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Excluded volume effect in statistics of self-avoiding walks

V I Alkhimov

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Abstract. The main results of investigations into the self-avoiding walk problem are reviewed. Different approaches to solving this problem are briefly discussed. Attention is paid to the asymptotic solution of the exact equation obtained for the distribution density $W_n(\boldsymbol{R})$, where $\boldsymbol{R}$ is the vector connecting the end points of an N-step self-avoiding walk. On the basis of the single-functional approach the problem of the diffusion of tracers in a random-velocity field is discussed in the general case. Also, approximate methods allowing one to obtain approximation equations together with the conditions for their validity are discussed. The case of plane parallel average flow is considered in detail and some peculiarities of statistical solutions are discussed for the simplest problem.

Two fixed ideas can no more exist together in the moral world than in the physical world two bodies can occupy the same place at the same time.
A S Pushkin *The Queen of Spades*

1. Introduction

The self-avoiding walk (SAW) problem arose in the statistical physics of linear polymers (macromolecules) in connection with the calculation of the mean spatial dimensions of a polymer and is an important topic in the modern science of polymers. Because of the extremely large number of degrees of freedom in macromolecules they can be looked upon as macroscopic systems and one can use statistical methods to determine the mean values of variables such as, for example, the spatial dimensions of a macromolecule. The results of numerous experiments and theoretical evaluations show that the small-scale structure of molecules becomes less and less essential for the explanation of macroscopic properties of such systems as the length of the macromolecules increases. Herein the leading role in the configuration statistics of macromolecules is played by both their chain structure and the attendant excluded volume effect, according to which a given volume element cannot contain more than one link (or monomer) of the polymer chain at any given time. It is precisely these factors that determine the typical features of the behaviour of macromolecules. At the same time it is the chain structure of the linear polymer and the path of a random particle that brings about the analogy between their descriptions. But to make the analogy complete one has to take into account the excluded volume effect in the theory of random walks. That is why the problem under consideration is known as that of the self-avoiding walk (SAW).

A little more than twenty years ago there came a new fruitful stage in the development of polymer statistics, brought about by the penetration of concepts and methods from modern theoretical physics. The functional integration and renormalization group methods, as well as efficient numerical methods for studying the statistics of the lattice polymer chain models proved to be highly useful for the understanding of the process and its connection with the physics of critical phenomena.

V I Alkhimov Moscow Pedagogical University, 10a ul. Radio, 107005 Moscow
Tel. (7-095) 261-0948

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Because of the equivalence between the polymer and SAW problems, functional integration methods were employed in the majority of theoretical approaches to these problems. This is well reflected in review articles and monographs (see, e.g. Refs [1, 2]), so here, in section 2, only a brief description will be given. It should, however, be remarked that the critical exponents for the SAW obtained by this method are only meaningful if the spatial dimensionality d is close to its formal value $d = 4$, and it is not as yet clear how to get results for real space in this way. Meanwhile there is another, less widely used, but not less efficient method of studying this problem, namely one based on the search for a solution to the exact equation for the probability density of the end-to-end distance of the random walk. Owing to the efficiency of this method in studying the SAW problem as well as in other fields of physics, it has gained a much wider recognition of late.

In view of the above it is reasonable to describe this approach in more detail than in the earlier review [3]. This will be done in sections 3–5 of this paper, while the second section contains a brief review of some other ways of dealing with the problem.

2. Statement of the problem and a brief review of results

2.1 Phenomenological approach

The main quantity in the SAW problem is usually taken to be the probability density $W_N(\boldsymbol{R})$ of the vector $\boldsymbol{R}$ joining the end points of the N-step random path ($N \gg 1$). Given the function $W_N(\boldsymbol{R})$ one could immediately deduce the mean square end-to-end distance $\langle R^2\rangle_N$, which is an important spatial characteristic of the path:

$$\langle R^2\rangle_N = \int R^2 W_N(\boldsymbol{R})\,\mathrm{d}^d R \ , \tag{2.1}$$

where d is dimensionality of the space. In the ideal case where there is no volume effect, the asymptotic form of $W_N(\boldsymbol{R})$ as $N \to \infty$ is

$$W_N^{(0)}(\boldsymbol{R}) \sim \left(\frac{d}{2\pi N l^2}\right)^{d/2} \exp\left(-\frac{dR^2}{2Nl^2}\right) \ , \tag{2.2}$$

where l is the length of the displacement of the point particle. Hence, the mean square end-to-end distance for the N-step path is

$$\langle R^2\rangle_N^{(0)} = Nl^2 \ . \tag{2.3}$$

The excluded volume problem in polymer chain statistics first arose in the thermodynamic theory of polymer solutions [4]. In this theory a model of the lattice liquid with either one solvent molecule, or one macromolecular monomer located at each cell was used. The change in the free energy produced by mixing the macromolecule and the solvent was determined and regarded as a function of the polymer chain swelling coefficient

$$\alpha_N = \left(\frac{\langle R^2\rangle_N}{\langle R^2\rangle_N^{(0)}}\right)^{1/2} \ , \tag{2.4}$$

where $\langle R^2\rangle_N$ and $\langle R^2\rangle_N^{(0)}$ are the mean square end-to-end distances for the N-member chain obtained with and without taking the excluded volume effect into account, respectively. A subsequent minimization of the free-energy with respect to the parameter α_N yields the following equation for the equilibrium value:

$$\alpha_N^5 - \alpha_N^3 = \frac{3\sqrt{3}}{2}\, v_1 N^{1/2} \ , \tag{2.5}$$

here $v_1 = (3/2\pi l^2)^{3/2} v_0$, v_0 is the excluded volume of the monomer, and l now stands for the distance between two neighbouring monomers in the chain. For sufficiently large N, Eqn (2.5) implies the classical result by Flory

$$\langle R^2\rangle_N \sim \text{const} \times (v_0 l^{-3})^{2/5} N^{6/5} l^2 \ . \tag{2.6}$$

More recently a number of different closed expressions for α_N similar to Eqn (2.5) have been suggested. They are generally based either upon computer processing experimental data, or upon a simulation of the problem by a self-avoiding walk on some spatial lattice.

A generalisation of Flory's formula (2.6) to arbitrary dimensionality d was first made by Fisher [5] in the form

$$\langle R^2\rangle_N \sim \text{const} \times (v_0 l^{-d})^{2\nu_F/3} N^{2\nu_F} l^2 \ , \tag{2.7}$$

where the critical exponent ν_F is given by

$$\nu_F = \begin{cases} 3(d+2)^{-1}, & d < 4 \ , \\ \frac{1}{2}, & d > 4 \ . \end{cases} \tag{2.8}$$

It follows from (2.8) that the value $d = 4$ is special, because when $d > 4$, the asymptotic behaviour of $\langle R^2\rangle_N$ in N is just the same as that in the absence of the volume effect, i.e.

$$\langle R^2\rangle_N \sim \mathrm{O}(Nl^2) \ .$$

Numerous experimental data and the results of computer calculations of $\langle R^2\rangle_N$ are well described by formula (2.7).

The excluded volume effect is essentially a long-range order phenomenon, since the spatial configuration of the macromolecule is mostly determined by the volume interaction between its monomers widely separated in the polymer chain. The interaction between any two monomers joined to each other by more than one link can be described in a simplified form by a short-range repulsive potential $U(\boldsymbol{R})$ with range r_0, where $r_0 < l$.

The quantity

$$v_0 = \int \left[1 - \exp(-U(\boldsymbol{R})T^{-1})\right] \mathrm{d}^d R$$

is then the excluded volume in the d-dimensional Euclidean space $\mathscr{R}^d$ and T is the absolute temperature expressed in terms of energy. In the simplest model, the monomer can be considered as a perfectly hard sphere of diameter r_0 and the polymer is a chain of $N+1$ monomers freely joined one to another by N identical links of length l. Let us denote by

$$U_N = \sum_{1 \leqslant i < j \leqslant N} \sum U(\boldsymbol{L}_{ij})$$

the potential energy of the polymer chain, where the vector

$$\boldsymbol{L}_{ij} = \sum_{i \leqslant m \leqslant j} \boldsymbol{l}_m$$

joins the geometric centres of the $i-1$st and jth monomers; the vector $\boldsymbol{l}_m$ ($|\boldsymbol{l}_m| = l, m = 1, 2, \ldots, N$) is drawn from the $m-1$st monomer to the mth monomer, and the 0th monomer coincides with the origin of the coordinate system in $\mathscr{R}^d$. It is easy to see that the volume interaction between the monomers of the polymer chain gives rise to a correlation between the spatial orientations of its links.

A convenient measure of this correlation is the mean cosine of the link-to-link angle:

$$\langle \cos\theta_{mn}\rangle = Q_N^{-1}\int (\boldsymbol{e}_m\cdot\boldsymbol{e}_n)\exp(-U_N T^{-1})\prod_{k=1}^{N} \mathrm{d}\Omega_k \ , \qquad (2.9)$$

where Q_N is the normalising factor given by

$$Q_N = \int \exp(-U_N T^{-1})\prod_{k=1}^{N} \mathrm{d}\Omega_k \ ,$$

θ_{mn} is the angle between the vectors $\boldsymbol{e}_m = \boldsymbol{l}_m/l$ and $\boldsymbol{e}_n = \boldsymbol{l}_n/l$ whose scalar product is denoted by $(\boldsymbol{e}_m\cdot\boldsymbol{e}_n)$, $\mathrm{d}\Omega = \omega^{-1}d\omega$, where $\mathrm{d}\omega$ is the area element on a unit sphere in $\mathscr{R}^d$,

$$\omega = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

is the total surface area of the sphere, and $\Gamma(x)$ is the Euler gamma function. Clearly, when $U(\boldsymbol{R}) \equiv 0$ so that $v_0 = 0$ we have $\langle\cos\theta_{mn}\rangle = 0$. It is natural to expect that $\langle\cos\theta_{mn}\rangle > 0$ when $v_0 > 0$. The obvious equality

$$\langle L_{ij}^2\rangle = (j-i+1)l^2 + 2l^2\sum_{i\leqslant m<n\leqslant j}\sum \langle\cos\theta_{mn}\rangle \qquad (2.10)$$

then shows that the excluded volume effect produces an increase in the mean square distance between the monomers in the polymer chain compared with the nonperturbed case. We shall now use relation (2.10) to construct the difference relation

$$\langle L_{ij}^2\rangle - \langle L_{i+1j}^2\rangle - \langle L_{ij-1}^2\rangle + \langle L_{i+1j-1}^2\rangle = 2l^2\langle\cos\theta_{ij}\rangle \qquad (2.11)$$

and then approximate it by a differential equation for large $t \equiv j-i+1$. As a result we see that the function $y(t) \equiv \langle L_{ij}^2\rangle/l^2$ satisfies

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \varepsilon(t) \ , \qquad (2.12)$$

where $\varepsilon(t) \equiv 2\langle\cos\theta_{ij}\rangle$. Taking into account the identity $\langle L_{1N}^2\rangle \equiv \langle R^2\rangle_N$ we let t tend to N and substitute relation (2.7) into the left-hand side of Eqn (2.12). We arrive at the asymptotic relation

$$\langle\cos\theta_{1N}\rangle \sim \mathrm{O}(N^{-(d-2)/2}) \ , \qquad (2.13)$$

which shows that the excluded volume effect determines the power of the decrease in the correlation coefficient between the directions of the chain links $\boldsymbol{l}_1$ and $\boldsymbol{l}_N$ as N increases. The exponent $s = (d-2)/2$ in relation (2.13) turns out to be less than one when $d < 4$ and greater than one when $d > 4$.

The self-consistent field method was used in the earliest attempts to solve this problem [6–11]. It involves the replacement of an individual macromolecule by an ideal polymer chain located in the so-called self-consistent field $\varphi(\boldsymbol{R})$. The effect of the latter on the ideal chain should in some sense be equivalent to the excluded volume effect in the real macromolecule. If it were possible to proceed along this line, i.e. to define the self-consistent field explicitly, the density $W_N(\boldsymbol{R})$ could be found with the help of the Fokker–Planck equation

$$\frac{\partial W_N(\boldsymbol{R})}{\partial N} = \frac{l^2}{2d}\Delta W_N(\boldsymbol{R}) - \frac{\varphi(\boldsymbol{R})}{T}W_N(\boldsymbol{R}) \qquad (2.14)$$

combined with the initial condition

$$W_0(\boldsymbol{R}) = \delta(\boldsymbol{R}) \ , \qquad (2.15)$$

where Δ is the Laplacian in $\mathscr{R}^d$ and $\delta(\boldsymbol{R})$ the Dirac delta-function [12–13].

One of the ways previously suggested of implementing the self-consisting field method in this problem is to apply the Kirkwood variational principle to the free energy of the polymer chain [8]. By this principle, in the macromolecular Gibbs distribution

$$\mathcal{D}_N = Q_N^{-1}\exp(-U_N T^{-1})\prod_{k=0}^{N-1}\delta(R_{k,k+1} - l) \ , \qquad (2.16)$$

where $R_{ij} = |\boldsymbol{R}_i - \boldsymbol{R}_j|$ is the distance between the ith and jth monomers, the polymer chain potential energy,

$$U_N = \sum_{0\leqslant i<j\leqslant N}\sum U(R_{ij}) \qquad (2.17)$$

is replaced by the sum

$$\sum_{k=0}^{N}\varphi(\boldsymbol{R}_k) \ , \qquad (2.18)$$

where the function $\varphi(\boldsymbol{R}_k)$ to be defined is the potential energy of the kth monomer in the self-consistent field. We then write down the expression

$$F = U - TS = \int (U_N + T\ln\mathcal{D}_N)\mathcal{D}_N\prod_{k=0}^{N}\mathrm{d}^d R_k \qquad (2.19)$$

for the chain free energy and equate to zero its variations with respect to each variable $\varphi(\boldsymbol{R}_k)$. The solution of the equation for $\varphi(\boldsymbol{R})$ thus obtained must give the 'best' possible approximation for the self-consistent field. But this solution contains both the single-particle and the two-particle coordinate distribution functions for the monomers forming the chain. The quantity $W_N(\boldsymbol{R})$ may be regarded as the single-particle distribution function for the Nth monomer, since the position of the 0th monomer is fixed at $\boldsymbol{R} = 0$. We are faced here with a situation analogous in a sense to that in the theory of dense gases and liquids: there is hierarchy of n-particle distribution functions connected by a system of integral equations. To obtain a closed equation for $W_N(\boldsymbol{R})$ one has to make some approximation to the n-particle distribution function, usually to factorise it. For example, the approximation of the two-particle distribution function which is equivalent to the Edwards approximation [6], yields the following equations for the self-consistent field

$$\varphi(\boldsymbol{R}) = Tv_0 c(\boldsymbol{R}) \ , \qquad (2.20)$$

where

$$c(\boldsymbol{R}) = \sum_{N\geqslant 0} W_N(\boldsymbol{R})$$

determines the density of monomers at the point $\boldsymbol{R}$. Analysing the consistency condition of Eqns (2.14) and (2.20), we can see that for $d = 3$ the field $\varphi(\boldsymbol{R})$ must depend on $\boldsymbol{R}$ in such a way that

$$\varphi(\boldsymbol{R}) \sim \mathrm{O}(R^{-4/3}) \ , \qquad (2.21)$$

when $R^2 < \langle R^2\rangle_N$. It follows that the chain mean square end-to-end distance $\langle R^2\rangle_N$ behaves as $\mathrm{O}(N^{6/5})$ when N is large enough.

A generalisation of the Edwards results to the d-dimensional case was provided by Gillis and Freed [14] who showed that the field $\varphi(\boldsymbol{R})$ satisfies

$$\varphi(\boldsymbol{R}) \sim \mathrm{O}(R^{-2(d-1)/3}) \ , \tag{2.22}$$

when $d < 4$ and decreases faster than $\mathrm{O}(R^{-2})$ when $d > 4$. Hence the critical exponent ν in the asymptotic formula $\langle R^2 \rangle_N / l^2 \sim \mathrm{O}(N^{2\nu})$, where $N \to \infty$, is equal to $\nu = \nu_{\mathrm{F}}$.

In conclusion it should be noted that the approach discussed above has not led to any important results for the density $W_N(\boldsymbol{R})$. It is likely that the inefficacy of the self-consistent field method in determining $W_N(\boldsymbol{R})$ lies primarily in the replacement of the substantially non-Markovian SAW process by a Markovian model. This method is described in more detail in monographs [15–17].

2.2 Self-avoiding walks on lattices

The excluded volume problem elicited numerous studies of the SAW for lattice models. The statement of the problem is straightforward and can be formulated as follows.

On the integral lattice

$$\mathscr{Z}^d = \{(z_1, z_2, \ldots, z_d) \colon \ z_i \in \mathscr{Z}\}$$

we consider self-avoiding paths each consisting of a point sequence $(\boldsymbol{R}_0, \boldsymbol{R}_1, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_N)$ in $\mathscr{Z}^d$ such that $\boldsymbol{R}_0 = 0$, $|\boldsymbol{R}_i - \boldsymbol{R}_{i-1}| = 1$ for $i = 1, 2, \ldots, N$ and $\boldsymbol{R}_i \neq \boldsymbol{R}_j$ when $i \neq j$. We shall denote by $\mathcal{N}(N)$ the number of all self-avoiding paths of length N.

The objective then is to obtain the asymptotic dependence of $\mathcal{N}(N)$ and $\langle R_N^2 \rangle$ on N for $N \to \infty$. Both computer calculations and analytic approaches have been used for this purpose. There are two kinds of numerical methods: some give ways for the direct sorting of all self-avoiding paths on some lattice, others involve different versions of the Monte-Carlo method. But owing to the limited capability of methods of the first kind they are so far restricted to self-avoiding paths whose lengths do not exceed 10^2, and the question arises to what extent the dependence of $\mathcal{N}(N)$ and $\langle R_N^2 \rangle$ on N thus obtained can be viewed as asymptotic. As for the Monte-Carlo method, it gives results whose degree of reliability is certainly determined by both the statistical nature of the method and the quality of the algorithm chosen to generate the self-avoiding paths.

The above statement of the SAW problem was first given by Hammersley [18, 19] who proved the inequality

$$\mathcal{N}(N_1 + N_2) \leqslant \mathcal{N}(N_1)\mathcal{N}(N_2) \ ,$$

which implies the existence of a nonnegative constant k (called the connection constant) such that

$$0 \leqslant k = \lim_{N \to \infty} N^{-1} \log \mathcal{N}(N) < \infty$$

(see also Refs [20–21]). It follows that when $N \to \infty$ the value of $\mathcal{N}(N)$ behaves as $\mathrm{O}(\tilde{z}^N)$, where $\tilde{z}$ is the effective coordination number for the SAW. The number $\tilde{z}$ depends on the choice of lattice. For example, for the plane square lattice we have $\tilde{z} \simeq 2.64$ while for the simple cubic lattice we have $\tilde{z} \simeq 4.68$.

On the basis of an analysis of the results of numerical calculations for various lattice models, Fisher and Sykes [22] suggested a more precise asymptotic formula

$$\mathcal{N}(N) \sim \mathrm{const} \times N^{\gamma-1} \tilde{z}^N, \quad N \to \infty \ , \tag{2.23}$$

where the exponent γ is no longer dependent on the lattice type and depends only on the spatial dimensionality d. According to the numerical results presented in Ref. [23], we have the approximate formula $\gamma \approx 1 + \frac{1}{6}(4 - d)$ when $2 \leqslant d \leqslant 4$.

As mentioned above, the result of repeated calculations of the mean square end-to-end distance for self-avoiding paths of length N can be described by $\langle R_N^2 \rangle \sim N^{2\nu}$ as $N \to \infty$; here $\nu \approx 3/(d+2)$ for $d < 4$ [24–26]. A more difficult problem is to find the asymptotic form of the distribution function $W_N(\boldsymbol{R})$ for the distance $\boldsymbol{R} = \boldsymbol{R}_N$ as $N \to \infty$.

Nevertheless, with the help of the scaling conjecture according to which the statistical properties of the SAW when $N \to \infty$ are determined only by the characteristic length

$$\bar{R}_N = \langle R_N^2 \rangle^{1/2} \sim N^{\nu} \ , \tag{2.24}$$

the following asymptotic form of the distribution function is established:

$$W_N(\boldsymbol{R}) \sim \bar{R}_N^{-d} \Psi(R \bar{R}_N^{-1}) \ , \quad 1 \ll R_N \ll N \ , \tag{2.25}$$

where the function $\Psi(t)$ is approximated by the expression

$$\Psi(t) = \psi(t) \exp(-t^{\delta}) \ , \tag{2.26}$$

based upon numerical calculations of $W_N(\boldsymbol{R})$, $\psi(t)$ is a power function in t, possibly different for the cases $t \to 0$ and $t \to \infty$ and δ is defined by $\delta = 1/(1 - \nu)$ [27–33]. Cloizeaux [34] showed that if

$$\psi(t) \sim \mathrm{O}(t^g) \ , \tag{2.27}$$

when $t \to 0$, the exponents g and γ are related by

$$\gamma = 1 + \nu g \ . \tag{2.28}$$

From this and the above expressions for γ and ν we obtain $g \approx \frac{1}{18}(4 - d)(d + 2)$ when $2 \leqslant d \leqslant 4$.

Thanks to the improvements both in computers, and in numerical methods for calculating $\mathcal{N}(N)$ (see Refs [35–48]) the following values of γ and ν have become accepted:

$$\gamma = 1.343\ldots, \quad \nu = 0.749\ldots, \quad \text{when } d = 2 \ ,$$

$$\gamma = 1.166\ldots, \quad \nu = 0.592\ldots, \quad \text{when } d = 3 \ .$$

For the case of spatial dimension $d = 4$, the relation

$$\bar{R}_N \sim N^{1/2} (\ln N)^{1/8}, \quad N \to \infty \ . \tag{2.29}$$

was obtained by Havlin and Ben-Avraham [49].

Finally, for $d > 4$, Bridges and Spencer [50] showed rigorously that the asymptotic form of the end-to-end distance distribution function for self-avoiding paths of length N is Gaussian, and that therefore in this case we have $\gamma = 1$ and $\nu = \frac{1}{2}$ [51].

The so-called n-vector model [52, 53] is of great importance for the analytic investigation of self-avoiding walks on lattices. In this model a spin-vector $\boldsymbol{\sigma}_z$ with n components

$$\sigma_{z1}, \sigma_{z2}, \ldots, \sigma_{zn} \ ,$$

normalised by

$$\boldsymbol{\sigma}_z^2 = \sum_{i=1}^{n} \sigma_{zi}^2 = n, \quad \forall z \in \mathscr{Z}^d \ . \tag{2.30}$$

is located at each node of the lattice $\mathscr{Z}^d$.

Only nearest neighbour spins interact in this model and the interaction energy is minimal when the spins are parallel. Hence the Hamiltonian $H(\boldsymbol{\sigma})$ of the module takes the form

$$H(\boldsymbol{\sigma}) = -J \sum_{(z,z')} \boldsymbol{\sigma}_z \cdot \boldsymbol{\sigma}_{z'} \ , \tag{2.31}$$

where $J > 0$, and the summation is taken over all pairs (z, z') of neighbouring nodes of the lattice $\mathscr{Z}^d$. The partition function of the system is

$$Q_n = \int \exp(-H(\boldsymbol{\sigma})T^{-1}) \prod_{z \in \mathscr{Z}^d} \mathrm{d}\Omega_z \ ; \tag{2.32}$$

where $\mathrm{d}\Omega = \omega^{-1}\,\mathrm{d}\omega$, $\mathrm{d}\omega$ is the area element on a unit sphere in $\mathscr{Z}^d$, and $\omega = 2\pi^{n/2}/\Gamma(n/2)$ is the total surface of the sphere.

A very important quantity in this model is its spin-to-spin correlation function

$$\langle \sigma_{zi}\sigma_{z'i}\rangle_n = Q_n^{-1} \int \sigma_{zi}\sigma_{z'i} \exp(-H(\boldsymbol{\sigma})T^{-1}) \prod_{z \in \mathscr{Z}^d} \mathrm{d}\Omega_z \ , \tag{2.33}$$

which is proportional to the magnetisation correlation function $\langle M(0)M(\boldsymbol{R})\rangle_n$ in the corresponding magnetic model, $i = 1, 2, \ldots, n$, where $M(\boldsymbol{R})$ is the local magnetisation at the point $\boldsymbol{R} = \boldsymbol{R}_{z'} - \boldsymbol{R}_z$. The knowledge of the correlation function $\langle \sigma_{zi}\sigma_{z'i}\rangle_n$ allows us to describe many properties of magnetic materials among them the susceptibility

$$\chi = \frac{1}{T} \sum_{z' \in \mathscr{Z}^d} \langle \sigma_{zi}\sigma_{z'i}\rangle_n \ . \tag{2.34}$$

De Gennes [54] was the first to point out that the n-vector model can be used for solving the SAW problem. But for that one needs to let n tend to zero. As n is in fact a natural number, the above procedure should be looked upon as only a mathematical device. Its essence is the following.

Using the formula

$$\exp(-H(\boldsymbol{\sigma})T^{-1}) = \prod_{(z,z')} \left[1 + \sum_{m=1}^{\infty} \frac{(J/T)^m}{m!} \left(\sum_{i=1}^{n} \sigma_{zi}\sigma_{z'i} \right)^m \right], \tag{2.35}$$

where the product is only defined for pairs of neighbouring nodes of the lattice $\mathscr{Z}^d$ we expand the exponential functions in (2.32) and (2.33) as power series and thus obtain for Q_n the expansion in powers of the interaction constant J/T whose coefficients are products of integrals of the form

$$\int \sigma_{zi}\sigma_{zj} \ldots \sigma_{zk}\,\mathrm{d}\Omega_z \ , \quad z \in \mathscr{Z}^d \ . \tag{2.36}$$

By symmetry, the integrals in (2.36) vanish when the integrand consists of an odd number of spin factors. Moreover, if we formally let n tend to zero, then, by the moment theorem (see, for example, Ref. [16]), all the integrals in (2.36) with more than two spin factors will vanish, whereas the integrals with two spin factors will be equal to

$$\int \sigma_{zi}\sigma_{zj}\,\mathrm{d}\Omega_z = \delta_{ij} \ , \tag{2.37}$$

where δ_{ij} is the Kronecker symbol. It follows that for the zero-component spin model one can use

$$\prod_{(z,z')} [1 + JT^{-1}\boldsymbol{\sigma}_z \cdot \boldsymbol{\sigma}_{z'} + \tfrac{1}{2}(JT^{-1})^2(\boldsymbol{\sigma}_z \cdot \boldsymbol{\sigma}_{z'})^2] \ . \tag{2.38}$$

instead of $\exp(-H(\boldsymbol{\sigma})/T)$ in expressions (2.32) and (2.33). We next expand expression (2.38) as a series and integrate it over all orientations of the spins $\boldsymbol{\sigma}_z$, $z \in \mathscr{Z}^d$. According to the moment theorem, only those integrals of products

$$(JT^{-1})^N \sigma_{zi}^2 \sigma_{z'i}^2 \sigma_{z''i}^2 \ldots \sigma_{z'''i}^2 \ , \tag{2.39}$$

do not vanish in which the sequence $z, z', z'', \ldots, z'''$ describes a closed chain consisting of N pairs of neighbouring nodes $(z, z'), (z', z''), \ldots, (z''', z)$ of the lattice $\mathscr{Z}^d$. If with each pair of neighbouring nodes we associate the line segment joining them, then expression (2.39) can be represented graphically as a closed self-avoiding path usually termed a loop. Performing the appropriate summation over all values of the spin components and taking integrals (2.37) into consideration, we see that the contribution of a single loop is equal to $n(JT^{-1})^N$, where N is the length of the loop. It follows that for $n = 0$ the contribution of all the loops to the partition function Q_0 vanishes, and we arrive at the trivial result

$$Q_0 = 1 \ . \tag{2.40}$$

We now substitute expression (2.38) for the exponential function in Eqn (2.33) and expand the product contained therein as a series as in the previous case, but we now have two external spin factors, σ_{zi} and $\sigma_{z'i}$, in the integrand in (2.33), as a result of which the products with nonvanishing integrals are

$$(JT^{-1})^N \sigma_{zi}(\sigma_{zi}\sigma_{z''i} \ldots \sigma_{z'''i}\sigma_{z'i})\sigma_{z'i} \ . \tag{2.41}$$

If we set $n = 0$ in this case, the self-avoiding paths joining the nodes z and z' will be the only paths to provide a nonzero contribution to the correlation function. As the index i in expression (2.41) is fixed we need only perform the summation over all self-avoiding paths joining the nodes z and z'. As a result we obtain the important formula

$$\langle \sigma_{zi}\sigma_{z'i}\rangle_0 = \sum_N \mathcal{N}_{zz'}(N)(JT^{-1})^N \ , \tag{2.42}$$

where $\mathcal{N}_{zz'}(N)$ is the number of all self-avoiding paths of length N joining the nodes z and z' on the lattice $\mathscr{Z}^d$. Formula (2.42) serves as the basis for the use of the zero-component spin model in the investigation of the SAW problem.

As an example, let us apply formula (2.42) to expression (2.34). We obtain

$$\chi = \frac{1}{T} \sum_N \mathcal{N}(N)(JT^{-1})^N \ , \tag{2.43}$$

where

$$\mathcal{N}(N) = \sum_{z' \in \mathscr{Z}^d} \mathcal{N}_{zz'}(N)$$

is the total number of self-avoiding paths of length N which start from the node $z \in \mathscr{Z}^d$.

Replacing the quantity $\mathcal{N}(N)$ in Eqn (2.43) by its asymptotic value (2.23) we obtain the approximate relation

$$\chi \approx \frac{\text{const}}{T} F\left(\frac{T_c}{T}, 1-\gamma\right) , \tag{2.44}$$

in which $T_c = \tilde{z}J$ is the critical temperature and

$$F(t, 1-\gamma) = \sum_{N=1}^{\infty} N^{\gamma-1} t^N, \quad |t| < 1 .$$

With the help of the relation

$$\lim_{t\to 1}(1-t)^{\gamma} F(t, 1-\gamma) = \Gamma(\gamma), \quad \gamma > 0$$

(see Ref. [55]) we obtain from relation (2.44) the following asymptotic expression for susceptibility:

$$\chi \sim \frac{\text{const}}{T_c} \tau^{-\gamma} , \tag{2.45}$$

when the quantity $\tau = (T - T_c)/T_c$ is positive and tends to zero. Thus, the parameter γ determines the degree of singularity of χ as $T \to T_c$, because of which γ is called the critical index of susceptibility.

Let us come back to formula (2.42) and clarify the behaviour of the correlation function when $T \to T_c$. For this purpose we use the definition

$$W_N(R) = \frac{\mathcal{N}_{zz'}(N)}{\mathcal{N}(N)} , \tag{2.46}$$

where $R = |\boldsymbol{R}_{z'} - \boldsymbol{R}_z|$, and the asymptotic relations (2.23) and (2.25).

For sufficiently small τ we then have

$$\langle \sigma_{zi}\sigma_{z'i}\rangle_0 \cong \sum_N \exp(-\tau N) N^{\gamma-1-\nu d} \Psi(RN^{-\nu}) , \tag{2.47}$$

which implies that τ and N are interrelated so that small values of τ are associated with large values of N. If we change summation in expression (2.47) for integration with respect to N, we obtain

$$\langle \sigma_{zi}\sigma_{z'i}\rangle_0 \cong R^{2-d-\eta} G(R\tau^{\nu}) , \tag{2.48}$$

where τ is close to the critical value $\tau = 0$, η is the new critical exponent given by

$$\gamma = (2-\eta)\nu , \tag{2.49}$$

and $G(x)$ is a dimensionless function that satisfies

$$G(x) \sim \begin{cases} 1, & x \to 0 , \\ x^{\eta}\exp(-x), & x \to \infty . \end{cases} \tag{2.50}$$

From relations (2.48) and (2.50) we conclude that according to this model

$$\bar{R} \sim \tau^{-\nu} \tag{2.51}$$

plays the role of the correlation length that determines the mean size of the region in $\mathcal{R}^d$ filled with self-avoiding paths. That is why the index ν is called the critical correlation length exponent. Thus it is sufficient to know only two independent critical exponents, e.g. η and ν, because all the other exponents can be expressed in terms of them.

2.3 Connection with the theory of critical phenomena

The early 1970s were marked by a significant achievement in the physics of critical phenomena which in turn had a powerful influence on the development of the statistical physics of polymers, in particular, on the investigation of the excluded volume problem. Wilson [56, 57] and others [58–60] developed the renormalization group method for calculating the critical exponents that determine the degree of singularity of different thermodynamic variables at the phase transition point. The conceptual basis for Wilson's method was provided by so-called universality hypothesis of Kadanoff [61, 62], according to which the critical exponents should be insensitive to the detailed behaviour of the interaction potential and should be determined mainly by large-scale properties such as the dimensionality of the system and the symmetry of its Hamiltonian.

For this hypothesis to be employed, Kadanoff suggested that identical cells of linear size L each should be formed from the nodes of the lattice $\mathcal{Z}^d$, to yield a new lattice with the same symmetry and with lattice constant now equal to L. The cells each contain L^d nodes of the initial lattice and their centres become the nodes of the new lattice. Since the correlation length $\bar{R}$ tends to infinity when $\tau \to 0$, as follows from the asymptotic formula (2.51), for sufficiently small values of τ there exists a number L such that

$$1 \ll L \ll \bar{R} . \tag{2.52}$$

Therefore, each cell contains L^d spins which are strongly correlated, since, by condition (2.52), they are situated within the correlation range. It follows that the majority of spins in any cell must have the same direction, owing to which the spin cell behaves as a single effective spin. Moreover, as the spins on the initial lattice interact with nearest neighbouring spins only, the spin cells will also interact with nearest cells. So we can construct a cell Hamiltonian similar to expression (2.31) but describing only the interaction between nearest spin cells. We define the cell spin by

$$\boldsymbol{\sigma}_{\bar{z}} = L^{-d} \sum_{z \in b(\bar{z})} \boldsymbol{\sigma}_z ,$$

where the summation is carried out over the nodes z within the cell $b(\bar{z})$ centred at the node $\bar{z}$ of the new lattice $\overline{\mathcal{Z}}^d$. The cell Hamiltonian

$$H'(\boldsymbol{\sigma}) = -J' \sum_{(\bar{z},\bar{z}') \in \bar{\mathcal{Z}}^d} \boldsymbol{\sigma}_{\bar{z}} \cdot \boldsymbol{\sigma}_{\bar{z}'} \tag{2.53}$$

with J' as the new parameter depending, of course, on J can be defined by

$$\exp(-H'(\boldsymbol{\sigma})T^{-1}) = \int \exp(-H(\boldsymbol{\sigma})T^{-1}) \prod_{\bar{z}} \delta\left(\boldsymbol{\sigma}_{\bar{z}} - L^{-d} \sum_{z \in b(\bar{z})} \boldsymbol{\sigma}_z\right) \prod_z \mathrm{d}\Omega_z , \tag{2.54}$$

where the variables z and $\bar{z}$ run through the nodes of the lattices $\mathcal{Z}^d$ and $\overline{\mathcal{Z}}^d$, respectively. It is readily seen that the partition function defined in (2.32) now takes the form

$$Q_n = \int \exp(-H'(\boldsymbol{\sigma})T^{-1}) \prod_{\bar{z} \in \bar{\mathcal{Z}}^d} \mathrm{d}\Omega_{\bar{z}} .$$

By the universality hypothesis, the correlation between the cell spins $\boldsymbol{\sigma}_z$ and $\boldsymbol{\sigma}_{\bar{z}}'$ must have the same structure as that between the spins $\boldsymbol{\sigma}_z$ and $\boldsymbol{\sigma}_{z'}$ in the initial model, and

thereby the scale parameter L turns out to be inessential. The construction of the cell Hamiltonian $H'(\boldsymbol{\sigma})$ in (2.54) from the initial lattice Hamiltonian $H(\boldsymbol{\sigma})$ is called the Kadanoff transformation and is denoted by

$$H'(\boldsymbol{\sigma})T^{-1} = \hat{K}_L H(\boldsymbol{\sigma})T^{-1} \ . \tag{2.55}$$

By means of $H'(\boldsymbol{\sigma})$ we can clearly construct another cell Hamiltonian $H''(\boldsymbol{\sigma})$ whose cell spins are defined on still bigger cells. In fact, as above, we construct a supercell from $(L')^d$, then define the corresponding cell spin, and finally define the cell Hamiltonian $H''(\boldsymbol{\sigma})$. The last procedure is described by the transformation

$$H''(\boldsymbol{\sigma})T^{-1} = \hat{K}_{L'} H'(\boldsymbol{\sigma})T^{-1} \ . \tag{2.56}$$

Combining expressions (2.55) and (2.56) and using the fact that the operators $\hat{K}_L$ and $\hat{K}_{L'}$ commute, we obtain

$$\hat{K}_L \hat{K}_{L'} = \hat{K}_{LL'} \ . \tag{2.57}$$

The set of operators $\hat{K}_L$ is a commutative semigroup, which is not a group, because the inverse operation is not defined. Eqn (2.57) means that if we construct a cell of L^d nodes of the lattice $\mathscr{Z}^{\epsilon d}$ and then construct a supercell consisting of $(L')^d$ cells, the result will be the same as if we had constructed a large cell from $(LL')^d$ nodes of the initial lattice immediately. The Kadanoff transformation played a significant role in Wilson's formulation of the renormalization group method in the theory of critical phenomena.

The transformation (2.55) implies a relation $K' = f_L(K)$ between the interaction constants $K = J/T$ and $K' = J'/T$. According to Kadanoff's hypothesis all the model systems obtained by successive partitioning of the initial lattice $\mathscr{Z}^{\epsilon d}$ into cells behave similarly near the critical value. Hence we can write

$$\bar{R}(K) = L\bar{R}(f_L(K)) \ , \tag{2.58}$$

That is, an L-fold increase in the cell dimension gives rise to an L-fold decrease in the correlation length. For convenience, we shall now assume the scale parameter L to be a positive real number and write $K_L = f_L(K)$. Further, we take into account Wilson's assumption that the relative change, for example in K_L, caused by the transition $L \to L'$ depends only on K_L, but cannot depend explicitly on L. Then the change in K_L corresponding to the infinitesimal change in the parameter $L, L' - L = \delta L$, is

$$K_{L+\delta L} - K_L \cong \left(\frac{\mathrm{d}K_L}{\mathrm{d}L}\right)\delta L \equiv \delta h \ ,$$

where h depends explicitly on K_L but not on L. As a result we obtain the differential equation

$$\frac{\mathrm{d}K_L}{\mathrm{d}L} = \frac{1}{L}h(K_L) \ , \tag{2.59}$$

which is called the *renormalization group equation* (or RG-equation). A crucial feature of this approach is the assumption that the function $h(K)$ is analytic even at the critical point $K = K_c$, where, by definition,

$$\bar{R}(K_c) = \infty \ .$$

By virtue of Eqn (2.58), at this point we also have $\bar{R}(K_L) = \infty$ for all finite L. Hence the solution of the RG-equation (2.59) with the initial condition $K = K_c$ must take the form $K_L = K_c$ for all L, which implies that

$$h(K_c) = 0 \ . \tag{2.60}$$

A straightforward analysis of Eqns (2.58) and (2.59) shows that the critical point K_c is an unstable fixed point of the RG-equation. If we now linearize the function $h(K_L)$ in the neighbourhood of the critical point, then expression (2.60) enables us to replace the RG- equation by its approximation

$$\frac{\mathrm{d}K_L}{\mathrm{d}L} = \frac{y}{L}(K_L - K_c) \ , \tag{2.61}$$

where

$$y = \left.\frac{\mathrm{d}h(K)}{\mathrm{d}K}\right|_{K=K_c} \ . \tag{2.62}$$

The solution of Eqn (2.61) with the initial condition $K_L = K$ for $L = 1$ has the following form:

$$(K_c - K_L) = (K_c - K)L^y \ . \tag{2.63}$$

Let us choose an arbitrary number of the form $K_L = \alpha K_c$, $0 < \alpha < 1$, and denote by L_α the value of L for which $K_L = \alpha Kc$. From Eqn (2.63) we get

$$L_\alpha = \left[\frac{(1-\alpha)K_c}{K_c - K}\right]^{1/y} \ .$$

Finally, we substitute the last expression for L in Eqn (2.58) and use the fact that $K_c - K \cong \tau K_c$ near the critical point to show that

$$\bar{R}(K) \cong \left(\frac{1-\alpha}{\tau}\right)^{1/y} \bar{R}(\alpha K_c) \ , \tag{2.64}$$

for all sufficiently small τ. The indices y and ν are therefore connected by $y = 1/\nu$.

For this scheme ultimately to yield numerical values for the critical exponents it turns out to be more efficient to use a continuous analogue of the n-vector model in which a spin vector $\boldsymbol{\sigma}(\boldsymbol{x})$ taking arbitrary values is assigned to each point $\boldsymbol{x}$ of the space $\mathscr{R}^d$. So, instead of the discrete set of the variables $\boldsymbol{\sigma}_z$, we now have a set of continuous spin variables, i.e. a spin field $\boldsymbol{\sigma}(\boldsymbol{x})$. In the new model the cell spin $\boldsymbol{\sigma}_{\boldsymbol{x}}$ similar to the average cell spin $\boldsymbol{\sigma}_{\bar{z}}$ of the lattice $\overline{\mathscr{Z}}^d$ is conveniently defined by

$$\boldsymbol{\sigma}_{\boldsymbol{x}} = \int_{k<\Lambda} \exp(-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{k})\boldsymbol{\sigma}_{\boldsymbol{k}} \frac{\mathrm{d}^d k}{(2\pi)^d} \ . \tag{2.65}$$

It follows that $\boldsymbol{\sigma}_{\boldsymbol{x}}$ is the smoothed variable describing the spin distribution on a scale of L up to $L \sim \Lambda^{-1}$. When choosing the Hamiltonian to represent the interaction between the cell spins $\boldsymbol{\sigma}_{\boldsymbol{x}}$ one is usually guided by considerations of simplicity and convenience. In this respect the most suitable Hamiltonian suggested by Ginsburg and Landau is in the form

$$\mathcal{H}[\boldsymbol{\sigma}] = \int [a\boldsymbol{\sigma}_{\boldsymbol{x}}^2 + b\boldsymbol{\sigma}_{\boldsymbol{x}}^4 + c(\nabla\boldsymbol{\sigma}_{\boldsymbol{x}})^2]\,\mathrm{d}^d x \ , \tag{2.66}$$

where $\mathcal{H}[\boldsymbol{\sigma}] = H[\boldsymbol{\sigma}]T^{-1}$, the coefficients a, b, c are analytic functions of T and possibly of other variables, and

$$\boldsymbol{\sigma}_{\boldsymbol{x}}^2 = \sum_{j=1}^{n} (\sigma_{xj})^2, \quad \boldsymbol{\sigma}_{\boldsymbol{x}}^4 = (\boldsymbol{\sigma}_{\boldsymbol{x}}^2)^2 \ ,$$

$$(\nabla\boldsymbol{\sigma}_{\boldsymbol{x}})^2 = \sum_{r=1}^{d}\sum_{j=1}^{n}\left(\frac{\partial\sigma_{xj}}{\partial x_r}\right)^2 \ ,$$

It is just the last expression that describes the interaction between the neighbouring cell spins $\boldsymbol{\sigma}_x$. From Eqns (2.65) and (2.66) we obtain

$$\mathcal{H}[\boldsymbol{\sigma}] = \int_{k<\Lambda} (a + ck^2)\boldsymbol{\sigma}_{\boldsymbol{k}}\cdot\boldsymbol{\sigma}_{-\boldsymbol{k}} \frac{\mathrm{d}^d k}{(2\pi)^d} + \frac{b}{(2\pi)^{3d}} \int_{(k_i<\Lambda)} (\boldsymbol{\sigma}_{\boldsymbol{k}_1}\cdot\boldsymbol{\sigma}_{\boldsymbol{k}_2})(\boldsymbol{\sigma}_{\boldsymbol{k}_3}\cdot\boldsymbol{\sigma}_{\boldsymbol{k}_4})\delta\left(\sum_{i=1}^{4} \boldsymbol{k}_i\right) \prod_{i=1}^{4} \mathrm{d}^d k_i \tag{2.67}$$

where the integration is performed over the vectors $\boldsymbol{k}, \boldsymbol{k}_i$ whose length does not exceed Λ.

The set of parameters

$$\mu = (a, b, c) \tag{2.68}$$

can be used to characterise the distribution function

$$W[\boldsymbol{\sigma}] = \frac{1}{Q} \exp(-\mathcal{H}[\boldsymbol{\sigma}]) \ . \tag{2.69}$$

These parameters form a three-dimensional parametric space every point μ of which is associated with a specific distribution function W. Then the transition from W to another distribution function W' corresponds to a transformation that maps the point μ into the point μ'. To discover this correspondence we examine the transition from W to W'.

Let us first integrate the function $W[\boldsymbol{\sigma}]$ over all possible values of $\sigma_{\boldsymbol{k}}$ such that $\Lambda' < k < \Lambda$ where $\Lambda' = \Lambda/L$, $L > 1$, and express the result in the form

$$W'[\boldsymbol{\sigma}] = (Q')^{-1} \exp(-\mathcal{H}'[\boldsymbol{\sigma}]) = Q^{-1} \int \exp(-\mathcal{H}[\boldsymbol{\sigma}]) \prod_{\Lambda'<k<\Lambda} \mathrm{d}^n \sigma_{\boldsymbol{k}} \ . \tag{2.70}$$

This smoothing operation transforms the original Hamiltonian $\mathcal{H}$ depending on the cut-off parameter Λ into a new Hamiltonian $\mathcal{H}'$ which will depend on the parameter Λ'. Next we subject the variable $\boldsymbol{k}$ to the scaling transformation

$$\boldsymbol{k} \to \boldsymbol{k}' = L\boldsymbol{k} \tag{2.71}$$

and at the same time renormalise the spin fields by

$$\sigma_{\boldsymbol{k}} \to \sigma'_{\boldsymbol{k}'} = \lambda_L^{-1} \sigma_{\boldsymbol{k}} \ . \tag{2.72}$$

The smoothing operation (2.70) reduces the domain of definition of the field $\sigma_{\boldsymbol{k}}$ from the interval $0 < k < \Lambda$ to $0 < k < \Lambda'$, while the scale transformation (2.71) expands the reduced interval up to the initial size. As a result of the transformations (2.71) and (2.72) the Hamiltonian $\mathcal{H}'[\boldsymbol{\sigma}']$ now describes the new field $\sigma'_{\boldsymbol{k}}$ in the initial domain of the variable $\boldsymbol{k}$, but the coefficients a', b', c' in the expression for $\mathcal{H}'[\boldsymbol{\sigma}']$ form a new set of parameters

$$\mu' = (a', b', c') \ . \tag{2.73}$$

Thus the transition from W to W' can be formally looked upon as a transformation of the parameter set (2.68) into the parameter set (2.73):

$$\hat{R}_L \mu = \mu' \ , \tag{2.74}$$

where the operators $\hat{R}_L$ as well as Kadanoff's transformations constitute a continuous semigroup of renormalization transformations. Hence the following condition applies

$$\hat{R}_L \hat{R}_{L'} \mu = \hat{R}_{LL'} \mu \ , \tag{2.75}$$

so that $\lambda_L \lambda_{L'} = \lambda_{LL'}$. But the last equality is true only when $\lambda_L = L^{\alpha}$, where the exponent α does not depend on L. The use of the distribution $W[\boldsymbol{\sigma}]$ is equivalent in a sense to the use of the distribution $W'[\boldsymbol{\sigma}']$. For example, for the Fourier transform of the spin–spin correlation function

$$D(\boldsymbol{k}, \mu) = \int \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}) \langle \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+R} \rangle_W \, \mathrm{d}^d R = \langle |\sigma_{\boldsymbol{k}}|^2 \rangle_W \tag{2.76}$$

we have

$$D(\boldsymbol{k}, \mu) = \lambda_L^2 D(L\boldsymbol{k}, R_L \mu) \ . \tag{2.77}$$

This shows that averaging a quantity which is a product of the transformed spin field components with respect to the distribtuion function W' gives the same result as averaging this quantity in its original form with respect to the original distribution function W.

The starting point for the use of the renormalization group in the study of critical phenomena is the fact that the Hamiltonian of the system under consideration is RG-invariant. In this connection we define a fixed point μ^* of the transformation $\hat{R}_L$ by

$$\hat{R}_L \mu^* = \mu^* \tag{2.78}$$

for all $L > 1$. Eqn (2.78) will clearly hold in the limit as $L \to \infty$. Suppose that there exists at least one fixed point μ^* and define its critical surface to be the set of points μ of the parametric space that satisfy the condition

$$\lim_{L\to\infty} \hat{R}_L \mu = \mu^* \ . \tag{2.79}$$

We next suppose that Eqn (2.74) in which the points μ and μ' are located near μ^* can be approximated by

$$\hat{R}_L^l \delta\mu = \delta\mu' \ , \tag{2.80}$$

where $\delta\mu = \mu - \mu^*$, $\delta\mu' = \mu' - u^*$, and $\hat{R}_L^l$ is a linear operator obtained by the linearisation of the operator $\hat{R}_L$. If we now denote by $\rho_j(L)$ and e_j the eigenvalues and the corresponding eigenvectors of the operator $\hat{R}_L^l$, then owing to the equalities

$$\hat{R}_L^l \hat{R}_{L'}^l e_j = \hat{R}_{LL'}^l e_j \ ,$$
$$\rho_j(L)\rho_j(L') = \rho_j(LL') \ ,$$

we have

$$\rho_j(L) = L^{y_j} \ , \tag{2.81}$$

where the exponent y_j does not depend on L. Further, having in mind that μ depends on T [or on $\tau = (T - T_c)/T_c$] we expand the variation $\delta\mu(\tau)$ in terms of the eigenvectors e_j of the operator $\hat{R}_L^l$

$$\delta\mu(\tau) = \mu(\tau) - \mu^* = \sum_j t_j(\tau) e_j \ . \tag{2.82}$$

From Eqns (2.80)–(2.82) we obtain

$$\delta\mu'(\tau) = \sum_j t_j(\tau) L^{y_j} e_j \ . \tag{2.83}$$

By the basic hypothesis of the theory of critical phenomena the value $\mu(0)$ belongs to the critical surface of the fixed point μ^* and therefore

$$\lim_{L\to\infty} \hat{R}_L \mu(0) = \mu^* \ ,$$

but $\mu(\tau)$ does not belong to this surface when $\tau \neq 0$. As $\mu(\tau)$ is an analytic function, the point $\mu(\tau)$ is clearly located

near the critical surface when τ is small. Then the point $\hat{R}_L\mu(\tau)$ might possibly be found within a small neighbourhood of the fixed point μ^* if L is large enough, but will start receding from μ^* with further increase in L. Certainly, the behaviour of the point $\hat{R}_L\mu(\tau)$ depends on the exponents y_j in expression (2.83). Let us suppose that one of these exponents, for example y_1, is positive while the others are negative. In that case for $L \gg 1$ we have

$$\hat{R}_L\mu(\tau) \cong \mu^* + \hat{R}_L^l\delta\mu(\tau) = \mu^* + t_1(\tau)L^{y_1}e_1 + \mathrm{O}(L^{y_2}) \ , \tag{2.84}$$

where $y_2 < 0$ is the maximal negative exponent. Using the equality $t_1(0) = 0$ and the analyticity of the function $t_1(\tau)$ we expand the latter as the series

$$t_1(\tau) = A_1\tau + A_2\tau^2 + \ldots, \quad A_1 > 0 \ . \tag{2.85}$$

From formulas (2.84) and (2.85) it follows that for small values of τ

$$\begin{aligned}\hat{R}_L\mu(\tau) &\cong \mu^* + A_1\tau L^{y_1}e_1 + \mathrm{O}(L^{y_2}) \\ &= \mu^* + (L\bar{R}^{-1})^{1/\nu}e_1 + \mathrm{O}(L^{y_2}) \ ,\end{aligned} \tag{2.86}$$

where by definition

$$\nu^{-1} = y_1, \quad \bar{R} = |A_1\tau|^{-\nu} \ .$$

To determine the dependence of the function $D(\boldsymbol{k}, \mu(\tau))$ on τ and the vector $\boldsymbol{k}$ we apply the expressions (2.86) and $\lambda_L = L^\alpha$ to Eqn (2.77). As a result we have

$$D(\boldsymbol{k}, \mu(\tau)) = L^{2\alpha}D(L\boldsymbol{k}, \mu^* + (L\bar{R}^{-1})^{1/\nu}e_1 + \mathrm{O}(L^{y_2})) \ . \tag{2.87}$$

which takes the form

$$D(\boldsymbol{k}, \mu(\tau)) = \bar{R}^{2\alpha}D(\bar{R}\boldsymbol{k}, \mu^* + e_1 + \mathrm{O}(\bar{R}^{y_2})) \ . \tag{2.88}$$

when $L = \bar{R}$. If we now neglect the term $\mathrm{O}(\bar{R}^{y_2})$ when $\tau \to 0$ in the last equality, the correlation function $D(\boldsymbol{k}, \mu(\tau))$ will become a product of $\bar{R}^{2\alpha}$ and the function of $\bar{R}\boldsymbol{k}$, which confirms the similarity hypothesis. We next set $\tau = 0$ ($\bar{R} \to \infty$) and $L = 1/k$ in Eqn (2.87) to obtain

$$D(\boldsymbol{k}, \mu(0)) \cong k^{-2\alpha} \ , \tag{2.89}$$

By the definition of the critical exponent η we then have

$$2\alpha = 2 - \eta \ .$$

Finally, we put $k = 0$ in expression (2.88) and for τ small enough obtain

$$D(0, \mu(\tau)) \cong \bar{R}^{2-\eta} \cong \tau^{-(2-\eta)\nu} \tag{2.90}$$

Thus, by the definition of the critical exponent γ we arrive at the similarity rule $\gamma = (2 - \eta)\nu$ mentioned previously.

Wilson and Fisher introduced the ε-expansion method of evaluating the critical exponents, where ε is the deviation of the spatial dimensionality d from $d = 4$. The latter occupies a special position in the sense that standard perturbation theory can be used in this model when $d > 4$. The case $d = 4$ has been investigated in detail by Larkin and Khmel'nitskii [63]. The ε-expansion procedure allows the determination of the critical exponent in the form of a power series in ε. Calculations show that the coefficients of the ε-series initially decrease, but then begin to increase rapidly with their ordinal number, which suggests that these series are asymptotic in character. Thus, the method can be effective only for spatial dimensionalities close to $d = 4$, although it is clear that only integer values of d are physically meaningful. Calculation of the critical exponents by means of ε-expansions are given in Refs [64] and [65] (see also Refs [59, 60], and [66]). Up to terms of $\mathrm{O}(\varepsilon^3)$ the expressions for the critical exponents η, ν, and γ are

$$\eta = \frac{n+2}{2(n+8)^2}\varepsilon^2 + \mathrm{O}(\varepsilon^3) \ , \tag{2.91}$$

$$\nu = \frac{1}{2} + \frac{n+2}{4(n+8)}\varepsilon + \frac{n+2}{8(n+8)^3}(n^2 + 23n + 60)\varepsilon^2 + \ldots \ , \tag{2.92}$$

$$\gamma = 1 + \frac{n+2}{2(n+8)}\varepsilon + \frac{n+2}{4(n+8)^3}(n^2 + 22n + 52)\varepsilon^2 + \ldots \tag{2.93}$$

Analysis of the structure of the ε-series shows that the factor $(n+2)$ appearing in expansions (2.91)–(2.93) is also present in all subsequent terms of the series. Because of this the
n-model under consideration can also describe the Gaussian case in which $\eta = 0$, $\nu = \frac{1}{2}$, $\gamma = 1$, if we formally let the number of components equal $n = 2$ [67, 68].

Let us now return to our problem and make use of the fact that the self-avoiding walk model is equivalent to a zero-component magnetic material. Letting $n = 0$ in expressions (2.91)–(2.93) we obtain the following values for the critical exponents of the SAW problem in the framework of ε-expansions:

$$\eta = \frac{\varepsilon^2}{64} + \mathrm{O}(\varepsilon^3) \ , \tag{2.94}$$

$$\nu = \frac{1}{2} + \frac{1}{16}\varepsilon + \frac{15}{512}\varepsilon^2 + \ldots \ , \tag{2.95}$$

$$\gamma = 1 + \frac{1}{8}\varepsilon + \frac{13}{256}\varepsilon^2 + \ldots \tag{2.96}$$

If we retain only the terms appearing explicitly in expressions (2.94)–(2.96), and let $\varepsilon = 1$, then for $d = 3$ the critical exponents will assume the following approximate values: $\eta \approx 0.016$, $\nu \approx 0.592$, $\gamma \approx 1.176$, which is in quite good agreement with the results of numerical calculations. However, the agreement gets substantially worse on taking into account subsequent terms in the above formulas so this might be no more than a coincidence.

The Kadanoff–Wilson method can be applied to the SAW problem directly in the form of the so-called decimation procedure [69] (see also Refs [16, 17]), in which the polymer chain consisting of N links is divided into $N_1 = N/g$ identical segments that contain g links each joined consecutively. Every such segment is looked upon as a new effective link with new length l_1 and new excluded volume parameter v_1. To determine l_1 we must take into account the interaction between all the monomers belonging to a given segment, and to find v_1 we must include both the interactions between two segments that are far enough from each other and the interactions within each of them. The result is a polymer chain with new parameters N_1, l_1, v_1 that certainly depend on the original values of N, l, and v. This procedure of arranging the links into segments is then performed repeatedly, and for each kth stage we must determine the renormalised quantities l_k and v_k. The relationship between two successive steps in this process is described by recurrence relations for l_k and v_k or for the dimensionless quantity $u_k = v_k/l_k^d$ where $u_k = f(u_{k-1})$. As

$k \to \infty$, the sequence of numbers u_k converges to a limit u^*, which is a root of the equation $u^* = f(u^*)$. This fact is a reflection of the universality conjecture. The approach described here has been implemented quantitatively only for $\varepsilon \ll 1$, and the result for ν agrees to the first order in ε with expression (2.95).

For real systems ($d = 2, 3$), the Wilson method gives only approximate values of the critical exponents, but at present its range of accuracy cannot be established.

The majority of exact results in the physics of critical phenomena has been obtained for two-dimensional lattice models. An example is the outstanding result of Nienhuis [70] concerning the critical exponents for the two-dimensional vector model [in other words, for the $\mathsf{O}(n)$-model on the hexagonal lattice]. The partition function for this system is conveniently expressed by

$$Q_{\mathsf{O}(n)} = \int \prod_{(z,z')} (1 + x\boldsymbol{\sigma}_z \cdot \boldsymbol{\sigma}_{z'}) \prod_z W(\boldsymbol{\sigma}_z)\, \mathrm{d}^n \boldsymbol{\sigma}_z \ , \tag{2.97}$$

where the first product under the integral sign is over all pairs of nearest neighbour nodes belonging to the hexagonal lattice. The spin distribution function $W(\boldsymbol{\sigma})$ in the partition function (2.97) is either isotropic in n-dimensional spin space, i.e. invariant under all rotations of the spin-vector, or invariant under cubic transformations, i.e. permutations and inversions of the spin components. In this case the function $W(\boldsymbol{\sigma})$ and the length of the spin-vector are normalised so that

$$\int W(\boldsymbol{\sigma})\, \mathrm{d}^n \boldsymbol{\sigma} = 1, \qquad \int W(\boldsymbol{\sigma})(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})\, \mathrm{d}^n \boldsymbol{\sigma} = n \ .$$

It is easy to show (see Ref. [71]) that the partition function (2.97) can be expanded in terms of diagrams that contain loops on the hexagonal lattice:

$$Q_{\mathsf{O}(n)} = \sum_D x^L n^c \ ; \tag{2.98}$$

here D is the set of such diagrams, c is the number of loops in a diagram, and L is the total length of these loops. A remarkable feature of the $\mathsf{O}(n)$-model is its universality. In fact, suppose the parameter n in this model can take any real values; by giving it specific values we obtain certain well-known particular cases. For example, setting $n = -2$, $n = 0$, $n = 1$, $n = 2$, we arrive respectively at the Gaussian model, the self-avoiding walk model, the Ising model, and the
XY-model. Moreover, by means of transformations preserving the expression (2.98) for the partition function, it can be shown that the $\mathsf{O}(n)$-model is equivalent to the two-dimensional Coulomb gas model, which makes it possible to obtain the main results. A detailed description of the Coulomb gas model and its application to the theory of phase transitions for $d = 2$ was given in the review by Nienhuis [72]. The basic relations taken from Nienhuis [70] that connect the quantity n, $-2 \leqslant n \leqslant 2$, with the critical exponents for the $\mathsf{O}(n)$-model have the following form:

$$n = -2\cos\frac{2\pi}{t} \ , \tag{2.99}$$

$$\frac{1}{\nu} = 4 - 2t \ , \tag{2.100}$$

$$2 - \frac{\eta}{2} = 1 + \frac{3}{4t} + \frac{t}{4} \ , \tag{2.101}$$

where $1 \leqslant t \leqslant 2$. In order to apply these results to a zero-component $\mathsf{O}(n)$-model we must let $t = \frac{4}{3}$ in expressions (2.99)–(2.101). The result is $\eta = \frac{5}{24}$, $\nu = \frac{3}{4}$, $\gamma = \frac{43}{32}$, which is in good agreement with the corresponding numerical calculations. Unfortunately, there are no exact results so far for three-dimensional lattice models.

2.4. Functional integration method

The RG method occupies a central place among approaches to the SAW problem. There are several renormalization group schemes, one of which is connected with Wilson's method described above. However, as already noted, the Wilson transformations actually form a semigroup. At the same time, for the direct investigation of polymer statistics or self-avoiding walks, there also exist renormalization schemes related to a true group, also called a renormalization group. Such a group was first discovered in quantum field theory by Stueckelberg and Peterman [73] and then actually used by Gell-Mann and Low [74] for obtaining the ultraviolet asymptotics of the photon Green function. More recently the RG method was developed by Bogolyubov and Shirkov [75, 76]. In order to distinguish this (true) group of renormalising transformations from the Wilson renormalization group the former is usually called the field renormalization group. Being a continuous group, it can be described by Lie differential equations, which are especially useful in practice. It turns out that the field RG method can also be used in various procedures for treating the SAW problem.

In this and subsequent sections I consider the continuous SAW models and use extensively both statistical analysis and the field RG method.

Here I shall briefly outline the continuous approach to our problem, a full coverage being given in the monograph by Freed [1]. The excluded volume problem was first formulated in terms of the functional integration method in the well-known work by Edwards [6], which stimulated an extensive development of this approach [77–90].

The continuous model of a polymer chain involves a passage to the limit as the number of links tends to infinity while the effective length of each link tends to zero in such a way that the entire length L of the chain remains finite. Any admissible spatial configuration of the polymer will then be described by the continuous-curve equation $\boldsymbol{r} = \boldsymbol{r}(t)$, where the coordinate t of a point on the curve varies from 0 to L. In this model the distribution density of the vector $\boldsymbol{R}$ joining the ends of the curve $\boldsymbol{r}(t)$ can be defined by

$$W(\boldsymbol{R}, L) = Z^{-1}(L) G(\boldsymbol{R}, L) \ , \tag{2.102}$$

where

$$Z(L) = \int G(\boldsymbol{R}, L)\, \mathrm{d}^d R \ , \tag{2.103}$$

$$G(\boldsymbol{R}, L) = \left\{ \int_{\boldsymbol{r}(0)=0}^{\boldsymbol{r}(L)=\boldsymbol{R}} \mathrm{D}[\boldsymbol{r}(t)] \exp(-H_0(\boldsymbol{r})) \right\}^{-1} \times \int_{\boldsymbol{r}(0)=0}^{\boldsymbol{r}(L)=\boldsymbol{R}} \mathrm{D}[\boldsymbol{r}(t)] \exp(-H(\boldsymbol{r})) \ , \tag{2.104}$$

the expressions

$$H_0(\boldsymbol{r}) = \frac{d}{2} \int_0^L \left[\frac{\mathrm{d}\boldsymbol{r}(t)}{\mathrm{d}t} \right]^2 \mathrm{d}t \ , \tag{2.105}$$

and

$$H(\boldsymbol{r}) = H_0(\boldsymbol{r}) + \frac{v_0}{2}\int^L \mathrm{d}t \int_{|t-t'|>\lambda}^L \mathrm{d}t'\,\delta(\boldsymbol{r}(t) - \boldsymbol{r}(t')) \ , \tag{2.106}$$

are the Hamiltonians of the continuous model in which the volume effect is neglected and included respectively. $\mathrm{D}[\boldsymbol{r}(t)]$ is the measure on the configuration manifold of the continuous curve $\boldsymbol{r}(t)$, $0 \leqslant t \leqslant L$, with end-points $\boldsymbol{r}(0) = 0$ and $\boldsymbol{r}(L) = \boldsymbol{R}$, and finally, the parameter λ in expression (2.106) is of the order of 1 on the chosen length scale. It is readily seen from the definition of the model that it ceases to be physically meaningful for distances of the order of λ, so it is reasonable to proceed to its macroscopic description, i.e. to distances much bigger than λ. With this in mind, a new scaling length parameter Λ, $\lambda \ll \Lambda$, is introduced. But it makes the macroscopic chain length $\tilde{L}$ on the new scale unequal to the microscopic length L. This fact enables us to employ the universality hypothesis when $L \to \infty$. For example, if we double L, the result will, of course, be a doubling of $\tilde{L}$, which means that there must be a linear relationship between L and $\tilde{L}$, i.e.,

$$\tilde{L} = XL \ , \tag{2.107}$$

where the factor X depends on the excluded volume v_0 (more precisely, on v_0/l^d; but for simplicity we assume that $l = 1$) and on the ratio λ/Λ. On the new length scale, the quantity describing the excluded volume effect must reflect the collective character of volume interactions over distances of the order of Λ. It follows that the macroscopic excluded volume is now the renormalised quantity v depending on v_0 and λ/Λ. However, it is more convenient to use the quantities $u_0 = v_0\Lambda^{\varepsilon/2}$ and $u = v\Lambda^{\varepsilon/2}$, so that, in view of the foregoing, we can write

$$u = u(u_0, \lambda\Lambda^{-1}) \ . \tag{2.108}$$

Let us now denote by $G_0(\boldsymbol{R}, L, v_0; \lambda)$ and $G(\boldsymbol{R}, \tilde{L}, u; \Lambda)$ the nonrenormalised distribution densities of the vector $\boldsymbol{R}$ in the microscopic and macroscopic approaches to the problem, respectively. Since the renormalization of the functions G_0 and G must lead to the same distribution density of the vector $\boldsymbol{R}$, these two functions are proportional, i.e.

$$G(\boldsymbol{R}, \tilde{L}, u; \Lambda) = Y^{-1}(u, \lambda\Lambda^{-1}) G_0(\boldsymbol{R}, L, v_0; \lambda) \ . \tag{2.109}$$

The relations (2.107)–(2.109) provide us with the basis for the application of the RG method to the continuous SAW model. In fact, if we rewrite expression (2.109) in the form

$$G_0(\boldsymbol{R}, L, v_0; \lambda) = Y(u, \lambda\Lambda^{-1}) \times G(\boldsymbol{R}, LX(u, \lambda\Lambda^{-1}), u(u_0, \lambda\Lambda^{-1}); \Lambda) \ , \tag{2.110}$$

and recall that the left-hand side of Eqn (2.110) is independent of Λ, i.e.,

$$\Lambda \frac{\partial}{\partial \Lambda} G_0(\boldsymbol{R}, L, v_0; \lambda)\bigg|_{L, v_0, \lambda} = 0 \ .$$

we obtain a differential equation for the function G:

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \beta(u)\frac{\partial}{\partial u} + \gamma_Y(u) + \gamma_X(u)\tilde{L}\frac{\partial}{\partial \tilde{L}}\right) \times G(\boldsymbol{R}, \tilde{L}, u; \Lambda) = 0 \ , \tag{2.111}$$

in which

$$\gamma_X(u) = \left(\Lambda \frac{\partial \ln X(u, \lambda\Lambda^{-1})}{\partial \Lambda}\right)_{L, v_0, \lambda} \ , \tag{2.112}$$

$$\gamma_Y(u) = \left(\Lambda \frac{\partial \ln Y(u, \lambda\Lambda^{-1})}{\partial \Lambda}\right)_{L, v_0, \lambda} \ , \tag{2.113}$$

and the quantity

$$\beta(u) = \left(\Lambda \frac{\partial u}{\partial \Lambda}\right)_{L, v_0, \lambda} \tag{2.114}$$

is known as the Gell-Mann–Low function. For simplicity we did not indicate here the dependence of the functions γ_X, γ_Y, and β on λ/Λ. Equation (2.111), called the RG equation, contains information on the functional dependence of the required function G on the macroscopic parameters of the model.

A solution of Equation (2.111) can be written in the most general form as

$$G(\boldsymbol{R}, \tilde{L}, u; \Lambda) = \exp\left(-\int^u \frac{\gamma_Y(t)}{\beta(t)}\,\mathrm{d}t\right) \times F\left(\boldsymbol{R}, \tilde{L}\exp\left(-\int^u \frac{\gamma_X(t)}{\beta(t)}\,\mathrm{d}t\right), \Lambda\exp\left(-\int^u \frac{\mathrm{d}t}{\beta(t)}\right)\right) \ , \tag{2.115}$$

in which the function F, to be defined, is differentiable with respect to its two last arguments. A simple analysis leads to the relation

$$G(\boldsymbol{R}, \tilde{L}, u; \Lambda) = r^{-d/2} G(\boldsymbol{R}r^{-1/2}, \tilde{L}r^{-1}, u; \Lambda r^{-1}) \ , \tag{2.116}$$

where r is a nonzero length parameter. If we now substitute Eqn (2.115) into relation (2.116) and then choose r to be

$$r = \tilde{L}\exp\left(-\int^u \frac{\gamma_X(t)}{\beta(t)}\,\mathrm{d}t\right) \ ,$$

the function G in the new variables

$$g(u) = \exp\left(-\frac{1}{2}\int^u \frac{\gamma_X(t)}{\beta(t)}\,\mathrm{d}t\right) \ ,$$

$$h(u) = \tilde{L}^{-d/2}\exp\left(-\int^u \frac{\gamma_Y(t) - (d/2)\gamma_X(t)}{\beta(t)}\,\mathrm{d}t\right) \ ,$$

$$\zeta = \left(\frac{2\pi\tilde{L}}{\Lambda}\right)^{\varepsilon/2}\exp\left(-\frac{\varepsilon}{2}\int^u \frac{\gamma_X(t) - 1}{\beta(t)}\,\mathrm{d}t\right)$$

will assume the form

$$G(\boldsymbol{R}, \tilde{L}, u; \Lambda) = h(u)\Phi(g(u)\boldsymbol{R}\tilde{L}^{-1/2}, \zeta) \ . \tag{2.117}$$

Using expression (2.104) and the definition of the mean square end-to-end distance $\langle R^2\rangle$ we get

$$\langle R^2\rangle = l\tilde{L}g^2(u)\varphi(\zeta) \ , \tag{2.118}$$

which leads to the following expression for the distribution density of the vector $\boldsymbol{R}$:

$$W(\boldsymbol{R}, \tilde{L}, u; \Lambda) = \Psi(\boldsymbol{R}\langle R^2\rangle^{-1/2}, \zeta) \ . \tag{2.119}$$

In the formula we focus our attention on the dependence of the probability density W on both $\boldsymbol{R}/\langle R^2\rangle^{1/2}$ and the parameter ζ. The scaling theories are usually free of the latter dependence.

If u depends more and more weakly on Λ as the parameter Λ increases and if this dependence vanishes in the limit when $\Lambda \to \infty$, i.e. if the model is asymptotically scale-invariant, the scaling concept can be put into practice. The limiting value u^* of u is called a fixed point and is determined by

$$\beta(u)\Big|_{u=u^*} = 0 \ . \tag{2.120}$$

The existence of the fixed point u^* enables us to simplify the calculation of the critical exponents. In fact, using the fixed point (2.120) we can transform Eqn (2.111) into the form

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + B + A\tilde{L}\frac{\partial}{\partial\tilde{L}}\right)G(\boldsymbol{R},\tilde{L},u^*;\Lambda) = 0 \ , \tag{2.121}$$

where

$$A = \left(\Lambda\frac{\partial \ln X}{\partial\Lambda}\right)_{v_0,\lambda,L|_{u=u^*}} \ ,$$

$$B = \left(\Lambda\frac{\partial \ln Y}{\partial\Lambda}\right)_{v_0,\lambda,L|_{u=u^*}} \ ,$$

and represent its general solution as

$$G(\boldsymbol{R},\tilde{L},u^*;\Lambda) = \Lambda^{-B}F(\boldsymbol{R},\Lambda\tilde{L}^{-1/A}) \ . \tag{2.122}$$

The scaling properties of the function G together with the asymptotic relations

$$\langle R^2\rangle \sim \tilde{L}^{2\nu}, \quad \tilde{L}\to\infty \ ,$$

$$Q_{\tilde{L}} = \int G\,\mathrm{d}^dR \sim \tilde{L}^{\gamma-1}\mu^{\tilde{L}}, \quad \tilde{L}\to\infty \ ,$$

imply the following relationship between the parameters A, B and the critical exponents ν,γ:

$$2\nu = (1-A)^{-1} \ ,$$

$$(\gamma-1)(2\nu)^{-1} = B \ .$$

Thus, dimensional analysis of the RG equation (2.121) under the scaling conditions yields

$$G(\boldsymbol{R},\tilde{L},u^*;\Lambda) = \tilde{L}^{\gamma-\nu d-1}\Lambda^{\nu d+1-\gamma-(d/2)}f(\boldsymbol{R}\langle R^2\rangle^{-1}) \ , \tag{2.123}$$

where

$$\langle \boldsymbol{R}^2\rangle \sim l\tilde{L}^{2\nu}\Lambda^{1-2\nu} \ . \tag{2.124}$$

It can be shown [1] that all the scaling functions depend on Λ only through the ratio $\tilde{L}/\Lambda$, which measures the intensity of the volume interaction. To calculate the values of ν and γ as well as the scaling function $f(\boldsymbol{R}/\langle R^2\rangle)$ in Eqn (2.123), we need to know the quantities X, Y, and u. Standard perturbation theory is used here to expand G in the Hamiltonian (2.106) as a power series in the parameter ε, i.e. close to $d=4$. In doing so, to avoid divergence of the integrals that determine the coefficients of this series, one has to apply regularisation procedures. The values of ν and γ obtained in this way coincide to the second order in ε with those in expressions (2.95) and (2.96). Thus, the functional integration scheme in combination with the RG method are only applicable for very small ε and hence do not allow extension of the results beyond a small neighbourhood of the spatial dimensionality $d=4$.

3. Basic equation

A method is necessary for cognition of the truth.
R Descartes *Rules for the Guidance of the Mind*, Rule 4.

3.1 Derivation of the equation

In the subsequent sections I shall describe the approach based on the renormalization group method to solve the exact equation for the probability density $W_N(\boldsymbol{R})$.

Consider a spherical particle of diameter r_0 which begins walking from the origin in $\mathscr{R}^d$ and assume that each step $\boldsymbol{r}_k$ of its geometric centre has a constant length $|\boldsymbol{r}_k| = l$, $k \geqslant 1$, and a random direction. Then the probability density of the step $\boldsymbol{r}_k$ can be written in the form

$$\tau(\mathbf{r}_k) = 2(\omega l^{2s})^{-1}\delta(r_k^2 - l^2) \ , \tag{3.1}$$

where

$$s = \tfrac{1}{2}(d-2) \ .$$

We now require that, after each step $\boldsymbol{r}_k$, $k \geqslant 1$, the walking particle avoids all regions of space it visited previously. It follows that $r_0 < l$ and that the probability density of the kth step $\boldsymbol{r}_k$ is proportional to

$$\tau(\boldsymbol{r}_k)\prod_{j=1}^{k-1}(1+f_{jk}) \ , \tag{3.2}$$

where

$$f_{jk} = f\left(\left|\sum_{i=j}^{k}\mathbf{r}_i\right|\right) \ ,$$
$$f(r) = \begin{cases} -1, & \text{if } r < r_0, \\ 0, & \text{if } r > r_0. \end{cases} \tag{3.3}$$

In accordance with the general statement of the problem, we seek the probability that, after N steps — $\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_N$ — the centre of the particle gets to the volume element d^dR in the neighbourhood of the point with radius-vector $\boldsymbol{R}$. We use the Markov method [91] to express the required probability density $W_N(\boldsymbol{R})$ for the end-to-end vector $\boldsymbol{R}$ of the N-step path by the following equations:

$$W_N(\boldsymbol{R}) = Q_N^{-1}w_N(\boldsymbol{R}) \ , \tag{3.4}$$

$$w_N(\boldsymbol{R}) = \int\delta\left(\boldsymbol{R} - \sum_{k=1}^{N}\boldsymbol{r}_k\right)P_{1N}\prod_{k=1}^{N}\tau(\boldsymbol{r}_k)\,\mathrm{d}^dr_k \ , \tag{3.5}$$

$$P_{1N} = \prod_{1\leqslant j<k\leqslant N}(1+f_{jk}) \ , \tag{3.6}$$

where the normalising factor Q_N is given by

$$Q_N = \int P_{1N}\prod_{k=1}^{N}\tau(\boldsymbol{r}_k)\,\mathrm{d}^dr_k \ . \tag{3.7}$$

Next, in formula (3.5) we use the Fourier representation of the Dirac δ-function

$$\delta(\boldsymbol{R}) = \int\exp(-\mathrm{i}\boldsymbol{R}\cdot\boldsymbol{\rho})\frac{\mathrm{d}^d\rho}{(2\pi)^d} \ ,$$

to express $w_N(\boldsymbol{R})$ as the Fourier integral

$$w_N(\boldsymbol{R}) = \int\exp(-\mathrm{i}\boldsymbol{R}\cdot\boldsymbol{\rho})a_N(\boldsymbol{\rho})\frac{\mathrm{d}^d\rho}{(2\pi)^d} \ , \tag{3.8}$$

where the Fourier transform $a_N(\boldsymbol{\rho})$ of the function $w_N(\boldsymbol{R})$ is given by

$$a_N(\boldsymbol{\rho}) = \int P_{1N} \prod_{k=1}^{N} \exp(\mathrm{i}\boldsymbol{\rho}\cdot\boldsymbol{r}_k)\tau(\boldsymbol{r}_k)\,\mathrm{d}^d r_k \ . \tag{3.9}$$

If we write the probability density $W_N(\boldsymbol{R})$ as the Fourier integral

$$W_N(\boldsymbol{R}) = \int \exp(-\mathrm{i}\boldsymbol{R}\cdot\boldsymbol{\rho}) A_N(\boldsymbol{\rho}) \frac{\mathrm{d}^d\rho}{(2\pi)^d} \ , \tag{3.10}$$

then by Eqns (3.4) and (3.8) its Fourier transform $A_N(\boldsymbol{\rho})$, usually called the characteristic function, is related to $a_N(\boldsymbol{\rho})$ by

$$A_N(\boldsymbol{\rho}) = Q_N^{-1} a_N(\boldsymbol{\rho}) \ . \tag{3.11}$$

Because of the spatial isotropy in the problem under consideration, the probability density $W_N(\boldsymbol{R})$ does not depend on the direction of the vector $\boldsymbol{R}$ and hence the functions $A_N(\boldsymbol{\rho})$ and $a_N(\boldsymbol{\rho})$ do not depend on the direction of the vector $\boldsymbol{\rho}$. Let us note finally that in the limiting case where $r_0 = 0$, the definition (3.3) implies the identity $f_{jk} \equiv 0$, which together with Eqns (3.6) and (3.7) in turn implies that $P_{1N} \equiv 1$ and $Q_N \equiv 1$.

We now pass to formula (3.9) and expand the product P_{1N} in the variables f_{jk}:

$$P_{1N} = 1 + \sum_{j<k} f_{jk} + \sum_{j<k}\sum_{l<m} f_{jk} f_{lm} + \ldots \tag{3.12}$$

By the definitions of f_{jk} and P_{1N} in Eqns (3.3) and (3.6) the expansion (3.12) is in essence an inversion formula, which is sometimes referred to as the inclusion and exclusion principle [92]. For simplicity we pass to a graphical representation of the series (3.12) and introduce the notation depicted in Fig. 1. Then the series (3.12) can be displayed as shown in Fig. 2. We call the factors f_{jk} and f_{lm} with $j < k$, $l < m$, and $j \leqslant l$ *connected* if $j \leqslant l < m \leqslant k$ or $j < l \leqslant k < m$, and disconnected if $j < k < l < m$. In Fig. 2 the first and second diagrams in square brackets correspond to connected factors while the third diagram corresponds to disconnected factors. Any diagram that represents a chain of connected factors is called a connected diagram. As is readily seen, the diagram corresponding to any term of the series (3.12) is either a connected diagram, or a product of connected diagrams. Therefore, it is natural to select all the connected diagrams and to add them together as shown in Fig. 3. We shall refer to the first diagrams in each row (with the minimal number of dotted lines) as simple connected diagrams. All the subsequent connected diagrams can clearly be obtained from the simple connected diagrams by inserting all possible diagrams into them. If we now take the sum of the diagrams in each individual row and then sum the results we obtain the series shown in Fig. 4, every term of which is the sum of connected diagrams in the corresponding row. It follows that

$$b_{jk} = f_{jk} P'_{jk} + \sum_{l<m} f_{jm} f_{lk} P_{jl} P_{lm} P_{mk} + \sum_{l<m<p<q}\sum (f_{jp} f_{lq} f_{mk} + f_{jm} f_{lq} f_{pk}) P_{jl} P_{lm} P_{mp} P_{pq} P_{qk} + \ldots , \tag{3.13}$$

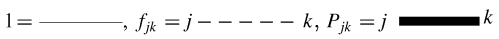

Figure 1.

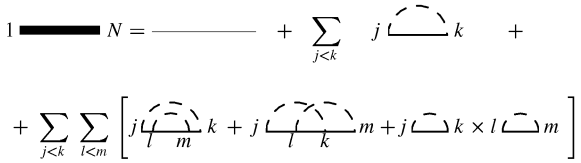

Figure 2.

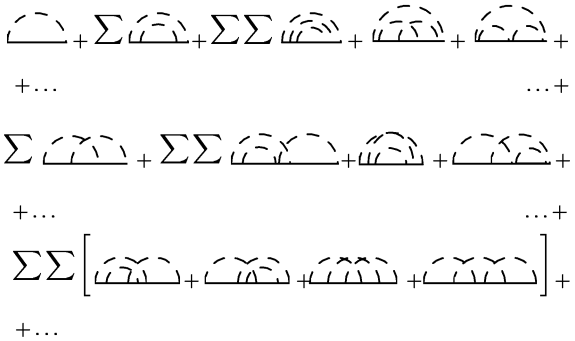

Figure 3.

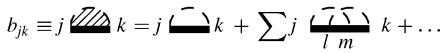

Figure 4.

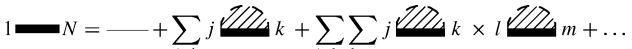

Figure 5.

where P'_{jk} denotes the product

$$P_{jk} = \prod_{j \leqslant l<m \leqslant k} (1 + f_{lm}) \ ,$$

in which the factor $(1 + f_{jk})$ is omitted, i.e.

$$P_{jk} = (1 + f_{jk}) P'_{jk} \ . \tag{3.14}$$

Next, we select and sum successively all the diagrams that can be represented as a product of two, three, etc. connected diagrams. The series in Fig. 2 will then take the form which can be written as

$$P_{1N} = 1 + \sum_{j<k} b_{jk} + \sum_{j<k<l<m}\sum b_{jk} b_{lm} + \ldots \tag{3.15}$$

Thus, the partial summation of the original series transforms it into a new series every terms of which is factorised.

Let us now substitute P_{1N} given by the series (3.15) into (3.9) and perform the integration in each term of the series thus obtained. Moreover, for all $n \geqslant 2$ define the function $b_n(\boldsymbol{\rho})$ by

$$b_n(\boldsymbol{\rho}) = \int b_{1n} \prod_{k=1}^{n} \exp(\mathrm{i}\boldsymbol{\rho}\cdot\boldsymbol{r}_k)\tau(\boldsymbol{r}_k)\,\mathrm{d}^d r_k \ , \tag{3.16}$$

where b_{1n} describes, as follows from expression (3.13), the family of all connected n-link diagrams. In this way we obtain the series

$$a_N(\boldsymbol{\rho}) = \Lambda_s^N(\rho l) + \sum_{n \geqslant 2}(N-n+1)\Lambda_s^{N-n}(\rho l)b_n(\boldsymbol{\rho}) + \sum_{n \geqslant 4}\left(\tfrac{1}{2}(N-n+1)(N-n+2)\Lambda_s^{N-n}(\rho l) \times \sum_{n_1+n_2=n}\sum b_{n_1}(\boldsymbol{\rho})b_{n_2}(\boldsymbol{\rho})\right) + \ldots , \tag{3.17}$$

which can be written in the more compact form

$$a_N(\boldsymbol{\rho}) = \sum_{m \geqslant 0}\sum_{n=0}^{N}\frac{(N-n+m)!}{(N-n)!\,m!}\Lambda_s^{N-n}(\rho l)B_n^{(m)}(\boldsymbol{\rho}) , \tag{3.18}$$

where

$$\Lambda_s(x) = \Gamma(s+1)\left(\frac{2}{x}\right)^s J_s(x) , \tag{3.19}$$

$$B_n^{(m)}(\boldsymbol{\rho}) = \sum_{n_1+\ldots+n_m=n}\cdots\sum b_{n_1}(\boldsymbol{\rho})\ldots b_{n_m}(\boldsymbol{\rho}) . \tag{3.20}$$

$J_s(x)$ is the Bessel function, and

$$\frac{(N-n+m)!}{(N-n)!\,m!} , \quad m \geqslant 1 ,$$

is the number of ways of arranging m segments consisting of $n_1, n_2, \ldots, n_m$ links, respectively, with $n_1 + n_2 + \ldots + n_m = n$, within a segment consisting of N $(n \leqslant N)$ links, in such a way that the order of the segments is preserved. By the well-known Cauchy theorem we have

$$\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \mathrm{d}z\, z^{n_1+\ldots+n_m-n-1} = \begin{cases} 1, & \text{if } \sum_{k=1}^m n_k = n , \\ 0, & \text{if } \sum_{k=1}^m n_k \neq n , \end{cases} \tag{3.21}$$

where the closed contour Γ encloses the origin of the coordinates on the complex z-plane. Using Eqn (3.21), one can easily reduce formula (3.20) to the form

$$B_n^{(m)}(\boldsymbol{\rho}) = \frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}z}{z^{n+1}} b^m(z,\boldsymbol{\rho}) , \tag{3.22}$$

where

$$b(z,\boldsymbol{\rho}) \equiv \sum_{n \geqslant 0} z^n b_n(\boldsymbol{\rho}) \tag{3.23}$$

is the generating function and the coefficients $b_0(\boldsymbol{\rho})$ and $b_1(\boldsymbol{\rho})$ of the last series are identically zero. By Eqns (3.22) and (3.18) we get the relation

$$a_N(\boldsymbol{\rho}) = \frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}z}{z^{N+1}}\sum_{m \geqslant 0}\sum_{n=0}^{N}\frac{(N-n+m)!}{(N-n)!\,m!} \times [z\Lambda_s(\rho l)]^{N-n} b^m(z,\boldsymbol{\rho}) , \tag{3.24}$$

in which the summation can be formally extended to all m from 0 to ∞ and all n from $-\infty$ to N without any effect on the final result. Applying this remark, together with formula

$$(1-x-y)^{-1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!}{m!\,n!}x^m y^n, \ |x+y| < 1 , \tag{3.25}$$

(see Ref. [93]) to relation (3.24), we get

$$a_N(\boldsymbol{\rho}) = \frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}z}{z^{N+1}}[1 - z\Lambda_s(\rho l) - b(z,\boldsymbol{\rho})]^{-1} , \tag{3.26}$$

where the closed contour Γ encloses the origin $z = 0$ and is chosen so that it satisfies

$$\left|z\Lambda_s(\rho l) + b(z,\boldsymbol{\rho})\right| < 1 .$$

Finally, if we define the generating function $a(z,\boldsymbol{\rho})$ by

$$a(z,\boldsymbol{\rho}) = \sum_{N \geqslant 0} z^N a_N(\boldsymbol{\rho}) , \tag{3.27}$$

Eqn (3.26) will imply that [94]

$$a^{-1}(z,\boldsymbol{\rho}) = a_0^{-1}(z,\boldsymbol{\rho}) - b(z,\boldsymbol{\rho}) , \tag{3.28}$$

where

$$a_0^{-1}(z,\boldsymbol{\rho}) = 1 - z\Lambda_s(\rho l) . \tag{3.29}$$

Eqn (3.28) is analogous to the well-known Dyson equation [95] and plays a fundamental role in our problem. For this reason we shall refer to it as the *basic equation*. To make Eqn (3.28) closed, we have to establish a relationship between the generating functions $b(z,\boldsymbol{\rho})$ and $a(z,\boldsymbol{\rho})$. For this purpose, we find b_{1n} from Eqn (3.13), substitute it into Eqn (3.16), and use the latter in Eqn (3.23). In addition, we represent the function f_{jk} in the form

$$f_{jk} = -\int v(\boldsymbol{\kappa})\exp\left(-\mathrm{i}\boldsymbol{\kappa}\cdot\sum_{m=j}^{k}\boldsymbol{r}_m\right)\frac{\mathrm{d}^d\kappa}{(2\pi)^d} , \tag{3.30}$$

where the Fourier transform $v(\boldsymbol{\kappa})$ of the function $-f(\boldsymbol{r})$ [see equalities (3.3)] is given by

$$v(\boldsymbol{\kappa}) = -\int \exp(\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{r})f(\boldsymbol{r})\,\mathrm{d}^d r = v_0\Lambda_{s+1}(r_0\kappa) , \tag{3.31}$$

and where the quantity

$$v_0 = \frac{(\pi r_0^2)^{s+1}}{\Gamma(s+2)} , \tag{3.32}$$

is, as before, called the excluded volume. Using Eqn (3.30), we obtain

$$\begin{aligned} b_n(\boldsymbol{\rho}) = &-\int v(\boldsymbol{\kappa})a_n'(\boldsymbol{\rho}-\boldsymbol{\kappa})\frac{\mathrm{d}^d\kappa}{(2\pi)^d} \\ &+ \sum_{1<l<m<n}\sum\int v(\boldsymbol{\kappa})v(\boldsymbol{\kappa}')a_l(\boldsymbol{\rho}-\boldsymbol{\kappa})a_{m-l}(\boldsymbol{\rho}-\boldsymbol{\kappa}-\boldsymbol{\kappa}') \\ &\times a_{n-m}(\boldsymbol{\rho}-\boldsymbol{\kappa}')\frac{\mathrm{d}^d\kappa\,\mathrm{d}^d\kappa'}{(2\pi)^{2d}} \\ &-\sum_{1<l<m<p<q<n}\cdots\sum\int v(\boldsymbol{\kappa})v(\boldsymbol{\kappa}')v(\boldsymbol{\kappa}'') \\ &\times [a_l(\boldsymbol{\rho}-\boldsymbol{\kappa})a_{m-l}(\boldsymbol{\rho}-\boldsymbol{\kappa}-\boldsymbol{\kappa}')a_{p-m}(\boldsymbol{\rho}-\boldsymbol{\kappa}-\boldsymbol{\kappa}'-\boldsymbol{\kappa}'') \\ &\times a_{q-p}(\boldsymbol{\rho}-\boldsymbol{\kappa}'-\boldsymbol{\kappa}'')a_{n-q}(\boldsymbol{\rho}-\boldsymbol{\kappa}'') \\ &+ a_l(\boldsymbol{\rho}-\boldsymbol{\kappa})a_{m-l}(\boldsymbol{\rho}-\boldsymbol{\kappa}-\boldsymbol{\kappa}')a_{p-m}(\boldsymbol{\rho}-\boldsymbol{\kappa}') \\ &\times a_{q-p}(\boldsymbol{\rho}-\boldsymbol{\kappa}'-\boldsymbol{\kappa}'')a_{n-q}(\boldsymbol{\rho}-\boldsymbol{\kappa}'')] \\ &\times \frac{\mathrm{d}^d\kappa\,\mathrm{d}^d\kappa'\,\mathrm{d}^d\kappa''}{(2\pi)^{3d}} + \ldots , \end{aligned} \tag{3.33}$$

where the function $a'_N(\boldsymbol{\rho})$ is defined by expression (3.9), in which the product P_{1N} must be replaced by P'_{1N}. Since P'_{1N} differs from P_{1N} only over the region

$$\left|\sum_{j=1}^{N} \mathbf{r}_j\right| < r_0 \ ,$$

the functions $a'_N(\boldsymbol{\rho})$ and $a_N(\boldsymbol{\rho})$ can behave differently only when ρ is large enough ($\rho \gtrsim r_0^{-1}$). But the latter region is inessential for the asymptotic behaviour of the probability density $W_N(\boldsymbol{R})$ as $N \to \infty$ and $R \gg l$. So henceforth we shall assume that the functions $a'_N(\boldsymbol{\rho})$ and $a_N(\boldsymbol{\rho})$ coincide everywhere. If we now substitute Eqn (3.33) into Eqn (3.23) and carry out the appropriate summation, we shall obtain the required relation

$$\begin{aligned} b(z,\boldsymbol{\rho}) = &-\int v(\boldsymbol{\kappa}) a(z,\boldsymbol{\rho}-\boldsymbol{\kappa}) \frac{\mathrm{d}^d\kappa}{(2\pi)^d} \\ &+ \int v(\boldsymbol{\kappa}) v(\boldsymbol{\kappa}') a(z,\boldsymbol{\rho}-\boldsymbol{\kappa}) a(z,\boldsymbol{\rho}-\boldsymbol{\kappa}-\boldsymbol{\kappa}') \\ &\times a(z,\boldsymbol{\rho}-\boldsymbol{\kappa}') \frac{\mathrm{d}^d\kappa\,\mathrm{d}^d\kappa'}{(2\pi)^{2d}} \\ &- \int v(\boldsymbol{\kappa}) v(\boldsymbol{\kappa}') v(\boldsymbol{\kappa}'') [a(z,\boldsymbol{\rho}-\boldsymbol{\kappa}) a(z,\boldsymbol{\rho}-\boldsymbol{\kappa}-\boldsymbol{\kappa}') \\ &\times a(z,\boldsymbol{\rho}-\boldsymbol{\kappa}-\boldsymbol{\kappa}'-\boldsymbol{\kappa}'') a(z,\boldsymbol{\rho}-\boldsymbol{\kappa}'-\boldsymbol{\kappa}'') a(z,\boldsymbol{\rho}-\boldsymbol{\kappa}'') \\ &+ a(z,\boldsymbol{\rho}-\boldsymbol{\kappa}) a(z,\boldsymbol{\rho}-\boldsymbol{\kappa}-\boldsymbol{\kappa}') a(z,\boldsymbol{\rho}-\boldsymbol{\kappa}') \\ &\times a(z,\boldsymbol{\rho}-\boldsymbol{\kappa}'-\boldsymbol{\kappa}'') a(z,\boldsymbol{\rho}-\boldsymbol{\kappa}'')] \\ &\times \frac{\mathrm{d}^d\kappa\,\mathrm{d}^d\kappa'\,\mathrm{d}^d\kappa''}{(2\pi)^{3d}} + \ldots \end{aligned} \tag{3.34}$$

The last equality and the basic equation (3.28) constitute a closed system of equations for the unknown function $a(z, \boldsymbol{\rho})$.

The infinite series (3.34) for the function $b(z, \boldsymbol{\rho})$ is conveniently represented graphically. This is done by assigning to each term of the series a diagram consisting of a certain number of vertices and a set of solid and wavy lines joining these vertices. In addition, each diagram contains two further external solid lines, one of which is ingoing and the other outgoing. Each line is assigned a particular vector ('momentum'), which must be conserved at every vertex where two solid lines and one wavy line converge. Finally, the quantity $a(z, \boldsymbol{\rho}-\boldsymbol{\kappa})$ is assigned to each internal solid line and $-v(\boldsymbol{\kappa})/(2\pi)^d$ is assigned to each wavy line. The integration is then carried out over all the wavy-line vectors. According to these rules, the series in (3.34) can be displayed as shown in Fig. 6.

Analysis of the individual terms in this series shows that the problem associated with the divergence of the integrals does not arise here. In fact, if in all the integrals we change to the configuration variables $\{\boldsymbol{r}\}$, we readily see that the integration with respect to $\{\boldsymbol{r}\}$ is carried out over the region with the lower limit $|\boldsymbol{r}| = r_0$, which is, or course, a reflection of the excluded volume effect, whereas at the upper limit (when $|\boldsymbol{r}| \to \infty$) the integrands tend exponentially to zero because of the number of steps that constitute the trajectory of the walking particle is finite. However, we still have a convergence problem for the series as a whole. To solve this problem we direct our attention to equalities (3.23), (3.16), and (3.13). Since the function f_{jk}, as defined in Eqns (3.3), assumes only the values -1 and 0, and since the product P_{jk} can be equal to 0 or 1 [see expression (3.6)], we conclude from equality (3.13) that b_{1n} has only three possible values, $-1, 0$, and 1. Together with Eqn (3.16) this implies that

$$|b_n(\boldsymbol{\rho})| \leqslant \int |b_{1n}| \prod_{k=1}^{n} |\exp(\mathrm{i}\boldsymbol{\rho}\cdot\boldsymbol{r}_k)| \tau(\boldsymbol{r}_k)\, \mathrm{d}^d r_k \leqslant 1$$

for $n \geqslant 2$ and all real values of ρ. Combining this inequality with the identity (3.33), we get

$$|b(z,\boldsymbol{\rho})| \leqslant \sum_{n \geqslant 0} |z|^n |b_n(\boldsymbol{\rho})| \leqslant |z|^2 (1-|z|)^{-1} \ ,$$

provided that $|z| < 1$ and $\operatorname{Im}\rho = 0$. Thus, for all real ρ, the series in (3.34) which represents the function $b(z, \boldsymbol{\rho})$ converges absolutely at least within the region $|z| < 1$.

An analogous assertion holds for the function $a(z, \boldsymbol{\rho})$. Indeed, by the definition of $a_N(\boldsymbol{\rho})$ in (3.9) we have

$$|a_N(\boldsymbol{\rho})| \leqslant \int |P_{1N}| \prod_{k=1}^{N} |\exp(\mathrm{i}\boldsymbol{\rho}\cdot\boldsymbol{r}_k)| \tau(\mathbf{r}_k)\, \mathrm{d}^d r_k \leqslant 1$$

for all real ρ. These inequalities together with Eqn (3.27) imply that

$$|a(z,\boldsymbol{\rho})| \leqslant \sum_{N \geqslant 0} |z|^N |a_N(\boldsymbol{\rho})| \leqslant (1-|z|)^{-1} \ ,$$

whenever $|z| < 1$ and $\operatorname{Im}\rho = 0$.

The above assertions can be regarded as proof of the closure of the basic equation. If a solution of this equation were known, we could use the Fourier transformation

$$w(z,\boldsymbol{R}) = \int \exp(-\mathrm{i}\boldsymbol{R}\cdot\boldsymbol{\rho}) a(z,\boldsymbol{\rho}) \frac{\mathrm{d}^d\rho}{(2\pi)^d} \tag{3.35}$$

to find the generating function

$$w(z,\boldsymbol{R}) \equiv \sum_{N \geqslant 0} z^N w_N(\boldsymbol{R}) \ , \tag{3.36}$$

and then use the inversion formula

$$w_N(\boldsymbol{R}) = \frac{1}{2\pi\mathrm{i}} \oint \frac{\mathrm{d}z}{z^{N+1}} w(z,\boldsymbol{R}) \ , \tag{3.37}$$

to determine $w_N(\boldsymbol{R})$. Finally, by Eqn (3.4) combined with the normalization condition

$$Q_N = \int w_N(\boldsymbol{R})\, \mathrm{d}^d R \ , \tag{3.38}$$

we would obtain the required probability density $W_N(\boldsymbol{R})$. However, the problem of solving the basic equation and subsequently determining the function $W_N(\boldsymbol{R})$ is very difficult. Here we are concerned with a more realistic problem, namely discovering the asymptotic behaviour of $W_N(\boldsymbol{R})$ as $N \to \infty$ and $R \gg l$.

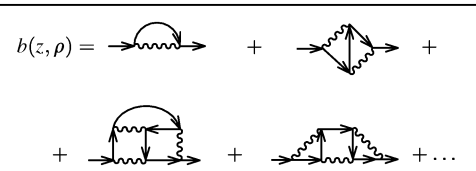

Figure 6.

3.2 Properties of the basic equation

We now apply formulae (3.37) and (3.35) in succession to Eqn (3.38) and integrate the result with respect to $\boldsymbol{R}$ to obtain

$$Q_N = \frac{1}{2\pi i}\oint \frac{dz}{z^{N+1}} a(z,0) \ , \tag{3.39}$$

wherefrom it follows that

$$a(z,0) = Q(z) \equiv \sum_{N \geqslant 0} z^N Q_N \ . \tag{3.40}$$

Combining Eqn (3.40) with Eqn (3.28), we arrive at the relation

$$Q^{-1}(z) = 1 - z - b(z,0) \ . \tag{3.41}$$

Suppose z_0 to be the singular point of the function $Q(z)$ closest to the origin. We then have by definition

$$1 - z_0 - b(z_0,0) \equiv 0 \ . \tag{3.42}$$

Hence it is readily seen that $z_0 = z_0(v_0/l^d)$, i.e. z_0 depends only on the dimensionless quantity v_0/l^d; moreover, $z_0(0) = 1$. The inclusion of the excluded volume effect thus leads to a shift of the singular point z_0 from its undisturbed position $z_0(0) = 1$. We know from the theory of analytic functions that any power series has the property that the boundary of its circle of convergence contains the singular point of its sum closest to the origin [96]. Moreover, the asymptotic behaviour of the function represented by a power series as the variable z approaches the boundary of its circle of convergence can be related to the asymptotic behaviour of its coefficient as their order numbers tend to infinity. Indeed, the well-known Tauberian theorem applied to the power series in expression (3.40) states that if the coefficients $Q_N z_0^N$ of the series constitute a monotonic sequence and if $0 < \alpha < \infty$, then the relations

$$Q(z) \sim \left(1 - \frac{z}{z_0}\right)^{-\alpha} \mathcal{L}\left(\frac{z_0}{z_0 - z}\right), \qquad \frac{z_0}{z} \to 1 - 0 \ ,$$

and

$$Q_N \sim \frac{N^{\alpha-1}}{\Gamma(\alpha) z_0^N} \mathcal{L}(N), \qquad N \to \infty \ ,$$

are equivalent; here $\mathcal{L}(t)$ is a slowly varying positive function (i.e. a function such that

$$\frac{\mathcal{L}(xt)}{\mathcal{L}(t)} \to 1, \qquad t \to \infty \ ,$$

for any fixed $x > 0$) [97]. The importance of the Tauberian theorem lies in its close relation to the problem of evaluating the asymptotic form of the probability density $W_N(\boldsymbol{R})$ for $N \to \infty$ and $R \gg l$.

With this in mind, we introduce the generating function

$$A(z,\boldsymbol{\rho}) = \sum_{N \geqslant 0} z^N A_N(\boldsymbol{\rho}) \tag{3.43}$$

for all z such that $|z| < R_A(\rho)$, where $R_A(\rho)$ is the radius of convergence of the series in (3.43), and hope to establish a relationship between $A(z,\boldsymbol{\rho})$ and $a(z,\boldsymbol{\rho})$. To do this, we find $a_N(\boldsymbol{\rho})$ from Eqn (3.11), substitute it into Eqn (3.27), to obtain

$$a(z,\boldsymbol{\rho}) = \sum_{N \geqslant 0} z^N Q_N A_N(\boldsymbol{\rho}) \ . \tag{3.44}$$

If we now apply the inversion formula to the series in Eqns (3.40) and (3.43), we arrive at the required relation

$$a(z,\boldsymbol{\rho}) = \frac{1}{2\pi i}\oint_\Gamma \frac{dz'}{z'} Q\left(\frac{z}{z'}\right) A(z',\boldsymbol{\rho}) \ , \tag{3.45}$$

in which the contour of integration Γ is chosen so that any point z' on this contour satisfies the inequalities $|z'| < R_A(\rho)$ and $|z/z'| < R_0$, where R_0 is the radius of convergence of the series in (3.40). In a similar way we get a formula that is inverse to (3.45). In fact, if we start from definition (3.43), then use Eqn (3.11), and apply the inversion formula to (3.27), we obtain

$$A(z,\boldsymbol{\rho}) = \frac{1}{2\pi i}\oint_{\tilde\Gamma} \frac{dz'}{z'} \tilde{Q}\left(\frac{z}{z'}\right) a(z',\boldsymbol{\rho}) \ , \tag{3.46}$$

where

$$\tilde{Q}(z) \equiv \sum_{N \geqslant 0} Q_N^{-1} z^N \ , \tag{3.47}$$

and the contour $\tilde\Gamma$ in (3.46) is chosen so that z' satisfies $|z'| < R_a(\rho)$ and $|z/z'| < \tilde{R}_0$; $R_a(\rho)$ and $\tilde{R}_0$ are the radii of convergence of the series in (3.44) and (3.47), respectively. The generating functions $Q(z)$ and $\tilde{Q}(z)$ are reciprocal in the sense of the equation

$$\frac{1}{2\pi i}\oint \frac{d\zeta}{\zeta} Q\left(\frac{\zeta}{z'}\right) \tilde{Q}\left(\frac{z}{\zeta}\right) = \left(1 - \frac{z}{z'}\right)^{-1} \ , \tag{3.48}$$

which is a particular case of formula (3.46) and follows from the latter if we put $\rho = 0$ and take into account that

$$A(z,0) = (1-z)^{-1}, \qquad |z| < 1 \ . \tag{3.49}$$

The inversion formulas show that, in order to find the asymptotic form of the density $W_N(\boldsymbol{R})$ for $N \to \infty$ and $R \gg l$, we have to know the asymptotic behaviour of the function $A(z,\boldsymbol{\rho})$ when $z \to z_A(\rho)$, $|z| < R_A(\rho)$ and ρ is within a small neighbourhood of the point $\rho = 0$, where $z_A(\rho)$ is the singular point of the function $A(z,\boldsymbol{\rho})$ closest to the origin. In particular, it follows from (3.49) that $z_A(0) = 1$. However, we cannot investigate the behaviour of the function $A(z,\boldsymbol{\rho})$ immediately, because the basic equation deals with the function $a(z,\boldsymbol{\rho})$. Our problem is therefore reduced to determining the asymptotic form of the function $a(z,\boldsymbol{\rho})$ when ρ is small, $z \to z_a(\rho)$, and $|z| < R_a(\rho)$, where $z_a(\rho)$ is the singular point of the function $a(z,\boldsymbol{\rho})$ closest to the origin. The relationship between the points $z_A(\rho)$ and $z_a(\rho)$ is described by the Hadamard multiplication theorem [96]: if $z_0^{(1)}, z_0^{(2)}, \ldots$ are the singular points of the function $Q(z)$ and $z_A^{(1)}(\rho), z_A^{(2)}(\rho), \ldots$ are the singular points of the function $A(z,\boldsymbol{\rho})$, then every singular point of the function $a(z,\boldsymbol{\rho})$ has the form $z_0^{(m)} z_A^{(n)}(\rho)$. By this theorem we have

$$z_a(\rho) = z_0 z_A(\rho) \ , \tag{3.50}$$

where $z_0 = z_a(0)$ [see the identity (3.42)]. In a similar manner, formula (3.46) implies that

$$z_A(\rho) = \tilde{z}_0 z_a(\rho) \ , \tag{3.51}$$

where $\tilde{z}_0$ is the singular point of $\tilde{Q}(z)$ closest to the origin. Combining Eqn (3.50) with (3.51), we see that $z_0 \tilde{z}_0 = 1$. Therefore, $z_a(\rho)/z_0$ behaves like $z_A(\rho)$ and hence it equals unity when $\rho = 0$. It is then reasonable to go over from z to another variable, for example, to z/z_0 or, more conveniently, to the variable $\zeta = z_0/z$.

Now introduce the notation

$$\mathcal{A}(\zeta,\rho)=\zeta^{-1}a(z,\boldsymbol{\rho}) \ , \tag{3.52}$$

$$B(\zeta,\rho l,\tilde{v};\mathcal{A})=\zeta b(z,\boldsymbol{\rho}) \ , \tag{3.53}$$

$$\tilde{v}=\zeta^2 v l^{-d}, \quad \tilde{v}_0=\zeta^2 v_0 l^{-d} \ . \tag{3.54}$$

Using the identity (3.42) we can rewrite Eqn (3.28) as follows:

$$\mathcal{A}^{-1}=\zeta-1+z_0[1-\Lambda_s(\rho l)] + B(1,0,\tilde{v};\mathcal{A})-B(\zeta,\rho l,\tilde{v};\mathcal{A}) \ . \tag{3.55}$$

Next, we denote by $\rho_1\equiv \mathrm{i}\xi$ $(\mathrm{Re}\,\xi>0)$ the root of the equation $\mathcal{A}^{-1}(\zeta,\rho)=0$ closest to the origin $\rho=0$ and put $\rho=\rho_1$ in Eqn (3.55). We then get the identity

$$\zeta-1+z_0[1-\Lambda_s(\mathrm{i}\xi l)] + B(1,0,\tilde{v};\mathcal{A})-B(\zeta,\mathrm{i}\xi l,\tilde{v};\mathcal{A})\equiv 0 \ , \tag{3.56}$$

connecting the variables ζ and ξ. Finally, with the help of expressions (3.56) and (3.55) we arrive at the following form of the basic equation:

$$\mathcal{A}^{-1}=\mathcal{A}_0^{-1}(\xi,\rho)+B(\zeta,\mathrm{i}\xi l,\tilde{v};\mathcal{A})-B(\zeta,\rho l,\tilde{v};\mathcal{A}) \ , \tag{3.57}$$

where

$$\mathcal{A}_0^{-1}(\xi,\rho)=z_0[\Lambda_s(\mathrm{i}\xi l)-\Lambda_s(\rho l)] \ . \tag{3.58}$$

The method adopted here for investigating the SAW problem is based on Eqn (3.57) combined with the identity (3.56).

An important special aspect of this problem is the evaluation of the momenta

$$\langle R^{2m}\rangle_N=\int R^{2m}W_N(\boldsymbol{R})\,\mathrm{d}^dR, \quad m\geqslant 1 \ , \tag{3.59}$$

By the Carleman theorem [97] these momenta determine the required probability density, provided that

$$\sum_{m\geqslant 1}\langle R^{2m}\rangle_N^{-1/2m}=\infty \ ,$$

i.e. the series diverges. From expressions (3.4), (3.37), (3.39), and (3.52) it follows that

$$W_N(\boldsymbol{R})=\oint_\Gamma \mathrm{d}\zeta\,\zeta^N\tilde{w}(\zeta,R)\left[\oint_\Gamma \mathrm{d}\zeta\,\zeta^N\mathcal{A}(\zeta,0)\right]^{-1} \ , \tag{3.60}$$

where

$$\tilde{w}(\zeta,R)=\int \exp(-\mathrm{i}\boldsymbol{R}\cdot\boldsymbol{\rho})\mathcal{A}(\zeta,\rho)\frac{\mathrm{d}^d\rho}{(2\pi)^d} \ , \tag{3.61}$$

and the closed contour of integration, Γ, is chosen so that the origin $\zeta=0$ and all the singular points of the integrand are situated inside Γ. We now substitute the right-hand side of Eqn (3.60) into Eqn (3.59) and then use the formula

$$\mathcal{A}(\zeta,\rho)=\omega\int_0^\infty \Lambda_s(\rho R)\tilde{w}(\zeta,R)R^{2s+1}\,\mathrm{d}R \ . \tag{3.62}$$

which is the inverse of (3.61), in conjunction with the equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}x^2}\right)^m\Lambda_s(x)=\left(-\frac{1}{4}\right)^m\frac{\Gamma(s+1)}{\Gamma(s+m+1)}\Lambda_{s+m}(x) \tag{3.63}$$

This yields

$$\langle R^{2m}\rangle_N=(-4)^m\frac{\Gamma(s+m+1)}{\Gamma(s+1)} \times\frac{\oint \mathrm{d}\zeta\,\zeta^N}{\oint \mathrm{d}\zeta\,\zeta^N\mathcal{A}(\zeta,0)}\left[\left(\frac{\partial}{\partial\rho^2}\right)^m\mathcal{A}(\zeta,\rho)\Big|_{\rho=0}\right] \ . \tag{3.64}$$

In particular, from Eqn (3.64) and (3.57) we obtain

$$\langle R^2\rangle_N=z_0l^2 \times\frac{\oint \mathrm{d}\zeta\,\zeta^N\mathcal{A}^2(\zeta,0)}{\oint \mathrm{d}\zeta\,\zeta^N\mathcal{A}(\zeta,0)}\left(1-\frac{2\,\mathrm{d}}{z_0}\frac{\partial B(\zeta,t,\tilde{v};\mathcal{A})}{\partial t^2}\Big|_{t=0}\right) \ , \tag{3.65}$$

where

$$\mathcal{A}^{-1}(\zeta,0)=\mathcal{A}_0^{-1}(\xi,0)+B(\zeta,\mathrm{i}\xi l,\tilde{v};\mathcal{A})-B(\zeta,0,\tilde{v};\mathcal{A}). \tag{3.66}$$

We now point out a very important property of Eqn (3.57), namely its invariance with respect to the multiplicative transformations

$$\mathcal{A}\to\mathcal{A}'=\alpha\mathcal{A}, \quad \mathcal{A}_0\to\mathcal{A}_0'=\alpha\mathcal{A}_0 \ , \quad v\to v'=\alpha^{-2}v \ , \tag{3.67}$$

where α is a nonzero continuously varying parameter [98]. These transformations constitute a continuous group, which is usually called the renormalization transformation group (RG). To understand the physical sense of the RG, consider the asymptotic case where $N\to\infty$ and $R\gg l$. It follows from Eqns (3.60) and (3.61) that in this case we need to know the behaviour of the function $\mathcal{A}(\zeta,\rho)$ within small neighbourhoods of the points $\zeta=1$ and $\rho=0$ to determine the asymptotic form of the density $W_N(\boldsymbol{R})$. In what follows we shall assume the values of ζ and ρ to belong to the above neighbourhoods. Then, by the identity (3.56) and

$$\Lambda_s(x)=1-\frac{x^2}{2d}+\mathrm{O}(x^4), \quad x\to 0 \ , \tag{3.68}$$

the values of ξ belong to the corresponding neighbourhood of the point $\xi=0$. If we now multiply the number of steps N in Eqn (3.60) by α, i.e. put

$$N\to N'=\alpha N, \quad \alpha>0 \ , \tag{3.69}$$

we obtain

$$W_N(\boldsymbol{R})=\oint \mathrm{d}\zeta'\,(\zeta')^{N'}\tilde{w}'(\zeta',R)\left[\oint \mathrm{d}\zeta'\,(\zeta')^{N'}\mathcal{A}'(\zeta',0)\right]^{-1} \ , \tag{3.70}$$

where

$$\zeta'=\zeta^{1/\alpha}, \quad \tilde{w}'=\alpha\tilde{w}, \quad \mathcal{A}'=\alpha\mathcal{A} \ . \tag{3.71}$$

Since for the values of ζ and ρ under consideration, Eqn (3.55) has the approximate form

$$\mathcal{A}^{-1}=\zeta-1+z_0\rho^2l^2(2d)^{-1} + B(1,0,\tilde{v};\mathcal{A})-B(\zeta,\rho l,\tilde{v};\mathcal{A}) \ , \tag{3.72}$$

the definitions (3.71) and the RG-properties of the above equation enable us to rewrite it in the following equivalent form:

$$(\mathcal{A}')^{-1} = \zeta' - 1 + z_0\rho^2 l_1^2 (2d)^{-1} + B(1,0,\tilde{v}';\mathcal{A}') - B(\zeta',\rho l_1,\tilde{v}';\mathcal{A}') \ , \quad (3.73)$$

where

$$l_1 = l\alpha^{-1/2} \ , \quad (3.74)$$

$$\tilde{v}' = \alpha^{s-1}\tilde{v} \ . \quad (3.75)$$

On comparing Eqn (3.60) with (3.70) and then Eqn (3.72) with (3.73), we conclude that the change in the number of steps N according to Eqn (3.69) is equivalent both to the change in the length l of the individual step according to Eqn (3.74), and to the change in the excluded volume v_0 according to Eqn (3.75). However, it should be stressed that this statement is true only for the asymptotic case $(N \to \infty, R \gg l)$ and appears symbolically as

$$W_N(Rl^{-1}, v_0 l^{-d}) = W_{\alpha N}(\alpha^{1/2} Rl^{-1}, \alpha^{s-1} v_0 l^{-d}) \ .$$

From this one can easily obtain the asymptotic form of the mean square end-to-end distance for the random walk:

$$\langle R^2\rangle_N \sim Nl^2 \chi(N^{1-s} v_0 l^{-d}) \ , \quad (3.76)$$

where $\chi(x)$ is a universal function still to be determined. The last formula shows that the dimensionality $d = 4$ $(s = 1)$ is special, because the excluded volume effect becomes weak when $d > 4$ $(s > 1)$, and in this case one can use perturbation theory to calculate $\langle R^2\rangle_N$.

3.3 Renormalization group method

The renormalization group method of solving the basic equation relies on the properties of this equation, which have been mentioned earlier. To develop this method, consider Eqns (3.56) and (3.57). The first establishes a relationship between ζ and ξ, which makes it possible to find the critical exponent ν, whereas the second determines the type of the singular point $\rho_1 \equiv \mathrm{i}\xi$ of the function $\mathcal{A}(\zeta,\rho)$, which enables us to evaluate the other critical exponent η. Clearly, we have to determine at the outset the character of the singularity of $\mathcal{A}(\zeta,\rho)$ at $\rho = \rho_1$. With this in mind we write $\mathcal{A}(\zeta,\rho)$ in the form

$$\mathcal{A} = \mathcal{A}_0 G \ . \quad (3.77)$$

For the new unknown quantity G we obtain from Eqn (3.57) the equation

$$G^{-1} = 1 + F(\xi^2 l^2, \rho^2 l^2, \tilde{v}; G) \ , \quad (3.78)$$

in which

$$F(\xi^2 l^2, \rho^2 l^2, \tilde{v}; G) = \mathcal{A}_0(\xi,\rho) \times [B(\zeta, \mathrm{i}\xi l, \tilde{v}; \mathcal{A}_0 G) - B(\zeta, \rho l, \tilde{v}; \mathcal{A}_0 G)] \ , \quad (3.79)$$

and the function $\mathcal{A}_0(\xi,\rho)$ displays the following behaviour for small values of ξ and ρ:

$$\mathcal{A}_0(\xi,\rho) = 2d\left[z_0(\xi^2+\rho^2)l^2\right]^{-1} \ . \quad (3.80)$$

If we put $\rho = 0$ in Eqn (3.78) and introduce the notation

$$G_0 = G\Big|_{\rho=0} \ ,$$

we can use the resulting equation for G_0 to rewrite Eqn (3.56) in the form

$$E^{-1} = G_0^{-1} + \mathcal{A}_0(\xi,0)[B(\zeta,0,\tilde{v};\mathcal{A}_0 G) - B(1,0,\tilde{v};\mathcal{A}_0 G)] \ , \quad (3.81)$$

where

$$E^{-1} = (\zeta - 1)\mathcal{A}_0(\xi,0) \quad (3.82)$$

is proportional to the ratio $(\zeta - 1)/\xi^2 l^2$ for small values of ξ. It is easily seen that Eqn (3.81) can be used for the direct evaluation of the critical exponent ν, once the function G has been determined.

Let us now return to Eqn (3.78) and postulate that

$$G\Big|_{\rho^2=\lambda} = 1 \ , \quad (3.83)$$

or, equivalently, that

$$F(\xi^2 l^2, \lambda l^2, \tilde{v}; G) = 0 \ .$$

Let us now multiply both sides of Eqn (3.78) by α^{-1} and use the RG-properties of this equation to write it down in the equivalent form

$$(G')^{-1} = 1 + F(\xi^2 l^2, \rho^2 l^2, \tilde{v}'; G') - (1 - \alpha^{-1}) \ , \quad (3.84)$$

where

$$G' = \alpha G, \quad \tilde{v}' = \alpha^{-2}\tilde{v} \ .$$

Next, choose the value of $\rho^2 = \lambda'$ such that

$$G'\Big|_{\rho^2=\lambda'} = 1 \ , \quad (3.85)$$

i.e. [see Eqn (3.84)]

$$F(\xi^2 l^2, \lambda' l^2, \tilde{v}'; G') = 1 - \alpha^{-1} \ .$$

It is clear that, in general, λ' depends on $\xi^2, l^2, \tilde{v}$, and α. Hence the equation for G can be represented in the form

$$G^{-1} = 1 + F(\xi^2 l^2, \rho^2 l^2, \tilde{v}; G) - F(\xi^2 l^2, \lambda l^2, \tilde{v}; G) \ , \quad (3.86)$$

invariant under the transformation

$$G \to G', \quad \tilde{v} \to \tilde{v}', \quad \lambda \to \lambda' \ . \quad (3.87)$$

It follows that it is proper to consider the dimensionless quantity G as a function of dimensionless variables:

$$G = G(\xi^2\lambda^{-1}, \rho^2\lambda^{-1}, \tilde{v}) \ ,$$

Eqn (3.83) written in the form

$$G(\xi^2\lambda^{-1}, 1, \tilde{v}) = 1 \ , \quad (3.88)$$

is called the *normalization condition*. The RG-property of Eqn (3.78) can now be expressed as

$$\alpha G(\xi^2\lambda^{-1}, \rho^2\lambda^{-1}, \tilde{v}) = G(\xi^2(\lambda')^{-1}, \rho^2(\lambda')^{-1}, \tilde{v}') \ , \quad (3.89)$$

which means that the multiplication of the function G by a nonzero number α is equivalent both to a change of the normalization point λ and to a renormalization of the excluded volume $\tilde{v}$. Setting $\rho^2 = \lambda'$ in Eqn (3.89) and using the normalization condition

$$G(\xi^2(\lambda')^{-1}, 1, \tilde{v}') = 1 \ ,$$

which corresponds to Eqn (3.85), we obtain

$$\alpha^{-1} = G(\xi^2\lambda^{-1}, \lambda'\lambda^{-1}, \tilde{v}) \ ,$$

so that in terms of the new variables

$$x = \frac{\xi^2}{\lambda}\ , \quad y = \frac{\rho^2}{\lambda}\ , \quad t = \frac{\lambda'}{\lambda}$$

Eqn (3.89) assumes the form

$$G(x,y,\tilde{v}) = G(x,t,\tilde{v})G\left(\frac{x}{t}\ ,\ \frac{y}{t}\ ,\ \tilde{v}G^2(x,t,\tilde{v})\right)\ . \tag{3.90}$$

Squaring both sides of Eqn (3.90) and then multiplying the result by $\tilde{v}$, we finally obtain for the RG-invariant quantity

$$V(x,y,\tilde{v}) = \tilde{v}G^2(x,y,\tilde{v}) \tag{3.91}$$

the functional equation

$$V(x,y,\tilde{v}) = V\big(xt^{-1}, yt^{-1}, V(x,t,\tilde{v})\big) \tag{3.92}$$

with the normalization condition

$$V(x,1,\tilde{v}) = \tilde{v}\ . \tag{3.93}$$

Eqn (3.92) is closed and can be solved in general form [76]. However, for practical purposes, it is more convenient to deal with the Lie differential equations corresponding to the continuous renormalization group. For example, to obtain the Lie equation for the function $V(x,y,\tilde{v})$, we differentiate both sides of Eqn (3.92) with respect to y and then put $t = y$. This gives

$$y\frac{\partial V(x,y,\tilde{v})}{\partial y} = \beta\left(\frac{x}{y}\ ,\ V(x,y,\tilde{v})\right)\ , \tag{3.94}$$

where

$$\beta(x,\tilde{v}) = \left.\frac{\partial V(x,y,\tilde{v})}{\partial y}\right|_{y=1}\ , \tag{3.95}$$

and the normalization condition (3.93) serves as the boundary condition for Eqn (3.94). On the other hand, if we differentiate Eqn (3.92) with respect to t and then put $t = 1$, we arrive at another form of the Lie equation:

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \beta(x,\tilde{v})\frac{\partial}{\partial \tilde{v}}\right)V(x,y,\tilde{v}) = 0.$$

In a similar way we can obtain differential equations for the function $G(x,y,\tilde{v})$ directly. In fact, if we differentiate both sides of expression (3.90) with respect to y and put $t = y$ we obtain

$$\frac{\partial \ln G(x,y,\tilde{v})}{\partial \ln y} = \gamma\left(\frac{x}{y}\ ,\ V(x,y,\tilde{v})\right)\ , \tag{3.96}$$

where

$$\gamma(x,\tilde{v}) = \left.\frac{\partial G(x,y,\tilde{v})}{\partial y}\right|_{y=1}\ , \tag{3.97}$$

and relation (3.88) serves as the boundary condition for Eqn (3.96). However, if we differentiate Eqn (3.90) with respect to t and then put $t = 1$, we obtain

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \beta(x,\tilde{v})\frac{\partial}{\partial \tilde{v}}\right)\ln G(x,y,\tilde{v}) = \gamma(x,\tilde{v})\ .$$

From expressions (3.95) and (3.97) we get the following relation between the functions $\beta(x,\tilde{v})$ and $\gamma(x,\tilde{v})$:

$$\beta(x,\tilde{v}) = 2\tilde{v}\gamma(x,\tilde{v})\ .$$

The above RG-equations are particularly useful when investigating the asymptotic properties of the function $G(x,y,\tilde{v})$. For example, to discover the behaviour of $G(x,y,\tilde{v})$ when $x \ll y$ and $y \to 0$, we choose a normalization point λ satisfying

$$\xi^2 \ll \rho^2 \lesssim \lambda$$

and proceed to the limit as $\xi^2 \to 0$ in Eqn (3.94). Putting

$$V(y,\tilde{v}) = \lim_{x\to 0} V(x,y,\tilde{v})\ , \tag{3.98}$$

$$\beta(\tilde{v}) = \left.\frac{\partial V(y,\tilde{v})}{\partial y}\right|_{y=1}\ , \tag{3.99}$$

we arrive at the equation

$$y\frac{\partial V(y,\tilde{v})}{\partial y} = \beta(V(y,\tilde{v}))\ , \tag{3.100}$$

subject to the boundary condition

$$V(1,\tilde{v}) = \tilde{v}\ . \tag{3.101}$$

Eqn (3.100) and the boundary condition (3.101) can also be written in the Gell-Mann–Low form

$$\int_{\tilde{v}}^{V(y,\tilde{v})} \beta^{-1}(t)\,\mathrm{d}t = \ln y\ , \tag{3.102}$$

which is exceedingly convenient in analyzing the behaviour of the invariant excluded volume $V(y,\tilde{v})$ for small values of y or for $\rho^2 \ll \lambda$.

Following the above derivation scheme for RG-equations and interchanging the roles of ξ^2 and ρ^2, i.e. looking upon ξ^2 as the leading variable, we obtain RG-equations which are analogous in form to the previous ones.

Thus, according to the RG-equations, the invariant excluded volume is an effective parameter characterizing the strength of the volume interaction in a small neighbourhood of the points $\xi = 0$ and $\rho = 0$. Hence, to determine the asymptotic behaviour of, for example, the function $G(y,\tilde{v})$ as $y \to 0$, we must know the associated behaviour of $V(y,\tilde{v})$. However, this behaviour is determined by the properties of the functions $\beta(V)$ given in Eqn (3.100).

In fact, if $\beta(V)$ is positive, the excluded invariant volume $V(y,\tilde{v})$ decreases as $y \to 0$. If for some $V = \tilde{v}_* < \tilde{v}$, the function $\beta(V)$ vanishes and the integral on the left-hand side of Eqn (3.102) diverges, then its right-hand side is also equal to infinity. In other words, we have $V(y,\tilde{v}) \to \tilde{v}_*$ as $y \to 0$, where $\beta(V)$ satisfies $\beta(\tilde{v}_*) = 0$. Note that $\tilde{v}_* = 0$ is necessarily a zero of the function $\beta(V)$.

On the other hand, if $\beta(V)$ is negative, the opposite applies. In fact, the function $V(y,\tilde{v})$ increases as $y \to 0$. If now $\beta(V)$ vanishes at the point $V = \tilde{v}_* > \tilde{v}$ and if the integral in Eqn (3.102) diverges, then $V(y,\tilde{v}) \to \tilde{v}_*$ as $y \to 0$. If, however, the function $\beta(V)$ has no zeros when $V > \tilde{v}$, we have $V(y,\tilde{v}) \to \infty$ as $y \to 0$.

It follows that there can be stable and unstable zeros of the function $\beta(V)$. When V is close to a stable zero $\tilde{v}_*$, the invariant excluded volume $V(y,\tilde{v})$ tends to $\tilde{v}_*$ as $y \to 0$. If $\tilde{v}_*$ is an unstable zero, the quantity $V(y,\tilde{v})$ moves away from $\tilde{v}_*$ and tends to zero or to infinity as $y \to 0$.

Since perturbation theory is commonly used for computing the function $\beta(V)$, we get real information on its

behaviour only in a small neighbourhood of the point $V = 0$, where $\beta(0) = 0$. Actually, if $\beta(V)$ is positive in this neighbourhood, the invariant excluded volume tends to zero as $y \to 0$. If, on the other hand, the function $\beta(V)$ is negative near the origin, then $V(y, \tilde{v})$ increases as $y \to 0$, and this takes us outside the range of validity of perturbation theory.

3.4 Diagrammatic description of the excluded volume effect

As already noted, in order to determine the asymptotic behaviour of the probability density $W_N(\boldsymbol{R})$ as $N \to \infty$ for $R \gg l$, we must know the behaviour of the function $\mathcal{A}(\zeta, \rho)$ near the points $\zeta = 1$ and $\rho = 0$ or, equivalently, the behaviour of the function $G(x, y, \tilde{v})$ near $x = 0$ and $y = 0$. The most significant ranges of integration in all terms of the series (3.34) will then be the regions in which the absolute values of the arguments of all the integrands are small, i.e., when $\kappa l \ll 1$ for every integration variable κ. For such values of κ we also have $\kappa r_0 \ll 1$, since $r_0 < l$, and this enables us to use the approximation

$$f(\boldsymbol{r}) = -v_0 \delta(\boldsymbol{r}) \ , \tag{3.103}$$

which is well known in the theory of polymers and is equivalent to the equality $\tilde{v} = \tilde{v}_0$, as can be seen from definitions (3.3) and (3.31).

On the other hand, as κ increases and attains values such that $\kappa \gtrsim l^{-1}$ (so that R decreases to $R \lesssim l$), the volume accessible to the walking particle decreases because of the increasing role of the excluded volume effect. Hence, the probability density $W_N(\boldsymbol{R})$ for $R \lesssim l$ and, correspondingly, the function $\mathcal{A}(\zeta, \kappa)$ for $\kappa \gtrsim l^{-1}$, should assume their low limiting values. The range of integration $\kappa \gtrsim l^{-1}$ in all terms of the series (3.34) will then provide a negligible contribution, and we are entitled to use the approximation $\tilde{v}(\kappa) = \tilde{v}_0$ in Eqn (3.34) for all κ.

Thus, substituting $\tilde{v}(\boldsymbol{\kappa}) = \tilde{v}_0$ and using the notation (3.52)–(3.54), we can write the series (3.34) in the form

$$\begin{aligned} B(\zeta, \rho l, \tilde{v}_0; \mathcal{A}) &= -v_1 \tilde{w}(\zeta, 0) \\ &\quad + v_1^2 \int \mathcal{A}(\zeta, \boldsymbol{\rho} - \boldsymbol{\kappa}) M(\zeta, \kappa) \frac{\mathrm{d}^d \kappa}{(2\pi)^d} \\ &\quad - 2 v_1^3 \int \mathcal{A}(\zeta, \boldsymbol{\rho} - \boldsymbol{\kappa}) M^2(\zeta, \kappa) \frac{\mathrm{d}^d \kappa}{(2\pi)^d} + \ldots , \end{aligned} \tag{3.104}$$

where

$$v_1 = \zeta^2 v_0 = \tilde{v}_0 l^d \ ,$$

and the functions $\tilde{w}(\zeta, 0)$ and $M(\zeta, \kappa)$ are given by

$$\tilde{w}(\zeta, 0) = \int \mathcal{A}(\zeta, \kappa) \frac{\mathrm{d}^d \kappa}{(2\pi)^d} \ , \tag{3.105}$$

$$\begin{aligned} M(\zeta, \kappa) &= \int \mathcal{A}(\zeta, \boldsymbol{\kappa} - \boldsymbol{\kappa}') \mathcal{A}(\zeta, \kappa') \frac{\mathrm{d}^d \kappa'}{(2\pi)^d} \\ &= \omega \int_0^\infty \Lambda_s(\kappa R) \tilde{w}^2(\zeta, R) R^{2s+1} \, \mathrm{d}R \ . \end{aligned} \tag{3.106}$$

in accordance with Eqn (3.61). As a result of the above approximation, some of the terms of the series (3.34), for example those corresponding to the third and fourth diagrams in Fig. 6, are now described in exactly the same way. Next, the integration with respect to the angles in all terms of the series (3.104) reduces the latter to the new form

$$\begin{aligned} B(\zeta, \rho l, \tilde{v}_0; \mathcal{A}) &= -v_1 \tilde{w}(\zeta, 0) \\ &\quad + \omega v_1^2 \int_0^\infty \Lambda_s(\rho R) \tilde{w}^3(\zeta, R) R^{2s+1} \, \mathrm{d}R \\ &\quad - 2\omega^2 v_1^3 \int_0^\infty \mathrm{d}R \, R^{2s+1} \Lambda_s(\rho R) \tilde{w}(\zeta, R) \\ &\quad \times \int_0^\infty \Lambda_s(R\kappa) M^2(\zeta, \kappa) \kappa^{2s+1} \frac{\mathrm{d}\kappa}{(2\pi)^d} + \ldots \end{aligned} \tag{3.107}$$

A diagrammatic representation of the series (3.107) can be deduced from the corresponding representation of the series (3.34) in Fig. 6 by 'shrinking' the wavy line to a point and assigning the factor $-\tilde{v}_0/(2\pi)^d$ to such points that the conservation of momentum is satisfied at it as before. The final result is the diagrammatic representation of the series shown in Fig. 7 where the square in the diagram $(\sum)$ represents the set of diagrams of Fig. 8, usually called the vertex part or the complete four-point vertex. It is very important here that a closed equation can be established [66] for the vertex part Y, and this equation, together with the basic equation, constitute a closed system.

For this purpose we denote by the diagram in Fig. 9 the sum of those diagrams in the complete four-point vertex Y that cannot be separated into two parts by a vertical straight line such that these parts are joined by only two lines. A connection between the four-point vertices X and Y can readily be established by means of the Bethe–Salpeter equation (Fig. 10). This equation can be inverted, i.e. the four-point vertex X can be formally found from it. In fact, using the iteration method we obtain the equation shown in Fig. 11, according to which the four-point vertex X can be expressed in terms of the complete four-point vertex Y. Let us now denote by K_{1234} (Fig. 12) the sum of those diagrams in the vertex part Y that satisfy the following condition: no vertical and no horizontal straight line cuts such a diagram into two parts joined by only two lines. We call the four-point vertex K *compact* if it has this property. The Bethe–Salpeter equation represented in Fig. 10 can be now rewritten diagrammatically as shown in Fig. 13 where the compact four-point vertex K can also be expressed in terms of the complete four-point vertex Y shown in Fig. 14. The only exception here is the first term, i.e. the point that represents the quantity $-\tilde{v}/(2\pi)^d$. The relationship between X and K is clearly that shown in Fig. 15. It follows that the equation in Fig. 13 is equivalent to the equation in Fig. 16. Finally, if we substitute the equations shown in Figs 11 and 14 into the equation shown in Fig. 16, we obtain for the vertex part of Y the closed equation shown in Fig. 17. At this point we conclude the presentation of general concepts and turn to the solution of the main problem.

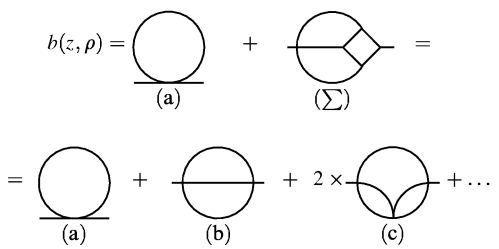

Figure 7.

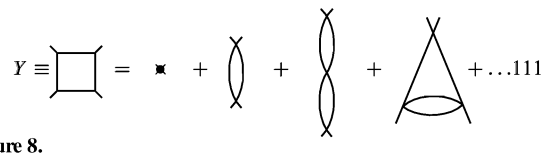

Figure 8.

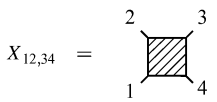

Figure 9.

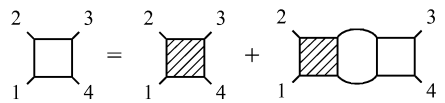

Figure 10.

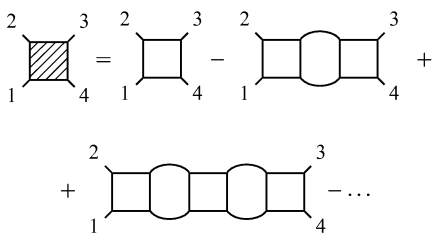

Figure 11.

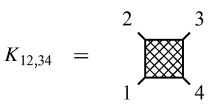

Figure 12.

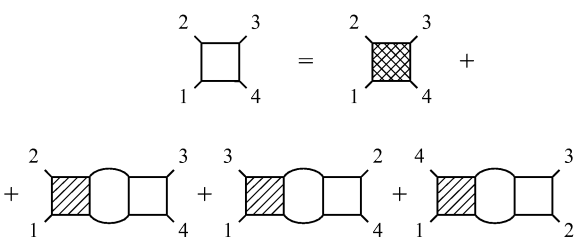

Figure 13.

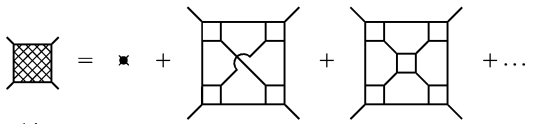

Figure 14.

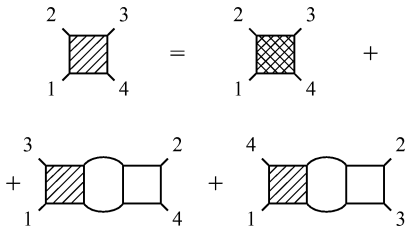

Figure 15.

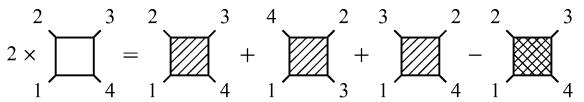

Figure 16.

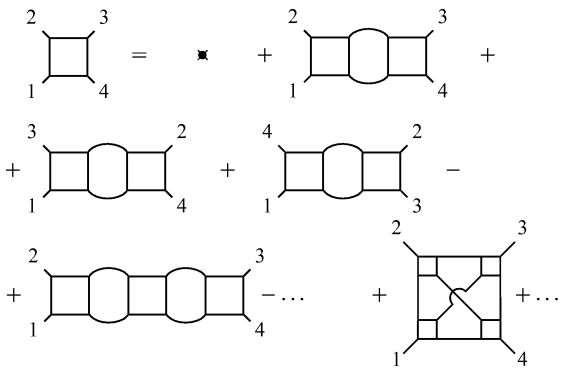

Figure 17.

4. Self-avoiding walks in a space of dimensionality $d \leqslant 4$

4.1 Small excluded volume. Perturbation theory

First we briefly consider a simpler case where the excluded volume $\tilde{v}_0$ for the random walk is so small that

$$\tilde{v}_0 N^{1-s} \ll 1 \tag{4.1}$$

for $d < 4$ ($s < 1$). The four-dimensional case will be treated subsequently in a different context. Condition (4.1) enables us to use the perturbation theory with the quantity $\tilde{v}_0 N^{1-s}$ acting as the small parameter. Whenever we calculate the approximation for the function $\mathcal{A}(\zeta, \kappa)$ in Eqn (3.57) by taking a finite number of terms in Eqn (3.107), we suppose this approximation has simple poles at the singular points $\rho_{1,2} = \pm i\xi$ ($\mathrm{Re}\,\xi > 0$) closest to the origin. We use this remark to obtain from Eqn (3.61) the following relation for large values of R ($R \gg l$):

$$\tilde{w}(\zeta, R) \sim 2d[(2\pi)^{d/2} l^2 \alpha(\xi)]^{-1} (\xi R^{-1})^s K_s(\xi R) \ , \tag{4.2}$$

where

$$\alpha(\xi) = z_0 \Lambda_{s+1}(\mathrm{i}\xi l) - 2d \frac{\partial}{\partial t^2} B(\zeta, t, \tilde{v}_0; \mathcal{A})\Big|_{t=\mathrm{i}\xi l} \tag{4.3}$$

and $K_s(z)$ is the MacDonald function [99]. If we substitute the expansion (3.107) into Eqn (4.3), we obtain

$$\begin{aligned}\alpha(\xi) &= z_0 \Lambda_{s+1}(\mathrm{i}\xi l) \\ &\quad + \omega v_1^2 l^{-2} \int_0^\infty \Lambda_{s+1}(\mathrm{i}\xi R) \tilde{w}^3(\zeta, R) R^{2s+3}\, \mathrm{d}R \\ &\quad - 2\omega^2 v_1^3 l^{-2} \int_0^\infty \mathrm{d}R\, R^{2s+3} \Lambda_{s+1}(\mathrm{i}\xi R) \tilde{w}(\zeta, R) \\ &\quad \times \int_0^\infty \Lambda_s(R\kappa) M^2(\zeta, \kappa) \kappa^{2s+1} \frac{\mathrm{d}\kappa}{(2\pi)^d} + \ldots \ ,\end{aligned} \tag{4.4}$$

which can be made a closed equation for $\alpha(\xi)$ by using relation (4.2).

To find the asymptotic form of the probability density $W_N(\boldsymbol{R})$ as $N \to \infty$ and $R \gg l$, we substitute expression (4.2) into Eqn (3.60) and then use identity (3.56) in the integral on the right-hand side of Eqn (3.60) to change from the variable ζ in the numerator to the variable ξ. If we also take into consideration that

$$K_s(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} \exp(-z), \qquad z \gg 1 \ , \tag{4.5}$$

we arrive at the asymptotic formula

$$\begin{aligned}\oint \mathrm{d}\zeta\, \zeta^N \tilde{w}(\zeta, R) &\sim \text{const} \times R^{-s-(1/2)} \\ &\quad \times \int \mathrm{d}\xi\, \xi^{s+(1/2)} \tilde{\alpha}(\xi) \alpha^{-1}(\xi) \exp(NH(\xi, RN^{-1})) \ ,\end{aligned} \tag{4.6}$$

in which

$$\tilde{\alpha}(\xi) = z_0 \Lambda_{s+1}(\mathrm{i}\xi l) + 2d \frac{\partial B(\zeta, \mathrm{i}\xi l, \tilde{v}_0; \mathcal{A})}{\partial (\xi l)^2}$$

and

$$H(\xi, t) = z_0 \Delta(\xi l) + B(\zeta, \mathrm{i}\xi l, \tilde{v}_0; \mathcal{A}) - B(1, 0, \tilde{v}_0; \mathcal{A}) - \xi t \ ,$$

$$\Delta(x) = \Lambda_s(\mathrm{i}x) - 1 \ .$$

If $N \to \infty$, with the ratio R/Nl fixed and $(R/Nl) \ll 1$, the integral in Eqn (4.6) can be estimated by the method of steepest descent. It enables us to represent the leading term in the asymptotic expansion of this integral in the form

$$\begin{aligned}\oint \mathrm{d}\zeta\, \zeta^N \tilde{w}(\zeta, R) &\sim \text{const} \times (Nl^2)^{-3/2} \alpha^{-1}(\xi_0) \\ &\quad \times (\xi_0 R^{-1})^{s-(1/2)} h^{-1/2}(RN^{-1}) \exp(NH(\xi_0, RN^{-1})) \ ,\end{aligned} \tag{4.7}$$

where

$$h(RN^{-1}) = \left| \frac{\partial^2 H(\xi, RN^{-1})}{\partial (\xi l)^2} \right|_{\xi = \xi_0}$$

and the stationary point $\xi = \xi_0$ is determined by

$$\xi_0 \tilde{\alpha}(\xi_0) = dR(Nl^2)^{-1} \ .$$

In view of expressions (3.60) and (4.7), the asymptotic form of the probability density $W_N(\boldsymbol{R})$ can be found from the normalization condition. For the cases $d = 3$ $(s = \frac{1}{2})$ and $d = 4$ $(s = 1)$ the asymptotic expressions for $W_N(\boldsymbol{R})$ to the second order in the small parameter $\tilde{v}_0 N^{1-s}$ are reported in Ref. [100].

In conclusion, we evaluate $\langle R^2 \rangle_N$ to the first order with respect to $\tilde{v}_0 N^{1-s}$ when $N \to \infty$, $\tilde{v}_0 \to 0$, and $\tilde{v}_0 N^{1-s} \ll 1$. For this purpose we use formula (3.65) and the fact that the function $B(\zeta, \rho l, \tilde{v}_0; \mathcal{A})$ given by the series (3.107) is independent of ρ to the first order with respect to $\tilde{v}_0 N^{1-s}$. Then the adopted degree of approximation, to formula (3.65) can be written as

$$\langle R^2 \rangle_N = l^2 \oint \mathrm{d}\zeta\, \zeta^N \Delta^{-2}(\xi l) \left[\oint \mathrm{d}\zeta\, \zeta^N \Delta^{-1}(\xi l) \right]^{-1} \ , \tag{4.8}$$

where

$$\zeta - 1 = z_0 \Delta(\xi l) + v_0(\tilde{w}(1, 0) - \tilde{w}(\zeta, 0)) \ . \tag{4.9}$$

Since $N \to \infty$, the biggest contribution to the integrals in Eqn (4.8) is provided by the neighbourhood of the point $\zeta = 1$ (or $\xi = 0$), and hence we can use the approximation

$$\zeta^N \cong \exp[N(\zeta - 1)] \ ,$$

$$\Delta(\xi l) \cong (\xi l)^2 (2d)^{-1} + \mathrm{O}(\xi^4 l^4) \ .$$

To determine the right-hand side of (4.9) in the neighbourhood of $\zeta = 1$ we apply the formula

$$\begin{aligned}\mathcal{K}_s(x) &\equiv 2^{1-s} \Gamma^{-1}(s) x^s K_s(x) \\ &= 1 - \frac{\Gamma(1-s)}{\Gamma(1+s)} \left(\frac{x}{2}\right)^{2s} + \mathrm{O}(x^2), \qquad x \to 0 \ ,\end{aligned} \tag{4.10}$$

where $0 < s < 1$. From Eqns (4.2) and (4.10) it follows that

$$\begin{aligned}\tilde{w}(\zeta, R) &\sim 2^s d\Gamma(s) z_0^{-1} (2\pi l^2)^{-d/2} \\ &\quad \times \left[\left(\frac{l}{R}\right)^{2s} - \frac{\Gamma(1-s)}{\Gamma(1+s)} \left(\frac{\xi l}{2}\right)^{2s} + \mathrm{O}\left((\xi l)^2 \left(\frac{R}{l}\right)^{2(1-s)} \right) \right]\end{aligned} \tag{4.11}$$

as $\zeta \to 1$ $(\xi \to 0)$. Introducing the notation

$$\tilde{w}(1, R) = \lim_{\zeta \to 1} \tilde{w}(\zeta, R)$$

we readily see that the difference

$$\tilde{w}(1, R) - \tilde{w}(\zeta, R) \tag{4.12}$$

for sufficiently small values of $\zeta - 1$ (or ξl) is almost independent of R. Therefore we equate the leading term in the expansion of the function (4.12) to the difference

$$\tilde{w}(1, 0) - \tilde{w}(\zeta, 0) \ ,$$

for which the relation (4.11) yields

$$\tilde{w}(1, 0) - \tilde{w}(\zeta, 0) \sim \frac{d\Gamma(s)\Gamma(1-s)}{z_0 (2\pi l^2)^{d/2} \Gamma(1+s)} \left(\frac{\xi^2 l^2}{2}\right)^s \ .$$

We now introduce the notation

$$g = [d(2\pi z_0 l^2)^{-1}]^{d/2} v_0 N^{1-s} \ ,$$

$$\lambda = z_0 N (\xi l)^2 (2d)^{-1}$$

and go over to the new integration variable λ in Eqn (4.8) given by

$$N(\zeta - 1) = \lambda + \Gamma(1-s) g \lambda^s s^{-1} \ .$$

Then the first approximation for $\langle R^2\rangle_N$ in terms of g can be represented in the form

$$\langle R^2\rangle_N = z_0 N l^2 \int \mathrm{d}\lambda(\exp\lambda)\lambda^{-2}\Phi(\lambda) \times\left[\int \mathrm{d}\lambda(\exp\lambda)\lambda^{-1}\Phi(\lambda)\right]^{-1} , \tag{4.13}$$

where

$$\Phi(\lambda) = 1 + \Gamma(1-s)g(\lambda^{s-1} + \lambda^s s^{-1}) .$$

Applying the identity

$$\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}\lambda \exp(-t\lambda)\lambda^{-\nu} = \frac{t^{\nu-1}}{\Gamma(\nu)}$$

to the evaluation of the integral in (4.13), and replacing $z_0 = z_0(\tilde{v}_0)$ by its limiting value $z_0(0) = 1$, we finally obtain

$$\langle R^2\rangle_N = N l^2\{1 + g[(1-s)(2-s)]^{-1} + \ldots\} , \tag{4.14}$$

which is used for calculating $\langle R^2\rangle_N$ in the phenomenological approach to our problem [3].

4.2 Dimensionality $d = 4$

Before proceeding to the four-dimensional case I shall make some general remarks essential to the investigation of the SAW problem for all $d \leqslant 4$.

To determine the value of F in Eqn (3.78), we use expression (3.79) and the approximation (3.107). Recall that this approximation for the function $B(\zeta, \rho l, \tilde{v}; \mathcal{A})$ was obtained under the condition that $\tilde{v}(\kappa) = \tilde{v}_0$ for all values of κ, owing to which the upper limit of integration with respect to this variable is now removed. Therefore, in order to avoid a possible divergence of the integral near the upper limit, we have to introduce a cut-off parameter into the theory. But, owing to the RG-properties of the basic equation, the final result cannot depend on the specific choice of this parameter. There are several ways to regularize divergent expressions (see, for example, Ref. [101]), but in our problem it turns out to be more convenient to employ a method which is essentially novel. Namely, when choosing the initial approximation for the function $\mathcal{A}(\zeta, \kappa)$ on the right-hand side of Eqn (3.78), we use the properties of the MacDonald function $K_\mu(z)$. First, we must take into account the asymptotic form of this function for small and large values of the argument z:

$$K_\mu(z) \sim \begin{cases} 2^{\mu-1}\Gamma(\mu)z^{-\mu}, & 0 < z \ll 1, \\ & \mu > 0, \\ \sqrt{\pi}(2z)^{-1/2}\exp(-z), & z \gg 1 . \end{cases} \tag{4.15}$$

Next, it is important to note that $K_\mu(z)$ is an integral function of the index μ for every fixed z ($z \neq 0$). As will be shown here, μ is directly related to the dimensionality of the space and, in addition, we have $K_{-\mu}(z) = K_\mu(z)$.

For example, instead of using $\mathcal{A}_0(\xi, \kappa)$, the 'unperturbed' function for $\mathcal{A}(\zeta, \kappa)$ can be given by the function

$$\mathcal{A}_1(\xi, \kappa) = 2dL\left(z_0 l^2\sqrt{\xi^2 + \kappa^2}\right)^{-1} K_1\left(L\sqrt{\xi^2 + \kappa^2}\right) , \tag{4.16}$$

where the cut-off parameter L^{-1} is such that $L \gtrsim l$. In fact, from expressions (4.15) and (4.16) it follows that for small values of κ $\mathcal{A}_1(\xi, \kappa)$ behaves like $\mathcal{A}_0(\xi, \kappa)$, and for large values of κ ($\kappa \gg L^{-1}$) $\mathcal{A}_1(\xi, \kappa)$ tends exponentially to zero.

Further, to simplify notation, we shall denote by C positive constants which may depend on the dimensionality of the space, i.e. C may assume different values in different formulas, but, if necessary, these values may be written in an explicit form.

We now turn to the SAW problem for space of dimensionality $d = 4$.

As the initial approximation for $\mathcal{A}$ in Eqn (3.78) we take the function

$$\mathcal{A}_1(\xi, \kappa) = 8L\left(z_0 l^2\sqrt{\xi^2 + \kappa^2}\right)^{-1} K_1\left(L\sqrt{\xi^2 + \kappa^2}\right) . \tag{4.17}$$

Since the functions $\mathcal{A}_0(\xi, \kappa)$ and $\mathcal{A}_1(\xi, \kappa)$ are practically equal to each other on the most essential part of their domain of definition, Eqn (3.77) shows that the initial approximation for G can be taken to be unity. Substituting expression (4.17) for (ζ, κ) in formula (3.61) and using the Sonine–Gegenbauer formula (see Ref. [99])

$$\int_0^\infty J_s(\alpha x)K_\mu\left(\beta\sqrt{x^2 + y^2}\right)(x^2 + y^2)^{-\mu/2}x^{s+1}\,\mathrm{d}x = \alpha^s\beta^{-\mu}y^{s-\mu+1}(\alpha^2 + \beta^2)^{(\mu-s-1)/2}K_{\mu-s-1}\left(y\sqrt{\alpha^2 + \beta^2}\right) , \tag{4.18}$$

where $\operatorname{Re} y > 0$, we obtain

$$\tilde{w}_1(\xi, R) = 2\xi\left(\pi^2 z_0 l^2\sqrt{R^2 + L^2}\right)^{-1} K_1\left(\xi\sqrt{R^2 + L^2}\right) . \tag{4.19}$$

In this approximation the quantity $F(\xi^2 l^2, \rho^2 l^2, \tilde{v}; G)$ can be represented, according to expressions (3.79) and (3.107), in the form

$$\begin{aligned} F(\xi^2 l^2, \rho^2 l^2, \tilde{v}; G_1) &= \mathcal{A}_0(\xi, \rho)\Big[2\pi^2 v_1^2\int_0^\infty \mathrm{d}R\, R^3\big(\Lambda_1(\mathrm{i}\xi R) - \Lambda_1(\rho R)\big) \\ &\times \tilde{w}_1^3(\xi, R) - \frac{1}{2}v_1^3\int_0^\infty \mathrm{d}R\, R^3\big(\Lambda_1(\mathrm{i}\xi R) - \Lambda_1(\rho R)\big) \\ &\times \tilde{w}_1(\xi, R)\int_0^\infty \mathrm{d}\kappa\,\kappa^3\Lambda_1(R\kappa)M_1^2(\xi, \kappa) + \ldots\Big] , \end{aligned} \tag{4.20}$$

where $G_1 = \mathcal{A}_1/\mathcal{A}_0$ and

$$M_1(\xi, \kappa) = 2\pi^2\int_0^\infty \Lambda_1(\kappa R)\tilde{w}_1^2(\xi, R)R^3\,\mathrm{d}R .$$

In order to find the asymptotic form of G when $\zeta = 1$ and $\rho \to 0$, we let $\xi = 0$ in the approximation (4.20) and use the relation

$$\tilde{w}_1(0, R) = 2[\pi^2 l^2(R^2 + L^2)]^{-1} ,$$

that follows from Eqn (4.19). To calculate the integrals in expression (4.20) we employ the formulas (see Ref. [99])

$$\int_0^\infty \frac{R^{s+1}J_s(\rho R)}{(R^2 + L^2)^{\mu+1}}\,\mathrm{d}R = \rho^\mu L^{s-\mu}[2^\mu\Gamma(\mu+1)]^{-1}K_{s-\mu}(\rho L) , \tag{4.21}$$

$$-1 < \operatorname{Re} s < 2\operatorname{Re}\mu + \tfrac{3}{2} ,$$

$$\int_0^\infty \frac{R^{s-\mu+1}}{R^2 + L^2}J_\mu(\kappa R)J_s(\rho R)\,\mathrm{d}R = L^{s-\mu}I_\mu(\kappa L)K_s(\rho L) , \tag{4.22}$$

$$-1 < \operatorname{Re} s < \operatorname{Re}\mu + 2, \quad \kappa \leqslant \rho ,$$

where $I_\mu(x)$ is the modified Bessel function of the first kind. After straightforward evaluations (see Appendix 1) we obtain

$$F(0,\rho^2 l^2,\tilde{v};G_1) = -\tfrac{1}{2}\tilde{v}_1^2 \ln(\rho L_1)^2 - \tilde{v}_1^3\left[\ln^2(\rho L_1)^2 - 3\ln(\rho L_1)^2\right] + \dots \ , \quad (4.23)$$

where $\tilde{v}_1 = 4\tilde{v}_0/\pi^2$, $L_1 = CL$. Finally, substitution of expression (4.23) into Eqn (3.78) yields

$$G^{-1}(0,y,\tilde{v}) = 1 - \tfrac{1}{2}\tilde{v}_1^2 \ln y - \tilde{v}_1^3(\ln^2 y - 3\ln y) + \dots \quad (4.24)$$

Further, from Eqns (3.99) and (4.24) we get the function $\beta\tilde{v}$ in the form

$$\beta(\tilde{v}) = \tilde{v}^3 + \mathrm{O}(\tilde{v}^4)$$

On substituting into the Gell-Mann–Low equation (3.102) and solving the latter we obtain the invariant excluded volume

$$V(y,\tilde{v}_1) = \tilde{v}_1(1 - 2\tilde{v}_1^2 \ln y)^{-1/2} \ . \quad (4.25)$$

It follows that $V(y,\tilde{v}_1) \to 0$ as $y \to 0$, i.e. the function $\beta(\tilde{v})$ has a stable zero at the point $\tilde{v} = 0$. Hence, for $\zeta = 1$ and $\rho \to 0$, from Eqn (4.25) and the definitions of G and V in (3.77) and (3.91) we obtain

$$\mathcal{A}^{-1}(1,\rho) \sim (\rho l)^2 \mathcal{L}(\rho l) \ , \quad (4.26)$$

where

$$\mathcal{L}(x) = (4\pi)^{-1}(4\tilde{v}_0^2 \ln x^{-1})^{1/4} \ .$$

Let us now assume that ζ belongs to a small neighbourhood point $\zeta = 1$. We now write the basic equation (3.55) in the form

$$\mathcal{A}^{-1}(\zeta,\rho) = \zeta - 1 + \mathcal{A}^{-1}(1,\rho) + B(1,\rho l,\tilde{v};\mathcal{A}) - B(\zeta,\rho l,\tilde{v};\mathcal{A}) \ , \quad (4.27)$$

where the function $\mathcal{A}^{-1}(1,\rho)$, according to its definition

$$\mathcal{A}^{-1}(1,\rho) = z_0\big(1 - \Lambda_1(\rho l)\big) + B(1,0,\tilde{v};\mathcal{A}) - B(1,\rho l,\tilde{v};\mathcal{A}) \ ,$$

exceeds in absolute value the difference

$$B(1,\rho l,\tilde{v};\mathcal{A}) - B(\zeta,\rho l,\tilde{v};\mathcal{A})$$

for all sufficiently small values of $|\zeta - 1|$. This difference can then be neglected, and the approximate equation for $\mathcal{A}(\zeta,\rho)$ will assume the form

$$\mathcal{A}^{-1}(\zeta,\rho) = \zeta - 1 + \mathcal{A}^{-1}(1,\rho) \ . \quad (4.28)$$

Using expression (4.26) in the right-hand side of this equation, we get the expression for $\mathcal{A}(\zeta,\rho)$ and substitute it into the formula

$$\tilde{w}(\zeta,R) = \frac{1}{4\pi^2 R}\int_0^\infty J_1(R\rho)\mathcal{A}(\zeta,\rho)\rho^2\,\mathrm{d}\rho \ . \quad (4.29)$$

To evaluate the integral in (4.29) for $R \gg l$, we take into consideration only the contribution provided by the roots of the equation $\mathcal{A}^{-1}(\zeta,\rho) = 0$ closest to the origin $\rho = 0$. These have the form $\rho_{1,2} \equiv \pm\mathrm{i}\xi$ ($\mathrm{Re}\,\xi > 0$), i.e.

$$(\xi l)^2 \mathcal{L}(\pm\mathrm{i}\xi l) = \zeta - 1 \ .$$

The result is

$$\tilde{w}(\zeta,R) \sim \xi[4\pi^2 l^2 R\mathcal{L}(\mathrm{i}\xi l)]^{-1} K_1(\xi R) \quad (4.30)$$

for all $R \gg l$. On substituting expression (4.30) into the inversion formula (3.60) and using the asymptotic form of the function $K_1(x)$ for large values of its argument, we obtain the following expression for the probability density $W_N(\boldsymbol{R})$:

$$W_N(\boldsymbol{R}) \sim C_N l^{-4}\left(\frac{l}{R}\right)^{3/2}\int \mathrm{d}t\, t^{3/2}\exp\left(NH\left(t,\frac{R}{Nl}\right)\right) \ , \quad (4.31)$$

where C_N is the normalization factor and

$$H(t,x) = t^2\mathcal{L}(\mathrm{i}t) - tx \ .$$

We can use the method of steepest descent to obtain the asymptotic expression for the integral in (4.31), provided that $N \to \infty$, the ratio R/N is fixed, and $(R/N) \ll 1$. A simple calculation [100] shows that

$$W_N(\boldsymbol{R}) \sim (\pi N_1 l^2)^{-2}[\ln(N_1 l R^{-1})]^{-1/2} \times \exp[-R^2\{N_1 l^2[\ln(N_1 l R^{-1})]^{1/4}\}^{-1}] \ , \quad (4.32)$$

where $N_1 = (\tilde{v}_1/2)^{1/2}N$. Hence, we can readily find the corresponding asymptotic form of the mean square end-to-end distance for the random particle:

$$\langle R^2\rangle_N \sim N_1 l^2(8\ln N_1)^{1/4} \ . \quad (4.33)$$

The last result is in good agreement with the numerical calculations of $\langle R^2\rangle_N$ when $d = 4$ [49].

4.3 Dimensionality $d < 4$

4.3.1 Asymptotic case: $\varepsilon \to 0$. We first consider the SAW problem for the space $\mathscr{R}^d$ of dimensionality $d = 4 - \varepsilon$ where $\varepsilon \to 0$. The initial approximation for $\mathcal{A}(\zeta,\kappa)$ on the right-hand side of Eqn (3.78) is taken to be the nonperturbed function $\mathcal{A}_1(\zeta,\kappa)$, given by Eqn (4.16). The presence or the auxiliary parameter L is only a matter of convenience when evaluating the integrals, and it can be eliminated from the final formulas by proceeding to the limit as $L \to 0$.

Substituting expression (4.16) for $\mathcal{A}(\zeta,\kappa)$ in Eqn (3.61) and using formula (4.18), we obtain

$$\tilde{w}_1(\xi,R) = 2\,\mathrm{d}\xi^s\left[(2\pi)^{s+1}z_0 l^2(R^2+L^2)^{s/2}\right]^{-1} \times K_s\left(\xi\sqrt{R^2+L^2}\right) \ . \quad (4.34)$$

If we now substitute the last expression for $\tilde{w}(\zeta,R)$ in Eqn (3.107), then use the result in Eqn (3.79) and finally proceed to the limit as $\zeta \to 1$ ($\xi \to 0$), we get

$$F(0,\rho^2 l^2,\tilde{v};G_1) = \mathcal{A}_0(0,\rho)\Big[\omega v_0^2\int_0^\infty\big[1 - \Lambda_s(\rho R)\big]\tilde{w}_1^3(R)R^{2s+1}\,\mathrm{d}R - 2\omega^2 v_0^3\int_0^\infty \mathrm{d}R\, R^{2s+1}\big(1 - \Lambda_s(\rho R)\big)\tilde{w}_1(R) \times \int_0^\infty \Lambda_s(R\kappa)M_1^2(\kappa)\kappa^{2s+1}\frac{\mathrm{d}\kappa}{(2\pi)^d} + \dots\Big] \ , \quad (4.35)$$

where $G_1 = \mathcal{A}_1/\mathcal{A}_0$, and

$$\tilde{w}_1(R) = \lim_{\xi\to 0}\tilde{w}_1(\xi,R) = 2^s\Gamma(s)d\left[(2\pi)^{s+1}l^2(R^2+L^2)^s\right]^{-1},$$

$$M_1(\kappa) = \lim_{\xi\to 0} \omega \int_0^\infty \Lambda_s(\kappa R)\tilde{w}_1^2(\xi, R) R^{2s+1}\,\mathrm{d}R \tag{4.36}$$
$$= C_1 L^{1-s}(l^4\kappa^{1-s})^{-1} K_{1-s}(\kappa L) \ ,$$
$$C_1 = 2d^2\Gamma^2(s)\left[(2\pi)^{s+1}\Gamma(2s)\right]^{-1} \ .$$

The value of the integral in Eqn (4.36) was found with the help of formula (4.21), and the integrals in (4.35) are calculated in Appendix 2. As a result we have

$$F(0, \rho^2 l^2, \tilde{v}; G_1) = \frac{1}{2\varepsilon}\tilde{v}_1^2(\rho l)^{-2\varepsilon} - \frac{8}{3\varepsilon^2}\tilde{v}_1^3(\rho l)^{-3\varepsilon} + \dots \ , \tag{4.37}$$

where $\tilde{v}_1 = v_0 d/\pi^2 l^d$. From this it is readily seen that the absolute value of $F(0, \rho^2 l^2, \tilde{v}; G_1)$ increases as $\rho \to 0$, whereas by Eqn (3.78), the function $G(0, y, \tilde{v})$ decreases. In what follows $\tilde{v}$ still denotes for brevity the renormalized coupling constant, which is proportional to the excluded volume, i.e. $\tilde{v} = Cv_0/l^d$. Defining the function $G(0, y, \tilde{v})$ by Eqns (3.78) and (4.37), we obtain the expansion

$$G^{-1}(0, y, \tilde{v}) = 1 - \frac{\tilde{v}^2}{2}\ln y + \frac{4\tilde{v}^3}{\varepsilon}\ln y - \dots \tag{4.38}$$

valid in a neighbourhood of the normalization point $y = 1$. The calculation of $\beta(\tilde{v})$ based on Eqns (3.99) and (3.38) yields

$$\beta(\tilde{v}) = \tilde{v}^3\left(1 - \frac{8}{\varepsilon}\tilde{v}\right) \ . \tag{4.39}$$

It follows that for sufficiently small values of ε the function $\beta(\tilde{v})$ can be negative. Because of this fact and the above remark concerning the behaviour of the function G when $\rho \to 0$, it is convenient to use the Gell-Mann–Low equation in the form

$$\int_{V(y,\tilde{v})}^{\tilde{v}} \left|\beta(t)\right|^{-1}\mathrm{d}t = \ln y^{-1} \ . \tag{4.40}$$

Since by Eqn (4.39) the function $\beta(\tilde{v})$ vanishes linearly at the point

$$\tilde{v} = \tilde{v}_* = \frac{\varepsilon}{8} \ , \tag{4.41}$$

the integral in Eqn (4.40) tends to infinity as $V(y, \tilde{v}) \to \tilde{v}_*$. Therefore, $\tilde{v}_*$ is the limiting value of the invariant excluded volume:

$$V(0, \tilde{v}) = \tilde{v}_* \ .$$

Finally, on performing the integration on the left-hand side of Eqn (4.40), we obtain the following transcendental equation for $V(y, \tilde{v})$:

$$\tfrac{1}{2}[V^{-2}(y, \tilde{v}) - \tilde{v}^{-2}] + \tilde{v}_*^{-1}[V^{-1}(y, \tilde{v}) - \tilde{v}^{-1}]$$
$$+\tilde{v}_*^{-2}\ln\left(\frac{\tilde{v}}{V(y, \tilde{v})}\,\frac{V(y, \tilde{v}) - \tilde{v}_*}{\tilde{v} - \tilde{v}_*}\right) = \ln y \ . \tag{4.42}$$

The simplest way of analyzing the behaviour of the function $V(y, \tilde{v})$ as $y \to 0$ in Eqn (4.42) is based on the approximation

$$\beta(\tilde{v}) \cong \frac{\varepsilon^2}{64}(\tilde{v}_* - \tilde{v}) \ . \tag{4.43}$$

On solving this equation we arrive at the relation

$$V(y, \tilde{v}) \cong \tilde{v}_* + C(\tilde{v} - \tilde{v}_*)y^{\varepsilon^2/64} \ , \tag{4.44}$$

which determines how $V(y, \tilde{v})$ converges to v_* as $y \to 0$. It follows that for $\varepsilon \to 0$ relation (4.44) transforms to

$$V(y, \tilde{v}) \sim C\tilde{v}y^{\varepsilon^2/64} \ , \tag{4.45}$$

and hence for the critical exponent η in the relation

$$\mathcal{A}(1, \rho) \sim \mathrm{O}\left((\rho l)^{\eta-2}\right), \quad \rho \to 0 \ ,$$

we have

$$\eta \cong \frac{\varepsilon^2}{64} \ , \tag{4.46}$$

which is in good agreement with the corresponding result obtained by De Gennes [54].

We now turn to the critical exponent ν that determines the asymptotic dependence

$$\xi l \sim \mathrm{O}\left(|\zeta - 1|^{\nu}\right), \quad \zeta \to 1 \ .$$

To find this exponent we evaluate the right-hand side of Eqn (3.81) to terms of the second order in $\tilde{v}_0$. In the approximation adopted, this equation can be written as

$$E^{-1} = 1 + 2d(\xi^2 l^2)^{-1}\Big\{-v_1\left[\tilde{w}_1(\xi, 0) - \tilde{w}_1(0, 0)\right]$$
$$+\omega v_1^2\int_0^\infty \left[\tilde{w}_1^3(\xi, R) - \tilde{w}_1^3(0, R))\right]R^{2s+1}\,\mathrm{d}R - \dots\Big\} \ , \tag{4.47}$$

where $\tilde{w}_1(\xi, R)$ is given by Eqn (4.34). Since the function $\tilde{w}_1(\xi, R)$ contains the auxiliary parameter L, it is necessary, as previously mentioned, to proceed to the limit as $L \to 0$ in all terms of the series (4.47). We then obtain the following asymptotic relations:

$$\tilde{w}_1(0, 0) - \tilde{w}_1(\xi, 0) \sim (\pi^2 l^2 \varepsilon)^{-1}\xi^{2-\varepsilon} \ , \tag{4.48}$$
$$\int_0^\infty \left[\tilde{w}_1^3(0, R) - \tilde{w}_1^3(\xi, R)\right]R^{2s+1}\,\mathrm{d}R$$
$$\sim 6\xi^{2-2\varepsilon}\left[(\pi l)^6\varepsilon^2\right]^{-1}, \quad \xi \to 0 \ . \tag{4.49}$$

The last integral is estimated in Appendix 2. Using relations (4.48) and (4.49) in Eqn (4.47) we obtain for the quantity

$$E = \xi^2 l^2\left[2d(\zeta - 1)\right]^{-1}$$

the equation

$$E^{-1} = 1 + \frac{2}{\varepsilon}\tilde{v}x^{-\varepsilon/2} - \frac{6}{\varepsilon^2}\tilde{v}^2 x^{-\varepsilon} + \dots \ , \tag{4.50}$$

in which $x = C\xi^2 l^2$ is the independent variable. Let us note that by Eqn (4.50) the absolute value of the function $E = E(x, v)$ decreases as $x \to 0$.

Let us now return to the initial equation (3.81) and use the RG-properties of Eqn (3.78). It is readily seen that the quantity

$$\bar{v}(x, \tilde{v}) = \tilde{v}E^{\varepsilon/2}(x, \tilde{v}) \tag{4.51}$$

is invariant under the RG-transformations. Hence the function $\bar{v}(x, \tilde{v})$ must obey the Gell-Mann–Low equation, which is conveniently written in the form

$$\int_{\bar{v}(x,\tilde{v})}^{\tilde{v}} \left|\bar{\beta}(t)\right|^{-1}\mathrm{d}t = \ln x^{-1} \ , \tag{4.52}$$

where

$$\bar{\beta}(\tilde{v}) = \left.\frac{\partial \bar{v}(x,\tilde{v})}{\partial x}\right|_{x=1} , \tag{4.53}$$

and the behaviour of the $\bar{v}(x,\tilde{v})$ in a neighbourhood of its normalization point $x = 1$ is determined by the expansion

$$\bar{v}(x,\tilde{v}) = \tilde{v}\left(1 + \frac{\varepsilon}{2}\tilde{v}\ln x - 4\tilde{v}^2 \ln x + \ldots\right) , \tag{4.54}$$

obtained by Eqns (4.51) and (4.50). Substituting Eqn (4.54) into Eqn (4.53) we find

$$\bar{\beta}(\tilde{v}) = 4\tilde{v}^2(\tilde{v}_* - \tilde{v}) , \tag{4.55}$$

which, like $\beta(\tilde{v})$ in the previous case, vanishes linearly at the point $\tilde{v}_* = \varepsilon/8$. After substituting Eqn (4.55) into Eqn (4.52) and performing the necessary integration we obtain

$$\tilde{v}_*[\bar{v}^{-1}(x,\tilde{v}) - \tilde{v}^{-1}] + \ln\left(\frac{\tilde{v}}{\bar{v}(x,\tilde{v})}\,\frac{\bar{v}(x,\tilde{v}) - \tilde{v}_*}{\tilde{v} - \tilde{v}_*}\right) = 4\tilde{v}_*^2 \ln x , \tag{4.56}$$

which is a transcendental equation for $\bar{v}(x,\tilde{v})$. However, to find the asymptotic form of the function $\bar{v}(x,\tilde{v})$ as $x \to 0$ we apply the approximation

$$\bar{\beta}(\tilde{v}) \cong \frac{\varepsilon^2}{16}(\tilde{v}_* - \tilde{v}) . \tag{4.57}$$

to Eqn (4.52) and obtain

$$\bar{v}(x,\tilde{v}) \cong \tilde{v}_* + C(\tilde{v} - \tilde{v}_*)x^{\varepsilon^2/16} , \tag{4.58}$$

which can be represented as

$$\bar{v}(x,\tilde{v}) \sim C\tilde{v}x^{\varepsilon^2/16} . \tag{4.59}$$

when $\varepsilon \to 0$. Using the definition (4.51) in the last formula to find the critical exponent ν, we obtain

$$\nu \cong \frac{1}{2} + \frac{\varepsilon}{16} , \tag{4.60}$$

which is also in agreement with the corresponding result (2.95) by De Gennes.

Since the above expressions for η and ν are obtained asymptotically as $d \to 4$, they cannot be used for real space. In the next section I shall present a method which permits calculation of the critical exponents for $d < 4$.

4.3.2 The general case. The fact, previously referred to, that formulas (4.46) and (4.60) have a restricted range of application is clearly connected with the use of the initial approximation $\mathcal{A}_1(\zeta,\rho)$ for $\mathcal{A}(\zeta,\rho)$. At the same time, the fact that the critical exponent η is not equal to zero shows that the singular points of the function $\mathcal{A}(\zeta,\rho)$ do not need to be poles. To extend the range of validity of the formulas for the critical exponents we must therefore start by choosing an approximation for $\mathcal{A}(\zeta,\rho)$ which contains singular points that are not poles.

We now note that when the variables ξ and ρ are sufficiently small, they appear in Eqn (3.57) only in the form of the combination $\xi^2 + \rho^2$, in accordance with Eqn (3.80). Moreover, the function $\mathcal{A}(\zeta,\rho)$ is even in ρ, and its singular points closest to the origin $\rho = 0$ are $\rho_{1,2} = \pm i\xi$. Hence we take the initial approximation for $\mathcal{A}(\zeta,\rho)$ to be the expression (see Ref. [102])

$$\tilde{\mathcal{A}}(\xi,\rho) = Cz_0^{-1}l^{-2}u^{2\mu}\left(\frac{L^2}{\xi^2+\rho^2}\right)^{(1-\mu)/2} K_{1-\mu}\left(L\sqrt{\xi^2+\rho^2}\right), \tag{4.61}$$

in which the parameters $u = u(\tilde{v}_0)$, $\mu = \mu(d)$ $(0 < \mu < 1)$ and $L \gtrsim l$ are to be determined. Since by assumption we have $\zeta \to 1$ $(\xi \to 0)$, it follows from Eqn (4.61) and the properties of the function $K_\lambda(t)$ that the quantity $\tilde{\mathcal{A}}(\xi,\rho)$ behaves as

$$\mathrm{O}\left(u^{2\mu}(\xi^2+\rho^2)^{\mu-1}\right)$$

when $\rho L \ll 1$, and is exponentially small when $\rho L \gg 1$. In accordance with the definition (3.77), the initial approximation for G will then be taken to be the function $\tilde{G} = \mathcal{A}/\tilde{\mathcal{A}}_0$, whose asymptotic behaviour as $\xi \to 0$ and $\rho \to 0$ can be described by:

$$\tilde{G}(\xi,\rho) \sim Cu^{2\mu}(\xi^2+\rho^2)^{\mu} . \tag{4.62}$$

It is important to emphasize here that the choice of a specific form for the initial approximation to the function $\mathcal{A}(\zeta,\rho)$ is of course, not unique if our main interest lies in the asymptotic form of this approximation as $\xi \to 0$ and $\rho \to 0$. Since the latter contains the desired information on the critical exponents in the SAW problem, the choice of the initial approximation in the form of expression (4.61) is prompted only by computational convenience. Our problem is now reduced to the evaluation of the parameters u, μ, and L for which the trial function $\tilde{\mathcal{A}}(\xi,\rho)$ is in a certain definite sense a solution of the basic equation.

Let us suppose that the values of u, μ, and L have been determined. Substituting the value of $\tilde{\mathcal{A}}(\xi,\rho)$ from expression (4.61) into formula (3.61) and using Eqn (4.18) we obtain

$$\tilde{w}(\xi,R) = Cz_0^{-1}l^{-2}u^{2\mu}\left(\frac{\xi^2}{R^2+L^2}\right)^{(\mu+s)/2} K_{\mu+s}\left(\xi\sqrt{R^2+L^2}\right), \tag{4.63}$$

which will be used later in the inversion formula

$$\tilde{w}_N(\boldsymbol{R}) = \frac{i}{2\pi z_0^N}\oint d\zeta\, \exp(N\ln\zeta)\tilde{w}(\xi,R) . \tag{4.64}$$

To find the asymptotic estimate for the integral in Eqn (4.64) when N and R tend to infinity, but their ratio R/N remains fixed and is much less than one, it is convenient to transform from ζ to the new integration variable ξ with the help of Eqn (3.56). Moreover, if we take the asymptotic properties of $K_\lambda(t)$ as $t \to \infty$ into account in expression (4.63), we can reduce formula (4.64) to the form

$$\tilde{w}_N(\boldsymbol{R}) \sim Cu^{2\mu}\left[z_0^{N+1}l^2R^{\mu+s+(\frac{1}{2})}\right]^{-1} \times \int d\xi\,\frac{d\zeta}{d\xi}\,\xi^{\mu+s-(\frac{1}{2})}\exp\left(ND\left(\xi,\frac{R}{N}\right)\right) , \tag{4.65}$$

where

$$D(\xi,t) = B(\zeta, i\xi l, \tilde{v}; \tilde{\mathcal{A}}) - B(1,0,\tilde{v};\tilde{\mathcal{A}}) - t\xi + \mathrm{O}(\xi^2 l^2) . \tag{4.66}$$

Finally, using the method of steepest descent for the asymptotic evaluation of the integral in expression (4.65) as $N \to \infty$, we obtain the required asymptotic form of the function $\tilde{w}_N(\boldsymbol{R})$.

We now proceed to evaluating $u, , \mu$, and L where μ is a dimensionless parameter connected with the critical exponent η by $\eta = 2\mu$, and where the parameters u and L have the dimension of length. As the exponent μ is the parameter that determines the degree of singularity of the function $\tilde{\mathcal{A}}(\xi, \rho)$, we should begin the solution of the problem with the evaluation of this parameter. Since the required parameters are independent of ξ and ρ we set $\xi = 0$ in Eqn (3.78) and, for simplicity, will not indicate the dependence of all the quantities on $\xi = 0$. In this case the function $\tilde{G}(\rho) = \tilde{\mathcal{A}}(\rho)/\mathcal{A}_0(\rho)$ has the following form near the point $\rho = 0$:

$$\tilde{G}(\rho) \sim C(u^2\rho^2)^{\mu} \ , \tag{4.67}$$

and this can be taken as the asymptotic solution of Eqn (3.78) for $\xi = 0$ and $\rho \to 0$, provided that

$$F\{\tilde{G}\} = \mathcal{A}_0(\rho)\left[B(1, 0, \tilde{v}; \tilde{\mathcal{A}}) - B(1, \rho l, \tilde{v}; \tilde{\mathcal{A}})\right]$$

$$\sim \sum_{m \geqslant 1} \left[-\mu \ln(u^2\rho^2)\right]^m (m!)^{-1} \ . \tag{4.68}$$

However, in accordance with the RG-method, to determine the asymptotic form of $G(\rho)$ when $\rho \to 0$ we only need to know the coefficient of the linear logarithmic term in the expansion for F

$$F\{\tilde{G}\} = 2d(z_0\rho^2 l^2)^{-1}\Big\{\omega v_1^2 \int_0^{\infty} [1 - \Lambda_s(\rho R)]\tilde{w}^3(R) R^{2s+1}\, \mathrm{d}R$$
$$-2\omega v_1^3 \int_0^{\infty} \mathrm{d}R\, R^{2s+1}\left[1 - \Lambda_s(\rho R)\right]\tilde{w}(\boldsymbol{R})$$
$$\times \int \tilde{w}^2(\boldsymbol{R} - \boldsymbol{r})\tilde{w}^2(r)\, \mathrm{d}^d r + \ldots\Big\} \tag{4.69}$$

in powers of $\ln(\rho^2 L^2)$, where

$$\tilde{w}(\boldsymbol{R}) = Cu^{2\mu}\left[z_0 l^2 (R^2 + L^2)^{\mu+s}\right]^{-1} \ . \tag{4.70}$$

The first term in the series (4.69) determining the contribution of diagram (b) in Fig. 7 to $F\{\tilde{G}\}$ will be written in the form

$$F_b(\rho) = \mathcal{A}_0(\rho)[B_b(0) - B_b(\rho)] \ , \tag{4.71}$$

where

$$B_b(\rho) = \omega v_1^2 \int_0^{\infty} \Lambda_s(\rho R)\tilde{w}^3(R) R^{2s+1}\, \mathrm{d}R$$
$$= Cz_0^{-3} v_1^2 l^{-6} u^{6\mu} (\rho L^{-1})^{3\mu+2s-1} K_{3\mu+2s-1}(\rho L) \tag{4.72}$$

represents the contribution of this diagram to $B(1, \rho l, \tilde{v}; \tilde{A})$. if we now substitute expression (4.72) into the right-hand side of Eqn (4.71) and take into account the formula

$$tK_1(t) = 1 + \frac{t^2}{2} \ln t + \mathrm{O}(t^2), \qquad t \to 0 \ , \tag{4.73}$$

we can readily see that the logarithmic behaviour of the function $F_b(\rho)$ for $\rho \to 0$ can be assured by putting $3\mu + 2s - 2 = 0$, i.e.

$$\mu = \tfrac{1}{3}(4 - d) \ . \tag{4.74}$$

In fact, it follows from Eqns (4.71)–(4.74) that

$$F_b(\rho) \sim -C(\tilde{v}_0 z_0^{-2})^2 (ul^{-1})^{6\mu} \ln \rho^2 L^2 \ . \tag{4.75}$$

It must be noted that when establishing the asymptotic formula (4.75) we have made no assumptions as yet about the value of $\varepsilon = 4 - d$. However, owing to the difficulties in evaluating the subsequent terms of the series, we have to use a small parameter in powers of which the function $F\{\tilde{G}\}$ may be expanded. In view of expression (4.68) it is natural to take $\mu = \varepsilon/3$ as such a parameter. In this connection we shall denote by C any positive number that may depend on the dimensionality d but does not tend to zero or to infinity as $d \to 4$.

We now turn to the second term in the series (4.69) which defines the contribution of diagram (c) in Fig. 7 to $F\{\tilde{G}\}$. The difference between diagrams (b) and (c) is that the latter contains an additional vertex and two lines that form a loop subdiagram. As a result the expressions

$$B_b(\rho) = v_1^2 \int \tilde{\mathcal{A}}(\boldsymbol{\rho} - \boldsymbol{\kappa}) M(\kappa) \frac{\mathrm{d}^d\kappa}{(2\pi)^d} \ , \tag{4.76}$$

and

$$B_c(\rho) = -2v_1^3 \int \tilde{\mathcal{A}}(\boldsymbol{\rho} - \boldsymbol{\kappa}) M^2(\kappa) \frac{\mathrm{d}^d\kappa}{(2\pi)^d} \ , \tag{4.77}$$

corresponding to diagrams (b) and (c) in Fig. 7 differ in that expression (4.77) contains the symmetry coefficient 2 and the additional integrand factor $-v_1 M(\kappa)$ in which

$$M(\kappa) = \int \tilde{\mathcal{A}}(\boldsymbol{\kappa} - \boldsymbol{\kappa}')\tilde{\mathcal{A}}(\kappa') \frac{\mathrm{d}^d\kappa'}{(2\pi)^d}$$
$$= Cu^{4\mu}(z_0^2 l^4 L^{\mu})^{-1}(\kappa L)^{\mu/2} K_{\mu/2}(\kappa L) \ . \tag{4.78}$$

By formula (4.15), for small values of κ ($\kappa \ll L^{-1}$), the function $M(\kappa)$ is almost independent of κ, and can be replaced by

$$M(0) = C\Gamma\left(\frac{\mu}{2}\right) \frac{u^{4\mu}}{z_0^2 l^4 L^{\mu}} \ , \tag{4.79}$$

whereas for large values of κ ($\kappa \gg L^{-1}$) it becomes negligible. The expression

$$-v_1 M(\kappa) = -v_1 M(0) \mathcal{K}_{\mu/2}(\kappa L) \tag{4.80}$$

in which the function $\mathcal{K}_{\mu/2}(x)$ as defined by Eqn (4.10) can also serve as an estimate for the contribution to $F\{\tilde{G}\}$ due to a transition from an arbitrarily chosen diagram to a diagram containing the subsequent vertex, since any such transition is always accompanied by the addition of one vertex and two lines. It should be noted here that for small values of μ and for $\kappa \ll L^{-1}$, the function $\mathcal{K}_{\mu/2}(\kappa L)$ [see Eqn (4.10)] behaves as follows:

$$\mathcal{K}_{\mu/2}(\kappa L) = -\frac{\mu}{2} \ln \kappa^2 L^2 + \mathrm{O}\left([\mu \ln(\kappa^2 L^2)]^2\right) \ . \tag{4.81}$$

To estimate the quantity

$$F_c(\rho) = \mathcal{A}_0(\rho)\left[B_c(0) - B_c(\rho)\right] \tag{4.82}$$

we represent it in the form

$$F_c(\rho) = 2v_1^3 M^2(0) \mathcal{A}_0(\rho)$$
$$\times \int [\tilde{\mathcal{A}}(\boldsymbol{\rho} - \boldsymbol{\kappa}) - \tilde{\mathcal{A}}(\kappa)][\mathcal{K}_{\mu/2}^2(\kappa L) - 1] \frac{\mathrm{d}^d\kappa}{(2\pi)^d} \ , \tag{4.83}$$

where we have used the identity

$$\int \tilde{\mathcal{A}}(\boldsymbol{\rho} - \boldsymbol{\kappa})\, \mathrm{d}^d\kappa = \int \tilde{\mathcal{A}}(\kappa)\, \mathrm{d}^d\kappa$$

combined with Eqns (4.77) and (4.78). By the estimate of the integral on the right-hand side of Eqn (4.83) given in Appendix III, we have for sufficiently small μ

$$F_c(\rho) = C\mu^{-2}(\tilde{v}_0 z_0^{-2})^3 (ul^{-1})^{9\mu}(uL^{-1})^{\mu} \times \left[\mu \ln(\rho L)^2 \sum_{k \geqslant 0} C_k(-\mu \ln \rho^2 L^2)^k + \mathrm{O}(1)\right], \quad (4.84)$$

where C_k, $k \geqslant 0$ are fully defined coefficients. Comparing expression (4.68) with (4.75) and taking into account the above remarks concerning Eqns (4.80) and (4.81), we come to the idea of defining u and L in such a way that

$$(\tilde{v}_0 z_0^{-2})^2 (ul^{-1})^{6\mu} = C\mu, \quad (uL^{-1})^{2\mu} = C\mu \ . \quad (4.85)$$

In this case the contributions of diagrams (b) and (c) to the asymptotic expression for $F\{\tilde{G}\}$ assume the form

$$F_b(\rho) \sim -C\mu \ln(u\rho)^2 \ , \quad (4.86)$$

$$F_c(\rho) \sim \mu \ln(u\rho)^2 \sum_{k \geqslant 0} C_k \left[-\mu \ln(u\rho)^2\right]^k \ , \quad (4.87)$$

where the coefficients of the last series are, for simplicity, denoted by the same symbols as those in the series (4.84). It is clear that the contributions to the asymptotic expression for $F\{\tilde{G}\}$ provided by diagram (d_1) and the subsequent diagrams that are similar to it and contain loop chains can also be represented as power series in $[-\mu \ln(u^2\rho^2)]$. But there are diagrams subsequent to (d_1) in which the loops are arranged differently, for example diagrams (d_2) and (d_3). To estimate the contributions of (d_2) and (d_3), the exact calculation of which is very difficult, we use the approximation $M(\kappa) \equiv M(0)$ in the expressions representing these contributions. To display this approximation graphically we replace a loop by a point and assign to the latter factor $-v_1 M(0)$. In this case diagrams (d_2) and (d_3) reduce to diagrams (c) and (b) whose contributions must be multiplied by $-v_1 M(0)$ and $[-v_1 M(0)]^2$ respectively. If we now add together the contributions of those diagrams that reduce to diagram (b) we obtain

$$(\tilde{v}_0 z_0^{-2})^2 (ul^{-1})^{6\mu} \left[\sum_{k \geqslant 0} (-1)^{k+1} C_k (v_1 M(0))^k\right] \ln(\rho L)^2 \ . \quad (4.88)$$

Because of relations (4.79) and (4.85) this expression is asymptotically proportional to $-\mu \ln(u^2\rho^2)$.

Next, on examining the structure of the terms of the series in the diagram in Fig. 7, one can readily notice that the above loop is the simplest of all possible loop subdiagrams. We call the set of these diagrams a *complete loop* and represent it in the graphic form as shown in Fig. 18, where the square represents, as usual, the complete four-point vertex. If, in addition, we define a *generalized loop* as shown in Fig. 19, then the series in the diagram in Fig. 7 can be transformed to the form shown in Fig. 20, resembling the initial series (Fig. 6) in which the wavy line is now replaced by a generalized loop. The coefficients $\alpha, \beta, \gamma, \ldots$ of the series in Fig. 20 are defined in such a way that after the replacement of every greneralized loop by the corresponding set of its subdiagrams we obtain the correct numbers of the diagrams represented in Fig. 7. Using the definition of the generalized loop

$$\mathcal{V} = \mathcal{V}(\zeta, \rho l, \tilde{v}_0; \mathcal{A})$$

Figure 18.

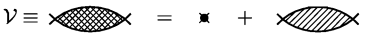

Figure 19.

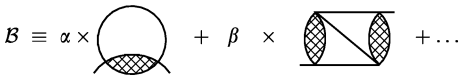

Figure 20.

we write the equality condition for the series in diagrams in Fig. 7 and Fig. 20 in the form

$$B(\zeta, \rho l, \tilde{v}_0; \mathcal{A}) = \mathcal{B}(\zeta, \rho l, \mathcal{V}; \mathcal{A}) \ .$$

By the last relation, the RG-property

$$\alpha^{-1} B(\zeta, \rho l, \tilde{v}_0; \mathcal{A}) = B(\zeta, \rho l, \alpha^{-2}\tilde{v}_0; \alpha\mathcal{A})$$

of the quantity B can be extended to $\mathcal{B}$, namely,

$$\alpha^{-1} \mathcal{B}(\zeta, \rho l, \mathcal{V}; \mathcal{A}) = \mathcal{B}(\zeta, \rho l, \alpha^{-2}\mathcal{V}; \alpha\mathcal{A})$$

and, moreover,

$$\alpha^{-2} \mathcal{V}(\zeta, \rho l, \tilde{v}_0; \mathcal{A}) = \mathcal{V}(\zeta, \rho l, \alpha^{-2}\tilde{v}_0; \alpha\mathcal{A}) \ .$$

If the function $\mathcal{V}$ displays weak dependence on ρ for $\rho \ll L^{-1}$ and if it is close to zero for $\rho \gg L^{-1}$, we can continue the reduction of the series in the diagram in Fig. 7 to that in Fig. 20, taking the latter as the starting point for the next step. In this case a generalized loop is approximately replaced by a point and the quantity $-\mathcal{V}_0/(2\pi)^d$, where $\mathcal{V}_0 \equiv \mathcal{V}|_{\rho=0}$ denotes the renormalized excluded volume, is assigned to this point. By repeated reduction we come to a diagrammatically represented series that differs from the previous one only by the factors $\alpha', \beta', \gamma', \ldots$ preceding its terms, where $\alpha' \leqslant \alpha, \beta' \leqslant \beta, \gamma' \leqslant \gamma, \ldots$. A decrease in the value of these factors after the reduction is caused by the fact that every generalized loop consists of an infinite number of loop subdiagrams taken from the corresponding terms of the series. Hence the reduction procedure leads to a gradual disappearance of certain terms from this series. The only exceptions here are the so-called irreducible diagrams whose vertices are connected by no more than one line. The simplest irreducible diagrams are represented in Fig. 21.

To estimate the quantity $\mathcal{V}$ we return to the Bethe–Salpeter equation in Fig. 10 and deduce from it the equality shown in Fig. 23 which has the analytic form

$$\int \mathcal{M}(\boldsymbol{\rho} - \boldsymbol{\kappa}, \boldsymbol{\kappa}) Y(\boldsymbol{\rho} - \boldsymbol{\kappa}, \boldsymbol{\kappa}; \boldsymbol{\rho} - \boldsymbol{\kappa}', \boldsymbol{\kappa}') \mathcal{M}(\boldsymbol{\rho} - \boldsymbol{\kappa}', \boldsymbol{\kappa}') \frac{\mathrm{d}^d\kappa\, \mathrm{d}^d\kappa'}{(2\pi)^{2d}}$$
$$= \int \mathcal{M}(\boldsymbol{\rho} - \boldsymbol{\kappa}, \boldsymbol{\kappa}) X(\boldsymbol{\rho} - \boldsymbol{\kappa}, \boldsymbol{\kappa}; \boldsymbol{\rho} - \boldsymbol{\kappa}', \boldsymbol{\kappa}') \times \mathcal{M}(\boldsymbol{\rho} - \boldsymbol{\kappa}', \boldsymbol{\kappa}') \frac{\mathrm{d}^d\kappa\, \mathrm{d}^d\kappa'}{(2\pi)^{2d}}$$
$$+ \int \mathcal{M}(\boldsymbol{\rho} - \boldsymbol{\kappa}, \boldsymbol{\kappa}) X(\boldsymbol{\rho} - \boldsymbol{\kappa}, \boldsymbol{\kappa}; \boldsymbol{\rho} - \boldsymbol{\kappa}', \boldsymbol{\kappa}') \mathcal{M}(\boldsymbol{\rho} - \boldsymbol{\kappa}', \boldsymbol{\kappa}') \times Y(\boldsymbol{\rho} - \boldsymbol{\kappa}', \boldsymbol{\kappa}'; \boldsymbol{\rho} - \boldsymbol{\kappa}'', \boldsymbol{\kappa}'') \mathcal{M}(\boldsymbol{\rho} - \boldsymbol{\kappa}'', \boldsymbol{\kappa}'') \frac{\mathrm{d}^d\kappa\, \mathrm{d}^d\kappa'\, \mathrm{d}^d\kappa''}{(2\pi)^{3d}} \ , \quad (4.89)$$

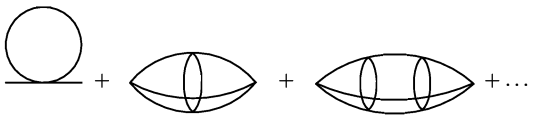

Figure 21.

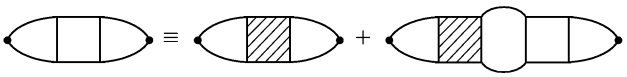

Figure 22.

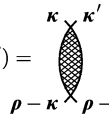

Figure 23.

where

$$\mathcal{M}(\xi;\boldsymbol{\rho}-\boldsymbol{\kappa},\boldsymbol{\kappa}) = \tilde{\mathcal{A}}(\xi,\boldsymbol{\rho}-\boldsymbol{\kappa})\tilde{\mathcal{A}}(\xi,\kappa) \tag{4.90}$$

and (for the sake of simplicity) the dependence of all functions on ξ (or ζ) is not indicated. We now take the quantity shown in Fig. 23 to be the initial approximation for the four-point vertex X that cannot be cut. This approximation can be viewed as a version of the so-called parquet diagram summation method. Suppose that the function $\mathcal{V}(\xi,\rho)$ depends weakly on ρ. Substituting the approximation $X \equiv \mathcal{V}_0 = \mathcal{V}(\xi,0)$ for X in Eqn (4.89) we then obtain the function

$$\mathcal{Y}(\xi,\rho) = \int \mathcal{M}(\xi;\boldsymbol{\rho}-\boldsymbol{\kappa},\boldsymbol{\kappa})Y(\xi;\boldsymbol{\rho}-\boldsymbol{\kappa},\boldsymbol{\kappa};\boldsymbol{\rho}-\boldsymbol{\kappa}',\boldsymbol{\kappa}') \times\mathcal{M}(\xi;\boldsymbol{\rho}-\boldsymbol{\kappa}',\boldsymbol{\kappa}')\frac{\mathrm{d}^d\kappa\,\mathrm{d}^d\kappa'}{(2\pi)^{2d}} \tag{4.91}$$

in the form

$$\mathcal{Y}(\xi,\rho) = \mathcal{V}_0 M^2(\xi,\rho)[1-\mathcal{V}_0 M(\xi,\rho)]^{-1} , \tag{4.92}$$

where

$$M(\xi,\rho) = \int \mathcal{M}(\xi;\boldsymbol{\rho}-\boldsymbol{\kappa},\boldsymbol{\kappa})\frac{\mathrm{d}^d\kappa}{(2\pi)^d} = \omega\int_0^\infty \Lambda_s(\rho R)\tilde{w}^2(\xi,R)R^{2s+1}\,\mathrm{d}R . \tag{4.93}$$

Using Eqn (4.63) together with the Nicolson and Sonine–Gegenbauer formulas [99] we can rewrite Eqn (4.93) as follows

$$M(\xi,\rho) = C\Gamma\left(\frac{\mu}{2}\right)u^{4\mu}(z_0^2 l^4 L^\mu)^{-1} \times\int_0^\infty \mathrm{d}t(1+t^2)^{(\mu-3)/2}\,\mathcal{K}_{\mu/2}\left[L\sqrt{\rho^2+4\xi^2(1+t^2)}\right] . \tag{4.94}$$

Since $\mathcal{K}_{\mu/2}(x)$ is a positive function decreasing when $x \geqslant 0$ and since $\mathcal{K}_{\mu/2}(0) = 1$, the integral in (4.94) does not exceed the value of the integral

$$\int_0^\infty (1+t^2)^{(\mu-3)/2}\,\mathrm{d}t = \sqrt{\pi}\Gamma\left(1-\frac{\mu}{2}\right)\left[2\Gamma\left(\frac{3}{2}-\frac{\mu}{2}\right)\right]^{-1} ,$$

which is close to unity for $\mu \ll 1$. Substituting expression (4.92) into the equation

$$\mathcal{V}(\xi,\rho) = -v_1 + v_1^2 M(\xi,\rho) + v_1^2\mathcal{Y}(\xi,\rho) , \tag{4.95}$$

which is a consequence of the diagram equations in Fig. 18 and Fig. 19, and introducing the new notation

$$\mathcal{V}(\xi,\rho) = -v_1\lambda(\xi,\rho) , \tag{4.96}$$

$$m(\xi,\rho) = v_1 M(\xi,\rho) , \tag{4.97}$$

we find, from formula (4.95) that

$$\lambda(\xi,\rho) = 1 - m(\xi,\rho)[1+\lambda_0(\xi)m(\xi,\rho)]^{-1} , \tag{4.98}$$

where $\lambda_0(\xi) = \lambda(\xi,0)$. Letting $\rho = 0$ in Eqn (4.98) and writing

$$m_0(\xi) = m(\xi,0) , \tag{4.99}$$

we obtain for $\lambda_0(\xi)$ the equation

$$\lambda_0 = 1 - m_0(1+m_0\lambda_0)^{-1} ,$$

whose nonnegative solution for $m_0 \leqslant 1$ is of the form

$$\lambda_0 = (2m_0)^{-1}\left[m_0 - 1 + (1+2m_0-3m_0^2)^{1/2}\right] . \tag{4.100}$$

By definitions (4.97) and (4.99) the inequality $m_0(0) \leqslant 1$ implies that

$$m(\xi,\rho) \leqslant m_0(\xi) \leqslant 1 .$$

Since the quantity

$$m_0(0) = v_0 M(0,0) = C$$

is determined by the constants appearing in Eqns (4.85), the condition $m_0(0) \leqslant 1$ clearly imposes some restriction on the choice of these constants. Thus, the assessment of $\mathcal{V}(\xi,\rho)$ in Eqns (4.96) and (4.98) is in good agreement with the above assumption about the behaviour of the function $\mathcal{V}(\xi,\rho)$.

We now continue the above reduction of the diagram series until we get a series consisting of irreducible diagrams whose vertices are occupied by generalized loops. These loops, obtained as a result of repeated reductions, will be denoted by the renormalized quantity

$$\tilde{\mathcal{V}}(\xi,\rho) = -Cv_1\tilde{\lambda}(\xi,\rho) ,$$

where the function $\tilde{\lambda}(\xi,\rho)$ is defined by

$$\tilde{\lambda}(\xi,\rho) = 1 - \tilde{m}(\xi,\rho)[1+\tilde{\lambda}_0(\xi)\tilde{m}(\xi,\rho)]^{-1} ,$$

in which

$$\tilde{m}(\xi,\rho) = Cm(\xi,\rho) ,$$

$$\tilde{\lambda}_0(\xi) = \tilde{\lambda}(\xi,0) .$$

Then the first (simplest) term of the series obtained has the form shown in Fig. 24, and the diagram next in order consists of five generalized loops and nine lines.

Figure 24.

The contribution to $F\{\tilde{G}\}$ of the diagram corresponding to the expression in Fig. 24 is equal to

$$\mathcal{A}_0(\rho)\big[\mathcal{B}_a(1,0,\mathcal{V};\tilde{\mathcal{A}})-\mathcal{B}_a(1,\rho l,\mathcal{V};\tilde{\mathcal{A}})\big]$$
$$= Cv_1\mathcal{A}_0(\rho)\int n(\kappa)\big[\tilde{\mathcal{A}}(\kappa)-\tilde{\mathcal{A}}(\boldsymbol{\rho}-\boldsymbol{\kappa})\big]\,\mathrm{d}^d\kappa\ , \quad (4.101)$$

where

$$n(\kappa)=\tilde{m}_0(0)\mathcal{K}_{\mu/2}(\kappa L)[1+\tilde{\lambda}_0(0)\tilde{m}_0(0)\mathcal{K}_{\mu/2}(\kappa L)]^{-1}\ .$$

As noted above, the condition $\tilde{\lambda}_0(0)\tilde{m}_0(0)<1$ can be satisfied if we choose the constants in Eqns (4.85) appropriately. This enables us to expand the integrand $n(\kappa)$ in Eqn (4.101) in powers of $\mathcal{K}_{\mu/2}(\kappa L)$, as a result of which we again arrive at the situation described earlier. Hence, letting $\mu\ll 1$ and integrating in Eqn (4.101) we finally obtain the representation of the required quantity as a series in powers of $-\mu\ln(\rho L)$.

Consider now the subsequent terms of the series in the diagram in Fig. 21 whose vertices are occupied by generalized loops. Let us write the contribution of each such term to $F\{\tilde{G}\}$ in the symbolic form

$$\mathcal{A}_0(\rho)\int\big[\tilde{\mathcal{V}}(0,\kappa)\big]^m\big\{\big[\tilde{\mathcal{A}}(0,\kappa)\big]^{2m-1}$$
$$-\big[\tilde{\mathcal{A}}(0,\boldsymbol{\rho}-\boldsymbol{\kappa})\big]^{2m-1}\big\}\big[\mathrm{d}^d\kappa(2\pi)^{-d}\big]^m\ , \quad (4.102)$$

where we assume that $m\geqslant 5$. By the definition of the functions $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{V}}$ the integration with respect to each variable $\{\boldsymbol{\kappa}\}$ in the integral

$$\int\big[\tilde{\mathcal{V}}(0,\kappa)\big]^m\big[\tilde{\mathcal{A}}(0,\kappa)\big]^{2m-1}\big[\mathrm{d}^d\kappa\big]^m$$

is in essence bounded above in absolute value by L^{-1}, and at the lower limit $\{|\boldsymbol{\kappa}|\}\geqslant\kappa_0$ the integral behaves as $\mathrm{O}(\kappa_0^{\tau})$, where $\kappa_0\to 0$ and

$$\tau=(2m-1)(2\mu-2)+md=2+(m-2)\mu\ .$$

It follows that the asymptotic form of the expression (4.102) as $\rho\to 0$ can be represented as a power series in $(\rho L)^{\mu}$ or $\mu\ln(\rho L)$.

In consequence, this approach allows us to conclude that the asymptotic form of the function $F\{\tilde{G}\}$ as $\rho\to 0$ is a series in powers of $-\mu\ln(\rho L)$. Using the remaining arbitrariness in the choice of constants in Eqns (4.85), we can ensure the coincidence of the terms in expressions (4.68) and (4.69) that are linear in $\ln(u^2\rho^2)$. Substituting the required expansion of the function $F\{\tilde{G}\}$ into the RG-equation (3.102), we readily obtain

$$G(\rho)\sim\tilde{G}(\rho),\qquad \rho\to 0\ . \quad (4.103)$$

It is precisely this result that justifies the choice of the function $\tilde{G}=\tilde{\mathcal{A}}/\mathcal{A}_0$ as the asymptotic solution of Eqn (3.78).

Since the values of the parameters μ, u, and L are now determined and consequently also the functions in Eqns (4.61) and (4.63), we can use formula (4.65) to find the asymptotic form of the function $\tilde{w}_N(\boldsymbol{R})$ for $N\to\infty$ and $R\gg l$.

Next, we take into account the connection between B and $\mathcal{B}$ and use the representation of the function $\mathcal{B}(\zeta,\rho l,\tilde{\mathcal{V}};\tilde{\mathcal{A}})$ as the series in Fig. 21 consisting of irreducible diagrams whose vertices are identified with generalized loops. $B(\zeta,\mathrm{i}\xi l,\tilde{v}_0;\tilde{\mathcal{A}})$ then has the asymptotic form:

$$B(\zeta,\mathrm{i}\xi l,\tilde{v}_0;\tilde{\mathcal{A}})=\int\tilde{\mathcal{V}}(\xi,\kappa)\tilde{\mathcal{A}}(\xi,\mathrm{i}\xi\boldsymbol{k}-\boldsymbol{\kappa})\frac{\mathrm{d}^d\kappa}{(2\pi)^d}$$
$$+\sum_{m\geqslant 5}\int\big[\tilde{\mathcal{V}}(\xi,\kappa)\big]^m\big[\tilde{\mathcal{A}}(\xi,\mathrm{i}\xi\boldsymbol{k}-\boldsymbol{\kappa})\big]^{2m-1}\left[\frac{\mathrm{d}^d\kappa}{(2\pi)^d}\right]^m,$$
$$|\boldsymbol{k}|=1\ . \quad (4.104)$$

It was noted above that in all the integrals in (4.104) the function $\tilde{\mathcal{V}}(\xi,\kappa)$ varies weakly over the most essential part $\kappa\lesssim L^{-1}$ of the region of integration. Hence we may use the approximation $\tilde{\mathcal{V}}=-Cv_1$ in evaluation of these integrals. In this case the first summand on the right-hand side of Eqn (4.104) takes the form

$$\mathcal{B}_a(\zeta,\mathrm{i}\xi l,\tilde{\mathcal{V}};\tilde{\mathcal{A}})=\int\tilde{\mathcal{V}}(\xi,\kappa)\tilde{\mathcal{A}}(\xi,\mathrm{i}\xi\boldsymbol{k}-\boldsymbol{\kappa})\frac{\mathrm{d}^d\kappa}{(2\pi)^d}$$
$$\cong -C_1\tilde{w}(\xi,0)\ ,$$

where

$$\tilde{w}(\xi,0)=\left(\frac{Cu^{2\mu}}{z_0 l^{d+2\mu}}\right)\left\{\left(\frac{l}{L}\right)^{2-\mu}\right.$$
$$\left.-\Gamma\left(\frac{\mu}{2}\right)\left[\Gamma\left(2-\frac{\mu}{2}\right)\right]^{-1}\left(\frac{\xi l}{2}\right)^{2-\mu}+\mathrm{O}\left((\xi l)^2\left(\frac{L}{l}\right)^{\mu}\right)\right\}$$

in accordance with expressions (4.63) and (4.10). It follows that

$$\mathcal{B}_a(\zeta,\mathrm{i}\xi l,\tilde{\mathcal{V}};\tilde{\mathcal{A}})-\mathcal{B}_a(1,0,\tilde{\mathcal{V}};\tilde{\mathcal{A}})$$
$$=C\,\tilde{v}_0(\mu z_0)^{-1}(ul^{-1})^{2\mu}(\xi l)^{2-\mu}+\mathrm{O}\big((\xi l)^2\big)\ . \quad (4.105)$$

Estimating each term of the series

$$\sum_{m\geqslant 5}\{-Cv_1\}^m\int\big\{\big[\tilde{\mathcal{A}}(\xi,\mathrm{i}\xi\boldsymbol{k}-\boldsymbol{\kappa})\big]^{2m-1}$$
$$-\big[\tilde{\mathcal{A}}(0,\kappa)\big]^{2m-1}\big\}\big(\mathrm{d}^d\kappa\big)^m$$

we get values of order $\mathrm{O}((\xi l)^{\tau})$, where $\tau=2+(m-2)\mu$, as shown above. Finally, with the help of Eqns (4.85) we obtain

$$B(\zeta,\mathrm{i}\xi l,\tilde{v}_0;\tilde{\mathcal{A}})-B(1,0,\tilde{v}_0;\tilde{\mathcal{A}})$$
$$=C(z_0\tilde{v}_0\mu^{-2})^{1/3}(\xi l)^{2-\mu}+\mathrm{O}\big((\xi l)^2\big)\ . \quad (4.106)$$

The difference on the left-hand side of Eqn (4.106) appears in the definition (4.66) of the function D. Using the notation

$$\tilde{v}_1=z_0\tilde{v}_0(1-s)^{-2} \quad (4.107)$$

and the formulas

$$2-\mu=\frac{d+2}{3}=\nu_{\mathrm{F}}^{-1}\ , \quad (4.108)$$

(4.66) and (4.106), we find that

$$D(\xi,RN^{-1})=C\tilde{v}_1^{1/3}(\xi l)^{1/\nu_{\mathrm{F}}}-R\xi N^{-1}+\mathrm{O}\big((\xi l)^2\big)\ . \quad (4.109)$$

Since the integrand in formula (4.65) has been determined, we can use the method of steepest descent to estimate the integral

in (4.65) as $N \to \infty$ and $R \gg l$. As a result we obtain the required asymptotic form of the probability density $W_N(\boldsymbol{R})$ [103]:

$$W_N(\boldsymbol{R}) \sim \begin{cases} R_{\mathrm{F}}^{-1}\delta(t-1), & d=1 \ , \\ CR_{\mathrm{F}}^{-d}t^p \exp(-t^q), & 1<d<4 \ , \end{cases} \tag{4.110}$$

where

$$t = RR_{\mathrm{F}}^{-1}, \quad R_{\mathrm{F}} l^{-1} = C(\tilde{v}_1^{1/3}N)^{\nu_{\mathrm{F}}} \ ,$$
$$p = \frac{(4-d)(d+2)}{6(d-1)} \ , \quad q = \frac{d+2}{d-1} \ . \tag{4.111}$$

For the mean-square end-to-end distance of the random walk we then have

$$\langle R^2 \rangle_N \sim \Gamma\left(\frac{d+p+2}{q}\right)\Gamma^{-1}\left(\frac{d+p}{q}\right)R_{\mathrm{F}}^2 \ , \tag{4.112}$$

which is, as was noted above, a generalization of the well-known Flory formula for d-dimensional space ($d<4$). When $\varepsilon = 4-d$ is small enough, we find from (4.108) that the critical exponent ν_{F} is given by

$$\nu_{\mathrm{F}} = \frac{1}{2} + \frac{\varepsilon}{12} + \mathrm{O}(\varepsilon^2)$$

which is not identical with the corresponding result obtained by De Gennes. The above method of solving the SAW problem is thus fundamentally different from the De Gennes approach and leads to a new ε-expansion, which now is found to be convergent.

From expression (4.110) it follows immediately that the asymptotic form of $W_N(\boldsymbol{R})$ has a pit for $R < R_{\mathrm{F}}$ owing to the excluded volume effect and that it is convex downward ($p = \frac{4}{3}$) and convex upward ($p = \frac{5}{12}$) for $d=2$ and $d=3$, respectively. When the density $W_N(\boldsymbol{R})$ is evaluated for $R < R_{\mathrm{F}}$, it is usual to replace p by a different critical exponent γ in calculations of the probability that the particle will return to the initial point. The relationship between γ and p is given by the Cloizeaux formula [34] $\gamma = 1 + \nu p$. If we replace ν in this formula by the expression for ν_{F} given by Eqn (4.108) and take into account the value of p from (4.111), we obtain $\gamma = 2$ for $d=2$ and $\gamma = \frac{5}{4}$ for $d=3$. We note, for comparison, that the values of γ obtained in some investigations (see, for example Refs [35–48]) by both analytic and computational methods are found to be somewhat lower than the above values. However, we must remember that expression (4.110) is an asymptotic formula obtained for $R \gg l$.

In terms of the notation commonly employed in the theory of critical phenomena we may write

$$\mathcal{A}(1,\rho) \sim \mathrm{O}((\rho l)^{\eta-2}) \quad \text{when } \rho \to 0$$

and

$$\mathcal{A}(\zeta,0) \sim \mathrm{O}((\zeta-1)^{-\gamma}) \quad \text{when } \zeta \to 1.$$

Taking expression (4.61) as the asymptotic solution of the basic equation when $\zeta \to 1$ and $\rho \to 0$, we find that $\eta = 2\varepsilon/3$ and the critical exponents $\gamma, \eta, \nu_{\mathrm{F}}$ are related by the usual formula $\gamma = (2-\eta)\nu_{\mathrm{F}}$. As to the critical exponent q, this is given by the well-known formula (4.111).

5. Basic equation with account taken of short-range correlations

In conclusion we shall consider the random walk for a more general case where the direction of each step of the particle depends on the direction of its previous step. Moreover, the length of an individual step is not fixed and has an arbitrary distribution.

Let the probability density of the kth step $\boldsymbol{r}_k$ be proportional to

$$\tau(\boldsymbol{r}_k)\sigma(\boldsymbol{n}_k, \boldsymbol{n}_{k-1}) \prod_{j=1}^{k-1}(1+f_{jk}), \qquad \boldsymbol{n}_k = \boldsymbol{r}_k r_k^{-1} \ ,$$

where $\tau(\boldsymbol{r})$ is the probability density of a single step $\boldsymbol{r}$ and $\sigma(\boldsymbol{n}_k, \boldsymbol{n}_{k-1}) = \sigma(\cos\theta_k)$ is the probability density of the angle θ_k between the directions of the kth and $(k-1)$th steps.

The normalization conditions for $\tau(\boldsymbol{r})$ and $\sigma(\boldsymbol{n}_k, \boldsymbol{n}_{k-1})$ will then be

$$\int \tau(\boldsymbol{r})\,\mathrm{d}^d r = 1 \ ,$$

$$\Gamma(s+1)\left[\sqrt{\pi}\Gamma\left(s+\frac{1}{2}\right)\right]^{-1}\int_0^{\pi}\sigma(\cos\theta)\sin^{2s}\theta\,\mathrm{d}\theta = 1 \ .$$

If we represent the required density $W_N(\boldsymbol{R})$ by (3.4), we obtain for the function $w_N(\boldsymbol{R})$ the equation

$$w_N(\boldsymbol{R}) = \int \delta\left(\boldsymbol{R} - \sum_{k=1}^{N}\boldsymbol{r}_k\right)P_{1N}\,\mathrm{d}\mu_{1N}(0) \ ,$$

where for brevity we use the notation

$$\mathrm{d}\mu_{lm}(\boldsymbol{\rho}) = \prod_{k=l+1}^{m}\sigma(\boldsymbol{n}_k, \boldsymbol{n}_{k-1})\prod_{k=l}^{m}\exp(\mathrm{i}\boldsymbol{\rho}\cdot\boldsymbol{r}_k)\tau(\boldsymbol{r}_k)\,\mathrm{d}^d r_k \ ,$$

and P_{1N} is given by (3.6). Writing the function $w_N(\boldsymbol{R})$ as the Fourier integral (3.8), we have for its Fourier transform $a_N(\boldsymbol{\rho})$ the representation

$$a_N(\boldsymbol{\rho}) = \int P_{1N}\,\mathrm{d}\mu_{1N}(\boldsymbol{\rho}) \ , \tag{5.1}$$

in which we first replace the product P_{1N} by its expansion (3.15), then expand the density $\sigma(\boldsymbol{n}_k, \boldsymbol{n}_{k-1}) = \sigma(\cos\theta_k)$ as a series in the Gegenbauer polynomials, and finally apply the addition theorem [99]

$$C_l^s(\boldsymbol{n}_k, \boldsymbol{n}_{k-1}) = \omega h^{-1}C_l^s(1)\sum_{m=1}^{h}S_l^m(\boldsymbol{n}_k)S_l^m(\boldsymbol{n}_{k-1}) \ ,$$

where the functions $S_l^m(n), m = 1, 2, \ldots, h$ constitute an orthogonal system of $h = 2(s+l)(2s+l-1)!/(2s)!\,l!$ real-valued spherical harmonics of degree l defined on the unit hypersphere. The expansion of the density $\sigma(\boldsymbol{n}_k, \boldsymbol{n}_{k-1})$ has the form

$$\sigma(\boldsymbol{n}_k, \boldsymbol{n}_{k-1}) = \sum_{l \geqslant 0}\sum_{m=1}^{h}\sigma_l S_l^m(\boldsymbol{n}_k)S_l^m(\boldsymbol{n}_{k-1}) \ , \tag{5.2}$$

where

$$\sigma_l = \frac{(4\pi)^s\Gamma(s)l!}{\Gamma(2s+l)}\int_{-1}^{1}\sigma(t)C_l^s(t)(1-t^2)^{s-(1/2)}\,\mathrm{d}t \ .$$

Next, define the operators $T(\boldsymbol{\rho})$ and $B_q(\boldsymbol{\rho})$ whose matrix elements are

$$\langle lm|T(\boldsymbol{\rho})|l'm'\rangle = (\sigma_l\sigma_{l'})^{1/2}\int \exp(\mathrm{i}\boldsymbol{\rho}\cdot\boldsymbol{r})\tau(\boldsymbol{r})S_l^m(\boldsymbol{n})S_{l'}^{m'}(\boldsymbol{n})\,\mathrm{d}^d r$$

and

$$\langle lm|B_q(\boldsymbol{\rho})|l'm'\rangle = (\sigma_l\sigma_{l'})^{1/2}\int b_{1q}S_l^m(\boldsymbol{n}_1)S_{l'}^{m'}(\boldsymbol{n}_q)\,\mathrm{d}\mu_{1q}(\boldsymbol{\rho})$$

respectively, and take into account that $\sigma_0 = \omega$ and $S_0^1(\boldsymbol{n}) = 1/\omega^{1/2}$. Then the initial terms of the series obtained by substituting Eqns (3.15) and (5.2) into formula (5.1) can be written as follows. The first term is

$$\int \mathrm{d}\mu_{1N}(\boldsymbol{\rho}) = \langle 01|T(\boldsymbol{\rho})|l_1m_1\rangle\langle l_1m_1|T(\boldsymbol{\rho})|l_2m_2\rangle\ldots$$
$$\times\langle l_Nm_N|T(\boldsymbol{\rho})|01\rangle = \langle 01|T^N(\boldsymbol{\rho})|01\rangle\ ,$$

where we assume the summation to be carried out with respect to repeated indices, the values of l and m running from 0 to ∞ and from 1 to h respectively. Before writing the operator form of the next term, we consider the chain of equations

$$\int b_{j+1k}\,\mathrm{d}\mu_{1N}(\boldsymbol{\rho}) = \langle 01|T^j(\boldsymbol{\rho})|l_jm_j\rangle$$
$$\times\langle l_jm_j|B_{k-j}(\boldsymbol{\rho})|l_{k+1}m_{k+1}\rangle\langle l_{k+1}m_{k+1}|T^{N-k}(\boldsymbol{\rho})|01\rangle$$
$$= \langle 01|T^{N+j-k}(\boldsymbol{\rho})B_{k-j}(\boldsymbol{\rho})|01\rangle\ ,$$

in which we have used the fact that the operators $T(\boldsymbol{\rho})$ and $B_q(\boldsymbol{\rho})$ commute. Hence, for the second term of our series we have

$$\sum_{1\leqslant j<k\leqslant N}\int b_{jk}\,\mathrm{d}\mu_{1N}(\boldsymbol{\rho})$$
$$= \sum_{q\geqslant 2}(N-q+1)\langle 01|T^{N-q}(\boldsymbol{\rho})B_q(\boldsymbol{\rho})|01\rangle\ .$$

In a similar way we can obtain the operator form of any term of the series that represents the function $a_N(\boldsymbol{\rho})$. We then arrive at the formula

$$a_N(\boldsymbol{\rho}) = \langle 01|A_N(\boldsymbol{\rho})|01\rangle\ , \tag{5.3}$$

in which the operator $A_N(\boldsymbol{\rho})$ is given by the series

$$A_N(\boldsymbol{\rho}) = \sum_{p\geqslant 0}\sum_{q\geqslant 0}\frac{(N-q+p)!}{(N-q)!\,p!}T^{N-q}(\boldsymbol{\rho})$$
$$\times\sum_{(q_1+\ldots+q_p=q)}\ldots\sum B_{q_1}(\boldsymbol{\rho})\ldots B_{q_p}(\boldsymbol{\rho})\ , \tag{5.4}$$

where the factor $(N-q+p)!/(N-q)!\,p!$ is the number of ways of choosing p places for the operators $B_{q_1}(\boldsymbol{\rho})$, $B_{q_2}(\boldsymbol{\rho}),\ldots,B_{q_p}(\boldsymbol{\rho})$ (with $q_1+q_2+\ldots+q_p=q$) out of $N-q+p$ free places so that the operators preserve their order. The last sum in Eqn (5.4) can be transformed, as in Eqn (3.22), into the form

$$\sum_{(q_1+\ldots+q_p=q)}\ldots\sum B_{q_1}(\boldsymbol{\rho})\ldots B_{q_p}(\boldsymbol{\rho}) = \frac{1}{2\pi\mathrm{i}}\oint_\Gamma B^p(z,\boldsymbol{\rho})\frac{\mathrm{d}z}{z^{1+q}}\ ,$$

where $B(z,\ \boldsymbol{\rho})$ is the generating operator defined by

$$B(z,\boldsymbol{\rho}) = \sum_{q\geqslant 0}z^qB_q(\boldsymbol{\rho})\ . \tag{5.5}$$

Eqn (5.4) will then take the form

$$A_N(\boldsymbol{\rho}) = \frac{1}{2\pi\mathrm{i}}\oint_\Gamma\frac{\mathrm{d}z}{z^{1+q}}\left[\sum_{p\geqslant 0}\sum_{q\geqslant 0}\frac{(N-q+p)!}{(N-q)!\,p!}\right.$$
$$\left.\times\left[zT(\boldsymbol{\rho})\right]^{N-q}B^p(z,\boldsymbol{\rho})\right]\ , \tag{5.6}$$

where the summation can be extended to all p from 0 to ∞ and all q from $-\infty$ to N with no influence on the result. If we now define the generating operator $A(z,\ \boldsymbol{\rho})$ by

$$A(z,\boldsymbol{\rho}) = \sum_{N\geqslant 0}z^NA_N(\boldsymbol{\rho})\ , \tag{5.7}$$

we deduce from (5.6) the important formula

$$A(z,\boldsymbol{\rho}) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(p+q)!}{p!\,q!}\left[zT(\boldsymbol{\rho})\right]^qB^p(z,\boldsymbol{\rho})\ . \tag{5.8}$$

Using the identity

$$(p+q)! = \sum_{n=0}^{\infty}\frac{n!}{2\pi\mathrm{i}}\oint_\Gamma t^{p+q-n-1}\,\mathrm{d}t$$

and commutativity of the operators $T(\boldsymbol{\rho})$ and $B(z,\ \boldsymbol{\rho})$, we express Eqn (5.8) in the form

$$A(z,\boldsymbol{\rho}) = \sum_{n=0}^{\infty}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\exp\Big(t\big[zT(\boldsymbol{\rho})+B(z,\boldsymbol{\rho})\big]\Big)\Big|_{t=0}\ .$$

It follows that the operator

$$A^{-1}(z,\boldsymbol{\rho}) = 1 - zT(\boldsymbol{\rho}) - B(z,\boldsymbol{\rho}) \tag{5.9}$$

is the inverse of $A(z,\ \boldsymbol{\rho})$. If we define the nonperturbed operator $A_0(z,\ \boldsymbol{\rho})$ by

$$A_0(z,\boldsymbol{\rho}) = \left[1-zT(\boldsymbol{\rho})\right]^{-1}\ ,$$

then Eqn (5.9) will assume the form of the basic equation

$$A^{-1}(z,\boldsymbol{\rho}) = A_0^{-1}(z,\boldsymbol{\rho}) - B(z,\boldsymbol{\rho})\ . \tag{5.10}$$

We shall establish a relationship between the operators $B(z,\ \boldsymbol{\rho})$ and $A(z,\ \boldsymbol{\rho})$ in order to close Eqn (5.10) with respect to the desired operator $A(z,\ \boldsymbol{\rho})$. For this purpose we use Eqns (3.13), (3.30), and the definition of the matrix elements of the operator $B_q(\boldsymbol{\rho})$. Taking into account the definition

$$\langle lm|A_q(\boldsymbol{\rho})|l'm'\rangle = (\sigma_l\,\sigma_{l'})^{1/2}\int P_{1q}S_l^m(\boldsymbol{n}_1)S_{l'}^{m'}(\boldsymbol{n}_q)\,\mathrm{d}\mu_{1q}(\boldsymbol{\rho})$$

we obtain the relation

$$\langle lm|B_q(\boldsymbol{\rho})|l'm'\rangle = -\int v(\boldsymbol{\kappa})\langle lm|A_q(\boldsymbol{\rho}-\boldsymbol{\kappa})|l'm'\rangle\frac{\mathrm{d}^d\kappa}{(2\pi)^d}$$
$$+\sum_{(q_1+q_2+q_3=q)}\ldots\sum\int v(\boldsymbol{\kappa})v(\boldsymbol{\kappa}')$$
$$\times\langle lm|A_{q_1}(\boldsymbol{\rho}-\boldsymbol{\kappa})A_{q_2}(\boldsymbol{\rho}-\boldsymbol{\kappa}-\boldsymbol{\kappa}')A_{q_3}(\boldsymbol{\rho}-\boldsymbol{\kappa}')|l'm'\rangle$$
$$\times\frac{\mathrm{d}^d\kappa\,\mathrm{d}^d\kappa'}{(2\pi)^{2d}}+\ldots\ , \tag{5.11}$$

which is analogous to Eqn (3.33). Substituting expression (5.11) into Eqn (5.5) and using the definition (5.7) of the operator $A(z,\ \boldsymbol{\rho})$, we find the desired connection:

$$B(z,\boldsymbol{\rho}) = -\int v(\boldsymbol{\kappa})A(z,\boldsymbol{\rho}-\boldsymbol{\kappa})\frac{\mathrm{d}^d\kappa}{(2\pi)^d}$$
$$+\int v(\boldsymbol{\kappa})v(\boldsymbol{\kappa}')A(z,\boldsymbol{\rho}-\boldsymbol{\kappa})A(z,\boldsymbol{\rho}-\boldsymbol{\kappa}-\boldsymbol{\kappa}')$$
$$\times A(z,\boldsymbol{\rho}-\boldsymbol{\kappa}')\frac{\mathrm{d}^d\kappa\,\mathrm{d}^d\kappa'}{(2\pi)^{2d}}+\ldots \tag{5.12}$$

The last formula is similar in appearance to formula (3.34). Hence the series (5.12) can also be represented by the

diagrams in Fig. 6, but the continuous lines are now associated with the operators $A(z, \boldsymbol{\rho}-\boldsymbol{\kappa})$.

Thus, the inclusion of the correlation between the directions of neighbouring steps of the walking particle leads us to the operator form of the basic equation in the SAW problem. Choosing the probability densities $\tau(\boldsymbol{r})$ and $\sigma(\cos\theta)$ appropriately, we can in principle produce any lattice model for this problem when $d = 2$.

In conclusion, we should note an important property of Eqn (5.10), namely its invariance under the group of multiplicative transformations

$$A \to A' = \alpha A\ , \quad A_0 \to A_0' = \alpha A_0\ ,$$
$$v \to v' = \alpha^{-2} v\ ,$$

which serves as the basis for the RG-method of solving this equation.

6. Conclusions

Among the methods of solving the SAW problem that have been considered here, the one presented in Sections 3–5 is based on the exact equation for the required probability density $W_N(\boldsymbol{R})$. This equation describing a non-Markovian process is an integral equation invariant under a continuous group of renormalizing transformations and hence the RG-method can be applied to it. The asymptotic expression for the density $W_N(\boldsymbol{R})$ when $N \to \infty$ and $R \gg l$ found by this method [see formula (4.110)] is similar to that established previously on the basis of scaling considerations. Thus, the problem is reduced to the determination of the critical exponents only. The calculations enable us to argue that it has now become clear how to obtain the generalized Flory formula for $\langle R^2\rangle_N$ from the microscopic theory. An essential fact here is that the expression for the exponent ν_{F} is continuous in the spatial dimensionality d when $1 \leqslant d < 4$, and moreover, this expression gives exact values when $d = 1, d = 2$, and provides at least a very good approximation when $d = 3$. As to the critical exponent p (or γ) which determines the behaviour of $W_N(\boldsymbol{R})$ for $R \ll R_{\mathrm{F}}$, its values, depending on d, were found under the additional assumption $R \gg l$. It is quite possible that the power dependence of $W_N(\boldsymbol{R})$ on R for $R \gtrsim l$ is completely different from that for $l \ll R \ll R_{\mathrm{F}}$, i.e. the value of p given by expression (4.111) can differ from that obtained by numerical methods.

Appendices

I. The values of some integrals in Section 4.2

1. $$S_1(\rho) = \int_0^\infty \left[1 - \Lambda_1(\rho R)\right] \tilde{w}_1^3(0, R) R^3\, \mathrm{d}R$$
$$= \frac{8}{\pi^6 l^6 z_0^3} \int_0^\infty \left[1 - \frac{2}{\rho R} J_1(\rho R)\right] \frac{R^3}{(R^2 + L^2)^3}\, \mathrm{d}R$$
$$= 2\left(\pi^6 l^6 L^2 z_0^3\right)^{-1} \left[1 - \rho L K_1(\rho L)\right]\ .$$

2. $$S_2(\rho) = \int_0^\infty \mathrm{d}R\, R^3 \left[1 - \Lambda_1(\rho R)\right] \tilde{w}_1(0, R) \times \int_0^\infty \Lambda_1(R\kappa) M_1^2(0, \kappa) \kappa^3\, \mathrm{d}\kappa$$
$$= \int_0^\infty \sigma(\rho, \kappa) M_1^2(0, \kappa) \kappa^3\, \mathrm{d}\kappa\ ,$$

where

$$M_1(0, \kappa) = 2\pi^2 \int_0^\infty \Lambda_1(\kappa R) \tilde{w}_1^2(0, R) R^3\, \mathrm{d}R$$
$$= 8\left(\pi^2 l^4 z_0^2\right)^{-1} K_0(\kappa L)\ ,$$

$$\sigma(\rho, \kappa) = \int_0^\infty \left[1 - \Lambda_1(\rho R)\right] \Lambda_1(\kappa R) \tilde{w}_1(0, R) R^3\, \mathrm{d}R$$
$$= \frac{4}{\pi^2 l^2 z_0 \kappa} \left[\int_0^\infty J_1(\kappa R) \frac{R^2}{R^2 + L^2}\, \mathrm{d}R - \frac{2}{\rho} \int_0^\infty J_1(\rho R) J_1(\kappa R) \frac{R}{R^2 + L^2}\, \mathrm{d}R \right]$$
$$= \frac{4}{\pi^2 l^2 z_0 \kappa^2} \left[\kappa L K_1(\kappa L) - \frac{2\kappa}{\rho} \times \begin{cases} I_1(\kappa L) K_1(\rho L), & \kappa < \rho \\ I_1(\rho L) K_1(\kappa L), & \rho < \kappa \end{cases} \right]\ .$$

Then

$$S_2(\rho) = 256\left(\pi^6 l^{10} L^2 z_0^5\right)^{-1} \left\{ \left[1 - \frac{2}{\rho L} I_1(\rho L)\right] \times \int_0^\infty K_0^2(t) K_1(t) t^2\, \mathrm{d}t + \frac{2}{\rho L} \int_0^{\rho L} K_0^2(t) \left[I_1(\rho L) K_1(t) - K_1(\rho L) I_1(t)\right] t^2\, \mathrm{d}t \right\}\ .$$

Substituting the expansions

$$I_1(t) = \frac{t}{2} + \frac{t^3}{16} + \mathrm{O}(t^5)\ ,$$
$$K_0(t) = -\ln t + \mathrm{O}(1)\ ,$$
$$K_1(t) = t^{-1} + \frac{1}{2} t \ln t + \mathrm{O}(t)\ ,$$

and letting $t \to 0$ we obtain

$$S_2(\rho) = 16(\pi^6 l^{10} z_0^5)^{-1} \rho^2 \{[\ln(L_1^2 \rho^2)]^2 - 3 \ln(L_1^2 \rho^2) + \mathrm{O}(1)\}\ ,$$

II. The values of some integrals in Section 4.3

1. $$S_1(\rho) = \int_0^\infty \Lambda_s(\rho R) \tilde{w}_1^3(R) R^{2s+1}\, \mathrm{d}R$$
$$= \Gamma(s+1) \left[\frac{\Gamma(s) d}{(2\pi)^{s+1} l^2 z_0}\right]^3 \left(\frac{16}{\rho}\right)^s \times \int_0^\infty J_s(\rho R) \frac{R^{s+1}}{(R^2 + L^2)^{3s}}\, \mathrm{d}R$$
$$= \frac{2^{s+1} d^3 \Gamma^3(s) \Gamma(s+1)}{(2\pi)^{3(s+1)} \Gamma(3s) l^6 z_0^3} \left(\frac{\rho}{L}\right)^{2s-1} K_{2s-1}(\rho L)\ .$$

It follows that

$$S_1(0) = \lim_{\rho \to 0} S_1(\rho) = \frac{2^{3s-1} d^3 \Gamma^3(s) \Gamma(s+1) \Gamma(2s-1)}{(2\pi)^{3(s+1)} \Gamma(3s) l^6 L^{2(2s-1)} z_0^3}\ .$$

We now form the difference $S_1(0) - S_1(\rho)$ and proceed to the limit as $L \to 0$. The result is

$$\lim_{L \to 0} \left[S_1(0) - S_1(\rho)\right] = \frac{2^{1-s} d^3 \Gamma^3(s) \Gamma(s+1) \Gamma(2s-1) \Gamma(2-2s)}{(2\pi)^{3(s+1)} \Gamma(2s) \Gamma(3s) l^6 z_0^3 \rho^{2-4s}}\ .$$

The last expression is taken as the value of the first integral on the right-hand side of Eqn (4.35), so that for small ε we have

$$\int_0^\infty \left[1-\Lambda_s(\rho R)\right]\tilde{w}_1^3(R)R^{2s+1}\,\mathrm{d}R \sim \rho^{2-2\varepsilon}\left(2\pi^6 l^6 z_0^3\varepsilon\right)^{-1} .$$

2. $$S_2(\rho) = \int_0^\infty \mathrm{d}R\, R^{2s+1}\Lambda_s(\rho R)\tilde{w}_1(R) \times \int_0^\infty \Lambda_s(R\kappa)M_1^2(\kappa)\kappa^{2s+1}\,\mathrm{d}\kappa$$
$$= 2^s\Gamma(s)\Gamma(s+2)C_1^2L^{2(1-s)}\left(\pi^{s+1}l^{10}z_0^5\right)^{-1} \times \int_0^\infty \mathrm{d}R\,\Lambda_s(\rho R)\frac{R^{s+1}}{(R^2+L^2)^s} \times \int_0^\infty J_s(\kappa R)K_{1-s}^2(\kappa L)\kappa^{3s-1}\,\mathrm{d}\kappa .$$

Using the formula

$$K_\lambda^2(x) = 2\int_0^\infty K_{2\lambda}(2x\cosh t)\,\mathrm{d}t$$

we can represent the last integral in the form

$$\int_0^\infty J_s(\kappa R)K_{1-s}^2(\kappa L)\kappa^{3s-1}\,\mathrm{d}\kappa = 2\int_0^\infty \mathrm{d}t\int_0^\infty \mathrm{d}\kappa\,\kappa^{3s-1}J_s(\kappa R)K_{2(1-s)}(2\kappa L\cosh t)$$
$$= 2^{5s-3}\Gamma(3s-1)L^{2(s-1)}R^s \times \int_0^\infty (\cosh t)^{2(s-1)}\left(R^2+4L^2\cosh^2 t\right)^{1-3s}\mathrm{d}t .$$

Further, letting

$$S_2(0) = \lim_{\rho\to 0} S_2(\rho) ,$$

we take the difference $S_2(0)-S_2(\rho)$ and proceed to the limit as $L\to 0$. By means of the equation

$$\int_0^\infty (\cosh t)^{-\varepsilon}\,\mathrm{d}t = \sqrt{\pi}\Gamma\left(\frac{\varepsilon}{2}\right)\left[2\Gamma\left(\frac{1}{2}+\frac{\varepsilon}{2}\right)\right]^{-1} ,$$

we then obtain

$$\lim_{L\to 0}\left[S_2(0)-S_2(\rho)\right] = \frac{2^{6s-4}\Gamma\left(\frac{1}{2}\varepsilon\right)\Gamma(s)\Gamma(s+2)\Gamma(3s-1)C_1^2}{\pi^{s+(\frac{1}{2})}\Gamma\left(\frac{1}{2}+\frac{1}{2}\varepsilon\right)l^{10}z_0^5} \times \int_0^\infty \left[1-\Lambda_s(\rho R)\right]R^{3-6s}\,\mathrm{d}R ,$$

where

$$\int_0^\infty [1-\Lambda_s(\rho R)]R^{3-6s}\,\mathrm{d}R = \frac{2^{3-6s}\Gamma(s+1)\Gamma(3-3s)}{(3s-2)\Gamma(4s-1)}\rho^{2-3\varepsilon}$$

Finally, for small ε we have

$$\int_0^\infty \mathrm{d}R\, R^{2s+1}[1-\Lambda_s(\rho R)]\tilde{w}_1(R)\int_0^\infty \Lambda_s(R\kappa)M_1^2(\kappa)\kappa^{2s+1}\,\mathrm{d}\kappa \sim 128\rho^{2-3\varepsilon}\left(3\pi^6 l^{10}z_0^5\varepsilon^2\right)^{-1} .$$

3. $$S_3(\xi) = \int_0^\infty \tilde{w}_1^3(\xi,R)R^{2s+1}\,\mathrm{d}R = \left[\frac{2^{1-s}(s+1)}{\pi^{s+1}l^2 z_0}\right]^3\int_0^\infty \bar{K}_s^3\left(\xi\sqrt{R^2+L^2}\right)\frac{R^{2s+1}}{(R^2+L^2)^{3s}}\,\mathrm{d}R ,$$

where

$$\bar{K}_s(x) = x^s K_s(x) .$$

Let us now form the difference $S_3(\xi)-S_3(0)$ and proceed to the limit

$$\lim_{L\to 0}\left[S_3(\xi)-S_3(0)\right] = \left[\frac{2^{1-s}(s+1)}{\pi^{s+1}l^2 z_0}\right]^3\Psi(\xi) ,$$

where

$$\Psi(\xi) = \int_0^\infty \left[\bar{K}_s^3(\xi R)-\bar{K}_s^3(0)\right]R^{1-4s}\,\mathrm{d}R .$$

Using the formula

$$\frac{\mathrm{d}}{x\,\mathrm{d}x}\bar{K}_s(x) = -x^{2(s-1)}\bar{K}_{1-s}(x) ,$$

we get

$$\frac{\mathrm{d}\Psi(\xi)}{\xi\,\mathrm{d}\xi} = -3\xi^{4(s-1)}\int_0^\infty \bar{K}_{1-s}(x)\bar{K}_s^2(x)x^{1-2s}\,\mathrm{d}x ,$$

where

$$\int_0^\infty \bar{K}_{1-s}(x)\bar{K}_s^2(x)x^{1-2s}\,\mathrm{d}x = 2^{-1}\bar{K}_{1-s}^2(0)\bar{K}_s(0) - 2^{-1}\int_0^\infty \bar{K}_{1-s}^3(x)x^{2s-1}\,\mathrm{d}x = 2^{-s-2}\Gamma(s)\Gamma^2(1-s)-C_s$$

and the absolute value of C_s is bounded for all $s\leqslant 1$. Thus, for the function $\Psi(\xi)$ we have a first order differential equation with the boundary condition $\Psi(0)=0$. The solution of this equation for small $\varepsilon = 2(1-s)$ is

$$\Psi(\xi) \sim -3(4\varepsilon^2)^{-1}\xi^{2(1-\varepsilon)} .$$

From this we obtain

$$\lim_{L\to 0}\left[S_3(0)-S_3(\xi)\right] \sim 6\left(\pi^6 l^6 z_0^3\varepsilon^2\right)^{-1}\xi^{2(1-\varepsilon)}$$

and then take the last expression as the value of the integral in Eqn (4.49).

III. Consider the integral

$$S_n(\rho) = \int \tilde{\mathcal{A}}(\rho-\kappa)\left[\mathcal{K}_{\mu/2}^n(\kappa L)-1\right]\mathrm{d}^d\kappa , \qquad \text{(III.1)}$$

in which n is a positive integer. It is obvious that the diagrams (b), (c), (d_1), etc. in Fig. 7 correspond to $n=1$, $n=2$, $n=3$ etc. In terms of the new variables $p=L\rho$, $q=L\kappa$ the integral (III.1) can be written as follows

$$S_n(\rho) = Cz_0^{-1}l^{-2}u^{2\mu}L^{2(1-\mu)-d} \times \int |p-q|^{\mu-1}K_{1-\mu}(|p-q|)\left[\mathcal{K}_{\mu/2}^n(q)-1\right]\mathrm{d}^d q . \qquad \text{(III.2)}$$

Next, we use the expansion

$$|p-q|^{\mu-1}K_{1-\mu}(|p-q|) = \Gamma(1-\mu)\left(\frac{2}{pq}\right)^{1-\mu} \times \sum_{m=0}^\infty (1-\mu+m)C_m^{1-\mu}(\cos\theta)I_{1-\mu+m}(p)K_{1-\mu+m}(q),\ p<q ,$$

where $C_m^{1-\mu}(t)$ is the Gegenbauer polynomial, θ is the angle between the vectors corresponding to p and q, and where the variables p and q must be interchanged when $q < p$. Taking into account the last remark and integrating with respect to the angles in (III.2), we get

$$S_n(\rho) = Cz_0^{-1} l^{-2} u^{2\mu} L^{2(1-\mu)-d} p^{\mu-1} \times \sum_{m=0}^{\infty} (1-\mu+2m)\chi_{2m} \left[\alpha_m(p) K_{1-\mu+2m}(p) + \beta_m(p) I_{1-\mu+2m}(p)\right] , \quad \text{(III.3)}$$

where

$$\chi_{2m} = \frac{\Gamma(s+1)\Gamma(1-\mu+m)\Gamma(1-\mu-s+m)}{m!\,\Gamma(1+s+m)\Gamma(1-\mu)\Gamma(1-s-\mu)} ,$$

$$\chi_{2m+1} = 0 ,$$

as follows from the equation

$$\chi_k = \omega^{-1} \int C_k^{1-\mu}(\cos\theta)\,\mathrm{d}\omega = \frac{\Gamma(s+1)}{\sqrt{\pi}\Gamma\left(s+\frac{1}{2}\right)} \int_{-1}^{1} C_k^{1-\mu}(t)(1-t^2)^{s-1/2}\,\mathrm{d}t$$

satisfied for all $k \geqslant 0$ and from the relation

$$C_k^{1-\mu}(-t) = (-1)^k C_k^{1-\mu}(t) .$$

Finally,

$$\alpha_m(p) = \int_0^p \left[\mathcal{K}_{\mu/2}^n(q) - 1\right] I_{1-\mu+2m}(q) q^{2s+\mu}\,\mathrm{d}q , \quad \text{(III.4)}$$

$$\beta_m(p) = \int_p^\infty \left[\mathcal{K}_{\mu/2}^n(q) - 1\right] K_{1-\mu+2m}(q) q^{2s+\mu}\,\mathrm{d}q , \quad \text{(III.5)}$$

where $n > 1$, since the case $n = 1$ has already been considered. To determine the asymptotic form of the integrals in (III.4) and (III.5) when $p \sim 0$ we use the equations

$$\bar{I}_\lambda(p) \equiv p^{-\lambda} I_\lambda(p) = 2^{-\lambda}\Gamma^{-1}(\lambda+1) + \mathrm{O}(p^2) ,$$

$$\bar{K}_\lambda(p) \equiv p^{\lambda} K_\lambda(p) = 2^{\lambda-1}\Gamma(\lambda)\mathcal{K}_\lambda(p) ,$$

$$\mathcal{K}_{\mu/2}^n(p) - 1 = \left[\mathcal{K}_{\mu/2}(p) - 1\right]\Sigma_n(p) ,$$

where

$$\Sigma_n(p) = 1 + \mathcal{K}_{\mu/2}(p) + \mathcal{K}_{\mu/2}^2(p) + \ldots + \mathcal{K}_{\mu/2}^{n-1}(p) . \quad \text{(III.6)}$$

By the definition of $\mathcal{K}_{\mu/2}(p)$ in Eqn (4.10), this function can be represented as a power series in $\mu \ln p$ when $\mu \ll 1$. From Eqn (III.6) we then obtain

$$\Sigma_n(p) = \sum_{k \geqslant 0} \lambda_k^{(n)} (-\mu \ln p)^k + \mathrm{O}(p^2) , \quad \text{(III.7)}$$

where $\lambda_k^{(n)}$, $k \geqslant 0$ are fully defined coefficients of the above series.

Next, introduce the notation

$$\Pi_\mu(p) = \left[1 - \mathcal{K}_{\mu/2}(p)\right] p^{-\mu} ,$$

so that

$$\Pi_\mu(p) = C + \mathrm{O}(p^{2-\mu}), \quad p \to 0 .$$

From (III.4) we obtain

$$\alpha_m(p) = -\int_0^p \bar{I}_{1-\mu+2m}(q)\Pi_\mu(q)\Sigma_n(q) q^{1+2s+\mu+2m}\,\mathrm{d}q = -Cp^{4-2\mu+2m}\left[1 + \mathrm{O}(p^\mu)\right] . \quad \text{(III.8)}$$

Now let us write expression (III.5) in the form

$$\beta_m(p) = -\int_p^\infty \bar{K}_{1-\mu+2m}(q)\Pi_\mu(q)\Sigma_n(q) q^{1-2m}\,\mathrm{d}q .$$

It follows that

$$\beta_0(p) = -\int_0^\infty \bar{K}_{1-\mu}(q)\Pi_\mu(q)\Sigma_n(q) q\,\mathrm{d}q + \mathrm{O}(p^2) . \quad \text{(III.9)}$$

The expression for the next coefficient is

$$\beta_1(p) = -\left\{\int_1^\infty + \int_p^1\right\} \bar{K}_{3-\mu}(q)\Pi_\mu(q)\Sigma_n(q) q^{-1}\,\mathrm{d}q .$$

Here, in the integral form from p to 1 we replace the function $\Sigma_n(q)$ by its expansion (III.7) to obtain

$$\int_p^1 \bar{K}_{3-\mu}(q)\Pi_\mu(q)\Sigma_n(q) q^{-1}\,\mathrm{d}q = \sum_{k \geqslant 0} \lambda_k^{(n)} (-\mu)^k \int_p^1 \bar{K}_{3-\mu}(q)\Pi_\mu(q)(\ln q)^k q^{-1}\,\mathrm{d}q + \mathrm{O}(1) .$$

Since

$$\int_p^1 \bar{K}_{3-\mu}(q)\Pi_\mu(q)(\ln q)^k q^{-1}\,\mathrm{d}q = -(k+1)^{-1}\bar{K}_{3-\mu}(p)\Pi_\mu(p)(\ln p)^{k+1} - (k+1)^{-1}\int_p^1 (\ln q)^{k+1}\,\mathrm{d}\left[\bar{K}_{3-\mu}(q)\Pi_\mu(q)\right] ,$$

where the last integral can be written as the difference of the integrals over the intervals $[0, 1]$ and $[0, p]$, we have

$$\int_p^1 \bar{K}_{3-\mu}(q)\Pi_\mu(q)(\ln q)^k q^{-1}\,\mathrm{d}q = -\frac{8}{k+1}\left[1 + \mathrm{O}(\mu)\right](\ln p)^{k+1} + \mathrm{O}(1) .$$

It follows that

$$\beta_1(p) = \ln p \sum_{k \geqslant 0} \Lambda_k^{(n)} (-\ln p)^k + \mathrm{O}(1) , \quad \text{(III.10)}$$

where $\Lambda_k^{(n)}, k \geqslant 0$ are coefficients of the series under consideration. Estimating the other coefficients, i.e. $\beta_m(p)$ for $m \geqslant 2$, we get

$$\beta_m(p) = -C\left[1 + \mathrm{O}(p^\mu)\right] p^{2-2m} , \quad \text{(III.11)}$$

where the constant C depends on $m \geqslant 2$ and $n \geqslant 1$. Let us now substitute expressions (III.8)–(III.11) into formula (III.3) and take into account that all the coefficients $\chi_{2m}, m \geqslant 1$, contain the factor μ. As a result we obtain

$$S_n(0) - S_n(\rho) = Cz_0^{-1} l^{-2} u^{2\mu} L^{2(1-\mu)-d} \times (\rho L)^2 \left\{\sum_{k \geqslant 0} \Lambda_k^{(n)} \left[-\mu \ln(\rho L)\right]^{k+1} + \mathrm{O}(1)\right\} . \quad \text{(III.12)}$$

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