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The article by Ya I Frenkel' on 'binding forces' and the theory of white dwarfs

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Abstract. The article by Ya I Frenkel' on 'binding forces' published in 1928 contained the main elements of the up-to-date theory of white dwarfs. Those results were almost unknown to astrophysicists and were later rederived by other authors.

1. Introduction

It is well known that white dwarfs are compact stars with masses of the order of the solar mass ($M \sim M_{\odot}$), and with small radii ($R \approx 5000 - 10000$ km). Their mean mass density is 10^5–10^7 g cm^{-3}, and their surface gravity is 10^8–10^9 cm s^{-2}. Powerful gravitational forces compressing such a compact star are outweighed by the pressure created by the degenerate electron gas. The properties of white-dwarf matter are outlined in Section 2.

According to the current theories, white dwarfs appear as a result of evolution of normal stars after the latter have exhausted the nuclear fuel in their central parts. White dwarfs have been successfully observed for a long time. The first estimate of a dwarf radius (of the star Sirius B) revealing that the object is extraordinarily compact was performed by Adams [1]. It is generally believed that the white dwarfs constitute a large fraction of the stars in the Galaxy (3% –10%, or even more).

The theory of the internal structure of white dwarfs is rather simple, well developed, and agrees with observations. The history of the theory's development is presented in textbooks, and is summarised in Section 3. The aim of this note is to draw attention to the article by Frenkel' [2] entitled "Application of the Pauli – Fermi theory of electron gas to the problem of binding forces" published in German in *Zeitschrift fur Physik* in 1928. This article is practically unknown among astrophysicists. Its contents are described briefly in Section 4. It is clear that Frenkel' developed almost all elements of the up-to-date theory of white dwarfs, and he did it earlier than indicated in textbooks.

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2. White-dwarf matter

If one is interested in the main features of the internal structure (but not the thermal evolution) of a white dwarf, one can neglect the rather thin outer envelope of non-degenerate gas and assume that dwarf matter consists of atoms and an almost-ideal degenerate gas of electrons and atoms at zero temperature. The atoms are almost fully ionised by the huge electron pressure: they are simply bare atomic nuclei. The nuclei, as a rule, form crystalline lattice. An elementary cell of this matter can be treated, with high accuracy, as a Wigner – Seitz cell of radius $a = (3/4\pi n_{\rm i})^{1/3}$, where $n_{\rm i}$ is the number density of the nuclei. Each cell contains a nucleus at its centre, and is filled with an electron gas of almost-uniform number density $n_{\rm e} = Z n_{\rm i}$ (Z being the atomic number of the nucleus). The density of the Helmholtz free energy U (including the rest mass contribution) and the pressure P can be written as

$$U = \rho c^2 = m_{\rm i} n_{\rm i} c^2 + U_{\rm e} + U_{\rm i} , \quad P = P_{\rm e} + P_{\rm i} , \tag{1}$$

where ρ is the mass density. The quantities $U_{\rm e}$ and $P_{\rm e}$ correspond to an ideal degenerate gas of electrons, while $U_{\rm i}$ and $P_{\rm i}$ are the Coulomb corrections due to the electrostatic interaction of electrons with electrons and with nuclei, and $m_{\rm i}$ is the mass of the nucleus. Eqns (1) determine the equation of state $P = P(\rho)$ of cold dense matter. The nuclei contribute mainly to the mass density, and the electrons to the pressure.

The state of the electrons is determined by the Fermi momentum $p_{\rm F} = \hbar(3\pi^2 n_{\rm e})^{1/3}$ or by the relativistic parameter $x = p_{\rm F}/m_{\rm e}c \approx 1.009(\rho_6/\mu_{\rm e})^{1/3}$, where $m_{\rm e}$ is the electron mass, ρ_6 is the density in units of 10^6 g cm^{-3}, and $\mu_{\rm e} = A/Z$ is the mean molecular weight (number of nucleons per electron). If $\rho \ll 10^6$ g cm^{-3}, the electron gas is nonrelativistic ($x \ll 1$), while for $\rho \gg 10^6$ g cm^{-3} it is ultrarelativistic ($x \gg 1$). The electron contribution in Eqns (1) is given by

$$U_e = P_0\left\{x(1+2x^2)(1+x^2)^{1/2} - \ln[x + (1+x^2)^{1/2}]\right\},$$
$$P_e = P_0\left\{x\left(\frac{2}{3}x^2 - 1\right)(1+x^2)^{1/2} + \ln[x + (1+x^2)^{1/2}]\right\}, \quad (2)$$

where $P_0 = m_e^4 c^5/8\pi^2\hbar^3 \approx 1.801 \times 10^{23}$ dyne cm^{-2}. In the extreme cases of nonrelativistic and ultrarelativistic electron gas the pressure is

$$P_e = \frac{8}{15}P_0 x^5 \quad (x \ll 1), \qquad P_e = \frac{2}{3}P_0 x^4 \quad (x \gg 1). \quad (3)$$

The Coulomb terms in Eqns (1) are

$$U_i = 3P_i = -\frac{9}{10}\frac{(Ze)^2}{a}n_i . \quad (4)$$

For not too low a density ($\rho \gg AZ$ g cm^{-3}), these quantities are indeed small. If, for instance, $\rho \gg 10^6$ g cm^{-3} we have $P_i/P_e \approx -0.0046 Z^{2/3} \approx -0.04$ (when $Z = 26$).

3. History from a textbook

Let us briefly recall the history of the development of the white-dwarf theory. To be specific, I shall refer to the well-known monograph of Shapiro and Teukolsky [3]. This is not entirely complete, and I shall add my own comments. In this section, we shall not mention the article of Frenkel' [2] which is discussed in Section 4.

The quantum statistics of fermions was formulated in a classical paper by Dirac [4], published in August 1926. Immediately after that, in December 1926, Fowler [5] published a paper putting forward the idea that the pres-sure in white dwarfs was produced by nonrelativistic degenerate electrons. Eqn (2) for P_e was then unknown for any degree of electron relativistic parameter x, but the extreme cases (3) could have been derived by anyone without difficulty.

Furthermore, according Shapiro and Teukolsky [3], the theory of white dwarfs was developed mainly by Chandrasekhar. First, in Ref. [6] Chandrasekhar (1931) showed that, in massive white dwarfs, the electron gas becomes relativistic (note that Shapiro and Teukolsky evidently present an incorrect reference). In 1931 Chandrasekhar [7] also predicted the existence of the upper limit of the white-dwarf mass; the same was done independently by Landau [8] somewhat later. Chandrasekhar was the first to construct accurate white-dwarf models [9, 10].

With the greatest respect to the works of Chandrasekhar one should also emphasise the fact that a very large contribution to white-dwarf theory was made by E C Stoner, an English physicist. Reading the old articles, one can see that the competition between Chandrasekhar and Stoner greatly stimulated the development of the theory. Both scientists were precise in referring to each other, but Stoner's name has been undeservedly forgotten. But it was Stoner who analysed the structure of not-too-massive white dwarfs (composed of nonrelativistic electrons) in his work [11] published in 1929. It must be admitted that he used a simplified stellar model with uniform density distribution. This model allows one to obtain qualitatively correct dependences of stellar mass on radius and central density, but it cannot yield the exact numerical coefficients which enter these dependences. Quantitatively correct models of not-too-massive white dwarfs were developed later by Chandrasekhar [12] based on the theory of polytropic stars.

According to Eqns (3) the equation of state of the nonrelativistic electron gas has the form $P_e = K\rho^{5/3}$, which corresponds to polytropic stars [$P = K\rho^{(n+1)/n}$] with the polytrope index $n = \frac{3}{2}$. With increasing central density ρ_c, the radius of white dwarfs composed of nonrelativistic electrons decreases ($R \propto \rho_c^{-1/6}$), and their mass increases ($M \propto \rho_c^{1/2}$).

When $\rho_c \sim 10^6$ g cm^{-3}, the electron gas at the centre of the white dwarf becomes relativistic. This happens when the dwarf mass $M \approx M_\odot$. If $\rho_c \gg 10^6$ g cm^{-3}, a white dwarf possesses a massive central core where the electrons are relativistic. The higher the value of ρ_c, the more massive the core is, and the less important is the dwarf's envelope made of nonrelativistic electrons. These massive white dwarfs differ strongly from the low-mass ones. The above conclusions were again made by Stoner (not by Chandrasekhar) in a work [13] of 1930 on the basis of the simplified uniform-density stellar model. Quantitatively accurate models of massive white dwarfs, based on the polytropic star theory were constructed later by Chandrasekhar [6].

According to Eqns (3), the equation of state of the relativistic electron gas is $P_e = K'\rho^{4/3}$. This is the polytropic law again, but with the index $n = 3$. The relativistic gas appears to be much 'softer' than the nonrelativistic one, and can hardly resist gravitational contraction. With growing ρ_c, the radius of such a dwarf decreases ($R \propto \rho_c^{-1/3}$), but its mass hardly increases and tends to the constant (limiting) value $M_c = 1.457(2\mu_e^{-1})^2 M_\odot$. The existence of the upper limit of the white-dwarf mass was first predicted in the above-mentioned work of Stoner [13]. In the same work Stoner also obtained a qualitatively correct estimate for M_c. The exact value of M_c, which is called the Chandrasekhar mass limit of white dwarfs, was calculated by Chandrasekhar [6]. To emphasise the importance of that result, Chandrasekhar published it additionally in a separate article [7] of only two pages. Independently, and almost simultaneously, M_c was calculated by Landau [8]. Moreover, Landau showed that the limiting mass could be written in the remarkably simple form,

$$M_c = 3.1\left(\frac{\hbar c}{2\pi G}\right)^{3/2}\left(\frac{4}{m_p \mu_e}\right)^2, \quad (5)$$

expressed through four fundamental physical constants (the gravitational constant G, the Planck constant $\hbar$, the proton mass m_p, and the velocity of light c), and a numerical factor of 3.1 composed of constants for the polytropic stars ($n = 3$).

The first models of white dwarfs of arbitrary mass $M \sim M_\odot$ were calculated by Chandrasekhar [9] as late as 1934. They appeared so late because the equation of state (2) of the degenerate electron gas for any degree of relativistic motion was still unknown. Eqn (2) for U_e was derived by Stoner [13] in (1930). Two years later, in 1932, Stoner also obtained [14] the expression for P_e. The equation of state derived by Stoner was used by Chandrasekhar [9, 10].

In the above-cited articles it has been assumed that $P = P_e$. The Coulomb corrections (4) to the equation of state are easily obtained with the aid of a simple model of dense matter as composed of Wigner–Seitz cells (Section 2). Shapiro and Teukolsky [3] do not mention the author of these corrections, and it is difficult to follow their history. The first astrophysicist who presented the correct expression for the Coulomb corrections was probably Kothari [15] (in a work published in 1938). Detailed models of white dwarfs that took into account those corrections as well as possible changes of nuclear composition owing to electron captures by atomic nuclei were constructed in the remarkable work of Hamada and Salpeter [16], which appeared in 1961. Although the Coulomb corrections are not large, they are important in massive white dwarfs ($M \approx M_c$): the relativistic electron gas is hardly able to resist the gravitational forces. Even a small decrease of the total pressure due to the Coulomb corrections noticeably reduces the upper mass limit. For instance, one has $M_c = 1.1 M_\odot$ for iron white dwarfs, instead of the classical Chandrasekhar value of $1.26 M_\odot$ (for $M_\odot/\mu_e = 2.15$).

The above works constitute the basis of the modern theory of white dwarfs. The theory is continuously developing, but the basic concept remains untouched.

4. The work by Ya I Frenkel' on 'binding forces'

Let us return to Frenkel's article (1928) [2] on 'binding forces' which are the forces that confine particles in material and determine its properties. The article consists of four sections which, at the first glance, are devoted to different aspects of physics. The section titles are as follows:

§1. Kinetic attraction forces and equilibrium conditions with allowance for the relativistic mechanics.
§2. A model of an atom according to Thomas–Fermi.
§3. Internal structure of an atomic nucleus.
§4. Superdense stars.

In spite of the wide range of topics covered, all sections are logically connected and related directly to §4, where the author considers superdense stars. There is no doubt that the latter are white dwarfs, although Frenkel' does not use this term. He studies stars where the pressure is mainly produced by degenerate electrons at zero temperature.

In §1 the author calculates the energy density and pressure of the degenerate electron gas with any degree of relativistic motion. His equations (3a) and (4b), up to trivial normalisation constants, fully coincide with Eqns (2) which were taken by Chandrasekhar [9, 10] from the works of Stoner [13, 14] for constructing the white-dwarf models (with $M_\odot$).

In §2 Frenkel' develops a traditional, classical Thomas–Fermi model for an atom with nonrelativistic electrons. As a limiting case, he considers a Wigner–Seitz cell of fixed radius (with an atomic nucleus in the cell centre) filled with electron gas of uniform density. Thus, the author solves two problems at a time. First, he analyses an elementary cell of white-dwarf matter with density $\rho \ll 10^6$ g cm^{-3}, when the electron gas is nonrelativistic. Second, he accurately derives the Coulomb correction to the equation of state (note that the correction is independent of the degree of relativistic motion of the electrons). Indeed, the sum of the last two terms in equation (12) of the cited work (provided the number of the electrons in the cell is $N = Z$) yields the Coulomb correction U_i [see Eqn (4) above]. As pointed out in Section 3, the Coulomb corrections for white-dwarf matter became known in the astrophysical literature long after Frenkel's work had been published.

In §3 Frenkel' proposes an original model of the atomic nucleus as a sphere filled with protons and electrons and confined by interactions between the magnetic moments of these particles. One should take into account that this (now naive) model was developed in 1928, four years before the discovery of neutrons. It is important that, while developing this model, the author also considered a Wigner–Seitz cell filled with an electron gas and compressed to very small sizes (about the radius of an atomic nucleus). The strongly compressed electron gas inevitably becomes relativistic. Thus, the author naturally comes to the model of elementary cells of white-dwarf matter with density $\rho \gg 10^6$ g cm^{-3}.

Finally, §4 is directly devoted to superdense stars (white-dwarfs). Frenkel' states that one should consider two cases, which correspond to superdense stars of two types. In the first case, stellar matter consists of elementary cells filled with nonrelativistic gas (§2). In the second case, the electron gas in the cells is relativistic (§3).

What is the difference between stars of these two types? Using simple estimates based on the stellar model with uniform density distribution, Frenkel' shows that the superdense stars of the second type should be more massive than those of the first type. For the electron gas to become relativistic, the stellar mass should be larger than the solar mass [$M > M_\odot$, equation (19b) of the Frenkel' article], and the mean stellar density should exceed 10^6 g cm^{-3} (the last, unnumbered equation of the article under discussion). These estimates underlie the current theory of white dwarfs.

5. Conclusions

The work of Ya I Frenkel' published in 1928 contained almost all the elements of the modern theory of white dwarfs. The work was done noticeably earlier than the classic works of other authors cited widely in the current astrophysical literature. Frenkel' predicted two types of superdense stars — those that consist of a nonrelativistic electron gas and those that consist of an ultrarelativistic electron gas — and correctly estimated that stars of the second type should be massive, $M \gtrsim M_\odot$. He obtained the equation of state of the degenerate electron gas for any degree of relativistic or nonrelativistic motion, the major element for constructing white-dwarf models. Frenkel' also obtained the Coulomb corrections to the electron pressure, which were rediscovered much later and used in the classic work of Hamada and Salpeter [16] in 1961.

Frenkel' failed to draw only one fundamental conclusion on the internal structure of white dwarfs — the existence of the upper mass limit M_c. However, it is impossible to criticise the author — his work is unique even without it. It contains almost everything that would have allowed one to build the modern quantitative theory of white dwarfs at the end of the 1920s, had it been continued by Frenkel' or his students.

However, the work had little impact. It was not used by astronomers and astrophysicists, and the theory of white dwarfs was developed (Section 3) later and independently by other authors. One can think of only one reason for the work not being a success, and that was the rather inopportune title of the work and its sections. In addition, the work was written by a theoretician who studied general problems of fundamental physics but not astrophysical problems. And although the paper was published in a journal with the highest standing, it did not attract the attention of the most appropriate readers — astronomers and astrophysicists. The importance of the results was probably not fully understood by the author himself; otherwise, he could have continued the work or significantly reconsidered it in subsequent articles.

All the same, Frenkel' was the first to develop many aspects of the theory of white dwarfs in 1928. His remarkable work should not be forgotten: it deserves to be cited among the classic works on the theory of white dwarfs.

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