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Transitional currents of spin $\frac{1}{2}$ particles

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Abstract. The problem of constructing explicitly covariant representations for transitional currents of $spin-\frac{1}{2}$ particles is solved for both the nonrelativistic theory and the relativistic theory with massive particles. In the massless case, such representations exist only for transitional currents that are nondiagonal in helicity. The diagonal currents have algebraic representations that are not explicitly covariant.

1. Introduction

In nonrelativistic quantum mechanics, the cross section of processes involving spin $\frac{1}{2}$ particles are determined by matrix elements containing transitional currents $\chi^+(\zeta')\chi(\zeta)$ or $\chi^+(\zeta')\sigma\chi(\zeta)$, where σ is the vector of Pauli matrices, and $\chi(\zeta')$ and $\chi(\zeta)$ are spinors describing particles with polarisations ζ', ζ . The dependence of currents on the vectors ζ', ζ is implicit. In order to determine the dependence of cross sections on particle polarisations the standard method is used (see, for example, Ref. [1]): after the absolute value of the amplitude has been squared, the spinors are removed by means of the technique of projection operators. As a result, an algebraic expression — which depends on the vectors ζ', ζ in an explicitly covariant way with respect to three-dimensional rotations-is obtained for the cross section. The question arises as to whether it is possible to substitute explicitly covariant expressions, which do not contain any reference to spinors, for transitional currents directly in amplitudes.

There exists an analogous problem in the relativistic theory. Cross sections and decay probabilities contain currents defined through the Dirac bispinors. The explicit

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Received 15 December 1993 Uspekhi Fizicheskikh Nauk **164** (6) 643–649 (1994) Submitted in English by the author; revised by J R Briggs dependence on momenta and on particle polarisation vectors remains hidden in currents of this type, hence certain problems in the asymptotics of the currents, the analysis of polarisation effects and other cases are not so transparent. Just as in nonrelativistic quantum mechanics, bispinors are removed in a covariant way from the squares of absolute values of the matrix elements (see, for example, Refs [2] or [3]). It seems to be easier in the case of a composite tensor structure to deal with explicitly covariant expressions for amplitudes rather than with transition probabilities. For example, the Compton effect amplitude is a contraction of a rank-two tensor with the polarisation vectors of two photons. The cross section of this process is quadratic in the amplitude and is determined by a rank-four tensor.

Explicitly covariant algebraic representations for transitional currents can be used in applications. This paper is devoted to the problem of the existence of such representations and the determination of their form (see also Refs [4-6]).

We start with the discussion of the appropriate characteristics of spinors in three-dimensional Euclidean space. It is shown in the next section that the products $\chi_{\alpha}(\zeta)\chi_{\beta}^{+}(\zeta')$ are expressed through the scalar product of the polarisation vectors ζ and ζ' with Pauli matrices up to a phase factor. This representation is then used to construct explicitly covariant (with respect to the rotation group) algebraic expressions for the currents $\chi^+(\zeta')\chi(\zeta)$ and $\chi^+(\zeta')\sigma\chi(\zeta)$. An analogous problem in the relativistic theory is considered in Section 3 for the massive and massless cases. Representations in the form of covariant contractions of the momenta pand p', and the polarisation vectors s and s' with Dirac gamma-matrices were found for the tensor forms $u_{\alpha}(\boldsymbol{p},s)\overline{u}_{\beta}(\boldsymbol{p}',s')$, constructed from Dirac bispinors describing massive particles. These representations are then used to construct the complete set of relativistic transitional currents in explicitly covariant form with respect to the Lorentz group. Thus the question of the possibility of constructing explicitly covariant representations is solved affirmatively in the nonrelativistic theory and in the relativistic theory with massive particles. In the massless case, as we shall demon-



strate, such representations exist only for transitional currents, nondiagonal in helicity. Diagonal currents have algebraic representations that are not explicitly covariant. Their form is established.

2. Transitional currents in nonrelativistic quantum mechanics

In nonrelativistic quantum mechanics the wave function of a spin $\frac{1}{2}$ particle, polarised in the direction of ζ , is described by a spinor satisfying

$$\hat{\zeta}\chi(\zeta) = \chi(\zeta), \quad \chi^+(\zeta)\chi(\zeta) = 1 \quad ,$$
 (2.1)

where $\hat{\boldsymbol{\zeta}} = \boldsymbol{\zeta} \cdot \boldsymbol{\sigma}$ and $|\boldsymbol{\zeta}|^2 = 1$.

In the particular case of a particle polarised in the direction of the axis $e_z = (0, 0, 1)$, it is easy to find the solution of Eqns (2.1). It has the following form:

$$\chi(\boldsymbol{e}_z) = \begin{pmatrix} 1\\ 0 \end{pmatrix} \,. \tag{2.2}$$

The given spinor is defined up to a phase factor. Having one solution, it is possible to construct a spinor with arbitrary polarisation ζ with the help of the rotation matrix $U(\zeta, e_z)$,

$$\chi(\boldsymbol{\zeta}) = U(\boldsymbol{\zeta}, \boldsymbol{e}_z)\chi(\boldsymbol{e}_z) \quad . \tag{2.3}$$

The rotation matrix $U(\boldsymbol{\zeta}, \boldsymbol{e}_{\tau})$ has the following properties:

$$\hat{\boldsymbol{\zeta}}U(\boldsymbol{\zeta},\boldsymbol{e}_z) = U(\boldsymbol{\zeta},\boldsymbol{e}_z)\hat{\boldsymbol{e}}_z, \quad U(\boldsymbol{\zeta},\boldsymbol{e}_z)U^+(\boldsymbol{\zeta},\boldsymbol{e}_z) = 1 \quad . \tag{2.4}$$

Owing to these properties, spinor (2.3) satisfies Eqns (2.1) identically.

The way in which the vector e_z is transformed into the vector $\boldsymbol{\zeta}$ is not unique. There exists an infinite set of matrices $U(\boldsymbol{\zeta}, \boldsymbol{e}_z)$ that implement this transformation for every pair of vectors \boldsymbol{e}_z and $\boldsymbol{\zeta}$ (see Fig. 1). However, all matrices $U(\boldsymbol{\zeta}, \boldsymbol{e}_z)$ satisfy Eqns (2.4) and hence define the same spinor up to a phase factor. This is due to the fact that Eqns (2.1) define in turn a spinor up to a phase factor.

We thus come to the conclusion that for two different matrices $U_1(\zeta, e_z)$ and $U_2(\zeta, e_z)$, transforming e_z into ζ , the following relation applies:

$$U_1(\boldsymbol{\zeta}, \boldsymbol{e}_z)\boldsymbol{\chi}(\boldsymbol{e}_z) = \exp(\mathrm{i}\boldsymbol{\phi})U_2(\boldsymbol{\zeta}, \boldsymbol{e}_z)\boldsymbol{\chi}(\boldsymbol{e}_z) , \qquad (2.5)$$

where ϕ is some phase.



It is possible to verify the correctness of the following equation directly:

$$\chi_{\alpha}(\boldsymbol{e}_{z})\chi_{\beta}^{+}(\boldsymbol{e}_{z}) = \left(\frac{1+\hat{e}_{z}}{2}\right)_{\alpha\beta} . \qquad (2.6)$$

Notice that the right-hand side defines the projection operator $P(\zeta) = \frac{1}{2}(1+\hat{\zeta})$ for which $P^2(\zeta) = P(\zeta)$ if $|\zeta|^2 = 1$.

Eqn (2.6) is acted on from the left with the matrix $U(\boldsymbol{\zeta}, \boldsymbol{e}_z)$ and from the right with the matrix $U^+(\boldsymbol{\zeta}', \boldsymbol{e}_z)$. As a result, we obtain

$$\chi_{\alpha}(\boldsymbol{\zeta})\chi_{\beta}^{+}(\boldsymbol{\zeta}') = \left[U(\boldsymbol{\zeta}, \boldsymbol{e}_{z}) \frac{1+\hat{\boldsymbol{e}}_{z}}{2} \frac{1+\hat{\boldsymbol{e}}_{z}}{2} U^{+}(\boldsymbol{\zeta}', \boldsymbol{e}_{z}) \right]_{\alpha\beta}$$
$$= \left[\frac{1+\hat{\boldsymbol{\zeta}}}{2} U(\boldsymbol{\zeta}, \boldsymbol{\zeta}') \frac{1+\hat{\boldsymbol{\zeta}}'}{2} \right]_{\alpha\beta}, \qquad (2.7)$$

where $U(\zeta, \zeta') = U(\zeta, e_z)U^+(\zeta', e_z)$ is the matrix transforming the vector ζ' into the vector ζ through the vector e_z . According to Eqn (2.5), any other choice of matrices $U(\zeta, e_z), U(\zeta', e_z)$ would affect only spinor phase factors, so to find up to a phase factor an explicit form of the righthand side of Eqn (2.7) one can substitute any matrix for $U(\zeta, \zeta')$, transforming ζ' into ζ .[†] Let $U(\zeta, \zeta') =$ $\exp(i\boldsymbol{\sigma} \cdot \boldsymbol{n}\theta/2)$ with $\boldsymbol{n} = (\zeta + \zeta')/|\zeta + \zeta'|$ and $\theta = \pi$. This matrix implements rotation around the vector \boldsymbol{n} , as shown in Fig. 2. It can be written in the form

$$U(\boldsymbol{\zeta},\,\boldsymbol{\zeta}') = \mathrm{i}\frac{\hat{\boldsymbol{\zeta}} + \hat{\boldsymbol{\zeta}'}}{\left(2 + 2\boldsymbol{\zeta} \cdot \boldsymbol{\zeta}'\right)^{1/2}} \,. \tag{2.8}$$

Relations (2.4) with fixed vectors $\boldsymbol{\zeta}$, $\boldsymbol{\zeta}'(=\boldsymbol{e}_z)$ can be considered as equations, fixing the rotation matrix $U(\boldsymbol{\zeta}, \boldsymbol{\zeta}')$. In the case of three-dimensional rotations, the explicit form of the matrix can be found by means of the exponential parametrisation $U(\boldsymbol{\zeta}, \boldsymbol{\zeta}') = \exp(i\boldsymbol{\sigma}\cdot\boldsymbol{n}\theta/2)$. Matrix (2.8), obtained in the same way, satisfies relations (2.4). However, in the case of Lorentz transformations exponential parametrisation is less effective. As will be shown in the next section, the corresponding matrix of Lorentz transforma-

[†] It is not necessary to demand that the vector $\boldsymbol{\zeta}'$ be transformed into $\boldsymbol{\zeta}$ through \boldsymbol{e}_z . To see this, first act on Eqn (2.6) with the matrix $U(\boldsymbol{\zeta}, \boldsymbol{e}_z)$ from the left and with the same matrix, but Hermitian conjugated, from the right. As a result, we obtain $\chi_{\alpha}(\boldsymbol{\zeta})\chi_{\beta}^{+}(\boldsymbol{\zeta}) = P_{\alpha\beta}(\boldsymbol{\zeta})$. Then act on the equation from the right with an arbitrary matrix $U^+(\boldsymbol{\zeta}', \boldsymbol{\zeta}) = U(\boldsymbol{\zeta}, \boldsymbol{\zeta}')$. The result takes the form of Eqn (2.7), since $P(\boldsymbol{\zeta})U(\boldsymbol{\zeta}, \boldsymbol{\zeta}') = P(\boldsymbol{\zeta})P(\boldsymbol{\zeta})U(\boldsymbol{\zeta}, \boldsymbol{\zeta}')$ because of Eqns (2.4).

tions can be found by solving a set of equations analogous to Eqns (2.4).

Substituting Eqn (2.8) into Eqn (2.7), we obtain

$$\chi_{\alpha}(\boldsymbol{\zeta})\chi_{\beta}^{+}(\boldsymbol{\zeta}') \doteq \frac{2}{(2+2\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}')^{1/2}} \left(\frac{1+\hat{\boldsymbol{\zeta}}}{2}\frac{1+\hat{\boldsymbol{\zeta}}'}{2}\right)_{\alpha\beta} .$$
(2.9)

The sign \doteq is used to emphasise the fact that two values (in this case, two matrices) are equal up to a phase factor.

We come to the conclusion that the explicitly noncovariant definition of spinors with the help of Eqn (2.3), the special role of the spinor $\chi(e_z)$, and the arbitrary choice of the way in which the vector e_z is transformed into the vector ζ , affect only the unobservable phase factor (which is independent of the indices α and β) on the right-hand side of Eqn (2.9), while the nontrivial dependence on the vectors ζ/ζ' has explicitly covariant form with respect to the rotation group O(3).

The transitional currents can be represented as follows:

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$$j(\zeta',\zeta) = \chi^{+}(\zeta')\chi(\zeta) = \operatorname{tr}\chi(\zeta)\chi^{+}(\zeta') \doteq \left(\frac{1+\zeta\cdot\zeta'}{2}\right)^{1/2},$$

$$j(\zeta',\zeta) = \chi^{+}(\zeta')\sigma\chi(\zeta) = \operatorname{tr}\sigma\chi(\zeta)\chi^{+}(\zeta') \doteq \frac{\zeta+\zeta'+\mathrm{i}\zeta\wedge\zeta'}{(2+2\zeta\cdot\zeta')^{1/2}}.$$
(2.10)

The complete set of two-dimensional Hermitian matrices consists of matrices 1 (the unit matrix) and σ , so the current enumeration listed above is exhaustive in the nonrelativistic theory.

It is possible to verify with Eqn (2.10) that, for example, the relation

$$j_{\alpha}(\boldsymbol{\zeta}',\,\boldsymbol{\zeta})j_{\beta}^{*}(\boldsymbol{\zeta}',\,\boldsymbol{\zeta}) = \operatorname{tr} \boldsymbol{\sigma}_{\alpha} \frac{1+\hat{\boldsymbol{\zeta}}}{2} \,\boldsymbol{\sigma}_{\beta} \frac{1+\hat{\boldsymbol{\zeta}}'}{2} \,, \qquad (2.11)$$

which is usually used to get rid of spinors in the squares of the absolute values of the matrix elements, is true.

Eqns (2.9) can be obtained in a shorter, but more formal way: act on the spinors $\chi(\zeta)$ with the projection operator $P(\zeta') = (1 + \hat{\zeta}')/2$. As a result, we obtain an arbitrary normalised spinor $\chi(\zeta')$. Having normalised $\chi(\zeta')$ to unity, we obtain

$$\chi(\boldsymbol{\zeta}') \doteq \frac{2}{\left(2 + 2\boldsymbol{\zeta} \cdot \boldsymbol{\zeta}'\right)^{1/2}} \frac{1 + \hat{\boldsymbol{\zeta}'}}{2} \,\chi(\boldsymbol{\zeta}) \quad (2.12)$$

Then we can reproduce relation (2.9) using Eqn (2.12) and the relation $\chi_{\alpha}(\zeta)\chi_{\beta}^{+}(\zeta) = P_{\alpha\beta}(\zeta)$.

3. Transitional currents in the relativistic theory

The above arguments allow quite evident relativistic extension to the case of massive spinor particles. In the massless case the extension is less trivial.

3.1. Massive case

To make the notation more compact, I shall define the symbol $\varepsilon = \pm 1$, with the help of which the bispinors u(p,s) and v(p,s)—describing particle and antiparticle—are written in the form $u(p,s,\varepsilon = +1)$ and $u(p,s,\varepsilon = -1)$.

Dirac bispinors with definite momentum p_{μ} and polarisation s_{μ} are constrained by the following equations:

$$\hat{p}u(p, s, \varepsilon) = \varepsilon m u(p, s, \varepsilon),$$

$$\gamma_5 s u(p, s, \varepsilon) = u(p, s, \varepsilon),$$

$$\bar{u}(p, s, \varepsilon) u(p, s, \varepsilon) = \varepsilon.$$
(3.1)

The first is the Dirac equation in momentum space, the second is the equation for eigenvalue +1 of the spin projection on the direction of unit vector s_{μ} ; and the third is the spinor covariant normalisation. The momentum and polarisation vectors satisfy the relations $p^2 = m^2$, $s^2 = -1$, $p \cdot s = 0$. Here $\hat{p} = p_{\mu} \gamma^{\mu}$, $\hat{s} = s_{\mu} \gamma^{\mu}$, and so on. The gammamatrices are defined as in Ref. [3].

First of all, as in the three-dimensional case, we construct all the linear independent solutions of Eqns (3.1) for some fixed momentum and polarisation, e.g. in the rest frame of the particle, where $p = \eta = (m, 0, 0, 0)$. The polarisation vector is taken to be $s = e_z = (0, 0, 0, 1)$. There exist two linear independent solutions of the form

$$u(\eta, e_z, +1) = \begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}, \quad u(\eta, e_z, -1) = \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} .$$
(3.2)

The bispinor corresponding to the momentum p_{μ} and polarisation s_{μ} can be obtained as a result of the action of the Lorentz transformation matrix on bispinors:

$$u(p, s, \varepsilon) = U(p, s; \eta, e_z)u(\eta, e_z, \varepsilon) .$$
(3.3)

The matrix $U(p, s; \eta, e_z)$ of Lorentz transformations has the following properties [cf. Eqn (2.4)]:

$$\hat{p}U(p, s; \eta, e_z) = U(p, s; \eta, e_z)\hat{\eta},
\hat{s}U(p, s; \eta, e_z) = U(p, s; \eta, e_z)\hat{e}_z,
U(p, s; \eta, e_z)\overline{U}(p, s; \eta, e_z) = 1,$$
(3.4)

as a result of which the bispinors defined in Eqn (3.3) satisfy relations (3.1) identically.

It is possible to verify the relation

$$u_{\alpha}(\eta, e_{z}, \varepsilon)\bar{u}_{\beta}(\eta, e_{z}, \varepsilon) = \left(\frac{\hat{\eta} + \varepsilon m}{2m} \frac{1 + \gamma_{5}\hat{e}_{z}}{2}\right)_{\alpha\beta}$$
(3.5)

by testing each component of the equality. Multiplying this equation by $U(p,s;\eta,e_z)$ from the left and by $\overline{U}(p',s';\eta,e_z)$ from the right, we obtain

$$u_{\alpha}(p, s, \varepsilon) \bar{u}_{\beta}(p', s', \varepsilon') = \left[\frac{\hat{p} + \varepsilon m}{2m} \frac{1 + \gamma_5 \hat{s}}{2} U(p, s; p', s') \frac{\hat{p'} + \varepsilon' m}{2m} \frac{1 + \gamma_5 \hat{s'}}{2}\right]_{\alpha\beta},$$
(3.6)

where $U(p,s;p's') = U(p,s;\eta,e_z)\overline{U}(p',s';\eta,e_z)$ is one of the Lorentz transformation matrices transforming the vectors p',s' into the vectors p,s; and $\varepsilon\varepsilon' = +1$. Ambiguity in the choice of matrices $U(p,s;\eta,e_z)$, $U(p',s';\eta,e_z)$ affects only the phase factors of bispinors defined by Eqn (3.3). The arguments here are the same as for Pauli spinors: owing to Eqns (3.4), bispinors (3.3) satisfy Eqns (3.1) identically. In turn these equations define spinors up to a phase factor. Therefore, if we are not interested in the phase factor, it is possible to choose any matrix transforming the vectors p',s'into the vectors p,s. Such a matrix was found in Ref. [3] as a particular solution of Eqns (3.4) with the substitution $\eta, e_z \rightarrow p', s'$:

$$U(p, s; p', s') = A_{+}(p, s; p', s') \times (m^{2} + \hat{p}\hat{p}' - m^{2}\hat{s}\hat{s}' + \hat{p}\hat{s}\hat{s}'\hat{p}') \quad . \tag{3.7}$$

It is possible to check that it satisfies the first two Eqns (3.4). The third equation imposes constraints on the form of U(p,s;p',s') and allows one to fix the normalising constant,

$$A_{+}(p,s;p',s') = \frac{1}{2m} [(m^{2} + p \cdot p')(1 - s \cdot s') + p \cdot s'p' \cdot s]^{-1/2}$$
(3.8)

Notice that the Lorentz transformation matrices $U = \exp(\omega_{\mu\nu}\gamma_{\mu} \wedge \gamma_{\nu})$ can be expanded over the matrices 1, $i\gamma_5$, and $\gamma_{\mu} \wedge \gamma_{\nu}$ with real coefficients. The matrix (3.7) can also be expanded over this basis.

Substituting Eqn (3.7) into Eqn (3.6), we obtain $(\epsilon\epsilon' = +1)$

$$u_{\alpha}(p, s, \varepsilon)\overline{u}_{\beta}(p', s', \varepsilon')$$

$$\doteq 4A_{+}(p, s; p', s') \left(\frac{\hat{p} + \varepsilon m}{2m} \frac{1 + \gamma_{5}\hat{s}}{2} \frac{\hat{p}' + \varepsilon' m}{2m} \frac{1 + \gamma_{5}\hat{s}'}{2}\right)_{\alpha\beta}.$$
(3.9)

The bilinear forms of the type $\varepsilon \varepsilon' = -1$ can be constructed in the following way. In the rest frame of a particle,

$$y_5 u(\eta, e_z, \varepsilon) = u(\eta, -e_z, -\varepsilon) \quad . \tag{3.10}$$

The Lorentz transformation matrices commutate with the matrix γ_5 , so that the following equation is correct in any reference frame:

$$\gamma_5 u(p, s, \varepsilon) = u(p, -s, -\varepsilon) . \tag{3.11}$$

We act on Eqn (3.9) from the left with the matrix γ_5 and perform the redesignations $s \to -s$, $\varepsilon \to -\varepsilon'$. As a result, one can obtain, for $\varepsilon\varepsilon' = -1$,

$$u_{\alpha}(p, s, \varepsilon)\overline{u}_{\beta}(p', s', \varepsilon) \\ \doteq 4A_{-}(p, s; p', s') \left(\frac{\hat{p} + \varepsilon m}{2m} \frac{1 + \gamma_{5}\hat{s}}{2} \gamma_{5} \frac{\hat{p}' + \varepsilon' m}{2m} \frac{1 + \gamma_{5}\hat{s}'}{2}\right)_{\alpha\beta},$$
(3.12)

where

$$A_{-}(p,s;p',s') = \frac{1}{2m} [(m^{2} + p \cdot p')(1 + s \cdot s') - p \cdot s'p' \cdot s]^{-1/2}$$
(3.13)

Explicit expressions for the transitional currents can be constructed with the help of

$$\bar{u}(p', s', \varepsilon')\Gamma u(p, s, \varepsilon) = \operatorname{tr} \Gamma u(p, s, \varepsilon)\bar{u}(p', s', \varepsilon') \quad (3.14)$$

For reference purposes I shall adduce the full list of explicitly covariant expressions for the transitional currents of massive spin $\frac{1}{2}$ particles in the relativistic theory:

$$\begin{split} & \varepsilon \varepsilon' = +1: \\ & \bar{u}(p', s', \varepsilon)u(p, s, \varepsilon) \dot{=} \frac{1}{4m^2} A_+(p, s; p', s')^{-1} , \\ & \bar{u}(p', s', \varepsilon')\gamma_5 u(p, s, \varepsilon) \\ & \dot{=} A_+(p, s; p', s')(m\varepsilon p' \cdot s - m\varepsilon' p \cdot s' + i\varepsilon_{\alpha\beta\mu\nu} p'_{\alpha} s'_{\beta} p_{\mu} s_{\nu}) \end{split}$$

$$\begin{split} \overline{a}(p', s', \varepsilon)\gamma_{\mu}u(p, s, \varepsilon) \\ \doteq A_{+}(p, s; p', s')[m(\varepsilon'p + \varepsilon p')_{\mu}(1 - s \cdot s') \\ &+ i\varepsilon_{\mu\alpha\beta\gamma}p'_{\alpha}p_{\beta}(s' + s)_{\gamma} + m\varepsilon p' \cdot ss'_{\mu} + m\varepsilon'p \cdot s's_{\mu}] \end{split}$$

$$\begin{split} \bar{u}(p',s',\varepsilon')\gamma_{\mu}\gamma_{5}u(p,s,\varepsilon) \\ &\doteq A_{+}(p,s;p',s')[(s'+s)_{\mu}(m^{2}+p\cdot p') , \\ &- \mathrm{i}m\varepsilon_{\mu\alpha\beta\gamma}s'_{\alpha}s_{\beta}(\varepsilon p'+\varepsilon' p)_{\gamma}-p'\cdot sp_{\mu}-p\cdot s'p'_{\mu})] \\ \bar{u}(p',s',\varepsilon')\sigma_{\mu\nu}u(p,s,\varepsilon) \\ &\doteq A_{+}(p,s;p',s')[(p'_{\mu}p_{\nu}-p'_{\nu}p_{\mu})(1-s\cdot s') \\ &-(s'_{\mu}s_{\nu}-s'_{\nu}s_{\mu})(m^{2}+p\cdot p')+(p'_{\mu}s_{\nu}-p'_{\nu}s_{\mu})s'\cdot p \\ &-(p_{\mu}s'_{\nu}-p_{\nu}s'_{\mu})s\cdot p'-\mathrm{i}m\varepsilon_{\mu\nu\alpha\beta}(\varepsilon p'+\varepsilon' p)_{\alpha}(s'+s)_{\beta}], \end{split}$$

$$\varepsilon \varepsilon' = -1; \tag{3.15}$$

$$\begin{split} & u(p, s, \varepsilon) u(p, s, \varepsilon) \\ & \doteq A_{-}(p, s, p', s') (m\varepsilon p' \cdot s - m\varepsilon' p \cdot s' - i\varepsilon_{\alpha\beta\mu\nu} p'_{\alpha} s'_{\beta} p_{\mu} s_{\nu}), \\ & \bar{u}(p', s', \varepsilon') \gamma_{5} u(p, s, \varepsilon) \doteq \frac{1}{4m^{2}} A_{-}(p, s; p', s')^{-1}, \end{split}$$

$$\begin{split} \bar{u}(p', s', \varepsilon') \gamma_{\mu} u(p, s, \varepsilon) \\ \doteq A_{-}(p, s; p', s') [(s' - s)_{\mu}(m^{2} + p \cdot p') \\ - \mathrm{i} m \varepsilon_{\mu\alpha\beta\gamma} s'_{\alpha} s_{\beta} (\varepsilon p' - \varepsilon' p)_{\gamma} + p' \cdot s p_{\mu} - p \cdot s' p'_{\mu}], \end{split}$$

$$\begin{split} \bar{u}(p',s',\varepsilon')\gamma_{\mu}\gamma_{5}u(p,s,\varepsilon) \\ &\doteq A_{-}(p,s;p',s')[m(\varepsilon'p-\varepsilon p')_{\mu}(1+s\cdot s') \\ &+ i\epsilon_{\mu\alpha\beta\gamma}p'_{\alpha}p_{\beta}(s'-s)_{\gamma}+m\varepsilon p'\cdot ss'_{\mu}-m\varepsilon'p\cdot s's_{\mu}], \\ \bar{u}(p',s',\varepsilon)\sigma_{\mu\nu}u(p,s,\varepsilon) \\ &\doteq A_{-}(p,s,p',s')\epsilon_{\mu\nu\alpha\beta}[p'_{\alpha}p_{\beta}(1+s\cdot s') \\ &+ s'_{\alpha}s_{\beta}(m^{2}+p\cdot p')-p'_{\alpha}s_{\beta}s'\cdot p+p_{\alpha}s'_{\beta}s\cdot p' \\ &+ \frac{i}{2}m\varepsilon_{\alpha\beta\tau\sigma}(\varepsilon p'-\varepsilon'p)_{\tau}(s'-s)_{\sigma}]. \end{split}$$
(3.16)

Notice that there are several symmetry relations. For $\varepsilon\varepsilon' = +1$ Eqn (3.14) can be written in the form

$$\bar{u}(p', s', \varepsilon')\Gamma u(p, s, \varepsilon)$$

$$\doteq 4A_+(p, s; p', s') \operatorname{tr} \Gamma \frac{\hat{p} + \varepsilon m}{2m} \frac{1 + \gamma_5 \hat{s}}{2} \frac{\hat{p}' + \varepsilon' m}{2m} \frac{1 + \gamma_5 \hat{s}'}{2}.$$

(3.17)

In the case $\varepsilon \varepsilon' = -1$ the analogous equation has the form

$$\bar{u}(p',s',\varepsilon')\Gamma u(p,s,\varepsilon)$$

$$\doteq 4A_{-}(p,s;p',s')\mathrm{tr}\Gamma\gamma_{5}\frac{\hat{p}-\varepsilon m}{2m}\frac{1-\gamma_{5}\hat{s}}{2}\frac{\hat{p}'+\varepsilon' m}{2m}\frac{1+\gamma_{5}\hat{s}'}{2}.$$

(3.18)

In comparing these two equations, one should notice that the scalar current $\varepsilon\varepsilon' = -1$ can be obtained from the pseudoscalar current $\varepsilon\varepsilon' = +1$ with the help of the substitution $\varepsilon \to -\varepsilon$, $s \to -s$, $A_+(p,s;p',s') \to$ $A_-(p,s;p',s') = A_+(p,-s;p',s')$. The same connection exists between the vector and pseudovector currents. The tensor current $\varepsilon\varepsilon' = -1$ is connected with the tensor current $\varepsilon\varepsilon' = +1$ through the identity $2i\gamma_5\sigma_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta}\sigma_{\alpha\beta}$.

These results can easily be extended for the case of transitional currents occuring in weak interactions, e.g. $p \rightarrow n$ and others with different particle masses in initial and final states. The bispinors, and hence the currents constructed from them, depend on velocities only; therefore, it is sufficient to make the substitution $p' \rightarrow mp'/m'$, in obtained expressions, where m' is the particle mass in the final state $(m' \neq m)$.

Eqns (3.9) and (3.12) can also be obtained with the help of the technique of projection operators. The bispinor $u(p', s', \varepsilon')$ can be presented in two ways:

$$\varepsilon\varepsilon' = +1:$$

$$u(p',s',\varepsilon') \doteq 4A_{+}(p,s;p',s')\frac{\hat{p}'+\varepsilon'm}{2m}\frac{1+\gamma_{5}\hat{s}'}{2}u(p,s,\varepsilon),$$

$$\varepsilon\varepsilon' = -1:$$

$$u(p',s',\varepsilon') \doteq 4A_{-}(p,s;p',s')\frac{\hat{p}'+\varepsilon'm}{2m}\frac{1+\gamma_{5}\hat{s}'}{2}\gamma_{5}u(p,s,\varepsilon).$$
(3.19)

The matrix γ_5 is introduced in the second expression to make the normalising coefficient nonsingular in the nonrelativistic limit. The bispinor $u(p', s', \varepsilon')$ satisfies Eqns (3.1), if the bispinor $u(p, s, \varepsilon)$ also satisfies them. Then relations (3.9) and (3.12) are reproduced for the tensor products $u_{\alpha}(p, s, \varepsilon)\overline{u}_{\beta}(p', s', \varepsilon')$.

3.2. Massless case

Dirac bispinors describing massless particles and antiparticles with fixed helicity have algebraically the same form, so from here on I shall drop the index $\varepsilon = \pm 1$ used above to distinguish particles from antiparticles.

The bispinors corresponding to particles with lefthanded (L) and right-handed (R) helicity are defined as solutions of the following equations:

$$\hat{k}u_{R}(k) = \hat{k}u_{L}(k) = 0,$$

$$\frac{1 - \gamma_{5}}{2}u_{R}(k) = \frac{1 + \gamma_{5}}{2}u_{L}(k) = 0,$$

$$\bar{u}_{R}(k)\gamma_{\mu}u_{R}(k) = \bar{u}_{L}(k)\gamma_{\mu}u_{L}(k) = 2k_{\mu} .$$
(3.20)

Here $k^2 = 0$. The normalisation condition is chosen in this form due to the fact that in the massless case the scalar current becomes identically zero: $\overline{u}_R(k)u_R(k) = \overline{u}_L(k)u_L(k) = 0$.

Now, we do not have at hand projection operators on the states with definite k. Unlike the massive case, the right-hand sides of the equalities

$$u_{L\alpha}(k)\bar{u}_{L\beta}(k) = \left(\hat{k}\frac{1+\gamma_5}{2}\right)_{\alpha\beta},$$

$$u_{R\alpha}(k)\bar{u}_{R\beta}(k) = \left(\hat{k}\frac{1-\gamma_5}{2}\right)_{\alpha\beta},$$
 (3.21)

being squared, vanish. Nevertheless, the relationships

$$u_{\rm L}(k') = C\hat{k'}u_{\rm R}(k),$$

$$u_{\rm R}(k') = C\hat{k'}u_{\rm L}(k),$$
(3.22)

exist, with the normalisation coefficient

$$C \doteq \frac{1}{(2k' \cdot k)^{1/2}} , \qquad (3.23)$$

which is singular in the limit $k' \rightarrow k$. The bispinors on the left-hand side of Eqns (3.22) obey Eqns (3.20) identically if the bispinors on the right-hand side obey Eqns (3.20). Coefficient (3.23) can be found by means of relations (3.21). The tensor products of bispinors of different helicity take the form

$$u_{\mathrm{L}\alpha}(k)\bar{u}_{\mathrm{R}\beta}(k') \doteq \frac{1}{\left(2k \cdot k'\right)^{1/2}} \left(\hat{k}\hat{k}'\frac{1-\gamma_{5}}{2}\right)_{\alpha\beta},$$

$$u_{\mathrm{R}\alpha}(k)\bar{u}_{\mathrm{L}\beta}(k') \doteq \frac{1}{\left(2k \cdot k'\right)^{1/2}} \left(\hat{k}\hat{k}'\frac{1+\gamma_{5}}{2}\right)_{\alpha\beta}.$$
(3.24)

With the help of these expressions, one can find explicitly covariant representations for transitional currents that are nondiagonal in helicity. First of all it is obvious that the currents containing an odd number of gamma matrices are identically equal to zero, since the right-hand sides of Eqns (3.24) have an even number of gamma matrices, whereas the trace of an odd number of gamma matrices equals zero. Below, the expressions are listed for nonvanishing currents:

$$\begin{split} \bar{u}_{\mathrm{R}}(k')u_{\mathrm{L}}(k) &\doteq \bar{u}_{\mathrm{L}}(k')u_{\mathrm{R}}(k) \doteq (2k \cdot k')^{1/2}, \\ \bar{u}_{\mathrm{R}}(k')\sigma_{\mu\nu}u_{\mathrm{L}}(k) &\doteq \left(\frac{2}{k \cdot k'}\right)^{1/2} (k'_{\mu}k_{\nu} - k'_{\nu}k_{\mu} + \mathrm{i}\varepsilon_{\mu\nu\tau\sigma}k'_{\tau}k_{\sigma}), \\ \bar{u}_{\mathrm{L}}(k')\sigma_{\mu\nu}u_{\mathrm{R}}(k) &\doteq \left(\frac{2}{k \cdot k'}\right)^{1/2} (k'_{\mu}k_{\nu} - k'_{\nu}k_{\mu} - \mathrm{i}\varepsilon_{\mu\nu\tau\sigma}k'_{\tau}k_{\sigma}). \end{split}$$

$$(3.25)$$

Consider now the currents that are diagonal in helicity. In the nonrelativistic theory, in the relativistic theory of massive particles, and in the massless case considered above, the currents have explicitly covariant representations, so it seems surprising that the currents diagonal in helicity do not have such representations.

Eqns (3.2) relate bispinors of different helicity. The only possibility to express, say, a left-handed bispinor with momentum k' in terms of a left-handed bispinor with momentum k is to introduce an auxiliary vector h into the right-hand side of Eqn (3.22). As a result we obtain

$$u_{\rm L}(k') = C'k'hu_{\rm L}(k), u_{\rm R}(k') = C'\hat{k'}\hat{h}u_{\rm R}(k)$$
(3.26)

The normalisation coefficient is given by

$$C' \doteq \frac{1}{\left[4k' \cdot h \, k \cdot h - 2k' \cdot k \, h^2\right]^{1/2}} \,. \tag{3.27}$$

The tensor products of bispinors of different helicity can now be written in the following form:

$$u_{\mathrm{L}\alpha}(k)\bar{u}_{\mathrm{L}\beta}(k') \doteq \frac{1}{\left(4k'\cdot h\,k\cdot h - 2k'\cdot k\,h^2\right)^{1/2}} \left(\hat{k}\hat{h}\hat{k}'\frac{1+\gamma_5}{2}\right)_{\alpha\beta},$$
$$u_{\mathrm{R}\alpha}(k)\bar{u}_{\mathrm{R}\beta}(k') \doteq \frac{1}{\left(4k'\cdot h\,k\cdot h - 2k'\cdot k\,h^2\right)^{1/2}} \left(\hat{k}\hat{h}\hat{k}'\frac{1-\gamma_5}{2}\right)_{\alpha\beta}$$
(3.28)

The right-hand side depends on h, whereas the left-hand side does not. The only explanation can be that on the right-hand side the h-dependence is such that the h variation affects only the phase factor. In this sense, the vector h fixing a certain frame does not break the covariance.

Let us consider a well-known example. Average over directions the product of two unit vectors $n_i n_j$. We have at hand only one tensor that does not violate the rotational symmetry—the Kronecker symbol—thus, the result can be constructed unambiguously as follows:

$$\int \frac{\mathrm{d}\Omega}{4\pi} n_i n_j = \frac{1}{3} \delta_{ij} \quad . \tag{3.29}$$

It follows from Eqns (3.28) that for quantities defined up to a phase factor, reasoning of this kind is not conclusive. In general there exist auxiliary vectors—apart from those already involved—by means of which the solution is expressed. These vectors break the initial symmetry (rotational, for example). Their variation, however, affects only the phase factor.

We are able now to construct explicit (in the algebraic sense) expressions for the vector currents. They contain the auxiliary vector h, so they are not explicitly covariant. Multiplying relations (3.28) by γ_{μ} and calculating the trace we find

$$\bar{u}_{\mathrm{L}}(k')\gamma_{\mu}u_{\mathrm{L}}(k) \doteq \frac{2(k'_{\mu}h\cdot k + k_{\mu}h\cdot k' - h_{\mu}k'\cdot k + \mathrm{i}\varepsilon_{\mu\nu\tau\sigma}k'_{\nu}k_{\tau}h_{\sigma})}{(4k'\cdot h\,k\cdot h - 2k'\cdot k\,h^{2})^{1/2}},$$

$$\bar{u}_{\mathrm{R}}(k')\gamma_{\mu}u_{\mathrm{R}}(k) \doteq \frac{2(k'_{\mu}h\cdot k + k_{\mu}h\cdot k' - h_{\mu}k'\cdot k - \mathrm{i}\varepsilon_{\mu\nu\tau\sigma}k'_{\nu}k_{\tau}h_{\sigma})}{(4k'\cdot h\,k\cdot h - 2k'\cdot k\,h^{2})^{1/2}}.$$
(3.30)

We thus conclude that the tensor-algebra tools are not sufficient to represent the left-hand side of Eqn (3.30) in algebraic form through the vectors k' and k. Any explicit expression which does not refer to some third vector is, in principle, nonexistent.

There are no other currents with odd numbers of gamma matrices. As for the currents with even numbers of gamma matrices, they are identically equal to zero. It also immediately follows that the right-hand sides of Eqns (3.28) can be written as the convolutions of currents (3.30) with the matrices $\gamma_{\mu}(1 \pm \gamma_5)$. That is why the phase factors on the right-hand side of Eqns (3.28) and (3.30) change in the same manner with respect to the *h*-vector variation.

One can verify [though it already follows from relations (3.28)] that the right-hand sides of Eqns (3.30) are indeed independent of h up to a phase factor. Consider the product of two left-handed currents $j_{L\mu}(k',k)j_{L\nu}^*(k',k)$. Using the first of the relations (3.30), we find after identical transformations that the vector h drops out from the final result. Moreover,

$$j_{L\mu}(k',k)j_{L\nu}^{*}(k',k) = \operatorname{tr} \gamma_{\mu}\hat{k}\gamma_{\nu}\hat{k}' \frac{1-\gamma_{5}}{2}$$
(3.31)

[cf relation (2.11)]. One can check identity (3.31) in the frame of reference where h = (1,0,0,0). In this frame, the space part of the left-handed current coincides within a factor of two with the nonrelativistic spin current (2.10) if one considers the vectors ζ, ζ' as the unit vectors in the direction k', k. When the indices μ, ν take on the space values, the right-hand side of Eqn (3.30) coincides with the right-hand side of Eqn (2.11) up to a normalisation factor, so for the values $\mu, \nu = 1, 2, 3$, the identity (3.31) may be thought of as being proved. Similar reasoning is valid for mixed components.

From the lack of an *h*-dependence of the product $j_{L\mu}(k',k)j_{L\nu}^*(k',k)$ at $\mu = \nu$, it follows that the absolute value of each of the current components is independent of *h*. From the absence of such dependence at $\mu \neq \nu$ it follows that all the components have the same phase factor. A similar line of reasoning is valid for the right-handed currents. Thus, a specific choice of the vector *h* can affect only a common vector-current phase factor. It depends neither on the index μ in Eqns (3.30) nor on the indices α, β in Eqns (3.28).

4. Concluding remarks

Thus, in the nonrelativistic theory and in the relativistic theory with massive particles the possibility of constructing explicitly covariant representations for the transition currents has been confirmed. In both cases, the full set of these currents has been listed [see Eqns (2.10), (3.15), and (3.16)]. For massless particles, the explicitly covariant representations exist only for the currents nondiagonal in helicity [see Eqn (3.25)]. The diagonal currents have algebraic representations that are not explicitly covariant [see Eqn (3.30)] because of their dependence on an auxiliary vector not entering the initial conditions. Its variation, however, affects only an unobservable common phase factor.

Considering massless particles as an example, we observed the breakdown of the well-known principle according to which the solution of a problem in the presence of a symmetry should contain only those vectors entering the statement of the problem. This principle does not work if the required quantity is defined up to a phase factor.

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