \hat{F}_p) = 0$ is satisfied. If the number of $\hat{F}_p(\varphi)$ functions is finite ($p = 1, 2, \ldots, P$) it should be possible [by an appropriate choice of the phase $\psi(\varphi)$] to satisfy these equalities for $F(\varphi)\exp[-\mathrm{i}\psi(\varphi)]$ in any function $F(\varphi)$, i.e. to ensure the amplitude approximability of the function $F(\varphi)$. The greater P and the narrower the $|F(\varphi)|$ pattern, the more broken will this phase be, and the greater will be the norm of the corresponding current. For $P = \infty$ this norm may reach infinity in narrow patterns, i.e. amplitude nonapproximability will occur.

It can be shown that for $P = \infty$ all the $(F, \hat{F}_p) = 0$ equalities can be satisfied by an appropriate choice of the phase if the $2n$ nonnegative functions $|F(\varphi - 2s\alpha)|$, $|F(2s\alpha - \varphi)|$, $s = 0, 1, \ldots, 2n-1$ have the following property for all φ: the greatest of them does not exceed the sum of the others. For example, for $n = 2$ (i.e. when the currents are distributed on two perpendicular straight lines) this conditions is satisfied for the Gaussian curve of

$$\exp\left[-A\sin^2\left(\frac{\varphi}{2} - \frac{\pi}{8}\right)\right]$$

(with its maximum along the bisector $\varphi = \pi/4$), i.e. amplitude approximability occurs only if $A < 1.76$, which corresponds to a half-width greater than 53°. The half-width of a similarly oriented Π-shaped pattern should be greater than 90°. Narrower patterns cannot be made approximable by an appropriate choice of the phase.

2.5 Determination of the shape of a body from its scattering pattern

By measuring only the scattering pattern without a knowledge of the illuminating field we can formulate some conclusions on the probable shape of the scattering body. First, we can establish the validity of the hypothesis that "the shape of the body is close to that of the special line $\hat{C}$ to which corresponds the function $\hat{F}(\varphi)$". If this hypothesis is valid we should have $|(f, \hat{F})| \ll 1$ [for $(f, f) = 1$] at any level of illumination. If $|(f, \hat{F})|$ is not small at even one level of illumination, the hypothesis is erroneous. Second, by measuring $f(\varphi)$ we can evaluate the function $\hat{F}(\varphi)$, which is close to it, i.e. such that $|(f, \hat{F})| \approx 1$. The contour of the body is not close to that of the special line corresponding to this function, $\hat{F}(\varphi)$. Third, from the measured $f(\varphi)$ pattern we can construct $\hat{F}(\varphi)$ functions orthogonal to it, i.e. such that $(f, \hat{F}) = 0$. The contour of the body may be close to the special line corresponding to one of these functions.

The information contributed by each of these conclusions increases if several measurements are carried out for different illuminations and different positions of the body. Because a smoothing (integral) functional (2.1) of $f(\varphi)$ is being calculated small errors in the measured diagram should not seriously affect the results, and, furthermore, incorrectly formulated methods of inferring the phase $f(\varphi)$ from the measured amplitude $|f(\varphi)|$ may be used. The method can be applied also to scattering on thin screens, which may be treated as nonclosed lines. Measurements of the scattering of nonmonochromatic signals may also be used.

3. Nonapproximability of near fields

3.1 Incompleteness of a system of near fields

A complete system of currents on the special line $\widehat{C}$ forms an incomplete system of fields not only at infinity but also on any contour Σ surrounding $\widehat{C}$.

There is a function $\widehat{W}(\sigma)$, where σ is the coordinate on the contour Σ, such that $(u, \widehat{W}) = 0$, where $u(\sigma)$ is the field generated on Σ by the line $\widehat{C}$, and the scalar product is determined as in Eqn (2.1) but the integration is carried out over σ rather than over φ. Thus, any function $U(\sigma)$ given on Σ can be approximated by the fields $u(\sigma)$ to within a precision not exceeding that defined by the inequality

$$\left[(U-u,\ U-u)\right]^{1/2} \geqslant \left|(U,\ \widehat{W})\right| , \tag{3.1}$$

which is analogous to expression (2.2). In inequality (3.1) the fields $u(\sigma)$ are not normalised, and $(\widehat{W}, \widehat{W}) = 1$. If the contour extends to infinity $\widehat{W}(\sigma)$ must be replaced by $\widehat{F}^*(\varphi)\exp(ikr)/\sqrt{kr}$.

To find the function $\widehat{W}(\sigma)$ we shall construct outside Σ a solution of the wave equation which satisfies the radiation condition and which takes the same value on Σ as the field $\widehat{u}(r, \varphi)$ whose zero line is $\widehat{C}$. We shall denote as $\widehat{w}(r, \varphi)$ a field equal to $\widehat{u}(r, \varphi)$ within Σ and equal to this solution outside it. The only singularity of the field $\widehat{w}(r, \varphi)$ is a discontinuity of its normal derivative on Σ, which we shall call $\widehat{W}^*(\sigma)$. Green's formula is then applied over the whole surface to the fields $u(r, \varphi)$ and $\widehat{w}(r, \varphi)$. Both these fields satisfy the radiation condition, so that their integral to infinity falls out, and two integrals are equal: that of the product of $j(s)$ by $\widehat{w}(r, \varphi)$ on $\widehat{C}$ and that of the product of $\widehat{W}^*(\sigma)$ by $u(\sigma)$ on Σ. The former is equal to zero according to Eqn (2.4), and this is a proof of the basic formula $(u, \widehat{W}) = 0$.

Not only $u(\sigma)$ but also $\partial n/\partial N$ form an incomplete system of functions on Σ. To prove this statement we must construct the field $\widehat{w}(r, \varphi)$ outside Σ not by the Dirichlet condition on Σ but by the Neumann condition (thus ensuring the continuity of the normal derivative), and again apply Green's formula over the whole surface.

3.2 The contour Σ and the singularities of the field $\widehat{u}(r,\varphi)$

The function $\widehat{W}(\sigma)$ is smoother than the function $\widehat{F}(\varphi)$ corresponding to the same field $\widehat{u}(r, \varphi)$. If Σ denotes the neighbourhood $(\sigma \equiv \varphi)$, the Fourier coefficients of the function $\widehat{W}(\varphi)$ decrease more slowly with increasing number than those of the function $\widehat{F}(\varphi)$, and if smooth functions $F(\varphi)$ and $U(\varphi)$ are given we must have $|(F, \widehat{F})| > |(U, \widehat{W})|$. Comparison of inequalities (2.2) and (3.1) shows that the nonapproximability decreases with the distance from the line on which the current is distributed. It can vanish entirely at infinity, as in the case of resonant rectangles and triangles, when $\widehat{F}(\varphi)$ becomes nonnormalisable (and there is no nonapproximability) whereas $\widehat{W}(\sigma)$, as can easily be verified, remains normalisable, so that the nonapproximability persists on any contour Σ at a finite distance.

A more significant situation is found when $\widehat{C}$ is the zero line of the field $\widehat{u}(r, \varphi)$, which has no singularities in a finite region only. We shall apply the name special and the symbol $\widehat{C}$ also to these fields. On any contour Σ lying in this region the system of fields is incomplete. If the contour Σ widens, and even a single singular point of the field $\widehat{u}(r, \varphi)$ falls within it, the system of fields on Σ becomes complete (unless $\widehat{C}$ is also the zero line of another field with a greater degree of analyticity). Under these conditions increasing the distance from Σ to $\widehat{C}$ not only decreases the defect of the system of fields generated by currents on $\widehat{C}$ (i.e. their lack of completeness) but eliminates it completely.

The arc of a nonresonant circle is an example of such a line. No field $\widehat{u}(r, \varphi)$ can be equal to zero in this region, and no field without singularities within the region can exist. If $\widehat{u}(r, \varphi)$ is zero on the arc of the circle it must be zero over the whole of the circle, because both the zero line and the circle are analytic curves, and two such curves having a common arc must coincide. However, a field equal to zero on a nonresonant circle must have some singularities within it. It can easily be shown that the centre of the circle is a singular point. Currents on the arc of a nonresonant circle give rise to an incomplete system of fields on any contour that does not include the centre of the circle; a complete system of fields is obtained when the centre falls within the contour.

3.3 Regeneration of the wave field from its zero line

Thus, the approximation problem gives rise to the following problem in the theory of the analytic properties of wave fields: we must establish whether a given line is the zero line of some field and if it is (i.e. if the line is special) we must find this field and in particular the region in which it has no singularities.

In any case, a line can be special only if it consists of arcs of analytic curves, and at the break point the angle between them must be a rational fraction of π. The closed special line should be a resonant contour. However, these two conditions are not sufficient.

If $\widehat{C}$ coincides with the line in the coordinate system in which Eqn (1.1) divides into branches, the field $\widehat{u}(r, \varphi)$ which satisfies condition (2.4) can easily be constructed in an explicit form. For example, if $\widehat{C}$ is an arc of the boundary, we can equate $\widehat{u}(r, \varphi)$ to $J_\nu(kr)\cos\nu\varphi$ or $N_\nu(kr)\cos\nu\varphi$, and find the index ν from the equation $J_\nu(ka) = 0$ or $N_\nu(ka) = 0$, where a is the radius of the circle. A field equal to zero on ellipses or hyperbolas can be constructed in a similar way.

If $\widehat{C}$ is not a closed line, it can be converted in many ways into a closed resonant contour C_0. The problem then becomes to define an analytic continuation of the eigenmode field outside C_0, i.e. to construct a field which is zero on C_0 and has the same value of the normal derivative. This is the Cauchy problem outside C_0, formulated for values of $\widehat{u}$ and $\partial\widehat{u}/\partial N$ on C_0 such that within C_0 this problem has a solution without singularities. An analytic continuation may not exist, it may have singularities at a finite distance from C_0 (C_0 is a curvilinear rectangle consisting of the arcs of two concentric nonresonant circles and parts of two radii) or at infinity (C_0 is a rectangle), or it may have no singularities over the entire surface (resonant circle). A closed line can be special only on a denumerable set of frequencies, an open line can be special only on a band of frequencies.

3.4 Analytic continuation of the eigenmode field

This problem has been discussed in several mathematical papers. For a given type of contours C_0, various authors used the methods of the analytic theory of the solution of the wave equation as a function of two complex variables $x = x' + \mathrm{i}x''$ and $y = y' + \mathrm{i}y''$ [6]. In the region of complex x and y the elliptic equation (1.1) (like a hyperbolic equation) has complex characteristics along which the singular points of the field can move. Special points with real coordinates can be treated as 'traces' of these characteristics.

Another method can be constructed by the following scheme. First we choose a nonresonant circle with its centre at the origin of the coordinate system and lying wholly within C_0. The field $\widehat{u}(r, \varphi)$ on it can be expressed as the series $\widehat{u}(a, \varphi) = \sum_n B_n \cos n\varphi$ (a is the radius of the circle), where the B_n are known or the field $\widehat{u}(r, \varphi)$ is known. At all the (r, φ) for which the series which differs from the series, for $\widehat{u}(a, \varphi)$ by the factors $J_n(kr)/J_n(ka)$, is convergent and differentiable by terms this series is the required analytic continuation, and the problem reduces to establishing the convergence region of this series. By applying the Debye asymptotic to $J_n(kr)$ and to $J_n(ka)$ we find that for any finite r this series converges simultaneously with the series $\sum_n B_n(r/a)^n$. If the number

$$l = \lim_{n\to\infty} |B_n|^{1/n}$$

is not zero, its general term is of the order of $(rl/a)^n$, so that the nearest singular point of the analytic continuation is at a distance a/l: nearer, the greater l is, i.e. the more slowly decreasing are the B_n coefficients. On the other hand if the B_n decrease so rapidly that $l = 0$, the analytic continuation has no singular points at a finite distance. To establish its convergence at infinity we must use the Hankel asymptotic. At $r \to \infty$ the series for the analytic continuation converges simultaneously with the series $\sum (a_0/a)^n$, where

$$a_0 = \frac{2}{\mathrm{e}k} \lim_{n\to\infty} \left(n|B_n|^{1/n}\right), \qquad \mathrm{e} = 2.7\ldots \tag{3.2}$$

(see Ref. 7, where problems of this type are discussed). The analytic continuation has singularities at infinity (the patterns are approximable) if $a_0 \geqslant a$, but not if $a_0 < a$ (when the patterns are nonapproximable).

The above methods are not properly posed (or, more strictly, they are correct only for analytic deformations of C_0). This is not a defect of the methods, but stems from the formulation of the special lines problem used above. However, as will be shown in Section 4, the two assertions "C is a special line" and "C is close to a special line" are from the physical point of view identical. In this sense the 'crudeness' of the method of calculation used below to construct $\widehat{u}(r, \varphi)$ from its zero line can be interpreted as an advantage of the approach.

The method is based on Kupradze's method of subsidiary sources in the variant developed, for example, in Ref. [8]. The field $\widehat{u}(r, \varphi)$ is expressed as a sum of cylindrical waves $a_n H_0^{(2)}(k\rho_n)$ $(n = 1, 3, \ldots, N)$ where ρ_n is the distance from the nth source; all the sources are located on some boundary Σ surrounding C. The amplitudes a_n are found from the requirement that $\widehat{u}(r, \varphi)$ must be zero at N points on C. If no singular points of this field are found within Σ, the system of N homogeneous equations for a_n has a nontrivial solution, stable at $N \to \infty$. If the line C is closed this solution is possible only at discrete resonant frequencies; the method can also be applied to open C curves.

It is easy to show that the function $\widehat{W}_N(\sigma)$, calculated from Σ by the equation

$$\widehat{W}_N(\sigma) = \frac{l_\Sigma}{N} \sum^{N} a_n \delta(\sigma - \sigma_n) , \tag{3.3}$$

(where l_Σ is the length of the boundary, and σ_n is the coordinate of the nth source) plays the role of a 'discrete orthogonal complement function' in the sense that the discrete orthogonality condition in the form $\sum_n u(\sigma_n)a_n = 0$ applies to the field $u(s)$ induced on Σ by arbitrarily chosen points on C. We can introduce the normalised (not containing δ-functions) $\widehat{W}_N(\sigma)$ function, equal in some sense to the limit of $\widehat{W}_N(\sigma)$ at $N \to \infty$, by stipulating that its value at $\sigma = \sigma_n$ is a_n. For $N \to \infty$ this function has the same significance as in Section 3.1.

This generalisation of the method [8] can be used to determine $\widehat{u}(r, \varphi)$ and $\widehat{F}(\varphi)$ even when $\widehat{C}$ is the zero line of a field which has no singularities over the entire surface. In this case $\widehat{u}(r, \varphi)$ must be expressed as a sum of plane (instead of cylindrical) waves:

$$\widehat{u}(r, \varphi) = \frac{2\pi}{N} \sum_{n=1}^{N} a_n \exp\left[\mathrm{i}kr\cos(\varphi - \varphi_n)\right] . \tag{3.4}$$

If $\widehat{u}(r, \varphi)$ is everywhere analytic, the system of equations for a_n has a nontrivial solution and is stable for $N \to \infty$. We can construct a function $\widehat{F}_N(\varphi)$ analogous to $\widehat{W}_N(\sigma)$ [Eqn (3.3), $l_\Sigma \to 2\pi$, $\sigma_n \to \sigma_n$] and find its 'limit' for $N \to \infty$, i.e. the function $\widehat{F}(\varphi)$ to be normalised, by imposing the condition $\widehat{F}(\varphi) = a_n$ at N points. This function has the same significance as in Section 2.

4. Current norm

The concept of approximability was introduced into the theory of fields as a reaction to the incorrect formulation of the realisability property. To a physicist there is no difference between realisability and nonrealisability of a given diagram if realisable diagrams as close to that diagram as necessary can be obtained in the latter case†. However, even the concept of approximability, which generalises the concept of realisability, is suspect from a physicist's point of view, though to a smaller extent. First, often a small (though finite) perturbation of the diagram can allow it to be approximated by a current with a small norm (1.4) $(N \approx 1)$, and under these conditions it is unimportant whether the initial diagram was nonapproximable $(N = \infty)$ or whether it was approximable but with a very large norm $N < \infty$ but $N \gg 1$). Second, a small perturbation of a special line can make it not special, i.e. it can change nonapproximable $(N = \infty)$ diagrams into approximable diagrams, (but with $N \gg 1M$ a situation almost indistinguishable from the first from the physical point of view). The concepts of optimum current synthesis and of region of influence of the special line, discussed in this section, are generalisations of the 'approximability' and 'special line' concepts and add physical reality to the mathematical model.

†The writer gratefully acknowledges the contribution of A F Chaplin, recently deceased, whose untiring support of these views stimulated much of the work in this direction.

4.1 Minimum current norm for a given precision of the approximation

The line C and the pattern $F(\varphi)$ are given; all the patterns discussed in the subsection are normalised to unity on the modulus. The problem of the optimum current synthesis consists essentially in replacing $F(\varphi)$ by the realisable pattern $\widetilde{F}(\varphi)$, separated from $F(\varphi)$ by not more than a distance δ and such that the norm $N(\delta)$ of the current that produces it is the smallest of the norms for all the currents giving rise to these patterns. If even for a small $\delta\,(\lesssim \frac{1}{2})$ the $N(\delta)$ value becomes small, it is convenient to construct $\widetilde{F}(\varphi)$ rather than $F(\varphi)$, whether $N(0) = \infty$ or whether $N(0) < \infty$ but $N(0) \gg 1$.

Of course, $N(\delta)$ is not a rising function. At small δ values its form depends only on whether or not the initial pattern $F(\varphi)$ is realisable by the line C. If $F(\varphi)$ is not realisable but is approximable we have $N(0) = \infty$, but for any $\delta > 0$ we have $N(\delta) < \infty$; the curve $N(\delta)$ has a vertical asymptote at $\delta = 0$. If C is a special line, and $F(\varphi)$ is not approximable $[(F, \widehat{F}) \neq 0]$, there are no realisable patterns in the range $0 < \delta < \Delta$; here the gap width Δ is given by formula (2.3) with $(f, \widehat{F})$ replaced by $(F, \widehat{F})$. This approximable function nearest to $F(\varphi)$ (not normalised) is $F(\varphi) - (F, \widehat{F})\,\widehat{F}(\varphi)$. If it is realisable, i.e. if the lower limit of the distance between $F(\varphi)$ and $\widetilde{F}(\varphi)$ is accessible, we have $N(\Delta) < \infty$. If it is unrealisable we find that $N(\Delta + \varepsilon) < \infty$, i.e. the curve $N(\delta)$ has a vertical asymptote $\delta = \Delta$.

For any line C we can construct a realisable pattern [which we shall call $\widetilde{F}_{\min}(\varphi)$] for which the norm of the generating current is a minimum among the norms of all the currents on C which produce patterns normalised to unity. When δ is greater than the distance between $F(\varphi)$ and $\widetilde{F}_{\min}(\varphi)$ the optimum current synthesis problem becomes meaningless, $N(\delta)$ no longer decreases as δ increases, and the optimum $\widetilde{F}(\varphi)$ function becomes equal to $\widetilde{F}_{\min}(\varphi)$.

4.2 Systems of orthogonal functions for currents and patterns

We shall now describe a mathematical procedure which provides a formal solution of the optimum current synthesis problem for any line C. We introduce [9] the two systems of functions $\psi_m(\varphi)$ and $j_n(s)$, full and orthonormalised respectively in the range $0 \leqslant \varphi \leqslant 2\pi$ and on C:

$$(\psi_m\,,\ \psi_q) = \delta_{mq}\,, \tag{4.1a}$$

$$(j_n\,,\ j_p) = \delta_{np}\,, \tag{4.1b}$$

such that each current $j_n(s)$ generates [by formula (1.3a)] a pattern proportional to one of the functions of the system $\psi_m(\varphi)$. The scalar product in Eqn (4.1b) is constructed as in Eqn (2.1), but the integration is taken over s rather than over φ.

We now write the relationship between the pattern and the current in the form $F = K_j$, where K is the integral operator in Eqns (1.3a) and (1.3b), and we introduce the operator K^C conjugate to it with respect to the scalar products which contribute to Eqns (4.1), i.e. an operator which converts a pattern into a current (or, more exactly, into a function on C), so that the identity

$$(K^C F, j) = (F, Kj)\,, \tag{4.2}$$

applies to any function $F(\varphi)$ and $j(s)$. The integral operator K^C differs from Eqns (1.3a) and (1.3b) in that the integration is carried out over φ instead of over s, and the kernel $\mathcal{K}$ is replaced by its complex conjugate.

The function $\psi_m(\varphi)$ and $j_n(s)$ will be defined as the eigenfunctions of the self-conjugated operators KK^C and K^CK respectively. They are associated with the eigenvalues μ_m and λ_n, nonnegative and tending towards zero at $m \to \infty$ and at $n \to \infty$. These numbers are equal in pairs in the sense that for every number m we find a number n [which we shall call $n(m)$] such that $\lambda_{n(m)} = \mu_m$; at the same time we define $m(n)$ so that $\mu_{m(n)} = \lambda_n$. It is easy to show that $Kj_{n(m)} = \sqrt{\mu_m}\psi_m(\varphi)$ and $K^C\psi_{m(n)} = \sqrt{\lambda_n}j_n(s)$.

Lines C for which the eigenvalues include zero values are of special interest in our problems. Therefore, we shall arrange the function $\psi_m(\varphi)$ and $j_n(s)$ not in the order of decreasing eigenvalues (since in that case the function corresponding to the zero eigenvalue would not have a number) but in the order of increasing complexity (for example, in the order of increasing integral of the square of the modulus of their derivative).

If the line C has the property that one of the numbers μ_m is equal to zero ($\mu_q = 0$), it is a special line. Any current on C gives rise to patterns which do not approximate to the function $F(\varphi)$ if $(F, \psi_q) \neq 0$. The symmetric situation can also be examined in this formalism: among the λ_n numbers one is equal to zero ($\lambda_p = 0$). Obviously, under these conditions a nonradiating current $\widehat{j}(s)$ can exist on C, i.e. a current such that $K\widehat{j} \equiv 0\,[\,\widehat{j}(s) = j_p(s)]$. This means that C is a resonant contour, as the current flowing on it at its eigenmode (and equal to $\partial\widehat{u}/\partial N$) does not generate a field outside C. Any cylindrical wave arriving from infinity generates a field on the resonant contour which does not approximate the function $J(s)$ if $(J, j_p) \neq 0$.

These two symmetry properties of the lines are independent, though in simple contours of the ellipse type they appear at equal frequencies. Thus, for an open special line $\mu_q = 0$, but there is no nonradiating current on it and $\lambda_n \neq 0$ for all n. If $\mu_q = 0$, but $\lambda_n \neq 0$, it means that $n(q) = \infty$: on this special line the 'nonradiating current' $\widehat{j}(s)$ changes infinitely rapidly along C, and therefore it does not generate a field. On the other hand, the resonant contour ($\lambda_p = 0$) formed by a nonanalytic line is not special, and $\mu_m \neq 0$ for all m. If $\lambda_p = 0$ but $\mu_m \neq 0$ we have $m(p) = \infty$, and for this resonanct contour the 'orthogonal complement function' $\widehat{F}(\varphi)$ changes infinitely rapidly as φ is increased and is therefore orthogonal to any function.

There is a simple relationship between the procedure described in this section and the procedure based on the analytic properties of the fields $u(r, \varphi)$ and $\widehat{u}(r, \varphi)$. If the kernel (1.3b) in the operator K is replaced by $H_0^{(2)}(k\rho)$, where ρ is the distance from a point on C to any point, the action of this operator on $j(s)$ gives the field $\widehat{u}(r, \varphi)$ over the whole surface. If in the kernel (1.3b) of the operator K^C we replace $r(\theta)$ by the coordinate of an arbitrary point, then the action of this operator on functions $\widehat{F}(\varphi)$ gives the field $\widehat{u}(r, \varphi)$ over the whole surface. The simultaneous use of both methods of analysis simplifies the study of the properties of both functions $\psi_m(\varphi)$ and $j_n(s)$ and the numbers μ_m, λ_n, and of the fields $u(r, \varphi)$ and $\widehat{u}(r, \varphi)$.

We constructed the systems $\psi_m(\varphi)$ and $j_n(s)$ by using the operator K, which translates the current into a pattern, and these systems are particularly convenient for solving the problem of synthesis from a given pattern. For problems associated with the near field we can use the same method to construct the systems of function on Σ and on C, with the use of an operator which translates the current into a field in the near region [10].

4.3 General solution of the problem of the optimum current synthesis

We shall denote the coefficients of the Fourier expansions $F(\varphi)$ and $\widetilde{F}(\varphi)$ into series of the function $\psi_m(\varphi)$ as A_m and $\widetilde{A}_m$ respectively, so that $A_m = (F, \psi_m)$ and $\widetilde{A}_m = (\widetilde{F}, \psi_m)$. The norm of the current which produces the pattern $F(\varphi)$ and the distance between $F(\varphi)$ and $\widetilde{F}(\varphi)$, expressed in terms of these coefficients, are

$$N = \left(\sum_m |\widetilde{A}_m|^2 \mu_m^{-1}\right)^{1/2}, \tag{4.3a}$$

$$\delta = \left(\sum_m |A_m - \widetilde{A}_m|^2\right)^{1/2}. \tag{4.3b}$$

According to Eqn (4.3a) the function $\psi_m(\varphi)$, corresponding to the maximum value of μ_m, is the pattern which we have called $\widetilde{F}_{\min}(\varphi)$.

The numbers A_m are given: we must find the numbers $\widetilde{A}_m$ which minimise the functional N [Eqn (4.3a)] for a given unitary norm of $\widetilde{F}(\varphi)$ and a given distance δ. Reducing this problem from a search for a nominal extremum by Lagrange's method to a search for an unconditional extremum of some more complex functional gives

$$\widetilde{A}_m = A_m \frac{\alpha}{1 + \beta\mu_m^{-1}}, \tag{4.4}$$

where α and β are real constants depending on all the A_m and on δ. They must be found by introducing Eqn (4.4) into condition (4.3b) and into the normalisation condition.

Replacing $F(\varphi)$ by $\widetilde{F}(\varphi)$ changes [according to Eqn (4.4)] all the Fourier coefficients: the coefficient corresponding to the greatest μ_m is the least affected, and the coefficients with small values of μ_m are the most affected, especially the higher terms of the series. If C is a special line, i.e. if $\mu_q = 0$, we have $\widetilde{A}_q = 0$, as must be the case for a realis-able $\widetilde{F}(\varphi)$ pattern. For $\delta < \Delta$ [Eqn (2.3)] there is no solution with a finite current norm and, for $\delta = \Delta$, $\widetilde{F}(\varphi)$ differs from $F(\varphi)$ only by the absence of the term $A_q\psi_q(\varphi)$ and by a proportional change in the other Fourier coefficients.

The problem of the optimum current synthesis can be solved by another method [11], expressing N, condition (4.3b), and the normalising condition directly as quadratic functionals of the current which produces the required pattern [$N^2 = (\widetilde{j}, \widetilde{j})$, etc.]. The Lagrange method gives the Euler equation for $\widetilde{j}(s)$:

$$\widetilde{j} + l_1 K^C K \widetilde{j} + l_2 (K^C K \widetilde{j} - K^C F) = 0, \tag{4.5}$$

where l_1 and l_2 are the Lagrange multipliers. They must be found by solving Eqn (4.5) simultaneously with condition (4.3b) and the normalisation condition. Eqn (4.5) can be solved as a series expansion of $\widetilde{j}(s)$ in terms of $j_n(s)$, which leads to the initial formula (4.4), or by various other methods. In this variant the introduction of K^C and $K^C K$ into the theory of operators is quite simple. Euler equations of the type of Eqn (4.5) can be obtained also for current norms more complex than Eqn (1.4), containing the integral with respect to $|\mathrm{d}\widetilde{j}/\mathrm{d}s|^2$.

These methods, based on formulas (4.4) and (4.3a), were used [1] to calculate $N(\delta)$ for several functions $F(\varphi)$ and for lines C in the form of nonresonant ($ka = 1$) and resonant ($ka = 3.83$) circles and in the form of arcs. For this type of resonant circle and for its arcs we have $\widehat{F}(\varphi) = \cos\varphi$.

As could have been expected, for $ka = 1$, $N(\delta)$ decreases more rapidly with δ the wider the pattern. For example, in a Π-shaped (i.e. nonrealisable) pattern [$N(0) = \infty$] of width $2\gamma = 3.0$ rad, $N^2 = 2.1$ even for $\delta^2 = 0.1$, and a narrower pattern of width $2\gamma = 2$ can be approached only to within approximately $\delta^2 = 0.3$ without large currents ($N^2 < 2$). The same is true of the Gaussian pattern $F(\varphi) \approx \exp(-A\sin^2\varphi/2)$. Although it is realisable by currents on the $ka = 1$ boundary only when $A < 2$, and narrower patterns with $A > 2$ are nonrealisable, even at $\delta^2 = 0.1$ the difference between the N^2 values for these patterns [$A = 1.51$, $N(0) < \infty$ and $A = 2.2$, $N(0) = \infty$] practically disappears. Even for a very narrow pattern [$A = 5.67$, $N(0) = \infty$] a pattern differing from it by $\delta^2 = 0.1$ can be generated by moderately large currents. The $N(\delta)$ curves for the arc of the circle have the same character as the curves for the whole circle. The current norm needed to approximate any pattern to within a given (finite) accuracy is, of course, greater for the arc than the circle.

In the case of a special line, i.e. for a resonant circle or for its arc, we find a more complicated state of affairs. The $N(\delta)$ curves for the Π-shaped pattern have vertical asymptotes $\delta = \Delta$, and the quantity Δ, depending on the product $(F, \widehat{F})$, decreases nonmonotonically with the width of the pattern. For example, $\Delta = 0.51$, 0.72, and 0.69 for $\gamma = 0.5$, 1.0, and 1.5 respectively (for the whole circle). As the pattern is widened from $\gamma = 0.5$ to $\gamma = 1.0$ the gap becomes wider, i.e. the vertical asymptote is shifted towards larger δ values. Although, generally speaking, the wider the pattern the more rapidly N decreases with increase in δ (for $\delta > \Delta$), this shift of the asymptote makes it possible for the narrower pattern to be approached to within a given distance δ with a lower current than the wider pattern.

If C is not a special line the $N(\delta)$ curves have the same asymptote $\delta = 0$ for all the Π-shaped patterns. However, if the curve C is geometrically close to the resonanct circle or to its arc the inversion of the distribution of the $N(\delta)$ curves for different γ is retained, at least up to moderately large δ values. In general, fields created by a line close to $\widehat{C}$ have some properties close to those of fields created by $\widehat{C}$.

Other formulations of the optimum current synthesis problem are possible. For example, we may stipulate a maximum concentration of the energy in a given solid angle [12] rather than closeness of the resulting pattern to the given pattern. If the irradiation takes place not in free space but in a waveguide, the corresponding requirement is the maximisation of the energy in a given group of waves in the waveguide (for a fixed distribution of the current). In the latter form of the problem we can use as a basis the eigenfunctions of an operator [13, 14] analogous to the operator K.

4.4 Region of influence of the special line

The smallest μ_m corresponding to the line C can be used as a characteristic of the nearness of the line C to any special line $\widehat{C}$, since the zero eigenvalue μ_q exists for $\widehat{C}$. However, the use of the systems $\psi_m(\varphi)$ and $j_n(s)$ is justified only in theoretical arguments, because in applied problems finding the functions $\psi_m(\varphi)$ and the numbers μ_m for a given line C requires cumbersome calculations. It is easier to use any simple orthonormalised system of functions $\chi_m(\varphi)$ (for example, trigonometric functions) not associated with the line C. In order to evaluate by this method the influence of the nearness of C to a special line without solving the integral equation for $j(s)$ we use the following inequality for the norm of the current giving rise to the $F(\varphi)$ pattern:

$$N^2 \geqslant \frac{|(\Phi, F)|^2}{(K^C\Phi, K^C\Phi)} . \tag{4.6}$$

This can easily be obtained by multiplying the equation $Kj = F$ by $\Phi(\varphi)$, with the use of definition (4.2) of the conjugated operator K^C and the Cauchy inequality $(f_1, f_1) \times (f_2, f_2) \geqslant |(f_1, f_2)|^2$. In inequality (4.6) $\Phi(\varphi)$ is any function. Writing it in the form

$$\Phi(\varphi) = \sum_m^M C_m \chi_m(\varphi) ,$$

converts inequality (4.6) into

$$N^2 \geqslant \left(\sum^M\sum A_m A_n^* C_n C_m^*\right)\left(\sum^M\sum \beta_{nm} C_n C_m^*\right)^{-1} , \tag{4.7}$$

where A_n are the coefficients of the expansion of $F(\varphi)$ in a series of $\chi_n(\varphi)$, and $\beta_{nm} = (K^C\chi_n, K^C\chi_m)$.

By appropriate choice of the functions $\Phi(\varphi)$, i.e. of the coefficients C_n ($n = 1, 2, \ldots, M$), we can maximise the right-hand side of inequality (4.7). As is well known, the maximum of this ratio of two quadratic forms, which we shall call $X^{(M)}$, is the largest root of the algebraic equation of the Mth order

$$\det \left| A_m A_n^* - X^{(M)}\beta_{mn} \right|_M = 0 . \tag{4.8}$$

Inequality (4.6) means that $N^2 \geqslant X^{(M)}$. As M increases, $X^{(M)}$ also increases (or, at least, does not decrease), and this inequality becomes more informative, i.e. it gives a more precise estimate of N^2. An important property of the inequality is that, as can easily be shown, it becomes an equality for $M \to \infty$. The quantity $X^{(M)}$ characterises the norm of the current which must be distributed on C in order to approximate the $F(\varphi)$ pattern.

If the system $\chi_m(\varphi)$ corresponds to $\psi_m(\varphi)$ we have $\beta_{nm} = \mu_m\delta_{nm}$, and if $\mu_q = 0$, i.e. if the line C is a special line, Eqn (4.8) has the root $X^{(M)} = \infty$ (if $A_q \neq 0$). This simply means that a pattern whose Fourier series contains $\psi_q(\varphi)$ is nonapproximable. The region surrounding the special line within which its influence is detectable (i.e. N is not large) depends on the complexity of the $F(\varphi)$ pattern, or, more precisely, on the relative values of its higher Fourier coefficients. If M is the number of the highest significant coefficient, i.e. of the term after which the series can be truncated with only a slight perturbation to $F(\varphi)$, the characteristic of this region is the number $X^{(M)}$.

We can determine a norm N independent of the particular form of the function $F(\varphi)$ and dependent only on this number M. Eqn (4.8) has the root $X^{(M)} = \infty$ if the condition $\det|\beta_{nm}|_M = 0$ is satisfied. Of course, for $M \to \infty$ the condition is satisfied for any line C, which simply means that large currents are needed to produce very complex patterns. However, if when M is increased this determinant becomes small for moderately large values of M, and its calculation remains stable, the line C is close to some special line.

A characteristic for N somewhat more precise than the value of $\det|\beta_{nm}|_M$, also depending only on M and not on the function $F(\varphi)$, can be obtained if inequality (4.6) is supplemented by replacing (Φ, F) by (Φ, Φ) in the numerator: in this case we must assume that $(F, F) = 1$. The highest value of the right-hand side becomes the largest root $Y^{(M)}$ of the Mth-order equation

$$\det \left| \delta_{nm} - Y^{(M)}\beta_{nm} \right|_M = 0 . \tag{4.9}$$

For large $Y^{(M)}$ the Mth-order polynomial on the left-hand side can be truncated after the second leading term. The $Y^{(M)}$ value is large for any line close to $\widehat{C}$, and decreases more slowly when M is large. The size of the region of influence of the special line depends on the class of the functions to be approximated. The simplest characteristic of this class is the number M, and the parameters $X^{(M)}$, $\det|\beta_{nm}|_M$, and $Y^{(M)}$ are explicitly dependent on this number.

5. The electromagnetic field. The Maxwell equations

5.1 Fundamental result

For the scalar problem it was shown that if a line C has one of the properties formulated below it also has the second property:

(a) All the currents distributed on $\widehat{C}$ induce on the closed line Σ surrounding $\widehat{C}$ an electromagnetic field u to which any function U given on Σ can be approximated with a precision not greater than that specified by inequality (3.1).

(b) There is a solution $\widehat{u}$ of the homogeneous Helmholtz equation which has no singularities within Σ, and such that condition (2.4) is satisfied on $\widehat{C}$.

All the theory developed above was based on this assertion, whose proof is elementary and which uses only Green's formula. It can be generalised to three-dimensional scalar problems by replacing the word 'line' by the word 'surface'.

The corresponding statement for the three-dimensional vector problem is the basis of the approximability theory of the vector field. Its proof is also elementary, but Green's formula is replaced by Lorentz's lemma. We shall not give this proof, but simply formulate the results. In spite of being outwardly cumbersome this formulation simply mirrors the formulation given above for the scalar problem. In the paragraph following formula (5.2) the con-struction described after inequality (3.1) is repeated, but the solution of Dirichlet's external problem is replaced by the solution of the first boundary condition for Maxwell's equations. The next paragraph gives an extension of the results from near fields to far fields.

If a surface $\widehat{C}$ has one of the two properties formulated below, it also has the other.

(a) Any current distributed on $\widehat{C}$ generates on the closed surface Σ surrounding $\widehat{C}$ an electric field $\boldsymbol{e}$ to which any pair of functions $E_1(\sigma)$, $E_2(\sigma)$ given on Σ (where σ is the coordinate of the point on Σ) can be approximated with a precision not greater than that specified by the inequality

$$\int_\Sigma \left(|E_1 - e_{t_1}|^2 + |E_2 - e_{t_2}|^2\right) \mathrm{d}\sigma \geqslant \left| \int_\Sigma \left(E_1\widehat{F}_1^* + E_2\widehat{F}_2^*\right) \mathrm{d}\sigma \right|^2 , \tag{5.1}$$

where t_1 and t_2 are two directions tangential to Σ, and the functions $\widehat{F}_1(\sigma)$ and $\widehat{F}_2(\sigma)$ are determined on Σ, independent of the currents, and normalised to the condition that the integral obtained by replacing E_1 and E_2 by $\widehat{F}_1$ and $\widehat{F}_2$, respectively, on the right-hand side of inequality (5.1) is equal to unity.

(b) There is a solution $\widehat{\boldsymbol{E}}$, $\widehat{\boldsymbol{H}}$ of the homogeneous Maxwell equation which has no singularities within Σ and such that on $\widehat{C}$ the two tangential components of the field $\widehat{\boldsymbol{E}}$ are both zero:

$$\widehat{\boldsymbol{E}}_{\text{tang}}\Big|_{\widehat{C}} = 0\,. \tag{5.2}$$

The functions $\widehat{F}_1(\sigma)$ and $\widehat{F}_2(\sigma)$ in inequality (5.1) and the field $\widehat{\boldsymbol{E}}$ in Eqn (5.2) are unambiguously related. These functions are equal to the difference between the values on Σ of the tangential components of two magnetic fields: the field $\widehat{\boldsymbol{H}}$ (within Σ) and another field obtained outside Σ from the problem of the electromagnetic field which satisfies the irradiation condition and which has (on Σ) the same value of the tangential component of the electric field as $\widehat{\boldsymbol{E}}$. But the field $\widehat{\boldsymbol{E}}$ within Σ is equal to the field produced by the currents with components, $\widehat{F}_2(\sigma)$ and $\widehat{F}_1(\sigma)$, distributed on Σ.

If the field $\widehat{\boldsymbol{E}}$, $\widehat{\boldsymbol{H}}$ having the property (5.2) has no singularities in the entire space, condition (5.1) can be applied not only to the field $\boldsymbol{e}$ at a finite distance from $\widehat{C}$ but also to the patterns, and therefore σ is replaced by (θ, φ) in inequality (5.1); $E_1(\theta, \varphi)$ and $E_2(\theta, \varphi)$ are the given pattern (i.e. its θ- and φ-components), and $\widehat{F}_1(\theta, \varphi)$, $\widehat{F}_2(\theta, \varphi)$, are the angular dependences of those components in the asymptotic for $r \to \infty$ of the field $\widehat{\boldsymbol{E}}$, which correspond to a converging spherical wave.

The existence of a field which satisfied condition (5.2) can be proved for many surfaces. The equivalence of conditions (5.2) and (5.1) means that these surfaces also display the property of nonapproximability.

5.2 Trivial generalisations

The result given in the last subsection is a reformulation for the three-dimensional vector problem of one of the results obtained above for the two-dimensional scalar model. In general this transformation involves only one basic difficulty, which we shall discuss in subsection 5.3. The automatic nature of this transposition is what makes it nontrivial. It means that similar results may be formulated for acoustic, seismic, and other fields described by linear equations. Essentially these results follow directly from the reciprocity theorem. They must be allowed for when the sources of the field are contained in a region smaller than the region occupied by the field.

We shall give two corollaries for surfaces which satisfy condition (5.2) and therefore also condition (5.1), i.e. such that the complete system of currents on these surfaces generates an incomplete system of patterns.

(a) The surface of the antenna should not be a special surface, or close to a special surface. When applied to a conical horn this means that one-half of the angle of divergence α should not be a root of Eqns (5.3a) or (5.3b):

$$P_n^m(\cos\alpha) = 0\,, \tag{5.3a}$$

$$\frac{\mathrm{d}P_n^m(\cos\theta)}{\mathrm{d}\theta}\bigg|_{\theta=\alpha} = 0\,, \tag{5.3b}$$

where P_n^m are the associated Legendre functions, m ($m = 0, 1, \ldots$) defines the dependence of the pattern ($\sim\cos m\varphi$) on the azimuthal angle φ, and n is a small integer. For $m = 0$ the 'forbidden' angles are $\alpha = 55°$ ($n = 2$); 39°, 63° ($n = 3$), etc. For $m = 1$ we find $\alpha = 63°$, 31° ($n = 3$), etc.

(b) If a surface $\widehat{C}$ has a high degree of symmetry it can show amplitude nonapproximability for narrow patterns, when no conceivable choice of phase can make it approximable by the patterns of currents distributed on $\widehat{C}$. For example, if $\widehat{C}$ consists of three mutually perpendicular planes and the pattern has an amplitude $\exp[-A\sin^2(\theta'/2)]$, where the angle θ' is measured from a direction making equal angles with all three lines of intersection of the planes, amplitude approximability occurs only for patterns with a half-width greater than 34°. This limitation was found to be less intrusive than in the corresponding two-dimensional problem [see Eqn (2.4)].

5.3 Properties of the special surfaces

Many properties of the special surfaces mirror the corresponding properties of the special lines. A closed special surface must be a resonant surface; the inverse does not always apply. A very large continuum of special surfaces can exist.

However, unlike the scalar case, it is known that for an arbitrary vector field $\widehat{\boldsymbol{E}}$ (even if real), there are no surfaces perpendicular to $\widehat{\boldsymbol{E}}$ at every point, i.e. such that both conditions (5.2) are satisfied on them. The conditions for which these surfaces exist contains $\operatorname{curl}\widehat{\boldsymbol{E}}$. For fields which obey Maxwell's equations it takes the form $\widehat{\boldsymbol{E}}\cdot\widehat{\boldsymbol{H}} = 0$. If it is satisfied over some volume, this is a sufficient as well as a necessary condition for the existence of special surfaces. The class which includes these fields studied by Khudak [15], is relatively limited. The opposite situation, in which $\widehat{\boldsymbol{E}}\cdot\widehat{\boldsymbol{H}} = 0$ only on some surface, is more typical. However, in that case this is only a necessary condition for making this a special surface.

Therefore, the problem of constructing $\widehat{C}$ from two given orthogonal complement functions $\widehat{F}_1(\theta, \varphi)$ and $\widehat{F}_2(\theta, \varphi)$ is much more complex than in the scalar case. If these functions are given independently, there will be no surfaces having the property (5.2) in the field $\widehat{\boldsymbol{E}}$ generated by them, even if $\widehat{\boldsymbol{E}}$ is real. For $\widehat{C}$ to exist we require that the conditions (5.2) should also be satisfied on the surface for which $\widehat{\boldsymbol{E}}\cdot\widehat{\boldsymbol{H}} = 0$. Only one of the functions $\widehat{F}_1$, $\widehat{F}_2$ can be fixed arbitrarily [and only with some limitations, similar to that stipulated above after formula (2.6)]. This fact complicates most seriously the realisation of one of the methods of identifying the surface of a scatterer from the measured graph described above. In the two variants of this method the orthogonal supplement functions (either close to the pattern or orthogonal to it) are found, and the appropriate special lines are then constructed. Under these conditions $\widehat{F}(\varphi)$ should experience only a very slight limitation, stipulating the reality of the field $\widehat{u}(r, \varphi)$. In the three-dimensional vector problems the functions $\widehat{F}_1(\theta, \varphi)$ and $\widehat{F}_2(\theta, \varphi)$ should be additionally related as discussed above so as to form an element of orthogonal complement space, i.e. a pair of functions corresponding to a special surface.

This relationship cannot be formulated analytically. We note that the same difficulty arose in an another problem: the constructive synthesis of resonator antennas [16]. This is typical of the easily formulated but basically difficult mathematical problems encountered in specific studies of high-frequency electrodynamics, associated with synthesis problems in different formulations.

References

1. Katsenelenbaum B Z, Shalukhin M Yu *Radiotekh. Elektron.* **33** 1878 (1988);
Shalukhin M Yu *Radiotekh. Elektron.* **35** 313 (1990)
2. Katsenelenbaum B Z, Shalukhin M Yu *Pis'ma Zh. Tekh. Fiz.* **14** 2012 (1988); [*Sov. Tech. Phys. Lett.* **14** 872 (1988); *Radiotekh. Elektron.* **34** 225 (1989); Katsenelenbaum B Z *Radiotekh. Elektron.* **34** 1370 (1989);
Pis'ma Zh. Tekh. Fiz. **15** (11) 17 (1989) [*Sov. Tech. Phys. Lett.* **15** 420 (1989)];
Shalukhin M Yu *Radiotekh. Elektron.* **35** 493 (1990);
Katsenelenbaum B Z, Shalukhin M Yu *Pis'ma Zh. Tekh. Fiz.* **16** (5) 60 (1990) [*Sov. Tech. Phys. Lett.* **16** 189 (1990)];
Katsenelenbaum B Z *Radiotekh. Elektron.* **35** 2427 (1990); **37** 71 (1992); **38** 998 (1993);
Katsenelenbaum B Z, Shalukhin M Yu *IEICE Trans. (Jpn.)* **74** 2910 (1991)
3. Apel'tsin U F, Kyurchan A G *Analiticheskie Svoistva Volnovykh Polei* (Analytic Properties of Wave Fields) (Moscow: Izd. Mosk. Univ., 1991)
4. Kolmogorov A N, Fomin S V *Elementy Teorii Funktsii i Funktsional'nogo Analiza* (Elements of the Theory of Functions and of Functional Analysis) (Moscow: Nauka, 1981)
5. Overfelt P, White D *IEEE Trans. Microw. Theory Tech.* **MTT-34** 161 (1986)
6. Kyurchan A G, Sternin B Yu, Shatalov V E *Radiotekh. Elektron.* **37** 777 (1992);
Savina T V, Sternin B Yu, Shatalov V E *Radiotekh. Elektron.* **38** 229 (1993);
Sternin B Yu, Shatalov V E *Springer Lect. Notes Math.* **1520** 237 (1992)
7. Katsenelenbaum B Z, Polishchuk I M *Radiotekh. Elektron.* **35** 2363 (1990)
8. Zaridze R S, Karkashadze D D, in *IX Vsesoyuznyi Simpozium po Difraktsii i Rasprostraneniyu Voln* (Ninth All-Union Symposium on the Diffraction and Propagation of Waves) (Tbilisi, 1985) Vol. 2, p. 287;
Popovidi-Zaridze R S, Karkashadze D D, Akhvlediani G Z, Khatiashvili D Sh *Radiotekh. Elektron.* **26** 254 (1981)
9. Inagaki N, Garbacz R, Posar D *IEEE Trans. Antennas Propag.* **AP-30** 571 (1982)
10. Lin D, Gorbacz R, Posar D *IEEE Trans. Antennas Propag.* **AP-38** 860 (1990)
11. Bakhrakh L D, Kremenetskii S D *Sintez Izluchayushchikh Sistem: Teoriya i Metody Rascheta* (Synthesis of Radiating Systems: Theory and Methods of Calculation) (Moscow: Sov. Radio, 1974);
Andreichuk M M, Voitovich N N *Radiotekh. Elektron.* **30** 276 (1985)
12. Bondarenko N G, Talanov V I *Izv. Vyssh. Uchebn. Zaved. Radiofiz.* **7** (2) (1964)
13. Talanov V I *Izv. Vyssh. Uchebn. Zaved. Radiofiz.* **28** 872 (1985)
14. Vdovicheva N K, Talanov V I, Fiks I Sh, Shereshevskii I A, in *Akustika Okeanskoi Sredy* (Acoustics of the Oceanic Environment) (Eds L M Brekhovskikh, I B Andreeva) (Moscow: Nauka, 1989)
15. Khudak Yu N *Radiotekh. Elektron.* **32** 225 (1987)
16. *Elektrodinamika Antenn s Poluprozrachnymi Poverkhnostyami. Metody Konstruktivnogo Sinteza* (Electrodynamics of Antennas with Semitransparent Surfaces. Constructive Synthesis Methods) (Eds B Z Katsenelenbaum, A N Sivov) (Moscow: Nauka, 1989)